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On the Numerical Solution of One-Dimensional Integral and Differential Equations

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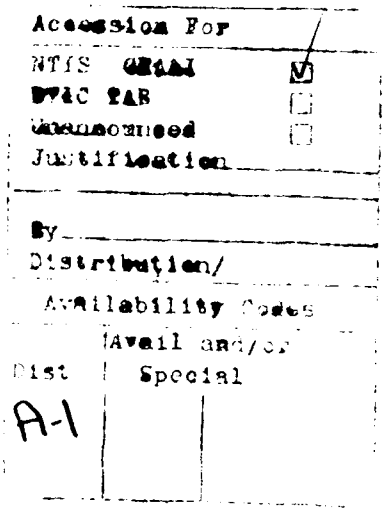
Research Report YALEU/DCS/RR-888
December 1991

YALE UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE

A dissertation presented to the faculty of the Graduate School of Yale University in candidacy for the degree of Doctor of Philosophy.



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Work supported in part by an IBM Graduate Fellowship, in part by the Office of Naval Research under Grant N00014-86-K-0310, and in part by the Defense Advance Research Projects Agency under grants DMS-9012751 and DMS-9114308.

Keywords: *Integral Equations, Two-Point Boundary Value Problems, Chebyshev Polynomials, Approximation Theory*

Abstract

On the Numerical Solution of One-Dimensional Integral and Differential Equations

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Many problems in mathematical physics can be formulated as one-dimensional integral equations. Examples include problems in electrostatics, crack problems in elastic bodies, and two-point boundary value problems for ordinary differential equations. Since most integral equations arising in applications do not have analytic solutions, there is considerable interest in the numerical solution of these problems. Unfortunately, discretization of integral equations leads to dense systems of linear algebraic equations, and the direct solution of a dense linear system of dimension N requires order $O(N^3)$ arithmetic operations. Alternatively, the solution to the linear system can be obtained using an iterative method such as the conjugate gradient algorithm or the conjugate residual algorithm. When the condition number of the linear system is small, the amount of work required is reduced to $O(N^2)$, and can be reduced further to $O(N)$ when the iterative method is combined with an algorithm such as the fast multipole method. However, many problems (including those formulated as first kind integral equations) yield ill-conditioned linear systems; for these problems, the cost of an iterative method is prohibitive, even when combined with an algorithm such as the fast multipole method. Recently, wavelet-like bases have been developed with the property that integral operators in these bases correspond to matrices which are sparse. When the inverses of these integral operators also correspond to sparse matrices, the Schulz method becomes highly effective and produces an $O(N \cdot \log^2 N)$ algorithm for solving a one-dimensional integral equation. Unfortunately, for first kind integral equations and other problems of interest, the integral operators do not have sparse inverses in these wavelet-like bases.

This thesis is based on the observation that one-dimensional integral operators can be recursively decomposed into sums of products of operators of low numerical rank. A complicated analytical apparatus is then constructed which allows for the direct solution of an integral equation in order $O(N)$ operations. The algorithms of this thesis permit the use of schemes with extremely high orders of convergence, and are quite insensitive to end-point singularities. The performance of the methods is illustrated with numerical examples.

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Chapter 1

Introduction

Many problems in mathematical physics can be formulated as one-dimensional integral equations. From an abstract viewpoint, the advantage of the integral equation formulation is that many properties of the solution are readily apparent; from a computational viewpoint, there exist extremely stable, high order numerical methods for the solution of integral equations. In addition, linear systems which arise from discretization of second kind integral equations are generally well-conditioned. On the other hand, linear systems arising from first kind integral equations generally have condition numbers of at least $O(N)$, where N is the number of points in the discretization.

Despite their advantages, integral equations are virtually never used as a numerical tool, since their discretization leads to dense systems of linear algebraic equations, and the solution of a dense linear system of dimension N requires order $O(N^3)$ arithmetic operations, with N the number of nodes in the discretization. This makes the use of integral equations extremely unattractive as a numerical tool, despite their desirable analytical properties.

In recent years, a number of algorithms has been developed for the fast application of integral operators [4], [21], the best known of which are the particle simulation algorithms developed by L. Greengard and V. Rokhlin. Each algorithm of this class exploits the special structure of a particular problem by combining interpolation of the function which defines the matrix elements with a divide-and-conquer strategy, leading to a scheme for applying the integral operator to an arbitrary vector for a cost proportional to N (or, sometimes, $N \cdot \log N$), where N is the number of elements in the discretization of the domain of the operator.

When such a scheme is combined with a conjugate gradient type procedure for the solution of the integral equation, the resulting algorithm requires (asymptotically) a finite number of iterations to converge, leading to an order $O(N)$ estimate for the solution of the original integral equation. Unfortunately, the actual number of iterations required is very sensitive to the conditioning of the problem being solved. In many cases of interest, this number is prohibitive.

In [5], orthonormal bases are developed with the property that integral operators in these bases correspond to matrices which are sparse. When their inverses also correspond to sparse matrices, the Schulz method becomes highly effective and produces an order

$O(N \cdot \log^2 N)$ algorithm for the solution of a second kind integral equation. While, formally speaking, the Schultz technique is an iterative one, in reality it behaves almost like a direct algorithm, since the number of iterations it requires is proportional to the logarithm of the condition number k of the matrix (as opposed to $\sqrt{k}$ and k for the conjugate gradient and conjugate residual techniques, respectively). However, this method can not be used for operators which do not have sparse inverses (for example, first kind integral operators).

In [22], a direct method was developed for the solution of second kind integral equations resulting from two-point boundary value problems of second order ordinary differential equations. For these problems, the integral operators can be recursively decomposed into sums of products of operators of low rank. A somewhat involved analytical apparatus is then constructed which allows for the direct solution of the integral equation in order $O(N)$ operations, with N the number of nodes on the interval.

In this thesis, we construct $O(N)$ algorithms for the direct solution of first and second kind integral equations. We first extend [22] to permit the fast, direct solution of two-point boundary problems of systems of first order ordinary differential equations. We then extend the observations of [5] to construct sparse representations of integral operators with either weakly singular kernels or a Cauchy kernel, and extend the techniques of [22] and [29] to apply the inverse of these operators using order $O(N)$ arithmetic operations.

The plan of this thesis is as follows: Chapter 2 describes the algorithms for two-point boundary value problems for systems of ordinary differential equations (this chapter has been published previously [29]); Chapter 3 describes the algorithms for integral equations with singularities; Chapter 4 briefly outlines some generalizations, and presents our conclusions.

Chapter 2

Two-Point Boundary Value Problems

Second kind integral equations have been a popular analytical tool in the study of ordinary differential equations for nearly a century. When boundary value problems are being considered, the integral equations which arise are of the Fredholm type. From an abstract viewpoint, the advantage of this formulation is that many properties of the solution are readily apparent; from a computational viewpoint, the linear systems which arise from discretization are generally well-conditioned. An ill-behaved differential equation can often be reduced to a perfectly tractable integral equation by means of an appropriate choice of the "background" Green's function (see Example 2.3 in Section 2.4 below). Standard finite difference and finite element methods, on the other hand, which discretize the original differential equation, encounter serious numerical difficulties when the solution possesses derivatives of large magnitude (boundary layers). A second advantage is that there exist extremely stable, high order numerical methods for the solution of second kind Fredholm equations, while the order of convergence of most practical schemes for the solution of ordinary differential equations tends to be limited, even if Richardson extrapolation and deferred correction approaches are considered.

Despite all these advantages, integral equations are virtually never used as a numerical tool for the solution of systems of two-point boundary value problems, since their discretization leads to dense systems of linear algebraic equations, and the solution of a dense linear system of dimension $N \cdot n$ requires order $O(N^3 \cdot n^3)$ arithmetic operations, with N the number of nodes in the discretization, and n the number of equations in the system. Finite difference and finite element schemes lead to banded systems of linear algebraic equations, and the solution of the latter requires order $O(N \cdot n^3)$ arithmetic operations. This makes the use of integral equations extremely unattractive as a numerical tool, despite their superior analytical properties. A similar difficulty is encountered when spectral methods are applied to boundary value problems. They yield high order accuracy, but result in dense systems of linear algebraic equations.

Recently, [22] presented a fast numerical algorithm for solving two-point boundary value

problems for second order differential equations. By solving the problems as second kind integral equations, one obtains the superior properties of integral equations over differential equations. By using the technique of [22], integral equations arising from boundary value problems are solved in order $O(N \cdot p^2)$ arithmetic operations, with N the number of nodes on the interval and p the desired order of convergence.

In this chapter, we extend the results of [22] by showing that integral equations arising from two-point boundary value problems for systems of ordinary differential equations can be solved in $O(N \cdot p^2 \cdot n^3)$ arithmetic operations, with n the number of equations in the system. We in addition present a Newton method for solving boundary value problems for nonlinear first order systems in which each Newton iterate is the solution of a second kind integral equation.

The plan of this chapter is as follows: in Section 2.1 we summarize both the theory of Green's functions for first order linear systems and the theory of Newton methods for first order nonlinear systems, in Section 2.2 we develop the analytical apparatus to be used, and in Section 2.3 we describe the numerical schemes themselves. The performance of the methods is illustrated in Section 2.4 with numerical examples.

The present chapter is similar to [22] in that while it is based on a sequence of fairly simple observations, the details of the algorithm are somewhat involved. We attempt in this chapter to present both cursory, qualitative descriptions as well as detailed, rigorous proofs.

2.1 Mathematical Preliminaries

In this section, we summarize the relevant properties of both the boundary value problems to be addressed and the second kind integral equations to be used for their solution. Most of the results are classical and can be found, for example, in [11] and [13]. The rest are straightforward generalizations to systems of ordinary differential equations of well-known facts concerning second order boundary value problems (see, for example, [14]).

2.1.1 Notation and Definitions

Definition 2.1 *A linear first order system of ordinary differential equations is an expression of the form*

$$\Phi'(x) + p(x) \cdot \Phi(x) = f(x), \tag{2.1}$$

with $\Phi : [a,c] \to \mathbf{R}^n$ *in* $C^1[a,c]$, $p : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ *and* $f : [a,c] \to \mathbf{R}^n$ *continuous, and* $\mathbf{L}(\mathbf{R}^{n\times n})$ *denoting the linear space of all linear operators* $\mathbf{R}^n \to \mathbf{R}^n$.

Definition 2.2 *If* $f(x) \equiv 0$, *(2.1) assumes the form*

$$\Phi'(x) + p(x) \cdot \Phi(x) = 0, \tag{2.2}$$

and is referred to as a linear homogeneous first order system of ordinary differential equations.

Definition 2.3 *A differentiable function $\Phi : [a,c] \to \mathbf{R}^n$ is a solution to a linear first order boundary value problem if it satisfies an equation of the form (2.1), subject to boundary conditions of the form*

$$A \cdot \Phi(a) + C \cdot \Phi(c) = \gamma. \tag{2.3}$$

with $A, C \in \mathbf{L}(\mathbf{R}^{n\times n})$, and $\gamma \in \mathbf{R}^n$.

Definition 2.4 *If $\gamma \equiv 0$, (2.3) becomes*

$$A \cdot \Phi(a) + C \cdot \Phi(c) = 0, \tag{2.4}$$

and is referred to as a set of homogeneous boundary conditions.

Definitions 2.5–2.6 are the nonlinear analogues to Definitions 2.1 and 2.3.

Definition 2.5 *A nonlinear first order system is defined as an expression*

$$\Phi'(x) = F(\Phi(x), x), \tag{2.5}$$

with $\Phi : [a,c] \to \mathbf{R}^n$ in $C^1[a,c]$, $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$ continuous.

Definition 2.6 *A differentiable function $\Phi : [a,c] \to \mathbf{R}^n$ is a solution to a nonlinear first order boundary value problem if it satisfies an equation of the form (2.5), subject to boundary conditions of the form*

$$A \cdot \Phi(a) + C \cdot \Phi(c) = \gamma, \tag{2.6}$$

with $A, C \in \mathbf{L}(\mathbf{R}^{n\times n})$, $\gamma \in \mathbf{R}^n$.

Definition 2.7 *A continuous function $G(x,t) : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is the Green's function for a boundary value problem (2.1), (2.4) if*

1. *$\frac{\partial G(x,t)}{\partial x}$ is continuous except at $x = t$,*
2. *$G(x+0,x) - G(x-0,x) = I_n$ for all $x \in [a,c]$,*
3. *$\frac{\partial}{\partial x}G(x,t) + p(x) \cdot G(x,t) = 0$ for all $x, t \in [a,c], x \neq t$,*
4. *$A \cdot G(a,t) + C \cdot G(c,t) = 0$ for all $t \in [a,c]$.*

Remark 2.1 Green's functions are the principal analytical tools which enable boundary value problems to be solved as second kind integral equations. However, Green's functions are known or computable for very few problems (2.1), (2.4). Fortunately, we can use one of the known Green's functions when constructing the second kind integral equation for a particular boundary value problem. When a Green's function unrelated to a problem (2.1), (2.4) is used to convert that problem to an integral equation, we will refer to this Green's function as a background Green's function. □

Definition 2.8 *A function* $\Upsilon : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ *is called a fundamental matrix for (2.2) if it is nonsingular and*

$$\Upsilon'(x) + p(x) \cdot \Upsilon(x) = 0 \tag{2.7}$$

for all $x \in [a,c]$.

We define boundary condition matrices D and D_N to be used in theorems in the remainder of Section 2.1.

Definition 2.9 *Given a fundamental solution matrix* Υ *of the system (2.2), and a pair of matrices* A, C *given by (2.3), the boundary condition matrices* $D, D_N \in \mathbf{L}(\mathbf{R}^{n\times n})$ *are defined by the formulae*

$$D = A \cdot \Upsilon(a) + C \cdot \Upsilon(c), \tag{2.8}$$

$$D_N = A + C. \tag{2.9}$$

We define a residual mapping K and Newton iterates δ_k to be used in a Newton method for nonlinear boundary value problems.

Definition 2.10 *Given functions* $G_0, p_0 : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$, *we define the* residual mapping $K : \mathbf{R}^{n+1} \to \mathbf{R}^n$ *by the formula*

$$\begin{aligned} K(\sigma(x), x) &= \sigma(x) - p_0(x) \cdot \int_a^c G_0(x,t) \cdot \sigma(t)\, dt \\ &\quad -F\left(\int_a^c G_0(x,t) \cdot \sigma(t)\, dt, x\right). \end{aligned} \tag{2.10}$$

Definition 2.11 *For any continuous* $\sigma_0 : [a,c] \to \mathbf{R}^n$, *we refer to the continuous functions* $\delta_k : [a,c] \to \mathbf{R}^n$ *as* Newton iterates *if for each* $k = 1, 2, \ldots,$

$$\delta_k = \sigma_{k+1}(x) - \sigma_k(x), \tag{2.11}$$

with each continuous $\sigma_k : [a,c] \to \mathbf{R}^n$ *recursively defined via the formula*

$$\frac{\partial K(\sigma_k(x), x)}{\partial \sigma_k} \cdot (\sigma_{k+1}(x) - \sigma_k(x)) = -K(\sigma_k(x), x), \quad k = 0, 1, \cdots. \tag{2.12}$$

Finally, we define a transposition operator to be used in the chapter.

Definition 2.12 *Given an interval* $[b_1, b_2] \subset \mathbf{R}$ *and an operator* $\chi : L^2[b_1, b_2] \to \mathbf{L}(\mathbf{R}^{n\times n})$, *the transpose* $\chi^T : (L^2[b_1, b_2])^n \to \mathbf{R}^n$ *of* χ *is defined by the formula*

$$\chi^T(\sigma) = \int_{b_1}^{b_2} \chi(t) \cdot \sigma(t)\, dt, \tag{2.13}$$

with $\sigma \in (L^2)^n$.

2.1.2 Green's Functions for First Order Systems

Theorems 2.1–2.8 provide the tools for the conversion of first order systems of differential equations into second kind integral equations. Theorems 2.1, 2.2, 2.5 and 2.6 are well known and can be found, for example, in [11] and [13]. The authors failed to locate the remaining theorems in the literature.

Theorems 2.1–2.2 provide conditions for the existence and uniqueness of solutions to (2.1), (2.4).

Theorem 2.1 *For any continuous function $p : [a, c] \to L(\mathbf{R}^{n\times n})$, the homogeneous first order system (2.2) has exactly n linearly independent solutions.*

Theorem 2.2 *If the matrix D defined by (2.8) is nonsingular, then there is a unique solution Φ to the equation (2.1) satisfying homogeneous boundary conditions (2.4). Furthermore, the solution to the homogeneous equation (2.2) satisfying homogeneous boundary conditions (2.4) is $\Phi(x) \equiv 0$.*

The purpose of the following two theorems is to permit the conversion of problems with inhomogeneous boundary conditions to those with homogeneous ones. Theorem 2.3 concerns linear problems of the form (2.1), (2.3); Theorem 2.4 concerns nonlinear problems of the form (2.5), (2.6).

Theorem 2.3 *If the boundary condition matrices D, D_N defined by (2.8), (2.9) are both nonsingular, then the solution to the problem (2.1), (2.3) is given by the formula*

$$\Phi(x) = \tilde{\Phi}(x) + \nu, \tag{2.14}$$

with $\nu \in \mathbf{R}^n$ given by the formula

$$\nu = (A + C)^{-1} \cdot \gamma, \tag{2.15}$$

and $\tilde{\Phi} : [a, c] \to \mathbf{R}^n$ in $C^1[a, c]$ the solution to the first order system

$$\tilde{\Phi}'(x) + p(x) \cdot \tilde{\Phi}(x) = f(x) - p(x) \cdot \nu, \tag{2.16}$$

satisfying homogeneous boundary conditions (2.4).

Proof. Since the matrix D is nonsingular, it immediately follows from Theorem 2.2 that there exists a unique $\tilde{\Phi}$ satisfying the equation (2.16). Substituting (2.14) into boundary conditions (2.3), we obtain

$$A \cdot (\tilde{\Phi}(a) + \nu) + C \cdot (\tilde{\Phi}(c) + \nu) = \gamma. \tag{2.17}$$

Now, (2.15) is easily obtained from the combination of (2.17) and (2.4), while (2.16) is a result of substituting (2.14) into (2.1).

Theorem 2.4 *If there exists a solution $\Phi : [a,c] \to \mathbf{R}^n$ to the problem (2.5), (2.6), and if the matrix D_N defined by (2.9) is nonsingular, then Φ is given by the formula*

$$\Phi(x) = \tilde{\Phi}(x) + \nu, \tag{2.18}$$

with $\nu \in \mathbf{R}^n$ given by

$$\nu = (A + C)^{-1} \cdot \gamma, \tag{2.19}$$

and $\tilde{\Phi} : [a,c] \to \mathbf{R}^n \in C^1[a,c]$ the solution to the nonlinear boundary value problem

$$\tilde{\Phi}'(x) = F(\tilde{\Phi} + \nu, x) \tag{2.20}$$

with homogeneous boundary conditions (2.4).

Proof. Substituting (2.18) into boundary conditions (2.6) we obtain

$$A \cdot (\tilde{\Phi}(a) + \nu) + C \cdot (\tilde{\Phi}(c) + \nu) = \gamma. \tag{2.21}$$

Now, (2.19) is easily obtained from the combination of (2.21) and (2.4), while (2.20) is a result of substituting (2.18) into (2.5).

Theorem 2.5 provides an explicit construction for the Green's function for a boundary value problem with a known fundamental matrix Υ. Given a Green's function for a homogeneous problem (2.2), (2.4), Theorem 2.6 provides an explicit solution for the inhomogeneous problem (2.1), (2.4).

Theorem 2.5 *If the matrix D defined by (2.8) is nonsingular, then there exists a unique Green's function $G : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ for (2.2), (2.4). G is given by the formula*

$$G(x,t) = \begin{cases} \Upsilon(x) \cdot (\Upsilon^{-1}(t) + J(t)) & (t \le x), \\ \Upsilon(x) \cdot J(t) & (t \ge x), \end{cases} \tag{2.22}$$

with $J : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ given by the formula

$$J(t) = -D^{-1} \cdot C \cdot \Upsilon(c) \cdot \Upsilon^{-1}(t), \tag{2.23}$$

and $\Upsilon : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ the fundamental matrix for (2.2) (see Definition 2.8).

Theorem 2.6 *Given a Green's function for the problem (2.2), (2.4), the solution Φ for the problem (2.1), (2.4) can be obtained via the formula*

$$\Phi(x) = \int_a^c G(x,t) \cdot f(t)\, dt. \tag{2.24}$$

The following two theorems are two of the principal analytical tools used in this chapter. Theorem 2.7 is used to reduce a linear boundary value problem (2.1), (2.4) to a second kind integral equation, even when the Green's function for the problem is not available; Theorem 2.8 is used in the same fashion to reduce nonlinear boundary value problems (2.5), (2.4) to nonlinear second kind integral equations.

Theorem 2.7 *Suppose $p_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is continuous, $\Upsilon_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is the fundamental matrix for the equation*

$$\Phi'(x) + p_0(x) \cdot \Phi(x) = 0, \tag{2.25}$$

and $G_0 : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is the Green's function for the boundary value problem (2.25), (2.4). Suppose further that the matrix D defined by (2.8) and the matrix $D_0 \in \mathbf{L}(\mathbf{R}^{n\times n})$ defined by the formula

$$D_0 = A \cdot \Upsilon_0(a) + C \cdot \Upsilon_0(c), \tag{2.26}$$

are both nonsingular. Then the solution Φ to the problem (2.1), (2.4) can be obtained via the formula

$$\Phi(x) = \int_a^c G_0(x,t) \cdot \sigma(t)\, dt, \tag{2.27}$$

with $\sigma : [a,c] \to \mathbf{R}^n$ the solution to the second kind integral equation

$$\sigma(x) + [p(x) - p_0(x)] \cdot \int_a^c G_0(x,t) \cdot \sigma(t)\, dt = f(x). \tag{2.28}$$

Proof. By Theorem 2.2, if matrices D, D_0 are nonsingular then the problems (2.1), (2.4) and (2.25), (2.4) have unique solutions, and therefore the background Green's function G_0 is also unique, and is defined by Theorem 2.4. Now, (2.28) is obtained by substituting (2.27) into (2.1).

Remark 2.2 If $p_0(x) = p(x)$, then the solution to equation (2.28) is trivially $\sigma = f$. Our working assumption is that for some background problem (2.25), (2.4), the Green's function is known or computable, but that for the original differential equation (2.1), (2.4) the Green's function is unavailable. □

Theorem 2.8 *Suppose $\Phi : [a,c] \to \mathbf{R}^n$ is a solution to (2.5), (2.4). Suppose further that $p_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is continuous, and $\Upsilon_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is a fundamental matrix for the equation*

$$\Phi'(x) + p_0(x) \cdot \Phi(x) = 0, \tag{2.29}$$

and $G_0 : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is the Green's function for the boundary value problem (2.25), (2.4). Suppose finally that the matrix D_0 defined by the formula

$$D_0 = A \cdot \Upsilon_0(a) + C \cdot \Upsilon_0(c) \tag{2.30}$$

is nonsingular. Then Φ can be obtained via the formula

$$\Phi(x) = \int_a^c G_0(x,t) \cdot \sigma(t)\, dt, \tag{2.31}$$

with $\sigma : [a,c] \to \mathbf{R}^n$ the solution to the second kind integral equation

$$\sigma(x) - p_0(x) \cdot \int_a^c G_0(x,t) \cdot \sigma(t)\, dt = F\left(\int_a^c G_0(x,t) \cdot \sigma(t)\, dt, x\right). \tag{2.32}$$

Proof. Since D_0 is nonsirgular, the background Green's function G_0 is unique, and therefore Φ can be obtained from (2.31). Now, (2.32) is obtained by substituting (2.31) into (2.5).

2.1.3 Green's Functions for Particular Equations

Lemmas 2.1–2.4 of this section provide fundamental matrices and Green's functions for two particular types of boundary value problems. Lemmas 2.1, 2.2 are easily verified by substituting formulae (2.34), (2.35) into (2.7), (2.22). Similarly, Lemmas 2.3, 2.4 are verified by substituting formulae (2.37), (2.38) into (2.7), (2.22).

Lemma 2.1 *A fundamental matrix Υ_0 for the equation*

$$\Phi'(x) = 0 \tag{2.33}$$

is given by the formula

$$\Upsilon_0(x) = I_n, \tag{2.34}$$

with n the dimensionality of the problem (2.33), and $x \in [a, c]$ (in accordance with standard practice, I_n denotes the unity operator $\mathbf{R}^n \to \mathbf{R}^n$).

Lemma 2.2 *The Green's function G_0 corresponding to the equation (2.33) subject to boundary conditions (2.4) is given by the formula*

$$G_0(x,t) = \begin{cases} I_n - (A+C)^{-1} \cdot C & (t \le x), \\ -(A+C)^{-1} \cdot C & (t \ge x). \end{cases} \tag{2.35}$$

Lemma 2.3 *For any $\lambda \in R$, a fundamental matrix Υ_0 for the equation*

$$\Phi'(x) + \lambda \cdot \Phi(x) = 0 \tag{2.36}$$

is given by the formula

$$\Upsilon_0(x) = e^{-\lambda x} \cdot I_n, \tag{2.37}$$

with n the dimensionality of the problem (2.36), and $x \in [0, 1]$.

Lemma 2.4 *The Green's function G_0 corresponding to the equation (2.36) subject to boundary conditions (2.4) is given by the formula*

$$G_0(x,t) = \begin{cases} e^{\lambda(t-x)} \cdot I_n - e^{\lambda(t-1)} \cdot (A + e^{-\lambda} \cdot C)^{-1} \cdot C & (t \le x), \\ -e^{\lambda(t-x-1)} \cdot (A + e^{-\lambda} \cdot C)^{-1} \cdot C & (t \ge x). \end{cases} \tag{2.38}$$

2.1.4 Linear Transformations for Problems with Singular D_0 or D_N

The purpose of Theorem 2.9 is to permit the conversion of a problem (2.1), (2.3) to a second kind integral equation (2.28). For most problems, Theorems 2.3 and 2.7 allow such a conversion, but Theorem 2.3 cannot be used when the matrix D_N defined by (2.9) is singular, while Theorem 2.7 cannot be used when the matrix D_0 defined by (2.26) is singular. We remove these obstacles in this section by providing a scheme which reduces a problem of the form (2.1), (2.3) with singular matrices D_0, D_N to a problem of the same form with nonsingular D_0, D_N.

Theorem 2.10 generalizes Theorem 2.9; it permits the conversion of nonlinear problems of the form (2.5), (2.6) to nonlinear integral equations of the form (2.32).

Remark 2.3 If only the matrix D_0 is singular, one can always choose a new background Green's function G_0 for which D_0 will be nonsingular. However, we have found that for most problems it is easier to develop a transformation of the type described in this section than to develop an alternate background Green's function. □

Theorem 2.9 *Suppose $\Phi : [a,c] \to \mathbf{R}^n$ is the unique solution to the problem (2.1), (2.4). Suppose further that $\Upsilon_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is a fundamental matrix for the background equation (2.25). Suppose finally that there exists $\Psi : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ such that $\Psi \in C^1[a,c]$, $\det \Psi(x) \neq 0$ for all $x \in [a,c]$, and the matrix*

$$\overline{D}_0 = A \cdot \Psi(a) \cdot \Upsilon_0(a) + C \cdot \Psi(c) \cdot \Upsilon_0(c) \tag{2.39}$$

is nonsingular. Then the equation

$$\Gamma'(x) + \Psi^{-1}(x) \cdot (\Psi'(x) + p(x) \cdot \Psi(x)) \cdot \Gamma(x) = \Psi^{-1}(x) \cdot f(x), \tag{2.40}$$

subject to boundary conditions

$$A \cdot \Psi(a) \cdot \Gamma(a) + C \cdot \Psi(c) \cdot \Gamma(c) = 0. \tag{2.41}$$

has a unique solution $\Gamma : [a,c] \to \mathbf{R}^n$, and

$$\Phi(x) = \Psi(x) \cdot \Gamma(x), \tag{2.42}$$

for all $x \in [a,c]$.

Proof. We immediately obtain (2.41) by substituting (2.42) into (2.4). Now, substituting (2.42) and its derivative into (2.1), we get

$$\Psi'(x) \cdot \Gamma(x) + \Psi(x) \cdot \Gamma'(x) + p(x) \cdot \Psi(x) \cdot \Gamma(x) = f(x), \tag{2.43}$$

and obtain (2.40) by combining (2.43) with the fact that $\Psi(x)$ is nonsingular for all $x \in [a,c]$.

Remark 2.4 Clearly, the transformed problem (2.40), (2.41) satisfies the conditions of Theorem 2.7, as $\overline{D}_0$ defined by (2.39) is nonsingular. However, for many problems, the boundary condition matrix $\overline{D}_N$ defined by the formula

$$\overline{D}_N = A \cdot \Psi(a) + C \cdot \Psi(c). \tag{2.44}$$

is singular, and therefore the transformed problem fails to satisfy the conditions of Theorem 2.3. If one needs to use the results of both Theorem 2.7 and 2.3, one must choose a transformation Ψ such that both $\overline{D}_0$ and $\overline{D}_N$ are nonsingular.

Of course, it is easier to choose transformations Ψ when $\overline{D}_0 = \overline{D}_N$. This is true when the background Green's function is chosen to correspond to the equation $\Phi' = 0$. By Lemma 2.1, the fundamental matrix for this equation is $\Upsilon \equiv I_n$; the equivalence for this fundamental matrix of (2.39) and (2.44) is readily apparent. □

Theorem 2.10 is the nonlinear analogue of Theorem 2.9; the proofs of the two theorems are nearly identical.

Theorem 2.10 *Suppose $\Phi : [a,c] \to \mathbf{R}^n$ is the unique solution to the problem (2.5), (2.4). Suppose further that $\Upsilon_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ is a fundamental matrix for the background equation (2.25). Suppose finally that there exists $\Psi : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ such that $\Psi \in C^1[a,c]$, $\det \Psi(x) \neq 0$ for all $x \in [a,c]$, and the matrix*

$$\overline{D}_0 = A \cdot \Psi(a) \cdot \Upsilon_0(a) + C \cdot \Psi(c) \cdot \Upsilon_0(c) \tag{2.45}$$

is nonsingular. Then the equation

$$\Gamma'(x) + \Psi^{-1}(x)\Psi'(x) \cdot \Gamma(x) = \Psi^{-1}(x) \cdot F(\Psi(x) \cdot \Gamma(x), x) \tag{2.46}$$

subject to boundary conditions

$$A \cdot \Psi(a) \cdot \Gamma(a) + C \cdot \Psi(c) \cdot \Gamma(c) = 0. \tag{2.47}$$

has a unique solution $\Gamma : [a,c] \to \mathbf{R}^n$, and

$$\Phi(x) = \Psi(x) \cdot \Gamma(x), \tag{2.48}$$

for all $x \in [a,c]$.

2.1.5 Newton's Method for Nonlinear Boundary Value Problems

Theorems 2.4, 2.8 of Section 2.1.2 reduce nonlinear boundary value problems of the form (2.5), (2.6) to nonlinear second kind integral equations of the form (2.32). In this section, we describe the convergence properties of the well-known Newton's method as applied to the latter (Theorem 2.12), and reduce each step of Newton's algorithm to the solution of a linear boundary value problem of the form (2.1), (2.4) (Theorem 2.11).

Theorem 2.11 permits each Newton iterate δ_k defined by (2.11) to be expressed as the solution to a second kind integral equation.

Theorem 2.11 *Suppose $K : \mathbf{R}^{n+1} \to \mathbf{R}^n$ in $C^1[a,c]$ defined by (2.10) is Fréchet differentiable at every point $x \in [a,c]$, and $\Phi_k : [a,c] \to \mathbf{R}^n$ is defined for all $k = 0, 1, \ldots$ via the formula*

$$\Phi_k(x) = \int_a^c G_0(x,t) \cdot \sigma_k(t)\, dt, \tag{2.49}$$

with $\sigma_k : [a,c] \to \mathbf{R}^n$ defined by (2.12), and $G_0 : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ the Green's function for (2.25), (2.4). Then the Newton iterates $\delta_k : [a,c] \to \mathbf{R}^n \in C^0[a,c]$ given by Definition 2.14 satisfy the equation

$$\delta_k(x) + \Omega_k(x) \cdot \int_a^c G_0(x,t) \cdot \delta_k(t)\, dt = g_k(x) \tag{2.50}$$

for all $k = 0, 1, \ldots$, *with* $\Omega_k : [a, c] \to \mathbf{L}(\mathbf{R}^{n \times n})$ *defined for all* $k = 0, 1, \ldots$ *by the formula*

$$\Omega_k(x) = -\frac{\partial F(\Phi_k(x), x)}{\partial \Phi_k} - p_0(x), \tag{2.51}$$

and $g_k : [a, c] \to \mathbf{L}(\mathbf{R}^{n \times n})$ *defined for all* $k = 0, 1, \ldots$ *by the formula*

$$g_k(x) = p_0(x) \cdot \Phi_k(x) + F(\Phi_k(x), x) - \sigma_k(x). \tag{2.52}$$

Proof. (2.50) is obtained by substituting the Fréchet derivative of the function K into (2.12), and substituting (2.49), (2.51), (2.52) into the resulting equation.

The convergence properties of Newton's method have been thoroughly studied. Theorem 2.12 is one fundamental result, and can be found, for example, in [23] (in a slightly different form).

Theorem 2.12 *Suppose* Φ *is the unique solution to (2.5), (2.4),* $\tilde{\sigma}$ *is the solution to (2.32), and* δ *is the unique solution to (2.50) (so that the linearization (2.50) to the equation (2.32) is nonsingular at* $\tilde{\sigma}$*). Then there exists* $\epsilon > 0$ *such that for any* $\sigma_0 : [a, c] \to \mathbf{R}^n$ *satisfying the condition*

$$\|\sigma_0 - \tilde{\sigma}\| \leq \epsilon \tag{2.53}$$

and Newton iterates $\sigma_k : [a, c] \to \mathbf{R}^n$ *defined by (2.11),*

1. $\|\sigma_k - \tilde{\sigma}\| \leq \epsilon$ *for all* $k = 1, 2, \ldots$,
2. $\lim_{k \to \infty} \sigma_k = \tilde{\sigma}$,
3. σ_k *converges to* $\tilde{\sigma}$ *quadratically.*

2.1.6 A Lemma from Linear Algebra

Given a perturbation of the unity operator $I_{(L^2)^n} : (L^2)^n \to (L^2)^n$, Lemma 2.5 provides its inverse. It is normally used when the rank of the perturbation is low, is a particular case of the Sherman-Morrison formula (see, for example, [19]), and is easy to verify directly.

Lemma 2.5 *For any two vectors* $U, V \in (L^2)^{n \times n}$ *such that* $V^T \cdot U \neq I_n$,

$$(I_{(L^2)^n} - U \cdot V^T)^{-1} = I_{(L^2)^n} + U \cdot (I_n - V^T \cdot U)^{-1} \cdot V^T. \tag{2.54}$$

2.2 The Analytical Apparatus

In the remainder of this chapter, we assume that the solution to the problem (2.1), (2.4) is being sought on the interval $[a, c]$, and that b is some intermediate point ($a < b < c$). The fundamental observation on which the algorithms of Section 2.3 are based is that the solution to the integral equation (2.28) on the entire domain $[a, c]$ can easily be constructed from the solutions of two independent integral equations, one defined on $[a, b]$ and one on $[b, c]$. This leads naturally to a recursive algorithm, in which independent solutions on a large number of subintervals are successively merged until the full solution is obtained. A precise formulation of the construction and the resulting numerical scheme will require some notation.

2.2.1 Notation

We will denote the subintervals $[a,b]$ and $[b,c]$ of $[a,c]$ by A and B, respectively. For convenience, we write the integral equation (2.28) in the form

$$\sigma(x) + \tilde{p}(x) \cdot \int_a^c G_0(x,t) \cdot \sigma(t)\, dt = f(x), \tag{2.55}$$

with $\tilde{p}(x) = p(x) - p_0(x)$, and $G_0 : [a,c] \times [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ the background corresponding to the equation (2.25) subject to boundary conditions (2.4).

We define the operator $P : (L^2[a,c])^n \to (L^2[a,c])^n$ corresponding to (2.55) by the formula

$$P(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_a^c G_0(x,t) \cdot \sigma(t)\, dt, \tag{2.56}$$

so that we have

$$P\sigma = f. \tag{2.57}$$

We will require the four operators

$$\begin{aligned} P_{AA} &: (L^2[a,b])^n \to (L^2[a,b])^n\,, \\ P_{AB} &: (L^2[b,c])^n \to (L^2[a,b])^n\,, \\ P_{BA} &: (L^2[a,b])^n \to (L^2[b,c])^n\,, \\ P_{BB} &: (L^2[b,c])^n \to (L^2[b,c])^n, \end{aligned}$$

defined by the formulae

$$P_{AA}(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_a^b G_0(x,t) \cdot \sigma(t)\, dt, \tag{2.58}$$

$$P_{AB}(\sigma)(x) = \tilde{p}(x) \cdot \int_b^c G_0(x,t) \cdot \sigma(t)\, dt, \tag{2.59}$$

$$P_{BA}(\sigma)(x) = \tilde{p}(x) \cdot \int_a^b G_0(x,t) \cdot \sigma(t)\, dt, \tag{2.60}$$

$$P_{BB}(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_b^c G_0(x,t) \cdot \sigma(t)\, dt. \tag{2.61}$$

We define the operator $Q : (L^2[a,c])^{n\times n} \to (L^2[a,c])^{n\times n}$ by the expression

$$Q(\chi)(x) = \chi(x) + \tilde{p}(x) \cdot \int_a^c G_0(x,t) \cdot \chi(t)\, dt. \tag{2.62}$$

We additionally require the four operators

$$\begin{aligned} Q_{AA} &: (L^2[a,b])^{n\times n} \to (L^2[a,b])^{n\times n}\,, \\ Q_{AB} &: (L^2[b,c])^{n\times n} \to (L^2[a,b])^{n\times n}\,, \\ Q_{BA} &: (L^2[a,b])^{n\times n} \to (L^2[b,c])^{n\times n}\,, \\ Q_{BB} &: (L^2[b,c])^{n\times n} \to (L^2[b,c])^{n\times n}, \end{aligned}$$

defined by the formulae

$$Q_{AA}(\chi)(x) = \chi(x) + \tilde{p}(x) \cdot \int_a^b G_0(x,t) \cdot \chi(t)\, dt, \tag{2.63}$$

$$Q_{AB}(\chi)(x) = \tilde{p}(x) \cdot \int_b^c G_0(x,t) \cdot \chi(t)\, dt, \tag{2.64}$$

$$Q_{BA}(\chi)(x) = \tilde{p}(x) \cdot \int_a^b G_0(x,t) \cdot \chi(t)\, dt, \tag{2.65}$$

$$Q_{BB}(\chi)(x) = \chi(x) + \tilde{p}(x) \cdot \int_b^c G_0(x,t) \cdot \chi(t)\, dt. \tag{2.66}$$

We also require the functions $\psi, v_L, v_R : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ defined by the formulae

$$\psi(x) = \tilde{p}(x) \cdot \Upsilon_0(x), \tag{2.67}$$

$$v_L(t) = \Upsilon_0^{-1}(t) + J_0(t), \tag{2.68}$$

$$v_R(t) = J_0(t), \tag{2.69}$$

with Υ_0 the fundamental matrix for equation (2.25), and $J_0 : [a,c] \to \mathbf{L}(\mathbf{R}^{n\times n})$ defined by the formula

$$J_0(t) = -D_0^{-1} \cdot C \cdot \Upsilon_0(c) \cdot \Upsilon_0^{-1}(t), \tag{2.70}$$

with the matrix D_0 defined by (2.26), and the matrix C given by (2.4).

Given a function $f \in (L^2[a,c])^n$, we will follow the convention of denoting its restriction to A and B by $f_{|A}$ and $f_{|B}$, respectively. Similarly, given a function $\psi \in (L^2[a,c])^{n\times n}$, we will denote its restriction to A and B by $\psi_{|A}$ and $\psi_{|B}$, respectively. Assuming that the operators P_{AA}, P_{BB} are nonsingular, we define the functions $\eta_A : A \to \mathbf{R}^n$, $\eta_B : B \to \mathbf{R}^n$ via the formulae

$$\eta_A = P_{AA}^{-1}(f_{|A}), \tag{2.71}$$

$$\eta_B = P_{BB}^{-1}(f_{|B}). \tag{2.72}$$

Similarly, assuming that the operators Q, Q_{AA}, Q_{BB} are nonsingular, we then define the operators

$$\begin{aligned} \chi &: [a,c]^{n\times n} \to \mathbf{L}(\mathbf{R}^{n\times n}), \\ \phi_A &: A^{n\times n} \to \mathbf{L}(\mathbf{R}^{n\times n}), \\ \phi_B &: B^{n\times n} \to \mathbf{L}(\mathbf{R}^{n\times n}), \end{aligned}$$

via the formulae

$$\chi = Q^{-1}(\psi), \tag{2.73}$$

$$\phi_A = Q_{AA}^{-1}(\psi_{|A}), \tag{2.74}$$

$$\phi_B = Q_{BB}^{-1}(\psi_{|B}). \tag{2.75}$$

Finally, we will define six matrices $\alpha_L^A, \alpha_R^A, \alpha_L^B, \alpha_R^B, \alpha_L, \alpha_R \in \mathbf{L}(\mathbf{R}^{n\times n})$ by the formulae

$$\alpha_L^A = \int_a^b v_L(t)\cdot\phi_A(t)\,dt\,, \tag{2.76}$$

$$\alpha_R^A = \int_a^b v_R(t)\cdot\phi_A(t)\,dt\,, \tag{2.77}$$

$$\alpha_L^B = \int_b^c v_L(t)\cdot\phi_B(t)\,dt\,, \tag{2.78}$$

$$\alpha_R^B = \int_b^c v_R(t)\cdot\phi_B(t)\,dt\,, \tag{2.79}$$

$$\alpha_L = \int_a^c v_L(t)\cdot\chi(t)\,dt\,, \tag{2.80}$$

$$\alpha_R = \int_a^c v_R(t)\cdot\chi(t)\,dt\,, \tag{2.81}$$

and six vectors $\delta_L^A, \delta_R^A, \delta_L^B, \delta_R^B, \delta_L, \delta_R \in \mathbf{R}^n$ via the formulae

$$\delta_L^A = \int_a^b v_L(t)\cdot\eta_A(t)\,dt, \tag{2.82}$$

$$\delta_R^A = \int_a^b v_R(t)\cdot\eta_A(t)\,dt, \tag{2.83}$$

$$\delta_L^B = \int_b^c v_L(t)\cdot\eta_B(t)\,dt, \tag{2.84}$$

$$\delta_R^B = \int_b^c v_R(t)\cdot\eta_B(t)\,dt, \tag{2.85}$$

$$\delta_L = \int_a^c v_L(t)\cdot\sigma(t)\,dt, \tag{2.86}$$

$$\delta_R = \int_a^c v_R(t)\cdot\sigma(t)\,dt, \tag{2.87}$$

with σ the solution to equation (2.57).

2.2.2 Analysis of the operators P_{AB}, P_{BA}

In this section, we observe that each of the operators P_{AB} and P_{BA} is of rank n, and give simple expressions for these operators.

Lemma 2.6 *In the notation of the preceding section,*

$$P_{AB} = \psi_{|A}\cdot v_R^T, \tag{2.88}$$

$$P_{BA} = \psi_{|B}\cdot v_L^T. \tag{2.89}$$

Proof. We obtain (2.88) by observing that $x \le t$ for any $x \in [a,b], t \in [b,c]$, and by using (2.59) and (2.69). Similarly, (2.89) follows from the combination of (2.60), (2.69) and observing that $x \ge t$ for any $x \in [b,c]$, $t \in [a,b]$.

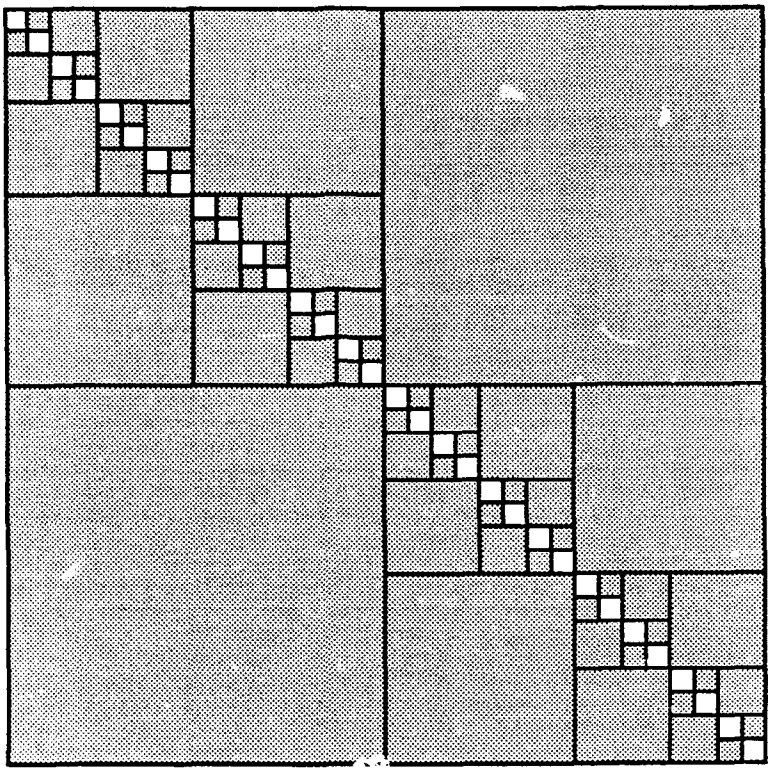

Figure 2.1: The shaded boxes are rank n matrices

Remark 2.5 Just as we decomposed the second kind operator P into two second kind integral operators (P_{AA}, P_{BB}) and two first kind integral operators (P_{AB}, P_{BA}), we can similarly decompose each of P_{AA}, P_{BB} into two second kind integral operators and two first kind integral operators. Clearly, this decomposition can be applied recursively to yield the structure shown in Figure 2.1, in which each block diagonal matrix is a second kind integral operator, while the remaining block matrices correspond to rank n operators. This immediately suggests a recursive algorithm to rapidly obtain solutions to the integral equation (2.57). We now present the principal lemmas for this algorithm. □

2.2.3 Recursive solution of the integral equation

We now consider the original integral equation (2.57)

$$P\sigma = f.$$

The main result of this section is the following lemma, which constructs the solution σ of equation (2.57) from η_A, η_B of equations (2.71) and (2.72).

Lemma 2.7 *If, in the notation of Section 2.2.1, all six operators P, P_{AA}, P_{BB},Q, Q_{AA}, Q_{BB} are nonsingular, and the matrices Δ_1, $\Delta_2 \in \mathbf{L}(\mathbf{R}^{n\times n})$ defined by the formulae*

$$\Delta_1 = I_n - \alpha_L^A \cdot \alpha_R^B, \tag{2.90}$$

$$\Delta_2 = I_n - \alpha_R^B \cdot \alpha_L^A, \tag{2.91}$$

are also nonsingular, then

$$\sigma_{|A} = \eta_A + \phi_A \cdot \Delta_2^{-1} \cdot (\alpha_R^B \cdot \delta_L^A - \delta_R^B), \tag{2.92}$$

$$\sigma_{|B} = \eta_B + \phi_B \cdot \Delta_1^{-1} \cdot (\alpha_L^A \cdot \delta_R^B - \delta_L^A). \tag{2.93}$$

Proof. Using definitions (2.56) - (2.61), the integral equation

$$P\sigma = f \tag{2.94}$$

can be rewritten in the form

$$P_{AA}(\sigma_{|A}) + P_{AB}(\sigma_{|B}) = f_{|A}, \tag{2.95}$$

$$P_{BA}(\sigma_{|A}) + P_{BB}(\sigma_{|B}) = f_{|B}. \tag{2.96}$$

The expansions (2.88) and (2.89) for P_{AB} and P_{BA}, respectively, can then be used to obtain an explicit solution to the coupled equations (2.95) and (2.96) in terms of the functions $\eta_{|A}$, η_B, ϕ_A, and ϕ_B defined by (2.71), (2.72), (2.74), (2.75), respectively. Indeed, applying the operator P_{AA}^{-1} to equation (2.95) and the operator P_{BB}^{-1} to equation (2.96), we have

$$\sigma_{|A} + P_{AA}^{-1} \cdot P_{AB}(\sigma_{|B}) = P_{AA}^{-1}(f_{|A}), \tag{2.97}$$

$$P_{BB}^{-1} \cdot P_{BA}(\sigma_{|A}) + \sigma_{|B} = P_{BB}^{-1}(f_{|B}). \tag{2.98}$$

Substituting (2.88) and (2.89) into (2.97) and (2.98) yields the formulae

$$\sigma_{|A} + P_{AA}^{-1} \cdot \psi_{|A} \cdot v_R^T \cdot \sigma_{|B} = \eta_A, \tag{2.99}$$

$$P_{BB}^{-1} \cdot \psi_{|B} \cdot v_L^T \cdot \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{2.100}$$

or

$$\sigma_{|A} + \phi_A \cdot v_R^T \cdot \sigma_{|B} = \eta_A, \tag{2.101}$$

$$\phi_B \cdot v_L^T \cdot \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{2.102}$$

where we have used the definitions (2.74), (2.75) for ϕ_A and ϕ_B, respectively. Now, multiplying (2.102) by $\phi_A \cdot v_R^T$ and subtracting it from (2.101), we obtain

$$(I_{(L^2)^n} - \phi_A \cdot v_R^T \cdot \phi_B \cdot v_L^T) \cdot \sigma_{|A} = \eta_A - \phi_A \cdot v_R^T \cdot \eta_B. \tag{2.103}$$

Similarly, multiplying (2.101) by $\phi_B \cdot v_L^T$ and subtracting it from (2.102) results in the equation

$$(I_{(L^2)^n} - \phi_B \cdot v_L^T \cdot \phi_A \cdot v_R^T) \cdot \sigma_{|B} = \eta_B - \phi_B \cdot v_L^T \cdot \eta_A. \tag{2.104}$$

Due to (2.76), (2.79), and (2.82), (2.85) we can rewrite these equations in the form

$$(I_{(L^2)^n} - \phi_A \cdot (\alpha_R^B \cdot v_L^T)) \cdot \sigma_{|A} = \eta_A - \phi_A \cdot \delta_R^B, \tag{2.105}$$

$$(I_{(L^2)^n} - \phi_B \cdot (\alpha_L^A \cdot v_R^T)) \cdot \sigma_{|B} = \eta_B - \phi_B \cdot \delta_L^A. \tag{2.106}$$

By application of Lemma 2.5, we obtain

$$\sigma_{|A} = (I_{(L^2)^n} + \phi_A \cdot (I_n - \alpha_R^B \cdot v_L^T \cdot \phi_A)^{-1} \cdot \alpha_R^B \cdot v_L^T) \cdot (\eta_A - \phi_A \cdot \delta_R^B), \tag{2.107}$$

$$\sigma_{|B} = (I_{(L^2)^n} + \phi_B \cdot (I_n - \alpha_L^A \cdot v_R^T \cdot \phi_B)^{-1} \cdot \alpha_L^A \cdot v_R^T) \cdot (\eta_B - \phi_B \cdot \delta_L^A). \tag{2.108}$$

The equations (2.92), (2.93) are now obtained from equations (2.107), (2.108) and equations (2.90), (2.91).

Remark 2.6 Suppose that b_1 and b_2 are a pair of real numbers such that $a < b_1 < b_2 < c$, and that the interval $[b_1, b_2]$ is denoted by C. We will denote by P_{CC} the restriction of the operator P to the interval C, and denote by Q_{CC} the restriction of the operator Q to the interval C. Assuming that P_{CC}, Q_{CC} are nonsingular, we define the functions $\eta_C : C \to \mathbf{R}^n, \phi_C : C \to \mathbf{L}(\mathbf{R}^{n \times n})$ by

$$\eta_C = P_{CC}^{-1}(f_{|C}), \tag{2.109}$$

$$\phi_C = Q_{CC}^{-1}(\psi_{|C}). \tag{2.110}$$

By applying the above lemma twice (once for the subinterval $[a, b_1]$ and once for $[a, b_2]$), we may easily observe that there exists $\lambda \in \mathbf{R}^n$ such that

$$\sigma(x) = \eta_C(x) + \phi_C(x) \cdot \lambda \tag{2.111}$$

for all $x \in C$. The exact expression for the vector λ is complicated, but irrelevant for the purposes of this chapter. The existence of a relation of the form (2.111), however, will be critically important in Section 2.3. □

The following corollary constructs the solution χ of equation (2.73) from ϕ_A, ϕ_B of equations (2.74) and (2.75).

Corollary 2.1 *If, in the notation of Section 2.2.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are nonsingular, then*

$$\chi_{|A} = \phi_A \cdot \Delta_2^{-1} \cdot (I_n - \alpha_R^B), \tag{2.112}$$

$$\chi_{|B} = \phi_B \cdot \Delta_1^{-1} \cdot (I_n - \alpha_L^A). \tag{2.113}$$

with the matrices α_L^A and α_R^B defined by equations (2.76) and (2.79), and the matrices Δ_1, Δ_2 defined by equations (2.90) and (2.91).

Proof. Substituting in equations (2.107), (2.108) the functions ϕ_A, ϕ_B defined by (2.74), (2.75) for the functions η_A, η_B defined by (2.71), (2.72), and the matrices α_L^A, α_R^B defined by (2.76), (2.79) for the vectors δ_L^A, δ_R^B defined by (2.82), (2.85), we obtain

$$\chi_{|A} = \phi_A - \phi_A \cdot \alpha_R^B + \phi_A \cdot \Delta_2^{-1} \cdot \alpha_R^B \cdot \alpha_L^A - \phi_A \cdot \Delta_2^{-1} \cdot \alpha_R^B \cdot \alpha_L^A \cdot \alpha_R^B, \quad (2.114)$$

$$\chi_{|B} = \phi_B - \phi_B \cdot \alpha_L^A + \phi_B \cdot \Delta_1^{-1} \cdot \alpha_L^A \cdot \alpha_R^B - \phi_B \cdot \Delta_1^{-1} \cdot \alpha_L^A \cdot \alpha_R^B \cdot \alpha_L^A. \quad (2.115)$$

The expressions (2.112), (2.113) are now easily obtained from the equations (2.114), (2.115).

2.2.4 Further Analytical Results

We now collect a number of identities which are necessary for Algorithm A, to be presented in Section 2.3.

Corollary 2.2 provides analytical expressions for the inner products δ_L and δ_R defined by (2.86), (2.87) in terms of the restricted inner products $\delta_L^A, \delta_L^B, \delta_R^A$ and δ_R^B defined by (2.82)–(2.85).

Corollary 2.2 *If, in the notation of Section 2.2.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are nonsingular, then*

$$\begin{aligned} \delta_L &= \int_a^c v_L(t) \cdot \sigma(t)\, dt = \int_a^b v_L(t) \cdot \sigma_{|A}(t)\, dt + \int_b^c v_L(t) \cdot \sigma_{|B}(t)\, dt \\ &= \delta_L^A + \delta_L^B + \alpha_L^A \cdot \Delta_2^{-1} \cdot (\alpha_R^B \cdot \delta_L^A - \delta_R^B) + \alpha_L^B \cdot \Delta_1^{-1} \cdot (\alpha_L^A \cdot \delta_R^B - \delta_L^A), \end{aligned} \quad (2.116)$$

$$\begin{aligned} \delta_R &= \int_a^c v_R(t) \cdot \sigma(t)\, dt = \int_a^b v_R(t) \cdot \sigma_{|A}(t)\, dt + \int_b^c v_R(t) \cdot \sigma_{|B}(t)\, dt \\ &= \delta_R^A + \delta_R^B + \alpha_R^A \cdot \Delta_2^{-1} \cdot (\alpha_R^B \cdot \delta_L^A - \delta_R^B) + \alpha_R^B \cdot \Delta_1^{-1} \cdot (\alpha_L^A \cdot \delta_R^B - \delta_L^A). \end{aligned} \quad (2.117)$$

Proof. Multiplying equation (2.92) by v_L^T and v_R^T, and equation (2.93) by v_L^T and v_R^T, we obtain

$$\int_a^b v_L(t) \cdot \sigma_{|A}(t)\, dt = \delta_L^A + \alpha_L^A \cdot \Delta_2^{-1} \cdot (\alpha_R^B \cdot \delta_L^A - \delta_R^B), \quad (2.118)$$

$$\int_b^c v_L(t) \cdot \sigma_{|B}(t)\, dt = \delta_L^B + \alpha_L^B \cdot \Delta_1^{-1} \cdot (\alpha_L^A \cdot \delta_R^B - \delta_L^A), \quad (2.119)$$

$$\int_a^b v_R(t) \cdot \sigma_{|A}(t)\, dt = \delta_R^A + \alpha_R^A \cdot \Delta_2^{-1} \cdot (\alpha_R^B \cdot \delta_L^A - \delta_R^B), \quad (2.120)$$

$$\int_b^c v_R(t) \cdot \sigma_{|B}(t)\, dt = \delta_R^B + \alpha_R^B \cdot \Delta_1^{-1} \cdot (\alpha_L^A \cdot \delta_R^B - \delta_L^A). \quad (2.121)$$

Now, expressions (2.116), (2.117) are easily obtained from (2.118)–(2.121).

Corollary 2.3 is similar to Corollary 2.2, but uses χ, the matrix valued function defined by (2.73), in place of σ, the vector valued function defined by (2.57). While the two corollaries concern different objects (the vectors δ_L, δ_R in Corollary 2.2, the matrices α_L, α_R in Corollary 2.3), their proofs are nearly identical.

Corollary 2.3 *If, in the notation of Section 2.2.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are nonsingular, then*

$$\begin{aligned}\alpha_L &= \int_a^c v_L(t)\cdot\chi(t)\,dt = \int_a^b v_L(t)\cdot\chi_{|A}(t)\,dt + \int_b^c v_L(t)\cdot\chi_{|B}(t)\,dt \\ &= \alpha_L^A\cdot\Delta_2^{-1}\cdot(I_n-\alpha_R^B)+\alpha_L^B\cdot\Delta_1^{-1}\cdot(I_n-\alpha_L^A),\end{aligned} \tag{2.122}$$

$$\begin{aligned}\alpha_R &= \int_a^c v_R(t)\cdot\chi(t)\,dt = \int_a^b v_R(t)\cdot\chi_{|A}(t)\,dt + \int_b^c v_R(t)\cdot\chi_{|B}(t)\,dt \\ &= \alpha_R^A\cdot\Delta_2^{-1}\cdot(I_n-\alpha_R^B)+\alpha_R^B\cdot\Delta_1^{-1}\cdot(I_n-\alpha_L^A).\end{aligned} \tag{2.123}$$

Finally, combining Lemma 2.7 with the expressions (2.112)–(2.113), we have

Corollary 2.4 *Suppose that in the notation of Section 2.2.1, all six operators P, P_{AA}, P_{BB}, Q,Q_{AA},Q_{BB} are nonsingular. Suppose further that the function F is defined by the formula*

$$F(x)=\chi\cdot\lambda+\sigma. \tag{2.124}$$

with $\lambda\in\mathbf{R}^n$. Then on the interval $[a,b]$,

$$F(x)=\phi_A(x)\cdot\mu+\eta_A(x), \tag{2.125}$$

with $\mu\in\mathbf{R}^n$ defined by the formula

$$\mu=\Delta_2^{-1}(\lambda-\alpha_R^B\cdot(\lambda-\delta_L^A)-\delta_R^B). \tag{2.126}$$

Similarly, on the interval $[b,c]$,

$$F(x)=\phi_B(x)\cdot\nu+\eta_B(x), \tag{2.127}$$

with $\nu\in\mathbf{R}^n$ defined via the formula

$$\nu=\Delta_1^{-1}\cdot(\lambda-\alpha_L^A\cdot(\lambda-\delta_R^B)-\delta_L^A). \tag{2.128}$$

Proof. Restricting (2.124) on the subintervals A, B of $[a,c]$, respectively, we have

$$F_{|A} = \chi_{|A}\cdot\lambda+\sigma_{|A}, \tag{2.129}$$

$$F_{|B} = \chi_{|B}\cdot\lambda+\sigma_{|B}. \tag{2.130}$$

Combining (2.129), (2.130) with (2.92), (2.93), (2.112), (2.113), we obtain

$$F_{|A} = \phi_A\cdot\Delta_2^{-1}\cdot(I_n-\alpha_R^B)\cdot\lambda+(\eta_A+\phi_A\cdot\Delta_2^{-1}\cdot(\alpha_R^B\cdot\delta_L^A-\delta_R^B)), \tag{2.131}$$

$$F_{|B} = \phi_B\cdot\Delta_1^{-1}\cdot(I_n-\alpha_L^A)\cdot\lambda+(\eta_B+\phi_B\cdot\Delta_1^{-1}\cdot(\alpha_L^A\cdot\delta_R^B-\delta_L^A)). \tag{2.132}$$

Now, the expressions (2.126), (2.128) immediately follow from the comparisons of (2.125), (2.127) with (2.131), (2.132), respectively.

2.3 Description of the Algorithms

We turn now to the construction of the fast algorithm for the solution of the integral equation (2.57)

$$P\sigma = f \ ,$$

based on the apparatus developed in Section 2.2. The main tool at our disposal is the ability to merge the solutions of restricted versions of the integral equation in adjacent subintervals (Lemma 2.7). As this suggests a recursive procedure, we begin by subdividing the whole interval $[a, c]$, on which the solution to (2.57) is sought, into a large number of subintervals. For the sake of simplicity, we assume that m is a positive integer and that $M = 2^m$ is the number of subintervals created. The boundary points of the subintervals are then defined by a strictly increasing sequence of numbers

$$b_1, b_2, \ldots, b_M, b_{M+1}, \tag{2.133}$$

with $b_1 = a$ and $b_{M+1} = c$. For each $i = 1, \ldots, M$, we define the interval B_i^m via the expression

$$B_i^m = [b_i, b_{i+1}], \tag{2.134}$$

and create a hierarchy of intervals B_i^j by recursively merging adjacent pairs. That is, for each $j = m-1, \ldots, 1, 0$, and $i = 1, \ldots, M$, we define

$$B_i^l = B_{2i-1}^{l+1} \bigcup B_{2i}^{l+1}. \tag{2.135}$$

We will refer to each fixed l as a *level*. We will also refer to the two intervals B_{2i-1}^{l+1} and B_{2i}^{l+1} as *children* and to the larger interval B_i^l as a *parent*.

It is obvious that

$$B_i^l = [b_{1+(i-1)\cdot 2^{m-l}}, b_{1+i\cdot 2^{m-l}}], \tag{2.136}$$

and that for each level l,

$$[a, c] = \bigcup_{i=1}^{2^l} B_i^l \ . \tag{2.137}$$

2.3.1 Notation

Generalizing the notation of Section 2.2, we will denote by $P_{i,l}$ the restriction to the interval B_i^l of the integral operator P, so that

$$P_{i,l}(\sigma)(x) \ = \ \sigma(x) + \tilde{p}(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} G_0(x,t) \cdot \sigma(t)\, dt \tag{2.138}$$

for any $\sigma \in L^2(B_i^l)^n$. Similarly, we will denote by $Q_{i,l}$ the restriction to the interval B_i^l of the integral operator Q, so that

$$Q_{i,l}(\chi)(x) \ = \ \chi(x) + \tilde{p}(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} G_0(x,t) \cdot \chi(t)\, dt \tag{2.139}$$

for any $\chi \in L^2(B_i^l)^{n\times n}$. For each B_i^l we will define the functions $\eta_{i,l} : B_i^l \to \mathbf{R}^n, \phi_{i,l} : B_i^l \to \mathbf{L}(\mathbf{R}^{n\times n})$ as the solutions of the equations

$$P_{i,l}(\eta_{i,l}) = f_{|B_i^l}, \tag{2.140}$$

$$Q_{i,l}(\phi_{i,l}) = \psi_{|B_i^l}, \tag{2.141}$$

provided the operators $P_{i,l}, Q_{i,l}$ are nonsingular.

Remark 2.7 Suppose now that the operators $P_{i,l}, Q_{i,l}$ are nonsingular on the interval B_i^l. Then, due to (2.111), there exists $\lambda^{i,l} \in \mathbf{R}^n$ such that

$$\sigma(x) = \eta_{i,l}(x) + \phi_{i,l}(x) \cdot \lambda^{i,l} \tag{2.142}$$

for all $x \in B_i^l$. □

For each $l = 0, 1, \ldots, m$, and $i = 1, 2, \ldots, 2^l$, we define the matrices $\alpha_L^{i,l}, \alpha_R^{i,l} \in \mathbf{L}(\mathbf{R}^{n\times n})$ by the formulae

$$\alpha_L^{i,l} = \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} v_{L|B_i^l}(t) \cdot \phi_{i,l}(t)\, dt, \tag{2.143}$$

$$\alpha_R^{i,l} = \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} v_{R|B_i^l}(t) \cdot \phi_{i,l}(t)\, dt, \tag{2.144}$$

and the vectors $\delta_L^{i,l}, \delta_R^{i,l} \in \mathbf{R}^n$ by the formulae

$$\delta_L^{i,l} = \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} v_{L|B_i^l}(t) \cdot \eta_{i,l}(t)\, dt, \tag{2.145}$$

$$\delta_R^{i,l} = \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} v_{R|B_i^l}(t) \cdot \eta_{i,l}(t)\, dt. \tag{2.146}$$

2.3.2 Discretization of the Restricted Integral Equations

Choosing an integer $p \geq 1$, we construct the p scaled Chebyshev nodes

$$\tau_i^j = \left(\frac{b_{i+1} - b_i}{2}\right) \cos\left[\frac{(2j-1)\pi}{2p}\right] + \left(\frac{b_{i+1} + b_i}{2}\right) \qquad j = 1, 2, \ldots, p \tag{2.147}$$

on each of the intervals B_i^m, $i = 1, 2, \ldots, M$. We then discretize the two integral equations (2.140), (2.141) via a Nyström algorithm based on $p-$point Chebyshev quadrature (see, for example, [22]). The resulting approximations to the functions $\eta_{i,l}$, $\phi_{i,l}$ at the nodes τ_i^j will be denoted by

$$\tilde{\eta}_{i,l} = (\tilde{\eta}_{i,l}^1, \tilde{\eta}_{i,l}^2, \ldots, \tilde{\eta}_{i,l}^p)\,,$$
$$\tilde{\phi}_{i,l} = (\tilde{\phi}_{i,l}^1, \tilde{\phi}_{i,l}^2, \ldots, \tilde{\phi}_{i,l}^p)\,,$$

respectively.

Remark 2.8 It is well-known that the order of convergence of the approximations $\tilde{\eta}_{i,l}, \tilde{\phi}_{i,l}$ to the functions $\eta_{i,l}, \phi_{i,l}$ is p. Since all subsequent steps in the construction of an approximate solution $\tilde{\sigma}$ to the integral equation (2.57) are analytic, the convergence rate of the full algorithm depends entirely on the parameter p. For example, by using 16 scaled Chebyshev points on each subinterval at the finest level, one obtains a sixteenth order method.

For many boundary value problems (2.1), (2.3), the matrix function $p : [a, c] \rightarrow \mathbf{L}(\mathbf{R}^{n \times n})$ is singular in the interval $[a, c]$ (see Example 2.3 in Section 2.4 below). For such problems, Chebyshev quadrature will yield acceptable convergence only if the singularity is removed by means of an appropriate choice of the "background" Green's function. □

Remark 2.9 The algorithm of this section makes extensive use of the apparatus of Chebyshev interpolation, quadratures, composite quadratures, etc. This apparatus is quite well-developed, and can be found in various forms in [16], [18], [20]. For a detailed description in the form most convenient for our purposes, we refer the reader to [22]. □

2.3.3 Informal Description of the Algorithm for Linear ODEs

We begin by directly solving the two integral equations (2.140), (2.141) on each subinterval B_i^m at the finest level, as discussed in the preceding section. Equation (2.142) then shows that σ restricted to B_i^m can be expressed as a linear combination of the two solutions $\eta_{i,m}$, $\phi_{i,m}$. Thus, it remains only to determine the coefficients $\lambda^{i,m} \in \mathbf{R}^n$ for each of the M subintervals B_i^m. Fortunately, this can be done recursively. To see this, suppose that, at some coarse level $l \leq m - 1$, we are given the coefficient $\lambda^{i,l}$ for the subinterval B_i^l. Then Corollary 2.4 provides formulae for the calculation of the corresponding coefficients $\lambda^{2i-1,l+1}, \lambda^{2i,l+1} \in \mathbf{R}^n$ for the two child intervals B_{2i-1}^{l+1} and B_{2i}^{l+1}, respectively. On the coarsest level, we observe that $\lambda^{0,1} = 0$, i.e. the solution of equation (2.140) on the whole interval $[a, c]$ is simply σ.

However, the formulae (2.126) and (2.128) of Corollary 2.4 contain the matrices $\alpha_L^{2i-1,l+1}$, $\alpha_R^{2i-1,l+1}$, $\alpha_L^{2i,l+1}$, $\alpha_R^{2i,l+1}$ and the vectors $\delta_L^{2i-1,l+1}$, $\delta_R^{2i-1,l+1}$, $\delta_L^{2i,l+1}$, $\delta_R^{2i,l+1}$. These quantities are also computed recursively but in the opposite direction, namely, from the finest level to the coarsest. They are certainly available at level m directly from the definitions (2.143)–(2.146). For the interval B_i^l at any coarser level $l \leq m - 1$, Corollaries 2.7 and 2.1 describe how matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ are obtained from the matrices α_L, α_R and vectors δ_L, δ_R of the two child intervals.

To summarize, the algorithm consists of three parts. First, a sufficiently fine subdivision $b_1, b_2, \ldots, b_{M+1}$ of the interval $[a, c]$ is chosen so that, on each of the intervals $B_{i,m}$, the functions $\eta_{i,m}, \phi_{i,m}$ can be accurately represented by a low order Chebyshev expansion. On each of the intervals $B_{i,m}$, the equations (2.140)–(2.141) are solved (approximately) by direct inversion of the linear system arising from a Nyström discretization. Second, the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ are computed in an upward sweep, beginning at the finest level m. Finally, the coefficients $\lambda^{i,l}$ are computed in a downward sweep, beginning at the coarsest level. The desired function σ is then recovered on each subinterval from equation (2.142).

The following is a more detailed description of the numerical procedure.

Algorithm A

Comment [Define the computational grid.]

Create $M = 2^m$ subintervals on $[a, c]$ by choosing a sequence of boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ with $b_1 = a$ and $b_{M+1} = c$. Choose the number p of Chebyshev nodes on each interval $B_i^m = [b_i, b_{i+1}]$ for $i = 1, \ldots, M$. Determine the locations of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m, and evaluate the functions f, ψ at these nodes, obtaining $f_{i,m}, \psi_{i,m}$.

Step 1.

Comment [Construct the approximate solutions $\tilde{\eta}_{i,m}, \tilde{\phi}_{i,m}$ on each interval B_i^m.]

do i = $1, 2, \ldots, M$

(1) Construct the two $p \cdot n \times p \cdot n$ linear systems on B_i^l obtained through a Nyström discretization of the corresponding integral equation.

(2) Solve the two $p \cdot n \times p \cdot n$ linear systems on B_i^l by Gaussian elimination, obtaining the values $\tilde{\eta}_{i,m}, \tilde{\phi}_{i,m}$.

end do

Step 2.

Comment [Construct the matrices $\alpha_L^{i,m}, \alpha_R^{i,m}$ and vectors $\delta_L^{i,m}, \delta_R^{i,m}$ on each interval B_i^m at the finest level.]

do i = $1, 2, \ldots, M$

Evaluate the matrices $\alpha_L^{i,m}, \alpha_R^{i,m}$ and vectors $\delta_L^{i,m}, \delta_R^{i,m}$ using the p–point Chebyshev quadrature formula.

end do

Step 3 (Upward Sweep).

Comment [Construct the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ for all intervals at all coarser levels $l = m-1, m-2, \ldots, 0$.]

do l= m-1, 0, -1

do i=1, 2^l

Compute the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ from the corresponding data in the two child intervals ($\alpha_L^{2i-1,l+1}, \alpha_R^{2i-1,l+1}, \alpha_L^{2i,l+1}, \alpha_R^{2i,l+1}, \delta_L^{2i-1,l+1}, \delta_R^{2i-1,l+1}, \delta_L^{2i,l+1}$, $\delta_R^{2i,l+1}$), using the results of Corollaries 2.2 and 2.3.

end do

end do

Step 4 (Downward Sweep).

Comment [Construct the coefficient $\lambda^{i,m}$ for all intervals at the finest level.]

Set $\lambda^{0,1} = 0$.

do l=0,m-1
  do i=1, 2^l
    Use Corollary 2.4 to compute the coefficients $\lambda^{l+1,2\cdot i-1}$, $\lambda^{2i,l+1}$,
    for the child intervals B_{2i}^{l+1} and B_{2i-1}^{l+1} from the coefficient $\lambda^{i,l}$ of the parent interval
    B_i^l.
  end do
end do

Step 5.

Comment [Compute the solution σ of equation (2.57) at the nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ for each interval B_i^m at the finest level.]

do i=1, M
  do j=1,p
    Determine the values of the solution σ of equation (2.57) at the node τ_i^j via
    formula (2.142).
  end do
end do

Step 6.

Comment [Compute the solution ϕ of equation (2.1) from the values of σ.]

Evaluate the integral (2.27), by using composite Chebyshev quadrature
(see Remark 2.11 below).

Remark 2.10 Inspection of the above algorithm shows that the amount of work required is of the order $O(M \cdot p^3 \cdot n^3)$. Step 1 involves solving two $(p \times n) \times (p \times n)$ linear systems for each of the M intervals. Steps 2–5 require no more than $O(M \cdot p \cdot n^2 \cdot (\log p + n))$ operations. Since $N = M \cdot p$ is the total number of nodes in the discretization of the interval $[a, c]$, we can write the CPU time estimate in the form $O(N \cdot p^2 \cdot n^3)$. The cost of evaluating the solution Φ of the differential equation (2.55) from the integral representation (2.27) is $O(N \cdot \log p \cdot n)$ (see Remark 2.11 below). □

Remark 2.11 The final step in the algorithm involves the evaluation of an integral of the form (2.27) at each of the Chebyshev nodes τ_i^j on each subinterval B_i^m, namely

$$\Phi(\tau_i^j) = \int_a^c G_0(\tau_i^j, t) \cdot \sigma(t)\, dt. \tag{2.148}$$

If these integrals were calculated independently for each τ_i^j, the amount of work required would be of the form $O(N^2 \cdot n)$, and would dominate the construction of the function Φ. In fact, this is unnecessary, for we may write

$$\Phi(\tau_i^j) = \Upsilon(\tau_i^j) \cdot \left[\int_a^{b_i} v_L(t) \cdot \sigma(t)\, dt + \int_{b_i}^{\tau_i^j} v_L(t) \cdot \sigma(t)\, dt \right. \tag{2.149}$$
$$\left. + \int_{\tau_i^j}^{b_{i+1}} v_R(t) \cdot \sigma(t)\, dt + \int_{b_{i+1}}^{c} v_R(t) \cdot \sigma(t)\, dt \right],$$

where we have used the representation (2.22) and the fact that τ_i^j lies in the interval $B_i^m = [b_i, b_{i+1}]$. Step 6 can then be written in detail as follows:

Step 6 (a).

Comment [Precompute the integrals of $v_L \cdot \sigma$ and $v_R \cdot \sigma$ on each subinterval B_i^m by Chebyshev quadrature. These integrals will be denoted I_L and I_R, respectively.]

do i=1, M
$I_L(B_i^m) = \int_{b_i}^{b_{i+1}} v_L(t) \cdot \sigma(t)\, dt.$

$I_R(B_i^m) = \int_{b_i}^{b_{i+1}} v_R(t) \cdot \sigma(t)\, dt.$
end do

Step 6 (b).

Comment [March across interval from a to c, computing Φ at each node in the discretization. The variables $\mathcal{J}_L$ and $\mathcal{J}_R$ will be used to accumulate the integrals $\int_a^{b_i} v_L(t) \cdot \sigma(t)\, dt$ and $\int_{b_{i+1}}^{c} v_R(t) \cdot \sigma(t)\, dt$, respectively.

Set $\mathcal{J}_R = \sum_{i=2}^{M} I_R(B_i^m)$.
Set $\mathcal{J}_L = 0$.

do i=1, M
do j=1,p
For each τ_i^j, compute
$\Phi(\tau_i^j) = \Upsilon(\tau_i^j) \cdot \left[\mathcal{J}_L + \int_{b_i}^{\tau_i^j} v_L(t) \cdot \sigma(t)\, dt + \int_{\tau_i^j}^{b_{i+1}} v_R(t) \cdot \sigma(t)\, dt + \mathcal{J}_R \right]$
end do
$\mathcal{J}_L = \mathcal{J}_L + I_L(B_i^m)$
$\mathcal{J}_R = \mathcal{J}_R - I_R(B_{i+1}^m)$
end do

Thus, the amount of work required in Step 6(a) is $O(N \cdot n)$. The integrals required on each subinterval in Step 6 (b) can be computed by spectral integration (see, for example, [22]) using $O(p \cdot \log p \cdot n)$ work. The total cost is therefore of the order $O(M \cdot p \cdot \log p \cdot n)$ or $O(N \cdot \log p \cdot n)$. □

2.3.4 Informal Description of a Simplified Algorithm for Linear ODEs

Figure 2.1 suggests a simpler version of Algorithm A, in which the solutions $\eta_{i,l}, \phi_{i,l}$ are obtained for all levels $l = m, m-1, \ldots, 0$, with the solution $\sigma \equiv \eta_{0,1}$. The time complexity of the resulting algorithm is $O(N \cdot p^2 \cdot q^3 + N \cdot \log_2(N) \cdot q^2)$; since this compares unfavorably with Algorithm A, we merely sketch the simplified algorithm below.

Algorithm A′

Comment [Define the computational grid.]

Create $M = 2^m$ subintervals on $[a, c]$ by choosing a sequence of boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ with $b_1 = a$ and $b_{M+1} = c$. Choose the number p of Chebyshev nodes on each interval $B_i^m = [b_i, b_{i+1}]$ for $i = 1, \ldots, M$. Determine the locations of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m, and evaluate the functions f, ψ at these nodes, obtaining $f_{i,m}, \psi_{i,m}$.

Step 1.

Comment [Construct the approximate solutions $\tilde{\eta}_{i,m}, \tilde{\phi}_{i,m}$ on each interval B_i^m.]

do i = $1, 2, \ldots, M$

(1) Construct the two $p \cdot n \times p \cdot n$ linear systems on B_i^l obtained through a Nyström discretization of the corresponding integral equation.

(2) Solve the two $p \cdot n \times p \cdot n$ linear systems on B_i^l by Gaussian elimination, obtaining the values $\tilde{\eta}_{i,m}, \tilde{\phi}_{i,m}$.

end do

Step 2.

Comment [Construct the matrices $\alpha_L^{i,m}, \alpha_R^{i,m}$ and vectors $\delta_L^{i,m}, \delta_R^{i,m}$ on each interval B_i^m at the finest level.]

do i = $1, 2, \ldots, M$

Evaluate the matrices $\alpha_L^{i,m}, \alpha_R^{i,m}$ and vectors $\delta_L^{i,m}, \delta_R^{i,m}$ using the p−point Chebyshev quadrature formula.

end do

Step 3 (Upward Sweep).

Comment [Construct the solutions $\eta_{i,l}, \phi_{i,l}$, matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ for all intervals at all coarser levels $l = m-1, m-2, \ldots, 0$.]

do l= m-1, 0, -1

do i=1, 2^l

(1) Compute the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and vectors $\delta_L^{i,l}, \delta_R^{i,l}$ from the corresponding data in the two child intervals ($\alpha_L^{2i-1,l+1}, \alpha_R^{2i-1,l+1}, \alpha_L^{2i,l+1}, \alpha_R^{2i,l+1}, \delta_L^{2i-1,l+1}, \delta_R^{2i-1,l+1}$, $\delta_L^{2i,l+1}, \delta_R^{2i,l+1}$), using the results of Corollaries 3.4 and 3.5. Alternatively, compute

the matrices $\alpha_l^{i,l}, \alpha_r^{i,l}$ vectors $\delta_L^{i,l}, \delta_R^{i,l}$ via formulae (2.143)-(2.146) using the p-point Chebyshev quadrature formula.

(2) Compute the solutions $\eta_{i,l}, \phi_{i,l}$ from data in the two child intervals ($\eta_{2i-1,l+1}$, $\eta_{2i,l+1}, \phi_{2i-1,l+1}, \phi_{2i,l+1}, \alpha_L^{2i-1,l+1}, \alpha_R^{2i,l+1}, \delta_L^{2i-1,l+1}, \delta_R^{2i,l+1}$), using the results of Lemma 2.7 and Corollary 2.1.

end do

end do

Step 4.

Comment [Compute the solution ϕ of equation (2.1) from the values of σ.]

Evaluate the integral (2.27), using composite Chebyshev quadrature.

2.3.5 Informal Description of the Algorithm for Nonlinear ODEs

The nonlinear algorithm is a straightforward application of Algorithm A described in Section 2.3.3. The solution is obtained using Newton's method for nonlinear ODEs; each Newton iterate is obtained by solving the linearized problem (2.50) via Algorithm A.

As with Algorithm A, we subdivide the interval $[a, c]$ into a large number of subintervals M; for simplicity we assume $M = 2^m$, with m a positive integer. As before the boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ are defined by (2.133), and the intervals $B_i^l, (1 \leq l \leq m), (1 \leq i \leq 2^l)$ by (2.134).

On the k^{th} step of the Newton process, Algorithm A is applied to the integral equation

$$P^k \delta_k = g_k, \tag{2.150}$$

with the operator $P^k : (L^2[a,c])^n \to (L^2[a,c])^n$ defined by the formula

$$P^k(\delta_k)(x) = \delta_k(x) + \Omega_k(x) \cdot \int_a^c G_0(x,t) \cdot \delta_k(t)\, dt, \tag{2.151}$$

with δ_k the solution of the integral equation (2.50), Ω_k given by (2.51), and g_k given by (2.52). The integral equations (2.140), (2.141) now assume the form

$$P_{i,l}^k(\eta_{i,l}) = g_{k|B_i^l}, \tag{2.152}$$

$$Q_{i,l}^k(\phi_{i,l}) = \Omega_{k|B_i^l}, \tag{2.153}$$

with the operator $P_{i,l}^k : (L^2[a,c])^n \to (L^2[a,c])^n$ defined by the formula

$$P_{i,l}^k(\delta_k)(x) = \delta_k(x) + \Omega_k(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} G_0(x,t) \cdot \delta_k(t)\, dt, \tag{2.154}$$

and the operator $Q_{i,l}^k : (L^2[a,c])^{n\times n} \to (L^2[a,c])^{n\times n}$ defined via the formula

$$Q_{i,l}^k(\chi)(x) = \chi(x) + \Omega_k(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} G_0(x,t) \cdot \chi(t)\, dt. \tag{2.155}$$

Once Algorithm A has computed the solution δ_k to (2.150), we obtain σ_{k+1} via (2.11), and Φ_{k+1} via (2.49).

The nonlinear algorithm requires an initial approximation Φ_0, Φ_0' to the solution Φ and its derivative Φ' of equation (2.5), and we assume that both are supplied by the calling program. σ_0 is obtained from Φ_0, Φ_0' via the identity

$$\sigma_0(x) = \Phi_0'(x) + p_0(x) \cdot \Phi_0(x). \tag{2.156}$$

The procedure is terminated when the stopping criterion

$$\frac{\|\delta_k\|_2}{\|\sigma_k\|_2} \leq \epsilon \tag{2.157}$$

is satisfied, with ϵ provided by the calling program. Since Newton's method frequently fails to converge, the calling program also permits a certain maximum number of iterations, after which the algorithm stops, signaling failure.

The following is a more detailed description of the numerical procedure.

Algorithm B

Comment [Define the computational grid.]

Create $M = 2^m$ subintervals on $[a, c]$ by choosing a sequence of boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ with $b_1 = a$ and $b_{M+1} = c$. Choose the number p of Chebyshev nodes on each interval $B_i^m = [b_i, b_{i+1}]$ for $i = 1, \ldots, M$. Determine the locations of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m, and use the initial approximations Φ_0, Φ_0' to evaluate the initial approximations Φ, σ at these nodes, obtaining $\Phi_{i,m}, \sigma_{i,m}$. Choose tolerance ϵ.

Step 1.

Comment [Use Algorithm A to compute Newton iterates Φ_k, obtaining the solution Φ of equation (2.5).]

repeat

(1) Set $\hat{\Phi} = \Phi, \hat{\sigma} = \sigma$.

(2) Evaluate the functions $\hat{\Omega}, \hat{g}$ at each of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m, obtaining $\hat{\Omega}_{(i,m)}, \hat{g}_{(i,m)}$.

(3) Apply Algorithm A to the discretized form of (2.50), obtaining $\hat{\delta}$.

(4) Set $\sigma = \hat{\sigma} + \hat{\delta}$.

(5) Compute the solution Φ of equation (2.49) from the values of σ, using composite Chebyshev quadrature (see Step 6 of Algorithm A and Remark 2.11 above).

until $\|\hat{\delta}\|_2 / \|\hat{\sigma}\|_2 \leq \epsilon$

2.4 Numerical Results

FORTRAN programs have been written implementing the algorithms described in the preceding section. In this section, we discuss several details of our implementation, and demonstrate the performance of the scheme with numerical examples.

The following technical details of our implementation appear to be worth mentioning.

1. The algorithms described in the preceding section require that the number M of elementary subintervals on the interval $[a, c]$ be a power of 2. Clearly, this is not an essential limitation and it can be removed by simple bookkeeping changes. In the version of the algorithms used for numerical experiments, these changes were made.

2. Algorithm A depends for its stability on the equations (2.140), (2.141) having unique solutions for all subintervals B_i^l ($l = 0, 1, \ldots, M$, $i = 1, \ldots, 2^l$), while Algorithm B depends on (2.152), (2.153) having unique solutions for all subintervals B_i^l and for all Newton iterates k. It is easy to construct examples for which these conditions are violated, even though equation (2.57) or equation (2.32) has a unique solution. In such cases, a different subdivision of the interval $[a, c]$ can be attempted, such that none of the subintervals B_i^k of the new subdivision coincides with an interval of the original one. This procedure can be viewed as a form of pivoting, and it is easy to show that it is always possible to make it work. It has not been implemented at this point, and we have not so far encountered a need for it.

3. We have, however, implemented a crude scheme for detecting high condition numbers in the algorithms. These can occur in two places: in the solution of the linear systems on each of the finest level subintervals (Step 1 of Algorithm A), and while computing coefficients Δ_1, Δ_2 defined by (2.90), (2.91) used when merging solutions on two consecutive subintervals (Step 3 of Algorithm A). In both cases, the condition number of the system being solved is estimated in the process of solution (we use a standard LINPACK routine), and the largest of these is returned to the user. When an extremely large condition number is detected by the LINPACK routine, the resulting solution of the original ODE should be viewed as suspect. It is easy to show that when the differential operator is positive definite, this cannot happen. A more complete treatment of this subject requires further study.

4. In the upward sweep (Step 3) of Algorithm A, we evaluate the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ for all intervals $B_{i,l}$ and use these matrices to evaluate the vectors $\delta_L^{i,l}, \delta_R^{i,l}$, the vectors $\lambda^{i,l}$, and, finally, the solution σ of the integral equation (2.57). But the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ do not depend on the right-hand side f of equation (2.57), and it is easy to see that their evaluation accounts for more than 90% of the work. Therefore, whenever the equation (2.57) has to be solved with multiple right-hand sides, we can precompute the matrices $\alpha_L^{i,l}, \alpha_R^{i,l}$ and store them, saving 90% of the cost of the evaluation of subsequent solutions.

The algorithms of this chapter have been applied to a variety of problems. Five experiments are described below, and their results are summarized in Tables 2.1–2.13.

Tables 2.1–2.11 are associated with examples for which analytic solutions are available. In each of these tables, the first column contains the total number N of nodes in the

discretization of the interval $[a, c]$. The second column contains the relative L^2 error of the numerical solution as compared with the analytically obtained one at 5000 equispaced points within the interval $[a, c]$, where Chebyshev interpolation has been used to evaluate the numerical solution at each of the 5000 points. The third column contains the maximum absolute error obtained at any of the 5000 points. The fourth column contains the CPU time required to solve the problem, excluding the time used to evaluate the solution at 5000 equispaced points, where in all cases the times are given for a SUN SPARCstation 1 computer. Tables 2.9–2.11, associated with a nonlinear example, have in addition a fifth column which contains the number of Newton steps taken before the stopping criterion (2.157) has been satisfied, with $\epsilon = 10^{-10}$.

Tables 2.12–2.13 are associated with an example for which we did not have analytic solutions. In this example, we compare each numerical solution with p Chebyshev nodes and n subintervals against the solution with p Chebyshev nodes and $2 \cdot n$ subintervals. In each of these tables, the first column contains the total number N of nodes in the discretization of the interval $[a, c]$. The second column contains the relative L^2 error of the numerical solution as compared with the numerical solution with twice the number of subintervals, where the comparison is made at each of 5000 equispaced points in the interval $[a, c]$, and where Chebyshev interpolation has been used to evaluate the numerical solution at each of the 5000 points. The third column contains the maximum absolute error obtained at any of the 5000 points. The fourth column contains the SPARCstation CPU time required to solve the problem, excluding the time used to evaluate the solution at 5000 equispace points.

Remark 2.12 In Example 2.3 below, we solve a system of boundary value problems of order 2; and in Example 2.5, we solve a problem of order 4. In both cases, the problems were reduced to canonical first order systems (see, for example, [13]), with the latter solved by means of algorithms A or B of the preceding section, as appropriate. □

Example 2.1 This example is taken from [6], where it is introduced as a stiff problem. The equation to be solved is given by the formulae

$$\phi_1'(x) - 998 \cdot \phi_1(x) - 1998 \cdot \phi_2(x) = 2 \cdot x, \tag{2.158}$$

$$\phi_2'(x) + 999 \cdot \phi_1(x) + 1999 \cdot \phi_2(x) = x, \tag{2.159}$$

subject to the boundary conditions

$$\phi_1(0) = 1, \tag{2.160}$$

$$\phi_2(1) = -6 \cdot e^{-1} + 5 \cdot e^{-1000} + .004 \cdot (.999 + .001 \cdot e^{-1000}). \tag{2.161}$$

We use the results of Theorem 2.3 to reduce the first order system (2.158)–(2.161) to one subject to homogeneous boundary conditio:

$$\phi_1(0) = 0, \tag{2.162}$$

$$\phi_2(1) = 0. \tag{2.163}$$

Table 2.1: Numerical results for Example 2.1, $p = 8$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (sec.)
16	0.962×10^{0}	0.685×10^{1}	0.150×10^{0}
32	0.244×10^{1}	0.216×10^{2}	0.300×10^{0}
64	0.700×10^{-1}	0.272×10^{1}	0.560×10^{0}
128	0.255×10^{-1}	0.125×10^{1}	0.108×10^{1}
256	0.667×10^{-2}	0.342×10^{0}	0.214×10^{1}
512	0.805×10^{-3}	0.536×10^{-1}	0.428×10^{1}
1024	0.330×10^{-4}	0.291×10^{-2}	0.846×10^{1}
2048	0.482×10^{-6}	0.529×10^{-4}	0.168×10^{2}
4096	0.316×10^{-8}	0.476×10^{-6}	0.337×10^{2}
8192	0.112×10^{-10}	0.170×10^{-8}	0.670×10^{2}
16384	0.115×10^{-11}	0.113×10^{-10}	0.137×10^{3}

We apply Algorithm A to this system using equispaced subintervals, with the number of Chebyshev nodes $p = 8, 16, 24$. For this experiment, the background Green's function is chosen to correspond to the equation

$$\Phi'(x) = 0, \tag{2.164}$$

subject to boundary conditions (2.162)–(2.163). The results of this experiment are presented in Tables 2.1–2.3.

Example 2.2 We solve the problem (2.158)–(2.161) defined in Example 2.1, but using an alternate division of the subintervals. Since the solution of this problem has a fairly sharp boundary layer near the left end of the interval $[0, 1]$, we construct the intervals $B_i^m = [b_i, b_{i+1}]$ via the formula

$$\begin{aligned} b_1 &= 0\,, \\ b_i &= \left(\frac{1}{2}\right)^{M+1-i} \quad \text{for } i = 2, \ldots, M+1\,, \end{aligned} \tag{2.165}$$

so that they become progressively smaller near the left end of the interval $[0, 1]$. As in Example 2.1, we reduce the problem (2.158)–(2.161) to a first order system subject to homogeneous boundary conditions (2.162)–(2.163). Algorithm A has been applied to this problem using the Green's function corresponding to the equation (2.164) subject to boundary conditions (2.162)–(2.163), and with the number of Chebyshev nodes $p = 16$ and 24. The results of this experiment appear in Tables 2.4–2.5, and are most satisfactory.

Example 2.3 The purpose of this example is to demonstrate the performance of the method when the coefficient p of the equation (2.2) is singular at the ends of its interval of definition, while the particular solution being sought is smooth. We solve the Bessel

Table 2.2: Numerical results for Example 2.1, $p = 16$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (sec.)
16	0.567×10^{0}	0.436×10^{1}	0.310×10^{0}
32	0.251×10^{1}	0.192×10^{2}	0.540×10^{0}
64	0.340×10^{-1}	0.139×10^{1}	0.950×10^{0}
128	0.744×10^{-2}	0.344×10^{0}	0.179×10^{1}
256	0.798×10^{-3}	0.389×10^{-1}	0.345×10^{1}
512	0.164×10^{-4}	0.930×10^{-2}	0.686×10^{1}
1024	0.364×10^{-7}	0.243×10^{-5}	0.137×10^{2}
2048	0.992×10^{-11}	0.817×10^{-9}	0.274×10^{2}
4096	0.942×10^{-13}	0.776×10^{-12}	0.532×10^{2}
8192	0.214×10^{-12}	0.164×10^{-11}	0.107×10^{3}

Table 2.3: Numerical results for Example 2.1, $p = 24$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (sec.)
24	0.511×10^{0}	0.373×10^{1}	0.690×10^{0}
48	0.251×10^{0}	0.244×10^{1}	0.116×10^{1}
96	0.591×10^{-2}	0.240×10^{0}	0.209×10^{1}
192	0.496×10^{-3}	0.214×10^{-1}	0.387×10^{1}
384	0.546×10^{-5}	0.258×10^{-3}	0.765×10^{1}
768	0.274×10^{-8}	0.129×10^{-6}	0.147×10^{2}
1536	0.105×10^{-12}	0.335×10^{-11}	0.295×10^{2}
3072	0.663×10^{-13}	0.624×10^{-12}	0.578×10^{2}

Table 2.4: Numerical results for Example 2.2, $p = 16$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (sec.)
16	0.567×10^{0}	0.436×10^{1}	0.310×10^{0}
32	0.251×10^{1}	0.192×10^{2}	0.540×10^{0}
64	0.790×10^{-2}	0.360×10^{0}	0.950×10^{0}
128	0.992×10^{-11}	0.818×10^{-9}	0.177×10^{1}
256	0.244×10^{-12}	0.261×10^{-11}	0.352×10^{1}
512	0.243×10^{-12}	0.261×10^{-11}	0.681×10^{1}

Table 2.5: Numerical results for Example 2.2, $p = 24$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (sec.)
24	0.511×10^{0}	0.373×10^{1}	0.710×10^{0}
48	0.251×10^{0}	0.244×10^{1}	0.117×10^{1}
96	0.496×10^{-3}	0.214×10^{-1}	0.209×10^{1}
192	0.293×10^{-12}	0.227×10^{-11}	0.387×10^{1}
384	0.294×10^{-12}	0.227×10^{-11}	0.758×10^{1}

equation system

$$\phi_\nu''(x) + \frac{x^2 - n^2 - n}{x^2} \cdot \phi_\nu(x) + \frac{1}{x} \cdot \phi_{\nu-1}(x) = 0, \tag{2.166}$$

$$\phi_{\nu-1}''(x) + \frac{x^2 - n^2 + n}{x^2} \cdot \phi_{\nu-1}(x) + \frac{1}{x} \cdot \phi_{\nu-2}(x) = 0, \tag{2.167}$$

$$\phi_{\nu-2}''(x) - \frac{1}{x} \cdot \phi_{\nu-1}(x) + \frac{x^2 - n^2 + 5 \cdot n - 6}{x^2} \cdot \phi_{\nu-2} = 0, \tag{2.168}$$

(see, for example, [1]) on the interval $[0, 600]$ with the boundary conditions

$$\phi_\nu(0) = \phi_{\nu-1}(0) = \phi_{\nu-2}(0) = 0, \tag{2.169}$$

$$\phi_\nu'(600) = 0.030598170290372796, \tag{2.170}$$

$$\phi_{\nu-1}'(600) = 0.015416721257492013, \tag{2.171}$$

$$\phi_{\nu-2}'(600) = -0.025526503991812874, \tag{2.172}$$

and $\nu = 100$. The difficulty of this problem is due to the fact that the two linearly independent solutions to each of equations (2.166), (2.167), (2.168) are $J_\nu(x)$, $Y_\nu(x)$; $J_{\nu-1}(x)$, $Y_{\nu-1}(x)$; and $J_{\nu-2}(x)$, $Y_{\nu-2}(x)$, respectively, (Bessel functions of the first and the second kinds). As is well known, $J_\nu(x), J_{\nu-1}$, and $J_{\nu-2}$ behave in the vicinity of zero like $x^\nu, x^{\nu-1}$, and $x^{\nu-2}$, respectively, while $Y_\nu(x), Y_{\nu-1}$, and $Y_{\nu-2}$ behave like $x^{-\nu}, x^{-(\nu-1)}$, and $x^{-(\nu-2)}$, respectively; most methods have trouble finding the decaying solution. In addition, this is a fairly large-scale calculation, since the the solution to (2.166)–(2.168) contains almost 100 wavelengths in the interval $[0, 600]$.

We reduce the problem (2.166)–(2.168) to a first order system, and apply Algorithm A to this system using equispaced subintervals, with the number of Chebyshev nodes $p = 16, 20$ and 24. For this experiment, the background Green's function is chosen to correspond to the equation

$$\Phi'(x) = 0, \tag{2.173}$$

subject to boundary conditions (2.169)–(2.172). The results of this experiment are presented in Tables 2.6–2.8.

Table 2.6: Numerical results for Example 2.3, $p = 16$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (*sec.*)
128	0.133×10^{1}	0.415×10^{0}	0.193×10^{2}
256	0.958×10^{0}	0.110×10^{0}	0.384×10^{2}
512	0.758×10^{-2}	0.779×10^{-3}	0.764×10^{2}
1024	0.203×10^{-5}	0.213×10^{-6}	0.152×10^{3}
2048	0.632×10^{-10}	0.661×10^{-11}	0.301×10^{3}
4096	0.823×10^{-11}	0.126×10^{-11}	0.609×10^{3}

Table 2.7: Numerical results for Example 2.3, $p = 20$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (*sec.*)
160	0.185×10^{1}	0.558×10^{0}	0.330×10^{2}
320	0.216×10^{1}	0.392×10^{0}	0.655×10^{2}
640	0.373×10^{-4}	0.388×10^{-5}	0.130×10^{3}
1280	0.493×10^{-9}	0.522×10^{-10}	0.260×10^{3}
2560	0.111×10^{-12}	0.220×10^{-13}	0.520×10^{3}
5120	0.142×10^{-10}	0.186×10^{-11}	0.104×10^{4}

Table 2.8: Numerical results for Example 2.3, $p = 24$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (*sec.*)
96	0.125×10^{1}	0.431×10^{0}	0.262×10^{2}
192	0.133×10^{1}	0.277×10^{0}	0.527×10^{2}
384	0.718×10^{-1}	0.797×10^{-2}	0.104×10^{3}
768	0.728×10^{-7}	0.765×10^{-8}	0.206×10^{3}
1536	0.308×10^{-12}	0.555×10^{-13}	0.409×10^{3}
3072	0.551×10^{-12}	0.397×10^{-13}	0.818×10^{3}

Example 2.4 We consider a system of Jacobian elliptic functions $sn, cn, dn : [0, 10 \cdot K] \rightarrow \mathbf{R}$ (see, for example, [1]) which are solutions to the equations

$$sn'(x) = cn(x) \cdot dn(x), \tag{2.174}$$
$$cn'(x) = -sn(x) \cdot dn(x), \tag{2.175}$$
$$dn'(x) = -m \cdot sn(x) \cdot cn(x), \tag{2.176}$$

with $m = \frac{1}{2}$ in our experiments, subject to the boundary conditions

$$sn(0) = 0, \tag{2.177}$$
$$cn(0) = 1, \tag{2.178}$$
$$dn(40 \cdot K) = 1, \tag{2.179}$$

with K given by the expression

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m \cdot \sin^2 \theta}}. \tag{2.180}$$

We use for an initial guess the solution to (2.176), (2.179) for $m = 0$, which is defined by the formulae

$$sn(x) = \sin\left(\frac{\pi}{2 \cdot K} \cdot x\right), \tag{2.181}$$
$$cn(x) = \cos\left(\frac{\pi}{2 \cdot K} \cdot x\right), \tag{2.182}$$
$$dn(x) = 1. \tag{2.183}$$

We use the results of Theorem 2.4 to reduce the system (2.174)–(2.176) to one subject to the homogeneous boundary conditions

$$sn(0) = 0, \tag{2.184}$$
$$cn(0) = 0, \tag{2.185}$$
$$dn(40 \cdot K) = 0, \tag{2.186}$$

and then apply Algorithm B to this system using equispaced subintervals, with the number of Chebyshev nodes $p = 8, 16$ and 32. For this experiment, the background Green's function is chosen to correspond to the equation

$$\Phi'(x) = 0,$$

subject to boundary conditions (2.184)–(2.186). The results of this experiment are presented in Tables 2.9–2.11.

Example 2.5 This example is taken from [28]. Its purpose is to demonstrate the performance of the method when the equation to be solved contains fourth order derivatives. The

Table 2.9: Numerical results for Example 2.4, $p = 8$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (*sec.*)	Steps
128	0.211×10^{-1}	0.551×10^{-1}	0.167×10^{2}	9
256	0.158×10^{-2}	0.416×10^{-2}	0.258×10^{2}	7
512	0.849×10^{-5}	0.226×10^{-4}	0.439×10^{2}	6
1024	0.106×10^{-7}	0.330×10^{-7}	0.883×10^{2}	6
2048	0.313×10^{-10}	0.984×10^{-10}	0.175×10^{3}	6
4096	0.147×10^{-12}	0.469×10^{-12}	0.348×10^{3}	6

Table 2.10: Numerical results for Example 2.4, $p = 16$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (*sec.*)	Steps
64	0.162×10^{0}	0.505×10^{0}	0.177×10^{2}	10
128	0.422×10^{-1}	0.108×10^{0}	0.246×10^{2}	7
256	0.181×10^{-3}	0.476×10^{-3}	0.409×10^{2}	6
512	0.441×10^{-7}	0.120×10^{-6}	0.851×10^{2}	6
1024	0.425×10^{-12}	0.125×10^{-11}	0.164×10^{3}	6
2048	0.115×10^{-12}	0.317×10^{-12}	0.324×10^{3}	6
4096	0.569×10^{-13}	0.152×10^{-12}	0.653×10^{3}	6

Table 2.11: Numerical results for Example 2.4, $p = 32$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (*sec.*)	Steps
256	0.148×10^{-3}	0.388×10^{-3}	0.163×10^{3}	6
512	0.124×10^{-9}	0.395×10^{-9}	0.322×10^{3}	6
1024	0.450×10^{-12}	0.119×10^{-11}	0.632×10^{3}	6
2048	0.266×10^{-12}	0.703×10^{-12}	0.126×10^{4}	6

deflection of a beam under a uniform load q, with the beam built in at the left end ($x = 0$) and simply supported at the right end, is given by the formula

$$y''''(x) + \frac{k}{E \cdot I} \cdot y(x) = \frac{q}{E \cdot I}, \tag{2.187}$$

subject to the boundary conditions

$$y(0) = y'(0) = y(L) = y''(L) = 0, \tag{2.188}$$

with k the force per unit deflection per unit length of beam, and $E \cdot I$ the flexural rigidity of the beam. The values of the constants used are

$$L = 1.2 \times 10^2 \text{ in.}, \tag{2.189}$$
$$k = 2.604 \times 10^3 \text{ psi}, \tag{2.190}$$
$$q = 4.34 \times 10^4 \text{ lbs/in.}, \tag{2.191}$$
$$E = 3.0 \times 10^7 \text{ psi}, \tag{2.192}$$
$$I = 3.0 \times 10^3 \text{ in.}^4, \tag{2.193}$$

(see [28], p. 174). The L^2 norm of y, the solution to (2.188), is approximately 10^6 times larger than the L^2 norm of y''''. Combined with the high number of derivatives in (2.187), this tends to present difficulties for finite difference methods. We reduce the problem (2.187), (2.188) to a first order system, and then use the results of Theorem 2.9 to express Φ by the formula

$$\Phi(x) = \Psi(x) \cdot \Gamma(x),$$

with $\Psi(x) : [0, L] \to \mathbf{L}(\mathbf{R}^{4\times 4})$ given by the formula

$$\Psi(x) = \begin{pmatrix} \frac{L-x}{L} & 0 & \frac{x}{L} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}. \tag{2.194}$$

and $\Gamma : [a, c] \to \mathbf{R}^n$ the solution to the equation

$$\Gamma'(x) + \Psi^{-1}(x) \cdot (\Psi'(x) + p(x) \cdot \Psi(x)) \cdot \Gamma(x) = \Psi^{-1}(x) \cdot f(x), \tag{2.195}$$

subject to boundary conditions

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \Psi(a) \cdot \Gamma(a) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \Psi(c) \cdot \Gamma(c) = 0, \tag{2.196}$$

with $p : [0, L] \to \mathbf{L}(\mathbf{R}^{4\times 4})$ defined by the formula

$$p(x) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{k}{E \cdot I} & 0 & 0 & 0 \end{pmatrix}, \tag{2.197}$$

Table 2.12: Numerical results for Example 2.5, $p = 4$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (sec.)
4	0.115×10^{0}	0.672×10^{-1}	0.110×10^{0}
8	0.755×10^{-2}	0.506×10^{-2}	0.220×10^{0}
16	0.474×10^{-3}	0.369×10^{-3}	0.400×10^{0}
32	0.269×10^{-4}	0.249×10^{-4}	0.780×10^{0}
64	0.185×10^{-5}	0.162×10^{-5}	0.153×10^{1}
128	0.117×10^{-6}	0.103×10^{-6}	0.303×10^{1}
256	0.891×10^{-8}	0.652×10^{-8}	0.603×10^{1}
512	0.306×10^{-8}	0.170×10^{-8}	0.120×10^{2}

Table 2.13: Numerical results for Example 2.5, $p = 8$.

n	$E^2(\Phi)$	$E^\infty(\Phi)$	t (sec.)
8	0.130×10^{-5}	0.816×10^{-6}	0.280×10^{0}
16	0.792×10^{-8}	0.510×10^{-8}	0.530×10^{0}
32	0.465×10^{-8}	0.235×10^{-8}	0.105×10^{1}
64	0.916×10^{-8}	0.463×10^{-8}	0.201×10^{1}

and $f : [0, L] \to \mathbf{R}^4$ defined via the formula

$$f(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{q}{EI} \end{pmatrix}. \tag{2.198}$$

We apply Algorithm A to this problem using equispace subintervals, with the number of Chebyshev nodes $p = 4$ and 8. For this experiment the background Green's function is chosen to correspond to the equation

$$\Phi'(x) = 0,$$

subject to boundary conditions (2.188). The results of this experiment are presented in Tables 2.12–2.13.

The following observations can be made from Tables 2.1–2.13, and are corroborated by our more extensive experiments.

1. The practical convergence rate of the method is consistent with the theoretical one. For larger p, the exact numerical verification of the order of convergence tends to be difficult, since the precision of calculations is exhausted before the behavior of the scheme becomes

asymptotic. However, this is often encountered when dealing with rapidly convergent algorithms.

2. For small-scale problems (such as in Example 2.5) and large p, the algorithm produces essentially exact results with a small number of nodes. For large-scale problems, double precision accuracy is achieved at approximately 20 nodes per wavelength with $p = 20$, at 12 nodes per wavelength with $p = 24$, and at 10 nodes per wavelength with $p = 32$. The optimal timings are achieved at p between 24 and 32 (provided that about 10–12 digits of accuracy are desired).

3. The condition number of a Nyström discretization of a second kind integral equation is asymptotically bounded, and our results reflect this fact. The relatively poor accuracy (8–11 digits) obtained in Example 2.5 is due to the ill-conditioning of the original ODE, as opposed to that of the numerical scheme used.

4. The algorithm is completely indifferent to the stiffness near the left end of the interval $[0, 1]$ of equations (2.158), (2.159) in Examples 2.1–2.2.

5. It is easy to use the algorithm in an adaptive manner, as demonstrated in Example 2.2. However, a fully adaptive version of the scheme has not been implemented. The intervals B_i^m in Example 2.2 were provided by the calling program (as opposed to having been constructed by the algorithm itself).

6. If the function $p : [a, c] \rightarrow \mathbf{L}(\mathbf{R}^{n\times n})$ given by (2.1) is singular in the interval $[a, c]$, then the choice of a background Green's function can dramatically affect the numerical results (see Remark 2.8). When p is not singular in the interval $[a, c]$, the numerical advantages of one background Green's function over another are usually minor. However, using the Green's function given in Lemma 2.2 results in a slightly faster algorithm. This is because this Green's function is constant in each of the intervals $(x \leq t), (x \geq t)$, which provides in Step 2 of Algorithm A faster evaluation of the matrices $\alpha_L^{i,m}, \alpha_R^{i,m}$ given by equations (2.143)–(2.144) and vectors $\delta_L^{i,m}, \delta_R^{i,m}$ given by equations (2.145)–(2.146), and provides in Step 6 a faster evaluation of the solution Φ of equation (2.27).

7. The algorithm can solve systems of high order equations with no numerical difficulty, as demonstrated by Example 2.5.

Chapter 3

One-Dimensional Integral Equations

3.1 Introduction

In this chapter, we consider the problem of determining a function $\sigma : [a, c] \to \mathbf{R}$ which satisfies the integral equation

$$\lambda \cdot \sigma(x) + \int_a^c k(x,t) \cdot \sigma(t)\, dt = f(x), \tag{3.1}$$

where the free term $f : [a, c] \to \mathbf{R}$ and the kernel $k : [a, c] \times [a, c] \to \mathbf{R}$ are known functions, and $0 \leq \lambda \leq 1$ (so that (3.1) is a first kind equation when $\lambda = 0$, and is a second kind equation when $\lambda = 1$). We assume that the kernel k contains a singularity of the form

$$s(x - t) = \log|x - t|, \tag{3.2}$$

or of the form

$$s(x - t) = |x - t|^{\alpha}, \quad (0 < |\alpha| < 1), \tag{3.3}$$

or of the form

$$s(x - t) = \frac{1}{x - t}. \tag{3.4}$$

When s is given by (3.2) or (3.3), equation (3.1) is a weakly singular integral equation; for a singularity given by (3.4), equation (3.1) is a Cauchy integral equation, and its integral is evaluated in the principal value sense.

For purposes of analysis, equations of the form (3.1) are usually divided into three classes:

(1) Second kind integral equations with singularities of the form (3.2) or (3.3).

(2) First and second kind integral equations with singularities of the form (3.4).

(3) First kind integral equations with second singularities of the form (3.2) or (3.3).

Generally speaking, integral operators for any of these classes have eigenfunctions with end-point singularities, so that an integral equation will have a smooth solution only if the solution is orthogonal to such eigenfunctions. Each class of equations also has its own set of theorems governing the existence and uniqueness of solutions. When second kind integral equations with weak singularities are considered, Fredholm's theorems govern the existence and uniqueness of solutions, and the linear systems arising from discretization are generally well-conditioned. When first and second kind integral equations with Cauchy singularities are considered, Fredholm's theorems apply in an extended sense (see Remark 3.3), and the discretized linear systems are also generally well-conditioned. Fredholm's theorems do not apply to first kind integral equations with weak singularities, although some first kind integral equations are known to have unique solutions. First kind integral equations also yield ill-conditioned linear systems: the condition number of the discretized system is at least $O(N)$ where N is the dimension of the problem.

When an integral equation is discretized using a Nyström scheme, the order of convergence of the method is equivalent to the order of convergence of the underlying quadrature formula. Standard quadrature formulas yield extremely poor convergence for the integral equations considered in this chapter, due to the kernel singularities in these equations. Special quadrature formulae have been developed for these types of integral equations. When the kernel has a Cauchy singularity, the methods presented in [8] and [9] yield quadrature formulas with superalgebraic convergence. When the kernel contains a weak singularity, the quadrature formulas developed in [27], [3] yield up to eighth-order convergence.

In this chapter, we construct two rapidly convergent, order $O(N)$ algorithms for the direct solution of first and second kind integral equations containing weak singularities or Cauchy singularities. The first algorithm is designed for integral equations with end-point singularities in the solution, while the second algorithm is designed for integral equations with smooth solutions. We extend the observations of [5] to construct sparse representations of the integral operators; we then extend the techniques of [22] and [29] to construct fast, direct, rapidly convergent solvers. In addition, we extend the methods of [27] and [3] to yield superalgebraically convergent quadrature formulas for equations with weak singularities.

The plan of this chapter is as follows: in Section 3.2 we review the relevant properties and tools of Chebyshev approximation, in Section 3.3 we apply Chebyshev analysis to the integral operators of interest, in Section 3.4 we develop superalgebraically convergent quadrature formula for integral operators with weak singularities; in Section 3.5 we develop the analytical apparatus for the algorithm for solutions with end-point singularities; in Section 3.6 we develop the analytical apparatus for the algorithm for smooth solutions; Sections 3.7 and 3.8 describes the numerical schemes for the algorithm for solutions with end-point singularities and the algorithm for smooth solutions, respectively; finally, in Section 3.9 we illustrate the performance of the algorithms with several practical examples.

3.2 Chebyshev Approximation

In this section, we define operators and summarize results from Chebyshev approximation theory. Most of the results are classical and can be found, for example, in [16], [18], [20]. Much of our discussion follows the presentation given in [22].

3.2.1 One-Dimensional Chebyshev Approximation

Given a function $f \in C^k[-1,1]$, we define the vector $V \in l^2$ of Chebyshev coefficients by the expression

$$V_i = \int_{-1}^{1} \frac{f(t) \cdot T_i(t)}{\sqrt{1-t^2}}\, dt, \tag{3.5}$$

where T_i denotes the ith Chebyshev polynomial. The function f can therefore be represented by the expression

$$f(x) = \Psi(x) \cdot V, \tag{3.6}$$

with $\Psi : l^2 \rightarrow L^2[-1,1]$ the Chebyshev interpolation operator given by

$$\Psi(x) \cdot V = \sum_{i=0}^{\infty} T_i(x) \cdot V_i. \tag{3.7}$$

Let $v \in \mathbf{R}^p$ be defined by

$$v_i = V_i,$$

for $i = 0, 1, \ldots, p-1$, and let $\psi : \mathbf{R}^p \rightarrow L^2[-1,1]$ be the interpolation operator given by

$$\psi(x) \cdot v = \sum_{i=0}^{p-1} T_i(x) \cdot v_i. \tag{3.8}$$

We denote the transpose interpolation operator $\psi^T : L^2[-1,1] \rightarrow \mathbf{R}^p$ by the expression

$$(\psi^T(x) \cdot f)_i = V_i, \tag{3.9}$$

for $i = 1, \ldots, p$, where V_i is given by (3.5).

The following lemma proves that a Chebyshev expansion converges rapidly for sufficiently smooth functions, and is proved, for example, in [20].

Lemma 3.1 *If $f \in C^k[-1,1]$, $v \in \mathbf{R}^p$ is given by (3.5) for $i = 0, 1, \ldots, p-1$, and $\psi : \mathbf{R}^p \rightarrow L^2[-1,1]$ is defined by (3.8), then for any $x \in [-1,1]$,*

$$\|f(x) - \psi(x) \cdot v\| = O\left(\frac{1}{p^{k-1}}\right). \tag{3.10}$$

Lemma 3.2 proves that, given a vector $\tilde{f} \in \mathbf{R}^p$ representing the function f discretized at the roots the pth Chebyshev polynomial, the vector $v \in \mathbf{R}^p$ given by (3.5) for $i = 0, 1, \ldots, p-1$ may be obtained from $\tilde{f}$ via a discrete cosine transform. A proof of this lemma may be found in [20].

Lemma 3.2 *Suppose that $f \in C^k[0,1]$, $v \in \mathbf{R}^p$ is given by (3.5) for $i = 0, 1, \ldots, p-1$, and $\tilde{f} \in \mathbf{R}^p$ is given by the expression*

$$\tilde{f}_i = f(t_i), \tag{3.11}$$

with t_i the ith root of the Chebyshev polynomial $T_p(x)$. Suppose further that $\alpha \in \mathbf{R}^p$ is given by the formula

$$\alpha = C(f), \tag{3.12}$$

where C denotes the discrete cosine transform of dimension p. Then

$$\|v_i - \alpha_i\| = O\left(\frac{1}{p^k}\right). \tag{3.13}$$

Remark 3.1 Since the discrete cosine transform of a function may be obtained via the Fast Fourier Transform, the vector of Chebyshev coefficients α may be obtained from the function values $\tilde{f}$ using $O(p \cdot \log p)$ arithmetic operations. □

While, strictly speaking, Chebyshev interpolation is defined only for the interval $[-1,1]$, we can define an interpolation operator $\psi_{[a,c]} : \mathbf{R}^p \to L^2[a,c]$ for the interval $[a,c]$ via the formula

$$\psi_{[a,c]}(x) = \frac{(c-a)}{2} \cdot \psi(t) + \frac{(a+c)}{2}, \tag{3.14}$$

so that

$$x \in [a,c] \Longleftrightarrow t \in [-1,1] \tag{3.15}$$

(see, for example, [15]). Similarly, we define the transpose operator $\psi^T_{[a,c]} : L^2[a,c] \to \mathbf{R}^p$ via the formula

$$\psi^T_{[a,c]}(x) = \frac{(c-a)}{2} \cdot \psi^T(t) + \frac{(a+c)}{2}. \tag{3.16}$$

Given a function $f_{[a,c]} : [a,c] \to \mathbf{R}$, we denote the Chebyshev coefficients for that function by $v_{[a,c]}$, so that

$$\psi_{[a,c]} \cdot v_{[a,c]} \tag{3.17}$$

is an approximation to $f_{[a,c]}$. Given the p roots t_i of the Chebyshev polynomial $T_p(t)$, we define roots $t_{i_{[a,c]}}$ for the interval $[a,c]$ via the formula

$$t_{i_{[a,c]}} = \frac{(c-a)}{2} \cdot t_i + \frac{(a+c)}{2}, \tag{3.18}$$

for $i = 0, 1, \ldots, p-1$.

Let $b = (a+c)/2$ denote the midpoint of $[a,c]$, and let $\psi_{[a,b]} : \mathbf{R}^p \to L^2[a,b]$, $\psi_{[b,c]} : \mathbf{R}^p \to L^2[b,c]$ denote the interpolation operators for the intervals $[a,b]$ and $[b,c]$, respectively. We consider the problem of obtaining the expansions $v_{[a,b]}$ and $v_{[b,c]}$, given the expansion $v_{[a,c]}$. If the function f associated with $v_{[a,c]}$ is sufficiently smooth, then we may use $v_{[a,c]}$ to obtain the approximations $\tilde{f}(t_{i_{[a,b]}})$ and $\tilde{f}(t_{i_{[b,c]}})$. Then, $v_{[a,b]}$ is obtained from $\tilde{f}(t_{i_{[a,b]}})$ via the cosine transform described in Lemma 3.2, and similarly $v_{[b,c]}$ is obtained from $\tilde{f}(t_{i_{[b,c]}})$. Thus, each of the mappings $v_{[a,c]} \to v_{[a,b]}$ and $v_{[a,c]} \to v_{[b,c]}$ is obtained via the application of

an interpolation operator and a cosine transform operator. Let $C_A, C_B : \mathbf{R}^p \to \mathbf{R}^p$ denote the operators which perform the mappings $v_{[a,c]} \to v_{[a,b]}$ and $v_{[a,c]} \to v_{[b,c]}$, respectively, so that

$$C_A \cdot v_{[a,c]} = v_{[a,b]}, \tag{3.19}$$
$$C_B \cdot v_{[a,c]} = v_{[b,c]}. \tag{3.20}$$

C_A, C_B are nonsingular (see, for example, [5]); we now define the inverse operators $C_D, C_U : \mathbf{R}^p \to \mathbf{R}^p$ via the expressions

$$C_D = C_A^{-1}, \tag{3.21}$$
$$C_U = C_B^{-1}. \tag{3.22}$$

3.2.2 Two-dimensional Chebyshev Approximation

Given a function $K(x,t) : [-1,1] \times [-1,1] \to \mathbf{R}$, the following three lemmas provide the methods by which K may be approximated via Chebyshev approximation, and define the circumstances under which these approximations are rapidly convergent. The lemmas are direct consequences of Lemma 3.1.

Lemma 3.3 *Given a function $K(x,t) : [-1,1] \times [-1,1] \to \mathbf{R}$, suppose that for fixed t, $K(x,t) \in C^k[-1,1]$. Let $M_1 : L^2[-1,1] \times L^2[-1,1] \to l^2 \times L^2[-1,1]$ be defined for all $t \in [-1,1]$ via the formula*

$$M_1(K,t)_i = \int_{-1}^{1} \frac{K(x,t) \cdot T_i(x)}{\sqrt{1-x^2}}\, dx, \tag{3.23}$$

so that for each t, $M_1(K,t)$ yields a Chebyshev expansion in x which approximates $K(x,t)$. Let $m_1 : L^2[-1,1] \times L^2[-1,1] \to \mathbf{R}^p \times L^2[-1,1]$ be given by the expression

$$m_1(K,t)_i = M_1(K,t)_i, \tag{3.24}$$

for $i = 0, 1, \ldots, p-1$ and for all $t \in [-1,1]$. Finally, let $\psi : \mathbf{R}^p \to L^2[-1,1]$ be the interpolation operator defined by (3.8). Then, for any $x, t \in [-1,1]$,

$$\|K(x,t) - \psi(x) \cdot m_1(K,t)\| = O\left(\frac{1}{p^{k-1}}\right). \tag{3.25}$$

Lemma 3.4 *Let $K(x,t) : [-1,1] \times [-1,1] \to \mathbf{R}$, and suppose that for fixed x, $K(x,t) \in C^k[-1,1]$. Let $M_2 : L^2[-1,1] \times L^2[-1,1] \to L^2[-1,1] \times l^2$ be defined for all $x \in [-1,1]$ via the formula*

$$M_2(K,x)_i = \int_{-1}^{1} \frac{K(x,t) \cdot T_i(t)}{\sqrt{1-t^2}}\, dt, \tag{3.26}$$

so that for each x, $M_2(K,x)$ yields a Chebyshev expansion in t which approximates $K(x,t)$. Let $m_2 : L^2[-1,1] \times L^2[-1,1] \to L^2[-1,1] \times \mathbf{R}^p$ be given by the expression

$$m_2(K,x)_i = M_2(K,x)_i, \tag{3.27}$$

for $i = 0, 1, \ldots, p-1$ and for all $x \in [-1,1]$. Finally, let $\psi^T : L^2[-1,1] \to \mathbf{R}^p$ be the interpolation operator defined by (3.9). Then, for any $x, t \in [-1,1]$,

$$\|K(x,t) - m_2(K,x) \cdot \psi^T(t)\| = O\left(\frac{1}{p^{k-1}}\right). \tag{3.28}$$

Lemma 3.5 *Given a function $K(x,t) : [-1,1] \times [-1,1] \to \mathbf{R}$, suppose that for fixed x, $K(x,t) \in C^k[-1,1]$. Suppose further that for each fixed t, $K(x,t) \in C^k[-1,1]$. Let $M_3 : L^2[-1,1] \times L^2[-1,1] \to l^2 \times l^2$ denote the two-dimensional Chebyshev expansion defined by the formula*

$$M_3(K)_{ij} = \int_{-1}^{1} \left(\int_{-1}^{1} \frac{K(x,t) \cdot T_j(t)}{\sqrt{1-t^2}}\, dt \right) \cdot \frac{T_i(x)}{\sqrt{1-x^2}}\, dx, \tag{3.29}$$

for each $i \times j \in \{0,1,\ldots\} \times \{0,1,\ldots\}$. Let $m_3 : L^2[-1,1] \times L^2[-1,1] \to \mathbf{R}^p \times \mathbf{R}^p$ be given by the expression

$$m_3(K)_{ij} = M_3(K)_{ij}, \tag{3.30}$$

for all $i = 0, 1, \ldots, p-1$ and for all $j = 0, 1, \ldots, p-1$. Finally, let $\psi : \mathbf{R}^p \to L^2[-1,1]$, $\psi^T : L^2[-1,1] \to \mathbf{R}^p$ be the interpolation operators defined by (3.8), (3.9), respectively. Then, for any $x, t \in [-1,1]$,

$$\|K(x,t) - \psi(x) \cdot m_3(K) \cdot \psi^T(t)\| = O\left(\frac{1}{p^{k-1}}\right). \tag{3.31}$$

Remark 3.2 Let $\tilde{K}(x_i, t_i) : \mathbf{R}^n \to \mathbf{R}^n \in \mathbf{L}(\mathbf{R}^{n \times n})$ denote an $n \times n$ discretization of K, and suppose that $n > p$. Suppose further that $\psi_n : \mathbf{R}^p \to \mathbf{R}^n$ is the operator mapping a Chebyshev expansion in x to function values at n points x_i, and suppose $\psi_n^T : \mathbf{R}^n \to \mathbf{R}^p$ is the operator mapping function values at n points in t_i to Chebyshev expansions in t. Then, using Lemmas 3.3–3.5, we may represent $\tilde{K}$ in one of four ways:

Case 1: If for fixed x, $K(x,t) \in C^k[-1,1]$, and also for fixed t, $K(x,t) \in C^k[-1,1]$, then by Lemma 3.5 we may represent $\tilde{K}$ by the formula

$$\tilde{K} = \psi_n \cdot m_3(K) \cdot \psi_n^T, \tag{3.32}$$

with m_3 defined by (3.30). Thus, K can be approximated by m_3, and ψ_n, ψ_n^T are merely used to obtain values of K at specific points x, t. We have reduced the n^2 representation $\tilde{K}$ to the p^2 representation m_3.

Case 2: If for fixed t, $K(x,t) \in C^k[-1,1]$, then by Lemma 3.3 we may represent $\tilde{K}$ by the formula

$$\tilde{K} = \psi_n \cdot \tilde{m}_1(K,t), \tag{3.33}$$

where $\tilde{m}_1 : \mathbf{R}^p \to \mathbf{R}^n$ denotes the operator m_1 given by (3.24), discretized at the n points t_i. $\tilde{K}$ can therefore be represented by $\tilde{m}_1$ using $n \cdot p$ coefficients; this is less than the n^2 coefficients used in $\tilde{K}$, but more than the p^2 coefficients used to represent m_3 in Case 1.

Case 3: If for fixed x, $K(x,t) \in C^k[-1,1]$, then by Lemma 3.4 we may represent $\tilde{K}$ by the formula

$$\tilde{K}(x,t) = \tilde{m}_2(K,x) \cdot \psi_n^T, \tag{3.34}$$

where $\tilde{m}_2 : \mathbf{R}^n \to \mathbf{R}^p$ denotes the operator m_2 given by (3.27), discretized at the n points x_i. The number of coefficients required to represent $\tilde{K}$ using $\tilde{m}_2$ is $p \cdot n$— the same as the number required in Case 2, but more than the number required in Case 1.

Case 4: If for fixed x, $K(x,t) \notin C^k[-1,1]$, and for fixed t, $K(x,t) \notin C^k[-1,1]$, then we cannot use Chebyshev approximation to reduce the number of coefficients needed to represent $\tilde{K}$. We require n^2 coefficients to represent $\tilde{K}$, more than the $p \cdot n$ coefficients required in Cases 2 and 3, and more than the p^2 coefficients required in Case 1. □

3.3 Chebyshev Approximation for Singular Integral Operators

In this section, we present techniques for the efficient representation of integral operators with singularities, with the result that a dimension N discretization of an integral operator with a singularity may be accurately represented using only $O(N)$ elements.

We assume the kernel $k : [a,c] \times [a,c] \to \mathbf{R}$ is of the form

$$k(x,t) = k_1(x,t) \cdot s(x-t) + k_2(x,t), \tag{3.35}$$

where the singularity s defined by one of (3.2)–(3.4), and $k_1, k_2 : [a,c] \times [a,c] \to \mathbf{R} \in C^k[a,c] \times [a,c]$. For convenience, we define the operator $P : L^2[a,c] \to L^2[a,c]$ by

$$P(\sigma)(x) = \lambda \cdot \sigma(x) + \int_a^c k(x,t) \cdot \sigma(t)\, dt, \tag{3.36}$$

so that the equation (3.1) assumes the form

$$P\sigma = f. \tag{3.37}$$

Remark 3.3 While P is defined as an operator yielding solutions in $L^2[a,c]$, the existence and uniqueness of solutions in $L^2[a,c]$ depends on the class of the integral operator under consideration. If P is a second kind integral operator with a weak singularity, then by the Fredholm Alternative, either there exist unique solutions $\sigma \in L^2[a,c]$, or the homogeneous equation

$$P\sigma = 0 \tag{3.38}$$

has a nontrivial solution.

For Cauchy integral equations, existence and uniqueness of solutions also depends on the index χ of the operator (as defined in [25]). For the integral operators under consideration, $\chi \in \{-1, 0, 1\}$. When $\chi = 0$, the operator is a Quasi-Fredholm operator, and the Fredholm Alternative is applicable. When $\chi = -1$, and the adjoint of the homogeneous equation has no nontrivial solutions, then a solution $\sigma \in L^2[a,c]$ exist, but is unique only up to a

constant. When $\chi = 1$, then the adjoint homogeneous equation has at least one nontrivial solution, and a solution $\sigma \in L^2[a,c]$ exists only if the free term f is orthogonal to the solutions of the adjoint homogeneous equation.

When the operator P is a weakly singular first kind integral operator, then there may not be a solution $\sigma \in L^2[a,c]$, since for this class of integral equation P is a compact operator. However, if such a solution exists, and the operator is positive definite, then the solution is also unique (see [26]). □

We will require the four operators

$$\begin{aligned} P_{AA} &: L^2[a,b] \to L^2[a,b], \\ P_{AB} &: L^2[b,c] \to L^2[a,b], \\ P_{BA} &: L^2[a,b] \to L^2[b,c], \\ P_{BB} &: L^2[b,c] \to L^2[b,c], \end{aligned}$$

defined by

$$P_{AA}(\sigma)(x) = \lambda \cdot \sigma(x) + \int_a^b K(x,t) \cdot \sigma(t)\, dt, \tag{3.39}$$

$$P_{AB}(\sigma)(x) = \int_b^c K(x,t) \cdot \sigma(t)\, dt, \tag{3.40}$$

$$P_{BA}(\sigma)(x) = \int_a^b K(x,t) \cdot \sigma(t)\, dt, \tag{3.41}$$

$$P_{BB}(\sigma)(x) = \lambda \cdot \sigma(x) + \int_b^c K(x,t) \cdot \sigma(t)\, dt. \tag{3.42}$$

The integral equation (3.37) can therefore be rewritten in the form

$$P_{AA}(\sigma_{|A}) + P_{AB}(\sigma_{|B}) = f_{|A}, \tag{3.43}$$

$$P_{BA}(\sigma_{|A}) + P_{BB}(\sigma_{|B}) = f_{|B}. \tag{3.44}$$

We denote subintervals $[c_1, c_2]$ and $[d_1, d_2]$ of $[a,c]$ by C and D, respectively, and assume that C and D are disjoint. We define $P_{CD} : L^2(D) \to L^2(C)$ via the formula

$$P_{CD}(\sigma)(x) = \int_{d_1}^{d_2} k(x,t) \cdot \sigma(t)\, dt, \tag{3.45}$$

so that P_{CD} is the operator P restricted to the domain $C \times D$.

Theorem 3.1 determines the circumstances under which P_{CD} may be represented via Chebyshev approximation. Proofs may be found in [21] and [5], with the most general proof located in [10].

Theorem 3.1 *Suppose $[c_1, c_2], [d_1, d_2]$ are disjoint subintervals of $[a,c]$, and let $c_l = c_2 - c_1, d_l = d_2 - d_1$, and $r = \min\{|c_1 - d_2|, |c_2 - d_1|\}$. Suppose further that P_{CD} is given by (3.45), with a kernel of the form (3.35), and the interpolation operators ψ, ψ^T are given*

by (3.8), (3.9), respectively. Suppose finally that Chebyshev nodes t_{i_C}, t_{i_D} are defined by (3.18). If m_1, m_2, m_3 given by Lemmas 3.3-3.5, then

1. If $c_l \leq r$ (the interval $[c_1, c_2]$ is well-separated from $[d_1, d_2]$), then

$$\|P_{CD} - \psi \cdot m_1(P_{CD})\| = O\left(\frac{1}{p^{k-1}}\right). \tag{3.46}$$

2. If $d_l \leq r$ (the interval $[d_1, d_2]$ is well-separated from $[c_1, c_2]$), then

$$\|P_{CD} - m_2(P_{CD}) \cdot \psi^T\| = O\left(\frac{1}{p^{k-1}}\right). \tag{3.47}$$

3. If $c_l \leq r$ and also $d_l \leq r$ (each of the intervals $[c_1, c_2]$ and $[d_1, d_2]$ is well-separated from the other), then

$$\|P_{CD} - \psi \cdot m_3(P_{CD}) \cdot \psi^T\| = O\left(\frac{1}{p^{k-1}}\right). \tag{3.48}$$

Let $b = (a + c)/2$ denote the midpoint of the interval $[a, c]$. We denote the subintervals $[a, b]$ and $[b, c]$ of $[a, c]$ by A and B, respectively.

Remark 3.4 The purpose of this section is to develop efficient representations for the discretized operators $\tilde{P}_{AB}, \tilde{P}_{BA}$. Since A and B are not well-separated intervals, we cannot apply Theorem 3.1 to the operators themselves. However, there are subintervals of A and B which are well-separated. We now proceed to decompose P_{AB}, P_{BA} into operators acting on smaller subintervals, so that Theorem 3.1 applies to these restricted operators. □

Suppose that we are given positive integers q_A, q_B, and that r_A, r_B are defined by the equations

$$r_A = \frac{(b-a)}{2^{q_A-1}}, \tag{3.49}$$

$$r_B = \frac{(c-b)}{2^{q_B-1}}. \tag{3.50}$$

If $q_A > 1$, then we denote the subintervals $[a, b - r_A]$ and $[b - r_A, b]$ of A by A_1 and A_0, respectively, and similarly if $q_B > 1$ we denote the subintervals $[b, b + r_B]$ and $[b + r_B, c]$ of B by B_0 and B_1, respectively. If $q_A > 2$, then we denote $q_A - 1$ subintervals $A_2, \ldots, A_{q_A}$ of A_1, via the formulae

$$\begin{aligned} A_i &= [b - r_A \cdot 2^{i-1}, b - r_A \cdot 2^{i-2}] \quad (i = 2, \ldots, q_A - 1), \\ A_{q_A} &= [a, b - r_A \cdot 2^{q_B-2}]. \end{aligned} \tag{3.51}$$

Similarly, if $q_B > 2$, then we denote the $q_B - 1$ subintervals $B_2, \ldots, B_{q_B}$ of B_1 using the formulae

$$\begin{aligned} B_i &= [b + r_B \cdot 2^{i-2}, b + r_B \cdot 2^{i-1}] \quad (i = 2, \ldots, q_B - 1), \\ B_{q_B} &= [b + r_B \cdot 2^{q_B-2}, c]. \end{aligned} \tag{3.52}$$

When $q_A = 1$, we define both A_1 and A_2 to be the interval $[b, b]$; when $q_B = 1$, we define B_1 and B_2 to be the interval $[b, b]$. When $q_A = 2$, we define A_2 to be equivalent to A_1, and similarly we define B_2 to be equivalent to B_1 when $q_B = 2$. Thus, for all values of q_A, q_B,

$$A_1 = \bigcup_{i=2}^{q_A} A_i, \tag{3.53}$$

$$B_1 = \bigcup_{i=2}^{q_B} B_i. \tag{3.54}$$

Given operators P_{AB}, P_{BA} defined by (3.39), (3.40), respectively, we may express P_{AB} and P_{BA} via the sums

$$\begin{aligned} P_{AB}(\sigma_{|B}) &= P_{A_0B_0}(\sigma_{|B_0}) + \sum_{i=2}^{q_A} P_{A_iB_0}(\sigma_{|B_0}) + \sum_{j=2}^{q_B} P_{A_0B_j}(\sigma_{|B_j}) \\ &\quad + \sum_{i=2}^{q_A}\sum_{j=2}^{q_B} P_{A_iB_j}(\sigma_{|B_j}), \end{aligned} \tag{3.55}$$

$$\begin{aligned} P_{BA}(\sigma_{|A}) &= P_{B_0A_0}(\sigma_{|A_0}) + \sum_{i=2}^{q_A} P_{B_iA_0}(\sigma_{|A_0}) + \sum_{j=2}^{q_B} P_{B_0A_j}(\sigma_{|A_j}) \\ &\quad + \sum_{i=2}^{q_B}\sum_{j=2}^{q_A} P_{B_iA_j}(\sigma_{|A_j}). \end{aligned} \tag{3.56}$$

The following theorem is the principal analytical tool of this chapter. It permits factorizations of P_{AB} and P_{BA} which approximate these operators to high accuracy. Corollary 3.1 then proves that for the discretizations of interest, few coefficients are required in the discretization of these factorizations.

Theorem 3.2 *Suppose* $\psi_{1_A} : L^2(A_0) \times \mathbf{R}^{p\cdot(q_A-1)} \to L^2(A)$ *is given by the expression*

$$\psi_{1_A} = \begin{pmatrix} \psi_{A_{q_A}} & & & & \\ & \psi_{A_{q_A-1}} & & & \\ & & \ddots & & \\ & & & \psi_{A_2} & \\ & & & & I_{A_0} \end{pmatrix}, \tag{3.57}$$

where I_{A_0} *denotes the identity operator for the interval* A_0, *and suppose further that* $\psi_{2_B} : L^2(B_0) \times \mathbf{R}^{p\cdot(q_B-1)} \to L^2(B)$ *is given by the expression*

$$\psi_{2_B} = \begin{pmatrix} I_{B_0} & & & & \\ & \psi_{B_2} & & & \\ & & \psi_{B_3} & & \\ & & & \ddots & \\ & & & & \psi_{B_{q_B}} \end{pmatrix}. \tag{3.58}$$

Then

$$\|P_{AB} - \psi_{1_A} \cdot M_{AB} \cdot \psi_{2_B}^T\| = O\left(\frac{1}{p^{k-1}}\right), \tag{3.59}$$

$$\|P_{BA} - \psi_{2_B} \cdot M_{BA} \cdot \psi_{1_A}^T\| = O\left(\frac{1}{p^{k-1}}\right), \tag{3.60}$$

where $M_{AB} : L^2(B_0) \times \mathbf{R}^{p\cdot(q_B-1)} \to L^2(A_0) \times \mathbf{R}^{p\cdot(q_A-1)}$ *is given by the expression*

$$M_{AB} = \begin{pmatrix} m_1(P_{A_{q_A}B_0}) & m_3(P_{A_{q_A}B_2}) & m_3(P_{A_{q_A}B_3}) & \cdots & m_3(P_{A_{q_A}B_{q_B}}) \\ m_1(P_{A_{q_A-1}B_0}) & m_3(P_{A_{q_A-1}B_2}) & m_3(P_{A_{q_A-1}B_3}) & \cdots & m_3(P_{A_{q_A-1}B_{q_B}}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1(P_{A_2B_0}) & m_3(P_{A_2B_2}) & m_3(P_{A_2B_3}) & \cdots & m_3(P_{A_2B_{q_B}}) \\ P_{A_0B_0} & m_2(P_{A_0B_2}) & m_2(P_{A_0B_3}) & \cdots & m_2(P_{A_0B_{q_B}}) \end{pmatrix}, \tag{3.61}$$

and where $M_{BA} : L^2(A_0) \times \mathbf{R}^{p\cdot(q_A-1)} \to L^2(B_0) \times \mathbf{R}^{p\cdot(q_B-1)}$ *is given by the expression*

$$M_{BA} = \begin{pmatrix} m_2(P_{B_0A_{q_A}}) & m_2(P_{B_0A_{q_A-1}}) & \cdots & m_2(P_{B_0A_2}) & P_{B_0A_0} \\ m_3(P_{B_2A_{q_A}}) & m_3(P_{B_2A_{q_A-1}}) & \cdots & m_3(P_{B_2A_2}) & m_1(P_{B_2A_0}) \\ m_3(P_{B_3A_{q_A}}) & m_3(P_{B_3A_{q_A-1}}) & \cdots & m_3(P_{B_3A_2}) & m_1(P_{B_3A_0}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_3(P_{B_{q_B}A_{q_A}}) & m_3(P_{B_{q_B}A_{q_A-1}}) & \cdots & m_3(P_{B_{q_B}A_2}) & m_1(P_{B_{q_B}A_0}) \end{pmatrix}, \tag{3.62}$$

and where $m_1 : L^2 \times L^2 \to \mathbf{R}^p \times L^2$, $m_2 : L^2 \times L^2 \to L^2 \times \mathbf{R}^p$, $m_3 : L^2 \times L^2 \to \mathbf{R}^p \times \mathbf{R}^p$ *are defined by (3.24), (3.27), (3.30), respectively.*

Proof. For each $i = 2, \ldots, q_A$, A_i is well-separated from B_0. Therefore, by Theorem 3.1 we may represent operators $P_{A_iB_0}$ and $P_{B_0A_i}$ to $O(1/p^{k-1})$ accuracy using the approximations

$$P_{A_iB_0} \approx \psi_{A_i} \cdot m_1(P_{A_iB_0}), \tag{3.63}$$

$$P_{B_0A_i} \approx m_2(P_{B_0A_i}) \cdot \psi_{A_i}^T, \tag{3.64}$$

for each $i = 2, \ldots, q_A$. In addition, each subinterval B_i is well-separated from A_0, for $i = 2, \ldots, q_B$. Therefore, by Theorem 3.1 we may represent operators $P_{A_0B_i}$ and $P_{B_iA_0}$ to $O(1/p^{k-1})$ accuracy using the approximations

$$P_{A_0B_i} \approx m_2(P_{A_0B_i}) \cdot \psi_{B_i}^T, \tag{3.65}$$

$$P_{B_iA_0} \approx \psi_{B_i} \cdot m_1(P_{B_iA_0}), \tag{3.66}$$

for each $i = 2, \ldots, q_B$. For each $i = 2, \ldots, q_A$ and for each $j = 2, \ldots, q_B$, each of the subintervals A_i and B_j are well-separated from the other. Therefore, by Theorem 3.1 we may represent operators $P_{A_iB_j}$ and $P_{B_jA_i}$ to $O(1/p^{k-1})$ accuracy using the approximations

$$P_{A_iB_j} \approx \psi_{A_i} \cdot m_3(P_{A_iB_j}) \cdot \psi_{B_j}^T, \tag{3.67}$$

$$P_{B_jA_i} \approx \psi_{B_j} \cdot m_3(P_{B_jA_i}) \cdot \psi_{A_i}^T, \tag{3.68}$$

for each $i = 2, \ldots, q_A$ and for each $j = 2, \ldots, q_B$. The approximations (3.59), (3.60) are now obtained from the approximations (3.63)-(3.68) and from equations (3.55), (3.56). □

The particular values of the integers q_A and q_B used in this section are discretization dependent. In this chapter, we will be using the following three discretizations:

Discretization C1: Both A and B contain n points, $n = p \cdot k$, where k is a positive integer. A_0 and B_0 contain p Chebyshev roots $t_{i_{A_0}}, t_{i_{B_0}}$, respectively. The remaining $n - p$ points for A and B are placed in A_2 and B_2 as follows:
1. p Chebyshev nodes are contained in the subinterval $[(a + b - r)/2, b - r]$ of A_2 and p nodes are contained in the subinterval $[b + r, (2b + r)/2]$.
2. For $i = 2, \ldots, k - 2$, both the subinterval $[(a + b - r)/2^i, (a + b - r)/2^{i-1}]$ of A_2 and the subinterval $[(2b + r)/2^{i-1}, (2b + r)/2^i]$ of B_2 contain p Chebyshev roots.
3. Finally, the subinterval $[a, (a + b - r)/2^{k-2}]$ of A_2 and the subinterval $[(2b + r)/2^{k-2}, c]$ each contain p Chebyshev roots.

Thus, for Discretization C1, $q_A = q_B = 2$.

Discretization C2: The structure of A_0 and A_2 are the same as in Discretization C1. However, there are only p points in the interval B, located at the p Chebyshev roots t_{i_B}. For this discretization, $q_A = 2$, and $q_B = 1$.

Discretization D: Both A and B contain n points, where $n = p \cdot 2^k$, for some integer k. The number of points in each of the subintervals A_i and B_i is proportional to the size of the subinterval. A_0 and B_0 contain the p Chebyshev roots $t_{i_{A_0}}$ and $t_{i_{B_0}}$, respectively, where the Chebyshev roots are defined by (3.18). A_2 and B_2 each contain the p Chebyshev roots $t_{i_{A_2}}, t_{i_{B_2}}$, respectively. For the subintervals A_3 and B_3, we have p Chebyshev roots in each half of the subinterval, for a total of $2p$ points in the subinterval. For the remaining subintervals A_i and B_i, for $i = 3, \ldots, q$, we divide each subinterval into 2^{q-1} pieces, and place the p Chebyshev roots in each piece (so that the total number of points is $p \cdot 2^{q-1}$). Thus, for Discretization C, $q_A = q_B = k + 1$.

Corollary (3.1) directly follows from Theorem 3.2 and from Discretizations C1, C2, D.

Corollary 3.1 *Suppose M_{AB} and M_{BA} are given by (3.61) and (3.62). Then,*

1. When M_{AB} and M_{BA} are discretized via Discretization C1, then the corresponding discretized operators $\tilde{M}_{AB}, \tilde{M}_{BA}$ require $4 \cdot p^2$ coefficients each.

2. When M_{AB} and M_{BA} are discretized via Discretization C2, then the corresponding discretized operators $\tilde{M}_{AB}, \tilde{M}_{BA}$ require $2 \cdot p^2$ coefficients each.
3. The discretization of both M_{AB} and M_{BA} via Discretization D requires $p^2 + p^2 \cdot k + p^2 \cdot k + p^2 \cdot k^2$ coefficients. Equivalently, each operator $\tilde{M}_{AB}, \tilde{M}_{BA}$ requires $p^2 \cdot (\log_2(n/p) + 1)^2$ coefficients for this discretization.

Remark 3.5 The analytical apparatus of this chapter strongly resembles the analytical apparatus of Chapter 2. This is largely because for both chapters the observations of

Remark 2.5 apply— that just as P was decomposed into operators P_{AA}, P_{BB} and low-rank operators P_{AB}, P_{BA}, so we can decompose each of P_{AA} and P_{BB} into four operators, two of which are of low numerical rank. If this process is continued, we obtain the structure shown in Figure 2.1. However, while the integral operators of Chapter 2 were of low analytic rank, the operators of the present chapter are of low rank only when discretized. □

3.4 Quadrature Formulae for Singular Integral Operators

In this section, we develop quadrature formulae to be used in the algorithms of this chapter. The results of this section can be used to construct $p \times p$ discretizations of singular or weakly singular integral operators. These discretizations are well-conditioned (the underlying quadrature weights are nearly all positive), and exhibit pth order convergence to the analytic integral operator.

3.4.1 Product Integration for Singular Functions

We first consider the numerical evaluation of integral equations of the form (3.1), with the kernel k given by (3.35), and the singularity s given by either (3.2) or (3.3).

Remark 3.6 The method of this section is not required for the Cauchy singularity (3.4), as high-order quadrature formulas for this singularity already exist (see, for example, [8],[9]). □

We denote an n-point discretization of the interval $[-1, 1]$ by $[x_1, x_2, \ldots, x_n]$. Given this discretization and quadrature weights w_{ij} $(1 \leq i, j \leq n)$, the Nyström algorithm replaces the integral equation (3.1) with the linear system

$$\lambda \cdot \sigma_i + \sum_{j=1}^{n} w_{ij} \cdot (k_1(x_i, x_j)s(x_j - x_i) + k_2(x_i, x_j)) \cdot \sigma_j = f(x_i) \quad (1 \leq i \leq n). \tag{3.69}$$

The solutions $\sigma_1, \ldots, \sigma_n$ are then viewed as approximations to the solution σ of (3.1) at the nodes $x_1, \ldots, x_n$. As is well-known, if (3.1) has a unique solution, then for a wide class of quadrature formulae and sufficiently large n, the system (3.69) has a unique solution, and the solution of (3.1) converges to the analytic solution at the same rate as the underlying quadrature formula.

Unfortunately, standard quadrature formulae (for example, end-point corrected trapezoidal quadrature formulae, or Chebyshev quadrature) exhibit extremely poor convergence for equations of the form (3.1), due to the singularity s. Appropriate quadrature formulae are usually obtained by some form of product integration (see, for example, [17]). Such formulae were obtained in [27], [3] for end-point corrected trapezoidal quadrature rules; we now use the method of these papers to construct weights for Chebyshev quadrature.

We assume that (3.1) has been discretized at the $2p$ zeroes $[x_1, \ldots, x_{2p}]$ of the $2p$th Chebyshev polynomial T_{2p}. Using the notation of [32], we define generalized moments of

the moment function s, with s singular at x_i, by the expression

$$m_j(x_i) = \int_{-1}^{1} T_{j-1}(t) \cdot s(t - x_i)\, dt \quad (1 \leq j \leq p). \tag{3.70}$$

For each x_i we now consider the following system of linear algebraic equations with respect to the unknowns w_{ij} $(1 \leq i \leq n)$:

$$\sum_{k=1}^{2p} T_{j-1}(x_k) \cdot w_{ik} = \int_{-1}^{1} T_{j-1}(t)\, dt, \tag{3.71}$$

for $j = 1, \ldots, p$, and

$$\sum_{k=1}^{2p} T_{j-p-1}(x_k) \cdot s(x_k - x_i) \cdot w_{ik} = m_{j-p}(x_i), \tag{3.72}$$

for $j = p+1, \ldots, 2p$. Obtaining the quadrature weights w_{ij} therefore requires the solution of a total of $2p$ linear systems.

Remark 3.7 While the $2p^2$ moments $m_j(x_i)$ can be determined analytically, this is quite burdensome for large p. Instead, we evaluate the moments numerically, using adaptive gaussian quadrature (see, for example, [30]). The expression (3.70) can be evaluated to high accuracy, provided the singularity is given by (3.2), or is given by (3.3) with a positive exponent.

If the singularity is given by (3.3) with negative exponent, then adaptive gaussian quadrature will yield high accuracy only if the location of the singularity is zero (for other locations of the singularity, the subinterval divisions can not be made sufficiently fine to yield high accuracy). However, the following expression is equivalent to (3.70), and can be evaluated accurately using adaptive Gaussian quadrature:

$$m_j(x_i) = \int_{x_i-1}^{x_i+1} T_{j-1}(x_i - t) \cdot s(t)\, dt \quad (1 \leq j \leq p). \tag{3.73}$$

□

The existence of unique solutions to the linear systems and the convergence rate of the quadrature formulae are given by the following two lemmas, which were presented in [3] (in a slightly different form).

Lemma 3.6 *For each x_i $(1 \leq i \leq 2p)$, the linear system (3.71)-(3.72) has a unique solution.*

Lemma 3.7 *For each x_i, the convergence rate of the quadrature weights w_{ij} $(1 \leq j \leq 2p)$ is at least p.*

Unfortunately, as the order of the quadrature formulae p increases, the linear systems given by (3.71)-(3.72) become increasingly ill-conditioned. As a result, the weights w_{ij} become large, to the point that substantial accuracy is lost when $p \geq 8$.

3.4.2 High-Order Quadrature Formulae with Well-Conditioned Weights

The algorithms of [27], [3] have favorable analytic properties, but produce unacceptably large quadrature weights for an order of convergence $p \geq 8$. In contrast, the algorithm we now present is a series of linear algebraic fixes to [27], [3]. We have no proofs concerning the performance of the algorithm in this section, but in practice it has been used to generate extremely high-order methods with quadrature weights that are nearly all positive.

We rewrite the system (3.71)–(3.72) in the form

$$M_1 w_1 = f_1, \tag{3.74}$$

where $M_1 \in \mathbf{R}^{2p \times 2p}$ is the system of Chebyshev polynomials and Chebyshev polynomials times a singular function, $w_1 \in \mathbf{R}^{2p}$ is the vector of quadrature weights, and $f \in \mathbf{R}^{2p}$ is the vector of analytic integrals. Assume that to some precision ϵ, the rank of the matrix $M_1 : f$ (the matrix M_1 with an additional column f) is n. Then, an orthonormal matrix $Q_1 \in \mathbf{R}^{2p \times 2p}$ can be constructed which maps the left nullspace to zero (to precision ϵ). More precisely, we obtain an orthonormal matrix Q_1 such that rows $n+1, \ldots, 2p$ of the matrix $Q \cdot M_1$ are zero, to precision ϵ. Let $Q_2 \in \mathbf{R}^{n \times 2p}$ denote the matrix with its n rows equivalent to the first n rows of Q_1. Then, to precision ϵ, the linear system

$$Q_2 M_1 w_1 = Q_2 f_1 \tag{3.75}$$

is equivalent to the linear system (3.74). For convenience, we rewrite the system (3.75) as the system

$$M_2 w_1 = f_2, \tag{3.76}$$

with $M_2 \in \mathbf{R}^{n \times 2p}$ defined by

$$M_2 = Q_2 \cdot M_1, \tag{3.77}$$

and $f_2 \in \mathbf{R}^n$ defined by

$$f_2 = Q_2 \cdot f_1. \tag{3.78}$$

By Lemma 3.6, there exist solutions to (3.76). We obtain w_1 by solving the following least squares problem with linear constraints (see, for example, [15]):

$$\text{Minimize} \quad \|w_1\|_2 \quad \text{subject to} \quad M_2 w_1 = f_2. \tag{3.79}$$

While solving the system (3.79) does provide solutions for $p \geq 8$, the weights w_1 obtained are still quite large. However, if we construct weights for points located at the zeroes of a Chebyshev polynomial $n \geq 2p$, these weights can be nearly all positive. The original equations (3.71)–(3.72) thus become, for each x_1:

$$\sum_{k=1}^{n} T_{j-1}(x_k) \cdot w_{ik} = \int_{-1}^{1} T_{j-1}(t)\, dt, \tag{3.80}$$

for $j = 1, \ldots, p$, and

$$\sum_{k=1}^{n} T_{j-p-1}(x_k) \cdot s(x_k - x_i) \cdot w_{ik} = m_{j-p}(x_i), \tag{3.81}$$

for $j = p+1, \ldots, 2p$. Our experiments indicate that when $n \geq 3p$, the weights are either all positive or nearly all positive. Unfortunately, we have no explanation for this behavior.

Remark 3.8 In [3], Alpert interpolates a function discretized at equispaced nodes to a Chebyshev discretization, in order to delay the growth in weights. A similar interpolation strategy can be used here to generate a $p \times p$ integral operator with pth order convergence, even when the number of quadrature weights required for a particular row is large (we use $3p$ weights per row). Assume an operator is to be applied to a function $f = [f(t_{1_{[a,c]}}), f(t_{2_{[a,c]}}), \ldots, f(t_{p_{[a,c]}})]$, with each $t_{i_{[a,c]}}$ given by (3.18), and assume furthermore that $f \in C^p[a,c]$. Denote by M the $p \times 3p$ matrix containing the discretized kernel function $K(t_{i_{[a,c]}}, y_j)$ (for each $t_{i_{[a,c]}}$, the points $y_1, y_2, \ldots, y_{3p}$ are chosen in some convenient fashion). Let $C : \mathbf{R}^p \rightarrow \mathbf{R}^p$ denote the discrete Chebyshev transform operator of dimension p. Then, $M \cdot f$ will yield the p Chebyshev coefficients of the function f. These coefficients can then be used to yield the interpolated values of f at the points $y_1, y_2, \ldots, y_{3p}$.

By itself, this interpolation procedure is of little interest. However, when combined with the algorithms of this chapter, the order of convergence of the discretized integral operator is improved from $p/3$ to p. □

3.5 The Analytical Apparatus for Singular Solutions

In the following two sections, we use the results of Section 3.3 to produce the apparatus for the rapid solution of the integral equation (3.1). By Corollary 3.1, the exact representation of the operators P_{AB} and P_{BA} defined by (3.40), (3.41) is discretization dependent. In this section, we present results for Discretizations C1 and C2— these discretizations are used when there is an end-point singularity in the solution of (3.1). In Section (3.6), we present results for Discretization D; this discretization is used when the solution to (3.1) is smooth.

The fast algorithms of this chapter are based on the fundamental observation that the solution to the integral equation (3.1) on the entire domain $[a,c]$ can easily be constructed from the solution of the two independent integral equations (3.43)–(3.44). one defined on $[a,b]$ and one on $[b,c]$. This leads naturally to a recursive algorithm, in which independent solutions on a large number of subintervals are successively merged until the full solution is obtained. A precise formulation of the construction and the resulting numerical scheme will require some notation.

3.5.1 Notation

We let $b = (a+c)/2$ denote the midpoint of $[a,c]$, and denote the subintervals $[a,b]$ and $[b,c]$ by A and B, respectively. In addition, we require the operators

$$\begin{aligned} P &: L^2[a,c] \rightarrow L^2[a,c], \\ P_{AA} &: L^2[a,b] \rightarrow L^2[a,b], \\ P_{AB} &: L^2[b,c] \rightarrow L^2[a,b], \\ P_{BA} &: L^2[a,b] \rightarrow L^2[b,c], \end{aligned}$$

$$P_{BB} \;:\; L^2[b,c] \to L^2[b,c],$$

defined by (3.36), (3.39), (3.40), (3.41), (3.42), respectively. Due to Theorem (3.2), if we are using Discretization C1 then we may approximate P_{AB} and P_{BA} via the expressions

$$P_{AB} \approx \psi_{1_A} \cdot M^1_{AB} \cdot \psi^T_{1_B}, \tag{3.82}$$
$$P_{BA} \approx \psi_{1_B} \cdot M^1_{BA} \cdot \psi^T_{1_A}, \tag{3.83}$$

with

$$\psi_{1_A} \;:\; \mathbf{R}^p \times L^2(A_0) \to L^2(A),$$
$$\psi_{1_B} \;:\; \mathbf{R}^p \times L^2(B_0) \to L^2(B),$$

the interpolation operators of the form (3.57), given by the expressions

$$\psi_{1_A} = \begin{pmatrix} \psi_{A_2} & \\ & I_{A_0} \end{pmatrix}, \tag{3.84}$$

$$\psi_{1_B} = \begin{pmatrix} \psi_{B_2} & \\ & I_{B_0} \end{pmatrix}, \tag{3.85}$$

with

$$M^1_{AB} \;:\; \mathbf{R}^p \times L^2(B_0) \to \mathbf{R}^p \times L^2(A_0),$$
$$M^1_{BA} \;:\; \mathbf{R}^p \times L^2(A_0) \to \mathbf{R}^p \times L^2(B_0),$$

given via the formulae

$$M^1_{AB} = \begin{pmatrix} m_1(P_{A_2B_0}) & m_3(P_{A_2B_2}) \\ P_{A_0B_0} & m_2(P_{A_0B_3}) \end{pmatrix}, \tag{3.86}$$

$$M^1_{BA} = \begin{pmatrix} m_2(P_{B_0A_2}) & P_{B_0A_0} \\ m_3(P_{B_2A_2}) & m_1(P_{B_2A_0}) \end{pmatrix}. \tag{3.87}$$

Similarly, if we are using Discretization C2 then using Theorem (3.2) we may approximate P_{AB} and P_{BA} via the expressions

$$P_{AB} \approx \psi_{1_A} \cdot M^2_{AB}, \tag{3.88}$$
$$P_{BA} \approx M^2_{BA} \cdot \psi^T_{1_A}, \tag{3.89}$$

with

$$M^2_{AB} \;:\; L^2(B) \to \mathbf{R}^p \times L^2(A_0),$$
$$M^2_{BA} \;:\; \mathbf{R}^p \times L^2(A_0) \to L^2(B),$$

given via the formulae

$$M^2_{AB} = \begin{pmatrix} m_1(P_{A_2B}) \\ P_{A_0B_0} \end{pmatrix}, \tag{3.90}$$

$$M^2_{BA} = \begin{pmatrix} m_2(P_{BA_2}) & P_{BA_0} \end{pmatrix}. \tag{3.91}$$

We define the operator

$$Q : L^2[a,c] \times \mathbf{R}^p \to L^2[a,c] \times \mathbf{R}^p$$

by the expression

$$Q(\chi)(x) = \chi(x) + \int_a^c k(x,t) \cdot \chi(t)\, dt. \tag{3.92}$$

We additionally require the four operators

$$\begin{aligned} Q_{AA} &: L^2[a,b] \times \mathbf{R}^p \to L^2[a,b] \times \mathbf{R}^p , \\ Q_{AB} &: L^2[b,c] \times \mathbf{R}^p \to L^2[a,b] \times \mathbf{R}^p , \\ Q_{BA} &: L^2[a,b] \times \mathbf{R}^p \to L^2[b,c] \times \mathbf{R}^p , \\ Q_{BB} &: L^2[b,c] \times \mathbf{R}^p \to L^2[b,c] \times \mathbf{R}^p , \end{aligned}$$

defined by the formulae

$$Q_{AA}(\chi)(x) = \chi(x) + \int_a^b k(x,t) \cdot \chi(t)\, dt, \tag{3.93}$$

$$Q_{AB}(\chi)(x) = \int_b^c k(x,t) \cdot \chi(t)\, dt, \tag{3.94}$$

$$Q_{BA}(\chi)(x) = \int_a^b k(x,t) \cdot \chi(t)\, dt, \tag{3.95}$$

$$Q_{BB}(\chi)(x) = \chi(x) + \int_b^c k(x,t) \cdot \chi(t)\, dt. \tag{3.96}$$

Given a function $f \in L^2[a,c]$, we will follow the convention of denoting its restriction to A and B by $f_{|A}$ and $f_{|B}$, respectively. Similarly, given a function $\psi \in L^2[a,c] \times \mathbf{R}^p$, we will denote its restriction to A and B by $\psi_{|A}$ and $\psi_{|B}$, respectively.

Given an interval $[b_1, b_2] \subset \mathbf{R}$ and operators $\psi : \mathbf{R}^n \to L^2[b_1,b_2]$, $\chi : \mathbf{R}^m \to L^2[b_1,b_2]$, and let $\psi_i(x)$ denote the ith component of $\psi(x)$, and let $\chi_j(x)$ denote the jth component of $\chi(x)$. Then we define the inner product $\alpha \in \mathbf{L}(\mathbf{R}^{m \times n})$ by the expression

$$(\alpha)_{ij} = \int_{b_1}^{b_2} \psi_i(t) \cdot \chi_j(t)\, dt, \tag{3.97}$$

where $(\alpha)_{ij}$ denotes the entry in the ith row and jth column of α. Similarly, we define the transpose of ψ by the formula

$$(\psi^T(\chi))_{ij} = \int_{b_1}^{b_2} \psi_i(t) \cdot \chi_j(t)\, dt, \tag{3.98}$$

Given interpolation operators

$$\begin{aligned} \psi_{[a,c]} &: \mathbf{R}^p \to L^2[a,c], \\ \psi_A &: \mathbf{R}^p \to L^2(A), \\ \psi_B &: \mathbf{R}^p \to L^2(B), \end{aligned}$$

defined by (3.14), we will in this section refer to them by ψ_3, ψ_{3_A}, and ψ_{3_B}, respectively. Given Discretizations C1 and C2, we require the interpolation operator

$$\psi_1 : \mathbf{R}^p L^2(B) \to L^2[a,c],$$

given by the formula

$$\psi_1 = \begin{pmatrix} \psi_A & \\ & I_B \end{pmatrix}. \tag{3.99}$$

We in addition require the zero operators

$$\begin{aligned} 0_A &: L^2[a,c] \to L^2(A), \\ 0_B &: \mathbf{R}^p \to L^2(B). \end{aligned} \tag{3.100}$$

The operators ψ_1 and ψ_3 are related to ψ_{3_A}, ψ_{3_B} via the expressions

$$\psi_{1_{|A}} = \psi_{3_A} \times 0_A \,, \tag{3.101}$$

$$\psi_{1_{|B}} = 0_B \times I_B \,, \tag{3.102}$$

$$\psi_{3_{|A}} = \psi_{3_A} \cdot C_D \,, \tag{3.103}$$

$$\psi_{3_{|B}} = \psi_{3_B} \cdot C_U \,, \tag{3.104}$$

where $C_D, C_U : \mathbf{R}^p \to R^p$ denote the shifting matrices defined by (3.21), (3.22), respectively.

Assuming that the operators P_{AA}, P_{BB} are nonsingular, we define the functions

$$\begin{aligned} \eta_A &: A \to \mathbf{R}, \\ \eta_B &: B \to \mathbf{R}, \end{aligned}$$

via the formulae

$$\eta_A = P_{AA}^{-1}(f_{|A}) \,, \tag{3.105}$$

$$\eta_B = P_{BB}^{-1}(f_{|B}) \,. \tag{3.106}$$

Similarly, assuming that the operators P, P_{AA}, P_{BB} are nonsingular, we then define the operators

$$\begin{aligned} \chi_1 &: \mathbf{R}^p \times L^2(B) \to L^2[a,c] \,, \\ \chi_3 &: \mathbf{R}^p \to L^2[a,c] \,, \\ \phi_{1_A} &: \mathbf{R}^p \times L^2(A_0) \to L^2(A) \,, \\ \phi_{3_A} &: \mathbf{R}^p \to L^2(A) \,, \\ \phi_{3_B} &: \mathbf{R}^p \to L^2(B) \,, \end{aligned}$$

via the formulae

$$\chi_1 = P^{-1}(\psi_1) , \tag{3.107}$$
$$\chi_3 = P^{-1}(\psi_3) , \tag{3.108}$$
$$\phi_{1_A} = P_{AA}^{-1}(\psi_{1_A}) , \tag{3.109}$$
$$\phi_{3_A} = P_{AA}^{-1}(\psi_{3_A}) , \tag{3.110}$$
$$\phi_{3_B} = P_{BB}^{-1}(\psi_{3_B}) . \tag{3.111}$$

Finally, given the transpose defined by (3.98), we define operators

$$\begin{aligned}
\alpha_{11}^A &: \mathbf{R}^p \times L^2(A_0) \to \mathbf{R}^p \times L^2(A_0) , \\
\alpha_{13}^A &: \mathbf{R}^p \to \mathbf{R}^p \times L^2(A_0) , \\
\alpha_{31}^A &: \mathbf{R}^p \times L^2(A_0) \to \mathbf{R}^p , \\
\alpha_{33}^A &: \mathbf{R}^p \to \mathbf{R}^p ,
\end{aligned}$$

$$\begin{aligned}
\alpha_{11} &: \mathbf{R}^p \times L^2(B) \to \mathbf{R}^p \times L^2(B) , \\
\alpha_{13} &: \mathbf{R}^p \to \mathbf{R}^p \times L^2(B) , \\
\alpha_{31} &: \mathbf{R}^p \times L^2(B) \to \mathbf{R}^p , \\
\alpha_{33} &: \mathbf{R}^p \to \mathbf{R}^p ,
\end{aligned}$$

by the formulae

$$\begin{aligned}
&\alpha_{11}^A = \psi_{1_A}^T \cdot \phi_{1_A} , \quad \alpha_{13}^A = \psi_{1_A}^T \cdot \phi_{3_A} , \\
&\alpha_{31}^A = \psi_{3_A}^T \cdot \phi_{1_A} , \quad \alpha_{33}^A = \psi_{3_A}^T \cdot \phi_{3_A} ,
\end{aligned} \tag{3.112}$$

$$\begin{aligned}
&\alpha_{11} = \psi_1^T \cdot \chi_1 , \quad \alpha_{13} = \psi_1^T \cdot \chi_3 , \\
&\alpha_{31} = \psi_3^T \cdot \chi_1 , \quad \alpha_{33} = \psi_3^T \cdot \chi_3 ,
\end{aligned} \tag{3.113}$$

and the functions

$$\begin{aligned}
\delta_1^A &\in \mathbf{R}^p \times L^2(A_0) , \\
\delta_3^A &\in \mathbf{R}^p ,
\end{aligned}$$

$$\begin{aligned}
\delta_1 &\in \mathbf{R}^p \times L^2(B) , \\
\delta_3 &\in \mathbf{R}^p ,
\end{aligned}$$

by the expressions

$$\begin{aligned}
&\delta_1^A = \psi_{1_A}^T \cdot \eta_A , \quad \delta_3^A = \psi_{3_A}^T \cdot \eta_A , \\
&\delta_1 = \psi_1^T \cdot \sigma , \qquad \delta_3 = \psi_3^T \cdot \sigma ,
\end{aligned} \tag{3.114}$$

where σ is the solution to equation (3.37).

3.5.2 Recursive solution of the integral equation

We now consider the original integral equation (3.37)

$$P\sigma = f.$$

The main results of this section are the following two lemmas, which construct the solutions σ of equation (3.37) from the solutions η_A, η_B of equations (3.105) and (3.106). Since the proofs are quite similar, we only present the proof for Lemma 3.8.

Lemma 3.8 *If Discretization C1 is used, so that operators P_{AB}, P_{BA} can be approximated via (3.88)-(3.89), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, and if furthermore the operator*

$$\Delta_1 : \mathbf{R}^p \times L^2(A_0) \to \mathbf{R}^p \times L^2(A_0)$$

given by

$$\Delta_1 = I_{\mathbf{R}^p \times L^2(A_0)} - \alpha_{11}^A \cdot M_{AB}^1 \cdot \alpha_{11}^B \cdot M_{BA}^1, \tag{3.115}$$

is also nonsingular, then to accuracy $O(1/p^{k-1})$,

$$\begin{aligned} \sigma_{|A} &= \eta_A + \phi_{1_A} \cdot M_{AB}^1 \cdot (\alpha_{11}^B \cdot M_{BA}^1 \cdot \Delta_1^{-1} \cdot \\ &\quad (\delta_1^A - \alpha_{11}^A \cdot M_{AB}^1 \cdot \delta_1^B) - \delta_1^B), \end{aligned} \tag{3.116}$$

$$\begin{aligned} \sigma_{|B} &= \eta_B + \phi_{1_B} \cdot M_{BA}^1 \cdot (\Delta_1^{-1} \cdot \alpha_{11}^A \cdot M_{AB}^1 \cdot \\ &\quad (\delta_1^B - \alpha_{11}^B \cdot M_{BA}^1 \cdot \delta_1^A) - \delta_1^A). \end{aligned} \tag{3.117}$$

Proof. Using definitions (3.39)-(3.42), the integral equation

$$P\sigma = f$$

can be rewritten in the form (3.43)-(3.44). The approximations (3.82) and (3.83) for P_{AB} and P_{BA}, respectively, can then be used to obtain an explicit solution to the coupled equations (3.43) and (3.44) in terms of the functions $\eta_A, \eta_B, \phi_{1_A}, \phi_{3_A}$ defined by (3.105), (3.106), (3.109), (3.110), respectively. Indeed, applying the operator P_{AA}^{-1} to equation (3.43) and the operator P_{BB}^{-1} to equation (3.44), we have

$$\sigma_{|A} + P_{AA}^{-1} \cdot P_{AB}(\sigma_{|B}) = P_{AA}^{-1}(f_{|A}), \tag{3.118}$$

$$P_{BB}^{-1} \cdot P_{BA}(\sigma_{|A}) + \sigma_{|B} = P_{BB}^{-1}(f_{|B}). \tag{3.119}$$

Substituting the approximations (3.82) and (3.83) into (3.118) and (3.119) yields the formulae

$$\sigma_{|A} + P_{AA}^{-1} \cdot \psi_{1_A} \cdot M_{AB}^1 \cdot \psi_{1_B}^T \cdot \sigma_{|B} = \eta_A, \tag{3.120}$$

$$P_{BB}^{-1} \cdot \psi_{1_B} \cdot M_{BA}^1 \cdot \psi_{1_A}^T \cdot \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{3.121}$$

or

$$\sigma_{|A} + \phi_{1_A} \cdot M_{AB}^1 \cdot \psi_{1_B}^T \cdot \sigma_{|B} = \eta_A, \tag{3.122}$$

$$\phi_{1_B} \cdot M_{BA}^1 \cdot \psi_{1_A}^T \cdot \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{3.123}$$

where we have used the definitions (3.109), (3.110) for ϕ_{1_A} and ϕ_{1_B}, respectively. Now, multiplying (3.123) by $\phi_{1_A} \cdot M_{AB}^1 \cdot \psi_{1_B}^T$ and subtracting it from (3.122), we obtain

$$\begin{aligned}(I_{L^2(A)} - \phi_{1_A} \cdot M_{AB}^1 \cdot \psi_{1_B}^T \cdot \phi_{1_B} \cdot M_{BA}^1 \cdot \psi_{1_A}^T) \cdot \sigma_{|A} \;&= \\ \eta_A - \phi_{1_A} \cdot M_{AB}^1 \cdot \psi_{1_B}^T \cdot \eta_B.&\end{aligned} \tag{3.124}$$

Similarly, multiplying (3.122) by $\phi_{1_B} \cdot M_{BA} \cdot \psi_{1_A}^T$ and subtracting it from (3.123) results in the equation

$$\begin{aligned}(I_{L^2(B)} - \phi_{1_B} \cdot M_{BA}^1 \cdot \psi_{1_A}^T \cdot \phi_{1_A} \cdot M_{AB}^1 \cdot \psi_{1_B}^T) \cdot \sigma_{|B} \;&= \\ \eta_B - \phi_{1_B} \cdot M_{BA}^1 \cdot \psi_{1_A}^T \cdot \eta_A.&\end{aligned} \tag{3.125}$$

Due to (3.112) and (3.180), we can rewrite these equations in the form

$$(I_{L^2(A)} - \phi_{1_A} \cdot M_{AB}^1 \cdot \alpha_{11}^B \cdot M_{BA}^1 \cdot (\psi_{1_A}^T)) \cdot \sigma_{|A} = \eta_A - \phi_{1_A} \cdot M_{AB}^1 \cdot \delta_1^B, \tag{3.126}$$

$$(I_{L^2(B)} - \phi_{1_B} \cdot M_{BA}^1 \cdot (\alpha_{11}^A \cdot M_{AB}^1 \cdot \psi_{1_B}^T)) \cdot \sigma_{|B} = \eta_B - \phi_{1_B} \cdot M_{BA}^1 \cdot \delta_1^A. \tag{3.127}$$

By application of Lemma 2.5, we obtain

$$\begin{aligned}\sigma_{|A} \;=\; &(I_{L^2(A)} + \phi_{1_A} \cdot M_{AB}^1 \cdot \alpha_{11}^B \cdot M_{BA}^1 \cdot \\ &(I_{\mathbf{R}^p \times L^2(A_0)} - \alpha_{11}^A \cdot M_{AB}^1 \cdot \alpha_{11}^B \cdot M_{BA}^1)^{-1} \cdot \psi_{1_A}^T) \cdot (\eta_A - \phi_{1_A} \cdot M_{AB}^1 \cdot \delta_1^B),\end{aligned} \tag{3.128}$$

$$\begin{aligned}\sigma_{|B} \;=\; &(I_{L^2(B)} + \phi_{1_B} \cdot M_{BA}^1 \cdot (I_{\mathbf{R}^p \times L^2(A_0)} - \alpha_{11}^A \cdot M_{AB}^1 \cdot \alpha_{11}^B \cdot M_{BA}^1)^{-1} \cdot \\ &\alpha_{11}^A \cdot M_{AB}^1 \cdot \psi_{1_B}^T) \cdot (\eta_B - \phi_{1_B} \cdot M_{BA}^1 \cdot \delta_1^A).\end{aligned} \tag{3.129}$$

The equations (3.116), (3.117) are now obtained from equations (3.128), (3.129) and equation (3.115). □

Lemma 3.9 *If Discretization C2 is used, so that operators P_{AB}, P_{BA} may be approximated via (3.82)-(3.83), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, and if furthermore the operator*

$$\Delta_2 : L^2(B) \to L^2(B)$$

given by

$$\Delta_2 = I_B - P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{11}^A \cdot M_{AB}^2, \tag{3.130}$$

is also nonsingular, with M_{AB}^2, M_{BA}^2 defined by (3.90)-(3.91), then to accuracy $O(1/p^{k-1})$,

$$\begin{aligned}\sigma_{|A} \;=\; &\eta_A + \phi_{1_A} \cdot M_{AB}^2 \cdot (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \\ &(\delta_1^A - \alpha_{11}^A \cdot M_{AB}^2 \cdot \eta_B) - \eta_B),\end{aligned} \tag{3.131}$$

$$\sigma_{|B} \;=\; \Delta_2^{-1} \cdot (\eta_B - P_{BB}^{-1} \cdot M_{BA}^2 \cdot \delta_1^A)\,. \tag{3.132}$$

3.5.3 Further Analytical Results for Discretization C2

We now collect a number of identities which are necessary for the algorithm to be presented in Section 3.7. First, we apply Lemma 3.9 to the particular cases $f = \psi_1$, $f = \psi_3$ to obtain analytical expressions for the functions χ_1 and χ_3 defined in equations (3.107) and (3.108). We omit the proofs here, as they are quite similar to the proofs for Lemmas 3.8–3.9

Corollary 3.2 *If Discretization C2 is used, so that operators P_{AB}, P_{BA} may be approximated via (3.82)-(3.83), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\begin{aligned} \chi_{1_{|A}} &= (\phi_{3_A} + \phi_{1_A} \cdot M_{AB}^2 \cdot \Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{13}^A) \times \\ &\quad (-\phi_{1_A} \cdot M_{AB}^2 \cdot (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{11}^A \cdot M_{AB}^2 \cdot P_{BB}^{-1} + P_{BB}^{-1})) \ , \end{aligned} \tag{3.133}$$

$$\chi_{1_{|B}} = (-\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{13}^A) \times (\Delta_2^{-1} \cdot P_{BB}^{-1}) \ , \tag{3.134}$$

$$\begin{aligned} \chi_{3_{|A}} &= \phi_{3_A} \cdot C_D + \phi_{1_A} \cdot M_{AB}^2 \cdot (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \\ &\quad (\alpha_{13}^A \cdot C_D - \alpha_{11}^A \cdot M_{AB}^2 \cdot \phi_{3_B} \cdot C_U) - \phi_{3_B} \cdot C_U) \ , \end{aligned} \tag{3.135}$$

$$\chi_{3_{|B}} = \Delta_2^{-1} \cdot (\phi_{3_B} \cdot C_U - P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{13}^A \cdot C_D) \ , \tag{3.136}$$

where the coefficients α_{ij}^A and are given by equation (3.112).

We will also require analytical expressions for the inner products δ_1, and δ_3 defined in (3.114) in terms of the restricted inner products δ_1^A and δ_3^A defined in (3.114). The proofs follow readily from Lemma 3.9.

Corollary 3.3 *If Discretization C2 is used, so that operators P_{AB}, P_{BA} may be approximated via (3.82)-(3.83), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\begin{aligned} \delta_1 &= \psi_1^T \cdot \sigma = (\psi_{3_A} \times 0_B)^T \cdot \sigma_{|A} + (0_A \times I_B)^T \cdot \sigma_{|B} \\ &= \begin{pmatrix} \delta_3^A + \alpha_{31}^A \cdot M_{AB}^2 \cdot (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot (\delta_1^A - \alpha_{11}^A \cdot M_{AB}^2 \cdot \eta_B) - \eta_B) \\ \sigma_{|B} \end{pmatrix} \ , \end{aligned} \tag{3.137}$$

$$\begin{aligned} \delta_3 &= C_U^T \cdot \psi_{3_B}^T \cdot \sigma_{|B} + C_D^T \cdot \delta_3^A + C_D^T \cdot \alpha_{31}^A \cdot M_{AB}^2 \cdot \\ &\quad (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot (\delta_1^A - \alpha_{11}^A \cdot M_{AB}^2 \cdot \eta_B) - \eta_B) \ . \end{aligned} \tag{3.138}$$

Special cases of Corollary 3.2 are obtained when $f = \psi_1$ or $f = \psi_3$. The proofs follow easily from the definitions of χ_1 and χ_3 in (3.107) and (3.108).

Corollary 3.4 *If Discretization C2 is used, so that operators P_{AB}, P_{BA} may be approximated via (3.82)-(3.83), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\alpha_{11} = \psi_1^T \cdot \chi_1 = \begin{pmatrix} M_{pp} \times N_{pp} \\ \chi_{1_{|B}} \end{pmatrix}, \tag{3.139}$$

with $M_{pp}, N_{pp} : \mathbf{R}^p \rightarrow \mathbf{R}^p$ given by

$$\begin{aligned} M_{pp} &= \alpha_{33}^A + \alpha_{31}^A \cdot M_{AB}^2 \cdot \Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{13}^A , \\ N_{pp} &= -\alpha_{31}^A \cdot M_{AB}^2 \cdot (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{11}^A \cdot M_{AB}^2 \cdot P_{BB}^{-1} + P_{BB}^{-1}) . \end{aligned}$$

Corollary 3.5 *If Discretization C2 is used, so that operators P_{AB}, P_{BA} may be approximated via (3.82)-(3.83), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\alpha_{13} = \psi_1^T \cdot \chi_3 = \begin{pmatrix} M_{pp} \\ \chi_{3_{|B}} \end{pmatrix}, \tag{3.140}$$

with $M_{pp} : \mathbf{R}^p \rightarrow \mathbf{R}^p$ given by

$$\begin{aligned} M_{pp} = {} & \alpha_{33}^A \cdot C_D + \alpha_{31}^A \cdot M_{AB}^2 \cdot (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \\ & (\alpha_{13}^A \cdot C_D - \alpha_{11}^A \cdot M_{AB} \cdot \phi_{3_B} \cdot C_U) - \phi_{3_B} \cdot C_U) . \end{aligned}$$

Corollary 3.6 *If Discretization C2 is used, so that operators P_{AB}, P_{BA} may be approximated via (3.82)-(3.83), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\begin{aligned} \alpha_{31} = {} & \psi_3^T \cdot \chi_1 = \\ & (C_U^T \cdot \psi_{3_B}^T \cdot \chi_{1_{|B}} + C_D^T \cdot \alpha_{33}^A + C_D^T \cdot \alpha_{31}^A \cdot M_{AB}^2 \cdot \Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{13}^A) \times \\ & (-C_D^T \cdot \alpha_{31}^A \cdot M_{AB}^2 \cdot (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \alpha_{11}^A \cdot M_{AB}^2 \cdot P_{BB}^{-1} + P_{BB}^{-1})) . \end{aligned} \tag{3.141}$$

Corollary 3.7 *If Discretization C2 is used, so that operators P_{AB}, P_{BA} may be approximated via (3.82)-(3.83), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\begin{aligned} \alpha_{33} = {} & \psi_3^T \cdot \chi_3 = \\ & C_U^T \cdot \psi_{3_B}^T \cdot \chi_{3_{|B}} + C_D^T \cdot \alpha_{33}^A \cdot C_D + C_D^T \cdot \alpha_{31}^A \cdot M_{AB}^2 \cdot (\Delta_2^{-1} \cdot P_{BB}^{-1} \cdot M_{BA}^2 \cdot \\ & (\alpha_{13}^A \cdot C_D - \alpha_{11}^A \cdot M_{AB}^2 \cdot \phi_{3_B} \cdot C_U) - \phi_{3_B} \cdot C_U). \end{aligned} \tag{3.142}$$

Finally, combining Lemma 3.9 with the expressions (3.133)-(3.136), we have

Corollary 3.8 *Suppose Discretization C2 is used, so that operators P_{AB}, P_{BA} may be approximated via (3.82)-(3.83), and suppose that in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular. Suppose further that the function $F : [a,c] \rightarrow \mathbf{R}$ is defined by the formula*

$$F(x) = \chi_1(x) \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \chi_3(x) \cdot \lambda_3 + \sigma(x), \tag{3.143}$$

with $\lambda_2 \in L^2(B)$, and $\lambda_1, \lambda_3 \in \mathbf{R}^p$. Then on the interval $[a,b]$, to accuracy $O(1/p^{k-1})$,

$$F(x) = \phi_{1_A}(x) \cdot \mu_1 + \phi_{3_A}(x) \cdot \mu_3 + \eta_A(x) \ , \tag{3.144}$$

with the functions $\mu_1 \in \mathbf{R}^p \times L^2(A_0)$, $\mu_3 \in \mathbf{R}^p$ defined by the formulae

$$\begin{aligned} \mu_1 &= M^2_{AB} \cdot (\Delta_1^{-1} \cdot P^{-1}_{BB} \cdot M^2_{BA} \cdot (\alpha^A_{13} \cdot (\lambda_1 + C_D \cdot \lambda_3) - \\ &\quad \alpha^A_{11} \cdot M^2_{AB} \cdot (P^{-1}_{BB} \cdot \lambda_3 + \phi_{3_B} \cdot C_U \cdot \lambda_3 + \eta_B) + \delta^A_1) - \\ &\quad P^{-1}_{BB} \cdot \lambda_2 - \phi_{3_B} \cdot C_U \cdot \lambda_3 - \eta_B) \ , \\ \mu_3 &= \lambda_1 + C_D \cdot \lambda_3. \end{aligned} \tag{3.145}$$

Similarly, on the interval $[b,c]$, to accuracy $O(1/p^{k-1})$,

$$F(x) = \chi_{1_{|B}}(x) \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \chi_{3_{|B}}(x) \cdot \lambda_3 + \sigma_{|B}(x). \tag{3.146}$$

Proof. Restricting (3.143) on the subintervals A, B of $[a,c]$, respectively, we have

$$F_{|A} = \chi_{1_{|A}} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \chi_{3_{|A}} \cdot \lambda_3 + \sigma_{|A} \ , \tag{3.147}$$

$$F_{|B} = \chi_{1_{|B}} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \chi_{3_{|B}} \cdot \lambda_3 + \sigma_{|B} \ . \tag{3.148}$$

Now, (3.148) is equivalent to (3.146), and the expression (3.145) results from combining (3.147) with (3.131), (3.133), (3.135), and comparing the resulting expression with the expressions (3.144). □

3.6 The Analytical Apparatus for Smooth Solutions

In this section, we present results for the solution of the integral equation (3.1), when the solution is smooth. Like Section 3.5, the techniques of the present section are used to merge independent solutions on a large number of subintervals until the full solution is obtained. In addition, the notation and theorems of the present section strongly resemble those of Section 3.5, but with two important differences. First, the underlying discretizations are different (Discretization D for the present section, Discretizations C1 and C2 for Section 3.5), resulting in different factorizations of P_{AB} and P_{BA}. Second, because the two sections assume different merging strategies, the relations (3.162)-(3.165) in the present section are different from the relations (3.101)-(3.104) of Section 3.5.

3.6.1 Notation

We let $b = (a + c)/2$ denote the midpoint of $[a, c]$, and denote the subintervals $[a, b]$ and $[b, c]$ by A and B, respectively. Given the integer q associated with Discretization D (see Corollary 3.1), we define $r = (c - a)/2^q$, and define subintervals

$$A_f = [a, a + r], \tag{3.149}$$
$$A_c = [b - r, b], \tag{3.150}$$
$$B_c = [b, b + r], \tag{3.151}$$
$$B_f = [c - r, c]. \tag{3.152}$$

In addition, we require the operators

$$\begin{aligned} P &: L^2[a,c] \to L^2[a,c], \\ P_{AA} &: L^2[a,b] \to L^2[a,b], \\ P_{AB} &: L^2[b,c] \to L^2[a,b], \\ P_{BA} &: L^2[a,b] \to L^2[b,c], \\ P_{BB} &: L^2[b,c] \to L^2[b,c], \end{aligned}$$

defined by (3.36), (3.39), (3.40), (3.41), (3.42), respectively. Due to Theorem (3.2), when we are using Discretization D we may approximate P_{AB} and P_{BA} via the expressions

$$P_{AB} \approx \psi_{1_A} \cdot M_{AB} \cdot \psi_{2_B}^T, \tag{3.153}$$
$$P_{BA} \approx \psi_{2_B} \cdot M_{BA} \cdot \psi_{1_A}^T, \tag{3.154}$$

with

$$\begin{aligned} \psi_{1_A} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c) \to L^2(A), \\ \psi_{1_B} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f) \to L^2(B), \\ \psi_{2_A} &: L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)} \to L^2(A), \\ \psi_{2_B} &: L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)} \to L^2(B), \end{aligned}$$

interpolation operators given by (3.57) and (3.58), and with

$$\begin{aligned} M_{AB} &: L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)} \to \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c), \\ M_{BA} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c) \to L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)}, \end{aligned}$$

defined by (3.61), (3.62), respectively.

We define the operator

$$Q : L^2[a,c] \times \mathbf{R}^{p\cdot q} \to L^2[a,c] \times \mathbf{R}^{p\cdot q},$$

by the expression

$$Q(\chi)(x) = \chi(x) + \int_a^c k(x,t) \cdot \chi(t)\, dt. \tag{3.155}$$

We additionally require the four operators

$$\begin{aligned} Q_{AA} &: L^2[a,b] \times \mathbf{R}^{p\cdot(q-1)} \to L^2[a,b] \times \mathbf{R}^{p\cdot(q-1)}, \\ Q_{AB} &: L^2[b,c] \times \mathbf{R}^{p\cdot(q-1)} \to L^2[a,b] \times \mathbf{R}^{p\cdot(q-1)}, \\ Q_{BA} &: L^2[a,b] \times \mathbf{R}^{p\cdot(q-1)} \to L^2[b,c] \times \mathbf{R}^{p\cdot(q-1)}, \\ Q_{BB} &: L^2[b,c] \times \mathbf{R}^{p\cdot(q-1)} \to L^2[b,c] \times \mathbf{R}^{p\cdot(q-1)}, \end{aligned}$$

defined by the formulae

$$Q_{AA}(\chi)(x) = \chi(x) + \int_a^b k(x,t)\cdot\chi(t)\,dt, \tag{3.156}$$

$$Q_{AB}(\chi)(x) = \int_b^c k(x,t)\cdot\chi(t)\,dt, \tag{3.157}$$

$$Q_{BA}(\chi)(x) = \int_a^b k(x,t)\cdot\chi(t)\,dt, \tag{3.158}$$

$$Q_{BB}(\chi)(x) = \chi(x) + \int_b^c k(x,t)\cdot\chi(t)\,dt. \tag{3.159}$$

Given interpolation operators

$$\begin{aligned} \psi_{[a,c]} &: \mathbf{R}^p \to L^2[a,c], \\ \psi_A &: \mathbf{R}^p \to L^2(A), \\ \psi_B &: \mathbf{R}^p \to L^2(B), \end{aligned}$$

defined by (3.14), we will in this section refer to them by ψ_3, ψ_{3_A}, and ψ_{3_B}, respectively. The operator ψ_3 is related to ψ_{3_A}, ψ_{3_B} via the expressions

$$\psi_{3_{|A}} = \psi_{3_A}\cdot C_D\,, \tag{3.160}$$

$$\psi_{3_{|B}} = \psi_{3_B}\cdot C_U\,, \tag{3.161}$$

with $C_D, C_U : \mathbf{R}^p \to \mathbf{R}^p$ given by (3.21), (3.22), respectively. We in addition require the zero operators

$$\begin{aligned} 0_{A_1} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(A_f) \to L^2(B), \\ 0_{A_2} &: \mathbf{R}^p \to L^2(B), \\ 0_{B_1} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f) \to L^2(A), \\ 0_{B_2} &: \mathbf{R}^p \to L^2(A). \end{aligned}$$

Given Discretization D, we require the interpolation operators

$$\begin{aligned} \psi_1 &: \mathbf{R}^{p\cdot q} \times L^2(B_f) \to L^2[a,c], \\ \psi_2 &: L^2(A_f) \times \mathbf{R}^{p\cdot q} \to L^2[a,c], \end{aligned}$$

given by the formulae

$$\psi_{1_{|A}} = \psi_{3_A} \times 0_{A_2} , \tag{3.162}$$
$$\psi_{1_{|B}} = 0_{B_1} \times \psi_{1_B} , \tag{3.163}$$
$$\psi_{2_{|A}} = \psi_{2_A} \times 0_{A_1} , \tag{3.164}$$
$$\psi_{2_{|B}} = 0_{B_2} \times \psi_{3_B} . \tag{3.165}$$

Assuming that the operators P_{AA}, P_{BB} are nonsingular, we define the functions

$$\eta_A : A \rightarrow \mathbf{R},$$
$$\eta_B : B \rightarrow \mathbf{R},$$

via the formulae

$$\eta_A = P_{AA}^{-1}(f_{|A}) , \tag{3.166}$$
$$\eta_B = P_{BB}^{-1}(f_{|B}) . \tag{3.167}$$

Similarly, assuming that the operators Q, Q_{AA}, Q_{BB} are nonsingular, we then define the operators

$$\chi_1 : \mathbf{R}^{p \cdot q} \times L^2(B_f) \rightarrow L^2[a,c] ,$$
$$\chi_2 : L^2(A_f) \times \mathbf{R}^{p \cdot q} \rightarrow L^2[a,c] ,$$
$$\chi_3 : \mathbf{R}^p \rightarrow L^2[a,c] ,$$
$$\phi_{1_A} : \mathbf{R}^{p \cdot (q-1)} \times L^2(A_c) \rightarrow L^2(A) ,$$
$$\phi_{2_A} : L^2(A_f) \times \mathbf{R}^{p \cdot (q-1)} \rightarrow L^2(A) ,$$
$$\phi_{3_A} : \mathbf{R}^p \rightarrow L^2(A) ,$$
$$\phi_{1_B} : \mathbf{R}^{p \cdot (q-1)} \times L^2(B_f) \rightarrow L^2(B) ,$$
$$\phi_{2_B} : L^2(B_c) \times \mathbf{R}^{p \cdot (q-1)} \rightarrow L^2(B) ,$$
$$\phi_{3_B} : \mathbf{R}^p \rightarrow L^2(B) ,$$

via the formulae

$$\chi_1 = Q^{-1}(\psi_1) , \tag{3.168}$$
$$\chi_2 = Q^{-1}(\psi_2) , \tag{3.169}$$
$$\chi_3 = Q^{-1}(\psi_3) , \tag{3.170}$$
$$\phi_{1_A} = Q_{AA}^{-1}(\psi_{1_A}) , \tag{3.171}$$
$$\phi_{2_A} = Q_{AA}^{-1}(\psi_{2_A}) , \tag{3.172}$$
$$\phi_{3_A} = Q_{AA}^{-1}(\psi_{3_A}) , \tag{3.173}$$
$$\phi_{1_B} = Q_{BB}^{-1}(\psi_{1_B}) , \tag{3.174}$$
$$\phi_{2_B} = Q_{BB}^{-1}(\psi_{2_B}) , \tag{3.175}$$
$$\phi_{3_B} = Q_{BB}^{-1}(\psi_{3_B}) . \tag{3.176}$$

Finally, given the transpose defined by (3.98), we will define operators

$$
\begin{aligned}
\alpha_{11}^{A} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c) \to \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c)\,, \\
\alpha_{12}^{A} &: L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)} \to \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c)\,, \\
\alpha_{13}^{A} &: \mathbf{R}^{p} \to \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c)\,, \\
\alpha_{21}^{A} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c) \to L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)}\,, \\
\alpha_{22}^{A} &: L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)} \to L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)}\,, \\
\alpha_{23}^{A} &: \mathbf{R}^{p} \to L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)}\,, \\
\alpha_{31}^{A} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c) \to \mathbf{R}^{p}\,, \\
\alpha_{32}^{A} &: L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)} \to \mathbf{R}^{p}\,, \\
\alpha_{33}^{A} &: \mathbf{R}^{p} \to \mathbf{R}^{p}\,,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{11}^{B} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f) \to \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f)\,, \\
\alpha_{12}^{B} &: L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)} \to \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f)\,, \\
\alpha_{13}^{B} &: \mathbf{R}^{p} \to \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f)\,, \\
\alpha_{21}^{B} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f) \to L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)}\,, \\
\alpha_{22}^{B} &: L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)} \to L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)}\,, \\
\alpha_{23}^{B} &: \mathbf{R}^{p} \to L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)}\,, \\
\alpha_{31}^{B} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f) \to \mathbf{R}^{p}\,, \\
\alpha_{32}^{B} &: L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)} \to \mathbf{R}^{p}\,, \\
\alpha_{33}^{B} &: \mathbf{R}^{p} \to \mathbf{R}^{p}\,,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{11} &: \mathbf{R}^{p\cdot q} \times L^2(B_f) \to \mathbf{R}^{p\cdot q} \times L^2(B_f)\,, \\
\alpha_{12} &: L^2(A_f) \times \mathbf{R}^{p\cdot q} \to \mathbf{R}^{p\cdot q} \times L^2(B_f)\,, \\
\alpha_{13} &: \mathbf{R}^{p} \to \mathbf{R}^{p\cdot q} \times L^2(B_f)\,, \\
\alpha_{21} &: \mathbf{R}^{p\cdot q} \times L^2(B_f) \to L^2(A_f) \times \mathbf{R}^{p\cdot q}\,, \\
\alpha_{22} &: L^2(A_f) \times \mathbf{R}^{p\cdot q} \to L^2(A_f) \times \mathbf{R}^{p\cdot q}\,, \\
\alpha_{23} &: \mathbf{R}^{p} \to L^2(A_f) \times \mathbf{R}^{p\cdot q}\,, \\
\alpha_{31} &: \mathbf{R}^{p\cdot q} \times L^2(B_f) \to \mathbf{R}^{p}\,, \\
\alpha_{32} &: L^2(A_f) \times \mathbf{R}^{p\cdot q} \to \mathbf{R}^{p}\,, \\
\alpha_{33} &: \mathbf{R}^{p} \to \mathbf{R}^{p}\,,
\end{aligned}
$$

by the formulae

$$\begin{aligned}
&\alpha_{11}^A = \psi_{1_A}^T \cdot \phi_{1_A} , \quad \alpha_{12}^A = \psi_{1_A}^T \cdot \phi_{2_A} , \quad \alpha_{13}^A = \psi_{1_A}^T \cdot \phi_{3_A} , \\
&\alpha_{21}^A = \psi_{2_A}^T \cdot \phi_{1_A} , \quad \alpha_{22}^A = \psi_{2_A}^T \cdot \phi_{2_A} , \quad \alpha_{23}^A = \psi_{2_A}^T \cdot \phi_{3_A} , \\
&\alpha_{31}^A = \psi_{3_A}^T \cdot \phi_{1_A} , \quad \alpha_{32}^A = \psi_{3_A}^T \cdot \phi_{2_A} , \quad \alpha_{33}^A = \psi_{3_A}^T \cdot \phi_{3_A} ,
\end{aligned} \tag{3.177}$$

$$\begin{aligned}
&\alpha_{11}^B = \psi_{1_B}^T \cdot \phi_{1_B} , \quad \alpha_{12}^B = \psi_{1_B}^T \cdot \phi_{2_B} , \quad \alpha_{13}^B = \psi_{1_B}^T \cdot \phi_{3_B} , \\
&\alpha_{21}^B = \psi_{2_B}^T \cdot \phi_{1_B} , \quad \alpha_{22}^B = \psi_{2_B}^T \cdot \phi_{2_B} , \quad \alpha_{23}^B = \psi_{2_B}^T \cdot \phi_{3_B} , \\
&\alpha_{31}^B = \psi_{3_B}^T \cdot \phi_{1_B} , \quad \alpha_{32}^B = \psi_{3_B}^T \cdot \phi_{2_B} , \quad \alpha_{33}^B = \psi_{3_B}^T \cdot \phi_{3_B} ,
\end{aligned} \tag{3.178}$$

$$\begin{aligned}
&\alpha_{11} = \psi_1^T \cdot \chi_1 , \quad \alpha_{12} = \psi_1^T \cdot \chi_2 , \quad \alpha_{13} = \psi_1^T \cdot \chi_3 , \\
&\alpha_{21} = \psi_2^T \cdot \chi_1 , \quad \alpha_{22} = \psi_2^T \cdot \chi_2 , \quad \alpha_{23} = \psi_2^T \cdot \chi_3 , \\
&\alpha_{31} = \psi_3^T \cdot \chi_1 , \quad \alpha_{32} = \psi_3^T \cdot \chi_2 , \quad \alpha_{33} = \psi_3^T \cdot \chi_3 ,
\end{aligned} \tag{3.179}$$

and the functions

$$\begin{aligned}
\delta_1^A &\in \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c) , \\
\delta_2^A &\in L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)} , \\
\delta_3^A &\in \mathbf{R}^p ,
\end{aligned}$$

$$\begin{aligned}
\delta_1^B &\in \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f) , \\
\delta_2^B &\in L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)} , \\
\delta_3^B &\in \mathbf{R}^p ,
\end{aligned}$$

$$\begin{aligned}
\delta_1 &\subset \mathbf{R}^{p\cdot q} \times L^2(B_f) , \\
\delta_2 &\in L^2(A_f) \times \mathbf{R}^{p\cdot q} , \\
\delta_3 &\in \mathbf{R}^p ,
\end{aligned}$$

by the expressions

$$\begin{aligned}
&\delta_1^A = \psi_{1_A}^T \cdot \eta_A , \quad \delta_2^A = \psi_{2_A}^T \cdot \eta_A , \quad \delta_3^A = \psi_{3_A}^T \cdot \eta_A , \\
&\delta_1^B = \psi_{1_B}^T \cdot \eta_B , \quad \delta_2^B = \psi_{2_B}^T \cdot \eta_B , \quad \delta_3^B = \psi_{3_B}^T \cdot \eta_B , \\
&\delta_1 = \psi_1^T \cdot \sigma , \qquad \delta_2 = \psi_2^T \cdot \sigma , \qquad \delta_3 = \psi_3^T \cdot \sigma ,
\end{aligned} \tag{3.180}$$

where σ is the solution to equation (3.37).

3.6.2 Recursive solution of the integral equation

We now consider the original integral equation (3.37)

$$P\sigma = f.$$

The main result of this section is the following lemma, which constructs the solution σ of equation (3.37) from the solutions η_A, η_B of equations (3.166) and (3.167).

Lemma 3.10 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, and the operators*

$$\begin{aligned} \Delta_1 &: \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c) \rightarrow \mathbf{R}^{p\cdot(q-1)} \times L^2(A_c)\ , \\ \Delta_2 &: L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)} \rightarrow L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)}\ , \end{aligned}$$

given by

$$\Delta_1 = I_{\mathbf{R}^{p\cdot(q-1)} \times L^2(A_c)} - \alpha_{11}^A \cdot M_{AB} \cdot \alpha_{22}^B \cdot M_{BA}, \tag{3.181}$$

$$\Delta_2 = I_{L^2(B_c) \times \mathbf{R}^{p\cdot(q-1)}} - \alpha_{22}^B \cdot M_{BA} \cdot \alpha_{11}^A \cdot M_{AB}, \tag{3.182}$$

are also nonsingular, then to accuracy $O(1/p^{k-1})$,

$$\sigma_{|A} = \eta_A + \phi_{1_A} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \delta_1^A - \delta_2^B)\ , \tag{3.183}$$

$$\sigma_{|B} = \eta_B + \phi_{2_B} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \delta_2^B - \delta_1^A)\ . \tag{3.184}$$

Proof. Using definitions (3.39)-(3.42), the integral equation

$$P\sigma = f$$

can be rewritten in the form (3.43)-(3.44). The approximation (3.59) and (3.60) for P_{AB} and P_{BA}, respectively, can then be used to obtain an explicit solution to the coupled equations (3.43) and (3.44) in terms of the functions η_A, $\eta_B, \phi_{1_A}, \phi_{2_A}, \phi_{3_A}, \phi_{1_B}, \phi_{2_B}$ and ϕ_{3_B} defined by (3.166), (3.167), (3.171), (3.172), (3.173), (3.174), (3.175), (3.176), respectively. Indeed, applying the operator P_{AA}^{-1} to equation (3.43) and the operator P_{BB}^{-1} to equation (3.44), we have

$$\sigma_{|A} + P_{AA}^{-1} \cdot P_{AB}(\sigma_{|B}) = P_{AA}^{-1}(f_{|A}), \tag{3.185}$$

$$P_{BB}^{-1} \cdot P_{BA}(\sigma_{|A}) + \sigma_{|B} = P_{BB}^{-1}(f_{|B}). \tag{3.186}$$

Substituting the approximations (3.59) and (3.60) into (3.185) and (3.186) yields the formulae

$$\sigma_{|A} + P_{AA}^{-1} \cdot w_{1_A} \cdot M_{AB} \cdot w_{2_B}^T \cdot \sigma_{|B} = \eta_A, \tag{3.187}$$

$$P_{BB}^{-1} \cdot \psi_{2_B} \cdot M_{BA} \cdot \psi_{1_A}^T \cdot \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{3.188}$$

or

$$\sigma_{|A} + \phi_{1_A} \cdot M_{AB} \cdot \psi_{2_B}^T \cdot \sigma_{|B} = \eta_A, \tag{3.189}$$

$$\phi_{2_B} \cdot M_{BA} \cdot \psi_{1_A}^T \cdot \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{3.190}$$

where we have used the definitions (3.171), (3.175) for ϕ_{1_A} and ϕ_{2_B}, respectively. Now, multiplying (3.190) by $\phi_{1_A} \cdot M_{AB} \cdot \psi_{2_B}^T$ and subtracting it from (3.189), we obtain

$$\begin{aligned}(I_{L^2(A)} - \phi_{1_A} \cdot M_{AB} \cdot \psi_{2_B}^T \cdot \phi_{2_B} \cdot M_{BA} \cdot \psi_{1_A}^T) \cdot \sigma_{|A} \;&= \\ \eta_A - \phi_{1_A} \cdot M_{AB} \cdot \psi_{2_B}^T \cdot \eta_B.&\end{aligned} \tag{3.191}$$

Similarly, multiplying (3.189) by $\phi_{2_B} \cdot M_{BA} \cdot \psi_{1_A}^T$ and subtracting it from (3.190) results in the equation

$$\begin{aligned}(I_{L^2(B)} - \phi_{2_B} \cdot M_{BA} \cdot \psi_{1_A}^T \cdot \phi_{1_A} \cdot M_{AB} \cdot \psi_{2_B}^T) \cdot \sigma_{|B} \;&= \\ \eta_B - \phi_{2_B} \cdot M_{BA} \cdot \psi_{1_A}^T \cdot \eta_A.&\end{aligned} \tag{3.192}$$

Due to (3.177), (3.178) and (3.180), we can rewrite these equations in the form

$$(I_{L^2(A)} - \phi_{1_A} \cdot M_{AB} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \psi_{1_A}^T)) \cdot \sigma_{|A} = \eta_A - \phi_{1_A} \cdot M_{AB} \cdot \delta_2^B, \tag{3.193}$$

$$(I_{L^2(B)} - \phi_{2_B} \cdot M_{BA} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \psi_{2_B}^T)) \cdot \sigma_{|B} = \eta_B - \phi_{2_B} \cdot M_{BA} \cdot \delta_1^A. \tag{3.194}$$

By application of Lemma 2.5, we obtain

$$\begin{aligned}\sigma_{|A} \;=\; &(I_A + \phi_{1_A} \cdot M_{AB} \cdot (I_{L^2(B_c) \times \mathbf{R}^{p(q-1)}} - \alpha_{22}^B \cdot M_{BA} \cdot \alpha_{11}^A \cdot M_{AB})^{-1} \cdot \\ &\alpha_{22}^B \cdot M_{BA} \cdot \psi_{1_A}^T) \cdot (\eta_A - \phi_{1_A} \cdot M_{AB} \cdot \delta_2^B),\end{aligned} \tag{3.195}$$

$$\begin{aligned}\sigma_{|B} \;=\; &(I_B + \phi_{2_B} \cdot M_{BA} \cdot (I_{\mathbf{R}^{p(q-1)} \times L^2(A_c)} - \alpha_{11}^A \cdot M_{AB} \cdot \alpha_{22}^B \cdot M_{BA})^{-1} \cdot \\ &\alpha_{11}^A \cdot M_{AB} \cdot \psi_{2_B}^T) \cdot (\eta_B - \phi_{2_B} \cdot M_{BA} \cdot \delta_1^A).\end{aligned} \tag{3.196}$$

The equations (3.183), (3.184) are now obtained from equations (3.195), (3.191) and equations (3.181), (3.182). □

3.6.3 Further Analytical Results

We now collect a number of identities which are necessary for the algorithm to be presented in Section 3.8. First, we apply Lemma 3.10 to the particular cases $f = \psi_1$, $f = \psi_2$, $f = \psi_3$ to obtain analytical expressions for the functions χ_1, χ_2 and χ_3 defined in equations (3.168) and (3.169).

Corollary 3.9 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\chi_{1_{|A}} = (\phi_{3_A} + \phi_{1_A} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{22}^B \cdot M_{BA} \cdot \alpha_{13}^A) \times (-\phi_{1_A} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{21}^B) , \quad (3.197)$$

$$\chi_{1_{|B}} = (-\phi_{2_B} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{13}^A) \times (\phi_{1_B} + \phi_{2_B} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{11}^A \cdot M_{AB} \cdot \alpha_{21}^B) , \quad (3.198)$$

$$\chi_{2_{|A}} = (\phi_{2_A} + \phi_{1_A} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{22}^B \cdot M_{BA} \cdot \alpha_{12}^A) \times (-\phi_{1_A} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{23}^B) , \quad (3.199)$$

$$\chi_{2_{|B}} = (-\phi_{2_B} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{12}^A) \times (\phi_{3_B} + \phi_{2_B} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{11}^A \cdot M_{AB} \cdot \alpha_{23}^B) , \quad (3.200)$$

$$\chi_{3_{|A}} = \phi_{3_A} \cdot C_D + \phi_{1_A} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \alpha_{13}^A \cdot C_D - \alpha_{23}^B \cdot C_U) , \quad (3.201)$$

$$\chi_{3_{|B}} = \phi_{3_B} \cdot C_U + \phi_{2_B} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \alpha_{23}^B \cdot C_U - \alpha_{13}^A \cdot C_D) , \quad (3.202)$$

where the coefficients α_{ij}^A and α_{ij}^B are given by equations (3.177) and (3.178).

Proof. We only prove the expressions (3.197), (3.198); the proofs for the remaining expressions (3.199), (3.200) and (3.201), (3.202) are nearly identical.

Substituting in equations (3.195), (3.191) the functions ϕ_{1_A}, ϕ_{1_B} defined by (3.171), (3.174) for the functions η_A, η_B defined by (3.166), (3.167), and the matrices $\alpha_{11}^A, \alpha_{22}^B$ defined by (3.177), (3.178) for the vectors δ_1^A, δ_2^B defined by (3.180), we obtain

$$\chi_{1_{|A}} = (I_{L^2(A)} + \phi_{1_A} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{22}^B \cdot M_{BA} \cdot \psi_{1_A}^T) \cdot (\phi_{1_A} - \phi_{1_A} \cdot M_{AB} \cdot \alpha_{22}^B), \quad (3.203)$$

$$\chi_{1_{|B}} = (I_{L^2(B)} + \phi_{2_B} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{11}^A \cdot M_{AB} \cdot \psi_{2_B}^T) \cdot (\phi_{2_B} - \phi_{2_B} \cdot M_{BA} \cdot \alpha_{11}^A). \quad (3.204)$$

The expressions (3.197), (3.198) are now easily obtained from the equations (3.203), (3.204). □

We will also require analytical expressions for the inner products δ_1, δ_2, and δ_3 defined in (3.180) in terms of the restricted inner products δ_1^A, δ_1^B, δ_2^A, δ_2^B, δ_3^A, and δ_3^B defined in (3.180).

Corollary 3.10 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P,*

P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,

$$\begin{aligned}\delta_1 &= \psi_1^T \cdot \sigma = (\psi_{3_A} \times 0_{A_2})^T \cdot \sigma_{|A} + (0_{B_1} \psi_{1_B})^T \cdot \sigma_{|B} \\ &= \begin{pmatrix} \delta_3^A + \alpha_{31}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \delta_1^A - \delta_2^B) \\ \\ \delta_1^B + \alpha_{12}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \delta_2^B - \delta_1^A) \end{pmatrix}, \end{aligned} \tag{3.205}$$

$$\begin{aligned}\delta_2 &= \psi_2^T \cdot \sigma = (\psi_{2_A} \times 0_{A_1})^T \cdot \sigma_{|A} + (0_{B_2} \times \psi_{3_B})^T \cdot \sigma_{|B} \\ &= \begin{pmatrix} \delta_2^A + \alpha_{21}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \delta_1^A - \delta_2^B) \\ \\ \delta_3^B + \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \delta_2^B - \delta_1^A) \end{pmatrix}, \end{aligned} \tag{3.206}$$

$$\begin{aligned}\delta_3 &= \psi_3^T \cdot \sigma = (\psi_{3_A} \cdot C_D)^T \cdot \sigma_{|A} + (\psi_{3_B} \cdot C_U)^T \cdot \sigma_{|B} \\ &= C_D^T \cdot \delta_3^A + C_U^T \cdot \delta_3^B + C_D^T \cdot \alpha_{31}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \delta_1^A - \delta_2^B) \\ &\quad + C_U^T \cdot \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \delta_2^B - \delta_1^A) . \end{aligned} \tag{3.207}$$

Proof. Multiplying equation (3.183) by $\psi_{2_A}^T$ and $\psi_{3_A}^T$, and equation (3.184) by $\psi_{1_B}^T$ and $\psi_{2_B}^T$, we obtain

$$\psi_{2_A}^T \cdot \sigma_{|A} = \delta_2^A + \alpha_{21}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \delta_1^A - \delta_2^B) , \tag{3.208}$$

$$\psi_{3_A}^T \cdot \sigma_{|A} = \delta_3^A + \alpha_{31}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \delta_1^A - \delta_2^B) , \tag{3.209}$$

$$\psi_{1_B}^T \cdot \sigma_{|B} = \delta_1^B + \alpha_{12}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \delta_2^B - \delta_1^A) , \tag{3.210}$$

$$\psi_{3_B}^T \cdot \sigma_{|B} = \delta_3^B + \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \delta_2^B - \delta_1^A) . \tag{3.211}$$

Now, expressions (3.205), (3.206) and (3.207) are easily obtained from (3.208)–(3.211). □

Special cases of Corollary 3.10 are obtained when $f = \psi_1$, $f = \psi_2$, or $f = \psi_3$. While the objects in Corollaries 3.11–3.19 are different from those of Corollary 3.10 (Corollary 3.10 is concerned with the vectors $\delta_1, \delta_2 \delta_3$, while Corollaries 3.11–3.19 are concerned with the matrices $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{31}, \alpha_{32}, \alpha_{33}$), the proof for each of Corollaries 3.11–3.19 is nearly identical to that of Corollary 3.10.

Corollary 3.11 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\alpha_{11} = \psi_1^T \cdot \chi_1 = \begin{pmatrix} M_{pp} & M_{pn} \\ M_{np} & M_{nn} \end{pmatrix}, \tag{3.212}$$

with

$$\begin{aligned} M_{pp} &: \mathbf{R}^p \rightarrow \mathbf{R}^p, \\ M_{pn} &: \mathbf{R}^{p \cdot (q-1)} \times L^2(B_f) \rightarrow \mathbf{R}^p, \\ M_{np} &: \mathbf{R}^p \rightarrow \mathbf{R}^{p \cdot (q-1)} \times L^2(B_f), \\ M_{nn} &: \mathbf{R}^{p \cdot (q-1)} \times L^2(B_f) \rightarrow \mathbf{R}^{p \cdot (q-1)} \times L^2(B_f), \end{aligned}$$

given by

$$\begin{aligned} M_{pp} &= \alpha^A_{33} + \alpha^A_{31} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha^B_{22} \cdot M_{BA} \cdot \alpha^A_{13} \ , \\ M_{pn} &= -\alpha^A_{31} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha^B_{21} \ , \\ M_{np} &= -\alpha^B_{12} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha^A_{13} \ , \\ M_{nn} &= \alpha^B_{11} + \alpha^B_{12} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha^A_{11} \cdot M_{AB} \cdot \alpha^B_{21} \ . \end{aligned}$$

Corollary 3.12 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\alpha_{12} = \psi_1^T \cdot \chi_2 = \begin{pmatrix} M_{pn} & M_{pp} \\ M_{nn} & M_{np} \end{pmatrix}, \tag{3.213}$$

with

$$\begin{aligned} M_{pn} &: L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)} \to \mathbf{R}^p, \\ M_{pp} &: \mathbf{R}^p \to \mathbf{R}^p, \\ M_{nn} &: L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)} \to \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f), \\ M_{np} &: \mathbf{R}^p \to \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f), \end{aligned}$$

given by

$$\begin{aligned} M_{pn} &= \alpha^A_{32} + \alpha^A_{31} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha^B_{22} \cdot M_{BA} \cdot \alpha^A_{12} \ , \\ M_{pp} &= -\alpha^A_{31} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha^B_{23} \ , \\ M_{nn} &= -\alpha^B_{12} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha^A_{12} \ , \\ M_{np} &= \alpha^B_{13} + \alpha^B_{12} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha^A_{11} \cdot M_{AB} \cdot \alpha^B_{23} \ . \end{aligned}$$

Corollary 3.13 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\alpha_{13} = \psi_1^T \cdot \chi_3 = \begin{pmatrix} M_{pp} \\ M_{np} \end{pmatrix}, \tag{3.214}$$

with

$$\begin{aligned} M_{pp} &: \mathbf{R}^p \to \mathbf{R}^p, \\ M_{np} &: \mathbf{R}^p \to \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f), \end{aligned}$$

given by

$$\begin{aligned} M_{pp} &= \alpha^A_{33} \cdot C_D + \alpha^A_{31} \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha^B_{22} \cdot M_{BA} \cdot \alpha^A_{13} \cdot C_D - \alpha^B_{23} \cdot C_U) \ , \\ M_{np} &= \alpha^B_{13} \cdot C_U + \alpha^B_{12} \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha^A_{11} \cdot M_{AB} \cdot \alpha^B_{23} \cdot C_U - \alpha^A_{13} \cdot C_D) \ . \end{aligned}$$

Corollary 3.14 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\alpha_{21} = \psi_2^T \cdot \chi_1 = \begin{pmatrix} M_{np} & M_{nn} \\ M_{pp} & M_{pn} \end{pmatrix}, \tag{3.215}$$

with

$$\begin{aligned} M_{np} &: \mathbf{R}^p \to L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)}, \\ M_{nn} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f) \to L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)}, \\ M_{pp} &: \mathbf{R}^p \to \mathbf{R}^p, \\ M_{pn} &: \mathbf{R}^{p\cdot(q-1)} \times L^2(B_f) \to \mathbf{R}^p, \end{aligned}$$

given by

$$\begin{aligned} M_{np} &= \alpha_{23}^A + \alpha_{21}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{22}^B \cdot M_{BA} \cdot \alpha_{13}^A , \\ M_{nn} &= -\alpha_{21}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{21}^B , \\ M_{pp} &= -\alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{13}^A , \\ M_{pn} &= \alpha_{31}^B + \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{11}^A \cdot M_{AB} \cdot \alpha_{21}^B . \end{aligned}$$

Corollary 3.15 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\alpha_{22} = \psi_2^T \cdot \chi_2 = \begin{pmatrix} M_{nn} & M_{np} \\ M_{pn} & M_{pp} \end{pmatrix}, \tag{3.216}$$

with

$$\begin{aligned} M_{nn} &: L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)} \to L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)}, \\ M_{np} &: \mathbf{R}^p \to L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)}, \\ M_{pn} &: L_2(A_f) \times \mathbf{R}^{p\cdot(q-1)} \to \mathbf{R}^p, \\ M_{pp} &: \mathbf{R}^p \to \mathbf{R}^p, \end{aligned}$$

given by

$$\begin{aligned} M_{nn} &= \alpha_{22}^A + \alpha_{21}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{22}^B \cdot M_{BA} \cdot \alpha_{12}^A , \\ M_{np} &= -\alpha_{21}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{23}^B , \\ M_{pn} &= -\alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{12}^A , \\ M_{pp} &= \alpha_{33}^B + \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{11}^A \cdot M_{AB} \cdot \alpha_{23}^B . \end{aligned}$$

Corollary 3.16 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\alpha_{23} = \psi_2^T \cdot \chi_3 = \begin{pmatrix} M_{np} \\ M_{pp} \end{pmatrix}, \tag{3.217}$$

with

$$\begin{aligned} M_{np} &: \mathbf{R}^p \rightarrow L^2(A_f) \times \mathbf{R}^{p\cdot(q-1)}, \\ M_{pp} &: \mathbf{R}^p \rightarrow \mathbf{R}^p, \end{aligned}$$

given by

$$\begin{aligned} M_{np} &= \alpha_{23}^A \cdot C_D + \alpha_{21}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \alpha_{13}^A \cdot C_D - \alpha_{23}^B \cdot C_U) , \\ M_{pp} &= \alpha_{33}^B \cdot C_U + \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \alpha_{23}^B \cdot C_U - \alpha_{13}^A \cdot C_D) . \end{aligned}$$

Corollary 3.17 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\begin{aligned} \alpha_{31} = \psi_3^T \cdot \chi_1 = & \\ & (C_D^T \cdot \alpha_{33}^A + C_D^T \cdot \alpha_{31}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{22}^B \cdot M_{BA} \cdot \alpha_{13}^A \\ & - C_U^T \cdot \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{13}^A) \times \\ & (-C_D^T \cdot \alpha_{31}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{21}^B + C_U^T \cdot \alpha_{31}^B + \\ & C_U^T \cdot \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{11}^A \cdot M_{AB} \cdot \alpha_{21}^B) . \end{aligned} \tag{3.218}$$

Corollary 3.18 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\begin{aligned} \alpha_{32} = \psi_3^T \cdot \chi_2 = & \\ & (C_D^T \cdot \alpha_{32}^A + C_D^T \cdot \alpha_{31}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{22}^B \cdot M_{BA} \cdot \alpha_{12}^A \\ & - C_U^T \cdot \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{12}^A) \times \\ & (-C_D^T \cdot \alpha_{31}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot \alpha_{23}^B + C_U^T \cdot \alpha_{33}^B + \\ & C_U^T \cdot \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot \alpha_{11}^A \cdot M_{AB} \cdot \alpha_{23}^B) . \end{aligned} \tag{3.219}$$

Corollary 3.19 *If Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)-(3.154), and if, in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular, then to accuracy $O(1/p^{k-1})$,*

$$\begin{aligned} \alpha_{33} &= \psi_3^T \cdot \chi_3 = \\ & C_D^T \cdot \alpha_{33}^A \cdot C_D + C_D^T \cdot \alpha_{31}^A \cdot M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot \alpha_{13}^A \cdot C_D - \alpha_{23}^B \cdot C_U) \\ & + C_U^T \cdot \alpha_{33}^B \cdot C_U + C_U^T \cdot \alpha_{32}^B \cdot M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot \alpha_{23}^B \cdot C_U - \alpha_{13}^A \cdot C_D). \end{aligned} \tag{3.220}$$

Finally, combining Lemma 3.10 with the expressions (3.197)–(3.202), we have

Corollary 3.20 *Suppose that Discretization D is used, so that operators P_{AB}, P_{BA} can be approximated via (3.153)–(3.154), and suppose further that in the notation of Section 3.5.1, all six operators P, P_{AA}, P_{BB}, Q, Q_{AA}, Q_{BB} are non-singular. Suppose finally that the function $F : [a, c] \to \mathbf{R}$ is defined by the formula*

$$F(x) = \chi_1(x) \cdot \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix} + \chi_2(x) \cdot \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \end{pmatrix} + \chi_3(x) \cdot \lambda_3 + \sigma(x), \tag{3.221}$$

with

$$\begin{aligned} \lambda_{11}, \lambda_{22}, \lambda_3 &\in \mathbf{R}^k, \\ \lambda_{12} &\in \mathbf{R}^{p \cdot (q-1)} \times L^2(B_f), \\ \lambda_{21} &\in L^2(A_f) \times \mathbf{R}^{p \cdot (q-1)}. \end{aligned}$$

Then on the interval $[a, b]$, to accuracy $O(1/p^{k-1})$,

$$F(x) = \phi_{1_A}(x) \cdot \mu_1 + \phi_{2_A}(x) \cdot \mu_2 + \phi_{3_A}(x) \cdot \mu_3 + \eta_A(x) \ , \tag{3.222}$$

with

$$\begin{aligned} \mu_1 &\in \mathbf{R}^{p \cdot (q-1)} \times L^2(A_c), \\ \mu_2 &\in L^2(A_f) \times \mathbf{R}^{p \cdot (q-1)}, \\ \mu_3 &\in \mathbf{R}^k, \end{aligned}$$

defined by the formulae

$$\begin{aligned} \mu_1 &= M_{AB} \cdot \Delta_2^{-1} \cdot (\alpha_{22}^B \cdot M_{BA} \cdot (\alpha_{13}^A \cdot \lambda_{11} + \alpha_{12}^A \cdot \lambda_{21} + \alpha_{13}^A \cdot C_D \cdot \lambda_3 + \delta_1^A) \\ &\quad - \alpha_{21}^B \cdot \lambda_{12} - \alpha_{23}^B \cdot (\lambda_{22} + C_U \cdot \lambda_3) - \delta_2^B) \ , \\ \mu_2 &= \lambda_{21} \ , \\ \mu_3 &= \lambda_{11} + C_D \cdot \lambda_3. \end{aligned} \tag{3.223}$$

Similarly, on the interval $[b, c]$, to accuracy $O(1/p^{k-1})$,

$$F(x) = \phi_{1_B}(x) \ \ \nu_1 + \phi_{2_B}(x) \cdot \nu_2 + \phi_{3_B}(x) \cdot \nu_3 + \eta_B(x) \ , \tag{3.224}$$

with

$$\begin{aligned} \nu_1 &\in \mathbf{R}^{p \cdot (q-1)} \times L^2(B_f), \\ \nu_2 &\in L^2(B_c) \times \mathbf{R}^{p \cdot (q-1)}, \\ \nu_3 &\in \mathbf{R}^p, \end{aligned}$$

defined by the formulae

$$\begin{aligned} \nu_1 &= \lambda_{12} \ , \\ \nu_2 &= M_{BA} \cdot \Delta_1^{-1} \cdot (\alpha_{11}^A \cdot M_{AB} \cdot (\alpha_{23}^B \cdot \lambda_{22} + \alpha_{21}^B \cdot \lambda_{12} + \alpha_{23}^B \cdot C_U \cdot \lambda_3 + \delta_2^B) \\ &\quad - \alpha_{12}^A \cdot \lambda_{21} - \alpha_{13}^A \cdot (\lambda_{11} + C_D \cdot \lambda_3) - \delta_1^A) \ , \\ \nu_3 &= \lambda_{22} + C_U \cdot \lambda_3. \end{aligned} \tag{3.225}$$

Proof. Restricting (3.221) on the subintervals A, B of $[a, c]$, respectively, we have

$$F_{|A} = \chi_{1_{|A}} \cdot \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix} + \chi_{2_{|A}} \cdot \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \end{pmatrix} + \chi_{3_{|A}} \cdot \lambda_3 + \sigma_{|A} , \tag{3.226}$$

$$F_{|B} = \chi_{1_{|B}} \cdot \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix} + \chi_{2_{|B}} \cdot \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \end{pmatrix} + \chi_{3_{|B}} \cdot \lambda_3 + \sigma_{|B} . \tag{3.227}$$

Now, the expressions (3.223), (3.225) follow by combining (3.226), (3.227) with (3.183), (3.184), (3.197), (3.198), (3.199), (3.200), and comparing the resulting expression with the expressions (3.222), (3.224). □

3.7 Description of the Algorithm for Singular Solutions

In this section, we present a merging strategy for integral equations with end-point singularities using Discretizations C1 and C2. We subdivide the whole interval $[a, c]$ into $2M$ subintervals, where M is a positive integer. The boundary points $b_1, b_2, \ldots, b_{M+1}$ are defined by the expressions

$$b_1 = a, \tag{3.228}$$

$$b_i = \frac{a \cdot (2^{M+2-i} - 1) + c}{2^{M+2-i}}, \quad (2 \leq i \leq M+1) \tag{3.229}$$

$$b_i = \frac{a + c \cdot (2^{2M-i+1})}{2^{2M-i+1}}, \quad (M+2 \leq i \leq 2M) \tag{3.230}$$

$$b_{2M+1} = c. \tag{3.231}$$

We now define subintervals $B_1, B_2, \ldots, B_{2M}$ by the formulae

$$B_i = (b_i, b_{i+1}), \quad (1 \leq i \leq M) \tag{3.232}$$

$$B_i = (b_{3M+1-i}, b_{3M+2-i}), \quad (M+1 \leq i \leq 2M) \tag{3.233}$$

so that B_1, B_2 and B_{M+1}, B_{M+2} are all the same length, and for all $i = 2, \ldots, M$, the subintervals B_i and B_{M+i} are equivalent in length, and are twice the length of B_{i-1} and We also define subintervals $A_2, A_3, \ldots, A_M, A_{M+2}, A_{M+3}, \ldots, A_{2M}$ (for notational convenience, we leave A_{M+1} undefined) by the formulae

$$\begin{aligned} A_2 &= B_1 \cup B_2, \\ A_i &= A_{i-1} \cup B_i, \quad (3 \leq i \leq M) \\ A_{M+1} &= B_M \cup B_{M+1}, \\ A_i &= A_{i-1} \cup B_i, \quad (M+2 \leq i \leq 2M) \end{aligned}$$

3.7.1 Notation

Generalizing the notation of Section 3.5, we will denote by P_{A_i}, P_{B_i} the restriction to the interval A_i, B_i of the integral operator P, respectively, so that

$$P_{B_i}(\sigma)(x) = \sigma(x) + \int_{b_i}^{b_{i+1}} k(x,t)\cdot\sigma(t)\,dt, \qquad (1 \le i \le M) \tag{3.234}$$

$$P_{B_i}(\sigma)(x) = \sigma(x) + \int_{b_{3M+1-i}}^{b_{3M+2-i}} k(x,t)\cdot\sigma(t)\,dt, \quad (M+1 \le i \le 2M) \tag{3.235}$$

$$P_{A_i}(\sigma)(x) = \sigma(x) + \int_{b_1}^{b_{i+1}} k(x,t)\cdot\sigma(t)\,dt, \qquad (2 \le i \le M) \tag{3.236}$$

$$P_{A_i}(\sigma)(x) = \sigma(x) + \int_{b_{3M+1-i}}^{b_{2M+1}} k(x,t)\cdot\sigma(t)\,dt, \quad (M+2 \le i \le 2M) \tag{3.237}$$

for any $\sigma \in L^2(A_i), \sigma \in L^2(B_i)$, respectively. Similarly, we will denote by Q_{A_i}, Q_{B_i} the restriction to the interval A_i, B_i of the integral operator Q, respectively so that

$$Q_{B_i}(\chi)(x) = \sigma(x) + \int_{b_i}^{b_{i+1}} k(x,t)\cdot\chi(t)\,dt, \qquad (1 \le i \le M) \tag{3.238}$$

$$Q_{B_i}(\chi)(x) = \sigma(x) + \int_{b_{3M+1-i}}^{b_{3M+2-i}} k(x,t)\cdot\chi(t)\,dt, \quad (M+1 \le i \le 2M) \tag{3.239}$$

$$Q_{A_i}(\chi)(x) = \sigma(x) + \int_{b_1}^{b_{i+1}} k(x,t)\cdot\chi(t)\,dt, \qquad (2 \le i \le M) \tag{3.240}$$

$$Q_{A_i}(\chi)(x) = \sigma(x) + \int_{b_{3M+1-i}}^{b_{2M+1}} k(x,t)\cdot\chi(t)\,dt, \quad (M+2 \le i \le 2M) \tag{3.241}$$

for any $\chi \in L^2(A_i)\times\mathbf{R}^p, \chi \in L^2(B_i)\times\mathbf{R}^p$, respectively. For each B_i we define the functions

$$\begin{aligned} \eta_{B_i} &: B_i \to \mathbf{R}\ , \\ \phi_{1_{B_i}} &: \mathbf{R}^p \times L^2(B_i) \to L^2(B_i)\ , \\ \phi_{3_{B_i}} &: \mathbf{R}^p \to L^2(B_i)\ , \end{aligned}$$

as the solutions of the equations

$$P_{B_i}(\eta_{B_i}) = f_{B_i}, \tag{3.242}$$

$$Q_{B_i}(\phi_{1_{B_i}}) = \psi_{1_{|B_i}}, \tag{3.243}$$

$$Q_{B_i}(\phi_{3_{B_i}}) = \psi_{3_{|B_i}}, \tag{3.244}$$

Similarly, we define the functions

$$\begin{aligned} \eta_{A_i} &: A_i \to \mathbf{R}\ , \\ \phi_{1_{A_i}} &: \mathbf{R}^p \times L^2(A_i) \to L^2(A_i)\ , \\ \phi_{3_{A_i}} &: \mathbf{R}^p \to L^2(A_i)\ , \end{aligned}$$

by the formulae

$$P_{A_i}(\sigma_{A_i}) = f_{A_i}, \tag{3.245}$$

$$Q_{A_i}(\chi_{1_{A_i}}) = \psi_{1_{|A_i}}, \tag{3.246}$$

$$Q_{A_i}(\chi_{3_{A_i}}) = \psi_{3_{|A_i}}, \tag{3.247}$$

provided the operators $P_{A_i}, P_{B_i}, Q_{A_i}, Q_{B_i}$ are nonsingular.

For each $i = 1, \ldots, 2M$, we define the operators and functions

$$\begin{aligned}
\alpha_{11}^{B_i} &: \mathbf{R}^p \times L^2(B_i) \to \mathbf{R}^p \times L^2(B_i)\,, \\
\alpha_{13}^{B_i} &: \mathbf{R}^p \to \mathbf{R}^p \times L^2(B_i)\,, \\
\alpha_{31}^{B_i} &: \mathbf{R}^p \times L^2(B_i) \to \mathbf{R}^p\,, \\
\alpha_{33}^{B_i} &: \mathbf{R}^p \to \mathbf{R}^p\,, \\
\delta_1^{B_i} &\in \mathbf{R}^p \times L^2(B_i)\,, \\
\delta_3^{B_i} &\in \mathbf{R}^p,
\end{aligned}$$

by the formulae

$$\begin{aligned}
&\alpha_{11}^{B_i} = \psi_{1_{B_i}}^T \cdot \phi_{1_{B_i}}\,, \quad \alpha_{13}^{B_i} = \psi_{1_{B_i}}^T \cdot \phi_{3_{B_i}}\,, \\
&\alpha_{31}^{B_i} = \psi_{3_{B_i}}^T \cdot \phi_{1_{B_i}}\,, \quad \alpha_{33}^{B_i} = \psi_{3_{B_i}}^T \cdot \phi_{3_{B_i}}\,,
\end{aligned} \tag{3.248}$$

$$\delta_1^{B_i} = \psi_{1_{B_i}}^T \cdot \eta_{B_i}\,, \quad \delta_3^{B_i} = \psi_{2_{B_i}}^T \cdot \eta_{B_i}\,.$$

For each $i = 2, \ldots, M, M+2, \ldots, 2M$ we define the operators and functions

$$\begin{aligned}
\alpha_{11}^{A_i} &: \mathbf{R}^p \times L^2(A_i) \to \mathbf{R}^p \times L^2(A_i), \\
\alpha_{13}^{A_i} &: \mathbf{R}^p \to \mathbf{R}^p \times L^2(A_i), \\
\alpha_{31}^{A_i} &: \mathbf{R}^p \times L^2(A_i) \to \mathbf{R}^p, \\
\alpha_{33}^{A_i} &: \mathbf{R}^p \to \mathbf{R}^p, \\
\delta_1^{A_i} &\in \mathbf{R}^p \times L^2(A_i), \\
\delta_3^{A_i} &\in \mathbf{R}^p,
\end{aligned}$$

by the formulae

$$\begin{aligned}
&\alpha_{11}^{A_i} = \psi_{1_{A_i}}^T \cdot \chi_{1_{A_i}}\,, \quad \alpha_{12}^{A_i} = \psi_{1_{A_i}}^T \cdot \chi_{2_{A_i}}\,, \\
&\alpha_{21}^{A_i} = \psi_{2_{A_i}}^T \cdot \chi_{1_{A_i}}\,, \quad \alpha_{22}^{A_i} = \psi_{2_{A_i}}^T \cdot \chi_{2_{A_i}}\,,
\end{aligned} \tag{3.249}$$

$$\delta_1^{A_i} = \psi_{1_{A_i}}^T \cdot \sigma_{A_i}\,, \quad \delta_2^{A_i} = \psi_{2_{A_i}}^T \cdot \sigma_{A_i}\,.$$

Finally, we define operators

$$\begin{aligned}
M_{B_1 B_2}^2 &: L^2(B_2) \to L^2(B_1)\,. \\
M_{B_2 B_1}^2 &: L^2(B_1) \to L^2(B_2)\,.
\end{aligned}$$

$$
\begin{aligned}
M^2_{B_{M+1}B_{M+2}} &: L^2(B_{M+2}) \to L^2(B_{M+1}) , \\
M^2_{B_{M+2}B_{M+1}} &: L^2(B_{M+1}) \to L^2(B_{M+2}) , \\
M^2_{A_i B_{i+1}} &: L^2(B_{i+1}) \to \mathbf{R}^p \times L^2(A_i) , \quad (i = 2, \ldots, M-1, M+2, \ldots, 2M-1), \\
M^2_{B_{i+1}A_i} &: \mathbf{R}^p \times L^2(A_i) \to L^2(B_{i+1}) , \quad (i = 2, \ldots, M-2, M+2, \ldots, 2M-1), \\
M^1_{A_M A_{2M}} &: \mathbf{R}^p \times L^2(A_{2M}) \to \mathbf{R}^p \times L^2(A_M) , \\
M^1_{A_{2M} A_M} &: \mathbf{R}^p \times L^2(A_M) \to \mathbf{R}^p \times L^2(A_{2M}) ,
\end{aligned}
$$

which are given by (3.86), (3.87), (3.90), (3.91).

3.7.2 Discretization of the Restricted Integral Equations

Choosing an integer $p \geq 1$, we construct the p scaled Chebyshev nodes $t_{j_{B_i}}$ defined by (3.18) on each of the intervals B_i, $i = 1, 2, \ldots, 2M$. We then discretize the three integral equations (3.242), (3.243), (3.244) via the Nyström algorithm based on $p-$point Chebyshev quadrature.

Remark 3.9 We use the standard Chebyshev quadratures (see [16], [18], [20]) when the particular operator being discretized is of the form $m_3(P_{AB}), m_3(P_{BA})$ (defined in Lemma 3.5) or of the form $m_2(P_{AB}), m_2(P_{BA})$ (defined in Lemmas 3.4). When the operator is of the form P_{AA}, P_{BB}, or of the form $m_1(P_{AB}), m_1(P_{BA})$ (defined in Lemma 3.3), then it is discretized via the singular Chebyshev quadrature rules described in Section 3.4. □

The resulting approximations to the functions $\eta_{B_i}, \phi_{1_{B_i}}, \phi_{2_{B_i}}$ at the nodes $t_{j_{B_i}}$ will be denoted by

$$
\begin{aligned}
\tilde{\eta}_{B_i} &= (\tilde{\eta}^1_{B_i}, \tilde{\eta}^2_{B_i}, \ldots, \tilde{\eta}^p_{B_i}) , \\
\tilde{\phi}_{1_{B_i}} &= (\tilde{\phi}^1_{1_{B_i}}, \tilde{\phi}^2_{1_{B_i}}, \ldots, \tilde{\phi}^p_{1_{B_i}}) , \\
\tilde{\phi}_{3_{B_i}} &= (\tilde{\phi}^1_{3_{B_i}}, \tilde{\phi}^2_{3_{B_i}}, \ldots, \tilde{\phi}^p_{3_{B_i}}) ,
\end{aligned}
$$

respectively. For each interval $A_i (1 \leq i \leq M-1)$, we do not construct approximations for $\sigma_{A_i}, \chi_{1_{A_i}}, \chi_{2_{A_i}}$ for the entire interval A_i, but only for the "rightmost" subinterval B_{i+1}. Similarly, for each interval $A_i (M+1 \leq i \leq 2M-1)$, we construct approximations for $\sigma_{A_i}, \chi_{1_{A_i}}, \chi_{2_{A_i}}$ on for the "leftmost" subinterval B_{i+1}. The resulting approximations to these functions at the nodes $t_{j_{B_{i+1}}}$ are denoted by

$$
\begin{aligned}
\tilde{\sigma}_{A_i|B_i} &= (\tilde{\sigma}^1_{A_i|B_i}, \tilde{\sigma}^2_{A_i|B_i}, \ldots, \tilde{\sigma}^p_{A_i|B_i}) \qquad (i = 2, \ldots, M, M+2, \ldots, 2M), \\
\tilde{\chi}_{1_{A_i|B_i}} &= (\tilde{\chi}^1_{1_{A_i|B_i}}, \tilde{\chi}^2_{1_{A_i|B_i}}, \ldots, \tilde{\chi}^p_{1_{A_i|B_i}}) \quad (i = 2, \ldots, M, M+2, \ldots, 2M), \\
\tilde{\chi}_{3_{A_i|B_i}} &= (\tilde{\chi}^1_{3_{A_i|B_i}}, \tilde{\chi}^2_{3_{A_i|B_i}}, \ldots, \tilde{\chi}^p_{3_{A_i|B_i}}) \quad (i = 2, \ldots, M, M+2, \ldots, 2M).
\end{aligned}
$$

Remark 3.10 It is well-known that the order of convergence of the approximations $\tilde{\eta}_{i,l}$, $\tilde{\phi}_{1,l}$, $\tilde{\phi}_{2,l}$, $\tilde{\phi}_{3,l}$ to the functions $\eta_{i,l}$, $\phi_{1,l}$, $\phi_{2,l}$, $\phi_{3,l}$ is p. Since all subsequent steps in the

construction of an approximate solution $\tilde{\sigma}$ to the integral equation (3.37) are analytic, the convergence rate of the full algorithm depends entirely on the parameter p. For example, by using 16 scaled Chebyshev points on each subinterval at the finest level, one obtains a sixteenth order method.

The parameter p also determines the order of Chebyshev polynomial used to approximate well-separated regions of the kernel. For example, choosing $p = 8$ results in single precision interpolation accuracy the kernels of interest, while choosing $p = 16$ results in double precision accuracy.

Thus, p determines both the *order of convergence* (for example, eighth order convergence for $p = 8$), and the *maximum accuracy* (for example, single precision accuracy for $p = 8$). □

The operators of the form $M_{AB}, M_{BA}, \alpha_{11}, \alpha_{13}, \alpha_{31}$ and functions of the form δ_1, λ_1 all have the property that they are composed of a finite rank function (corresponding to Chebyshev approximation for well-separated intervals) and an L^2 function defined for an interval not well-separated. The L^2 portion of each of these operators is discretized at p Chebyshev nodes; we will refer to the discretizations of these operators as $\tilde{M}_{AB}, \tilde{M}_{BA}, \tilde{\alpha}_{11}, \tilde{\alpha}_{13}, \tilde{\alpha}_{31}, \tilde{\delta}_1, \tilde{\lambda}_1$, respectively. We will also refer to the discretizations $\tilde{\alpha}_{33}$, $\tilde{\delta}_3$, $\tilde{\lambda}_3$, which are equivalent to $\alpha_{33}, \delta_3, \lambda_3$, respectively, since the operators $\alpha_{33}, \delta_3, \lambda_3$ are finite dimensional.

3.7.3 Informal Description of the Algorithm for Singular Solutions

We begin by directly solving the integral equations (3.242), (3.243), (3.244) on each of the subintervals B_i. The algorithm then proceeds with an upward sweep for computing the operators $\alpha_{11}^{A_i}, \alpha_{13}^{A_i}, \alpha_{31}^{A_i}, \alpha_{33}^{A_i}$, and functions $\delta_1^{A_i}, \delta_3^{A_i}$; and a downward sweep for computing $\tilde{\lambda}_1^{A_i} \in \mathbf{R}^{2p}$ and $\tilde{\lambda}_3^{A_i} \in \mathbf{R}^p$. Using Corollaries 3.3–3.7, we first obtain data of the form $\alpha_{st}^{A_1}, \delta_s^{A_1}$ from $\alpha_{st}^{B_1}, \delta_s^{B_1}$, $\alpha_{st}^{B_2}, \delta_s^{B_2}$, and also obtain $\alpha_{st}^{A_M}, \delta_s^{A_M}$ from $\alpha_{st}^{B_{M+1}}, \delta_s^{B_{M+1}}$, $\alpha_{st}^{B_{M+2}}, \delta_s^{B_{M+2}}$. Then, using Lemma 3.8 and Corollaries 3.2–3.7, we obtain for each $i = 3, \ldots, M, M+3, \ldots, 2M$ the functions $\sigma_{A_i|B_i}$, $\chi 1_{A_i|B_i}$, $\chi 3_{A_i|B_i}$, the operators $\alpha_{st}^{A_i}$, and the functions $\delta_s^{A_i}$ from the functions η_{B_i}, $\phi_{1_{B_i}}, \phi_{3_{B_i}}$, the operators $\alpha_{st}^{A_{i-1}}, \alpha_{st}^{B_i}$ and the functions $\delta_s^{A_{i-1}}, \delta_s^{B_i}$.

The splitting process proceeds in the reverse order. First, since $[a, c] = A_M \cup A_{2M}$, we can use Lemma 3.8 to construct $\lambda_s^{A_M}, \lambda_s^{A_{2M}}$ (by inspection, λ_1^A is the expression to which ϕ_{1_A} is applied in equation (3.116), λ_1^B is the expression to which ϕ_{1_B} is applied in equation (3.117), and $\lambda_3^A = 0, \lambda_3^B = 0$). Using (3.146) from Corollary 3.8, we can immediately obtain the solution $\sigma_{|B_M}$ using the functions $\sigma_{A_M|B_M}$, $\chi 1_{A_M|B_M}$, $\chi 2_{A_M|B_M}$ and the vectors $\lambda_1^{A_M}$, $\lambda_2^{A_M}$. Similarly, we obtain the solution $\sigma_{|B_{2M}}$ using the functions $\sigma_{A_{2M}|B_{2M}}$, $\chi 1_{A_{2M}|B_{2M}}$, $\chi 2_{B_{2M}|B_{2M}}$ and the vectors $\lambda_1^{A_{2M}}$, $\lambda_2^{A_{2M}}$. Corollary 3.8 also provides the formula for the calculation of $\lambda_1^{A_i}, \lambda_3^{A_i}$ given $\lambda_1^{A_{i+1}}, \lambda_3^{A_{i+1}}$. Therefore, we compute the $\lambda_s^{A_i}, \lambda_s^{A_{M+i}}$ in the order $i = M-1, M-2, \ldots, 2$, and subsequently use equation (3.146) from Corollary 3.8 to determine the solutions $\sigma_{|B_i}, \sigma_{|B_{M+i}}$. We then calculate four final functions, $\tilde{\lambda}_1^{B_1}, \lambda_3^{B_1}$, $\tilde{\lambda}_1^{B_{M+1}}, \lambda_3^{B_{M+1}} \in \mathbf{R}^p$, from the vectors $\tilde{\lambda}_1^{A_2}, \lambda_3^{A_2}, \tilde{\lambda}_1^{A_{M+2}}, \lambda_3^{A_{M+2}}$. Using equation (3.143) from Corollary 3.8, we obtain the solutions $\sigma_{|B_1}, \sigma_{|B_M}$.

To summarize, the algorithm consists of three parts. First, a sufficiently fine subdivision $b_1, b_2, \ldots, b_{2M+1}$ of the interval $[a, c]$ is chosen so that, on each of the intervals B_i, the functions $\eta_{B_i}, \phi 1_{B_i}, \phi 3_{B_i}, \sigma_{A_i|B_i}, \chi 1_{A_i|B_i}, \chi 3_{A_i|B_i}$ can be accurateiy represented by a low order Chebyshev expansion (the latter three functions are not computed for both $i = 1$ and $i = M+1$). On each of the intervals B_i, the equations (3.242)–(3.244) are solved (approximately) by direct inversion of the linear system arising from a Nyström discretization. Second, the functions $\sigma_{A_i|B_i}$, $\chi 1_{A_i|B_i}, \chi 3_{A_i|B_i}$, matrices $\alpha_{st}^{A_i}$ and vectors $\delta_s^{i,l}$ are computed for $i = 2, \ldots, M, M+2, \ldots, 2M$. The vectors $\lambda_1^{A_i}, \lambda_3^{A_i}$, are computed for $i = M, M-1, \ldots, 2$ and also for $i = 2M, 2M-1, \ldots, M+2$, and finally the vectors $\lambda_1^{B_1}, \lambda_3^{B_1}$, $\lambda_1^{B_{M+1}}, \lambda_3^{B_{M+1}}$. The desired function σ is then recovered on each subinterval B_i.

The following is a more detailed description of the numerical procedure.

Algorithm C

Comment [Define the computational grid.]

Create $2M$ subintervals on $[a, c]$ by using the sequence of boundary points $b_1, b_2, \ldots, b_{2M}, b_{2M+1}$ given by (3.228)–(3.231). Create subintervals $B_1, \ldots, B_{2M}$ defined by (3.232)–(3.233), and choose the number p of Chebyshev nodes on each B_i. Determine the locations of the scaled Chebyshev nodes $t_{1_{B_i}}, t_{2_{B_i}}, \ldots, t_{p_{B_i}}$ on each interval B_i, and evaluate the functions f, ψ_1, ψ_2 at these nodes, obtaining $f_{B_i}, \psi_{1_{B_i}}, \psi_{3_{B_i}}$. Compute the discretized operators $\tilde{M}_{B_1B_2}, \tilde{M}_{B_2B_1}, \tilde{M}_{B_{M+1}B_{M+2}}, \tilde{M}_{B_{M+2}B_{M+1}}, \tilde{M}_{A_iB_{i+1}}, \tilde{M}_{B_{i+1}A_i}$ $(i = 2, \ldots, M-1, M+2, \ldots, 2M-1)$, $\tilde{M}_{A_MA_{2M}}, \tilde{M}_{A_{2M}A_M}$, using one-dimensional and two-dimensional Chebyshev transforms, as appropriate (see Theorem 3.2 and Lemmas 3.3–3.5).

Step 1.

Comment [Construct the approximate solutions $\tilde{\eta}_{B_i}, \tilde{\phi}_{1_{A_i}}, \tilde{\phi}_{3_{A_i}}$ on each interval B_i.]

do i = 1,2,..., 2M

(1) Construct the two $p \times p$ matrices on B_i obtained through a Nyström discretization of the corresponding integral equation.

(2) Solve the three $p \times p$ linear systems on B_i^l by Gaussian elimination, obtaining the values $\tilde{\eta}_{B_i}, \tilde{\phi}_{1_{B_i}}, \tilde{\phi}_{3_{B_i}}$.

end do

Step 2.

Comment [Construct the four matrices $\tilde{\alpha}_{11}^{B_i}, \tilde{\alpha}_{13}^{B_i}, \tilde{\alpha}_{31}^{B_i}, \tilde{\alpha}_{33}^{B_i}$ and the two vectors $\tilde{\delta}_1^{B_i}, \tilde{\delta}_3^{B_i}$ on each interval B_i.]

do i = 1,2,..., 2M

Evaluate the four matrices $\tilde{\alpha}_{11}^{B_i}, \tilde{\alpha}_{13}^{B_i}, \tilde{\alpha}_{31}^{B_i}, \tilde{\alpha}_{33}^{B_i}$ and the two vectors $\tilde{\delta}_1^{B_i}, \tilde{\delta}_3^{B_i}$ using the p-point Chebyshev quadrature formula (see Remark 3.9).

end do

Step 3 (Upward Sweep).

Comment [Construct the three functions $\tilde{\sigma}_{A_i|B_i}, \tilde{\chi}1_{A_i|B_i}, \tilde{\chi}3_{A_i|B_i}$, the four matrices $\tilde{\alpha}_{11}^{A_i}$, $\tilde{\alpha}_{13}^{A_i}$, $\tilde{\alpha}_{31}^{A_i}$, $\tilde{\alpha}_{33}^{A_i}$ and the two vectors $\tilde{\delta}_1^{A_i}, \tilde{\delta}_3^{A_i}$ for $i = 2, \ldots, M, M+2, \ldots, 2M$.]

(1) Compute the three functions $\tilde{\sigma}_{A_2|B_2}, \tilde{\chi}1_{A_2|B_2}, \tilde{\chi}3_{A_2|B_2}$, the four matrices $\tilde{\alpha}_{11}^{A_2}, \tilde{\alpha}_{13}^{A_2}, \tilde{\alpha}_{31}^{A_2}, \tilde{\alpha}_{33}^{A_2}$, and the two vectors $\tilde{\delta}_1^{A_2}, \tilde{\delta}_3^{A_2}$ from the data $\tilde{M}_{B_1B_2}, \tilde{M}_{B_2B_1}, \tilde{\eta}_{B_2}, \tilde{\phi}_{1_{B_2}}, \tilde{\phi}_{3_{B_2}}, \tilde{\alpha}_{11}^{B_1}, \tilde{\alpha}_{13}^{B_1}, \tilde{\alpha}_{31}^{B_1}$ $\tilde{\alpha}_{33}^{B_1}, \tilde{\alpha}_{11}^{B_2}, \tilde{\alpha}_{13}^{B_2}, \tilde{\alpha}_{31}^{B_2}, \alpha_{33}^{B_2}, \tilde{\delta}_1^{B_1}, \tilde{\delta}_3^{B_1}, \tilde{\delta}_1^{B_2}, \tilde{\delta}_3^{B_2}$, using the results of Lemma 3.8 and Corollaries 3.2–3.7. Similarly, compute the three functions $\tilde{\sigma}_{A_{M+2}|B_{M+2}}, \tilde{\chi}1_{A_{M+2}|B_{M+2}}$, $\tilde{\chi}3_{A_{M+2}|B_{M+2}}$, the four matrices $\tilde{\alpha}_{11}^{A_{M+2}}, \tilde{\alpha}_{13}^{A_{M+2}}, \tilde{\alpha}_{31}^{A_{M+2}}, \tilde{\alpha}_{33}^{A_{M+2}}$, and the two vectors $\tilde{\delta}_1^{A_{M+2}}$, $\tilde{\delta}_3^{A_{M+2}}$ from the data $\tilde{M}_{B_{M+1}B_{M+2}}, \tilde{M}_{B_{M+2}B_{M+1}}, \tilde{\eta}_{B_{M+2}}, \tilde{\phi}_{1_{B_{M+2}}}, \tilde{\phi}_{3_{B_{M+2}}}, \tilde{\alpha}_{11}^{B_{M+1}}, \tilde{\alpha}_{13}^{B_{M+1}}$, $\tilde{\alpha}_{31}^{B_{M+1}}, \tilde{\alpha}_{33}^{B_{M+1}}, \tilde{\alpha}_{11}^{B_{M+2}}, \tilde{\alpha}_{13}^{B_{M+2}}, \tilde{\alpha}_{31}^{B_{M+2}}, \tilde{\alpha}_{33}^{B_{M+2}}, \tilde{\delta}_1^{B_{M+1}}, \tilde{\delta}_3^{B_{M+1}}, \tilde{\delta}_1^{B_{M+2}}, \tilde{\delta}_3^{B_{M+2}}$.

(2) **do i= 3,4,...,M,M+3,M+4,...,2M**

Compute the three functions $\tilde{\sigma}_{A_i|B_i}, \tilde{\chi}1_{A_i|B_i}, \tilde{\chi}3_{A_i|B_i}$, the four matrices $\tilde{\alpha}_{11}^{A_i}, \tilde{\alpha}_{13}^{A_i}, \tilde{\alpha}_{31}^{A_i}, \tilde{\alpha}_{33}^{A_i}$, and the two vectors $\tilde{\delta}_1^{A_i}, \tilde{\delta}_3^{A_i}$ from the data $\tilde{M}_{A_{i-1}B_i}, \tilde{M}_{B_iA_{i-1}}$ $\tilde{\eta}_{B_i}, \tilde{\phi}_{1_{B_i}}, \tilde{\phi}_{3_{B_i}}, \tilde{\alpha}_{11}^{A_{i-1}}, \tilde{\alpha}_{13}^{A_{i-1}}$, $\tilde{\alpha}_{31}^{A_{i-1}}, \tilde{\alpha}_{33}^{A_{i-1}}, \tilde{\alpha}_{11}^{B_i}, \tilde{\alpha}_{13}^{B_i}, \tilde{\alpha}_{31}^{B_i}, \tilde{\alpha}_{33}^{B_i}, \tilde{\delta}_1^{A_{i-1}}, \tilde{\delta}_3^{A_{i-1}}, \tilde{\delta}_1^{B_i}, \tilde{\delta}_3^{B_i}$, using the results of Lemma 3.8 and Corollaries 3.2–3.7.

end do

Step 4 (Downward Sweep).

Comment [Construct the two vectors $\tilde{\lambda}_1^{A_i}, \lambda_3^{A_i}$ for all intervals A_i. Compute the vectors $\tilde{\lambda}_1^{B_1}$, $\tilde{\lambda}_3^{B_1}$, $\tilde{\lambda}_1^{B_{M+1}}, \tilde{\lambda}_3^{B_{M+1}}$.]

(1) Set $\tilde{\lambda}_3^{A_M} = 0, \tilde{\lambda}_3^{A_{2M}} = 0$. Use the results of Lemma 3.8 to construct $\tilde{\lambda}_1^{A_M}, \tilde{\lambda}_1^{A_{2M}}$ from the data $\tilde{\alpha}_{11}^{A_M}, \tilde{\alpha}_{11}^{A_{2M}}, \tilde{\delta}_1^{A_M}, \tilde{\delta}_1^{A_{2M}}$.

(2) **do i=2M-1,2M-2,...,2M,M-2,M-3,...,2**

Use Corollary 3.8 to compute the vectors $\tilde{\lambda}_1^{A_i}, \tilde{\lambda}_3^{A_i}$ from the vectors $\tilde{lambda}_1^{A_{i+1}}, \tilde{\lambda}_3^{A_{i+1}}$.

end do

(3) Use Corollary 3.8 to compute the vectors $\tilde{\lambda}_1^{B_1}, \tilde{\lambda}_3^{B_1}$ from the vectors $\tilde{\lambda}_1^{A_2}, \tilde{\lambda}_3^{A_2}$, and to compute the vectors $\tilde{\lambda}_1^{B_{M+1}}, \tilde{\lambda}_3^{B_{M+1}}$ from the vectors $\tilde{\lambda}_1^{A_{M+2}}, \tilde{\lambda}_3^{A_{M+2}}$

Step 5.

Comment [Compute the solution σ of equation (3.37) at the nodes $t_{1_{B_i}}, t_{2_{B_i}}, \ldots, t_{p_{B_i}}$ for each interval B_i.]

(1) Determine the values of the the solution $\sigma_{|B_1}, \sigma_{|B_{M+1}}$ using equation (3.144).

(2) **do i=2,3,...,M,M+2,M+3,...,2M**

Determine the values of the solution $\sigma_{|B_i}$ using equation (3.145).

end do

Remark 3.11 Inspection of the above algorithm shows that the amount of work required is of the order $O(M \cdot p^3)$. Step 1 involves solving three $p \times p$ linear systems for each of the $2M$ intervals. Step 2 requires for each interval the application of two $p \times p$ matrices to two $p \times p$ matrices each (a total of four matrix-matrix multiplies), and also requires the application of the same two $p \times p$ matrices to a length p vector. Steps 3–5 require no more than $O(M \cdot p^2)$ operations. Since $N = 2M \cdot p$ is the total number of nodes in the discretization of the interval $[a, c]$, we can write the CPU time estimate in the form $O(N \cdot p^2)$. □

3.8 Description of the Algorithm for Smooth Solutions

We turn now to the construction of the fast algorithm for the solution of the integral equation (3.37)

$$P\sigma = f \ ,$$

using Discretization D, and based on the apparatus developed in Section 3.6. The main tool at our disposal is the ability to merge the solutions of restricted versions of the integral equation in adjacent subintervals (Lemma 3.10). As this suggests a recursive procedure, we begin by subdividing the whole interval $[a, c]$, on which the solution to (3.37) is sought, into a large number of subintervals. For the sake of simplicity, we assume that m is a positive integer and that $M = 2^m$ is the number of subintervals created. The boundary points of the subintervals are then defined by a strictly increasing sequence of numbers

$$b_1, b_2, \ldots, b_M, b_{M+1}, \tag{3.250}$$

with $b_1 = a$ and $b_{M+1} = c$. For each $i = 1, \ldots, M$, we define the interval B_i^m via the expression

$$B_i^m = [b_i, b_{i+1}], \tag{3.251}$$

and create a hierarchy of intervals B_i^j by recursively merging adjacent pairs. That is, for each $j = m-1, \ldots, 1, 0$, and $i = 1, \ldots, M$, we define

$$B_i^l = B_{2i-1}^{l+1} \bigcup B_{2i}^{l+1}. \tag{3.252}$$

We will refer to each fixed l as a *level.* We will also refer to the two intervals B_{2i-1}^{l+1} and B_{2i}^{l+1} as *children* and to the larger interval B_i^l as a *parent.*

It is obvious that

$$B_i^l = [b_{1+(i-1)\cdot 2^{m-l}}, b_{1+i\cdot 2^{m-l}}], \tag{3.253}$$

and that for each level l,

$$[a, c] = \bigcup_{i=1}^{2^l} B_i^l \ . \tag{3.254}$$

Furthermore, since in this section we are using discretization D, so that the number of points in a subinterval is proportional to the size of the interval), the dimension of the finite rank portions of M_{AB}, M_{BA} given by (3.61), (3.61), for an interval B_i^l is $p \cdot (m - l)$.

3.8.1 Notation

Generalizing the notation of Section 3.6, we will denote by $P_{i,l}$ the restriction to the interval B_i^l of the integral operator P, so that

$$P_{i,l}(\sigma)(x) = \sigma(x) + \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} k(x,t)\cdot\sigma(t)\,dt \tag{3.255}$$

for any $\sigma \in L^2(B_i^l)$. Similarly, we will denote by $Q_{i,l}$ the restriction to the interval B_i^l of the integral operator Q, so that

$$Q_{i,l}(\chi)(x) = \chi(x) + \int_{b_{1+(i-1)\cdot 2^{m-l}}}^{b_{1+i\cdot 2^{m-l}}} k(x,t)\cdot\chi(t)\,dt \tag{3.256}$$

for any $\chi \in L^2(B_i^l) \times \mathbf{R}^{p\cdot(m-l)}$. For each B_i^l we will define the functions

$$\eta_{i,l} : B_i^l \to \mathbf{R}, \tag{3.257}$$
$$\phi_{1_{i,l}} : \mathbf{R}^{p\cdot(m-l)} \times L^2(B_i^l) \to L^2(B_i^l), \tag{3.258}$$
$$\phi_{2_{i,l}} : L^2(B_i^l) \times \mathbf{R}^{p\cdot(m-l)} \to L^2(B_i^l), \tag{3.259}$$
$$\phi_{3_{i,l}} : \mathbf{R}^p \to L^2(B_i^l), \tag{3.260}$$

as the solutions of the equations

$$P_{i,l}(\eta_{i,l}) = f_{|B_i^l}, \tag{3.261}$$
$$Q_{i,l}(\phi_{1_{i,l}}) = \psi_{1_{|B_i^l}}, \tag{3.262}$$
$$Q_{i,l}(\phi_{2_{i,l}}) = \psi_{2_{|B_i^l}}, \tag{3.263}$$
$$Q_{i,l}(\phi_{3_{i,l}}) = \psi_{3_{|B_i^l}}, \tag{3.264}$$

provided the operators $P_{i,l}, Q_{i,l}$ are nonsingular.

For each $l = 0, 1, \ldots, m$, and $i = 1, 2, \ldots, 2^l$, we define the operators

$$\alpha_{11}^{i,l} : \mathbf{R}^{p\cdot(m-l)} \times L^2(B_i^l) \to \mathbf{R}^{p\cdot(m-l)} \times L^2(B_i^l),$$
$$\alpha_{12}^{i,l} : L^2(B_i^l) \times \mathbf{R}^{p\cdot(m-l)} \to \mathbf{R}^{p\cdot(m-l)} \times L^2(B_i^l),$$
$$\alpha_{13}^{i,l} : \mathbf{R}^p \to \mathbf{R}^{p\cdot(m-l)} \times L^2(B_i^l),$$
$$\alpha_{21}^{i,l} : \mathbf{R}^{p\cdot(m-l)} \times L^2(B_i^l) \to L^2(B_i^l) \times \mathbf{R}^{p\cdot(m-l)},$$
$$\alpha_{22}^{i,l} : L^2(B_i^l) \times \mathbf{R}^{p\cdot(m-l)} \to L^2(B_i^l) \times \mathbf{R}^{p\cdot(m-l)},$$
$$\alpha_{23}^{i,l} : L^2(B_i^l) \times \mathbf{R}^p \to \mathbf{R}^{p\cdot(m-l)},$$
$$\alpha_{31}^{A_i} : \mathbf{R}^{p\cdot(m-l)} \times L^2(B_i^l) \to \mathbf{R}^p,$$
$$\alpha_{32}^{A_i} : L^2(B_i^l) \times \mathbf{R}^{p\cdot(m-l)} \to \mathbf{R}^p,$$
$$\alpha_{33}^{A_i} : \mathbf{R}^p \to \mathbf{R}^p,$$

by the formulae

$$\begin{aligned}
&\alpha_{11}^{i,l} = \psi_{1_{i,l}}^T \cdot \phi_{1_{i,l}} , \quad \alpha_{12}^{i,l} = \psi_{1_{i,l}}^T \cdot \phi_{2_{i,l}} , \quad \alpha_{13}^{i,l} = \psi_{1_{i,l}}^T \cdot \phi_{3_{i,l}} , \\
&\alpha_{21}^{i,l} = \psi_{2_{i,l}}^T \cdot \phi_{1_{i,l}} , \quad \alpha_{22}^{i,l} = \psi_{2_{i,l}}^T \cdot \phi_{2_{i,l}} , \quad \alpha_{23}^{i,l} = \psi_{2_{i,l}}^T \cdot \phi_{3_{i,l}} , \\
&\alpha_{31}^{i,l} = \psi_{3_{i,l}}^T \cdot \phi_{1_{i,l}} , \quad \alpha_{32}^{i,l} = \psi_{3_{i,l}}^T \cdot \phi_{2_{i,l}} , \quad \alpha_{33}^{i,l} = \psi_{3_{i,l}}^T \cdot \phi_{3_{i,l}} ,
\end{aligned} \tag{3.265}$$

and the functions

$$\begin{aligned}
\delta_1^{i,l} &\in \mathbf{R}^{p\cdot(m-l)} \times L^2(B_i^l), \\
\delta_2^{i,l} &\in L^2(B_i^l) \times \mathbf{R}^{p\cdot(m-l)}, \\
\delta_3 &\in \mathbf{R}^p,
\end{aligned}$$

by the formulae

$$\delta_1^{i,l} = \psi_{1_{i,l}}^T \cdot \eta_{i,l} , \quad \delta_2^{i,l} = \psi_{2_{i,l}}^T \cdot \eta_{i,l} , \quad \delta_3^{i,l} = \psi_{3_{i,l}}^T \cdot \eta_{i,l} . \tag{3.266}$$

Finally for each $l = 0, 1, \ldots, m-1$, and $i = 1, 2, \ldots, 2^l$, we define the operators

$$\begin{aligned}
M_{B_{2i-1}^{l+1} B_{2i}^{l+1}} &: L^2(B_{2i}^{l+1}) \times \mathbf{R}^{p\cdot(m-l)} \to \mathbf{R}^{p\cdot(m-l)} \times L^2(B_{2i-1}^{l+1}), \\
M_{B_{2i}^{l+1} B_{2i-1}^{l+1}} &: \mathbf{R}^{p\cdot(m-l)} \times L^2(B_{2i-1}^{l+1}) \to L^2(B_{2i}^{l+1}) \times \mathbf{R}^{p\cdot(m-l)},
\end{aligned}$$

defined by (3.61) and (3.62).

3.8.2 Discretization of the Restricted Integral Equations

Choosing an integer $p \geq 1$, we construct the p scaled Chebyshev nodes $t_{j_{B_i^m}}$ defined by (3.18) on each of the intervals B_i^m, $i = 1, 2, \ldots, M$. We then discretize the two integral equations (3.261), (3.264) via the Nyström algorithm based on $p-$point Chebyshev quadrature (see Remark 3.9). The resulting approximations to the functions $\eta_{i,l}$, $\phi_{1_{i,l}}$, $\phi_{2_{i,l}}, \phi_{3_{i,l}}$ at the nodes τ_i^j will be denoted by

$$\begin{aligned}
\tilde{\eta}_{i,l} &= (\tilde{\eta}_{i,l}^1, \tilde{\eta}_{i,l}^2, \ldots, \tilde{\eta}_{i,l}^p) , \\
\tilde{\phi}_{1_{i,l}} &= (\tilde{\phi}_{1_{i,l}}^1, \tilde{\phi}_{1_{i,l}}^2, \ldots, \tilde{\phi}_{1_{i,l}}^p) , \\
\tilde{\phi}_{2_{i,l}} &= (\tilde{\phi}_{2_{i,l}}^1, \tilde{\phi}_{2_{i,l}}^2, \ldots, \tilde{\phi}_{2_{i,l}}^p) , \\
\tilde{\phi}_{3_{i,l}} &= (\tilde{\phi}_{3_{i,l}}^1, \tilde{\phi}_{3_{i,l}}^2, \ldots, \tilde{\phi}_{3_{i,l}}^p) ,
\end{aligned}$$

respectively.

The operators of the form $M_{AB}, M_{BA}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{31}, \alpha_{32}$ and functions of the form $\delta_1, \delta_2, \lambda_1, \lambda_2$ all have the property that they are composed of a finite rank function

(corresponding to Chebyshev approximation for well-separated intervals) and an L^2 function defined for an interval not well-separated. The L^2 portion of each of these operators is discretized at p Chebyshev nodes; we will refer to the discretizations of these operators as $\tilde{M}_{AB}, \tilde{M}_{BA}, \tilde{\alpha}_{11}, \tilde{\alpha}_{12}, \tilde{\alpha}_{13}, \tilde{\alpha}_{21}, \tilde{\alpha}_{22}, \tilde{\alpha}_{23}, \tilde{\alpha}_{31}, \tilde{\alpha}_{32}, \tilde{\alpha}_{33}, \tilde{\delta}_1, \tilde{\delta}_2, \tilde{\lambda}_1, \tilde{\lambda}_2$, respectively. We will also refer to the discretizations $\tilde{\alpha}_{33}, \tilde{\delta}_3, \tilde{\lambda}_3$, which are equivalent to $\alpha_{33}, \delta_3, \lambda_3$, respectively, since the operators $\alpha_{33}, \delta_3, \lambda_3$ are finite dimensional.

3.8.3 Informal Description of the Algorithm for Smooth Solutions

We begin by directly solving the two integral equations (3.261), (3.264) on each subinterval B_i^m at the finest level, as discussed in the preceding section. Theorem (3.20) then shows that σ restricted to B_i^m can be expressed as a linear combination of the four solutions $\eta_{i,m}, \phi_{1_{i,m}}, \phi_{2_{i,m}}, \phi_{3_{i,m}}$. Thus, it remains only to determine the vectors $\tilde{\lambda}_1^{i,m}, \tilde{\lambda}_2^{i,m} \in \mathbf{R}^{p\cdot(m+1-i)}$, $\tilde{\lambda}_3^{i,m} \in \mathbf{R}^p$ for each of the M subintervals B_i^m. Fortunately, this can be done recursively. To see this, suppose that, at some coarse level $l \leq m-1$, we are given the vectors $\tilde{\lambda}_1^{i,l}, \tilde{\lambda}_2^{i,l}, \tilde{\lambda}_3^{i,l}$ for the subinterval B_i^l. Then Corollary 3.20 provides formulae for the calculation of the corresponding vectors $\tilde{\lambda}_1^{2i-1,l+1}, \tilde{\lambda}_2^{2i-1,l+1}, \tilde{\lambda}_1^{2i,l+1}, \tilde{\lambda}_2^{2i,l+1} \in \mathbf{R}^{p\cdot(m-l)}$, $\tilde{\lambda}_3^{2i-1,l+1}, \tilde{\lambda}_3^{2i-1,l+1} \in \mathbf{R}^p$ for the two child intervals B_{2i-1}^{l+1} and B_{2i}^{l+1}, respectively. On the coarsest level, we observe that $\tilde{\lambda}_1^{0,1} = 0$, $\tilde{\lambda}_2^{0,1} = 0$, $\tilde{\lambda}_3^{0,1} = 0$, i.e. the solution of equation (3.261) on the whole interval $[a,c]$ is simply σ.

However, the formulae (3.223) and (3.225) of Corollary 3.6 contain the eighteen matrices $\tilde{\alpha}_{st}^{2i-1,l+1}, \tilde{\alpha}_{st}^{2i,l+1}$ $(1 \leq s,t \leq 3)$ and the six vectors $\tilde{\delta}_s^{2i-1,l+1}, \tilde{\delta}_s^{2i,l+1} (1 \leq s \leq 3)$. These quantities are also computed recursively but in the opposite direction, namely, from the finest level to the coarsest. They are certainly available at level m directly from the definitions (3.265)–(3.266). For the interval B_i^l at any coarser level $l \leq m-1$, Corollary 3.10 and Corollaries 3.11–3.19 describe how the nine matrices $\tilde{\alpha}_{st}^{i,l} (1 \leq s,t \leq 3)$ and the three vectors $\tilde{\delta}_s^{i,l} (1 \leq s \leq 3)$ are obtained from the eighteen matrices $\tilde{\alpha}_{st}$, and six vectors $\tilde{\delta}_s$ of the two child intervals.

To summarize, the algorithm consists of three parts. First, a sufficiently fine subdivision $b_1, b_2, \ldots, b_{M+1}$ of the interval $[a,c]$ is chosen so that, on each of the intervals $B_{i,m}$, the functions $\eta_{1_{i,m}}, \phi_{1_{i,m}}, \phi_{2_{i,m}}, \phi_{3_{i,m}}$ can be accurately represented by a low order Chebyshev expansion. On each of the intervals $B_{i,m}$, the equations (3.261)–(3.264) are solved (approximately) by direct inversion of the linear system arising from a Nyström discretization. Second, the matrices $\tilde{\alpha}_{st}^{i,l}$ and three vectors $\tilde{\delta}_s^{i,l}$ are computed in an upward sweep, beginning at the finest level m. Finally, the vectors $\tilde{\lambda}_1^{i,l}, \tilde{\lambda}_2^{i,l}, \tilde{\lambda}_3^{i,l}$ are computed in a downward sweep, beginning at the coarsest level. The desired function σ is then recovered on each subinterval from equations (3.221)–(3.222).

The following is a more detailed description of the numerical procedure.

Algorithm D

Comment [Define the computational grid.]

Create $M = 2^m$ subintervals on $[a, c]$ by choosing a sequence of boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ with $b_1 = a$ and $b_{M+1} = c$. Choose the number p of Chebyshev nodes on each interval $B_i^m = [b_i, b_{i+1}]$ for $i = 1, \ldots, M$. Determine the locations of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m, and evaluate the functions $f, \psi_1, \psi_2, \psi_3$ at these nodes, obtaining $f_{i,m}, \psi_{1_{i,m}}, \psi_{2_{i,m}}, \psi_{3_{i,m}}$. For each $l = 0, 1, \ldots, m-1$, and $i = 1, 2, \ldots, 2^l$, compute the discretized operators $\tilde{M}_{B_{2i-1}^{l+1} B_{2i}^{l+1}}$, $\tilde{M}_{B_{2i}^{l+1} B_{2i-1}^{l+1}}$, using one-dimensional and two-dimensional Chebyshev transforms, as appropriate (see Theorem 3.2 and Lemmas 3.3–3.5).

Step 1.

Comment [Construct the approximate solutions $\tilde{\eta}_{i,m}, \tilde{\phi}_{1_{i,m}}, \tilde{\phi}_{2_{i,m}}, \tilde{\phi}_{3_{i,m}}$ on each interval B_i^m.]

do i = 1,2,..., M

(1) Construct the two $p \times p$ matrices on B_i^l obtained through a Nyström discretization of the corresponding integral equation.

(2) Solve the four $p \times p$ linear systems on B_i^l by Gaussian elimination, obtaining the values $\tilde{\eta}_{i,m}, \tilde{\phi}_{1_{i,m}}, \tilde{\phi}_{2_{i,m}}, \tilde{\phi}_{3_{i,m}}$.

end do

Step 2.

Comment [Construct the nine matrices $\tilde{\alpha}_{st}^{i,m}$ and six vectors $\tilde{\delta}_s^{i,m}$ on each interval B_i^m at the finest level.]

do i = 1,2,..., M

Evaluate the nine matrices $\tilde{\alpha}_{st}^{i,m}$ and three vectors $\tilde{\delta}_s^{i,m}$ using p–point Chebyshev quadrature formula (see Remark 3.9).

end do

Step 3 (Upward Sweep).

Comment [Construct the matrices $\tilde{\alpha}_{st}^{i,l}$ and vectors $\tilde{\delta}_s^{i,l}$ for all intervals at all coarser levels $l = m-1, m-2, \ldots, 0$.]

do l= m-1, 0, -1

do i=1, 2^l

Compute the nine matrices $\tilde{\alpha}_{st}^{i,l}$ and three vectors $\tilde{\delta}_s^{i,l}$ from the corresponding data in the two child intervals ($\tilde{M}_{B_{2i-1}^{l+1} B_{2i}^{l+1}}$, $\tilde{M}_{B_{2i}^{l+1} B_{2i-1}^{l+1}}$, $\tilde{\alpha}_{st}^{2i-1,l+1}$, $\tilde{\alpha}_{st}^{2i,l+1}$, $\tilde{\delta}_s^{2i-1,l+1}$, $\tilde{\delta}_s^{2i,l+1}$), using the results of Corollaries 3.4 and 3.5.

end do

end do

Step 4 (Downward Sweep).

Comment [Construct the three vectors $\lambda_s^{i,m}$ for all intervals at the finest level.]

Set $\tilde{\lambda}_1^{0,1} = 0, \tilde{\lambda}_2^{0,1} = 0, \tilde{\lambda}_3^{0,1} = 0.$

do l=0,m-1
  do i=1, 2^l
    Use Corollary 3.20 to compute the vectors $\tilde{\lambda}_s^{l+1,2\cdot i-1}, \tilde{\lambda}_s^{2i,l+1} (1 \le s \le 3)$,
    for the child intervals B_{2i}^{l+1} and B_{2i-1}^{l+1} from the vectors $\tilde{\lambda}_s^{i,l} (1 \le s \le 3)$ of the parent
    interval B_i^l.
  end do
end do

Step 5.

Comment [Compute the solution σ of equation (3.37) at the nodes $t_{1_{B_i^m}}, t_{2_{B_i^m}}, \ldots, t_{p_{B_i^m}}$ for each interval B_i^m at the finest level.]

do i=1, M
  do j=1,p
    Determine the values of the solution σ of equation (3.37) at the node $t_{j_{B_i^m}}$ via
    formulae (3.221)–(3.222).
  end do
end do

Remark 3.12 Inspection of the above algorithm shows that the amount of work required is of the order $O(M \cdot p^3)$. Step 1 involves solving four $p \times p$ linear systems for each of the M intervals. Step 2 requires for each interval the application of three $p \times p$ matrices to three $p \times p$ matrices each (a total of nine matrix-matrix multiplies), and also requires the application of the same three $p \times p$ matrices to a length p vector. In step 3, the asymptotic cost for each B_i^l is bounded by the cost of multiplying the largest matrices from the previous level ($\tilde{\alpha}_{st}^{2i-1,l+1}, \tilde{\alpha}_{st}^{2i,l+1}$, $(1 \le s,t \le 3)$, each of which have dimensions $p\cdot(m-l) \times p\cdot(m-l)$). The asymptotic total cost is given by the series

$$\sum_{l=0}^{m-1} p^3 \cdot 2^{l+1} \cdot (m-l) = 4 \cdot p^3 \cdot (2^m - 1) - 2 \cdot p^3 \cdot m. \tag{3.267}$$

Steps 4–5 require no more than $O(M \cdot p^2)$ operations. Since $N = M \cdot p$ is the total number of nodes in the discretization of the interval $[a,c]$, we can write the CPU time estimate in the form $O(N \cdot p^2)$. □

3.9 Numerical Results

FORTRAN programs have been written implementing the algorithms described in this chapter. In this section, we discuss several details of our implementation, and demonstrate the performance of the scheme with numerical examples.

The following technical details of our implementation appear to be worth mentioning.

1. Algorithm C depends for its stability on (3.242)–(3.247) having unique solutions for all subintervals A_i and B_i, while Algorithm D depends on the equations (3.261)–(3.264) having unique solutions for all subintervals B_i^l ($l = 0, 1, \ldots, M$, $i = 1, \ldots, 2^l$). It is easy to construct examples for which these conditions are violated, even though equation (3.37) has a unique solution. In such cases, a different subdivision of the interval $[a, c]$ can be attempted, such that none of the subintervals of the new subdivision coincides with an interval of the original one. This procedure can be viewed as a form of pivoting, and it is easy to show that it is always possible to make it work. However, since the numerical ranks of the discretized operators of the form P_{AB} or P_{BA} are sensitive to the sizes of the subintervals A, B, an implementation of this pivoting scheme would be somewhat involved. It has not been implemented at this point, and we have not so far encountered a need for it.

2. We have, however, implemented a crude scheme for detecting high condition numbers in the algorithms. For Algorithm C, these can occur in the solution of the linear systems on each of the subintervals B_i (Step 1), while computing inverses of the matrices Δ_1 defined by (3.130) used when merging solutions (Step 3), and while computing the inverse Δ_2 defined by (3.115) (Step 4.1). For Algorithm D, these can occur in the solution of the linear systems on each of the finest level subintervals (Step 1), and while computing inverses of the matrices Δ_1, Δ_2 defined by (3.181), (3.182) (Step 3). In all cases, the condition number of the system being solved is estimated in the process of solution (we use a standard LINPACK routine), and the largest of these is returned to the user. When an extremely large condition number is detected by the LINPACK routine, the resulting solution of the original integral equation should be viewed as suspect. It is easy to show that when the differential operator is positive definite, this cannot happen. A more complete treatment of this subject requires further study.

The algorithms of this chapter have been applied to a variety of problems. Five experiments are described below, and their results are summarized in Tables 3.2–3.14.

In each of these tables, the first column contains the total number N of nodes in the discretization of the interval $[a, c]$. The second column contains the relative L^2 error of the numerical solution as compared with the analytically obtained one at 5000 equispaced points within the interval $[a, c]$, where Chebyshev interpolation has been used to evaluate the numerical solution at each of the 5000 points. The third column contains the maximum absolute error obtained at any of the 5000 points. The fourth column contains the CPU time required to solve the problem, excluding the time used to evaluate the solution at 5000 equispaced points, where in all cases the times are given for a Sun SPARCstation 2 computer.

Table 3.1: LU Factorization timings

n	t (*sec.*)
20	0.100×10^{-1}
40	0.300×10^{-1}
80	0.180×10^{0}
160	0.127×10^{1}
320	0.948×10^{1}
640	0.723×10^{2}
1280	0.578×10^{3}
2560	0.462×10^{4} (est)

Table 3.2: Numerical results for Example 3.1, $p = 5$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (*sec.*)
20	0.184×10^{0}	0.952×10^{0}	0.500×10^{-1}
40	0.112×10^{0}	0.687×10^{0}	0.150×10^{0}
80	0.260×10^{-1}	0.152×10^{0}	0.260×10^{0}
160	0.292×10^{-3}	0.125×10^{-2}	0.510×10^{0}
320	0.241×10^{-3}	0.471×10^{-3}	0.106×10^{1}

Remark 3.13 For comparison, Table 3.1 shows times for solving linear systems of comparable sizes using the LINPACK LU factorization routines on a Sun SPARCstation 2 computer. □

Example 3.1 The following problem was presented in [24] for the purpose of modeling the intrinsic viscosities of flexible macromolecules. The equation to be solved is given by the formulae

$$\int_{-1}^{1} \frac{\sigma(t)}{\sqrt{|x-t|}}\, dt = \frac{3}{8}x^2 + \frac{1}{4}. \tag{3.268}$$

The closed form solution of this problem (due to [7]) is given by the formula

$$\sigma(x) = \frac{x^2}{\pi \cdot \sqrt{2} \cdot (1-x^2)^{\frac{1}{4}}}. \tag{3.269}$$

We solve (3.268) using Algorithm C with the number of Chebyshev nodes $p = 5, 10, 20$. The results of this experiment are presented in Tables 3.2–3.4.

Example 3.2 The following problem was presented in [12] and [31] (in a slightly different form) for the purpose of modeling an antiplane elasticity problem of a crack terminating

Table 3.3: Numerical results for Example 3.1, $p = 10$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (sec.)
40	0.112×10^{0}	0.686×10^{0}	0.230×10^{0}
80	0.540×10^{-1}	0.335×10^{0}	0.550×10^{0}
160	0.207×10^{-1}	0.166×10^{0}	0.123×10^{1}
320	0.346×10^{-7}	0.240×10^{-6}	0.252×10^{1}
640	0.789×10^{-8}	0.213×10^{-7}	0.508×10^{1}

Table 3.4: Numerical results for Example 3.1, $p = 20$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (sec.)
80	0.541×10^{-1}	0.335×10^{0}	0.138×10^{1}
160	0.263×10^{-1}	0.149×10^{0}	0.343×10^{1}
320	0.988×10^{-2}	0.720×10^{-1}	0.738×10^{1}
640	0.345×10^{-8}	0.268×10^{-7}	0.154×10^{2}
1280	0.138×10^{-10}	0.116×10^{-9}	0.319×10^{2}

perpendicularly at a bimaterial interface [31]. The equation to be solved is given by the formulae

$$\int_0^1 \sigma(t) \left\{ \frac{1}{x-t} + \frac{1}{x+t} \right\} dt = 4x - 2\sqrt{x+x^2}. \tag{3.270}$$

The closed form solution of this problem (due in part to [26]) is given by the formula

$$\sigma(x) = \frac{2}{\pi}\sqrt{x-x^2}. \tag{3.271}$$

We solve (3.270) using Algorithm C with the number of Chebyshev nodes $p = 5, 10, 20$. The results of this experiment are presented in Tables 3.5–3.7.

Table 3.5: Numerical results for Example 3.2, $p = 5$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (sec.)
20	0.482×10^{-1}	0.102×10^{0}	0.500×10^{-1}
40	0.148×10^{-1}	0.482×10^{-1}	0.120×10^{0}
80	0.100×10^{-2}	0.619×10^{-2}	0.230×10^{0}
160	0.288×10^{-3}	0.161×10^{-2}	0.460×10^{0}
320	0.155×10^{-2}	0.115×10^{-1}	0.102×10^{1}

Table 3.6: Numerical results for Example 3.2, $p = 10$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (*sec.*)
40	0.129×10^{-1}	0.469×10^{-1}	0.240×10^{0}
80	0.390×10^{-2}	0.201×10^{-1}	0.550×10^{0}
160	0.277×10^{-3}	0.190×10^{-2}	0.119×10^{1}
320	0.550×10^{-6}	0.244×10^{-5}	0.237×10^{1}
640	0.181×10^{-6}	0.124×10^{-5}	0.489×10^{1}

Table 3.7: Numerical results for Example 3.2, $p = 20$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (*sec.*)
80	0.350×10^{-2}	0.201×10^{-1}	0.131×10^{1}
160	0.102×10^{-2}	0.680×10^{-2}	0.336×10^{1}
320	0.699×10^{-4}	0.464×10^{-3}	0.728×10^{1}
640	0.144×10^{-6}	0.862×10^{-6}	0.151×10^{2}
1280	0.854×10^{-10}	0.564×10^{-9}	0.311×10^{2}

Example 3.3 This example resembles an experiment presented in [5]. The equation to be solved solved is given by the formulae

$$\lambda \cdot \sigma(x) + \int_0^1 [(1 + \sin(25x)) \log(|x - t|) + \cos(25xt)] \cdot \sigma(t)\, dt = \lambda \cdot \sin(mx) + q_1(x) \cdot (1 + \sin(25x)) + q_2(x), \tag{3.272}$$

with $m = 250$, $\lambda \in \{0, 1\}$, and q_1, q_2 given by the expressions

$$q_1(x) = \frac{1}{m} \cdot [\log(x) - \cos(m) \log(1 - x) - \cos(mx) \\ [\mathrm{Ci}(mx) - \mathrm{Ci}(m(1 - x))] - \sin(mx)[\mathrm{Si}(mx) + \mathrm{Si}(m(1 - x))]], \tag{3.273}$$

$$q_2(x) = \frac{\cos(25x + m)(25x - m) - \cos(25x - m)(25x + m) + 2m}{-2(25x + m)(25x - m)}, \tag{3.274}$$

where Ci and Si are the cosine integral and sine integral, respectively. The solution to this equation is given by

$$\sigma(x) = \sin(mx). \tag{3.275}$$

We first solve (3.272) setting $\lambda = 1$ (a second kind integral equation), applying Algorithm D to this equation with the number of Chebyshev nodes $p = 5, 10, 20$. The results of this experiment are presented in Tables 3.8–3.10. We then solve (3.272) with $\lambda = 0$ (a first kind integral equation), applying Algorithm D to this equation with the number of Chebyshev nodes $p = 10, 20$ (due to the high condition number, $p = 5$ yields no accuracy for this problem). The results of this second experiment are presented in Tables 3.11–3.12.

Table 3.8: Numerical results for Example 3.3, $\lambda = 1, p = 5$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (sec.)
20	0.728×10^{1}	0.142×10^{2}	0.600×10^{-1}
40	0.271×10^{1}	0.589×10^{1}	0.310×10^{0}
80	0.125×10^{1}	0.218×10^{1}	0.920×10^{0}
160	0.201×10^{0}	0.345×10^{0}	0.228×10^{1}
320	0.945×10^{-2}	0.163×10^{-1}	0.532×10^{1}
640	0.121×10^{-2}	0.234×10^{-2}	0.122×10^{2}
1280	0.113×10^{-2}	0.190×10^{-2}	0.259×10^{2}
2560	0.112×10^{-2}	0.188×10^{-2}	0.553×10^{2}

Table 3.9: Numerical results for Example 3.3, $\lambda = 1, p = 10$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (sec.)
40	0.280×10^{1}	0.641×10^{1}	0.390×10^{0}
80	0.127×10^{1}	0.273×10^{1}	0.159×10^{1}
160	0.947×10^{-1}	0.138×10^{0}	0.495×10^{1}
320	0.231×10^{-3}	0.333×10^{-3}	0.135×10^{2}
640	0.297×10^{-6}	0.539×10^{-6}	0.329×10^{2}
1280	0.885×10^{-7}	0.146×10^{-6}	0.751×10^{2}
2560	0.885×10^{-7}	0.146×10^{-6}	0.164×10^{3}

Table 3.10: Numerical results for Example 3.3, $\lambda = 1, p = 20$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (sec.)
80	0.123×10^{1}	0.217×10^{1}	0.236×10^{1}
160	0.217×10^{-1}	0.354×10^{-1}	0.105×10^{2}
320	$0.21. \times 10^{-6}$	0.288×10^{-6}	0.344×10^{2}
640	0.321×10^{-12}	0.467×10^{-12}	0.943×10^{2}
1280	0.107×10^{-13}	0.340×10^{-13}	0.237×10^{3}
2560	0.112×10^{-13}	0.310×10^{-13}	0.558×10^{3}

Table 3.11: Numerical results for Example 3.3, $\lambda = 0, p = 10$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (sec.)
40	0.102×10^{1}	0.165×10^{1}	0.420×10^{0}
80	0.927×10^{0}	0.146×10^{1}	0.171×10^{1}
160	0.425×10^{0}	0.640×10^{1}	0.537×10^{1}
320	0.740×10^{-3}	0.135×10^{-1}	0.143×10^{2}
640	0.313×10^{-3}	0.791×10^{-2}	0.351×10^{2}
1280	0.861×10^{-3}	0.200×10^{-1}	0.801×10^{2}
2560	0.202×10^{-2}	0.414×10^{-1}	0.174×10^{3}

Table 3.12: Numerical results for Example 3.3, $\lambda = 0, p = 20$.

n	$E^2(\sigma)$	$E^\infty(\sigma)$	t (sec.)
80	0.952×10^{0}	0.160×10^{1}	0.242×10^{1}
160	0.207×10^{-1}	0.417×10^{-1}	0.108×10^{2}
320	0.150×10^{-5}	0.423×10^{-4}	0.355×10^{2}
640	0.345×10^{-7}	0.699×10^{-6}	0.978×10^{2}
1280	0.200×10^{-7}	0.436×10^{-6}	0.245×10^{3}
2560	0.330×10^{-6}	0.958×10^{-5}	0.575×10^{3}

The following observations can be made from Tables 3.1–3.12, and are corroborated by our more extensive experiments.

1. For Algorithm D, the practical convergence rate of the method is consistent with the theoretical one.

2. While we have not analyzed the theoretical convergence rate for Algorithm C, in practice the convergence is determined by the number of subintervals used, as opposed to the total number of points in the discretization. While double precision arithmetic permits a maximum of 104 subintervals, our experiments indicate that maximum accuracy is achieved by using approximately 64 subintervals.

3. Example 3.3 with $\lambda = 0$ is an extremely ill-conditioned problem, which substantially reduces the accuracy of the computed results compared to the relatively well-conditioned Example 3.3 with $\lambda = 1$. Because Algorithm D is a direct method, the timings for $\lambda = 0$ and for $\lambda = 1$ are the same, for equivalent number of points n and order of method p.

4. For both Algorithm C and Algorithm D, most of the computational effort is devoted to merging the $\tilde{\alpha}$ matrices (Step 3 in both Algorithm C and D). However, the size of the matrices is fixed for Algorithm C, while for Algorithm D the matrices increase in size for coarser levels. As a result, Algorithm C is from 3–7 times faster than Algorithm D, for an equivalent number of points n and order of convergence p.

Chapter 4

Generalizations and Conclusions

4.1 Generalizations

In Chapter 3, we decomposed a one-dimensional integral operator P into four operators P_{AA}, P_{AB}, P_{BA}, P_{BB}, and then constructed low rank factorizations of P_{AB} and P_{BA}. The specific factorization of P_{AB} and P_{BA} in Chapter 3 involved subdividing each of the intervals A and B into a number of smaller subintervals, and decomposing P_{AB} and P_{BA} into a number of operators acting on these smaller subintervals. Each of the smaller operators either acted on subintervals which were well-separated from each other (the operator thus can be approximated by a low-order Chebyshev polynomial), or acted on subintervals which, when discretized, contained few points (the discretization of the operator thus being of low dimension). Such factorizations can also be applied to integral operators corresponding to integral equations on a curve, and to operators corresponding to integral equations of two and three dimensions.

Two problems arise when this method is applied to integral equations on curves. First, it is more difficult to subdivide A and B into a number of smaller well-separated intervals. For example, suppose that the curve is described by a polynomial, and suppose further that one needs to determine the rank of the interaction between two sections of the curve. It is not clear how to determine whether the two sections are well-separated. A second problem is that the choice of boundary separating subintervals A and B can dramatically affect the ranks of the operators P_{AB} and P_{BA}. As an example, consider an integral equation for an ellipse in which one of the axes of the ellipse is much longer than the other. If the short axis is chosen to be the boundary separating subintervals A and B, then there are few points in the discretization which are close to this boundary; P_{AB} and P_{BA} will therefore be low rank operators. On the other hand, if the long axis is chosen to be the boundary separating A and B, then nearly all the points in the discretization will be close to the boundary, and the ranks of P_{AB} and P_{BA} will be quite high.

When integral operators of two and three dimensions are considered, the rank of the operators P_{AB} and P_{BA} is largely determined by the number of points on the boundary separating A and B. For an $N \times N$ discretization in two dimensions, the boundary separating

A and B is a line containing N points, and the rank of P_{AB} and P_{BA} is $N \log N$. Thus a direct solver for a two dimensional integral equation would require order $O(N^3)$ arithmetic operations, and would be of considerable practical interest. On the other hand, for an $N \times N \times N$ discretization in three dimensions, the boundary separating A and B is a square containing $N \times N$ points, and the rank of P_{AB} and P_{BA} for this operator is $N^2 \log N$. A direct solver for a three dimensional integral equation would require order $O(N^6)$ arithmetic operations, and would be prohibitively expensive.

4.2 Conclusions

Algorithms have been presented for the solution of two-point boundary value problems for ordinary differential equations, and for the solution of one-dimensional first and second kind integral equations of potential theory. All algorithms have CPU time requirements proportional to $N \cdot p^2$, with N the number of nodes in the discretization, and p the desired order of convergence. In addition, the time requirements are insensitive to the condition number of the discretized linear system, The methods permit the use of schemes with extremely high orders of convergence, and are quite insensitive to end-point singularities.

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