# NAVAL POSTGRADUATE SCHOOL

## Monterey, California



# THESIS

**SEQUENTIAL ESTIMATION OF OPTIMAL AGE REPLACEMENT POLICIES WHEN DISTRIBUTION OF LIFETIMES IS PHASE TYPE**

by

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March, 1992

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

# REPORT DOCUMENTATION PAGE

**1a. REPORT SECURITY CLASSIFICATION:** Unclassified

**1b. RESTRICTIVE MARKINGS:**

**2a. SECURITY CLASSIFICATION AUTHORITY:**

**2b. DECLASSIFICATION/DOWNGRADING SCHEDULE:**

**3. DISTRIBUTION/AVAILABILITY OF REPORT:** Approved for public release; distribution is unlimited.

**4. PERFORMING ORGANIZATION REPORT NUMBER(S):**

**5. MONITORING ORGANIZATION REPORT NUMBER(S):**

**6a. NAME OF PERFORMING ORGANIZATION:** Naval Postgraduate School

**6b. OFFICE SYMBOL (If applicable):** OR

**7a. NAME OF MONITORING ORGANIZATION:** Naval Postgraduate School

**6c. ADDRESS (City, State, and ZIP Code):** Monterey, CA 93943-5000

**7b. ADDRESS (City, State, and ZIP Code):** Monterey, CA 93943-5000

**8a. NAME OF FUNDING/SPONSORING ORGANIZATION:**

**8b. OFFICE SYMBOL (If applicable):**

**9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER:**

**8c. ADDRESS (City, State, and ZIP Code):**

**10. SOURCE OF FUNDING NUMBERS**

| Program Element No | Project No. | Task No | Work Unit Accession Number |
|---|---|---|---|
| | | | |

**11. TITLE (Include Security Classification):** Sequential Estimation of Optimal Age Replacement Policies when Distribution of lifetimes is Phase Type

**12. PERSONAL AUTHOR(S):** Patchara Pumpiched

| 13a. TYPE OF REPORT | 13b. TIME COVERED | 14. DATE OF REPORT (year, month, day) | 15. PAGE COUNT |
|---|---|---|---|
| Master's Thesis | From To | March 1992 | 73 |

**16. SUPPLEMENTARY NOTATION:** The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

**17. COSATI CODES**

| FIELD | GROUP | SUBGROUP |
|---|---|---|
| | | |
| | | |

**18. SUBJECT TERMS (continue on reverse if necessary and identify by block number):** Seqential Estimation Procedure, Age replacement Policy, Optimal Replacement

**19. ABSTRACT (continue on reverse if necessary and identify by block number)**


Optimal age replacement policies are designed to cut down system failures and minimize maintenance cost. By scheduling planned replacements, a system is replaced at a schedule time or at failure, whichever comes first, and the cost of replacement before failure (planned) is less than the cost after failure (unplanned). In this thesis, the distribution of lifetimes is a known, increasing failure rate phase type distribution. To find the optimal age of replacement, the parameters of underlying phase type distribution must be estimated.

An optimal age sequential estimation procedure is developed. In particular, the phase type distributions parameters are estimated using a matching moments nonlinear programming approach. Since there are many parameters associated with phase type distributions and the distributions include matrix exponential terms, the parameters are in general difficult to estimate. A specific case where the phase type distribution has initial probability vector = (1, 0, 0) is studied for different sample sizes and compared with a similar nonparametric procedure.


**20. DISTRIBUTION/AVAILABILITY OF ABSTRACT:** [X] UNCLASSIFIED/UNLIMITED [ ] SAME AS REPORT [ ] DTIC USERS

**21. ABSTRACT SECURITY CLASSIFICATION:** Unclassified

**22a. NAME OF RESPONSIBLE INDIVIDUAL:** Lyn R. Whitaker

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**22c. OFFICE SYMBOL:** OR/Wh

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

Unclassified

Approved for public release; distribution is unlimited.

# Sequential Estimation of Optimal Age Replacement Policies When Distribution of lifetimes is Phase Type

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Submitted in partial fulfillment
of the requirements for the degree of

MASTER OF SCIENCE IN OPERATION RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
March 1992

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## ABSTRACT


Optimal age replacement policies are designed to cut down system failures and minimize maintenance cost. By scheduling planned replacements, a system is replaced at age $\phi^*$ or at failure, whichever comes first, and the cost of replacement before failure (planned) is less than the cost after failure (unplanned). In this thesis, the distribution of lifetimes is a known, increasing failure rate phase type distribution. To find the optimal age of replacement, the parameters of the underlying phase type distribution must be estimated.

An optimal age sequential estimation procedure is developed. In particular, the phase type distributions parameters are estimated using a matching moments nonlinear programming approach. Since there are many parameters associated with phase type distributions and the distributions include matrix exponential terms, the parameters are in general difficult to estimate. A specific case where the phase type distribution has initial probability vector $\alpha=(1,0,0)$ is studied for different sample sizes and compared with a similar nonparametric procedure.

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## ACKNOWLEDGMENTS

I owe a considerable debt to a number of individuals. Foremost among these is Assistant Professor Lyn R. Whitaker, who was never too busy to advise, to offer helpful suggestions and thorough review. I also would like to thank the following people for their helpful, guidance and encouragement:

◦ Professor Peter Purdue

◦ Assistant Professor Michael P. Bailey

◦ Assistant Professor Siriphong Lawphongpanich

◦ Dennis M. Mar

Especially, my wife "Khanitha" for her patience and support through the past two years.

# I. INTRODUCTION

Normally a system or component is replaced when it fails and $C_1$, the cost of replacement before failure (planned replacement), is often less than $C_2$, the cost after failure (unplanned replacement). The problem of determining when to repair and when to replace failing systems is the concern of management resources. Inefficient management due to the use of nonoptimal maintenance policies can lead to significant system maintenance cost. In general, optimal maintenance policies are designed to cut down the number of system failures and minimize maintenance costs. In this thesis, we study the problem of estimating a type of optimal maintenance policy (the optimal age replacement policy) when the underlying system life distribution is phase-type. In particular, we consider an adaptive estimation scheme where the estimated optimal replacement age is updated each time the system is replaced.

Maintenance is defined to be all activities taken to keep the system in serviceable condition or to bring it back to serviceability. There are two types of maintenance, corrective and preventive maintenance [Ref. 5: pp. 1-8]. Corrective maintenance is used after a failure. This does not necessarily mean that such action has not been foreseen. Preventive maintenance aims to reduce the probability of failure and includes:

-Planned (scheduled) maintenance in which specified components are replaced (usually at regular intervals). The maintenance time is usually based on component lifetime distributions.

-Unplanned (condition-based) maintenance in which the decision to replace or not to replace is made according to the outcome of a diagnostic study.

The policies that are designed to reduce the number or the probability of system failure and maintenance costs are called maintenance policies [Ref. 21: pp. 1-3]. We concentrate on age replacement policies where a system is replaced at age T or failure, whichever comes first. For this problem to make sense, $C_1$ must be less than $C_2$ and the system must age with time i.e. the failure rate of the system must be increasing with age. It is not wise to replace the equipment before failure when the system has a constant or decreasing failure rate such as when the system failures occur according to an exponential distribution. In this case, replacement will not reduce the probability of system failure occurring in the next instance of time. When using an age replacement policy the question always asked is, "At what age should the equipment be replaced". If the replacement occurs too frequently the maintenance costs will be high. If replacement is too infrequent, the system will fail more often than necessary and again, the maintenance cost will be too high. Thus it is desirable to find an "optimal" replacement age. Here, the optimal age is defined as the age which yields the minimum long run expected maintenance costs per unit time.

We will assume that the maintenance action returns the system to the good-as-new condition, thus the same services are provided as before replacement. To accomplish this we assume that the systems used for replacement have lifetimes that are independent and identically distributed according to a phase-type distribution. The class of phase type distributions is large and includes Exponential and Gamma distributions along with convolutions and mixtures of these distributions. This fact, along with the fact that phase type distributions have a physical interpretation make them particularly well suited for modeling system lifetimes.

The optimal replacement age depends on the underlying phase type distribution. This is in general unknown and must be estimated. Estimation for phase type distributions based on iid lifetimes has been studied by Neuts and Meier (1980). Estimation of the optimal age of replacement for phase type distributions is new. However both parametric and nonparametric estimation based on iid observations have been considered by Arunkumar (1972), Ingram and Schaeffer (1976), Bergman (1977), and Barlow (1978). Here we use a sequential approach for estimating the optimal replacement age. At each replacement the estimate is updated and the new system is subject to an age replacement policy based on the optimal age estimated so far. This type of sequential approach has been studied parametrically by Oclay (1990) and nonparametricly by Bather (1977), Frees and Ruppert (1985), Aras and Whitaker (1991), Wu (1990) and in a decision theoretic framework by Glazebrook, Bailey and Whitaker (1991).

Phase type distributions are described in Chapter II. The sequential estimation procedure to estimate the optimal age of replacement is given in Chapter III, and in Chapter IV we present results comparing the sequential estimation procedure assuming an underlying phase type distribution with a nonparametric procedure. Conclusion and recommendation are given in Chapter V.

## II. PHASE TYPE DISTRIBUTION

### A. GENERAL

A phase type distribution is defined as the distribution of the time until absorption in a finite state Markov process. To determine the distribution of the absorption time, consider an m+1 state, continuous time Markov chain whose infinitesimal generator **Q** has the form

$$Q = \begin{vmatrix} T & T^{\circ} \\ 0 & 0 \end{vmatrix}$$

where **T** is a nonsingular m × m matrix with negative diagonal elements and nonnegative off-diagonal elements, **T°** is vector of length m with nonnegative elements and **0**=(0,0,...,0). Moreover,

$$\mathbf{Te} + \mathbf{T}^{\circ} = \mathbf{0}'$$

where $\mathbf{e} = (1,1,\ldots,1)'$.

By the construction of **Q** the state m+1 is absorbing and all other states are transient. The necessary and sufficient condition for this is that the inverse $\mathbf{T}^{-1}$ exists [Ref. 15: pp. 446]. Let the initial probability vector of the Markov chain be given by ( $\alpha$, $\alpha_{m+1}$ ) with $\alpha\mathbf{e} + \alpha_{m+1} = 1$ and $\alpha$ being the m dimensional initial probability vector of transient states such that $0 < \alpha\mathbf{e} \leq 1$. Let X be the time until absorption into the $(m+1)^{st}$ state. The probability distribution F(x) of the time until absorption in the state m+1 corresponding to the initial probability vector ($\alpha$, $\alpha_{m+1}$ ) is given by

$$F(x) = 1 - \alpha \exp(Tx)e , \quad \text{for } x \geq 0 , \tag{2.1}$$

where exp(A) is the matrix exponential of the matrix **A** defined in Ref. 8.

The probability distribution F(x) on [0, ∞) is said to be of phase type (PH-distribution) and the pair (α, T) is called a representation of F(x). If $\alpha_{m+1} > 0$ the distribution F(x) has a jump of height $\alpha_{m+1}$ at x = 0.

On (0, ∞), the probability mass is distributed according to a density given by

$$f(x) = \alpha \exp(Tx)T^{\circ} , \text{ for } x > 0 , \tag{2.2}$$

The noncentral moments $\mu_i = E[X^i]$ of F(x) are all finite and given by

$$\mu_i = (-1)^i i!(\alpha T^{-i} e) , \text{ for } i \geq 1 \text{ [Ref. 15: pp. 446]} . \tag{2.3}$$

## B. COST FUNCTION

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) positive random variables with distribution function F. The sequence $X_1, X_2, \ldots$ represents the sequence of system lifetimes that would be observable if the system were replaced at failure. Let $C_1$, $C_2$ ($C_2 > C_1$) and $\phi$ be the respective cost of planned replacement; unplanned replacement and the replacement age. The long run expected cost per unit time can be verified [Ref. 1: pp. 87] to be

$$C(\phi) = \frac{C_2 F(\phi) + C_1 \overline{F}(\phi)}{\int_0^{\phi} \overline{F}(u)\, du} \tag{2.4}$$

where $\overline{F}(u) = 1 - F(u)$ is the survival function. A sufficient condition that guarantees a unique and finite $\phi^*$ that minimizes $C(\phi)$, is that F have strictly increasing failure rate.

When the distribution F is phase type with representation ($\alpha$, T), the long run expected cost function is

$$C(\phi) = \frac{C_2[1-\boldsymbol{\alpha}\exp(\boldsymbol{T}\phi)\boldsymbol{e}] + C_1[\boldsymbol{\alpha}\exp(\boldsymbol{T}\phi)\boldsymbol{e}]}{\int_0^{\phi}\boldsymbol{\alpha}\exp(\boldsymbol{T}u)\boldsymbol{e}du} \quad (2.5)$$

$$= \frac{C_2 - \boldsymbol{\alpha}\exp(\boldsymbol{T}\phi)\boldsymbol{e}[C_2-C_1]}{\boldsymbol{\alpha}\boldsymbol{T}^{-1}[\exp(\boldsymbol{T}\phi) - \boldsymbol{I}]\boldsymbol{e}} \quad (2.6)$$

where I is an identity matrix.

The phase type distribution does not necessarily have increasing failure rate, so the cost function C($\phi$) does not necessarily have a unique finite minimum. In the m=3 case even when the initial probability of the transient states is fixed the minimum can be unique and finite or can occur at infinity as shown in Figure 1, where $\alpha$ = (1, 0, 0) and

$$\boldsymbol{T}_1 = \begin{vmatrix} -15 & 13 & 2 \\ 3 & -15 & 12 \\ 2 & 5 & -15 \end{vmatrix}, \qquad \boldsymbol{T}_2 = \begin{vmatrix} -20 & 10 & 10 \\ 10 & -20 & 10 \\ 10 & 10 & -25 \end{vmatrix}.$$

In Figure 2, the same phenomena is shown when F has the same nonsingular 3×3 matrix

$$\boldsymbol{T}_3 = \begin{vmatrix} -18 & 13 & 5 \\ 10 & -25 & 15 \\ 4 & 4 & -24 \end{vmatrix}$$

but the initial prubabilities are different.

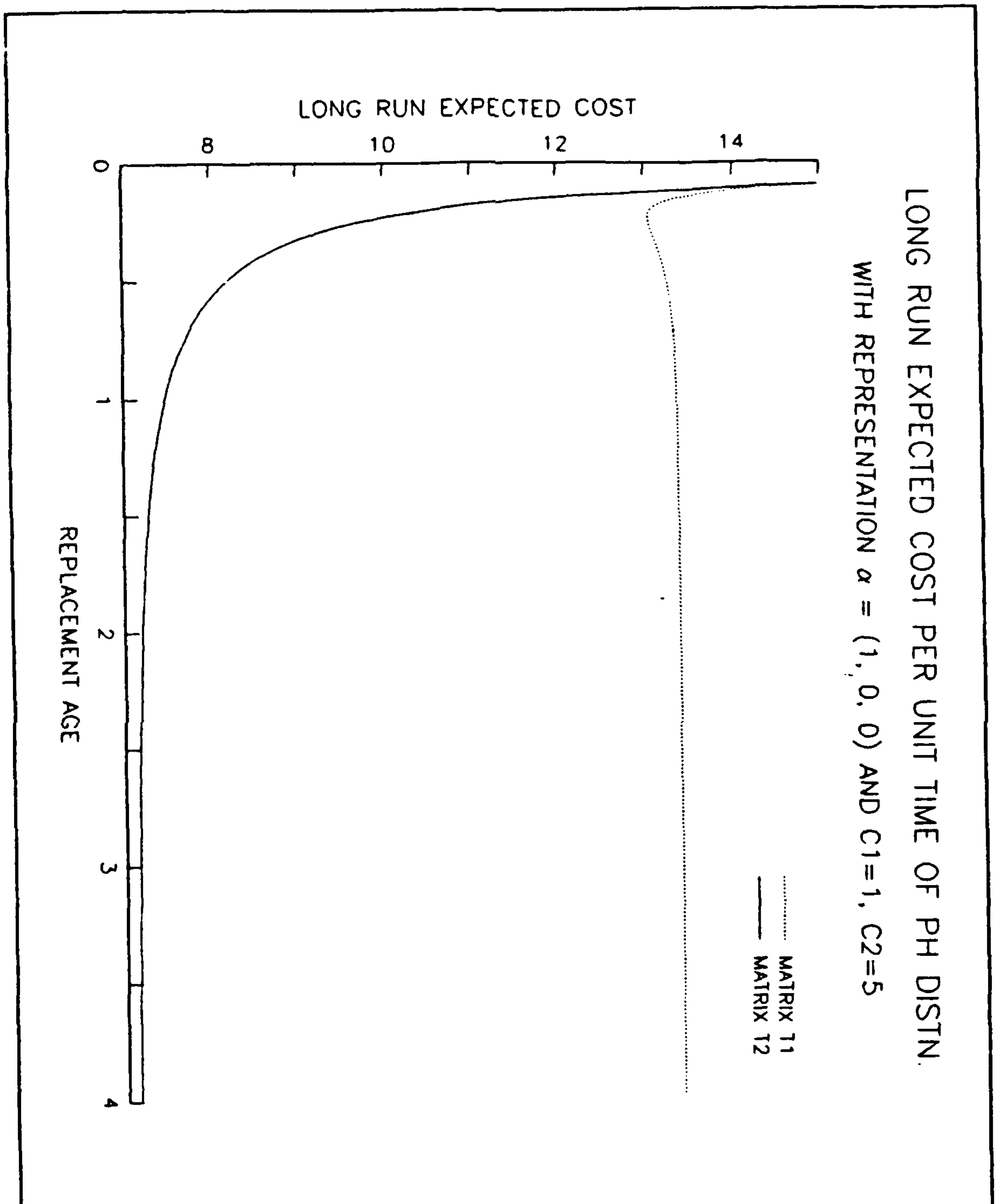


**Figure 1** The long run expected cost per unit time curves of PH distribution with the same $\alpha$ and different $T = T_1$ or $T_2$

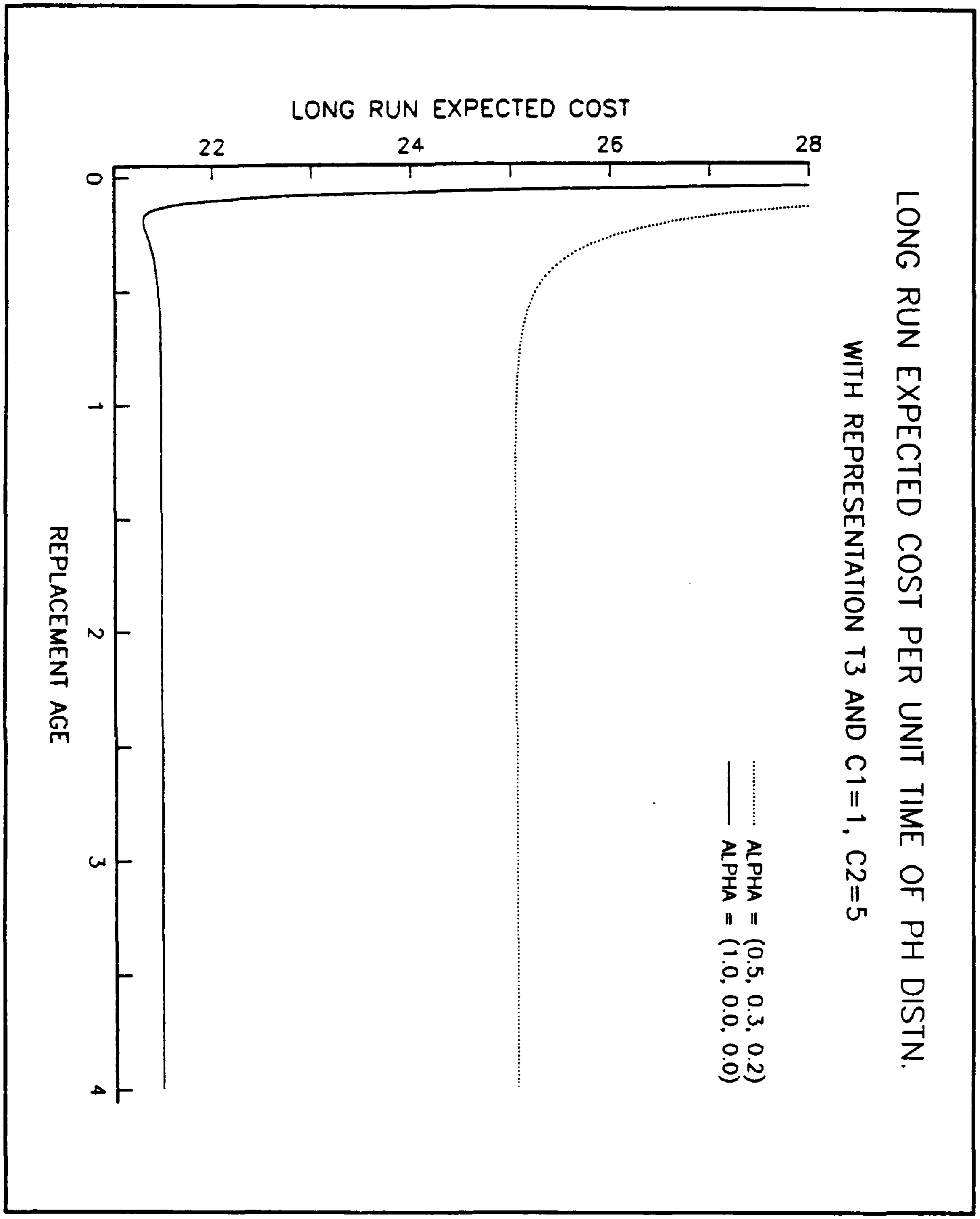


**Figure 2** The long run expected cost per unit time curves of PH distribution with the same T and different $\alpha$

## III. THE SEQUENTIAL ESTIMATION PROCEDURE

In theory, any distribution of a nonnegative random variable can be approximated arbitrarily closely by a PH distribution [Ref. 11: pp. 1-2]. This means that the PH distributions are dense in $\mathcal{F}$ where $\mathcal{F}$ is the set of distributions with support on $[0, \infty)$. The paper of Johnson and Taaffe (1988) shows that the finite mixtures of Erlang distributions and PH distributions with a Coxian representation are both dense in $\mathcal{F}$. Since both families are subsets of the family of PH distributions, this implies that PH distribution are dense in $\mathcal{F}$ [Ref. 10: pp. 1-8]. Thus, the PH distribution can be used to approximate any unknown distribution of lifetimes (lifetime is always greater than or equal 0). Another feature that makes PH distributions a desirable choice to model system lifetimes is that they may have a physical interpretation. The absorbing state represents system failure and the transient states may represent different levels of a system's ability to function.

### A. PARAMETER ESTIMATION

There are many ways to estimate the parameters of a distribution including the method of maximum likelihood (MLE), and the method of moments. The PH distributions have special properties which make estimation difficult. First, the number of parameters is not fixed, it varies with the number of transient states, e.g. a PH distribution with 2 transient states has 6 parameters, a PH distribution with 3 transient states has 12 parameters etc. Also the probability distribution and density function include the exponential of a matrix. This makes it hard to work with the likelihood function. In this thesis, the moment matching method is used to estimate the parameters of a phase type distribution but the MLE method and its associated problems are also discussed . In addition, the number of transient states, m, is assumed to be known.

### 1. The Maximum Likelihood Method

Let $X_1$, $X_2$,... represent the sequence of system lifetimes, where $X_1$, $X_2$,... are iid PH distributions with representation ($\alpha$, T). After N observations the data available for estimation will be $(Z_1,\delta_1),(Z_2,\delta_2),\ldots,(Z_N,\delta_N)$ where $Z_i=\min(X_i,\hat{\phi}^*_{i-1})$, $X_i$ is the $i^{th}$ lifetime and $\hat{\phi}^*_{i-1}$ is the optimal age replacement estimated after i-1 observations and $\delta_i=1$ if $X_i \leq \hat{\phi}^*_{i-1}$ and $\delta_i=0$ otherwise. Assuming system lifetimes have a PH distribution the likelihood for the replacement ages $Z_1,\ldots,Z_N$ and types of replacement $\delta_1$, $\delta_2,\ldots,\delta_N$ is

$$L(\boldsymbol{\alpha},\boldsymbol{T}) = \prod_{i=1}^{N} [\boldsymbol{\alpha}\exp(\boldsymbol{T}Z_i)\boldsymbol{T}^{\circ}]^{\delta_i}[\boldsymbol{\alpha}\exp(\boldsymbol{T}Z_i)\boldsymbol{e}]^{1-\delta_i}$$

$$= \prod_{i=1}^{N} [-\boldsymbol{\alpha}\exp(\boldsymbol{T}Z_i)\boldsymbol{Te}]^{\delta_i}[\boldsymbol{\alpha}\exp(\boldsymbol{T}Z_i)\boldsymbol{e}]^{1-\delta_i} \quad .$$

It is too difficult to maximize this likelihood directly by differentiating L($\alpha$,T) and solving for the MLE's. There are several reasons for this. The likelihood and its derivatives include the exponential of a matrix form that cannot be written in closed form and the parameters are subject to numerous constraints. In addition, there are many likelihood equations due to the number of parameters (3 transient states will have up to

12 equations). An alternate approach is to find approximate MLE's using nonlinear programming algorithms i.e.:

$$\mathbf{MAX}\ L(\boldsymbol{\alpha},\boldsymbol{T}) = \prod_{i=1}^{N} [-\boldsymbol{\alpha}\exp(\boldsymbol{T}Z_i)\boldsymbol{Te}]^{\delta_i}[\boldsymbol{\alpha}\exp(\boldsymbol{T}Z_i)\boldsymbol{e}]^{1-\delta_i}$$

$$\begin{array}{llll} \text{S.T.} & \alpha & \geq 0 & \\ & \alpha e & = 1 & \\ & t_{ii} & < 0 & \text{for } i = 1,2,\ldots,m \\ & t_{ij} & \geq 0 & \text{for } i \neq j \\ & \sum_{i\neq j} t_{ij} & \leq -t_{ii} & \text{for } i = 1,2,\ldots,m \\ & \det(T) & \neq 0 & \end{array}$$

where m = the number of transient states of the PH distribution. Even when m is assumed to be known, there is still the problem of approximating the exponential of a matrix and the optimization software to take care of this problem is not available. Thus, it is very difficult to use this approach to estimate the PH distribution parameters.

### 2. The Moment Matching Method.

The PH distribution is a complicated distribution, there are many parameters and there are difficulties with computing the exponential of a matrix. One property, the existence of $T^{-1}$, implies that all noncentral moments of a PH distribution are finite. The $k^{th}$ moment is given by

$$\mu_k = (-1)^k k!\alpha T^{-k} e. \tag{3.1}$$

Moment matching is a common method for estimating. Using moment matching bypasses the problem of evaluating the exponential of a matrix [Ref. 12: pp.3].

Let m be the number of transient states of the PH distribution. There are m(m+1) parameters that need to be estimated, thus the m(m+1) equations from the moments that need to be solved are

$$\hat{\mu}_1 = -\alpha T^{-1} e$$

$$\hat{\mu}_2 = 2!\alpha T^{-1} e$$

$$\vdots$$

$$\hat{\mu}_{m(m+1)} = (-1)^{m(m+1)}(m^2+m)!\alpha T^{-1} e,$$

where $\hat{\mu}_k$ is the $k^{th}$ sample moment.

This method is still inappropriate due to the large number of equations and the fact that when the dimension of matrix T is big and the moments are of high order, substantial amount of error is introduced when trying to solve these equations. Johnson and Taaffe have shown that the moment matching method and nonlinear programming approaches can be used to estimate the parameters of a PH distribution. They match only the second and third standardized moments [Ref. 11: pp. 3-11].

Let $\overline{\mu}_i$ be the $i^{th}$ central moment. The second standardized moment, the coefficient of variation (C), defined as

$$C = \frac{standard\ deviation}{mean},$$

can be written as

$$C = \frac{(\overline{\mu}_2)^{1/2}}{\mu_1} \quad . \tag{3.2}$$

The third standardized moment, the coefficient of skewness, ($\gamma$) is defined as

$$\gamma = \frac{\overline{\mu}_3}{(\overline{\mu}_2)^{3/2}} \quad . \tag{3.3}$$

And the $t^{th}$ standardized moment is $\overline{\mu}_t/(\overline{\mu}_2)^{t/2}$ for t = 3,4,... .

The reason for matching standardized moments is that the standardized moments do not change with scale changes. So, the shape of a PH distribution with representation ($\alpha$,T) will not change if it is multiplied by a nonnegative number.

The nonlinear programming formulation for matching C and $\gamma$ to a continuous PH distribution is given by

MIN. $w_1(C-C(0))^2 + w_2(\gamma-\gamma(0))^2$

S.T.

$\alpha \geq 0$

$\alpha e = 1$

$t_{ii} < 0$ for i = 1,2,...,m

$t_{ij} \geq 0$ for $i \neq j$

$\sum_{i \neq j} t_{ij} \leq -t_{ii}$ for i = 1,2,...,m

$\det(T) \neq 0$

where

C(0) = the second standardized moment of the current solution.

$\gamma(0)$ = the third standardized moment of the current solution.

m = the number of transient states of the PH distribution.

$t_{ij}$ = the elements of row i and column j of nonsingular matrix **T**.

$w_1$,$w_2$ are positive weights chosen to guide the search and $w_1 \geq 3w_2$.

The moments are matched when the objective function value is zero. The solutions which match the target moments are called "moment-matching solutions" [Ref. 11: pp. 5].

Even with fixed dimension, this procedure may have many solutions and these solutions are unpredictable. There are some practical points that need to be made associated with the properties of PH distribution and the problem of using nonlinear programming to get the moment matching solutions:

- Different combinations of initial solutions and algorithm parameters may lead to different moment-matching solutions. Also, some combinations of initial solution and algorithm parameters may not lead to a moment-matching solution and the nonlinear programming algorithm may not converge.

- For a given dimension we do not know how many solutions exist or how to guide the search toward a preferable solution.

- The moment-matching solutions do not guarantee that the estimated cost function will have a finite minimum. When choosing among several solutions, a solution which gave a cost function with a finite minimum was always chosen.

- The values of $w_1$ and $w_2$ can be modified to guide the search; generally $w_1 \geq 1$, $w_2 \geq 1$ and $w_1 \geq 3w_2$. The magnitude of $w_1$ and $w_2$ can be increased to get more or sufficient accuracy [Ref. 11: pp. 6].

- If the feasible solutions are known, parameters can be bounded to guide the search. However, some bounds on the parameters may not lead to a moment-matching solution or to a solution whose cost function does not have a unique finite minimum.

- User interaction is often necessary in obtaining the appropriate or preferable solutions.

## B. THE SEQUENTIAL ESTIMATION PROCEDURE

Let $X_1, X_2, \ldots, X_N$ be the sequence of lifetimes of the system and $\{\hat{\phi}^*_N\}$ be the sequence of estimators of $\phi^*$ where $\hat{\phi}^*_N$ is the estimator after N replacements. Then after N observations, we have the right censored data $(Z_1,\delta_1),(Z_2,\delta_2),\ldots,(Z_N,\delta_N)$.

where $\quad Z_i = \min (X_i, \hat{\phi}^*_{i-1})$

$$\delta_i = 1 \quad \text{if } X_i \le \hat{\phi}^*_{i-1} \text{ otherwise } \delta_i = 0 .$$

Because the data are right censored the usual method of moments approach for estimation needs to be modified. Rather than use sample moments calculated from an empirical distribution, we use moments from a nonparametric estimator of F based on the censored data. From the right censored data after N observations, the procedure to compute the estimators $\{\hat{\phi}^*_i\}$ is developed as follows.

1. Use the right censored data sequence $(Z_1,\delta_1),(Z_2,\delta_2), \ldots,(Z_N,\delta_N)$ to estimate moments with $\overline{F}=1-F$ by the product-limit estimator:

$$\hat{\bar{F}}(t) = \prod_{(i: Z_{(i)} \le t)} \left(\frac{N-i}{N-i+1}\right)^{\delta_{(i)}} \tag{3.4}$$

where $Z_{(1)}, Z_{(2)}, \ldots, Z_{(N)}$ are the order statistics of $Z_1, Z_2, \ldots, Z_N$ and $\delta_{(1)}, \delta_{(2)}, \ldots, \delta_{(N)}$ are ordered according to the ordering of $Z_{(1)}, Z_{(2)}, \ldots, Z_{(N)}$. Then calculate probabilities $P_i$, associated with $Z_{(i)}$, by

$$P_i = \hat{\bar{F}}(Z_{(i)}^-) - \hat{\bar{F}}(Z_{(i)}) \,. \tag{3.5}$$

Note, since $\bar{F}$ has discontinuities only at $Z_{(i)}$ where $\delta_{(i)} = 1$, that $P_i = 0$ when $\delta_{(i)} = 0$. Then estimates of the $k^{th}$ moment are given by

$$\hat{E}[X^k] = \sum_{i=1}^{N} P_i Z^k_{(i)} \tag{3.6}$$

The estimate of the $2^{nd}$ standardized moment is

$$\hat{C} = \frac{\sqrt{\hat{E}[X^2] - (\hat{E}[X])^2}}{\hat{E}[X]} \tag{3.7}$$

and the estimate of the $3^{rd}$ standardized moment is

$$\hat{\gamma} = \frac{\hat{E}[X^3] - 3\hat{E}[X]\hat{E}[X^2] + 2(\hat{E}[X])^3}{(\sqrt{\hat{E}[X^2] - (\hat{E}[X])^2})^3} \,. \tag{3.8}$$

2. Estimate parameters of PH distribution $(\alpha, T)$, by matching the second and third standardized moments using the nonlinear programming approach.

When executing the nonlinear programming algorithm using the model described in the previous section, the det(T) $\neq$ 0 constraint is replaced by a constraint on the expected lifetime. The reason for doing this is that we could not formulate an algebraic constraint equivalent to the constraint det(T) $\neq$ 0 . The problem can be solved by a constraint on the expected lifetime, i.e. by taking

$$\boldsymbol{\alpha} \boldsymbol{T}^{-1} \boldsymbol{e} = \hat{E}[X] \tag{3.9}$$

where $\hat{E}[X]$ is calculated in step 1. This constraint will make the search for preferable solutions easier. The initial solution is chosen by the user. Some initial solutions may or may not lead to a moment matching solution. Typically, the initial vector $\alpha$ is (1/m,1/m,...,1/m) or (1,0,...,0) and an initial matrix **T** consists of $t_{i,i}$ = -1 and $t_{i,j}$ = 1/m for i $\neq$ j, i = 1,2,...,m. In this thesis the initial vector $\alpha$ is (1,0,...,0) and an initial matrix **T** is taken to have $t_{i,i}$ = -1 and $t_{i,j}$ = 1/m [Ref. 11: pp. 9] .

3. Using the parameters estimated in step 2, the estimated cost function is

$$\hat{C}(\phi) = \frac{C_2 - \hat{\boldsymbol{\alpha}} \exp(\hat{\boldsymbol{T}}\phi) \boldsymbol{e} [C_2 - C_1]}{\hat{\boldsymbol{\alpha}} \hat{\boldsymbol{T}}^{-1} [\exp(\hat{\boldsymbol{T}}\phi) - \boldsymbol{I}] \boldsymbol{e}} \; . \tag{3.10}$$

The new estimate of the optimal age of replacement $\hat{\phi}^*_N$ is taken to be the $\phi$ that minimizes (3.10) by enumerative search.

4. Compare the $\hat{\phi}^*_N$ with $X_{N+1}$ then repeat Step 1.

The initial solutions for matrix **T** and vector $\alpha$ in step 2 are taken to be the previously estimated values of **T** and $\alpha$. In case the moment matching solution cannot be met, the initial solution in step 2 will be taken to be the original initial solution and the previous optimal age replacement is used for step 3.

The replacement cost for $i^{th}$ system is $C_2$ if $X_i < \hat{\phi}^*_{i-1}$; otherwise the replacement cost is $C_1$. The actual total replacement cost for the first N systems that are observed is

$$c_N = \sum_{i=1}^{N} [C_2 \times \delta_i + C_1 \times (1-\delta_i)] \tag{3.11}$$

and the total operating time for the N systems is

$$t_N = \sum_{i=1}^{N} Min(X_i, \hat{\Phi}^*_{i-1}) = \sum_{i=1}^{N} Z_i \ . \tag{3.12}$$

So, the actual average cost (AAC) after N replacement is computed by

$$AAC_N = \frac{c_N}{t_N} \tag{3.13}$$

In this thesis, the moment matching nonlinear programming approach for estimating parameters of PH distributions uses the GAMS program as shown in Appendix A (for a PH distribution with 3 transient states). The Fortran programs are written to estimate the optimal age of replacement, second and third standardized moments and for data preparation as shown in Appendix B.

## IV. COMPARISON WITH NONPARAMETRIC PROCEDURE

### A. NONPARAMETRIC PROCEDURE

An alternate approach for estimating $\phi^*$ is to estimate nonparametrically, as in Aras and Whitaker (1991). The procedure is much simpler computationally than the parametric procedure described in the previous section for PH distributions. In the nonparametric procedure after N replacements the product limit estimator $\hat{\bar{F}}$ of $\bar{F}$ given in (3.4) is used to estimate the cost function $C(\phi)$ as

$$\hat{C}_N(\phi) = \frac{C_2\hat{F}(\phi) + C_1\hat{\bar{F}}(\phi)}{\int_0^{\phi}\hat{\bar{F}}(u)\,du} \tag{4.1}$$

where $\hat{F} = 1 - \hat{\bar{F}}$ is the estimator of the cdf F, the estimator $\hat{\phi}^*_N$ of $\phi^*$ is then found by minimizing $\hat{C}_N(\phi)$. In general one would expect parametric procedures to do much better than nonparametric procedures. However, with the numerical difficulties involved with estimating parameters for PH distribution and the fact that the family of PH distributions is so large it is not obvious which procedure is best. The criterion for comparison is the actual average cost per unit time, $AAC_N$, after N replacements.

## B. COMPARISON OF NONPARAMETRIC AND PARAMETRIC PROCEDURE FOR THE PH DISTRIBUTION

Simulation was used to compare sequential estimation based on PH distributions with nonparametric sequential estimation, $C_1 = 100$, $C_2 = 500$. System lifetimes were generated by an increasing failure rate PH distribution with representation $(T_{41}, \alpha_{41})$ where,

$$T_{41} = \begin{vmatrix} -5 & 4 & 1 \\ 1 & -6 & 5 \\ 1 & 1 & -7 \end{vmatrix} , \qquad \alpha_{41} = (1.0, 0.0, 0.0)$$

For this representation the long run expected cost per unit time is given in Figure 3.

To give the parametric procedure a chance the first 25 system lifetimes are uncensored. After the first 25 observations, the parameters of the PH distribution are estimated sequentially as in Chapter III Section B. The actual average costs (AAC) are calculated and compared for small, medium and large sample sizes with planned cost $C_1 = 100$ and unplanned cost $C_2 = 500$. The 20 initial data sets are simulated as shown in the Appendix A.

The programs are separated into 5 parts. The first one is written in GAMS as shown in Appendix A, to estimate the parameters of a PH distribution by the nonlinear programming approach. The rest are written in Fortran as shown in Appendix B, to estimate the optimal age of replacement based on the estimated PH distribution from the GAMS program; to simulate the system lifetime; to prepare the right censored data; to estimate the expected lifetime, second and third standardized moments; and finally to tabulate the results and to prepare the initial data for the next estimation.

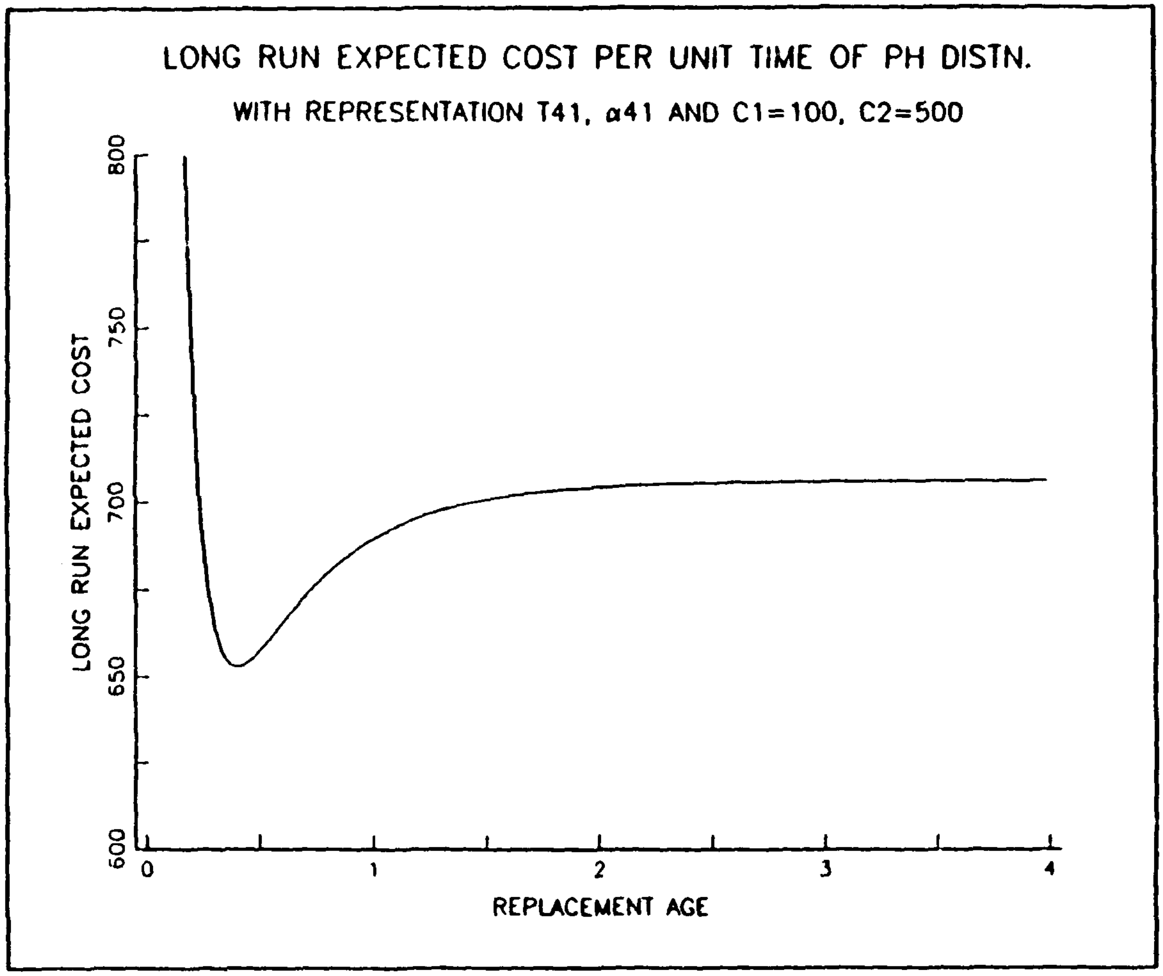


**Figure 3** Long run expected cost per unit time curve of PH distribution with representation $T_{41}$ and $\alpha_{41}$

In this simulation, the number of transient states is fixed at 3 and the initial probability vector $\alpha = (1, 0, 0)$. By fixing $\alpha = (1, 0, 0)$ we reduce the number of parameters to be estimated and use a model in which all systems start in the same state, when states are thought of as the level of system repair, this choice better reflects the idea that replacement systems are as good as new. Bounded parameters and user interaction are used in guiding each estimation during the simulation to a preferable solution.

Tables 1, 2 and 3 summarize the simulation results of the actual average cost resulting from sequential estimation of PH and the nonparametric procedure. Also given are the signed ranks of the differences in the actual average costs. These are used in the Wilcoxon Signed-Rank test to test the hypothesis:

$$H_o: E[AAC_{PH}] = E[AAC_{NONP}]$$

$$H_a: E[AAC_{PH}] < E[AAC_{NONP}]$$

by using $T^+$ the sum of the positive ranks as the test statistic .

Tables 1, 2 and 3 give the statistic $T^+$ as 35, 10 and 2 respectively. When compared to the value in Table 9 [Ref. 13: pp. 780] at N=20 all p-values are less than 0.005 which implies that the sequential estimation of a PH distribution is better than sequential estimation of nonparametric for all sample sizes (small, medium and large). The box plots in Figures 5 and 6 of actual average cost versus the number of replacements at different sample sizes show that the actual average costs decrease as the number of replacements increase and that $AAC_{PH}$ for the parametric procedure decrease more than the $AAC_{NONP}$ for the nonparametric procedure.

A second, PH distribution was chosen to compare the parametric PH procedure with the nonparametric procedure. The second PH distribution has an average long run expected cost function that is more shallow than the first PH distribution. It has representation $T_{42}$ and $\alpha_{42}$

where

$$T_{42} = \begin{vmatrix} -5 & 4 & 1 \\ 1 & -6 & 5 \\ 1 & 2 & -7 \end{vmatrix} \quad , \quad \alpha_{42} = (1.0, 0.0, 0.0) \quad ,$$

and the long run expected cost per unit time is given in Figure 4.

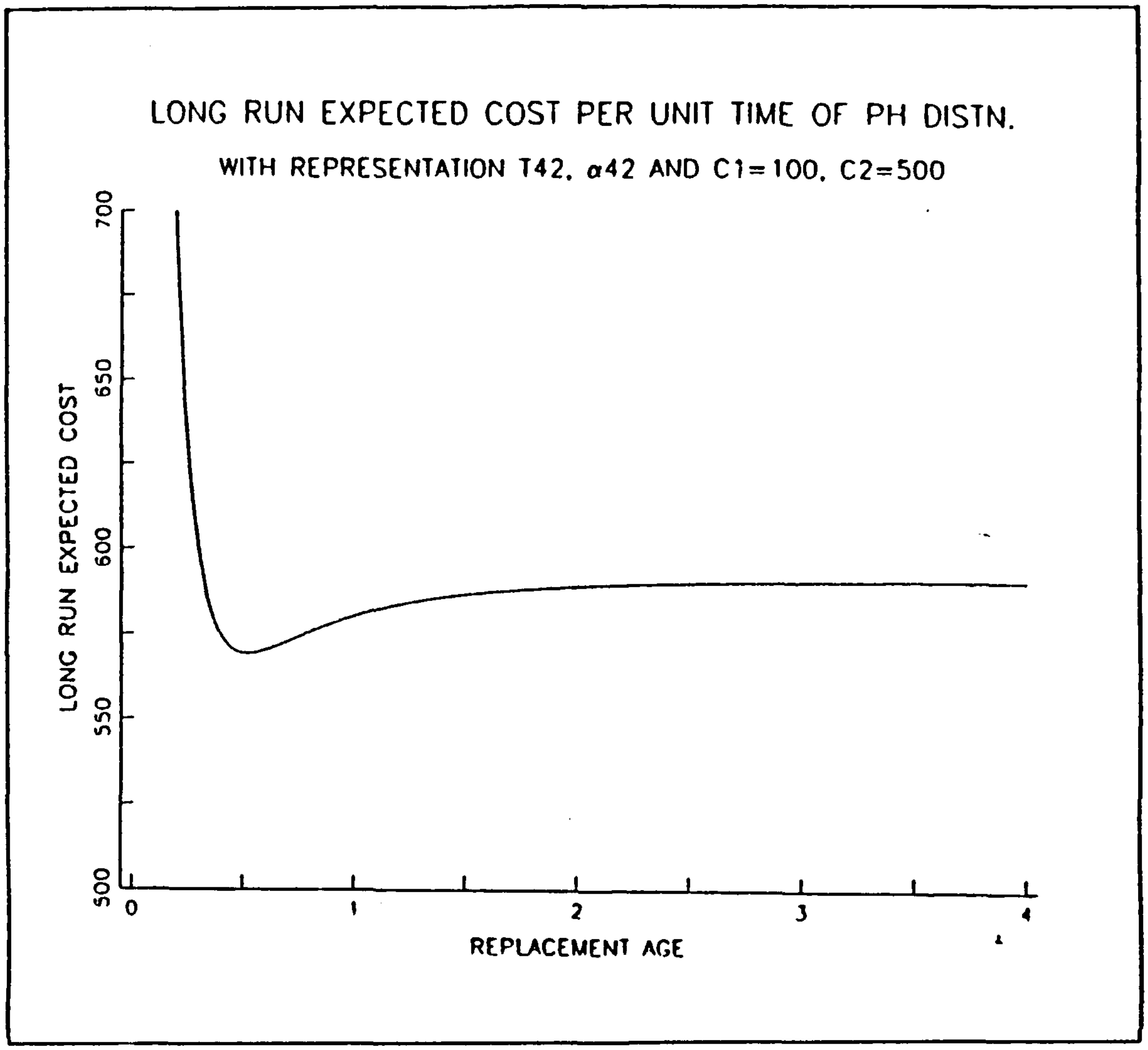


Figure 4 Long run expected cost per unit time curve of PH distribution with representation $T_{42}$ and $\alpha_{42}$

Tables 4, 5 and 6 summarize the result and give the statistic $T^+$ of 93, 81 and 47 respectively. The p-value taken from table 9 [Ref. 13: pp. 780] indicated that the parametric procedure is not better than the nonparametric procedure for small and moderate sample sizes. In fact, for some cases (e.g. runs 16 and 20 in Table 4 and runs 1, 5 and 16 in Table 5) the actual average cost resulting from using the nonparametric procedure can be quite a bit less than the actual average cost resulting from the parametric procedure. This is due to the fact that $C(\phi)$ is shallow at $\phi^*$, thus even though $\hat{C}(\phi)$ may be a reasonable estimator of $C(\phi)$, the variance of its minimizing value $\hat{\phi}^*$ is higher than for the previous PH distribution. This means that $\phi^*$ is often underestimated. Because $C(\phi)$ increases so rapidly to the left of $\phi^*$, underestimating $\phi^*$ can increase the actual average costs dramatically. Another difficulty is that the estimated $\hat{C}(\phi)$ may be relatively flat around $\phi^*$. This causes numerical difficulties with the nonlinear programming algorithm. For large sample sizes the parametric procedure does do better than the nonparametric procedure. Here $\hat{C}(\phi)$ estimates $C(\phi)$ more closely and it is easier for the nonlinear programming algorithm to identify the correct solution.

In Figures 5 and 6, the improvement in actual average cost with sample size is shown for both the parametric and nonparametric procedure for the PH distribution with representation $\mathbf{T}_{41}$ and $\alpha_{41}$. As expected, both procedures improve (actual average costs decrease) with sample size. However, the actual average cost for the parametric procedure decreases faster than for the nonparametric procedure. Both procedures exhibit considerable variability. The solutions have got still mixed that we should do more study on another representation until the more precise solutions come up.

**Table 1 COMPARISON OF ACTUAL AVERAGE COSTS FOR SMALL SAMPLE SIZES (N = 50) OF PH DISTRIBUTION WITH $T_{41}$ AND $\alpha_{41}$**

| Run | $AAC_{PH}$ | $AAC_{NONP}$ | $AAC_{RF}$ | DIFFERENCE | RANK |
|---|---|---|---|---|---|
| 1 | 690.095 | 754.555 | 759.142 | -64.46 | 15 |
| 2 | 714.728 | 775.712 | 780.872 | -60.444 | 13 |
| 3 | 700.342 | 684.379 | 688.578 | 15.963 | 4 * |
| 4 | 666.014 | 725.331 | 738.987 | -59.317 | 12 |
| 5 | 714.330 | 757.528 | 735.058 $ | -43.198 | 8 |
| 6 | 544.322 | 621.505 | 621.507 | -77.183 | 19 |
| 7 | 721.505 | 810.149 | 810.149 | -88.644 | 20 |
| 8 | 565.854 | 617.458 | 636.556 | -51.604 | 11 |
| 9 | 676.511 | 725.394 | 702.676 $ | -48.883 | 10 |
| 10 | 698.572 | 766.182 | 809.629 | -67.61 | 17 |
| 11 | 814.989 | 767.586 | 801.150 | 47.403 | 9 |
| 12 | 644.921 | 667.553 | 672.116 | -22.632 | 5 |
| 13 | 761.516 | 686.794 | 679.892 ! | 74.722 | 18 * |
| 14 | 608.477 | 601.215 | 617.711 | 7.262 | 1 * |
| 15 | 706.591 | 767.782 | 782.198 | -61.191 | 14 |
| 16 | 629.294 | 667.805 | 682.719 | -38.511 | 6 |
| 17 | 688.572 | 755.475 | 748.353 $ | -66.903 | 16 |
| 18 | 636.963 | 624.132 | 665.344 | 12.831 | 3 * |
| 19 | 583.226 | 625.161 | 609.036 $ | -41.935 | 7 |
| 20 | 644.065 | 654.691 | 676.005 | -10.626 | 2 |

$AAC_{RF}$ = Actual average cost replacing when failure occurs

DIFFERENCE = $AAC_{PH}$ - $AAC_{NONP}$

* positive rank

$T^{+} = 35$ ;

p-value $< 0.005$

$ $AAC_{RF}$ is less than only $AAC_{NONP}$

! $AAC_{RF}$ is less than $AAC_{PH}$ and $AAC_{NONP}$

**Table 2 COMPARISON OF ACTUAL AVERAGE COSTS FOR MEDIUM SAMPLE SIZES (N = 100) OF PH DISTRIBUTION WITH $T_{41}$ AND $\alpha_{41}$**

| Run | $AAC_{PH}$ | $AAC_{NONP}$ | $AAC_{RF}$ | DIFFERENCE | RANK |
|---|---|---|---|---|---|
| 1 | 648.127 | 723.170 | 732.263 | -75.043 | 18 |
| 2 | 690.971 | 786.554 | 789.767 | -95.583 | 20 |
| 3 | 649.886 | 644.710 | 683.933 | 5.176 | 2 * |
| 4 | 696.858 | 747.640 | 755.982 | -50.782 | 11 |
| 5 | 690.195 | 688.749 | 682.840 ! | 1.446 | 1 * |
| 6 | 577.742 | 607.414 | 610.850 | -29.672 | 6 |
| 7 | 686.414 | 766.365 | 762.245 $ | -79.951 | 19 |
| 8 | 550.106 | 612.046 | 638.286 | -61.94 | 14 |
| 9 | 637.268 | 696.249 | 689.552 $ | -58.981 | 12 |
| 10 | 684.253 | 730.138 | 786.920 | -45.885 | 10 |
| 11 | 802.039 | 790.698 | 807.530 | 11.341 | 4 * |
| 12 | 615.087 | 687.719 | 695.174 | -72.632 | 17 |
| 13 | 677.302 | 666.119 | 674.830 | 11.183 | 3 |
| 14 | 586.838 | 628.687 | 638.451 | -41.849 | 8 |
| 15 | 726.213 | 749.507 | 737.792 $ | -23.294 | 5 |
| 16 | 626.605 | 694.232 | 667.462 $ | -67.627 | 16 |
| 17 | 671.908 | 732.053 | 758.912 | -60.145 | 13 |
| 18 | 632.954 | 667.156 | 727.346 | -34.202 | 7 |
| 19 | 635.989 | 679.399 | 667.141 $ | -43.41 | 9 |
| 20 | 590.881 | 657.064 | 701.735 | -66.183 | 15 |

$AAC_{RF}$ = Actual average cost replacing when failure occurs
DIFFERENCE = $AAC_{PH}$ - $AAC_{NONP}$
* positive rank
$T^{+} = 10$ ;
p-value $< 0.005$
$ $AAC_{RF}$ is less than only $AAC_{NONP}$
! $AAC_{RF}$ is less than $AAC_{PH}$ and $AAC_{NONP}$

**Table 3 COMPARISON OF ACTUAL AVERAGE COSTS FOR LARGE SAMPLE SIZES (N = 200) OF PH DISTRIBUTION WITH $T_{41}$ AND $\alpha_{41}$**

| Run | $AAC_{PH}$ | $AAC_{NONP}$ | $AAC_{RF}$ | DIFFERENCE | RANK |
|---|---|---|---|---|---|
| 1 | 622.560 | 655.891 | 655.865 $ | -33.331 | 7 |
| 2 | 658.640 | 711.831 | 708.966 $ | -53.191 | 11 |
| 3 | 623.823 | 635.065 | 671.611 | -11.242 | 1 |
| 4 | 678.141 | 749.866 | 754.490 | -71.725 | 16 |
| 5 | 674.548 | 714.291 | 707.926 $ | -39.743 | 9 |
| 6 | 585.435 | 645.920 | 647.927 | -60.485 | 13 |
| 7 | 647.302 | 729.159 | 733.860 | -81.857 | 20 |
| 8 | 581.711 | 607.068 | 649.386 | -25.357 | 5 |
| 9 | 638.465 | 678.078 | 719.904 | -81.439 | 19 |
| 10 | 655.709 | 717.569 | 766.020 | -61.86 | 14 |
| 11 | 760.478 | 779.362 | 762.986 $ | -18.884 | 3 |
| 12 | 589.088 | 649.545 | 681.004 | -60.457 | 12 |
| 13 | 696.081 | 679.816 | 712.761 | 16.265 | 2 * |
| 14 | 602.032 | 643.703 | 649.832 | -47.8 | 10 |
| 15 | 685.977 | 767.133 | 755.331 $ | -81.156 | 18 |
| 16 | 646.752 | 723.719 | 663.489 $ | -76.967 | 17 |
| 17 | 680.457 | 713.146 | 754.616 | -32.689 | 6 |
| 18 | 674.300 | 697.304 | 740.394 | -23.004 | 4 |
| 19 | 636.649 | 674.845 | 670.146 $ | -38.196 | 8 |
| 20 | 588.468 | 654.994 | 670.194 | -66.526 | 15 |

$AAC_{RF}$ = Actual average cost replacing when failure occurs
DIFFERENCE = $AAC_{PH}$ - $AAC_{NONP}$
* positive rank
$T^{+} = 2$ ;
p-value $< 0.005$
$ $AAC_{RF}$ is less than only $AAC_{NONP}$

**Table 4 COMPARISON OF ACTUAL AVERAGE COSTS FOR SMALL SAMPLE SIZES (N = 50) OF PH DISTRIBUTION WITH $T_{42}$ AND $\alpha_{42}$**

| Run | $AAC_{PH}$ | $AAC_{NONP}$ | $AAC_{RF}$ | DIFFERENCE | RANK |
|---|---|---|---|---|---|
| 1 | 618.463 | 561.050 | 545.871 ! | 57.413 | 16 * |
| 2 | 604.098 | 684.564 | 686.650 | -80.466 | 19 |
| 3 | 674.353 | 664.878 | 637.614 ! | 9.475 | 6 * |
| 4 | 605.528 | 559.966 | 575.315 | 45.562 | 15 * |
| 5 | 580.054 | 521.136 | 535.683 | 58.918 | 17 * |
| 6 | 547.534 | 553.458 | 556.857 | -5.924 | 4 |
| 7 | 632.508 | 625.711 | 625.711 | 6.797 | 5 * |
| 8 | 632.510 | 647.089 | 651.264 | -14.579 | 7 |
| 9 | 490.099 | 578.531 | 566.695 $ | -88.432 | 20 |
| 10 | 629.150 | 668.268 | 670.786 | -39.118 | 13 |
| 11 | 535.032 | 536.771 | 535.194 | -1.739 | 3 |
| 12 | 605.358 | 644.458 | 645.689 | -39.100 | 12 |
| 13 | 519.537 | 552.440 | 552.443 | -32.903 | 10 |
| 14 | 532.469 | 553.664 | 569.868 | -21.195 | 8 |
| 15 | 671.880 | 672.507 | 672.160 $ | -0.627 | 1 |
| 16 | 645.125 | 571.650 | 576.024 | 73.475 | 18 * |
| 17 | 495.932 | 533.334 | 539.149 | -37.402 | 11 |
| 18 | 648.741 | 677.909 | 682.898 | -29.168 | 9 |
| 19 | 577.580 | 576.101 | 600.152 | 1.479 | 2 * |
| 20 | 560.629 | 519.418 | 532.841 | 41.211 | 14 * |

$AAC_{RF}$ = Actual average cost replacing when failure occurs
DIFFERENCE = $AAC_{PH} - AAC_{NONP}$
* positive rank
$T^+ = 93$ ;
p-value > 0.05
$ $AAC_{RF}$ is less than only $AAC_{NONP}$
! $AAC_{RF}$ is less than $AAC_{PH}$ and $AAC_{NONP}$

**Table 5 COMPARISON OF ACTUAL AVERAGE COSTS FOR MEDIUM SAMPLE SIZES (N = 100) OF PH DISTRIBUTION WITH $T_{42}$ AND $\alpha_{42}$**

| Run | $AAC_{PH}$ | $AAC_{NONP}$ | $AAC_{RF}$ | DIFFERENCE | RANK |
|---|---|---|---|---|---|
| 1 | 573.310 | 548.751 | 539.868 ! | 24.559 | 13 * |
| 2 | 588.501 | 633.798 | 623.661 $ | -45.297 | 16 |
| 3 | 618.891 | 626.148 | 608.227 ! | -7.257 | 2 |
| 4 | 621.602 | 590.532 | 601.378 | 31.070 | 14 * |
| 5 | 557.189 | 521.040 | 535.590 | 36.149 | 15 * |
| 6 | 549.717 | 562.877 | 569.251 | -13.160 | 5 |
| 7 | 651.656 | 629.631 | 630.823 | 22.025 | 10 * |
| 8 | 615.798 | 624.193 | 628.347 | -8.395 | 4 |
| 9 | 475.757 | 574.707 | 567.688 $ | -98.95 | 20 |
| 10 | 602.762 | 626.458 | 628.035 | -23.696 | 12 |
| 11 | 535.426 | 537.073 | 549.888 | -1.647 | 1 |
| 12 | 580.480 | 598.790 | 590.995 $ | -18.310 | 7 |
| 13 | 525.770 | 597.208 | 597.210 | -71.438 | 19 |
| 14 | 518.669 | 540.978 | 553.372 | -22.309 | 11 |
| 15 | 634.978 | 627.571 | 633.019 | 7.407 | 3 * |
| 16 | 675.027 | 616.239 | 618.041 | 58.788 | 18 * |
| 17 | 535.911 | 585.289 | 587.998 | -49.378 | 17 |
| 18 | 617.194 | 636.912 | 645.975 | -19.718 | 9 |
| 19 | 550.468 | 564.214 | 579.831 | -13.746 | 6 |
| 20 | 539.256 | 520.034 | 531.189 | 19.222 | 8 * |

$AAC_{RF}$ = Actual average cost replacing when failure occurs
DIFFERENCE = $AAC_{PH}$ - $AAC_{NONP}$
* positive rank
$T^+ = 81$ ;
p-value > 0.05
$ $AAC_{RF}$ is less than only $AAC_{NONP}$
! $AAC_{RF}$ is less than $AAC_{PH}$ and $AAC_{NONP}$

Table 6 COMPARISON OF ACTUAL AVERAGE COSTS FOR LARGE SAMPLE SIZES (N = 200) OF PH DISTRIBUTION WITH $T_{42}$ AND $\alpha_{42}$

| Run | $AAC_{PH}$ | $AAC_{NONP}$ | $AAC_{RF}$ | DIFFERENCE | RANK |
|---|---|---|---|---|---|
| 1 | 556.373 | 550.131 | 544.302 ! | 6.242 | 3 * |
| 2 | 574.206 | 616.553 | 621.391 | -42.347 | 17 |
| 3 | 578.172 | 591.186 | 606.516 | -13.014 | 7 |
| 4 | 594.573 | 579.915 | 588.547 | 14.658 | 8 * |
| 5 | 580.261 | 551.976 | 560.463 | 28.285 | 14 * |
| 6 | 526.351 | 558.604 | 573.294 | -32.253 | 15 |
| 7 | 632.238 | 622.079 | 624.275 | 10.159 | 6 * |
| 8 | 612.338 | 621.511 | 610.932 ! | -9.173 | 5 |
| 9 | 460.437 | 567.069 | 563.305 $ | -106.632 | 20 |
| 10 | 557.424 | 599.181 | 602.122 | -41.757 | 16 |
| 11 | 555.711 | 539.303 | 551.773 | 16.408 | 9 * |
| 12 | 560.534 | 584.128 | 589.078 | -23.594 | 12 |
| 13 | 576.939 | 621.856 | 621.857 | -44.917 | 18 |
| 14 | 568.925 | 595.619 | 599.305 | -26.694 | 13 |
| 15 | 562.097 | 583.057 | 595.668 | -20.960 | 11 |
| 16 | 584.778 | 604.211 | 606.979 | -19.433 | 10 |
| 17 | 549.979 | 607.061 | 610.878 | -57.082 | 19 |
| 18 | 631.719 | 628.295 | 639.159 | 3.424 | 2 * |
| 19 | 578.172 | 577.936 | 607.344 | 0.236 | 1 * |
| 20 | 538.683 | 530.002 | 570.516 | 8.681 | 4 * |

$AAC_{RF}$ = Actual average cost replacing when failure occurs
DIFFERENCE = $AAC_{PH}$ - $AAC_{NONP}$
* positive rank
$T^{+} = 47$ ;
$0.01 < \text{p-value} < 0.025$
$ $AAC_{RF}$ is less than only $AAC_{NONP}$
! $AAC_{RF}$ is less than $AAC_{PH}$ and $AAC_{NONP}$

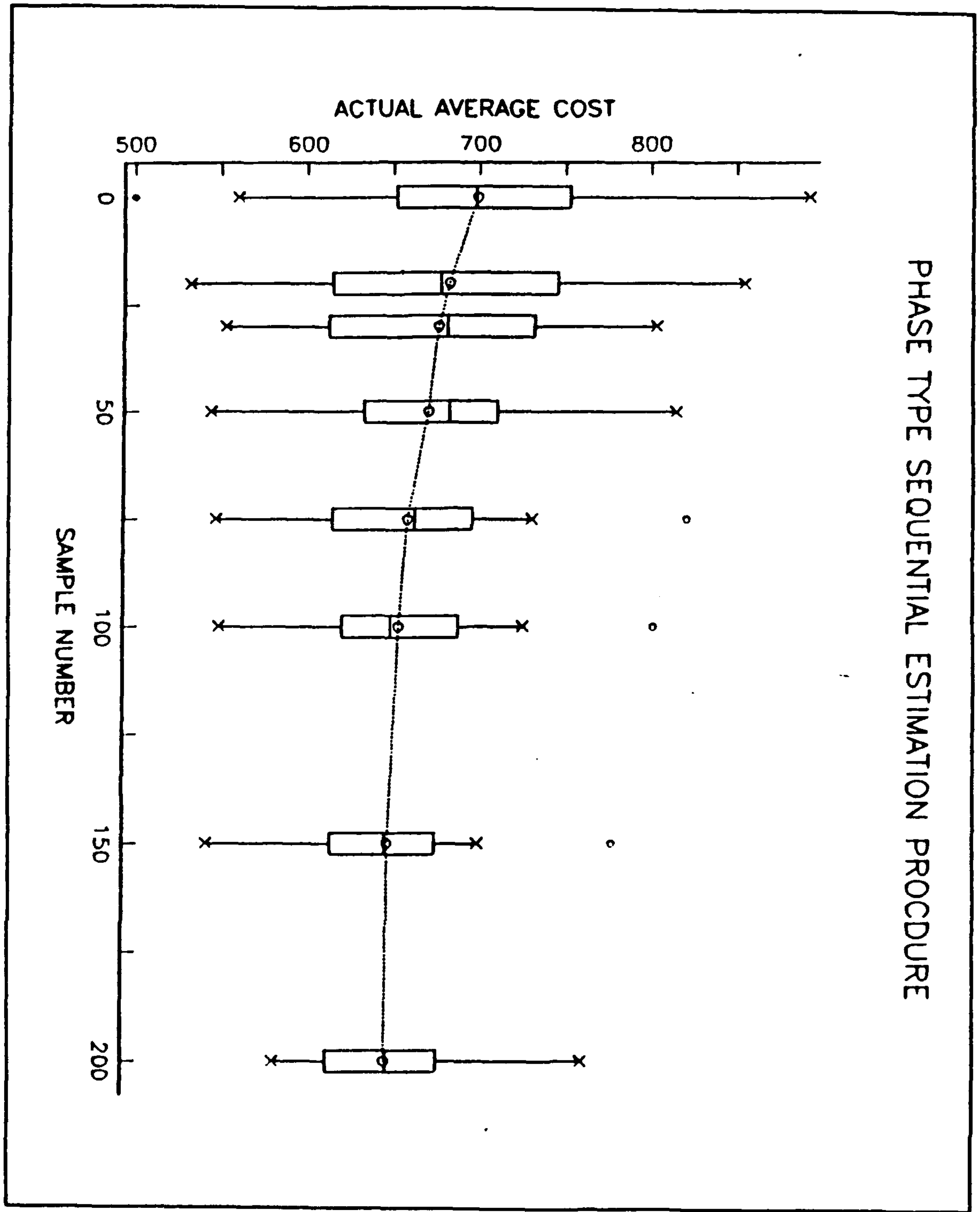


**Figure 5** The box plot of actual average cost for different sample sizes of phase type sequential procedure for $T_{41}$ and $\alpha_{41}$

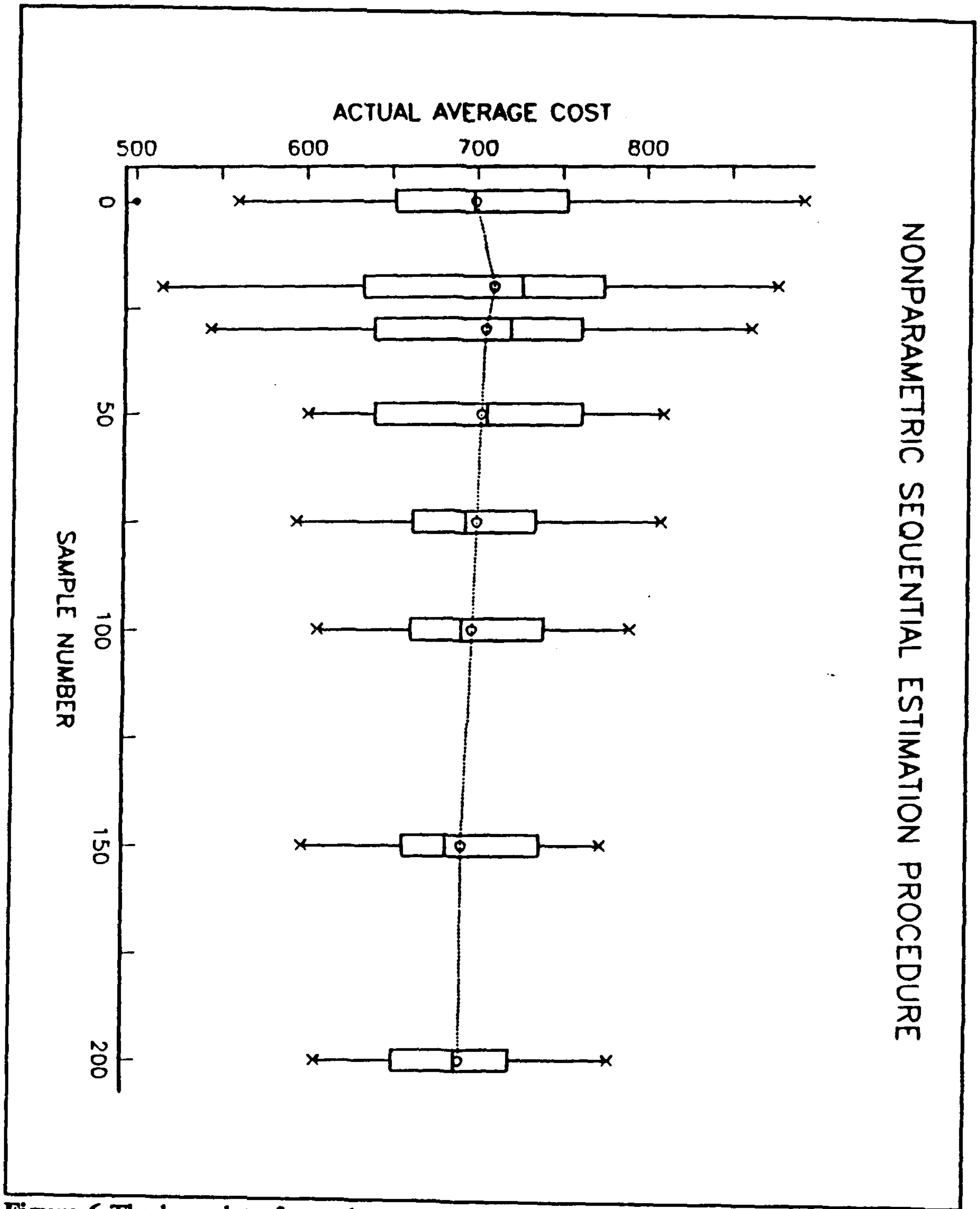


**Figure 6** The box plot of actual average cost for different sample sizes of nonparametric sequential procedure for $T_{41}$ and $\alpha_{41}$

## V. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, the age replacement policy has been considered. A system is replaced at the time of failure or at a scheduled replacement age whichever comes first. The replacement system is assumed to be as good as new. The objective is to achieve the minimum long run expected maintenance cost per unit time. The system lifetimes are assumed to have PH distributions. Estimating the parameters of a PH distribution is difficult because it may have many parameters and its density function involves a matrix exponential. Moment matching with a nonlinear programming approach is used for estimating the parameters. This method does not guarantee that the estimated cost function will have a unique and finite minimum. In this thesis we restric our attention to the case with initial probability vector $\alpha = (1, 0, 0)$. Sequential estimation using the phase type procedure gives smaller average costs than the nonparametric procedure for both small and large sample sizes when the underlying long run average cost function has a very distinct minimum at $\phi^*$. When this cost function is shallow around $\phi^*$, the parametric procedure does not do better than the nonparametric procedure for small samples. In fact, for cost functions that are shallow, it is better not to use a maintenance policy i.e. to take $\hat{\phi}^* = \infty$, until the sample sizes are large enough to give reasonable estimates of the underlying parameters. The reason for this is that for such cost functions, overestimating $\phi^*$ does not increase the long run average cost much, at the same time the decrease in the amount of censoring with over estimating $\phi^*$ greatly improves estimation of the underlying distribution function F. Conversely underestimating $\phi^*$ for these cost functions causes a drastic increase in long run average costs and increase censoring making estimation very difficult.

**Recommendation for the future research**

The following forms a list of future research initiatives:

-Examine the impact of the product-limit estimator in estimating standardized moments on the rest of procedure.

-Do more experimentation on different representations of phase type distributions to get more precise conclusion.

-Look for other policies using phase-sensitive estimate and modified replacement costs over time.

-The phase-type sequential estimations we have been so far assume the underlying lifetimes are iid and after replacement the system is new. To be more realistic, we can modify the initial probability vector, increase transition rates between states, or both, over time.

-Use phase-type sequential estimation when the underlying life distribution F comes from Gamma, Weibull etc., that have increasing failure rate and compare with the nonparametric procedure. Since we can use the denseness of phase type distribution property to approximate the set of distribution with support on $[0, \alpha)$ i.e., the set of lifetimes, we may get a better method when the distribution of lifetimes is not known.

# APPENDIX A.

```
$TITLE    PH DISTRIBUTION ESTIMATION PARAMETERS
*
*-----------GAMS AND DOLLAR CONTROL OPTIONS-----------------------------------
*            (See Appendices B & C)

OPTIONS
   WORK = 100000,
   LIMCOL = 0 , LIMROW  = 0 , SOLPRINT = OFF , DECIMALS = 2
   RESLIM = 100, ITERLIM =100000, OPTCR = 0.0 , SEED  = 3141;

*------------DEFINITIONS AND DATA-----------------------------------------------
* This program uses nonlinear programming to estimate the 3-transient states of phase
* type distribution parameters. Matching the second and third standardized moments are
* used. All adjustable data are prepared by the "FILE SCALAR2" and uses the
* "$INCLUDE" statement bring to the program.

$INCLUDE "FILE SCALAR2"

*---------------------------MODEL-----------------------------------------------------------
   VARIABLE
       Z   objective function value
       A   first entry of the matrix
       B   second entry of the matrix
       C   third entry of the matrix
       D   fourth entry of the matrix
       E   fifth entry of the matrix
       F   sixth entry of the matrix
       G   seventh entry of the matrix
       H   eighth entry of the matrix
       I   ninth entry of the matrix
      AL1  initial probability of being in state 1
      AL2  initial probability of being in state 2
      AL3  initial probability of being in state 3 ;

 EQUATION
    OBJ  Define objective function
    EQ1  Set initial probability of being in state 1(nonnegative)
    EQ2  Set initial probability of being in state 2(nonnegative)
```

```
    EQ3  Set initial probability of being in state 3(nonnegative)
    EQ4  Sum over all of initial prob. less than or equal 1.0
    EQ5  Set the first diagonal entry to be nonpositive
    EQ6  Set the second diagonal entry to be nonpositive
    EQ7  Set the third diagonal entry to be nonpositive
    EQ8  Set off diagonal entries in the first row to be positive
    EQ9  Set off diagonal entries in the first row to be positive
    EQ10 Set sum off diagonal of the first row less than or equal first diagonal
    EQ11 Set off diagonal entries in the second row to be positive
    EQ12 Set off diagonal entries in the second row to be positive
    EQ13 Set sum off diagonal of the second row less than or equal second diagonal
    EQ14 Set off diagonal entries in the third row to be positive
    EQ15 Set off diagonal entries in the third row to be positive
    EQ16 Set sum off diagonal of the third row less than or equal third diagonal
    EQ17 Define the expected lifetime  ;

* MINIMIZE
      OBJ..
        Z =E=

***  3*SQR(C - C(0))  ***

3*POWER((((2*AL1*(POWER((-(F*H)+E*I),2)+(C*H-B*I)*(F*G-D*I)
      +(-(C*E)+B*F)*(-(E*G)+D*H)
+(-(F*H)+E*I)*(C*H-B*I)+(C*H-B*I)*(-(C*G)+A*I)+(-(C*E)+B*F)*(B*G-A*H)
+(-(F*H)+E*I)*(-(C*E)+B*F)+(C*H-B*I)*(C*D-A*F)+(-(C*E)+B*F)*(-(B*D)+
A*E))

+ 2*AL2*((F*G-D*I)*(-(F*H)+E*I)+(-(C*G)+A*I)*(F*G-D*I)+(-(E*G)+D*H)*
      (C*D-A*F)
+(F*G-D*I)*(C*H-B*I)+POWER((-(C*G)+A*I),2)+(C*D-A*F)*(B*G-A*H)
+(F*G-D*I)*(-(C*E)+B*F)+(-(C*G)+A*I)*(C*D-A*F)+(C*D-A*F)*(-(B*D)+A*E))

+ 2*AL3*((-(E*G)+D*H)*(-(F*H)+E*I)+(B*G-A*H)*(F*G-D*I)+(-(B*D)+A*E)*
      (-(E*G)+D*H)
+(-(E*G)+D*H)*(C*H-B*I)+(B*G-A*H)*(-(C*G)+A*I)+(-(B*D)+A*E)*
(B*G-A*H)+(-(E*G)+D*H)*(-(C*E)+B*F)+(B*G-A*H)*(C*D-A*F)+
POWER((-(B*D)+A*E),2)))/

POWER((-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I),2)

-POWER((-(AL1*(-F*H+E*I+C*H-B*I-(C*E)+B*F)
+AL2*(F*G-D*I-C*G+ A*I+C*D-A*F)+AL3*(-E*G+D*H+B*G-A*H-B*D+A*E))/
```

```
(-(C*E*G)+B*F*G+C*D*H-A*F*H-B*D*I+A*E*I)),2))**0.5 /

(-(AL1*(-(F*H)+E*I+C*H-B*I-(C*E)+B*F)
+AL2*(F*G-D*I-C*G+A*I+C*D-A*F)
+AL3*(-E*G+D*H+B*G-A*H-B*D+A*E))/
(-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I)) - MM2),2)

***  W2*SQR(GAMMA - GAMMA(0))  ***

+ POWER(((((-6*AL1*((POWER((-(F*H)+E*I),2)+(C*H-B*I)*(F*G-D*I)+
(-(C*E)+B*F)*(-(E*G)+D*H))*(-(F*H)+E*I)

+((-(F*H)+E*I)*(C*H-B*I)+(C*H-B*I)*(-(C*G)+A*I)+(-(C*E)+B*F)*  (B*G-A*H))*
     (F*G-D*I)

+((-(F*H)+E*I)*(-(C*E)+B*F)+(C*H-B*I)*(C*D-A*F)+(-(C*E)+B*F)*
     (-(B*D)+A*E))*(-(E*G)+D*H)

+(POWER((-(F*H)+E*I),2)+(C*H-B*I)*(F*G-D*I)+(-(C*E)+B*F)*  (-(E*G)+D*H))*
     (C*H-B*I)

+((-(F*H)+E*I)*(C*H-B*I)+(C*H-B*I)*(-(C*G)+A*I)+(-(C*E)+B*F)*(B*G-A*H))*
     (-(C*G)+A*I)

+((-(F*H)+E*I)*(-(C*E)+B*F)+(C*H-B*I)*(C*D-A*F)+(-(C*E)+B*F)*
     (-(B*D)+A*E))*(B*G-A*H)

+(POWER((-(F*H)+E*I),2)+(C*H-B*I)*(F*G-D*I)+(-(C*E)+B*F)*
     (-(E*G)+D*H))*(-(C*E)+B*F)

+((-(F*H)+E*I)*(C*H-B*I)+(C*H-B*I)*(-(C*G)+A*I)+(-(C*E)+B*F)*
     (B*G-A*H))*(C*D-A*F)

+((-(F*H)+E*I)*(-(C*E)+B*F)+(C*H-B*I)*(C*D-A*F)+(-(C*E)+B*F)*
     (-(B*D)+A*E))*(-(B*D)+A*E))

-6*AL2*(((F*G-D*I)*(-(F*H)+E*I)+(-(C*G)+A*I)*(F*G-D*I)+(-(E*G)+D*H)*
     (C*D-A*F))*(-(F*H)+E*I)

+((F*G-D*I)*(C*H-B*I)+POWER((-(C*G)+A*I),2)+(C*D-A*F)*(B*G-A*H))*
     (F*G-D*I)

+((F*G-D*I)*(-(C*E)+B*F)+(-(C*G)+A*I)*(C*D-A*F)+(C*D-A*F)*
```

(-(B*D)+A*E))*(-(E*G)+D*H)

+((F*G-D*I)*(-(F*H)+E*I)+(-(C*G)+A*I)*(F*G-D*I)+(-(E*G)+D*H)*
(C*D-A*F))*(C*H-B*I)

+((F*G-D*I)*(C*H-B*I)+POWER((-(C*G)+A*I),2)+(C*D-A*F)*(B*G-A*H))*
(-(C*G)+A*I)

+((F*G-D*I)*(-(C*E)+B*F)+(-(C*G)+A*I)*(C*D-A*F)+(C*D-A*F)*
(-(B*D)+A*E))*(B*G-A*H)

+((F*G-D*I)*(-(F*H)+E*I)+(-(C*G)+A*I)*(F*G-D*I)+(-(E*G)+D*H)*
(C*D-A*F))*(-(C*E)+B*F)

+((F*G-D*I)*(C*H-B*I)+POWER((-(C*G)+A*I),2)+(C*D-A*F)*(B*G-A*H))*
(C*D-A*F)

+((F*G-D*I)*(-(C*E)+B*F)+(-(C*G)+A*I)*(C*D-A*F)+(C*D-A*F)*
(-(B*D)+A*E))*(-(B*D)+A*E))

-6*AL3*(((-(E*G)+D*H)*(-(F*H)+E*I)+(B*G-A*H)*(F*G-D*I)+(-(B*D)+A*E)*
(-(E*G)+D*H))*(-(F*H)+E*I)

+((-(E*G)+D*H)*(C*H-B*I)+(B*G-A*H)*(-(C*G)+A*I)+(-(B*D)+A*E)*
(B*G-A*H))*(F*G-D*I)

+((-(E*G)+D*H)*(-(C*E)+B*F)+(B*G-A*H)*(C*D-A*F)+
POWER((-(B*D)+A*E),2))*(-(E*G)+D*H)

+((-(E*G)+D*H)*(-(F*H)+E*I)+(B*G-A*H)*(F*G-D*I)+(-(B*D)+A*E)*
(-(E*G)+D*H))*(C*H-B*I)

+((-(E*G)+D*H)*(C*H-B*I)+(B*G-A*H)*(-(C*G)+A*I)+(-(B*D)+A*E)*
(B*G-A*H))*(-(C*G)+A*I)

+((-(E*G)+D*H)*(-(C*E)+B*F)+(B*G-A*H)*(C*D-A*F)+
POWER((-(B*D)+A*E),2))*(B*G-A*H)

+((-(E*G)+D*H)*(-(F*H)+E*I)+(B*G-A*H)*(F*G-D*I)+(-(B*D)+A*E)*
(-(E*G)+D*H))*(-(C*E)+B*F)

+((-(E*G)+D*H)*(C*H-B*I)+(B*G-A*H)*(-(C*G)+A*I)+(-(B*D)+A*E)*
(B*G-A*H))*(C*D-A*F)

+((-(E*G)+D*H)*(-(C*E)+B*F)+(B*G-A*H)*(C*D-A*F)+
POWER((-(B*D)+A*E),2))*(-(B*D)+A*E)))

/POWER((-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I),3)

+3*((AL1*(-(F*H)+E*I+C*H-B*I-(C*E)+B*F)+AL2*(F*G-D*I-C*G+A*I+C*D-A*F)+AL3*(-E*G+D*H+B*G-A*H-B*D+A*E))/(-C*E*G+B*F*G+C*D*H-A*F*H-B*D*I+A*E*I))*

*** expected of lifetime square ***

((2*AL1*(POWER((-(F*H)+E*I),2)+(C*H-B*I)*(F*G-D*I)+(-(C*E)+B*F)*(-(E*G)+D*H)+(-(F*H)+E*I)*(C*H-B*I)+(C*H-B*I)*(-(C*G)+A*I)+(-(C*E)+B*F)*(B*G-A*H)+(-(F*H)+E*I)*(-(C*E)+B*F)+(C*H-B*I)*(C*D-A*F)+(-(C*E)+B*F)*(-(B*D)+A*E))

+2*AL2*((F*G-D*I)*(-(F*H)+E*I)+(-(C*G)+A*I)*(F*G-D*I)+(-(E*G) +D*H)*(C*D-A*F)+(F*G-D*I)*(C*H-B*I)+POWER((-(C*G)+A*I),2)+(C*D-A*F)*(B*G-A*H)+(F*G-D*I)*(-(C*E)+B*F)+(-(C*G)+A*I)*(C*D-A*F)+(C*D-A*F)*(-(B*D)+A*E))

+2*AL3*((-(E*G)+D*H)*(-(F*H)+E*I)+(B*G-A*H)*(F*G-D*I)+(-(B*D)+A*E)*(-(E*G)+D*H)+(-(E*G)+D*H)*(C*H-B*I)+(B*G-A*H)*(-(C*G)+A*I)+(-(B*D)+A*E)*(B*G-A*H)+(-(E*G)+D*H)*(-(C*E)+B*F)+(B*G-A*H)*(C*D-A*F)+POWER((-(B*D)+A*E),2)))/

POWER((-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I),2))

*** 2 time cube of the expected lifetime ***

-2*POWER(((AL1*(-(F*H)+E*I+C*H-B*I-(C*E)+B*F)+AL2*(F*G-D*I-C*G+A*I+C*D-A*F)+AL3*(-E*G+D*H+B*G-A*H-B*D+A*E))/(-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I)),3))

/((2*AL1*(POWER((-(F*H)+E*I),2)+(C*H-B*I)*(F*G-D*I)+(-(C*E)+B*F)*(-(E*G)+D*H)+(-(F*H)+E*I)*(C*H-B*I)+(C*H-B*I)*(-(C*G)+A*I)+(-(C*E)+B*F)*(B*G-A*H)+(-(F*H)+E*I)*(-(C*E)+B*F)+(C*H-B*I)*(C*D-A*F)+(-(C*E)+B*F)*(-(B*D)+A*E))

+2*AL2*((F*G-D*I)*(-(F*H)+E*I)+(-(C*G)+A*I)*(F*G-D*I)+(-(E*G)+D*H)*(C*D-A*F)+(F*G-D*I)*(C*H-B*I)+POWER((-(C*G)+A*I),2)+(C*D-A*F)*(B*G-A*H)+(F*G-D*I)*(-(C*E)+B*F)+(-(C*G)+A*I)*(C*D-A*F)+(C*D-A*F)*(-(B*D)+A*E))

```
+ 2*AL3*((-(E*G)+D*H)*(-(F*H)+E*I)+(B*G-A*H)*(F*G-D*I)+
(-(B*D)+A*E)*(-(E*G)+D*H)+(-(E*G)+D*H)*(C*H-B*I)+(B*G-A*H)*
(-(C*G)+A*I)+(-(B*D)+A*E)*(B*G-A*H)+(-(E*G)+D*H)*(-(C*E)+B*F)
+(B*G-A*H)*(C*D-A*F)+POWER((-(B*D)+A*E),2)))/

(POWER((-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I),2))

-POWER(((AL1*(-(F*H)+E*I+C*H-B*I-(C*E)+B*F)
         +AL2*(F*G-D*I-C*G+A*I+C*D-A*F)
         +AL3*(-E*G+D*H+B*G-A*H-B*D+A*E))/
 (-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I)),2))**1.5) - MM3),2);
*
* SUBJECT TO
*
   EQ1..
      AL1 =G= 0 ;
   EQ2..
      AL2 =G= 0 ;

   EQ3..
      AL3 =G= 0 ;
   EQ4..
      AL1 + AL2 + AL3 =L= 1 ;
   EQ5..
      A   =L= 0  ;
   EQ6..
      E   =L= 0  ;
   EQ7..
      I   =L= 0  ;
   EQ8..
      B   =G= 0 ;
   EQ9..
      C   =G= 0 ;
   EQ10..
     B + C =L= -A ;
   EQ11..
      D   =G= 0 ;
   EQ12..
      F   =G= 0 ;
   EQ13..
     D + F =L= -E ;
   EQ14..
      G   =G= 0 ;
```

```
  EQ15..
      H   =G= 0 ;
  EQ16..
     G + H =L= -I ;
  EQ17..
      -(AL1*(-(F*H)+E*I+C*H-B*I-(C*E)+B*F)
      +AL2*(F*G-D*I-C*G+A*I+C*D-A*F)
      +AL3*(-E*G+D*H+B*G-A*H-B*D+A*E))/
   (-(C*E*G)+B*F*G+C*D*H-A*F*H-B*D*I+A*E*I) =E= EX ;
*
*------------------------REPORT------------------------------------------------------
*
  MODEL MATCHM /ALL/ ;

* Parameters bounding

 A.LO = -7.0; B.LO = 3.0; C.LO = 0.0; D.LO = 0.0; E.LO =-7.0;
 A.UP = -5.0; B.UP = 5.0; C.UP = 2.0; D.UP = 1.0; E.UP =-5.0;
 F.LO =  4.0; G.LO = 0.0; H.LO = 0.0; I.LO = -8.0;
 F.UP =  5.0; G.UP = 1.0; H.UP = 2.0; I.UP = -7.0;
 AL1.LO = 1.0; AL2.LO = 0.0; AL3.LO = 0.0;
 AL1.UP = 1.0; AL2.UP = 0.0; AL3.UP = 0.0;

* The initial solution

   A.L = AA ; B.L = BB ; C.L = CC; D.L = DD; E.L = EE ;
   F.L = FF ; G.L = GG ; H.L = HH ; I.L = II ;
   AL1.L = ALP1 ; AL2.L = ALP2 ; AL3.L = ALP3 ;

   SOLVE MATCHM USING NLP MINIMIZING Z ;
   DISPLAY Z.L,A.L , B.L, C.L, D.L, E.L, F.L, G.L, H.L, I.L,
            AL1.L, AL2.L, AL3.L;

* Write output to the CMS file

FILE OUT1 /FILE OBJ A / ;
PUT OUT1 ;
PUT Z.L ;
FILE OUT2 /FILE ALPHA A / ;
PUT OUT2 ;
PUT A.L ;
FILE OUT3 /FILE BRAVO A / ;
PUT OUT3 ;
```

```
PUT B.L ;
FILE OUT4 /FILE CHAREE A / ;
PUT OUT4 ;
PUT C.L ;
FILE OUT5 /FILE DELTA A / ;
PUT OUT5 ;
PUT D.L ;
FILE OUT6 /FILE ECHO A / ;
PUT OUT6 ;
PUT E.L ;
FILE OUT7 /FILE FOXTROT A / ;
PUT OUT7 ;
PUT F.L ;
FILE OUT8 /FILE GOLF A / ;
PUT OUT8 ;
PUT G.L ;
FILE OUT9 /FILE HOTEL A / ;
PUT OUT9 ;
PUT H.L ;
FILE OUT10 /FILE INDIA A / ;
PUT OUT10 ;
PUT I.L ;
FILE OUT11 /FILE AL1 A / ;
PUT OUT11 ;
PUT AL1.L ;
FILE OUT12 /FILE AL2 A / ;
PUT OUT12 ;
PUT AL2.L ;
FILE OUT13 /FILE AL3 A / ;
PUT OUT13 ;
PUT AL3.L ;
```

**THE INITIAL COMPLETE SYSTEM LIFETIME SET 1-5**

| N | SET 1 | SET 2 | SET 3 | SET 4 | SET 5 |
|---|---|---|---|---|---|
| 1 | 0.1226 | 0.0740 | 0.0855 | 0.0528 | 0.1143 |
| 2 | 0.1455 | 0.1413 | 0.2157 | 0.1139 | 0.1999 |
| 3 | 0.2662 | 0.2261 | 0.2209 | 0.2412 | 0.2002 |
| 4 | 0.2925 | 0.2629 | 0.2839 | 0.2892 | 0.2422 |
| 5 | 0.3205 | 0.2831 | 0.2852 | 0.3118 | 0.3164 |
| 6 | 0.4214 | 0.3340 | 0.3116 | 0.3485 | 0.3398 |
| 7 | 0.4394 | 0.3911 | 0.3403 | 0.3973 | 0.3534 |
| 8 | 0.4685 | 0.4617 | 0.4517 | 0.4442 | 0.3893 |
| 9 | 0.5804 | 0.4761 | 0.4691 | 0.4686 | 0.4410 |
| 10 | 0.5871 | 0.4909 | 0.5254 | 0.4777 | 0.4766 |
| 11 | 0.6228 | 0.5290 | 0.5536 | 0.4885 | 0.4964 |
| 12 | 0.6534 | 0.5818 | 0.5993 | 0.4970 | 0.5074 |
| 13 | 0.7220 | 0.5962 | 0.6575 | 0.5114 | 0.5240 |
| 14 | 0.7590 | 0.6233 | 0.6623 | 0.6073 | 0.5825 |
| 15 | 0.7618 | 0.6419 | 0.6855 | 0.6433 | 0.6033 |
| 16 | 0.8206 | 0.7407 | 0.7156 | 0.7576 | 0.6255 |
| 17 | 0.8245 | 0.7682 | 0.7690 | 0.8057 | 0.7391 |
| 18 | 0.8317 | 0.8312 | 0.7866 | 0.8740 | 0.7723 |
| 19 | 1.0056 | 0.8693 | 0.8852 | 1.0004 | 0.9128 |
| 20 | 1.0259 | 0.9727 | 0.9473 | 1.0424 | 1.0832 |
| 21 | 1.0853 | 1.1084 | 1.0478 | 1.0634 | 1.1936 |
| 22 | 1.1659 | 1.1272 | 1.1155 | 1.0659 | 1.2003 |
| 23 | 1.1852 | 1.2037 | 1.2936 | 1.2238 | 1.8976 |
| 24 | 1.2040 | 1.3319 | 1.3906 | 1.6404 | 2.0769 |
| 25 | 1.3069 | 2.5362 | 1.6715 | 3.0033 | 2.2631 |
| **E[X]** | 0.705 | 0.704 | 0.679 | 0.735 | 0.742 |
| $2^{nd}$ MM | 0.47917 | 0.70936 | 0.57558 | 0.80566 | 0.77290 |
| $3^{rd}$ MM | 0.03190 | 1.88276 | 0.73612 | 2.23985 | 1.40912 |

**THE INITIAL COMPLETE SYSTEM LIFETIME SET 6-10**

| N | SET 6 | SET 7 | SET 8 | SET 9 | SET 10 |
|---|---|---|---|---|---|
| 1 | 0.1337 | 0.0402 | 0.1759 | 0.0830 | 0.0761 |
| 2 | 0.1538 | 0.0634 | 0.2031 | 0.1101 | 0.0913 |
| 3 | 0.2452 | 0.1939 | 0.2106 | 0.2083 | 0.2235 |
| 4 | 0.2876 | 0.1954 | 0.2376 | 0.2339 | 0.3309 |
| 5 | 0.3591 | 0.1958 | 0.3043 | 0.2408 | 0.3470 |
| 6 | 0.4589 | 0.2353 | 0.3719 | 0.2726 | 0.3662 |
| 7 | 0.5161 | 0.2775 | 0.3850 | 0.2777 | 0.4091 |
| 8 | 0.6030 | 0.3185 | 0.3892 | 0.2894 | 0.4523 |
| 9 | 0.6181 | 0.3579 | 0.4032 | 0.3137 | 0.5581 |
| 10 | 0.6761 | 0.3922 | 0.4199 | 0.3227 | 0.5800 |
| 11 | 0.6851 | 0.3985 | 0.6029 | 0.3390 | 0.5805 |
| 12 | 0.7738 | 0.5206 | 0.6406 | 0.3873 | 0.5993 |
| 13 | 0.8541 | 0.5233 | 0.6506 | 0.3878 | 0.6354 |
| 14 | 0.8654 | 0.6261 | 0.7156 | 0.4055 | 0.6571 |
| 15 | 0.9539 | 0.6336 | 0.7584 | 0.4076 | 0.7529 |
| 16 | 1.0193 | 0.6776 | 0.7677 | 0.4333 | 0.7874 |
| 17 | 1.0352 | 0.8705 | 0.8754 | 0.6178 | 0.7991 |
| 18 | 1.0677 | 1.1124 | 0.8836 | 0.6245 | 0.8572 |
| 19 | 1.1196 | 1.1228 | 0.9575 | 0.6455 | 0.8920 |
| 20 | 1.1995 | 1.1742 | 0.9805 | 0.9824 | 0.9024 |
| 21 | 1.2301 | 1.1872 | 1.3367 | 1.0054 | 0.9699 |
| 22 | 1.3043 | 1.1887 | 1.4040 | 1.1143 | 0.9734 |
| 23 | 2.0273 | 1.4849 | 1.5844 | 1.2279 | 1.0452 |
| 24 | 3.2891 | 1.7218 | 1.6250 | 1.2420 | 1.1461 |
| 25 | 3.5360 | 2.5301 | 1.8392 | 2.1319 | 1.1594 |
| **E[X]** | 1.000 | 0.722 | 0.749 | 0.572 | 0.648 |
| 2nd MM | 0.82457 | 0.81263 | 0.63261 | 0.80948 | 0.46844 |
| 3rd MM | 1.89734 | 1.25687 | 0.79102 | 1.69217 | 0.14707 |

**THE INITIAL COMPLETE SYSTEM LIFETIME SET 11-15**

| N | SET 11 | SET 12 | SET 13 | SET 14 | SET 15 |
|---|---|---|---|---|---|
| 1 | 0.0268 | 0.0433 | 0.0470 | 0.1730 | 0.1101 |
| 2 | 0.0745 | 0.1285 | 0.0922 | 0.2556 | 0.1386 |
| 3 | 0.1235 | 0.2052 | 0.1295 | 0.3124 | 0.1709 |
| 4 | 0.1800 | 0.2435 | 0.2064 | 0.3125 | 0.1998 |
| 5 | 0.2526 | 0.2924 | 0.2924 | 0.3447 | 0.2075 |
| 6 | 0.2704 | 0.3335 | 0.2957 | 0.3894 | 0.2588 |
| 7 | 0.2972 | 0.3645 | 0.3110 | 0.4044 | 0.2669 |
| 8 | 0.3697 | 0.4936 | 0.3209 | 0.5555 | 0.2737 |
| 9 | 0.3811 | 0.4984 | 0.4173 | 0.5796 | 0.2905 |
| 10 | 0.3990 | 0.5064 | 0.4577 | 0.5803 | 0.3506 |
| 11 | 0.4070 | 0.5778 | 0.5168 | 0.5928 | 0.3604 |
| 12 | 0.4855 | 0.6699 | 0.5179 | 0.6056 | 0.4117 |
| 13 | 0.6402 | 0.7442 | 0.5258 | 0.6525 | 0.5809 |
| 14 | 0.7087 | 0.7502 | 0.5456 | 0.7299 | 0.5981 |
| 15 | 0.7408 | 0.8656 | 0.5547 | 0.7355 | 0.6558 |
| 16 | 0.8496 | 0.8831 | 0.5559 | 0.8122 | 0.7306 |
| 17 | 0.9290 | 1.0607 | 0.5633 | 0.9379 | 0.7388 |
| 18 | 0.9873 | 1.1003 | 0.5692 | 0.9433 | 0.7400 |
| 19 | 0.9983 | 1.1834 | 0.6501 | 1.0197 | 0.7786 |
| 20 | 1.0318 | 1.2235 | 0.7978 | 1.2675 | 0.7936 |
| 21 | 1.0937 | 1.3370 | 0.8750 | 1.3842 | 1.1532 |
| 22 | 1.1137 | 1.5754 | 1.0843 | 1.6525 | 1.2444 |
| 23 | 1.2706 | 1.8085 | 1.1683 | 1.8276 | 1.2781 |
| 24 | 1.4875 | 1.9023 | 1.1881 | 1.8545 | 1.3428 |
| 25 | 1.4958 | 1.9894 | 1.3242 | 2.8764 | 1.5937 |
| E[X] | 0.665 | 0.831 | 0.560 | 0.872 | 0.611 |
| $2^{nd}$ MM | 0.64986 | 0.66778 | 0.60540 | 0.71398 | 0.68553 |
| $3^{rd}$ MM | 0.32858 | 0.60712 | 0.69085 | 1.51250 | 0.77568 |

THE INITIAL COMPLETE SYSTEM LIFETIME SET 16-20

| N | SET 16 | SET 17 | SET 18 | SET 19 | SET 20 |
|---|---|---|---|---|---|
| 1 | 0.1677 | 0.0565 | 0.0398 | 0.1798 | 0.0202 |
| 2 | 0.1819 | 0.1811 | 0.1402 | 0.2037 | 0.1920 |
| 3 | 0.2139 | 0.1941 | 0.1506 | 0.2197 | 0.2205 |
| 4 | 0.3436 | 0.2810 | 0.1792 | 0.3159 | 0.2209 |
| 5 | 0.3641 | 0.3315 | 0.1922 | 0.3464 | 0.3406 |
| 6 | 0.4393 | 0.4147 | 0.3787 | 0.3629 | 0.3645 |
| 7 | 0.4800 | 0.4249 | 0.4660 | 0.3744 | 0.3941 |
| 8 | 0.5328 | 0.4629 | 0.4944 | 0.3793 | 0.4130 |
| 9 | 0.5643 | 0.4939 | 0.5952 | 0.3870 | 0.5525 |
| 10 | 0.7220 | 0.5382 | 0.6198 | 0.3875 | 0.5729 |
| 11 | 0.7393 | 0.5776 | 0.6269 | 0.4011 | 0.5787 |
| 12 | 0.7666 | 0.6003 | 0.6276 | 0.4906 | 0.5835 |
| 13 | 0.7766 | 0.6055 | 0.6608 | 0.7436 | 0.6138 |
| 14 | 0.7964 | 0.6230 | 0.7183 | 0.7690 | 0.6389 |
| 15 | 0.7981 | 0.6477 | 0.7970 | 0.7824 | 0.6609 |
| 16 | 0.8228 | 0.7631 | 0.7997 | 0.8226 | 0.6624 |
| 17 | 0.8390 | 0.7980 | 0.8065 | 0.8492 | 0.7091 |
| 18 | 0.9422 | 0.7988 | 0.8065 | 0.9542 | 0.7178 |
| 19 | 1.0115 | 0.8518 | 0.9682 | 1.2333 | 0.7447 |
| 20 | 1.0381 | 0.9091 | 0.9939 | 1.3601 | 1.2027 |
| 21 | 1.1195 | 1.0447 | 1.0028 | 1.4489 | 1.3247 |
| 22 | 1.2716 | 1.1973 | 1.0875 | 1.5989 | 1.4412 |
| 23 | 1.3426 | 1.2331 | 1.2722 | 2.2702 | 1.6155 |
| 24 | 1.6082 | 1.2339 | 1.4913 | 2.2971 | 1.7162 |
| 25 | 1.7724 | 1.3386 | 1.8663 | 3.1519 | 2.1912 |
| E[X] | 0.786 | 0.664 | 0.711 | 0.893 | 0.748 |
| $2^{nd}$ MM | 0.52328 | 0.51958 | 0.60066 | 0.83695 | 0.70314 |
| $3^{rd}$ MM | 0.55452 | 0.32242 | 0.66254 | 1.46868 | 1.13073 |

## APPENDIX B.

```
      PROGRAM COST
C
C    THIS PROGRAM PROVIDES THE OPTIMAL AGE REPLACEMENT OF THE
C    PHASE TYPE DISTRIBUTION. IT WORKS ONLY SPECIFIC CASE OF 3
C    PHASES OF TRANSIENT STATES. THE PARAMETERS ARE AS FOLLOWS:
C      T(I,J)  = ENTRIES OF THE TRANSITION MATRIX OF PH DISTN.
C      AL(I)   = INITIAL PROBABILITY OF BEING IN STATE I
C      C1      = PLANNED MAINTENANCE COST
C      C2      = UNPLANNED MAINTENANCE COST
C      L       = MAXIMUM LIFETIME THAT WANT TO CALCULATE
C      DET     = DETERMINANT OF TRANSITION MATRIX
C    SMALL     = THE OPTIMAL AGE REPLACEMENT
C       CS     = AGE REPLACEMENT COSTS
*
* INITIAL INPUT
*
      INTEGER I,J,K,N
      REAL C1, C2, TI(3,3), T(3,3), DET, AL(3), L, OBJ
      REAL TS(3,3), SMALL, PI, A, IDEN(3,3), S, NUMER
      REAL DENOM, P(3,3), Q(3,3), CS(400), EXPTS(3,3)
      DATA C1,C2,L,TS/100.0,500.0,4.0,9*0.0 /
      N = 0
      S = 0.01
      OPEN(UNIT =11,FILE = 'ALPHA')
      OPEN(UNIT =12,FILE = 'BRAVO')
      OPEN(UNIT =13,FILE = 'CHAREE')
      OPEN(UNIT =14,FILE = 'DELTA')
      OPEN(UNIT =15,FILE = 'ECHO')
      OPEN(UNIT =16,FILE = 'FOXTROT')
      OPEN(UNIT =17,FILE = 'GOLF')
      OPEN(UNIT =18,FILE = 'HOTEL')
      OPEN(UNIT =19,FILE = 'INDIA')
      OPEN(UNIT =20,FILE = 'AL1')
      OPEN(UNIT =21,FILE = 'AL2')
      OPEN(UNIT =22,FILE = 'AL3')
      OPEN(UNIT =23,FILE = 'OBJ')
      READ(11,*)T(1,1)
      READ(12,*)T(1,2)
      READ(13,*)T(1,3)
```

```
      READ(14,*)T(2,1)
      READ(15,*)T(2,2)
      READ(16,*)T(2,3)
      READ(17,*)T(3,1)
      READ(18,*)T(3,2)
      READ(19,*)T(3,3)
      READ(20,*)AL(1)
      READ(21,*)AL(2)
      READ(22,*)AL(3)
      READ(23,*)OBJ
      IF(OBJ .GT. 0.05) GOTO 555
*
* CALCULATION COST FOR EACH REPLACEMENT TIME
*
      DET = -(T(1,3)*T(2,2)*T(3,1))
     /      + T(1,2)*T(2,3)*T(3,1)
     /      + T(1,3)*T(2,1)*T(3,2)
     /      - T(1,1)*T(2,3)*T(3,2)
     /      - T(1,2)*T(2,1)*T(3,3)
     /      + T(1,1)*T(2,2)*T(3,3)
      PRINT*,'DET = ',DET
      TI(1,1) = (-(T(2,3)*T(3,2)) + T(2,2)*T(3,3))/DET
      TI(1,2) = (T(1,3)*T(3,2) - T(1,2)*T(3,3))/DET
      TI(1,3) = (-(T(1,3)*T(2,2)) + T(1,2)*T(2,3))/DET
      TI(2,1) = (T(2,3)*T(3,1) - T(2,1)*T(3,3))/DET
      TI(2,2) = (-(T(1,3)*T(3,1)) + T(1,1)*T(3,3))/DET
      TI(2,3) = (T(1,3)*T(2,1) - T(1,1)*T(2,3))/DET
      TI(3,1) = (-(T(2,2)*T(3,1)) + T(2,1)*T(3,2))/DET
      TI(3,2) = (T(1,2)*T(3,1) - T(1,1)*T(3,2))/DET
      TI(3,3) = (-(T(1,2)*T(2,1)) + T(1,1)*T(2,2))/DET
      DO 10 I = 1,3
        DO 5 J = 1,3
          IF(I.EQ.J)THEN
            IDEN(I,J) = 1.0
          ELSE
            IDEN(I,J) = 0.0
          ENDIF
    5   CONTINUE
   10 CONTINUE
*
  100 CONTINUE
```

```
      N = N+1
      DO 20 I = 1,3
         DO 15 J = 1,3
            TS(I,J) = S*T(I,J)
 15      CONTINUE
 20   CONTINUE
*
* FIND THE EXPONENTIAL OF MATRIX TS
*
      CALL EXPON(TS,EXPTS,PI)
      A = 0.0
      DO 21 I = 1,3
         DO 22 J = 1,3
            A = A + AL(I)*EXPTS(I,J)
 22      CONTINUE
 21   CONTINUE
      NUMER = C2 - A*(C2-C1)
      DO 30 I = 1,3
         DO 25 J = 1,3
            P(I,J) = EXPTS(I,J) - IDEN(I,J)
 25      CONTINUE
 30   CONTINUE
      DO 45 I = 1,3
         DO 40 J = 1,3
            Q(I,J) = 0.0
            DO 35 K = 1,3
               Q(I,J) = Q(I,J) +TI(I,K)*P(K,J)
 35         CONTINUE
 40      CONTINUE
 45   CONTINUE
      DENOM = 0.0
      DO 55 I = 1,3
         DO 50 J = 1,3
            DENOM = DENOM + AL(I)*Q(I,J)
 50      CONTINUE
 55   CONTINUE
      CS(N) = NUMER/DENOM
      S = S+0.01
      IF(S.LE.L)GOTO 100
*
* FIND THE TIME THAT HAS MINIMUM COST
*
```

```
      K = 1
  200 IF(K.EQ.N)THEN
        SMALL = K*0.01
      ELSEIF(CS(K+1).GT.CS(K))THEN
        SMALL = K*0.01
      ELSEIF(CS(K+1).LE.CS(K))THEN
        SMALL = (K+1)*0.01
        K = K + 1
        GOTO 200
      ENDIF
*
* OUTPUT OPTIMAL REPLACEMENT AGE
*
      OPEN(UNIT = 3,FILE = 'OPTAGE')
      WRITE(3,333)SMALL , PI
  333 FORMAT(1X,F6.3,4X,E11.4)
      OPEN(UNIT = 4,FILE = 'OPTCOST')
      WRITE(4,*)CS(K)
  555 STOP
      END

***********************************************************************
      SUBROUTINE EXPON(A,EXPA,PI)
C
C  THIS SUBROUTINE USES TO FIND THE EXPONENTIAL OF A 3X3 MATRIX.
C  IT WORK WITH COUPLE OF IMSL SUBROUTINES
C  ('EVCRG','LINCG','EPIRG').
C
      INTEGER LDA,LDEVEC,N,NOUT,I,J,K
      PARAMETER(N=3,LDA=N,LDEVEC=N,LDUINV=N,LD=N,LDU=N)
      REAL A(LDA,N),PI,EXPA(3,3)
      COMPLEX EVAL(N), EVEC(LDEVEC,N), ELD(3,3), UINV(3,3),
      COMPLEX C(3,3), D(3,3)
      OPEN(UNIT=1,FILE='EXPONENT')
*
* CALCULATE THE EIGENVALUES AND EIGENVECTERS
*
      CALL EVCRG(N,A,LDA,EVAL,EVEC,LDEVEC)
      DO 15 I = 1,3
        DO 10 J = 1,3
            ELD(I,J) = 0.0
   10   CONTINUE
   15 CONTINUE
```

```
      ELD(1,1) = CEXP(EVAL(1))
      ELD(2,2) = CEXP(EVAL(2))
      ELD(3,3) = CEXP(EVAL(3))
*
* CALCULATE THE INVERSE EIGENVECTERS' MATRIX
*
      CALL LINCG(N,EVEC,LDU,UINV,LDUINV)
      DO 30 I = 1,3
        DO 25 J = 1,3
          C(I,J) = 0.0
          DO 20 K = 1,3
            C(I,J) = C(I,J) + ELD(I,K)*UINV(K,J)
   20     CONTINUE
   25   CONTINUE
   30 CONTINUE
      DO 45 I = 1,3
        DO 40 J = 1,3
          D(I,J) = 0.0
          DO 35 K = 1,3
            D(I,J) = D(I,J) + EVEC(I,K)*C(K,J)
   35     CONTINUE
   40   CONTINUE
   45 CONTINUE
      DO 55 I = 1,3
        DO 50 J = 1,3
          EXPA(I,J) = REAL(D(I,J))
   50   CONTINUE
   55 CONTINUE
C
C CALCULATE THE PERFORMANCE INDEX OF EIGENVALUES AND
C EIGENVECTERS IF LESS THAN 1 'EXCELLENT',IF IT IS BETWEEN 1
C AND 100 'GOOD', IF IT IS GREATER THAN 100 'THE SOLUTION IS
C SUSPECTED'
C
      PI = EPIRG(N,N,A,LDA,EVAL,EVEC,LDEVEC)
      RETURN
      END
```

```
      PROGRAM SIMLIF
C
C    THIS PROGRAM PROVIDES A RANDOM NUMBER OF A PHASE TYPE
C    DISTRIBUTION THAT HAVE SPECIFIC REPRESENTATION WITH AN
C    INITIAL PROBABILITIES VECTOR(ALPHA) AND A MATRIX OF
C    TRANSITION  RATES BETWEEN TRANSIENT STATES. THIS PROGRAMS
C    WORKS WITH SUBROUTINE 'RANNUM' AND MORE SPECIFIC CASE
C    ONLY 4 STATES DISTRIBUTION. THE VARIABLES ARE AS FOLLOWS:
C      LD(I,J)= TRANSITION RATE FROM STATE I TO STATE J
C      P(I,J)  = TRANSITION PROBABILITY FROM STATE I TO STATE J
C      AL(I)   = INITIAL PROBABILITY OF BEING IN STATE I
C      ELIF    = THE SUPPOSED UPPER BOUND EXPECTED LIFETIME
*
* INITIALIZATION
*
      INTEGER S,SEED,N,N1
      REAL U,EXP,LIF,ELIF
      REAL LD12,LD13,LD14,LD21,LD23,LD24,LD31,LD32,LD34
      REAL P12,P13,P21,P23,P31,P32,AL1,AL2,AL3,D
      DATA LD12,LD13,LD14,LD21,LD23,LD24,LD31,LD32,LD34
     +    /5.0,5.0,5.0,6.0,6.0,6.0,7.0,7.0,7.0/
      DATA P12,P13,P21,P23,P31,P32,AL1,AL2,AL3,ELIF
     +     /0.8,0.2,0.167,0.833,0.143,0.286,1.0,0.0,0.0,10.0/
      CALL EXCMS('FILEDEF 10 DISK FILE LFTEST (DISP MOD')
      OPEN(UNIT = 1,FILE ='LIFETIME')
      OPEN(UNIT = 2,FILE ='SEED')
      READ(2,*)SEED
      CLOSE(2)
      LIF = 0.0
*
* GENERATE UNIFORM 0 AND 1 FOR INITIAL PROBABILITY
*
      CALL RANNUM(1,SEED,0.0,1.0,0,U)
      IF (U.LE.AL1) THEN
        S = 1
      ELSEIF(U.LE.AL1+AL2)THEN
        S = 2
      ELSE
        S = 3
      ENDIF
*
* SIMULATE LIFETIME WHEN IS IN STATE 1
*
```

```
      5 CONTINUE
        IF ((S.EQ.1) .AND. (LIF.LT.ELIF)) THEN
          CALL RANNUM(1,SEED,0.0,1.0,0,U)
          IF(U.GT.P12+P13)THEN
             S = 4
             CALL RANNUM(3,SEED,LD14,0,0,EXP)
             LIF = LIF + EXP
          ELSEIF(U.GT.P12)THEN
             S = 3
             CALL RANNUM(3,SEED,LD13,0,0,EXP)
             LIF = LIF + EXP
          ELSE
             S = 2
             CALL RANNUM(3,SEED,LD12,0,0,EXP)
             LIF = LIF + EXP
          ENDIF
        ENDIF
*
* SIMULATE LIFETIME WHEN IS IN STATE 2
*
        IF((S.EQ.2) .AND. (LIF.LT.ELIF))THEN
          CALL RANNUM(1,SEED,0.0,1.0,0,U)
          IF(U.GT.P21+P23)THEN
             S = 4
             CALL RANNUM(3,SEED,LD24,0,0,EXP)
             LIF = LIF + EXP
          ELSEIF(U.GT.P21)THEN
             S = 3
             CALL RANNUM(3,SEED,LD23,0,0,EXP)
             LIF = LIF + EXP
          ELSE
             S = 1
             CALL RANNUM(3,SEED,LD21,0,0,EXP)
             LIF = LIF + EXP
          ENDIF
        ENDIF
*
* SIMULATION LIFETIME WHEN IS IN STATE 3
*
        IF((S.EQ.3) .AND. (LIF.LT.ELIF))THEN
          CALL RANNUM(1,SEED,0.0,1.0,0,U)
          IF(U.GT.P31+P32)THEN
             S = 4
```

```
            CALL RANNUM(3,SEED,LD34,0,0,EXP)
            LIF = LIF + EXP
         ELSEIF(U.GT.P31)THEN
            S = 2
            CALL RANNUM(3,SEED,LD32,0,0,EXP)
            LIF = LIF + EXP
         ELSE
            S = 1
            CALL RANNUM(3,SEED,LD31,0,0,EXP)
            LIF = LIF + EXP
         ENDIF
      ENDIF
      IF((S.LT.4) .AND. (LIF.LT.ELIF))GOTO 5
*
* OUTPUT TO 'FILE LIFETIME A'
*
      WRITE(10,*)LIF
      WRITE(1,*)LIF
      OPEN(UNIT = 2,FILE ='SEED')
      WRITE(2,*)SEED
      STOP
      END

***********************************************************************
      SUBROUTINE RANNUM(DISTN,SEED,RPARM1,RPARM2,IPARM,X)
C
C   THIS SUBROUTINE PROVIDES A RANDOM NUMBER OF 7 DISTRIBUTION.
C   IT INTERFACES WITH THE LLRANDOMII ROUTINES PROVIDED IN THE
C   NONIMSL LIBRARY. THE PARAMETER AND CALLING PROCEDURE ARE
C   AS FOLLOWS:
C   DIST = DISTRIBUTION TYPE YOU WANT TO SELECT AN INTEGER
C          BETWEEN 1 AND 7.
C   SEED = THE RANDOM NUMBER SEED YOU WISH TO USE.
C   RPARM1, RPARM2, AND IPARM ARE REAL AND INTEGER PARAMETERS.
C    PASSED TO THE ROUTINE WITH MEANINGS WHICH VARY WITH THE
C   TYPE OF DISTRIBUTION YOU DESIRE.
C   X = THE RETURNED RANDOM NUMBER, IT IS ALWAYS REAL.
C    DISTRIBUTION NUMBERS AND THE ASSOCIATED PARM DEFINITIONS
C   1 = UNIFORM ON THE INTERVAL RPARM1 TO RPARM2
C   2 = NORMAL WITH MEAN RPARM1 AND VARIANCE RPARM2
C   3 = EXPONENTIAL WITH RATE RPARM1
C   4 = COUCHY WITH A = RPARM1 AND B = RPARM2
C   5 = GAMMA WITH SHAPE RPARM2 AND RATE RPARM1
```

```
C   6 = POISSON WITH RATE RPARM1
C   7 = GEOMETRIC WITH P = RPARM1
C
      REAL RPARM1,RPARM2,X
      INTEGER DISTN,SEED,IPARM,N
      REAL TEMP,VARIAT(1)
      IF(DISTN.LE.0 .OR. DISTN.GE.8)THEN
        WRITE(10,*)'DISTN DOES NOT EXIT'
        STOP
      ENDIF
*
      GOTO (10,20,30,40,50,60,70),DISTN
*
*   GENERATE A UNIFORM BETWEEN RPARM1 AND RPARM2
*
   10 CONTINUE
      IF(RPARM1-RPARM2.EQ.0)THEN
        WRITE(10,*)'ILLEGAL EQUAL RPARMS IN RANNUM'
        STOP
      ENDIF
      IF(RPARM1.GT.RPARM2)THEN
        TEMP = RPARM1
        RPARM1 = RPARM2
        RPARM2 = TEMP
      ENDIF
      CALL LRND(SEED,VARIAT,1,1,0)
      VARIAT(1) = RPARM1 + (RPARM2-RPARM1)*VARIAT(1)
      GOTO 99
*
*   GENERATE A NORMAL WITH MEAN RPARM1 AND STD. DEV. RPARM2
*
   20 CALL LNORM(SEED,VARIAT,1,1,0)
      WRITE(10,*)'NORMAL(0,1)  ',VARIAT(1)
      VARIAT(1) = (VARIAT(1)*RPARM2) + RPARM1
      GOTO 99
*
*   GENERATE AN EXPONENTIAL WITH RATE RPARM1 (1/MEAN)
*
   30 CONTINUE
      IF(RPARM1.EQ.0.0)THEN
        WRITE(10,*)'ILLEGAL ZERO RATE IN RANNUM'
        STOP
      ENDIF
```

```
      CALL LEXPN(SEED,VARIAT,1,1,0)
      VARIAT(1) = VARIAT(1)/RPARM1
      GOTO 99
*
*     GENERATE A COUCHY WITH A = RPARM1 AND B = RPARM2
*
   40 CONTINUE
      IF(RPARM2.LE.0.0)THEN
        WRITE(10,*)'ILLEGAL COUCHY SPREAD IN RANNUM, B= ',RPARM2
        STOP
      ENDIF
      CALL LCCHY(SEED,VARIAT,1,1,0)
      VARIAT(1) = (VARIAT(1)*RPARM2) + RPARM1
      GOTO 99
*
*     GENERATE A GAMMA WITH SHAPE RPARM2 AND RATE RPARM1
*
   50 CONTINUE
      IF(RPARM1.LE.0.0)THEN
        WRITE(10,*)'ILLEGAL NONPOSITIVE GAMMA RATE IN RANNUM'
        STOP
      ENDIF
      IF(RPARM2.LE.0.0)THEN
        WRITE(10,*)'ILLEGAL SHAPE PARAMETER IN RANNUM'
        STOP
      ENDIF
      CALL LGAMA(SEED,VARIAT,1,1,0,RPARM2)
      VARIAT(1) = VARIAT(1)*(1.0/RPARM1)
      GOTO 99
*
*     GENERATE A POISSON WITH RATE RPARM1
*
   60 CONTINUE
      IF(RPARM1.LE.0.0)THEN
        WRITE(10,*)'ILLEGAL POISSON RATE IN RANNUM'
        STOP
      ENDIF
      CALL LPOIS(SEED,VARIAT,1,1,0,RPARM1)
      GOTO 99
*
*     GENERATE A GEOMETRIC WITH P = RPARM1
*
   70 CONTINUE
```

```
      IF(RPARM1.LE.0.0)THEN
        WRITE(10,*)'ILLEGAL GEOM. PROB. IN RANNUM'
        STOP
      ENDIF
      CALL LGEOM(SEED,VARIAT,1,1,0,RPARM1)
      GOTO 99
*
   99 CONTINUE
      X = VARIAT(1)
      RETURN
      END
```

```
      PROGRAM MOMENT
C
C     THIS PROGRAM USES TO COMPUTE THE SECOND AND THIRD
C     STANDARDIZED MOMENTS THAT SUPPORTS MOMENT MATCHING IN
C     GAMS PROGRAM.
*
* INITIALIZE DATA
*
      INTEGER LIMIT,N,N1,I,J,COUNT
      REAL LIF, NEW(2), REPAGE, A, B, C, EX, EXSQ, EXC, EX3
      REAL STD, SECOND, THIRD, C1, C2, AAC, OBJ
      REAL RAWDAT(1000,2), FB(1000), P(1000), TOT, TCOST
      DATA C1,C2,TOT,TCOST,AAC/100.0,500.0,0.0,0.0,0.0/
      OPEN(UNIT = 1,FILE ='LIFETIME',STATUS ='OLD')
      OPEN(UNIT = 2,FILE ='OPTAGE',STATUS ='OLD')
      OPEN(UNIT = 3,FILE ='RAWDATA',STATUS ='OLD')
      OPEN(UNIT = 4,FILE ='MOMENT')
      OPEN(UNIT = 5,FILE ='COUNT',STATUS ='OLD')
      READ(1,*)LIF
      READ(2,*)REPAGE
      READ(5,*)COUNT
      PRINT*,'COUNT=',COUNT
      N1 = 25 + COUNT
      DO 5 I = 1,1000
        P(I) = 0.0
        FB(I) = 0.0
    5 CONTINUE
      DO 44 I = 1,N1-1
        READ(3,11)RAWDAT(I,1),RAWDAT(I,2)
   11   FORMAT(1X,F7.4,4X,F3.1)
   44 CONTINUE
      CLOSE(3)
      OPEN(UNIT=3,FILE='RAWDATA')
*
* CALCULATION
*
      IF(LIF.LE.REPAGE)THEN
        NEW(1) = LIF
        NEW(2) = 1.0
      ELSE
        NEW(1) = REPAGE
        NEW(2) = 0.0
      ENDIF
```

```
      CALL INSERT(N1,NEW,RAWDAT)
      DO 55 I = 1,N1
          WRITE(3,33) RAWDAT(I,1),RAWDAT(I,2)
33        FORMAT(1X,F7.4,4X,F3.1)
55    CONTINUE
      IF(RAWDAT(1,2) .NE. 0.0)THEN
        FB(1) = (N1-1.0)/N1
      ELSE
        FB(1) = 1.0
      ENDIF
      DO 10 I = 2,N1
        IF(RAWDAT(I,2).EQ.0.0)THEN
            FB(I) = FB(I-1)
        ELSEIF(I.LT.N1)THEN
            FB(I) = ((N1-I)/(N1+1.0-I))*FB(I-1)
        ELSE
            FB(N1) = 0.0
        ENDIF
10    CONTINUE
*
      P(1) = 1.0-FB(1)
      DO 22 I = 2,N1
        P(I) = FB(I-1)-FB(I)
22    CONTINUE
      EX = 0.0
      EXSQ = 0.0
      EXC  = 0.0
      EX3 = 0.0
      DO 15 J = 1,N1
        TOT = TOT + RAWDAT(J,1)
        TCOST = TCOST + C2*RAWDAT(J,2) + C1*(1-RAWDAT(J,2))
        A = RAWDAT(J,1)*P(J)
        B = (RAWDAT(J,1)**2)*P(J)
        C = (RAWDAT(J,1)**3)*P(J)
        EX = EX + A
        EXSQ = EXSQ + B
        EXC = EXC + C
15    CONTINUE
      AAC = TCOST/TOT
      STD = (EXSQ - EX**2)**0.5
      EX3 = EXC + (2*EX**3) - (3*EX*EXSQ)
      SECOND = STD/EX
      THIRD = EX3/(STD**3)
```

```
*
* OUTPUT
*
      WRITE(4,100)SECOND,THIRD,AAC,EX
  100 FORMAT(2X,F11.8,4X,F11.8,4X,F8.3,4X,F6.3)
*     CLOSE(5)
*     OPEN(UNIT = 5,FILE ='COUNT',STATUS ='OLD')
*     WRITE(5,*)COUNT+1
      STOP
      END

*****************************************************************
      SUBROUTINE INSERT(LIMIT,NEW,X)
C     THIS SUBROUTINE USES TO INSERT A PAIR 'NEW' DATA INTO THE
C     DATA FILE 'X' IN ASCENDING ORDER.
      INTEGER LIMIT,J,K
      REAL NEW(2),X(1000,2)
      LOGICAL DONE
      DONE = .FALSE.
      J = 1
    5 IF((J.LT.LIMIT).AND.(.NOT.DONE))THEN
        IF(X(J,1).LT.NEW(1))THEN
           J =J+1
        ELSE
           DONE = .TRUE.
        ENDIF
      GOTO 5
      ENDIF
      IF(J.EQ.LIMIT)THEN
        X(J,1) = NEW(1)
        X(J,2) = NEW(2)
      ELSE
        IF(J.LT.LIMIT)THEN
          DO 10 K = LIMIT,J+1,-1
             X(K,1) = X(K-1,1)
             X(K,2) = X(K-1,2)
   10     CONTINUE
          X(J,1) = NEW(1)
          X(J,2) = NEW(2)
        ENDIF
      ENDIF
      RETURN
      END
```

```
      PROGRAM DATA
C
C     THIS PROGRAM PROVIDES THE RESULT SEQUENTIAL ESTIMATED
C     ENTRIES TRANSITION RATES T(I,J) AND INITIAL STATE PROB.
C     ALPAH(I),AND ALSO ADJUST, PREPARE ALL THE INITIAL DATA THAT
C     WILL BE USED FOR THE NEXT SEQUENTIAL ESTIMATION.
C     THE VARIABLE ARE AS FOLLOWS:
C      A,B,C
C      D,E,F   = THE ESTIMATED TRANSITION MATRIX
C      G,H,I
C      AL1,AL2,AL3 = THE ESTIMATED INITIAL STATES PROBABILITIES
C      X  = RANDOM VARIABLE OF LIFETIME
C      OPTAGE = OPTIMAL AGE REPLACEMENT
C      SECOND = THE SECOND STANDARDIZED MOMENT
C      THIRD = THE THIRD STANDARDIZED MOMENT
C      OBJ = OBJECTIVE FUNCTION VALUE OF GAMS PROGRAM
*
*    INPUT
*
      INTEGER COUNT
      REAL A,B,C,D,E,F,G,H,I,AL1,AL2,AL3,X
      REAL EX,OPTCS,OPTAGE,Z,SECOND,THIRD,DI,OBJ,AAC,PI
      OPEN(UNIT = 10,FILE ='OBJ',STATUS ='OLD')
      OPEN(UNIT = 11,FILE ='ALPHA',STATUS ='OLD')
      OPEN(UNIT = 12,FILE ='BRAVO',STATUS ='OLD')
      OPEN(UNIT = 13,FILE ='CHAREE',STATUS ='OLD')
      OPEN(UNIT = 14,FILE ='DELTA',STATUS ='OLD')
      OPEN(UNIT = 15,FILE ='ECHO',STATUS ='OLD')
      OPEN(UNIT = 16,FILE ='FOXTROT',STATUS ='OLD')
      OPEN(UNIT = 17,FILE ='GOLF',STATUS ='OLD')
      OPEN(UNIT = 18,FILE ='HOTEL',STATUS ='OLD')
      OPEN(UNIT = 19,FILE ='INDIA',STATUS ='OLD')
      OPEN(UNIT = 20,FILE ='AL1',STATUS ='OLD')
      OPEN(UNIT = 21,FILE ='AL2',STATUS ='OLD')
      OPEN(UNIT = 22,FILE ='AL3',STATUS ='OLD')
      OPEN(UNIT = 23,FILE ='OPTAGE',STATUS ='OLD')
      OPEN(UNIT = 24,FILE ='LIFETIME',STATUS ='OLD')
      OPEN(UNIT = 25,FILE ='MOMENT',STATUS ='OLD')
      OPEN(UNIT = 26,FILE ='COUNT',STATUS ='OLD')
      OPEN(UNIT = 27,FILE ='OPTCOST',STATUS ='OLD')
      CALL EXCMS('FILEDEF 50 DISK FILE ESTDAT1 (DISP MOD')
      CALL EXCMS('FILEDEF 60 DISK FILE ESTDAT2 (DISP MOD')
      CALL EXCMS('FILEDEF 70 DISK FILE ESTDAT3 (DISP MOD')
```

```
OPEN(UNIT = 30,FILE ='SCALAR2')
READ(10,*)OBJ
IF(OBJ .GT. 0.05)THEN
  A = -1.0
  B = 0.3
  C = 0.3
  D = 0.3
  E = -1.0
  F = 0.3
  G = 0.3
  H = 0.3
  I = -1.0
ELSE
  READ(11,*)A
  READ(12,*)B
  READ(13,*)C
  READ(14,*)D
  READ(15,*)E
  READ(16,*)F
  READ(17,*)G
  READ(18,*)H
  READ(19,*)I
ENDIF
READ(20,*)AL1
READ(21,*)AL2
READ(22,*)AL3
READ(23,*)OPTAGE,PI
READ(24,*)X
READ(25,*)SECOND,THIRD,AAC,EX
READ(26,*)COUNT
READ(27,*)OPTCS
CLOSE(26)
PRINT*,'OBJ=',OBJ
*
IF(OPTAGE .GT. X)THEN
 Z = X
 DI = 1.0
ELSE
 Z = OPTAGE
 DI = 0.0
ENDIF
```

```
*
* OUTPUT
*
      OPEN(UNIT = 26,FILE ='COUNT',STATUS ='OLD')
      WRITE(26,*)COUNT+1
      WRITE(50,200)COUNT,A,B,C,D,E,F,G,H,I
      WRITE(60,300)COUNT,OPTCS,OPTAGE,X,Z,DI,AAC
      WRITE(70,100)COUNT,AL1,AL2,AL3,SECOND,THIRD,OBJ,PI

*
* PREPARE THE DATA FOR "GAMS" PROGRAM
*
      WRITE(30,*)'   SCALAR'
      WRITE(30,400)'        AA  INITIAL SOLUTION OF A /',A,' /'
      WRITE(30,400)'        BB  INITIAL SOLUTION OF B /',B,' /'
      WRITE(30,400)'        CC  INITIAL SOLUTION OF C /',C,' /'
      WRITE(30,400)'        DD  INITIAL SOLUTION OF D /',D,' /'
      WRITE(30,400)'        EE  INITIAL SOLUTION OF A /',E,' /'
      WRITE(30,400)'        FF  INITIAL SOLUTION OF B /',F,' /'
      WRITE(30,400)'        GG  INITIAL SOLUTION OF C /',G,' /'
      WRITE(30,400)'        HH  INITIAL SOLUTION OF D /',H,' /'
      WRITE(30,400)'        II  INITIAL SOLUTION OF D /',I,' /'
      WRITE(30,500)'        ALP1  INITIAL SOLUTION OF AL1 /',AL1,' /'
      WRITE(30,500)'        ALP2  INITIAL SOLUTION OF AL2 /',AL2,' /'
      WRITE(30,500)'        ALP3  INITIAL SOLUTION OF AL3 /',AL3,' /'
      WRITE(30,600)'        MM2 SECOND STANDARD MOMENT /',SECOND,'/'
      WRITE(30,600)'        MM3 THIRD  STANDARD MOMENT /',THIRD,'/'
      WRITE(30,700)'        EX  EXPECTED VALUE OF LIFETIME /',EX,' / ;'
  100 FORMAT(1X,I3,3(2X,F5.2),2(2X,F8.5),2X,F4.2,2X,E11.4)
  200 FORMAT(1X,I3,2X,9(2X,F6.2))
  300 FORMAT(1X,I3,2X,F8.3,2X,F5.2,2(4X,F7.4),4X,F3.1,4X,F7.3)
  400 FORMAT(A35,F6.2,A2)
  500 FORMAT(A39,F5.2,A2)
  600 FORMAT(A36,F11.8,A1)
  700 FORMAT(A40,F6.3,A4)
      STOP
      END
```

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