LIMIT THEOREMS FOR CUMULATIVE PROCESSES

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Peter W. Glynn and Ward Whitt

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by

*Peter W. Glynn**

Department of Operations Research
Stanford University
Stanford, CA 94305-4022

and

Ward Whitt

AT&T Bell Laboratories
Murray Hill, NJ 07974-0636

Abstract

Necessary and sufficient conditions are established for cumulative processes (associated with regenerative processes) to obey several classical limit theorems; e.g., a strong law of large numbers, a law of the iterated logarithm and a functional central limit theorem. The key random variables are the integral of the regenerative process over one cycle and the supremum of the absolute value of this integral over all possible initial segments of a cycle. The tail behavior of the distribution of the second random variable determines whether the cumulative process obeys the same limit theorem as the partial sums of the cycle integrals. Interesting open problems are the necessary conditions for the weak law of large numbers and the ordinary central limit theorem.

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1. Introduction

In this paper we establish necessary and sufficient (N&S) conditions for several limits to hold for appropriately normalized cumulative processes (associated with regenerative processes), with the emphasis being on the necessity. The limits we have in mind are the limits in the strong law of large numbers (SLLN), the law of the iterated logarithm (LIL), the weak law of large numbers (WLLN), the central limit theorem (CLT) and functional generalizations of these, denoted by FSLLN and so forth; we define the versions we consider precisely in §2. The topic of this paper is very close to classic results, e.g., see Gnedenko and Kolmogorov (1968). Hence, there is considerable related literature. In particular, our paper extends Smith (1955), Chung (1967), Iglehart (1971), Brown and Ross (1972), Serfozo (1972, 1975), Whitt (1972), Glynn and Whitt (1987, 1988a,b) and Asmussen (1987).

We use the "classical" definition of regenerative process throughout; i.e., the process splits into i.i.d. cycles; cf. p. 125 of Asmussen (1987). For the necessity results, this is without loss of generality. Let $0 \le T(0) < T(1) < \ldots$ denote the regeneration times, with $T(-1) = 0$. Consider a stochastic process $\{X(t) : t \ge 0\}$ with general state space and a measurable real-valued function f. We assume that the process $\{X(t) : t \ge 0\}$ is regenerative with respect to these regeneration times, and we focus on the associated cumulative process $C \equiv \{C(t) : t \ge 0\}$, defined by

$$C(t) = \int_0^t f(X(s))\,ds\ , \ t \ge 0\ . \tag{1.1}$$

The key random variables associated with the cycles are

$$\begin{aligned} \tau_i &= T(i) - T(i-1)\ , \\ Y_i(f) &\equiv \int_{T(i-1)}^{T(i)} f(X(s))\,ds\ , \\ W_i(f) &\equiv \sup_{0 \le s \le \tau_i} \left| \int_0^s f(X(T(i-1)+u))\,du \right|\ . \end{aligned} \tag{1.2}$$

By "regenerative structure," we mean that for any suitable f the three-tuples $(\tau_i, Y_i(f), W_i(f))$ are i.i.d. for $i \geq 1$. We also assume throughout that $E\tau_1 < \infty$. In addition, we assume throughout that

$$\int_0^t |f(X(s))|\, ds < \infty \quad \text{w.p.1 for each } t\,, \tag{1.3}$$

which implies that the cumulative process C has continuous sample paths w.p.1.

We shall consider the given function f and a centered function f_c defined by $f_c(x) = f(x) - \alpha$ for a constant α, both of which are assumed to satisfy (1.3). When we write Y_1 or W_1 we understand the function f to be the given one.

We are interested in N&S conditions for the cumulative process to obey the classical limit theorems. For this purpose, it is natural to represent the cumulative process as a random sum of i.i.d. summands plus two remainder terms. In particular,

$$C(t) \equiv \int_0^t f(X(s))\, ds = S_{N(t)} + R_1(t) + R_2(t)\,,\ t \geq 0\,, \tag{1.4}$$

where

$$S_n = Y_1 + \ldots + Y_n\,,\ n \geq 1\,, \tag{1.5}$$

with $S_0 = 0$, $N \equiv \{N(t) : t \geq 0\}$ is the (possibly delayed) renewal counting process associated with the regeneration times, i.e.,

$$N(t) = \max\{i : T(i) \leq t\}\,,\ t \geq 0\,, \tag{1.6}$$

and $R_i \equiv \{R_i(t) : t \geq 0\}$ are the remainder processes, defined by

$$R_1(t) = \int_0^{\min\{t, T_0\}} f(X(s))\, ds \quad \text{and} \quad R_2(t) = \int_{T(N(t))}^t f(X(s))\, ds\,,\ t \geq 0\,. \tag{1.7}$$

Since $E\tau_1 < \infty$, we have

$$t^{-1} N(t) \to \lambda \equiv 1/E\tau_1 \text{ as } t \to \infty \text{ w.p.1}\,, \tag{1.8}$$

which we will exploit frequently. Since $|R_1(t)| \le W_0$, we see that the first remainder term $R_1(t)$ in (1.7) is trivially dispensed with in limit theorems since it is bounded by a random variable that does not depend on t. A significant part of the analysis is finding what knocks out the second remainder term $R_2(t)$ in (1.7). Of course, the key relation here is

$$|R_2(t)| \le W_{N(t)} \; , \; t \ge 0 \; . \tag{1.9}$$

From (1.9) it is evident that we could just as well impose conditions on the supremum over the integral from s to the end of the cycle instead of on $W_1(f)$. (This is to be expected since our definition of regenerative process is time reversible.)

Given (1.4), it is interesting to compare N&S conditions for limit theorems for the cumulative process $C(t)$ with N&S conditions in the corresponding limit theorem for the random sums $S_{N(t)}$. In turn it is interesting to compare the N&S conditions in the limit theorems for the random sums $S_{N(t)}$ with the N&S conditions in the corresponding limit theorem for the ordinary partial sums S_n in (1.5). We state our main result in §3 so as to make these connections clear.

Here is how the rest of the paper is organized. In §2 we specify precisely what we mean by the classical limit theorems. (It is important to note that there are several possible definitions.) After we state the main results in §3, we establish some supporting propositions in §4. We establish N&S conditions for the WLLN and a joint CLT for C and N in the case f is nonnegative in §5. We then prove the main result in §6.

2. The Classical Limit Theorems

In this section we indicate precisely what we mean by the classical limit theorems. For this purpose, consider a general stochastic process $Z \equiv \{Z(t) : t \ge 0\}$ with real-valued sample paths having limits from the left and right. (Note that we consider only one process rather than a sequence of processes.) By (1.3), the cumulative process C actually has continuous sample paths, but the random sums $S_N(t)$ and the partial sums S_n do not. Discrete-time processes can be regarded as the special case in which $Z(t) = Z([t])$, where $[t]$ is the greatest integer less than or equal to t.

We say that Z obeys a SLLN if there exists a constant α such that $t^{-1}Z(t) \to \alpha$ as $t \to \infty$ w.p.1. We say that Z obeys a FSLLN if there exists a constant α such that, for each T with $0 < T < \infty$,

$$\sup_{0 \le t \le T} |n^{-1}Z(nt) - \alpha t| \to 0 \quad \text{as } n \to \infty \text{ w.p.1} . \tag{2.1}$$

As in Theorem 4 of Glynn and Whitt (1988), such a FSLLN is actually equivalent to the ordinary SLLN above, so we do not discuss it further. (To verify this, we use the existence of left and right limits to conclude that $\sup_{0 \le s \le t} |Z(s)| < \infty$ w.p.1 for all t; e.g., see p. 110 of Billingsley (1968).)

We say that Z obeys an LIL if there exist constants α and β such that $\beta \ge 0$ and

$$[Z(t) - \alpha t]/\sqrt{2tL_2t} \rightsquigarrow [-\sqrt{\beta}, \sqrt{\beta}] \text{ w.p.1} , \tag{2.2}$$

where $Lx = \max\{1, \log_e x\}$, $L_k x = L_{k-1}(Lx)$ and $\rightsquigarrow$ denotes that the set on the left is relatively compact with the set on the right being the set of all limit points of convergent subsequences (with $t_k \to \infty$ as $k \to \infty$).

For the FLIL and FCLT we work in the function space $D[0, \infty)$ with the usual Skorohod (J_1) topology, see Billingsley (1968), Whitt (1980) and Ethier and Kurtz (1986). Following Strassen (1964), we say that Z obeys a FLIL if there exist constants α and β with $\beta \ge 0$ and a compact set C in $D[0, \infty)$ such that

$$[Z(nt) - \alpha nt]/\sqrt{2nL_2n} \rightsquigarrow \sqrt{\beta}\, C \text{ w.p.1} \tag{2.3}$$

where convergence of a subsequence is understood to be in $D[0, \infty)$ and the limit set C is the set of all functions $\{x(t) : t \geq 0\}$ that are absolutely continuous with respect to Lebesgue measure with derivative $x'(t)$ satisfying $\int_0^\infty x'(t)^2 dt \leq 1$. (This is the standard limit set associated with partial sums of i.i.d. random variables.)

We say that Z obeys a WLLN if there exists a constant α such that $t^{-1} Z(t) \Rightarrow \alpha$ as $t \to \infty$, where $\Rightarrow$ denotes convergence in law, which coincides with convergence in probability in this case because α is deterministic. We say that Z obeys a FWLLN if there exists a constant α such that

$$[Z(nt) - \alpha nt]/n \Rightarrow 0 \quad \text{in } D[0,\infty) \text{ as } n \to \infty . \tag{2.4}$$

We say that Z obeys a CLT if there exist constants α and β with $\beta \geq 0$ such that

$$[Z(t) - \alpha t]/\sqrt{t} \Rightarrow \sqrt{\beta}\, N(0,1) \text{ as } t \to \infty , \tag{2.5}$$

where $N(0,1)$ denotes a standard (mean 0, variance 1) normal random variable. We say that Z obeys a FCLT if there exist constants α and β with $\beta \geq 0$ such that

$$[Z(nt) - \alpha nt]/\sqrt{n} \Rightarrow \sqrt{\beta}\, B(t) \text{ in } D[0,\infty) \text{ as } n \to \infty , \tag{2.6}$$

where $B(t)$ is standard (drift 0, diffusion coefficient 1) Brownian motion.

It is significant that in all the limit theorem above we have stipulated fixed normalization constants. We always translate $Z(t)$ by αt. In the LLN, LIL and CLTs we always divide $Z(t) - \alpha t$ by t, $\sqrt{2tL_2 t}$ and $\sqrt{t}$, respectively. Moreover in the CLT we have specified that the limit be standard normal. For partial sums of i.i.d. random variables, these assumptions are known to significantly restrict the range of possibilities; e.g., see Gnedenko and Kolmogorov (1968). For example, for partial sums of i.i.d. random variables, the CLT involves the domain of normal attraction of the normal law, for which the N&S condition is for the underlying distribution to have finite second moment; see p. 181 of Gnedenko and Kolmogorov (1968).

3. The Main Result

In this section we state, wherever possible, N&S conditions for the three processes S_n, $S_{N(t)}$ and $C(t)$ defined in (1.1), and (1.4)–(1.6) to obey the seven limit theorems: SLLN, LIL, FLIL, WLLN, FWLLN, CLT and FCLT. (We have indicated that the FSLLN is equivalent to the SLLN in the setting of §2.)

To relate the limit theorems for the partial sums to the random sums and cumulative processes, we assume that the summands Y_i are of the form $Y_i(f_c)$ for an appropriate centering constant α. When $E|Y_1(f_c)| < \infty$, the parameter α will be chosen so that $E\ Y_1(f_c) = 0$.

We prove the following in §6. More results in the case f is nonnegative appear in §5.

Theorem 3.1. *(a) For the WLLN and CLT, the N&S conditions for the random sums $S_{N(t)}$ and the cumulative process $C(t)$ are the same. For all other theorems, the N&S conditions for the partial sums S_n and the random sums $S_{N(t)}$ are the same.*

(b) The specific N&S conditions for the partial sums S_n and the cumulative process $C(t)$ are given in Table 1, with a question mark indicating that the answer is unknown. Each established N&S condition for the cumulative process is the N&S condition for the partial sum plus the indicated extra condition.

(c) For the WLLN and CLT, the N&S condition for the partial sums S_n is sufficient for the random sums $S_{N(t)}$ and the cumulative process $C(t)$. Moreover, these conditions are necessary in the sense that there are examples for which the random sum and cumulative process limits do not exist when these conditions are violated. (See Remark 3.2 below.)

(d) For the WLLN and the FWLLN, the centering constant α is necessarily the limit of $E[Y_1; |Y_1| \le t]$ as $t \to \infty$. In all other cases it is necessarily EY_1, which is consequently finite.

(e) The normalizing constant β in the LIL, FLIL, CLT and FCLT must always be the variance

$\mathrm{Var}(Y_1)$, *which is necessarily finite for those limits.*

Remark (3.1) We conjecture that the N&S conditions for the partial sums S_n in the WLLN and CLT are also N&S conditions for the random sums $S_{N(t)}$ and the cumulative process $C(t)$. This would follow if the WLLN and the CLT in (2.5) for $S_{N(t)}$ were equivalent to the FWLLN in (2.4) and the FCLT in (2.6), respectively, for $S_{N(t)}$, which we also conjecture to be true.

(3.2) The partial necessity result in part (c) of Theorem 1 is easily explained as follows: For any distribution of Y_1, we can construct a regenerative process such that $N(t) = [t]$, $C([t]) = S_{N(t)} = S_{[t]}$, and

$$C(t) = (1 - t + [t])C[t] + (t - [t])C([t] + 1)\ , \ t \geq 0\ , \tag{3.1}$$

where $[t]$ is the greatest integer less than or equal to t; in particular, just let

$$X(t) = Y_{[t]}\ , \ t \geq 0\ . \tag{3.2}$$

Hence, for the WLLN and CLT, the cumulative process $C(t)$ and the random sum $S_{N(t)}$ are equivalent to the partial sum $S_{[t]}$. For such examples, the N&S condition for the partial sums also obviously is the N&S condition for $S_{N(t)}$ and $C(t)$.

(3.3) The SLLN result is due to Smith (1955); see Theorem 3.1 on p. 136 of Asmussen (1987). The standard sufficient condition for the CLT is $\mathrm{Var}\ Y_1(f) < \infty$ and $\mathrm{Var}\ \tau_1 < \infty$, see Theorem 3.2 on p. 136 of Asmussen (1987), which is stronger than our sufficient condition, because we do not require that $\mathrm{Var}\ \tau_1 < \infty$; see Proposition 2 below. To see that we could have $\mathrm{Var}\ \tau_1 = \infty$, suppose that $Y_i(f) = \tau_i + U_i$ where $\mathrm{Var}\ U_i < \infty$. Then $Y_i(f_c) = U_i$ and $\mathrm{Var}\ Y_1(f_c) < \infty$ for $\alpha = 1$.

(3.4) The sufficient condition for the WLLN is weaker than $E|Y_1| < \infty$. Since $E|Y_1| = \int_0^\infty P(|Y_1| > t)\,dt$, $E|Y_1| < \infty$ implies that $tP(|Y_1| > t) \to 0$ as $t \to \infty$ For example, if Y_1 has a symmetric distribution with $P(Y_1 > t) = A/t(\log t)^p$ for $p \leq 1$, then the conditions hold with $E|Y_1| = \infty$.

(3.5) To see that the established conditions on $W_1(f_c)$ are needed in addition to the conditions on Y_1, consider the following example. Let $P(\tau_1 = 2) = 1$ and let $f(t) = Z_k$ for $2k-2 \le t < 2k-1$ and $f(t) = -Z_k$ for $2k-1 \le t < 2k$, where $\{Z_k : k \ge 1\}$ is a sequence of i.i.d. random variables. Then $P(Y_1 = 0) = P(S_n = 0$ for all $n) = 1$, while $C(2k-1) = Z_k = W_k$. Then apply Propositions 5-8 below.

4. Supporting Propositions

In this section we present several basic propositions that help establish and interpret Theorem 1. The first four propositions show how the conditions on $Y_1(f_c) \equiv Y_1(f-\alpha) \equiv Y_1(f) - \alpha\tau_1$ in Table 1 relate to conditions on $Y_1(f)$, τ_1 and α.

Proposition 1. *If $E|Y_1(f_c)| < \infty$ holds for some α, then it holds for all α, in which case $E\,Y_1(f_c) = EY_1(f) - \alpha E\tau_1$.*

Proof. Note that

$$\begin{aligned} E|Y_1(f) - \alpha_1\tau_1| &= E|Y_1(f) - \alpha_2\tau + (\alpha_2 - \alpha_1)\tau_1| \\ &\le E|Y_1(f) - \alpha_2\tau| + |\alpha_2 - \alpha_1|E\tau_1 \end{aligned}$$

and recall that $E\tau_1 < \infty$. ■

Proposition 2. *A sufficient (but not necessary) condition for $E|Y_1(f_c)|^p < \infty$ for $p > 1$ is to have $E|Y_1(f)|^p < \infty$ and $E\tau_1^p < \infty$.*

Proof. By Minkowski's inequality, p. 47 of Chung (1974),

$$\begin{aligned} (E|Y_1(f_c)|^p)^{1/p} &= (E|Y_1(f) - \alpha\tau_1|^p)^{1/p} \\ &\le (E|Y_1(f)|^p)^{1/p} + \alpha(E\tau_1^p)^{1/p} . \end{aligned}$$

To see that the condition is not necessary, suppose that $Y_1(f) = \alpha\tau_1$. ■

Proposition 3. *A WLLN holds for the partial sums of $Y_i(f_c)$ for one α if and only if it does for all α. Moreover, the limit is γ for $Y_i(f - \alpha_1)$, if and only if it is $\gamma - (\alpha_2 - \alpha_1)\lambda^{-1}$ for*

$Y_i(f - \alpha_2)$.

Proof. Suppose that the WLLN holds for the partial sums of $Y_i(f - \alpha_1)$. Since $E\tau_1 < \infty$, the τ_i obey a WLLN too. By Theorem 4.4 of Billingsley (1968),

$$n^{-1}\left(\sum_{i=1}^{n}(Y_i(f) - \alpha_1\tau_i),\ \sum_{i=1}^{n}\tau_i\right) \Rightarrow (\gamma, \lambda^{-1}) \text{ as } n \to \infty ,$$

so that by the continuous mapping theorem with the function $h(x, y) = x - (\alpha_2 - \alpha_1)y$, we obtain

$$n^{-1}\sum_{i=1}^{n}(Y_i(f) - \alpha_2\tau_i) \Rightarrow \gamma - (\alpha_2 - \alpha_1)\lambda^{-1} \text{ as } n \to \infty . \quad \blacksquare$$

As a corollary to Proposition 3, we obtain the following property of the N&S conditions in the WLLN for the partial sums. A direct proof is also possible.

Proposition 4. *(a) If*

$$t\,P(|Y_1(f_c)| > t) \to 0 \text{ as } t \to \infty$$

for some α, then it holds for all α.

(b) If

$$E[Y_1(f) - \alpha_1\tau_1\ ;\ |Y_1(f) - \alpha_1\tau_1| \le t] \to \gamma \text{ as } t \to \infty ,$$

then

$$E[Y_1(f) - \alpha_2\tau_1\ ;\ |Y_1(f) - \alpha_2\tau_1| \le t] \to \gamma - (\alpha_2 - \alpha_1)\lambda^{-1} \text{ as } t \to \infty .$$

Proof. We use Proposition 3 plus the fact that the conditions in Proposition 4 are known to be N&S for the WLLN for partial sums of i.i.d. random variables; see p. 235 of Feller (1971). ■

The conditions on $W_1(f_c)$ in Table 1 can be established, explained and applied via the following propositions.

Proposition 5. *Let $\{Z_i : i \ge 1\}$ be a sequence of i.i.d. random variables and let $\phi(t)$ be a* *deterministic function of t such that* $\phi(t) \to \infty$ *as* $t \to \infty$. *Then*

$$\phi(n)^{-1} \max_{1 \le i \le n} \{|Z_i|\} \Rightarrow 0 \quad \text{as } n \to \infty \tag{4.1}$$

if and only if

$$t\, P(|Z_1| > \phi(t)) \to 0 \quad \text{as } t \to \infty\ . \tag{4.2}$$

Proof. Note that

$$P(\max_{1 \le i \le n} \{|Z_i|\} > \varepsilon\phi(n)) = 1 - F(\varepsilon\phi(n))^n\ ,$$

where $F(x) = P(|Z_1| < x)$, so that (4.1) holds if and only if, for each $\varepsilon > 0$, $F(\varepsilon\phi(n))^n \to 1$ or, equivalently,

$$n \log(1 - F^c(\varepsilon\phi(n)) \to 0 \text{ as } n \to \infty\ , \tag{4.3}$$

where $F^c(x) = 1 - F(x)$. Since $F^c(\varepsilon\phi(n)) \to 0$ as $n \to \infty$, we can apply Taylor's theorem to obtain

$$\log(1 - F^c(\varepsilon\phi(n))) = -F^c(\varepsilon\phi(n)) + O(F^c(\varepsilon\phi(n))^2)\ .$$

Thus, (4.3) holds if and only if $nF^c(\varepsilon\phi(n)) \to 0$ as $n \to \infty$. ∎

The following is a consequence of the Borel-Cantelli lemma; see Theorems 4.2.2 and 4.2.4 of Chung (1974).

Proposition 6. *Let* $\{Z_i : i \ge 1\}$ *be a sequence of i.i.d. random variables and let* a_n *be constants such that* $a_n \to \infty$ *as* $n \to \infty$. *Then the following are equivalent:*

(i) $Z_n/a_n \to 0$ *w.p.1 as* $n \to \infty$

(ii) $\max_{1 \le k \le n} \{\text{[illegible]}\}/a_n \to 0$ *w.p.1 as* $n \to \infty$

(iii) $\sum_{n=1}^{\infty} P(|Z_1| > a_n) < \infty$.

If these properties do not hold, then $\overline{\lim_{n \to \infty}} \{Z_n/a_n\} = \infty$ *w.p.1.*

As a consequence of Proposition 6, we have

Proposition 7. *In the setting of Proposition 6, if $a_n = n$, then a further equivalent property is* $E|Z_1| < \infty$.

Proof. As in Theorem 3.2.1 of Chung (1974),

$$\sum_{n=1}^{\infty} P(|Z_1| \geq n) \leq E|Z_1| \leq 1 + \sum_{n=1}^{\infty} P(|Z_1| \geq n) . \quad \blacksquare$$

Proposition 8. *Let c be a constant,* $0 < c < 1$. *For any positive random variable Z,*

$$P(Z^2 > nL_2 n) \leq P\left(\frac{Z^2}{L_2 Z} > n\right) \leq P(Z^2 > cnL_2 n)$$

for all sufficiently large n, so that

$$\sum_{n=1}^{\infty} P(Z > \sqrt{n\, L_2 n}\,) < \infty$$

if and only if

$$\sum_{n=1}^{\infty} P\left(\frac{Z^2}{L_2 Z} > n\right) < \infty ,$$

Proof. If $(Z^2/L_2 Z) > n$, then $Z^2 > n$, so that $2LZ > Ln$, $L_2 Z > L_2 n - L2$ and

$$Z^2 > nL_2 Z > n(L_2 n - L2) > cnL_2 n$$

for all suitably large n. Next, note that $g(x) = x^2/L_2 x$ is increasing for all x large. Hence, if $Z^2 > nL_2 n$, then

$$g(Z) = \frac{Z^2}{L_2 Z} > g(\sqrt{nL_2 n}\,) = \frac{nL_2 n}{L_2\sqrt{nL_2 n}} = \frac{nL_2 n}{L\frac{1}{2}(Ln + L_3 n)} > \frac{nL_2 n}{L_2 n} = n$$

for all suitably large n. Hence, we have the desired inequalities. $\blacksquare$

We now show that the second remainder term $R_2(t)$ in (1.7) is asymptotically negligible in the setting of the WLLN and CLT, because $E\tau_1 < \infty$. The asymptotic negligibility follows from convergence without further normalization, for which we must distinguish between the lattice and nonlattice cases. Recall that the distribution of τ is *lattice* if $\sum_{h=0}^{\infty} P(\tau = k\delta) = 1$ for some δ, with the largest such δ being the period; otherwise it is *nonlattice*.

Proposition 9. *(a) If τ has a nonlattice distribution, then $R_2(t) \Rightarrow R_2(\infty)$ as $t \to \infty$, where $R_2(t)$ is in (1.7) and $R_2(\infty)$ is a proper random variable with distribution function*

$$P(R_2(\infty) \le x) = \lambda \int_0^{\infty} P(R_2(t) \le x \,;\, \tau_1 > t)\,dt \,. \tag{4.4}$$

(b) If τ has a lattice distribution with period δ, then $R_2(k\delta + y) \Rightarrow R_y(\infty)$ as $k \to \infty$ for each y, $0 \le y < \delta$, and $\sup_{0 \le y < \delta} \{|R_2(k\delta + y)|\} \Rightarrow R'(\infty)$ as $k \to \infty$, where $R_y(\infty)$ and $R'(\infty)$ are all proper random variables.

Proof. (a) We apply the key renewal theorem; see p. 120 of Asmussen (1987). For this purpose, let g be a continuous nonnegative real-valued function of a real variable with $g(t) \le M$ for all t. Note that $E[g(R_2(t))]$ satisfies a renewal equation, i.e.,

$$E[g(R_2(t))] = E[g(R_2(t)) 1_{\{\tau_1 > t\}}] + \int_0^t E[g(R_2(t-u))] P(\tau_1 \in du) \,, \tag{4.5}$$

Where 1_A is the indicator function of the set A. Let $z(t) \equiv E[g(R_2(t)) 1_{\{\tau_1 > t\}}]$. We now show that z is directly Riemann integrable, so that we can apply the key renewal theorem. For this purpose, we apply Proposition 4.1(ii) on p. 119 of Asmussen (1987). Since $z(t) \le M$, the function z is bounded. Moreover, $b(t) \equiv g(R_2(t)) 1_{\{\tau_1 > t\}}$ as a function of t has a single discontinuity at τ_1 for each sample path. Hence, the function b is continuous w.p.1 at all points t for which $P(\tau_1 = t) = 0$. By the bounded convergence theorem, $z(t) \equiv Eb(t)$ is thus continuous at all t for which $P(\tau_1 = t) = 0$. Since $P(\tau_1 = t) = 0$ for all but countably many t, z is continuous almost everywhere with respect to Lebesgue measure. Next, let

$$\bar{z}_h(t) = \sup\{z(y) : kh \le y \le (k+1)h\} \text{ for } kh \le t < (k+1)h \tag{4.6}$$

as on p. 118 of Asmussen (1987). Since $z(t) \le P(\tau_1 > t)$,

$$\int_0^\infty \bar{z}_h(t)\,dt \le \sum_{k=0}^{\infty} P(\tau_1 > h) < \infty$$

by Proposition 7 above. Hence, we have shown that z is indeed directly Riemann integrable. The key renewal theorem thus implies that $Eg(R_2(t)) \to \lambda \int_0^\infty z(u)\,du$ as $t \to \infty$. However, all bounded continuous nonnegative functions g determine convergence, so indeed $R_2(t) \Rightarrow R_2(\infty)$ as $t \to \infty$. Moreover, we can characterize the limiting distribution using these functions g, so that (4.4) holds.

(b) The argument is essentially the same; we apply discrete-time renewal theory along subsequences; see pages 8 and 121 of Asmussen (1987). Note that we have the renewal equations

$$P(R_2(k\delta + y) > x) = P(R_2(k\delta + y) > x\,,\ \tau_1 > k) + \sum_{j=0}^{k} P(R_2((k-j)\delta + y)P(\tau_1 = j)$$

and

$$P(\sup_{0 \le y < \delta}\{|R_2(k\delta + y)|\} > x) = P(\sup_{0 \le y < \delta}\{|R_2(k\delta + y)|\} > x\,,\ \tau_1 > k)$$
$$+ \sum_{j=0}^{k} P(\sup_{0 \le y < \delta}\{|R_2((k-j)\delta + y)|\} > x)P(\tau_1 = j)\,,$$

where

$$P(R(k\delta + y) > x\,,\ \tau_1 > k) \le P(\tau_1 > k)$$

and

$$P(\sup_{0 \le y < \delta}\{|R(k\delta + y)|\} > x\,,\ \tau_1 > k) \le P(\tau_1 > k)$$

with $\sum_{k=0}^{\infty} P(\tau_1 > k) < \infty$ since $E\tau_1 < \infty$. ■

Under our i.i.d. conditions, functional versions of the WLLN and CLT for the partial sums are equivalent to the ordinary versions. For this purpose, we can apply Theorem 2.7 of Skorohod (1957), which we now quote.

Proposition 10. *(Skorohod (1957)) Let* $\{U_{ni} : i \geq 1\}$ *be i.i.d. for each* n *and let* $Z_n(t) = \sum_{i=1}^{[nt]} U_{ni}$, $t \geq 0$. *Then*

$$Z_n(t) \Rightarrow Z(t) \text{ as } n \to \infty \quad \text{in } D[0, \infty) ,$$

where Z *has stationary independent increments, if and only if* $Z_n(t) \Rightarrow Z(t)$ *as* $n \to \infty$ *in* $\mathbb{R}$ *for each* t.

5. A Joint Central Limit Theorem

In this section we consider a joint CLT for the cumulative process C and the counting process N. We obtain a necessity result in the case $E|Y_1| < \infty$, which holds when f is nonnegative; necessity in the general case remains open.

Remark (5.1) Even without the i.i.d. conditions, limits for the counting process N alone hold if and only if the corresponding limit holds for the associated partial sums; see §7 of Whitt (1980), Theorems 3 and 6 of Glynn and Whitt (1988a) and Theorem 1 of Glynn and Whitt (1988b). For example, as a consequence, N satisfies a CLT if and only if $E\tau_1^2 < \infty$. For this we apply Theorem 6 of Glynn and Whitt (1988) and Theorem 4, p. 181, of Gnedenko and Kolmogorov (1968). ■

We start with a necessary condition for the WLLN when f is nonnegative.

Theorem 2. *Suppose that* f *is nonnegative. Then a N&S condition for the WLLNs for* S_n, $S_{N(t)}$ *and* $C(t)$ *is* $E|Y_1| < \infty$.

Proof. If f is nonnegative, then $Y_1 = W_1$, so that $E|Y_1| < \infty$ is sufficient for the three SLLNs by Theorem 1. If f is nonnegative, then the N&S condition for the WLLN for S_n in Table 1 is equivalent to $E|Y_1| < \infty$. By Proposition 9, the WLLNs for $S_{N(t)}$ and $C(t)$ are equivalent. Hence, suppose that $C(t)$ obeys a WLLN; i.e.,

$$t^{-1} \int_0^t f(X(s))\,ds \Rightarrow \alpha \text{ as } t \to \infty . \tag{5.1}$$

On the other hand, suppose that $E|Y_1| = \infty$. Since we have $f \geq 0$, we can apply the SLLN to conclude that

$$n^{-1} \sum_{i=1}^{n} Y_i(f) \to \infty \text{ w.p.1 as } n \to \infty \tag{5.2}$$

(see Exercise 1 on p. 130, of Chung (1974)), from which we can deduce from the SLLN proof in Theorem 1 that

$$t^{-1} \int_0^t f(X(s))\,ds \to \infty \quad \text{w.p.1 as } t \to \infty . \tag{5.3}$$

(Recall that $W_1 = Y_1$ when $f \geq 0$.) Hence, (5.1) cannot hold and we must have $E|Y_1(f)| < \infty$. ■

Remark 5.2 An alternative approach to Theorem 5.2 (pointed out by A. Pukholskii) is to note that the WLLN for $C(t)$ implies the FWLLN because the sample paths are nondecreasing. This argument also depends critically on f being nonnegative.

We now state N&S conditions for the joint CLT.

Theorem 3. If $E[\tau_1^2] < \infty$ and $E[Y_1(f)^2] < \infty$, then $(C(t), N(t))$ obeys a joint CLT, i.e.,

$$t^{-1/2}(C(t) - \alpha t\, ,\, N(t) - \lambda t) \Rightarrow N(0, \Sigma) \; as \; t \to \infty \; in \; \mathbb{R}^2 \tag{5.4}$$

where $\lambda = 1/E\tau_1$, $\alpha = \lambda E Y_1(f)$ and $N(0, \Sigma)$ is a bivariate normal distribution with covariance matrix elements $\Sigma_{11} = \lambda E[Y_1(f_c)^2]$, $\Sigma_{22} = \lambda^3 \operatorname{Var} \tau_1$ and $\Sigma_{12} = \lambda^2 E[Y_1(f_c)\tau]$.

(b) If $E|Y_1| < \infty$, then the joint CLT (5.4) implies that $E[Y_1(f)^2] < \infty$ and $E[\tau_1^2] < \infty$.

Proof. (a) the sufficiency is a minor extension of Theorem 1 of Glynn and Whitt (1987). First, by the multivariate version of Donsker's theorem, the conditions imply a joint FCLT for the partial sums of $(Y_i(f), \tau_i)$, i.e.,

$$n^{-1/2}\left(\sum_{i=1}^{[nt]}(Y_i(f) - E[Y_1(f)) \,,\, \sum_{i=1}^{[nt]}(\tau_i - E\tau_1)\right) \Rightarrow (B_1(t), B_2(t)) \tag{5.5}$$

in $D[0,\infty)\times D[0,\infty)$, where (B_1, B_2) is Brownian motion. Then by continuous mapping arguments we obtain a joint FCLT for $S_{N(t)}$ and $N(t)$, i.e.,

$$n^{-1/2}\left(\sum_{i=1}^{N(nt)}(Y_i(f) - EY_i(f)) \,,\, N(nt) - \lambda nt\right) \Rightarrow (\bar{B}_1(t), \bar{B}_2(t)) \tag{5.6}$$

as $n \to \infty$ in $D[0,\infty)\times D[0,\infty)$. The joint FCLT (5.6) is obtained in Glynn and Whitt (1987). It in turn implies an ordinary CLT by applying the continuous mapping theorem with the projection. This ordinary CLT is

$$t^{-1/2}\left(\sum_{i=1}^{N(t)}(Y_i(f) - EY_1(f)) \,,\, N(t) - \lambda t\right) \Rightarrow N(0,\Sigma) \tag{5.7}$$

for the stated Σ. The result (5.7) then is equivalent to (5.4) because the remainder terms are asymptotically negligible; i.e., by Proposition 9, $t^{-1/2}R_2(t) \Rightarrow 0$ as $t \to \infty$. Hence, (5.4) follows from (5.7) and the converging-together theorem, Theorem 4.1 of Billingsley (1968).

(b) Turning to the necessity, we reverse the argument and note that (5.4) implies (5.7), because the difference is asymptotically negligible, by virtue of Proposition 9. Now we apply Theorem 7(a) of Glynn and Whitt (1988a), for which we use the assumption that $E|Y_1| < \infty$. It implies that

$$t^{-1/2}\left(\sum_{k=1}^{N(t)} Y_k(f_c) - \sum_{k=1}^{[\lambda t]} Y_k(f_c)\right) \Rightarrow 0 \text{ as } t \to \infty\,, \tag{5.8}$$

which with the converging-together theorem, Theorem 4.1 of Billingsley (1968), implies that the partial sums of $Y_i(f_c)$ obey a CLT. As before, Theorem 4 on p. 181 of Gnedenko and Kolmogorov (1968) then implies that $E[Y_1(f_c)^2] < \infty$. By Remark 5.1, the CLT for N implies that $E[\tau_1^2] < \infty$. ■

6. Proof of Theorem 1

(a) *SLLN*

The condition $E|Y_1| < \infty$ is well known to be N&S for the partial sums S_n in (1.5) to obey the SLLN; see p. 126 of Chung (1974). Since

$$\frac{N(t)}{t}\frac{S_{N(t)}}{N(t)} = \frac{S_{N(t)}}{t}\,, \tag{6.1}$$

(1.8) implies that the same condition is N&S for the random suns $S_{N(t)}$. Since

$$|C(t) - S_{N(t)}| \le |R_1(t)| + |R_2(t)| \le |R_1(t)| + W_{N(t)+1} \tag{6.2}$$

by (1.4) and (1.7), and

$$\frac{N(t)+1}{t}\frac{W_{N(t)+1}}{N(t)+1} = \frac{W_{N(t)+1}}{t}\,, \tag{6.3}$$

Proposition 7 implies that $E|Y_1| < \infty$ and $EW_1 < \infty$ are sufficient for the cumulative process $C(t)$ to obey the SLLN.

Now we establish the necessity for $C(t)$. Suppose that $t^{-1}C(t) \to \gamma$ w.p.1 as $t \to \gamma$, where $0 < \gamma < \infty$. First, since

$$\frac{C(T_k)}{T_k} = \frac{R_1(T_k)}{T_k} + \frac{S_k}{k}\frac{k}{T_k}\,, \tag{6.4}$$

and (1.8) is equivalent to $k^{-1}T_k \to \lambda^{-1}$ w.p.1 as $k \to \infty$, we see that then $n^{-1}S_n \to \lambda^{-1}\gamma$ w.p.1 as $n \to \infty$, which implies that $E|Y_1| < \infty$ and $\gamma = \lambda E[Y_1$. Next suppose that $EW_1 = \infty$. Then, by Proposition 7,

$$\overline{\lim_{n\to\infty}}\, n^{-1}W_n > 0 \quad \text{w.p.1}\,, \tag{6.5}$$

(indeed, even $\overline{\lim_{n\to\infty}}\, n^{-1}W_n = \infty$ w.p.1) so that there is a sequence of random times $\{\beta_k : k \ge 1\}$

such that $T_{n_k} \leq \beta_k < T_{n_k+1}$ and

$$\overline{\lim_{k \to \infty}} \; \beta_k^{-1} \int_{t_{n_k}}^{\beta_k} f(X(s))ds > 0 , \tag{6.6}$$

so that

$$\overline{\lim_{k \to \infty}} \; \beta_k^{-1} \int_0^{\beta_k} f(X(s)ds > \lim_{k \to \infty} T_k^{-1} \int_0^{T_k} f(X(s))ds = EY_1 \; ; \tag{6.7}$$

i.e., then $t^{-1} C(t)$ fails to converge w.p.1 as $t \to \infty$, so that assuming $EW_1 = \infty$ leads to a contradiction. Hence, the SLLN for $C(t)$ implies that $EW_1 < \infty$.

(b) *LIL*

The condition $E[Y_1^2(f_c)] < \infty$ is well known to be N&S for the partial sums of $Y_i(f_c)$ to obey the LIL with $\beta = \text{Var}\, Y_1(f_c)$; see Strassen (1966), Heyde (1968) and pp. 297-8 of Stout (1974). Since

$$\frac{S_{N(t)}}{\sqrt{2N(t)L_2N(t)}} = \frac{S_{N(t)}}{\sqrt{2t\,L_2\,t}} \left(\frac{t\,L_2\,t}{N(t)\;L_2\;N(t)} \right) \tag{6.8}$$

and (1.8) implies that

$$\frac{t\,L_2 t}{N(t)L_2N(t)} \to \lambda^{-1} \quad \text{w.p.1 as } t \to \infty , \tag{6.9}$$

the LIL holds for the partial sums if and only if it does for the random sums; for $S_{N(t)}$, $\beta = \lambda\, \text{Var}\, Y_1(f_c)$. By (6.2), we establish sufficiency for the cumulative process if we show that

$$\frac{W_{N(t)+1}(f_c)}{\sqrt{t\,L_2\,t}} \to 0 \quad \text{w.p.1 as } t \to \infty . \tag{6.10}$$

By (6.9), it suffices to show that

$$\frac{W_n(f_c)}{\sqrt{n\,L_2 n}} \to 0 \text{ w.p.1. as } n \to \infty . \tag{6.11}$$

To establish (6.11), we use Propositions 6-8. Proposition 7 with the condition on $W_1(f_c)$ implies that $W_n(f_c)/nL_2 W_n(f_c) \rightarrow 0$ w.p.1 as $n \rightarrow \infty$. Propositions 6-8 then imply (6.11).

Turning to the necessity, from the LIL for $C(t)$, we obtain the LIL for the partial sums themselves by considering the subsequence of times $\{T(n) : n \geq 1\}$. By the known converse of the LIL for the partial sums, we deduce that we must have $E[Y_1^2(f_c)] < \infty$. Finally, if the moment condition on $W_1(f_c)$ is violated, then, by Proposition 7,

$$\sum_{n=1}^{\infty} P\left[\frac{W_n(f_c)}{L_2 W_n(f_c)} > n\right] = \infty . \tag{6.12}$$

By Proposition 8, (6.12) implies that

$$\sum_{n=1}^{\infty} P\left[W_n(f_c) > \sqrt{n\, L_2\, n}\right] = \infty , \tag{6.13}$$

i.e., by Borel-Cantelli,

$$\overline{\lim_{n \rightarrow \infty}} \frac{W_n(f_c)}{\sqrt{n\, L_2\, n}} > 0 \quad \text{w.p.1.} \tag{6.14}$$

As in the necessity for the SLLN in (6.6) and (6.7), (6.14) implies that there are random times β_k with $T_{n_k} \leq \beta_k < T_{n_k+1}$ such that

$$\overline{\lim_{k \rightarrow \infty}} \frac{1}{\sqrt{\beta_k L_2 \beta_k}} \int_0^{\beta_k} f_c(X(s))\, ds > \overline{\lim_{k \rightarrow \infty}} \frac{1}{\sqrt{T_k\, L_2\, T_k}} \int_0^{T_k} f_c(X(s))\, ds = \text{Var}\ Y_1(f_c)\ . \tag{6.15}$$

(c) *FLIL*

In our i.i.d. setting, the sufficient condition for the LIL implies the FLIL for the partial sums of $Y_i(f_c)$; see Strassen (1964). Since the FLIL implies the ordinary LIL, by virtue of the continuous mapping applied to the projection at time 1, the N&S condition for the LIL for the partial sums is N&S for the FLIL for the partial sums. By (1.8) the SLLN holds for $N(t)$. As before, the SLLN for $N(t)$ is equivalent to a FSLLN of the form

$$\frac{N(nt)}{n} \to \lambda t \text{ w.p.1 in } D[0, \infty) \text{ as } n \to \infty . \tag{6.16}$$

Using the random time change by $N(nt)/n$ in $D[0, \infty)$, which is a continuous map (see §17 of Billingsley (1968) or Whitt (1980)), we obtain

$$\frac{S_{N(nt)}}{\sqrt{2nL_2 n}} \to C' \text{ w.p.1 in } D[0, \infty) \text{ as } n \to \infty , \tag{6.17}$$

where C' is the set of y in $D[0, \infty)$ such that $y(t) = x(\lambda t)$, $t \geq 0$, for x in C, and C is the limit set associated with the partial sums. As before, the FLIL for the random sums implies the ordinary LIL, which we saw in part (b) implies the LIL for the partial sums.

To establish the FLIL for the cumulative process $C(t)$, we apply the moment condition on $W_1(f_c)$. With ... noment condition, Propositions 6-8 imply that

$$(nL_2 n)^{-1/2} \max_{1 \leq k \leq n} \{ W_k(f_c) \} \to 0 \text{ w.p.1 as } n \to \infty . \tag{6.18}$$

Then (6.16) and (6.18) imply that

$$(nL_2 n)^{-1/2} \max_{1 \leq k \leq N(n)+1} \{ W_k(f_c) \} \to 0 \text{ w.p.1 as } n \to \infty . \tag{6.19}$$

Given the FLIL for the random sums, (6.2) and (6.19) imply the FLIL for $C(t)$.

Turning to the necessity for the cumulative process, we obtain the FLIL for the partial sums by considering the times $T(nt)/n$. (The first remainder term is obviously asymptotically negligible.) Hence, $E[Y_1^2(f_c)] < \infty$ is a necessary condition. Since

$$\frac{T(N(nt))}{n} \to t \text{ in } D[0, \infty) \text{ w.p.1 as } n \to \infty , \tag{6.20}$$

we have the joint limit

$$(nL_2 n)^{-1/2} \left(\int_0^{nt} f_c(X(s))\, ds \, , \int_0^{T(N(nt))} f_c(X(s))\, ds \right) \rightsquigarrow (C, C) \text{ w.p.1} \tag{6.21}$$

as $n \to \infty$ in $D[0, \infty) \times D[0, \infty)$. Hence, the normalized difference converges to 0, i.e.,

$$(nL_2 n)^{-\frac{1}{2}} \int_{T(nt)}^{nt} f_c(X(s))\,ds \to 0 \text{ in } D[0,\infty) \text{ w.p.1 as } n \to \infty \tag{6.22}$$

or, equivalently, (6.19) holds, which in turn is equivalent to (6.18) given (6.16). By Propositions 6-8, (6.18) implies the moment condition on $W_1(f_c)$.

(d) *WLLN*

The stated conditions for the partial sums in Table 1 are known to be N&S; see Theorem 1 on p. 235 of Feller (1971). By Proposition 10, the WLLN implies the FWLLN for the partial sums in this setting. Alternatively, it is not difficult to show that the conditions are N&S for the FWLLN directly. Since the FWLLN implies the WLLN, we only need demonstrate sufficiency. Instead of (7.4) and (7.5) on p. 234, 235 of Feller (1971), we write

$$\begin{aligned} P(\sup_{0 \le t \le 1} |S_{[nt]} - tm'_n| > nx) &\le P(\max_{1 \le k \le n} |S'_k - km'_n| > nx) \\ &\quad + P(S_k \ne S'_k \text{ for some } k,\ 1 \le k \le n) \\ &\le \frac{E(X_1'^2)}{nx^2} + nP(|X_1| > s_n) \end{aligned}$$

using Kolmogorov's inequality in the second step. The rest of the argument is the same.

The FWLLN for the partial sums in turn implies the FWLLN for the random sums, by virtue of a random-time change argument (as in §17 of Billingsley (1968) or §3 of Whitt (1980)). The FWLLN for the random sums implies the ordinary WLLN. By applying the continuous projection map at $t = 1$. (Alternatively, the WLLN for the random sums follows directly from the WLLN for the partial sums; see Theorem 10.1 on p. 148 of Revesz (1968).)

Finally, the WLLN for the random sums is equivalent to the WLLN for the cumulative processes by (6.2) and Proposition 9. In particular, since $|R_1(t)| \le W_0$, $R_1(t)/t \Rightarrow 0$ as $t \to \infty$; Proposition 9 implies that $R_2(t)/t \Rightarrow 0$.

(e) *FWLLN*

The sufficiency for the partial sums and random sums follows from the argument in part (d). Given the FWLLN for the random sums, the FWLLN for the cumulative process follows from the extra condition on $W_1(f_c)$, Proposition 5 with $Z_i = W_i(f_c)$ and $\phi(t) = t$, and (6.2). In particular, the extra condition on $W_1(f_c)$ and Proposition 5 imply that

$$n^{-1} \max_{1 \le k \le n} W_k(f_c) \Rightarrow 0 \text{ as } n \to \infty \tag{6.24}$$

or, equivalently,

$$n^{-1} W_{[nt]}(f_c) \Rightarrow 0 \text{ as } n \to \infty \text{ } in \text{ } D[0, \infty) \ . \tag{6.25}$$

Then, by a random time change argument,

$$n^{-1} W_{N(nt)+1}(f_c) \Rightarrow 0 \text{ as } n \to \infty \text{ in } D[0, \infty) \tag{6.26}$$

or, equivalently,

$$n^{-1} \max_{1 \le k \le N(n) + 1} \{ W_k(f_c) \} \Rightarrow 0 \text{ as } n \to \infty \ , \tag{6.27}$$

but, by (6.2),

$$\sup_{0 \le t \le 1} \{ | n^{-1} S_{(nt)} - n^{-1} C(nt) | \} \le n^{-1} W_0 + n^{-1} \max_{1 \le k \le N(n) + 1} \{ W_k(f_c) \} \ . \tag{6.28}$$

We now turn to necessity. Given the FWLLN for the random sums, we obtain the FWLLN for the partial sums by applying the converse to continuity for composition, i.e., Theorem 3.3 of Whitt (1980). A direct application yields the FWLLN for the partial sums in $D((0, \infty))$, with an open interval at the left, but this implies convergence in $D[a, b]$ for all a and b with $0 < a < b < \infty$, which in turn implies convergence in $D[0, \infty)$ for the partial sums of i.i.d. random variables. We have already seen that the FWLLN for the partial sums implies the condition on $Y_1(f_c)$.

Turning to necessity he cumulative process, first we apply a random time change argument to get the FWLLN for the partial sums, which in turn implies the condition on $Y_1(f_c)$.

In particular, $n^{-1}T(nt) \to \lambda t$ as $n \to \infty$ in $D[0, \infty)$, so that

$$n^{-1}S_{[nt]} = n^{-1}\int_0^{T(nt)} f_c(X(s))\,ds - n^{-1}R_1(nt) \Rightarrow 0 \text{ as } n \to \infty \text{ in } D[0, \infty) \ . \quad (6.29)$$

Finally, to establish the condition on $W_1(f_c)$, we note that $T(N(nt) \to t$ as $n \to \infty$ in $D[0, \infty)$, so that

$$n^{-1}\Big(\int_0^{nt} f_c(X(s))\,ds\ ,\ \int_0^{T(N(nt))} f_c(X(s))\,ds\Big) \Rightarrow (0, 0) \text{ as } n \to \infty \quad (6.30)$$

in $D[0, \infty) \times D[0, \infty)$ and

$$n^{-1}\Big[\int_0^{nt} f_c(X(s))\,ds - \int_0^{T(N(nt))} f_c(X(s))\,ds\Big] \Rightarrow 0 \text{ as } n \to \infty \text{ as } n \to \infty \text{ in } D[0, \infty) (6.31)$$

which in turn implies that

$$n^{-1}\max_{0 \le k \le N(nt)} W_k(f_c) \le \sup_{0 \le t \le 1}\Big\{n^{-1}\Big|\int_{T(N(nt))}^{nt} f_c(X(s))\,ds\Big|\Big\} \Rightarrow 0 \text{ as } n \to \infty \quad (6.32)$$

and then, reversing the argument from (6.24) to (6.27), we obtain (6.24), which by Proposition 5 implies the condition on $W_1(f_c)$.

(f) *CLT*

By p. 181 of Gnedenko and Kolmogorov (1968), $E[Y_1^2(f_c)] < \infty$ and $E[Y_1(f_c)] = 0$ is N&S for the CLT for the partial sums. Donsker's theorem or Proposition 10 implies that this condition is also N&S for the FCLT.

Given the FCLT for partial sums, we obtain the FCLT for random sums by a random time change argument as in §17 of Billingsley. As usual, the FCLT for the random sums implies the ordinary CLT for random sums by applying the continuous mapping theorem with the projection at $t = 1$. Just as in part (d), the CLT for the random sums is equivalent to the CLT for the cumulative process, because $R_1(t)/\sqrt{t} \Rightarrow 0$ and $R_2(t)/\sqrt{t} \Rightarrow 0$ as $t \to \infty$, the last by Proposition 9.

(g) *FCLT*

The necessity and sufficiency for the partial sums and the sufficiency for the random sums follows from the argument in part (f). The rest of the argument is just as in part (e).

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limit theorem	partial sums S_n	cumulative process $C(t) \equiv \int_0^t f(X(s))\,ds$
SLLN	$E\|Y_1\| < \infty$	$+ \; EW_1 < \infty$
LIL	$E[Y_1(f_c)^2] < \infty$	$+ \; E[W_1(f_c)/L_2 W_1(f_c)] < \infty$
FLIL	$E[Y_1(f_c)^2] < \infty$	$+ \; E[W_1(f_c)/L_2 W_1(f_c)] < \infty$
WLLN	$tP(\|Y_1\| > t) \to 0$ and $E[Y_1; \|Y_1\| \le t] \to \alpha$ as $t \to \infty$	?
FWLLN	$t\,P(\|Y_1\| > t) \to 0$ and $E[Y_1; \|Y_1\| \le t] \to \alpha$ as $t \to \infty$	$+ \; t\,P(W_1(f_c) > t) \to 0$ as $t \to \infty$
CLT	$E[Y_1(f_c)^2] < \infty$	?
FCLT	$E[Y_1(f_c)^2] < \infty$	$+ \; t^2 P(W_1(f_c) > t) \to 0$ as $t \to \infty$

Table 1. Necessary and sufficient conditions for the processes to obey the indicated limit theorem. For the established cumulative process results, the condition is the stated one plus the condition for the partial sums at the left.

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Peter W. Glynn and Ward Whitt

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Department of Operations Research - 4022
Terman Engineering Center
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LIMIT THEOREMS FOR CUMULATIVE PROCESSES

by

*Peter W. Glynn**

Department of Operations Research
Stanford University
Stanford, CA 94305-4022

and

Ward Whitt

AT&T Bell Laboratories
Murray Hill, NJ 07974-0636

Abstract

Necessary and sufficient conditions are established for cumulative processes (associated with regenerative processes) to obey several classical limit theorems; e.g., a strong law of large numbers, a law of the iterated logarithm and a functional central limit theorem. The key random variables are the integral of the regenerative process over one cycle and the supremum of the absolute value of this integral over all possible initial segments of a cycle. The tail behavior of the distribution of the second random variable determines whether the cumulative process obeys the same limit theorem as the partial sums of the cycle integrals. Interesting open problems are the necessary conditions for the weak law of large numbers and the ordinary central limit theorem.

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