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Variational Formulation and Finite Element Implementation of Pagano's Theory of Laminated Plates

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Pagano's theory of laminated plates is restated in the form of a set of self-adjoint differential equations. Identifying consistent boundary operators, general variational principle is stated following standard procedures for linear coupled self-adjoint systems of equations. Extensions to relax requirements of differentiability of various field variables are indicated along with specializations to reduce the number of free field variables. One specialization is implemented in a finite element computer program and used to solve several example problems of free-edge delamination specimens.

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FOREWORD

The research reported herein is part of a research program supported by the Air Force System Command/ASD under Grant No. F33615-85-C-3213 to the Ohio State University, and describes one of the several alternative approaches developed to evaluate stresses in free-edge delamination specimens in order to define cumulative damage. Dr. George P. Sendeckyj of Wright Laboratory/Flight Dynamics Directorate was the Program Manager. At the Ohio State University, the research was carried out, under the supervision of Professors Ranbir S. Sandhu and William E. Wolfe, by Hui-Huang Chyou, Graduate Research Associate in the Department of Civil Engineering. The work reported is essentially based on Mr. Chyou's doctoral dissertation. The Instruction and Research Computer Center (IRCC) at The Ohio State University and The Ohio Supercomputer Center (OSC) provided the computational and documentation facilities.

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SECTION I

INTRODUCTION

1.1 BACKGROUND

Most criteria for development of failure in materials are based upon stress/strain/energy distribution. Therefore, in order to model cumulative damage processes in laminated composites, we feel that reliable procedures for evaluation of distribution of stress/strain/energy in the material must be available. This requires an adequate theory governing the behavior of laminated composites along with appropriate methods for solution of the boundary value problem. The current research program covered development of theoretical framework as well as approximate solutions. This report covers one of the alternative approaches investigated viz., Pagano's theory of linear elastic composite laminates based on the assumption of linear variation of in-plane stresses over the thickness of a layer or sublayer (each lamina being further divided into sublayers, if desired), satisfying equilibrium equations exactly, evaluating strains from stresses, and evaluating generalized displacements by integration of strain components and their moments up to a certain order. This section provides an introduction to the problem and describes the scope of the work as well as the organization of this report.

1.2 Introduction

Considerable research effort has been devoted to the development of analytical procedures for the analysis of the behavior of composite materials. This has resulted in a variety of laminated plate theories and solution methods including, among others, classical thin plate theory [Stavsky 1961], higher order theories [Whitney 1973, Nelson 1974, Lo 1977, Reddy 1984] and discrete laminate theories [Srinivas 1973, Sun 1973, Pagano 1978, 1983].

Classical thin plate theory (CPT) based on Kirchhoff hypothesis assumes that transverse shear deformation is negligible. For analysis of laminated composites, it is well known [Whitney 1969, Pagano 1969,1970, Srinivas 1970] that use of CPT leads to underprediction of the transverse deflection. This is due to the fact that the ratio of shear to Young's modulus is lower in most composite materials than in conventional isotropic materials. Also, the error grows with an increase in plate thickness.

Higher order theories [Whitney 1973,1974] which include higher order shear modes lead to improved estimates of in-plane stress distributions. However, higher order theories based on assumption of second and higher order polynomial distribution of in-plane displacements over the depth of the plate, have two critical deficiencies. The first is their lack of capability to describe local deformation precisely. Due to this, it is difficult to avoid error in calculating in-plane stresses around laminar interfaces, especially, when shear rigidities of adjacent laminae are quite different [Sun 1973, Lo 1977]. The other deficiency is the violation of equilibrium of the plate because stress continuity at the interface is, in general, not satisfied. The need to eliminate these deficiencies has motivated the development of several discrete laminated plate theories [Srinivas 1973, Sun 1973] in which variation of directional properties within the laminate is properly incorporated. As the discrete laminate theory not only removes the drawbacks of higher order theories noted above, but also allows different boundary conditions to be specified in each layer, it has been able to accurately describe the mechanical behavior of most laminated plates. The discrete laminate theory results in better estimates of in-plane stress distribution [Sun 1973]. However, this theory, in general, involves a large number of field equations, and consequently makes the problems quite complicated.

Since the boundary value problem of a structure constructed with composite laminates is extremely complex, approximate techniques are often used to obtain solutions. The approximate techniques based on the discrete laminated plate theory can be classified as displacement-based or stress-based approaches. The analytical solutions of displacement-based approach include the approximate elasticity solutions [Puppo and Evensen 1970, Pagano and Pipes 1973], modified higher order theory [Pagano 1974], boundary layer theory [Tang and Levy 1975], Pagano's theory based on the restatement of Reissner's theory for each lamina [Pagano 1978], and its simplication the global-local model [Pagano 1983]. Numerical solutions include use of the finite difference method [Pipes and Pagano 1970, Altus et al. 1980] and the finite element method [Wang and Crossman 1977,1978; Raju and Crews 1981; Whitcomb et al. 1982]. The stress-based finite element approaches include Pian [1969], Rybicki [1971], and Spilker [1980].

Recently, Chang [1987] developed a finite element procedure based on minimization of potential energy and ensuring continuity of displacements as well as tractions. This development was for stress analysis of free-edge delamination specimens under uniform longitudinal strain and is not applicable to damage cumulation in laminated plates of arbitrary geometrical configuration. Hong [1988] developed a consistent shear deformation theory in which the shear forces in each lamina depend upon the shear deformation of all the laminae. Hong's analysis was for dynamic response of laminates and ensured continuity of tractions at interfaces. However, Hong assumed the transverse displacement to be independent of x_3 coordinate i.e. the thickness of the lamina is assumed to stay constant.

The objective of the current research program is to develop a procedure capable of providing good estimates of interlayer and intra-layer stresses so that cumulative damage criteria based on stress distribution can be used to predict damage. Pagano's theory [1978] assuming the in-plane stresses to vary linearly over a layer or sublayer had been shown to give excellent results for four layer laminates. However, the soultion procedure employed by Pagano could not be applied to a larger number of layers. The finite element method has proved to be a powerful tool for obtaining numerical solutions to boundary value problems including the bending and stretching of the plates. To apply this method to obtain numerical solutions and to develop alternative solution strategies for Pagano's theory we felt it necessary to write the governing equations in a self-adjoint form so that the general procedures for development of variational principles for coupled linear problems could be applied. A brief review of earlier work on the laminate theory is given in Section II. Section III contains a summary of Pagano's equations. In Section IV it is shown that Pagano's equations must be restated in a modified form to constitute a self-adjoint system. A modification, involving a reduction in the number of field variables is proposed. Variational principles governing Pagano's theory are developed including various extensions and certain useful specializations. Section V describes a finite element formulation. Section VI contains some examples of application.

SECTION II

LITERATURE REVIEW

2.1 Theory of Laminated Composite Plates

In essence, plates are three-dimensional solids. The advantage of being able to treat them as two-dimensional problems has been the primary motivation for the construction of plate theories. As the use of fiber-reinforced composite laminates is being extended to various engineering fields, a considerable amount of work has been done over the past few decades to develop a reliable theory of laminated plate. Recently, Al-Ghothani [1986] and Hong [1988] presented reviews of the earlier work on this subject. Existing theories may be categorized into three groups:

1. Classical thin plate theory
2. Higher order theories
3. Discrete laminate theories

The first two theories assume the displacements in a single power expansion of the out-of-plane coordinate through the thickness of the laminate, whatever the number of layers. On the other hand, the third group of theories treats each layer as a homogeneous, anisotropic plate and combines field equations of each layer through proper continuity conditions between layers.

In this Section, a brief review of some of the laminate theories described above is presented as an update complementary to the ones by [Al-Ghothani 1986] and Hong [1988]. Throughout, standard index notation is used, in which Latin indices take on the range of values 1,2,3 and Greek indices take on values 1 and 2.

2.1.1 Classical Thin Plate Theory (CPT)

Classical plate theory of composite laminates follows the same philosophy as employed in the homogeneous isotropic thin plate theory. In developing the theory, the displacement field through the thickness of laminate is assumed to be such that the plane of crosssection before bending remains plane and perpendicular to the midsurface of the plate during deformation. In addition, it is assumed that the variation of lateral displacement through the thickness and the stress normal to the midsurface are negligible. A mathematical representation of these assumptions is

$$u_\alpha(x_i) = u^\circ_\alpha(x_\beta) - x_3 w_{,\alpha} \tag{1}$$

$$u_3(x_i) = w(x_\beta) \tag{2}$$

$$\sigma_{33} = 0 \tag{3}$$

where u°_α are the components of in-plane displacements of the midplane. With this displacement field, the kinematic relations are

$$\epsilon_{\alpha\beta} = \frac{1}{2}(u^\circ_{\alpha,\beta} + u^\circ_{\beta,\alpha}) - x_3 w_{,\alpha\beta} \tag{4}$$

$$\epsilon_{\alpha 3} = 0 \tag{5}$$

$$\epsilon_{33} = 0 \tag{6}$$

Apparently, the first complete classical laminated plate theory is due to Reissner [1961]. In analyzing an angle-ply laminate of two layers, it was noticed that coupling of bending and stretching exists unless the material properties are symmetric with respect to the midplane. Stavsky [1961] further investigated this phenomenon for a multi-layer plate. Dong, Pister and Taylor [1962] extended this approach to the analysis of anisotropic laminated shells.

The classical laminated plate theory neglects the effect of transverse shear deformation, implying infinite shear rigidity, i.e., leading to overestimation of plate stiffness. As a result, the theory gives an underprediction of lateral deflection.

Naturally, the error becomes larger as the thickness of plate increases. It was pointed out [Pagano 1969, 1970, Srinivas 1970] that the transverse shear effect is more pronounced in composite laminates because the ratio of shear to Young's modulus is lower for these materials than for conventional isotropic materials.

2.1.2 Higher Order Theory

As noted above, it is clear that realistic transverse shear variations cannot be achieved by theories based on the assumption of plane cross section. This assumption has resulted in an inaccurate prediction of in-plane stress distribution, especially, for a laminate made of layers with different material properties or differently oriented material axes. The need to include the effects of such local deformation has stimulated development of theories that use higher order terms in the assumed displacement field.

Most of these theories are based on an assumed displacement field. As indicated by Al-Ghothani [1986] these theories can be discussed within the framework defined by the following displacement field assumption.

$$u_i(x_j) = u_i^o(x_\beta) + x_3\,\phi_i(x_\beta) + x_3^2\,\psi_i(x_\beta) + x_3^3\,\xi_i(x_\beta) \tag{7}$$

In the expression (7), it is worth r[illegible]g that the terms including even and odd powers of x_3, represent the symmetric and antisymmetric modes, respectively, of the in-plane displacements through the thickness. Of course, powers higher than the third can be included. Theoretically, as the degree of the polynomial increases, the displacement components can be approximated as closely as one wishes. However, there are practical limitations to the degree of the polynomial that can be used. Also we shall see that in composite laminates there is in general discontinuity in gradient of displacements at the interfaces of dissimilar layers. This makes polynomial approximation very difficult.

Whitney [1973] first proposed a higher order theory with quadratic polynomial functions for in-plane displacements (u_α) and linear functions for out-of-plane displacement (u_3) to represent the first antisymmetric shear mode and non-zero normal strain. Whitney [1974] also presented another higher-order theory, in which linear variation of in-plane displacements and quadratic variation of normal displacement through the thickness were assumed. This was used to analyze laminated cylindrical shells with moderate radius-to-thickness ratio under static loading. Nelson [1974] used quadratic functions for both the in-plane and the out-of-plane displacements. Such a displacement field would model the effect of normal strain more precisely. In the higher-order theory proposed by Lo [1977], cubic functions for in-plane displacements and quadratic function for normal displacement were assumed. With this displacement field, it was claimed that the level of truncation is consistent in the sense that the transverse shear strains due to in-plane displacements and normal displacement are of the same order in x_3. From the application of the theory to thick laminated plates in cylindrical bending, it was shown that accuracy of in-plane stress distribution through the thickness could be considerably improved, except at the interfaces of the laminate.

Although higher order theories were, to some extent, successful in incorporating the effect of higher shear modes, the solution process was costly because more field variables were involved. To overcome such difficulties, a simple higher order theory was developed [Levinson 1980] using higher order terms for the in-plane displacements, but assuming u_3 to be constant over the thickness of the plate, i.e.

$$u_\alpha(x_i) = u^o_\alpha(x_\beta) + x_3\,\phi_\alpha(x_\beta) + x_3^2\psi_\alpha(x_\beta) + x_3^3\,\xi_\alpha(x_\beta) \tag{8}$$

$$u_3(x_i) = w(x_\beta) \tag{9}$$

This idea was first proposed by Levinson [1980] for the homogeneous, isotropic plate and extended to the laminated composite plates by Bert [1984]. Imposing the stress free plate surface conditions on this displacement field, two field variables

ψ_α and ξ_α were eliminated to give

$$u_\alpha = u_\alpha^o + x_3\left[\phi_\alpha - \frac{4}{3}(\frac{x_3}{h})^2(\phi_\alpha + w_{,\alpha})\right] \tag{10}$$

$$u_3 = w(x_\beta) \tag{11}$$

where h denotes the thickness of laminate. Equations (10) and (11) still include the effect of higher-order terms. This theory was used [Bert 1984, Reddy 1984] to analyze angle-ply and cross-ply laminated plates and it was reported that accurate results were obtained in predicting transverse shear stresses. A comparison of various theories is given, in summary form, in Table 1.

Table 1: A Comparison of the Assumptions in Some Higher Order Theories

	In-plane strains	Out-of-plane shear strain	Out-of-plane normal strain
Whitney et al. [1973]	Quadratic	Linear	constant
Whitney et al. [1974]	Linear	Quadratic	Linear
Nelson et al. [1974]	Quadratic	Quadratic	Linear
Lo et al. [1977]	Cubic	Quadratic	Linear

The higher order theory has two critical drawbacks. First, it is incapable of accounting for local deformation accurately because local deformation depends on the stacking sequences of layers and elastic properties of each layer [Sun 1973]. Second, it violates equilibrium of the plate because stress continuity in the interfaces of the laminate is not satisfied. This theoretical deficiency can be fatal in certain applications, e.g., in-plane stress analysis or evaluation of natural frequencies of a hybrid laminate.

2.1.3 Discrete Laminate Theory

The need to eliminate the deficiencies of higher order theory has motivated development of several discrete laminated plate theories [Sun 1973, Srinivas 1973, Pagano 1978]. Common procedure for derivation of those theories is to treat each layer as a homogeneous, anisotropic plate and to combine the governing equations of each layer by using interlaminar continuity conditions to obtain global governing equations of the laminated plate. For a homogeneous, orthotropic laminated plate, exact three-dimensional elasticity solutions [Pagano 1969, 1970] revealed that in-plane displacements are, layerwise, almost linear through thickness, thickness stretch is zero even for thick laminates, and the transverse shear stresses are close to parabolic over each layer.

Basically, the discrete laminate theories described above are based upon the same philosophy, i.e., assumption of layer-wise linear variation of the in-plane displacements even though the final form of the governing equations is different. Also they do not satisfy plate equilibrium. Stating that theories based on assumed-displacement field are not reliable for stress analysis of laminates especially where the stress gradient is large, Pagano [1978] developed a discrete laminate theory based on the assumed-stress field using Reissner's variational principle. In-plane normal stresses were assumed to be linear through the thickness of each layer and other stress components were obtained from three-dimensional equilibrium equations. In this theory, all six stress components are, in general, non-zero and continuity of stresses in the interfaces of the laminate is exactly satisfied. Displacement continuity conditions are also satisfied.

In practical applications, a laminate may be composed of numerous layers. Even though discrete theory may provide a reliable tool in predicting precise local behavior, the problem becomes intractable as the number of layers become large. To overcome this difficulty, Pagano [1983] developed the global-local model in which the cross section of a laminated plate is divided into local and global portions. For the local domain Pagano's theory [1978] is used while a higher order theory is adopted for the global domain. A variational principle was used to obtain governing equations of the plate. This dual model is expected to relieve the burden of handling extremely complicated problem, giving precise stress resolution in the local domain [Pagano 1983]. However, in dividing the region into the local and the global domains, technical difficulty remains because the critical portion is not always known in advance.

As an improvement upon the discrete laminate theory, Hong [1988] developed a consistent shear deformation theory of laminated plates and applied it to vibration and transient response. Hong [1988] ensured interlaminar traction continuity in his theory.

2.2 Analysis of Free-Edge Delamination

Due to the presence of singular interlaminar stresses near the laminate free-boundary, edge delamination is observed to occur under incremental axial strain. Delamination can be simply interpreted as separation of laminae from each other in the laminate, and can occur under static, impact or fatigue loading conditions. Based on the discrete laminated plate theory, there are many computational techniques developed to calculate the stress components in laminated composites based on either displacement-based or stress-based approaches to predict free-edge delamination.

Investigations of the free-edge problem were carried out by Puppo and Evensen [1970] using a composite model essentially consisting of a set of anisotropic layers separated by isotropic adhesive layers. It was assumed that the isotropic layers, developed only interlaminar shear stresses, between the anisotropic layers. It was reported that a sharp rise of the interlaminar shear stress could be observed in finite width laminates. However, the simplicity of these elastic formulations precluded calculation of the transverse normal stress and the problem became more complicated when more layers were involved.

In an attempt to approximate the interlaminar normal stress, a simplified formulation was developed by Pagano and Pipes [1973] for the free-edge problem in laminate elasticity. The strategy was to use solutions along the longitudinal midplane of the laminate based upon classical laminated plate theory in conjunction with an assumed distribution of σ_{33}. One could then compute the force and moment resultants caused by the interlaminar stresses on any plane z=constant through consideration of static equilibrium. The maximum interlaminar normal stress at the free-edge was thus expressed in terms of the transverse stress in the y-direction calculated from the laminated plate theory. This assumed distribution, however, was based solely on statics considerations and contained no description of the influence of material and geometric parameters on the interlaminar normal stress [Pagano 1973].

Another approximate elasticity solution proposed by Pipes and Pagano [1974] was based upon displacement-equilibrium equations for an anisotropic elastic medium. Assuming the transverse stresses in the y, z directions to vanish, the equations were written in terms of the single variable U (axial displacement function). This yielded components of displacement, strain as well as remaining stress fields in the form of sinusoidal-hyperbolic series. However, violation of stress equilibrium in the transverse directions as well as neglect of the interlaminar normal stress constituted major drawbacks of this scheme.

Pagano [1974] derived another approximate method for determination of distribution of the interlaminar normal stress only along the midplane of a symmetric, finite width laminate. The approach was based upon a modified version of a higher order theory proposed by Whitney and Sun [1973], which recognized the effect of shear deformation through inplane rotations as well as the thickness strain in the assumed displacement field. However, like the approximate theories discussed previously, none of them was able to determine the complete stress field near the free-edge.

A boundary layer theory for laminated composites in plane stress was developed by Tang and Levy [1975] from the three-dimensional theory of anisotropic elasticity. By expanding the stresses, displacements, body forces and surface tractions in power series of the half-thickness of a lamina in the equations of equilibrium, compatibility and boundary conditions, a sequence of systems of equations was obtained. The complete solution was obtained by combining solutions of the interior domain based on the classical lamination theory and those from boundary layer and matching a set of appropriate boundary conditions. This formulation provided a way to obtain analytical solution for estimating interlaminar normal as well as shear stress distribution, but became too complicated with increasing number of layers.

In order to have displacement as well as stress continuity, a mixed formulation is sometimes used. Unlike the elastic approximations discussed previously, Pagano [1978] developed an approximate theory for a general composite laminate based upon an application of Reissner's variational principle. The assumption was that the inplane stresses are linear in the thickness coordinate while the transverse stresses derived from equilibrium consideration are cubic. Substitution of stress components based on the differential equations of equilibrium and the strain energy density of an elastic anisotropic body into the Reissner's variational principle, integration with respect to z, and setting the first variations equal to zero yields the appropriate field equations and the boundary conditions. The field equations, which consist of the elastic constitutive relations and the differential equations of equilibrium, must be satisfied within each layer. If a laminate or a single lamina is viewed as an assembly of N sheets, each having a finite thickness and required to satisfy force and moment equilibrium, the analysis leads to a set of 23N algebraic and ordinary differential equations which had to be solved simultaneously. Based upon the assumption that the stress field in a free-edge delamination coupon is independent of the longitudinal axis, Pagano [1978] further specialized the theory to the free-edge problem by reducing the stress field determination to the solution of a one-dimensional problem. However, the number of layers considered in the solution process could not exceed six. The manner in which singular behavior was described could not consider larger number of layers.

Pagano [1983] introduced a global-local model, which was able to define detailed response functions in a particular, predetermined region of interest while representing the remainder of the domain by effective properties. This reduced the number of variables in a given problem. In this model, for the global region of the laminate, potential energy was utilized, and the displacement components were based upon the assumption given by Whitney and Sun [1973]. The Reissner variational principle described in Pagano [1978], however, was applied for the local region in which a thickness distribution of the stress field satisfying equilibrium equation within each layer was assumed. A variational principle was then used to derive the governing equations of equilibrium for the whole system. However, for application to delamination of laminated composites, it is sometimes hard to identify the location where the delamination will occur in the delamination process.

Pipes and Pagano [1970] used the classical theory of linear elasticity to formulate the problem of free-edge delamination of a strip under uniform axial strain. Allowing for material symmetries and uniform extension, the transverse components of displacement were assumed to be independent of the longitudinal coordinate. The three coupled elliptic equations for the displacement functions were solved using a finite difference solution technique to approximate the interlaminar stresses. Delamination was assumed to be primarily due to the high shear stress near the free-edge and the interlaminar stress field was found to be an edge effect which was restricted to a boundary region approximately equal to the laminate thickness.

A three-dimensional finite difference analysis was carried out by Altus, Rotem and Shmueli [1980] to examine the free-edge stress field. The displacement equilibrium equation was solved by using central difference method while for displacement or traction-free boundary conditions as well as interfacial continuity conditions, either forward or backward difference scheme was applied. Convergence of the solution was expected providing a reasonable displacement field was assumed initially. Although a complete stress field was available due to three-dimensional characteristics, an iteration scheme could be a serious inconvenience.

Wang and Crossman [1977] used 392 constant strain triangular elements with 226 nodal points to model the laminate boundary region through a crosssection of quasi-three-dimensional boundary value problem with orthotropic material properties only. The functional dependence of the assumed displacement field was of the same type as in Pipes and Pagano's analysis [1970]. The traction-free boundary conditions cannot be satisfied in this analysis.

A quasi-three-dimensional finite element analysis was carried out by Raju and Crew [1981] using eight-noded isoparametric elements. In order to approximate the stress singularities, a polar mesh was introduced near the intersection of interface and free-edge and a log-linear relationship between stress components of σ_{xz} and σ_{zz} and the radial distance for the singular power was postulated. The power of singularity was determined by fitting a straight line to log-linear plots of stresses calculated from several mesh refinements near the interface of the free-edge and the radial distance.

Whitcomb, Raju and Goree [1982] further pointed out that the disagreement for both magnitude and sign of the interlaminar normal stress distribution among various numerical methods could be attributed to the unsymmetric stress tensor at the singularity. In their approach, the problem was modeled by eight-noded isoparametric elements. It was concluded that finite element displacement models were capable of giving accurate stress distributions everywhere except in the region within two elements of a stress singularity. In this analysis, the traction-free boundary conditions cannot be satisfied.

Chang [1987] used pseudo-three-dimensional finite element analysis to solve composite coupons by satisfying the traction-free boundary conditions, continuity of displacements and tractions at interfaces. However, the stress-equilibrium relations were not satisfied. Dandan [1988] developed a finite element analysis of laminated composite axisymmetric solids and used it to solve for stress distribution in laminated composite coupons. However, in this work, continuity of tractions was not satisfied.

Rybicki [1971] used a three-dimensional equilibrium finite element analysis procedure, based upon minimization of complementary energy, to solve the free-edge stress problem. However, this method involved very large matrices and was computationally expensive, and even at that did not yield a continuous stress field.

In Pian's hybrid model [1969], stress equilibrium in the interior of the elements as well as displacement continuity along interelement boundaries are ensured, but the interelement stress continuity is satisfied only in a weighted integral sense. Following Pian's formulation, Spilker [1980] developed a special hybrid element for the edge-stress problem in cross-ply laminates. In his work, the assumed stress field was made to satisfy exactly the continuity of traction across interlayer boundaries as well as traction-free conditions along exterior planes of the laminate. A comparison of various methods for solving the FED (Free-Edge Delamination) problems is given in Table 2.

Table 2: A Comparison of Various Methods for Solving the FED Problems

Ref	Author	method of analysis	calculated stresses
28	Puppo & Evensen	Elastic approximation	$\sigma_x\sigma_y\tau_{xz}\tau_{xy}$
26	Pipes & Pagano	Approximate elastic solution	$\sigma_x\tau_{xy}\tau_{xz}\tau_{yz}$
41	Tang & Levy	Boundary Layer theory	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$
20	Pagano	Reissner's variational principle--mixed method	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$
22	Pagano & Soni	Global-local model	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$
25	Pipes & Pagano	Finite difference method	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$
43	Wang & Crossman	Finite element method: constant strain triangle	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$
45	Whitcomb et al.	Finite element method: 8-noded isoparametric element	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$
32	Rybicki	Finite element method: equilibrium stress approach	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$
36	Spilker	Finite element method: hybrid assumed stress model	$\sigma_y\sigma_z\tau_{yz}$
5	Chang	Finite element method: Q23 element	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$
6	Dandan	Finite element method: Axisymmetric element	$\sigma_x\sigma_y\sigma_z\tau_{yz}\tau_{xz}\tau_{xy}$

2.3 Need for Solution Procedures

Recent development in the analysis of composite laminate coupons under uniform extension indicated that the high interlaminar stresses near the free edge are mainly responsible for delamination failure, [Pagano and Pipes 1973]. Before delamination can be predicted on the basis of a stress-based failure criterion, it is essential that a highly reliable estimate of interlaminar stresses be available for the given situation. However, it has been difficult to obtain solutions for the stress field because of the anisotropy as well as heterogeneity of the material, and the difficulty in satisfying traction-free boundary condition in a solution procedure based on the displacement formulation.

Some of the solution techniques are only applicable under certain conditions. For this reason, a complete stress distribution is hard to obtain. Although results calculated from various approaches have demonstrated similarities in some cases, discrepancies do exist in the magnitude as well as sign of the computed interlaminar stresses near the free edge of laminate coupons. One possible source of these discrepancies is that, in these methods, the continuity conditions for displacements and tractions across laminate interfaces along with traction-free boundary condition along free-edges characteristic can only be approximated to a limited extent. Chang [1987] solved the problem of a free-edge specimen by a pseudo-three-dimensional finite element procedure based on minimization of potential energy formulation while satisfying continuity of tractions and displacement. However, this theory is not general enough to apply to a laminated plate. Also the stress-equilibrium relation of composite layers is violated. Dandan [1988] used axisymmetric elements to solve the problem of free-edge coupons while continuity of tractions was not satisfied. Table 3 gives a comparison of theory of Pagano [1978] with those of Chang [1987], Hong [1988], and Dandan [1988].

In Hong [1988] theory, the interfacial transverse stresses are not considered as the direct stresses in the variational formulation. The tractions at the interfaces are assumed to be continuous in the sense of interpolating traction components from one interface to another. The traction-free boundary conditions also are not satisfied. Pagano's [1978] approximate theory for a general composite laminate, based on an application of Reissner's variational principle to define the six stress components, has been the basis of solutions to some problems and has been used as bench mark for

Table 3: A Comparison of Theories by Pagano, Chang, Hong, and Dandan

	Displacement Continuity	Traction Continuity	Application to Coupon	Application to Laminated Plate
Pagano [1978]	yes	yes	yes	yes
Pagano [1983]	yes	yes	yes	yes
Chang [1987]	yes	yes	yes	no
Hong [1988]	yes	yes	yes	yes
Dandan [1988]	yes	no	yes	no

validation of finite element procedures. This theory appears to be an excellent candidate for successful determination of stresses in composite laminates. The finite element method can handle a problem of complex boundary conditions in a straight-forward manner and modern computers can easily handle a large number of algebraic equations. To apply the finite element method effectively and to develop alternative strategies, it is desirable to write the governing equations in self-adjoint form so that the general procedure for development of variational principles for coupled linear problems can be applied.

SECTION III

PAGANO'S THEORY OF LAMINATED PLATES

3.1 Introduction

In this section, equations of Pagano's theory [1978] of laminated plates are summarized. Starting from the equilibrium of an elastic solid, equations of generalized equilibrium in two dimensions are introduced. Pagano's theory [1978], based on an assumed equilibrium stress field, is stated along with derivation of constitutive relations for generalized displacements.

3.2 Equilibrium of an Elastic Solid

For a three-dimensional solid, the differential equations of equilibrium for linear elastostatics are

$$\sigma_{ij,j} + f_i = 0 \tag{12}$$

where σ_{ij} are the components of the symmetric Cauchy stress tensor and f_i are the components of the body force vector per unit volume.

3.2.1 Generalized Equilibrium of a Two Dimension Plate

We consider a plate of uniform thickness h in which the plate is assumed to be homogeneous, linear elastic. For the Cartesian reference frame used, the origin is located in the midsurface of the plate (x_1-x_2 axes) with x_3 axis normal to this plane, but the range of x_3 is limited to the thickness of the plate i.e., $x_3 = \pm\frac{h}{2}$.

To reduce the equilibrium to an equality in two dimensions, the equilibrium equations are integrated over the transverse dimension. Equation (12) and its first moment are integrated over the plate thickness. Neglecting the body force components:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta,\beta} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha 3,3} dx_3 = 0 \tag{13}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{3\alpha,\alpha} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{33,3} dx_3 = 0 \tag{14}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta,\beta} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha 3,3} x_3 dx_3 = 0 \tag{15}$$

Defining

$$V_\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{3\alpha} dx_3 \tag{16}$$

$$N_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} dx_3 \tag{17}$$

$$M_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta}\, x_3\, dx_3 \tag{18}$$

$$\sigma_{i3}^{+} = \sigma_{i3}\left(\frac{h}{2}\right) \tag{19}$$

$$\sigma_{i3}^{-} = \sigma_{i3}\left(-\frac{h}{2}\right) \tag{20}$$

The integral form of the generalized equilibrium equations (13), (14), and (15) may be written as

$$N_{\alpha\beta,\beta} + (\sigma_{\alpha 3}^{+} - \sigma_{\alpha 3}^{-}) = 0 \tag{21}$$

$$V_{\alpha,\alpha} + (\sigma_{33}^{+} - \sigma_{33}^{-}) = 0 \tag{22}$$

$$M_{\alpha\beta,\beta} + \frac{h}{2}(\sigma^{+}_{\alpha 3} + \sigma^{-}_{\alpha 3}) - V_{\alpha} = 0 \tag{23}$$

where V_{α}, $N_{\alpha\beta}$, $M_{\alpha\beta}$ are components of the force resultants.

3.3 Kinematics

For small deformation, the kinematic relations for linear elasticity are:

$$\epsilon_{ij} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}) \equiv u_{(i,j)} \tag{24}$$

where ϵ_{ij} is the symmetric strain tensor. For a laminated plate subject to bending and stretching, in order to reduce the problem to one in two dimensions, the functional dependence of the displacements upon the transverse coordinate x_3 is made explicit. Often, the in-plane displacements are assumed to vary linearly within the plate. Mathematically, for a plate, this can be expressed as

$$u_{\alpha}(x_i) = \bar{v}_{\alpha}(x_{\beta}) + x_3 \bar{\Phi}_{\alpha}(x_{\beta}) \tag{25}$$

where u_{α} are the components of inplane displacement vector; $\bar{v}_{\alpha}$ are the displacements at the midsurface of the plate; and $\bar{\Phi}_{\alpha}$ are the rotations of the crosssection of the plate. Substituting (25) into (24), the strain-displacement relations for the plate become

$$\epsilon_{\alpha\beta} = \epsilon^{(0)}_{\alpha\beta} + x_3 \kappa_{\alpha\beta} \tag{26}$$

$$\epsilon_{\alpha 3} = \frac{1}{2}(u_{\alpha,3} + u_{3,\alpha}) \tag{27}$$

$$\epsilon_{33} = u_{3,3} \tag{28}$$

where we define:

$$\epsilon^{(0)}_{\alpha\beta} \equiv \frac{1}{2}(\bar{v}_{\alpha,\beta} + \bar{v}_{\beta,\alpha}) \equiv \bar{v}_{(\alpha,\beta)} \tag{29}$$

$$\kappa_{\alpha\beta} \equiv \frac{1}{2}(\bar{\Phi}_{\alpha,\beta} + \bar{\Phi}_{\beta,\alpha}) \equiv \bar{\Phi}_{(\alpha,\beta)} \tag{30}$$

Many approximate theories use (25) or higher order polynomials in x_3 as the starting point. In Pagano's theory, (24) is used as the basis of evaluation of generalized displacements from strains defined through constitutive relations by the assumed equilibrium stress field in each lamina. No independent assumptions regarding variation of displacements over a layer or over the laminate are made. This is discussed in Section 3.6.

3.4 Constitutive Relations of a Monoclinic Material

For linear elastic material, the general stress-strain relationship is

$$\sigma_{ij} = E_{ijkl}\epsilon_{kl} \tag{31}$$

where ϵ_{kl}, E_{ijkl} are the components of the infinitesimal strain tensor and the isothermal elasticity tensor, respectively. In the absence of body couples, due to symmetry of σ_{ij} and ϵ_{kl} and the existence of an energy function

$$E_{ijkl} = E_{jikl} = E_{klij} = E_{ijlk} \tag{32}$$

and the number of independent constants is 21. For a monoclinic plate, with symmetry about $x_3 = 0$, the number of independent constants is 13 and the reduced stress-strain relationship can be written in the form

$$\left.\begin{aligned} \sigma_{\alpha\beta} &= E_{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta} + E_{\alpha\beta 33}\epsilon_{33} \\ \sigma_{\alpha 3} = \sigma_{3\alpha} &= 2E_{\alpha 3\beta 3}\epsilon_{\beta 3} \\ \sigma_{33} &= E_{33\alpha\beta}\epsilon_{\alpha\beta} + E_{3333}\epsilon_{33} \end{aligned}\right| \tag{33}$$

Inverse relationship of (33) can be written as

$$\left.\begin{aligned} \epsilon_{\alpha\beta} &= S_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} + S_{\alpha\beta 33}\sigma_{33} \\ \epsilon_{\alpha 3} = \epsilon_{3\alpha} &= 2S_{\alpha 3\beta 3}\sigma_{\beta 3} \\ \epsilon_{33} &= S_{33\alpha\beta}\sigma_{\alpha\beta} + S_{3333}\sigma_{33} \end{aligned}\right| \tag{34}$$

where S_{ijkl} are the components of the elastic compliance tensor with the properties

$$S_{ijkl} = S_{klij} = S_{jikl} = S_{ijlk}$$

3.5 An Approximate theory

3.5.1 Equilibrium Stress Field

Equations (16) through (18) express V_α , $N_{\alpha\beta}$, $M_{\alpha\beta}$ as force resultants of the components of the stress tensor, determined over the plate thickness. The inverse relationship, i.e., stress distribution for given V_α , $N_{\alpha\beta}$, $M_{\alpha\beta}$ is not uniquely defined. However, if an assumption is made regarding distribution of some of the components of σ_{ij} , the distribution of the others can be determined. For homogeneous plate, Reissner assumed linear distribution of $\sigma_{\alpha\beta}$ over the thickness. If $\sigma_{\alpha\beta} = \bar{\sigma}_{\alpha\beta} + H_{\alpha\beta}x_3$, substitution in (17) and (18) give

$$\bar{\sigma}_{\alpha\beta} = \frac{N_{\alpha\beta}}{h} \tag{35}$$

and

$$H_{\alpha\beta} = \frac{12M_{\alpha\beta}}{h^3} \tag{36}$$

i.e.

$$\sigma_{\alpha\beta} = \bar{\sigma}_{\alpha\beta} + \frac{12M_{\alpha\beta}x_3}{h^3} \tag{37}$$

The first two equilibrium equations, in the absence of body force, in (12) are

$$\sigma_{\alpha\beta,\beta} + \sigma_{\alpha3,3} = 0 \tag{38}$$

Integrating (38) over the thickness of the plate,

$$\sigma_{\alpha3} = \int_{-\frac{h}{2}}^{x_3} (-\sigma_{\alpha\beta,\beta})dx_3 + \sigma^-_{\alpha3} \tag{39}$$

Substitution of (37) into (39) gives

$$\sigma_{\alpha3} = \int_{-\frac{h}{2}}^{x_3} (-\frac{12x_3}{h^3}M_{\alpha\beta,\beta})dx_3 + \sigma^-_{\alpha3} - (x_3 + \frac{h}{2})\,\bar{\sigma}_{\alpha\beta,\beta} \tag{40}$$

Substitution of $M_{\alpha\beta,\beta}$ from (23) into (40), we have

$$\sigma_{\alpha 3} = \int_{-\frac{h}{2}}^{x_3} -\frac{12x_3[V_\alpha - \frac{h}{2}(\sigma^+_{\alpha 3} + \sigma^-_{\alpha 3})]}{h^3} dx_3 + \sigma^-_{\alpha 3} - (x_3 + \frac{h}{2})\,\bar{\sigma}_{\alpha\beta,\beta} \tag{41}$$

From (41), we have

$$\sigma_{\alpha 3} = \frac{3}{2h}[V_\alpha - \frac{h}{2}S_\alpha][1 - (\frac{2x_3}{h})^2] + \sigma^-_{\alpha 3} + (x_3 + \frac{h}{2})\frac{P_\alpha}{h} \tag{42}$$

Here we define

$$P_\alpha = \sigma^+_{\alpha 3} - \sigma^-_{\alpha 3}$$

and

$$S_\alpha = \frac{h}{2}(\sigma^+_{\alpha 3} + \sigma^-_{\alpha 3})$$

From the third equilibrium equation in (12), integrating along the x_3 axis yields

$$\sigma_{33} = \sigma^-_{33} - \int_{-\frac{h}{2}}^{x_3} (\sigma_{\alpha 3,\alpha})dx_3 \tag{43}$$

Substitution of (42) into (43), equation (43), combining with (43), yields

$$\sigma_{33} = \frac{(\sigma^+_{33} + \sigma^-_{33})}{2} - \{(\sigma^+_{\alpha 3,\alpha} - \sigma^-_{\alpha 3,\alpha})(\frac{x_3^2}{2h} - \frac{h}{8}) + (\sigma^+_{\alpha 3,\alpha} + \sigma^-_{\alpha 3,\alpha})[\frac{x_3^3}{h^2} - \frac{x_3}{4}]$$

$$+ \frac{3}{2}(\sigma^-_{33} - \sigma^+_{33})[\frac{x_3}{h} - \frac{4x_3^3}{3h^3}]\} \tag{44}$$

Following Reissner's assumption that in-plane stress components are linear functions of x_3 coordinate, Pagano assumed the in-plane stress distribution to be given by (37) with (35) i.e.,

$$\sigma_{\alpha\beta} = \frac{N_{\alpha\beta}}{h} + \frac{12x_3M_{\alpha\beta}}{h^3} \tag{45}$$

but derived expressions for $\sigma_{\alpha 3}$ and σ_{33} in a form somewhat different from Reissner's.

3.6 Pagano's Equations of Equilibrium

Defining

$$N_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{33}\, dx_3 \tag{46}$$

$$M_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{33}\, x_3\, dx_3 \tag{47}$$

substitution of (44) into (46) and (47), with consideration of (22), yields

$$N_{33} = \frac{h}{2}(\sigma^{+}_{33} + \sigma^{-}_{33}) + \frac{h^2}{12}(\sigma^{+}_{\alpha 3,\alpha} - \sigma^{-}_{\alpha 3,\alpha}) \tag{48}$$

$$M_{33} = \frac{h^3}{120}(\sigma^{+}_{\alpha 3,\alpha} + \sigma^{-}_{\alpha 3,\alpha}) + \frac{h^2}{10}(\sigma^{+}_{33} - \sigma^{-}_{33}) \tag{49}$$

Equation (41), with (35) and (22), gives

$$\sigma_{\alpha 3} = (\sigma^{+}_{\alpha 3} - \sigma^{-}_{\alpha 3})\frac{x_3}{h} + \frac{(\sigma^{+}_{\alpha 3} + \sigma^{-}_{\alpha 3})}{4}\,(12\frac{x_3^2}{h^2} - 1) + \frac{3V_\alpha}{2h}(1-\frac{4x_3^2}{h^2}) \tag{50}$$

Substituting (48) and (49) into (44), we have

$$\sigma_{33} = \frac{1}{4}(\sigma^{+}_{33} + \sigma^{-}_{33})\,[\frac{12x_3^2}{h^2}-1] + \frac{1}{4}(\sigma^{+}_{33} - \sigma^{-}_{33})\,[\frac{40x_3^3}{h^3}-\frac{6x_3}{h}]$$

$$+ \frac{3N_{33}}{2h}[1-\frac{4x_3^2}{h^2}] + \frac{15M_{33}}{h^2}[\frac{2x_3}{h}-\frac{8x_3^3}{h^3}] \tag{51}$$

Adding (22) and $\frac{20}{h^2}$ times (49)

$$V_{\alpha,\alpha} + \frac{20M_{33}}{h^2} + (\sigma^{-}_{33} - \sigma^{+}_{33}) - \frac{h}{6}(\sigma^{+}_{\alpha 3,\alpha} + \sigma^{-}_{\alpha 3,\alpha}) = 0 \tag{52}$$

Adding (22) and $\frac{60}{h^2}$ times (49)

$$V_{\alpha,\alpha} + \frac{60M_{33}}{h^2} + 5(\sigma^{-}_{33} - \sigma^{+}_{33}) - \frac{h}{2}(\sigma^{+}_{\alpha 3,\alpha} + \sigma^{-}_{\alpha 3,\alpha}) = 0 \tag{53}$$

These two equations may be used to replace (22) and (49). Summarizing, Pagano's equations of plate equilibrium are:

$$N_{\alpha\beta,\beta} + (\sigma^{+}_{\alpha 3} - \sigma^{-}_{\alpha 3}) = 0 \tag{54}$$

$$V_{\alpha,\alpha} + \frac{20M_{33}}{h^2} + (\sigma^{-(k)}_{33} - \sigma^{+(k)}_{33}) - \frac{h}{6}(\sigma^{+(k)}_{\alpha 3,\alpha} + \sigma^{-(k)}_{\alpha 3,\alpha}) = 0 \tag{55}$$

$$M_{\alpha\beta,\beta} - V_{\alpha} + \frac{h}{2}(\sigma^{-}_{\alpha 3} + \sigma^{+}_{\alpha 3}) = 0 \tag{56}$$

We note here the introduction of stress resultant M_{33}. This and the force resultant N_{33} are defined by (47), (46) and, for the assumption of $\sigma_{\alpha\beta}$ linear over the plate, by approximation equations (48) and (49). Equation (49), as we have shown, may be replaced by (53) yielding the following equations as completely defining N_{33}, M_{33} for Pagano's theory:

$$N_{33} - \frac{h(\sigma^{-}_{33} + \sigma^{+}_{33})}{2} + \frac{h^2}{12}(\sigma^{-}_{\alpha 3,\alpha} - \sigma^{+}_{\alpha 3,\alpha}) = 0 \tag{57}$$

$$V_{\alpha,\alpha} + \frac{60M_{33}}{h^2} + 5(\sigma^{-}_{33} - \sigma^{+}_{33}) - \frac{h}{2}(\sigma^{-}_{\alpha 3,\alpha} + \sigma^{+}_{\alpha 3,\alpha}) = 0 \tag{58}$$

3.7 Constitutive Relations for Generalized Displacements

Substitution of (45), (50), and (51), taking account of (34), into the variational equation of Reissner Complementary Energy and integration with respect to x_3 leads to the appropriate field equations and boundary conditions. In the derivation of the governing equations, the integration with respect to x_3 gives rise to weighted average displacements at the surface of each layer. Pagano introduced the notation

$$
\left.
\begin{aligned}
\bar{f} &= \int_{-h/2}^{h/2} f \, \frac{2}{h} \, dx_3 \\
f^{*} &= \int_{-h/2}^{h/2} f \, \frac{2x_3}{h} \, \frac{2}{h} \, dx_3 \\
\hat{f} &= \int_{-h/2}^{h/2} f \, \frac{4x_3^2}{h^2} \, \frac{2}{h} \, dx_3
\end{aligned}
\right| \qquad (59)
$$

where f may represent any function of x_3 The constitutive relations given by Pagano [1978], neglecting the expansional strains, were derived by integrating with respect to x_3 the quantities $\epsilon_{\alpha\beta}$, $x_3\epsilon_{\alpha\beta}$, ϵ_{33}, $x_3\epsilon_{33}$, $x_3^2\epsilon_{33}$, $x_3^3\epsilon_{33}$, $\epsilon_{\alpha 3}$, $x_3\epsilon_{\alpha 3}$, and $x_3^2\epsilon_{\alpha 3}$. In evaluation of the integrals, repeated use of integration by parts and substitution for stresses $\sigma_{\alpha 3}$, σ_{33} for equation (50), (51) was required. The resulting equations are:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \epsilon_{\alpha\beta} dx_3 \;=>\; \bar{u}_{(\alpha,\beta)} = \frac{2}{h}(S_{\alpha\beta\gamma\delta}N_{\gamma\delta} + S_{\alpha\beta 33}N_{33}) \tag{60}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\epsilon_{\alpha\beta} dx_3 \;=>\; u^{*}_{(\alpha,\beta)} = \frac{4}{h^2}(S_{\alpha\beta\gamma\delta}M_{\gamma\delta} + S_{\alpha\beta 33}M_{33}) \tag{61}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \epsilon_{33} dx_3 \;=>\; u_3^+ - u_3^- = S_{33\alpha\beta}N_{\alpha\beta} + S_{3333}N_{33} \tag{62}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\epsilon_{33} dx_3 \;=>\; \bar{u}_3 = u_3^+ + u_3^- - \frac{2}{h}S_{33\alpha\beta}M_{\alpha\beta} - \frac{2}{h}S_{3333}M_{33} \tag{63}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3^2\epsilon_{33} dx_3 \;=>\; 6u_3^{*} = -S_{33\alpha\beta}N_{\alpha\beta} - \frac{h}{5}S_{3333}(\sigma_{33}^+ + \sigma_{33}^-) - \frac{3}{5}S_{3333}N_{33}$$

$$+ 3(u_3^+ - u_3^-) \tag{64}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3^3 \epsilon_{33}\, dx_3 \;=>\; \hat{u}_3 = \frac{1}{3}(u_3^+ + u_3^-) - \frac{2}{5h} S_{33\alpha\beta} M_{\alpha\beta} - \frac{2}{7h} S_{3333} M_{33}$$

$$- \frac{h}{105} S_{3333}(\sigma_{33}^+ - \sigma_{33}^-) \tag{65}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \epsilon_{\alpha 3}\, dx_3 \;=>\; \bar{u}_{3,\alpha} = \frac{8}{h} S_{\alpha 3\beta 3} V_\beta - \frac{2}{h}(u_\alpha^+ - u_\alpha^-) \tag{66}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \epsilon_{\alpha 3}\, dx_3 \;=>\; \frac{h}{4} u^*_{3,\alpha} - \frac{1}{2}\bar{u}_\alpha = \frac{h}{3} S_{\alpha 3\beta 3}(\sigma_{\beta 3}^+ - \sigma_{\beta 3}^-) - \frac{1}{2}(u_\alpha^+ + u_\alpha^-) \tag{67}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3^2 \epsilon_{\alpha 3}\, dx_3 \;=>\; \hat{u}_{3,\alpha} = \frac{8}{15} S_{\alpha 3\beta 3}(\sigma_{\beta 3}^+ + \sigma_{\beta 3}^-) + \frac{8}{5h} S_{\alpha 3\beta 3} V_\beta + \frac{4}{h} u^*_\alpha$$

$$- \frac{2}{h}(u_\alpha^+ - u_\alpha^-) \tag{68}$$

The above equations contain quantities u_i^+ and u_i^- which are apriori unknowns. Combining (63) and (65) to eliminate $u_3^+ + u_3^-$ gives

$$3\hat{u}_3 - \bar{u}_3 = \frac{4}{5h} S_{33\alpha\beta} M_{\alpha\beta} + \frac{8}{7h} S_{3333} M_{33} - \frac{h}{35} S_{3333}(\sigma_{33}^+ - \sigma_{33}^-) \tag{69}$$

Combining (62) and (64) to eliminate $u_3^+ - u_3^-$ yields

$$6u_3^* = 2S_{33\alpha\beta} N_{\alpha\beta} + \frac{12}{5} S_{3333} N_{33} - \frac{h}{5} S_{3333}(\sigma_{33}^+ + \sigma_{33}^-) \tag{70}$$

Combining (66) and (68) to eliminate $u_\alpha^+ - u_\alpha^-$ gives

$$\hat{u}_{3,\alpha} - \bar{u}_{3,\alpha} - \frac{4}{h} u_\alpha^* = \frac{8}{15} S_{\alpha 3\beta 3}(\sigma_{\beta 3}^+ + \sigma_{\beta 3}^-) - \frac{32}{5h} S_{\alpha 3\beta 3} V_\beta \tag{71}$$

Equations (60), (61), (69), (70), and (71) are Pagano's constitutive equations for displacement functions in terms of force resultants and surface tractions.

3.7.1 Equations of Displacement at the interfaces and Interface Conditions

Equations (46) through (71) apply to each lamina in a laminate. The positive or negative signs of the superscript denote, respectively, the top or the bottom surface of the lamina.

3.7.1.1 Interface Displacement Equations

The displacement at the interface can be obtained [Pagano 1978] by combining some of the equations (60) through (71). Equation (67) gives $u_\alpha^+ + u_\alpha^-$ in terms of displacement functions and surface tractions as

$$u_\alpha^+ + u_\alpha^- = -\frac{h}{2}\dot{u}_{3,\alpha} + \bar{u}_\alpha + \frac{2h}{3}S_{\alpha3\beta3}(\sigma_{\beta3}^+ - \sigma_{\beta3}^-) \tag{72}$$

Equations (66) and (68) both involve $u_\alpha^+ - u_\alpha^-$. Combining these two equations gives

$$u_\alpha^+ - u_\alpha^- = 2h[-\frac{3}{8}\hat{u}_{3,\alpha} + \frac{1}{8}\bar{u}_{3,\alpha} + \frac{3}{2h}\dot{u}_\alpha] + \frac{2h}{5}S_{\alpha3\beta3}(\sigma_{\beta3}^+ + \sigma_{\beta3}^-) - \frac{4}{5}S_{\alpha3\beta3}V_\beta \tag{73}$$

From (72) and (73), the in-plane displacements at the interfaces are:

$$u_\alpha^+ = -h(\frac{3}{8}\hat{u}_{3,\alpha} - \frac{1}{8}\bar{u}_{3,\alpha} - \frac{3}{2h}\dot{u}_\alpha) - (\frac{h}{4}\dot{u}_{3,\alpha} - \frac{1}{2}\bar{u}_\alpha) + 4S_{\alpha3\beta3}[\frac{(4\sigma_{\beta3}^+ - \sigma_{\beta3}^-)h}{30} - \frac{V_\beta}{10}] \tag{74}$$

$$u_\alpha^- = h(\frac{3}{8}\hat{u}_{3,\alpha} - \frac{1}{8}\bar{u}_{3,\alpha} - \frac{3}{2h}\dot{u}_\alpha) - (\frac{h}{4}\dot{u}_{3,\alpha} - \frac{1}{2}\bar{u}_\alpha) - 4S_{\alpha3\beta3}[\frac{(4\sigma_{\beta3}^- - \sigma_{\beta3}^+)h}{30} - \frac{V_\beta}{10}] \tag{75}$$

Clearly, other combinations of (66) and (68) are possible. Combining (62) and (64) to eliminate $N_{\alpha\beta}$ yields

$$u_3^+ - u_3^- = 3\dot{u}_3 - \frac{1}{5}S_{3333}N_{33} + \frac{h}{10}S_{3333}(\sigma_{33}^+ + \sigma_{33}^-) \tag{76}$$

Combining (63) and (65) to eliminate $M_{\alpha\beta}$ leads to

$$u_3^+ + u_3^- = \frac{15}{2}\hat{u}_3 - \frac{3}{2}\bar{u}_3 - \frac{6}{7h}S_{3333}M_{33} + \frac{h}{14}S_{3333}(\sigma_{33}^+ - \sigma_{33}^-) \tag{77}$$

From (76) and (77), out-of-plane displacements at the interfaces are:

$$u_3^+ = \frac{3}{4}(5\hat{u}_3 - \bar{u}_3) + \frac{3}{2}u_3^* + \frac{S_{3333}}{70h}[(6\sigma_{33}^+ + \sigma_{33}^-)h^2 - 7hN_{33} - 30M_{33}] \tag{78}$$

$$u_3^- = \frac{3}{4}(5\hat{u}_3 - \bar{u}_3) - \frac{3}{2}u_3^* - \frac{S_{3333}}{70h}[(6\sigma_{33}^- + \sigma_{33}^+)h^2 - 7hN_{33} + 30M_{33}] \tag{79}$$

3.7.1.2 Interface Continuity

Using superscripts in parentheses to denote the identifying lamina number, the condition of continuity of displacement and traction at the interfaces implies that for k=1, 2, 3, ., . . . N-1,

$$\sigma_{i3}^{-(k)} = \sigma_{i3}^{+(k+1)} \tag{80}$$

$$u_i^{(k)}(-t_k/2) = u_i^{(k+1)}(t_{k+1}/2) \tag{81}$$

where t_k is the thickness of the kth layer.

3.7.1.3 Prescribed Tractions

On interfacial planes, tractions or displacements may be specified [Pagano 1978] in the case of a cracked or unbonded interfacial region. Symbolically, if tractions are specified,

$$\left.\begin{aligned} \sigma_{i3}^{-(k)} &= \tilde{\sigma}_{i3}^{-(k)} \\ \sigma_{i3}^{+(k+1)} &= \tilde{\sigma}_{i3}^{+(k+1)} \end{aligned}\right| \tag{82}$$

for k=1, 2, 3, ., . . . N-1

3.7.1.4 Prescribed Displacements

Pagano [1978] allowed for specification of displacements at interfaces i.e., for k=1, 2, 3, ., . . . N-1,

$$\left.\begin{aligned} u_i^{(k)}(-t_k/2) &= \tilde{u}_i^{(k)} \\ u_i^{(k+1)}(t_{k+1}/2) &= \tilde{u}_i^{(k+1)} \end{aligned}\right\} \tag{83}$$

The field equations, which consist of the elastic constitutive realtions, equations of equilibrium, and equations of continuity in displacements and tractions, must be satisfied within each layer for the composite laminated plate.

SECTION IV

VARIATIONAL FORMULATION OF PAGANO'S THEORY OF LAMINATED PLATES

4.1 Introduction

To implement the theory described in Section III in a finite element analysis, a self-adjoint form of the governing equations along with consistent boundary conditions is desirable so that Ritz type variational formulation can be employed. In this section we outline the basic procedure developed by Sandhu and Salaam [1975] and Sandhu [1976] for variational formulation of linear problems and then proceed to apply it to Pagano's theory of laminated composites.

It is shown that Pagano's equations as originally stated and described in Section III do not readily lend themselves to a self-adjoint formulation. A modification, reducing the number of field variables, is introduced to ensure a self-adjoint formulation. A general variational principle for the problem is stated. Extended variational principles are developed using self-adjointness, in the sense of the bilinear mapping used, of the operator matrix. Specializations to reduce the number of field variables by requiring that some of the field equations be satisfied exactly are derived. The general approach provides a basis for development of consistent finite element approximation procedures.

4.2 Basic Variational Principles

In order to develop variational formulations for the multivariate problem of laminated composites, it is necessary to introduce some definitions and procedures. We summarize here the approach introduced by Sandhu [1976] for a self-adjoint linear operator.

4.2.1 Boundary Value Problem

A typical boundary value problem is defined by the set of equations

$$\left.\begin{aligned} A(u) &= f \qquad \text{on } R\,, \\ C(u) &= g \qquad \text{on } S \end{aligned}\right| \tag{84}$$

where R is an open connected bounded region in an euclidean space, S is the boundary of R , and $\bar{R}$ is the closure of R. A and C are linear bounded operators. The operator A is the field operator and C is the boundary operators

$$A : W_R \rightarrow V_R \tag{85}$$

$$C : W_S \rightarrow V_S \tag{86}$$

where V_R, V_S are linear spaces defined over the region R and S and W_R, W_S are in general, dense subsets in V_R, V_S, respectively. Operator A and C are linear implies

$$A(au+bv) = aA(u) + bA(v) \qquad \text{for all } u\,,\, v \in W_R \tag{87}$$

$$C(au+bv) = aC(u) + bC(v) \qquad \text{for all } u\,,\, v \in W_S \tag{88}$$

where a , b are arbitrary scalars.

4.2.2 Bilinear Mapping

A variational formulation of the problem seeks to set up an equivalent problem so that the search for $u_0 \in W$ for known f corresponds to the search for a function F whose stationary points are solutions to the given equations. The function is based on use of suitable bilinear mapping B_R and B_S such that

$$B_R : V_R \times V_R \rightarrow S \tag{89}$$

$$B_S : V_S \times V_S \rightarrow S \tag{90}$$

where V_R, V_S and S are linear vector spaces. To each ordered pair of vectors $u , v \in V$, B assigns a point $B(u , v) \in S$, such that:

$$B(u , v) = B(v , u) \tag{91}$$

$$B(au_1+bu_2 , v) = aB(u_1 , v)+bB(u_2 , v) \tag{92}$$

$$B(u , av_1+bv_2) = aB(u , v_1)+bB(u , v_2) \tag{93}$$

where a, b are scalars. B_R is said to be nondegenerate if

$$B_R(u , v) = 0 \quad \text{for all } v \in V \text{ if and only if } u = 0 \tag{94}$$

A variety of bilinear mapping have been used [Sandhu and Salaam, 1975].

4.2.3 Self-Adjoint Operator

An operator A is said to be adjoint of operator $A:W \rightarrow V$ with respect to a bilinear mapping $B_R(\; , \;) : V \times V \rightarrow S$ if

$$B_R(u , Av) = B_R(v , A^*u) + D_S(v , u) \text{ for all } v \in W , u \in V \tag{95}$$

where $D_S(v , u)$ represents quantities associated with the boundary S of R. If A , A^* are linear, $D_S(u , v)$ is bilinear in its arguments. If $A^* = A$, this A is said to be self-adjoint. If A is a self-adjoint operator, $D_S(u , v)$ is antisymmetric, i.e.,

$$D_S(u , v) = -D_S(v , u) \tag{96}$$

A self-adjoint operator A on V is symmetric with respect to the bilinear mapping if $W = V$ and

$$B_S(u , Av) = B_S(v , Au) \tag{97}$$

Two operators A , E are equal on V if they have the same domain and range and

$$A(u) = E(u) \text{ for all } u \in V$$

4.2.4 Gateaux Differential of A Function

The Gateaux differential of a function $F : V \rightarrow S$ is defined as

$$\Delta_v F(u) = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda}[F(u+\lambda v)-F(u)] \tag{98}$$

provided the limit exists. The quantity $v \in V$ is referred to as the path, λ is a scalar, and if $u, v \in V$ then $u+\lambda v \in V$. If $\Delta_v F(u)$ exists at each point in a neighborhood of u, (98) can be equivalently written as

$$\Delta_v F(u) = \frac{d}{d\lambda} F(u+\lambda v)_{|\lambda=0} \tag{99}$$

If the Gateaux differential is linear in v and there exists $G(u) \in W$, a linear vector space, and a bilinear mapping $B : V \times W \rightarrow S$ such that

$$\Delta_v F(u) = B(v\ ,\ G(u)) \tag{100}$$

G(u) may be regarded as the gradient of F at u. If B is continuous and non-degenerate and G(u) can be identified with the residual P(u) at u, F(u) is the potential of the operator P. This is the basis for setting up variational formulations.

4.2.5 Variational Principles For Linear Operators

For boundary value problem with homogeneous boundary conditions, Miklhin showed that for self-adjoint, positive definite operator A, the unique solution u_0 minimizes the function

$$\Omega(u) = B_R(u\ ,\ Au) - B_R(u\ ,\ 2f) \tag{101}$$

Here the inner product was used as the bilinear mapping. Conversely, u_0 which minimizes the function of (101) is the solution of the problem defined by the set of (84). Gateaux differential of (101) yields

$$\begin{aligned} \Delta_v \Omega(u) &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda}[B(u+\lambda v\ ,\ A(u+\lambda v)) - B(u+\lambda v\ ,\ 2f) - B(u\ ,\ Au) + B(u\ ,\ 2f)] \\ &= B(u\ ,\ Av) + B(v\ ,\ Au) - 2B(v\ ,\ f) \\ &= 2B(v\ ,\ Au-f) = 0 \end{aligned} \tag{102}$$

if A is symmetric

In addition to the symmetry of the bilinear mapping, only linearity and self-adjointness of the operator A are assumed in writing (102). The Gateaux differential evidently vanishes at the solution u_0 such that $Au_0-f = 0$ In order that vanishing of the Gateaux differential at $u = u_0$ imply $Au_0-f = 0$, the bilinear mapping has to be non-degenerate. To prove the minimization property, the bilinear mapping has to be into the real line and the operator must be positive. However, in general, it is only necessary to use vanishing of the Gateaux differential as equivalent to (84) being satisfied. For nonhomogeneous boundary conditions, Sandhu [1975] showed that for a linear self-adjoint A and C consistent with A, an equivalent function to the set of field equations is stated as

$$\Omega(u) = B_R(u\ ,\ Au-2f)+B_S(u\ ,\ Cu-2g) \tag{103}$$

Consistency of boundary operators with field operators is considered in the following sub-section. Sandhu [1975] pointed out that appropriate boundary terms should be included in the governing function even if they are homogeneous.

4.2.6 Coupled Problems

If u is not a single field variable but consists of n dependent field variables, then a linear coupled boundary value problem may be written explicitly as

$$\sum_{j=1}^{n} A_{ij}u_j = f_i \quad \text{on } R \qquad i = 1, 2, \ldots n \tag{104}$$

$$\sum_{j=1}^{n} C_{ij}u_j = g_i \quad \text{on } S \qquad i = 1, 2, \ldots n \tag{105}$$

Here A_{ij} is an element of the matrix of field operators and C_{ij} is an element of the matrix of boundary operators such that:

$$A_{ij} : W_{R_j} \rightarrow V_{R_i} \qquad i, j = 1, 2, \ldots n \tag{106}$$

$$C_{ij} : W_{S_j} \rightarrow V_{S_i} \qquad i, j = 1, 2, \ldots n \tag{107}$$

where W_{R_i} and W_{S_i} are subspaces of V_{R_i} and V_{S_i}, respectively. (103) is completely analogous to (104) through (105) if u is regarded as an n-tuple $u=\{u_i, i=1,\ldots n\}$ and A is a matrix of operators. Similarly C, g, f have extended definitions. Then $u \in V_R$ where V_R is the direct sum $V_R = V_{R_1}+V_{R_2}+ \ldots V_{R_n}$. A bilinear mapping B_R on V_R is defined by

$$B_R(u , v) = \sum_{i=1}^{n} B_{R_i}(u_i , v_i) \tag{108}$$

where B_{R_i} is a bilinear mapping defined on V_{R_i}. The matrix of operators A_{ij} is self-adjoint with respect to the bilinear mapping if

$$\sum_{j=1}^{n} B_{R_i}(u_j , A_{ji}v_i)=B_{R_i}(v_i , \sum_{j=1}^{n} A_{ij}u_j) + D_S(v_i , u_j) , \qquad i = 1, 2, \ldots n \tag{109}$$

The matrix of boundary operators is said to be consistent [Sandhu 1976] with the self-adjoint matrix of field operators if D_S in (109) satisfies

$$D_S(v_i , u_j) = B_{S_i}(v_i , \sum_{j=1}^{n} C_{ij}u_j) - \sum_{j=1}^{n} B_{S_j}(u_j , C_{ji}v_i) \tag{110}$$

Substitution of (110) into (109) results in

$$\sum_{j=1}^{n} B_{R_j}(u_j , A_{ji}v_i) = B_{R_i}(v_i , \sum_{j=1}^{n} A_{ij}u_j) + B_{S_i}(v_i , \sum_{j=1}^{n} C_{ij}u_j) - \sum_{j=1}^{n} B_{S_j}(u_j , C_{ji}v_i) \tag{111}$$

Sandhu [1975] showed that the Gateaux differential of the function defined by (103), along with the extended definitions (108) through (111), vanishes if and only if the (104) and (105) are satisfied. The functions approximating the field variables are required to obey certain continuity requirements so that they are admissible as possible solutions of (104) and (105) i.e. each function u_j lies in the domain of the set of operators $\{A_{ij} , i = 1, 2, \ldots n \}$. However, in seeking approximation to the correct solution by the finite element method, the region under consideration is discretized into a finite number of elements and the field variables are represented by functions which satisfy the continuity conditions only piecewise within each element. If the continuity conditions along the interelement boundary are not satisfied, internal discontinuity conditions, [Sandhu 1976] need to be introduced in the form

$$\sum_{j=1}^{n} (C_{ij} u_j)' = g'_i \quad \text{on} \quad R_i \tag{112}$$

where a superscripted prime denotes the internal jump discontinuity along element boundary R_I embedded in the domain R and g' are the specified values of the jump discontinuities. Sandhu and Salaam [1975] showed that this condition can be included explicitly in the governing function by simply adding a term and defining the bilinear map over R as the sum of maps over individual elements.

4.3 Self-adjointness of Pagano's. Equations

4.3.1 Introduction

The field equations for a single lamina include the equilibrium equations and the constitutive relationships. These are given, respectively, by (21) through (23) and by (60), (61) along with (69) through (71). There are five equations of equilibrium and ten constitutive relationships. For a laminate, however, the interfacial displacements and tractions are additional, apriori unknown, field variables. The six surface displacement components are given in terms of mechanical and kinematic variables by (74), (75), (78), and (79). The field variables must also satisfy the continuity requirement expressed by (80) and (81). Satisfying the constitutive relations (74), (75), (78), (79), the continuity equations are replaced by relationships between mechanical and kinematic field variables. In the following we restate Pagano's equations for a laminate and examine their self-adjointness.

4.3.2 Equilibrium Equations

From (21), (22), and (23), we have, for the kth layer,

$$V_{r,r}^{(k)} + (\sigma_{33}^{+(k)} - \sigma_{33}^{-(k)}) = 0 \tag{113}$$

$$\frac{1}{2}\Gamma_1 N_{\alpha\beta}^{(k)} + (\sigma_{r3}^{+(k)} - \sigma_{r3}^{-(k)}) = 0 \tag{114}$$

$$\frac{1}{2}\Gamma_1 M_{\alpha\beta}^{(k)} - V_r^{(k)} + \frac{t_k}{2}(\sigma_{r3}^{+(k)} + \sigma_{r3}^{-(k)}) = 0 \tag{115}$$

where

$$\Gamma_1 = (\delta_{\alpha r}\frac{\partial}{\partial\beta} + \delta_{\beta r}\frac{\partial}{\partial\alpha}) \tag{116}$$

4.3.3 Constitutive Equations

Equations (60) through (68) are nine sets of constitutive equations for the plate. Equations (62) and (64) are used to set up (70) and (76) and can therefore be replaced by the latter. Similarly (63) and (65) can be replaced by (69), (77) and (66), (68) by (71), (73). Restating (67) as (72) and replacing (72) and (73) by (74), (75), Pagano used (60), (61), (69) through (71), and (74) through (77) as the 16 constitutive equations. Equations (60), (61), and (71) rewritten, for the kth layer, are:

$$\frac{1}{2}\Gamma_2(\frac{\bar{u}_r^{(k)}}{2}) = \frac{1}{t_k}S_{\mu\rho\alpha\beta}^{(k)}N_{\alpha\beta}^{(k)} + \frac{1}{t_k}S_{\mu\rho33}^{(k)}N_{33}^{(k)} \tag{117}$$

$$\frac{1}{2}\Gamma_2(\frac{3u_r^{*(k)}}{t_k})=\frac{12}{t_k^3}S_{\mu\rho\alpha\beta}^{(k)}M_{\alpha\beta}^{(k)}+\frac{12}{t_k^3}S_{\mu\rho33}^{(k)}M_{33}^{(k)} \tag{118}$$

$$\frac{3}{4}(\bar{u}_3^{(k)}-\hat{u}_3^{(k)})_{,\rho}+3\frac{u_\rho^{*(k)}}{t_k} = \frac{24}{5t_k}S_{\rho3r3}^{(k)}V_r^{(k)} - \frac{2}{5}S_{\rho3r3}^{(k)}(\sigma_{r3}^{+(k)} + \sigma_{r3}^{-(k)}) \tag{119}$$

where

$$\Gamma_2 = (\delta_{\mu r}\frac{\partial}{\partial\rho} + \delta_{\rho r}\frac{\partial}{\partial\mu}) \tag{120}$$

4.3.4 Operator Matrix Form of the Field Equations

Combining the equilibrium equations (113) through (115), the definitions (48), (49), and the constitutive equations (117) through (119) along with (69), (70), appropriately modified to apply to the kth layer, the complete set of equations for the kth layer can be written as:

$$[A']^{(k)}\{u'\}^{(k)} + [B']^{(k)}\{\sigma'\}^{-(k)} + [C']^{(k)}\{\sigma'\}^{+(k)} = 0 \qquad k=1\ ,\ 2\ ,\ 3\ ,\ .\ .\ ,\ .\ N \tag{121}$$

where $[A']^{(k)}$, $[B']^{(k)}$, $[C']^{(k)}$ are operator matrices and $\{u'\}^{(k)}$, $\{\sigma'\}^{+(k)}$, $\{\sigma'\}^{-(k)}$ are sets of field variables, respectively. Explicitly,

$$
[A']^{(k)} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{2}\Gamma_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\Gamma_1 & 0 & -\delta_{\gamma\alpha} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial\gamma} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{4S^{(k)}_{33\alpha\beta}}{5t_k} & -\frac{8S^{(k)}_{3333}}{7t_k} & 0 \\
0 & 0 & 0 & 0 & 1 & -2S^{(k)}_{33\alpha\beta} & -\frac{12}{5}S^{(k)}_{3333} & 0 & 0 & 0 \\
-\frac{1}{2}\Gamma_2 & 0 & 0 & 0 & 0 & \frac{S^{(k)}_{\mu\rho\alpha\beta}}{t_k} & \frac{S^{(k)}_{\mu\rho 33}}{t_k} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -\frac{1}{2}\Gamma_2 & 0 & 0 & 0 & 0 & 0 & \frac{12}{t_k^3}S^{(k)}_{\mu\rho\alpha\beta} & \frac{12}{t_k^3}S^{(k)}_{\mu\rho 33} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -\delta_{\gamma\rho} & -\frac{\partial}{\partial\rho} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{24}{5t_k}S^{(k)}_{\rho 3\gamma 3}
\end{pmatrix}
\qquad (122)
$$

$$
[B']^{(k)} = \begin{bmatrix}
-\delta_{\gamma o} & 0 \\
\frac{t_k}{2} & 0 \\
0 & -1 \\
0 & -\frac{t_k}{35} S_{3333}^{(k)} \\
0 & -\frac{t_k}{5} S_{3333}^{(k)} \\
0 & 0 \\
\frac{t_k^2}{12}\frac{\partial}{\partial\gamma} & -\frac{t_k}{2} \\
0 & 0 \\
-\frac{t_k^3}{120}\frac{\partial}{\partial\gamma} & \frac{t_k^2}{10} \\
-\frac{2}{5} S_{\rho 3\gamma 3}^{(k)} & 0
\end{bmatrix} \tag{123}
$$

$$[C']^{(k)} = \begin{bmatrix} \delta_{\gamma\alpha} & 0 \\ \frac{t_k}{2} & 0 \\ 0 & 1 \\ 0 & \frac{t_k}{35} S^{(k)}_{3333} \\ 0 & \frac{t_k}{5} S^{(k)}_{3333} \\ 0 & 0 \\ -\frac{t_k^2}{12}\frac{\partial}{\partial\gamma} & -\frac{t_k}{2} \\ 0 & 0 \\ -\frac{t_k^3}{120}\frac{\partial}{\partial\gamma} & -\frac{t_k^2}{10} \\ -\frac{2}{5} S^{(k)}_{\rho 3\gamma 3} & 0 \end{bmatrix} \tag{124}$$

$$\{\sigma'\}^{\pm(k)} = \begin{Bmatrix} \sigma^{\pm(k)}_{\gamma 3} \\ \sigma^{\pm(k)}_{33} \end{Bmatrix} \qquad k = 1, 2, 3, \ldots, N \tag{125}$$

$$
\{u'\}^{(k)} = \left\{ \begin{array}{c} \dfrac{\bar{u}_\gamma^{(k)}}{2} \\ 3\dfrac{\overset{*}{u}_\gamma^{(k)}}{t_k} \\ \dfrac{3}{4}(\bar{u}_3^{(k)} - \hat{u}_3^{(k)}) \\ 3\hat{u}_3^{(k)} - \bar{u}_3^{(k)} \\ 6\overset{*}{u}_3^{(k)} \\ N_{\alpha\beta}^{(k)} \\ N_{33}^{(k)} \\ M_{\alpha\beta}^{(k)} \\ M_{33}^{(k)} \\ V_\gamma^{(k)} \end{array} \right\} \tag{126}
$$

The ten sets of equations in (121) are, respectively, (114), (115), (113), (69), (70), (117), (48), (118), (49), and (119).

4.3.5 Displacement Continuity Conditions

From the first two equations in (81), we have

$$
u_\alpha^{-(k)} = u_\alpha^{+(k+1)} \tag{127}
$$

Substitution of (74) and (75) into (127) leads to

$$
t_k(\frac{3}{8}\hat{u}_{3,\alpha}^{(k)} - \frac{1}{8}\bar{u}_{3,\alpha}^{(k)} - \frac{3}{2t_k}\overset{*}{u}_\alpha^{(k)}) - (\frac{t_k}{4}\overset{*}{u}_{3,\alpha}^{(k)} - \frac{1}{2}\bar{u}_\alpha^{(k)}) - 4S_{\alpha3\beta3}^{(k)}[\frac{(4\sigma_{\beta3}^{-(k)} - \sigma_{\beta3}^{-(k-1)})t_k}{30} - \frac{V_\beta^{(k)}}{10}]
$$

$$= \quad -t_{k+1}\left(\frac{3}{8}\hat{u}_{3,\alpha}^{(k+1)} - \frac{1}{8}\bar{u}_{3,\alpha}^{(k+1)} - \frac{3}{2t_{k+1}}\dot{u}_{\alpha}^{(k+1)}\right) - \left(\frac{t_{k+1}}{4}\dot{u}_{3,\alpha}^{(k+1)} - \frac{1}{2}\bar{u}_{\alpha}^{(k+1)}\right)$$

$$+ \; 4S_{\alpha 3\beta 3}^{(k+1)}\left[\frac{(4\sigma_{\beta 3}^{-(k)} - \sigma_{\beta 3}^{\cdot(k+1)})t_{k+1}}{30} - \frac{V_{\beta}^{(k+1)}}{10}\right] \tag{128}$$

From the third equations in (81), we have

$$u_3^{-(k)} = u_3^{+(k+1)} \tag{129}$$

Substitution of (78) and (79) into (129) leads to

$$\frac{3}{4}(5\hat{u}_3^{(k)} - \bar{u}_3^{(k)}) - \frac{3}{2}\dot{u}_3^{(k)} - \frac{S_{3333}^{(k)}}{70t_k}[(6\sigma_{33}^{-(k)} + \sigma_{33}^{-(k-1)})t_k^2 - 7t_kN_{33}^{(k)} + 30M_{33}^{(k)}]$$

$$= \frac{3}{4}(5\hat{u}_3^{(k+1)} - \bar{u}_3^{(k+1)}) + \frac{3}{2}\dot{u}_3^{(k+1)}$$

$$+ \; \frac{S_{3333}^{(k+1)}}{70t_{k+1}}[(6\sigma_{33}^{\cdot(k)} + \sigma_{33}^{-(k+1)})t_{k+1}^2 - 7t_{k+1}N_{33}^{(k+1)} - 30M_{33}^{(k+1)}] \tag{130}$$

Adding displacement continuity equations (128), (130) to (121) above, the complete set of equations can be written compactly in the form

$$[L]^{(k)}\{r\}^{(k)} = \{0\} \tag{131}$$

where

$$[L]^{(k)} = \begin{bmatrix} [C']^{(k)} & [A']^{(k)} & [B']^{(k)} & [0] & [0] \\ [H_1]^{(k-1)} & [H_2]^{(k)} & [H_3]^{(k)} & [H_4]^{(k+1)} & [H_5]^{(k+1)} \end{bmatrix} \tag{132}$$

$$\{r\}^{(k)} = \begin{Bmatrix} \{\sigma'\}^{-(k-1)} \\ \{u'\}^{(k)} \\ \{\sigma'\}^{-(k)} \\ \{u'\}^{(k+1)} \\ \{\sigma'\}^{-(k+1)} \end{Bmatrix} \tag{133}$$

with

$$[H_1]^{(k-1)} = \begin{bmatrix} -\frac{2}{15} t_k S^{(k)}_{r3\rho 3} & 0 \\ 0 & \frac{1}{70} t_k S^{(k)}_{3333} \end{bmatrix} \tag{134}$$

$$[H_2]^{(k)} = \begin{bmatrix} -\delta_{\gamma\rho} & \frac{t_k}{2} & 0 & -\frac{t_k}{8} & \frac{t_k}{24}\delta_{\alpha\gamma}\delta_{\beta\rho}\frac{\partial}{\partial r} & 0 & 0 & 0 & 0 & -\frac{2}{5}S^{(k)}_{r3\rho 3} \\ 0 & 0 & -1 & -\frac{3}{2} & \frac{1}{4} & 0 & -\frac{S^{(k)}_{3333}}{10} & 0 & \frac{3S^{(k)}_{3333}}{7t_k} & 0 \end{bmatrix} \tag{135}$$

$$[H_3]^{(k)} = \begin{bmatrix} \frac{8}{15}(t_k S^{(k)}_{r3\rho 3} + t_{k+1} S^{(k+1)}_{r3\rho 3}) & 0 \\ 0 & \frac{3}{35}(t_k S^{(k)}_{3333} + t_{k+1} S^{(k+1)}_{3333}) \end{bmatrix} \tag{136}$$

$$[H_4]^{(k+1)} = \begin{bmatrix} \delta_{\gamma\rho} & \frac{t_{k+1}}{2} & 0 & -\frac{t_{k+1}}{8} & -\frac{t_{k+1}}{24}\delta_{\alpha\gamma}\delta_{\beta\rho}\frac{\partial}{\partial r} & 0 & 0 & 0 & 0 & -\frac{2}{5}S^{(k+1)}_{r3\rho 3} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{4} & 0 & -\frac{S^{(k+1)}_{3333}}{10} & 0 & -\frac{3S^{(k+1)}_{3333}}{7t_{k+1}} & 0 \end{bmatrix} \tag{137}$$

$$[H_5]^{(k+1)} = \begin{bmatrix} -\frac{2}{15} t_{k+1} S^{(k+1)}_{r3\rho 3} & 0 \\ 0 & \frac{1}{70} t_{k+1} S^{(k+1)}_{3333} \end{bmatrix} \tag{138}$$

The two sets of equations (132) correspond, respectively, to (121), and to the pair (128), (130). For the laminate, combining equations for the layers (k), (k+1) and (k+2), we get

$$
[L]^{(k)}=\begin{bmatrix}
[H_3]^{(k-1)} & [H_4]^{(k)} & [H_5]^{(k)} & [0] & [0] & [0] & [0] \\
[C']^{(k)} & [A']^{(k)} & [B']^{(k)} & [0] & [0] & [0] & [0] \\
[H_1]^{(k-1)} & [H_2]^{(k)} & [H_3]^{(k)} & [H_4]^{(k+1)} & [H_5]^{(k+1)} & [0] & [0] \\
[0] & [0] & [C']^{(k+1)} & [A']^{(k+1)} & [B']^{(k+1)} & [0] & [0] \\
[0] & [0] & [H_1]^{(k)} & [H_2]^{(k+1)} & [H_3]^{(k+1)} & [H_4]^{(k+2)} & [H_5]^{(k+2)} \\
[0] & [0] & [0] & [0] & [C']^{(k+2)} & [A']^{(k+2)} & [B']^{(k+2)} \\
[0] & [0] & [0] & [0] & [H_1]^{(k+1)} & [H_2]^{(k+2)} & [H_3]^{(k+2)}
\end{bmatrix} \tag{139}
$$

$$
\{r\}^{(k)} = \begin{Bmatrix}
\{\sigma'\}^{-(k-1)} \\
\{u'\}^{(k)} \\
\{\sigma'\}^{-(k)} \\
\{u'\}^{(k+1)} \\
\{\sigma'\}^{-(k+1)} \\
\{u'\}^{(k+2)} \\
\{\sigma'\}^{-(k+2)}
\end{Bmatrix} \tag{140}
$$

For this matrix of operators to be self-adjoint, a sufficient condition is that $[A']^{(k)}$ and $[H_3]^{(k)}$ be self-adjoint and $[B']^{(k)}$ be the adjoint of $[H_2]^{(k)}$, $[H_4]^{(k)}$ of $[C']^{(k)}$ and $[H_1]^{(k)}$ of $[H_5]^{(k+1)}$. This is not true for the above formulation. As the operators are all linear, it would be possible to follow Tonti's [1967] approach and write the variational formulation as a generalization of Mikhlin [1965] least square method. However, this type of formulation when used with the finite element procedure, would require base functions with a high degree of regularity. This would be prohibitively expensive to use. An alternative is to use $\bar{u}_3^{(k)}$, and $\hat{u}_3^{(k)}$ as field variables instead of the combinations $\frac{3}{4}\,(\bar{u}_3^{(k)}-\hat{u}_3^{(k)})$ and $3\hat{u}_3^{(k)}-\bar{u}_3^{(k)}$. This would yield a self-adjoint form.

However, a simpler strategy, based on reduction of the number of field variables, is presented in the following section. This approach will lead to a generalized Ritz type variational formulation convenient for development of finite element approximations.

4.3.6 Discussion

Pagano [1978] used seven equilibrium equations, (113) through (115), and (48), (49), ten constitutive equations, (60), (61), (69), (70), (71), and six interfacial continuity equations (80), (81), to solve for 23 field variables as given in (125) and (126). Actually, there are only five equations of equilibrium and, therefore, there can only be five corresponding displacement field variables. The quantities N_{33} and M_{33} must be regarded as entities introduced for convenience and completely defined by (48) and (49). Using (48) and (49) to eliminate N_{33} and M_{33}, the number of local mechanical variables reduces to eight requiring exactly eight constitutive equations. Noting that $\hat{u}_3$, u_3^* may be regarded as defined completely by (69) and (70), substitution in the remaining eight constitutive equations viz. (60), (61) and (71) gives exactly eight equations for the eight local kinematic variables viz. $\bar{u}_{(\alpha,\beta)}$, $u^*_{(\alpha,\beta)}$ and $\bar{u}_{3,\alpha}$ derived from the five global variables $\bar{u}_\alpha$, u^*_α and $\bar{u}_3$. The total number of field equations [Pagano 1978] is thus reduced to 19 in five independent displacement field variables, eight mechanical quantities and six interfacial tractions and displacement components. The set of interfacial displacement continuity equations and field equations constitutes a self-adjoint system. The precise form is derived in the following section.

4.4 Self-adjoint Form of the Field Equations

In this section we restate the governing equations of Pagano's theory, after elimination of N_{33} and M_{33}, in a self-adjoint form. Essentially, this consists of using (48) and (49) as definitions for N_{33} and M_{33} and to eliminate them as field variables from the system of equations.

4.4.1 Displacements at the Interfaces

Substituting for N_{33} from (48) into (70) leads to

$$\frac{3}{2}u^{*}_{3}=\frac{1}{2}S_{33\alpha\beta}N_{\alpha\beta}+\frac{h}{4}S_{3333}(\sigma^{+}_{33} + \sigma^{-}_{33}) + \frac{h^{2}}{20}S_{3333}(\sigma^{+}_{r3,r} - \sigma^{-}_{r3,r}) \tag{141}$$

Differentiation with respect to x_ρ gives

$$\frac{h}{4}u^{*}_{3,\rho} = \frac{h}{12}S_{33\alpha\beta}N_{\alpha\beta,\rho}+\frac{h^{2}}{24}S_{3333}(\sigma^{+}_{33,\rho} + \sigma^{-}_{33,\rho})+ \frac{h^{3}}{120}S_{3333}(\sigma^{+}_{r3,r\rho}-\sigma^{-}_{r3,r\rho}) \tag{142}$$

Substituting for M_{33} in (69) using (49) gives

$$\frac{3}{2}(3\hat{u}_{3} - \bar{u}_{3})=\frac{6}{5h}S_{33\alpha\beta}M_{\alpha\beta}+\frac{9h}{70}S_{3333}(\sigma^{+}_{33} - \sigma^{-}_{33})+\frac{h^{2}}{70}S_{3333}(\sigma^{+}_{r3,r} + \sigma^{-}_{r3,r}) \tag{143}$$

Differentiating with respect to x_ρ gives

$$\begin{aligned} h\left(\frac{3}{8}\hat{u}_{3,\rho}-\frac{1}{8}\bar{u}_{3,\rho}\right) &= \frac{1}{10}S_{33\alpha\beta}M_{\alpha\beta,\rho} + \frac{3}{280}h^{2}S_{3333}(\sigma^{+}_{33,\rho} - \sigma^{-}_{33,\rho}) \\ &+ \frac{h^{3}}{840}S_{3333}(\sigma^{+}_{r3,r\rho} + \sigma^{-}_{r3,r\rho}) \end{aligned} \tag{144}$$

Substituting (142), (144) in (74) and regrouping terms:

$$\begin{aligned} u^{+}_{\rho} = {}& \frac{1}{2}\bar{u}_{\rho} + \frac{3}{2}u^{*}_{\rho} - \frac{h}{12}S_{33\alpha\beta}N_{\alpha\beta,\rho} - \frac{1}{10}S_{33\alpha\beta}M_{\alpha\beta,\rho} - \frac{2}{5}S_{\rho3r3}V_{r} \\ & - \frac{h^{3}}{120}S_{3333}(\sigma^{+}_{r3,r\rho} - \sigma^{-}_{r3,r\rho}) + \frac{h}{3}S_{\rho3r3}(\sigma^{+}_{r3}-\sigma^{-}_{r3}) \\ & - \frac{h^{3}}{840}S_{3333}(\sigma^{+}_{r3,r\rho} + \sigma^{-}_{r3,r\rho}) + \frac{h}{5}S_{\rho3r3}(\sigma^{+}_{r3}+\sigma^{-}_{r3}) \\ & - \frac{3}{280}h^{2}S_{3333}(\sigma^{+}_{33,\rho} - \sigma^{-}_{33,\rho}) - \frac{h^{2}}{24}S_{3333}(\sigma^{+}_{33,\rho} + \sigma^{-}_{33,\rho}) \end{aligned} \tag{145}$$

Similarly, (75) with (142) and (144) gives

$$u_\rho^- = \frac{1}{2}\bar{u}_\rho - \frac{3}{2}u_\rho^* - \frac{h}{12}S_{33\alpha\beta}N_{\alpha\beta,\rho} + \frac{1}{10}S_{33\alpha\beta}M_{\alpha\beta,\rho} + \frac{2}{5}S_{\rho 3r3}V_r$$

$$- \frac{h^3}{120}S_{3333}(\sigma_{r3,r\rho}^+ - \sigma_{r3,r\rho}^-) + \frac{h}{3}S_{\rho 3r3}(\sigma_{r3}^+ - \sigma_{r3}^-)$$

$$+ \frac{h^3}{840}S_{3333}(\sigma_{r3,r\rho}^+ + \sigma_{r3,r\rho}^-) - \frac{h}{5}S_{\rho 3r3}(\sigma_{r3}^+ + \sigma_{r3}^-)$$

$$+ \frac{3}{280}h^2S_{3333}(\sigma_{33,\rho}^+ - \sigma_{33,\rho}^-) - \frac{h^2}{24}S_{3333}(\sigma_{33,\rho}^+ + \sigma_{33,\rho}^-) \tag{146}$$

Substituting for N_{33} and M_{33} from (48) and (49), with (141) and (143), (78) and (79) give

$$u_3^+ = \frac{3}{4}(\bar{u}_3 - \hat{u}_3) + \frac{1}{2}S_{\alpha\beta 33}N_{\alpha\beta} + \frac{6}{5h}S_{33\alpha\beta}M_{\alpha\beta} + \frac{h}{4}S_{3333}(\sigma_{33}^+ + \sigma_{33}^-)$$

$$+ \frac{17}{140}hS_{3333}(\sigma_{33}^+ - \sigma_{33}^-)$$

$$+ h^2S_{3333}[\frac{3}{280}(\sigma_{r3,r}^+ + \sigma_{r3,r}^-) + \frac{1}{24}(\sigma_{r3,r}^+ - \sigma_{r3,r}^-)] \tag{147}$$

$$u_3^- = \frac{3}{4}(\bar{u}_3 - \hat{u}_3) - \frac{1}{2}S_{\alpha\beta 33}N_{\alpha\beta} + \frac{6}{5h}S_{33\alpha\beta}M_{\alpha\beta} - \frac{h}{4}S_{3333}(\sigma_{33}^+ + \sigma_{33}^-)$$

$$+ \frac{17}{140}hS_{3333}(\sigma_{33}^+ - \sigma_{33}^-)$$

$$+ h^2S_{3333}[\frac{3}{280}(\sigma_{r3,r}^+ + \sigma_{r3,r}^-) - \frac{1}{24}(\sigma_{r3,r}^+ - \sigma_{r3,r}^-)] \tag{148}$$

Following Pagano [1978], we introduce $\bar{v}_\rho^{(k)}$ and $\bar{\phi}_\rho^{(k)}$ through the relationship

$$\bar{v}_\rho^{(k)} = \frac{\bar{u}_\rho^{(k)}}{2} \tag{149}$$

$$\bar{\phi}_\rho^{(k)} = \frac{3u_\rho^{*(k)}}{t_k} \tag{150}$$

Using (149) and (150), (145) through (148) may be rewritten as:

$$u_\rho^{(k)}(\frac{t_k}{2}) = \bar{v}_\rho^{(k)} + \frac{t_k}{2}\bar{\phi}_\rho^{(k)} - \frac{t_k}{12}S_{33\alpha\beta}^{(k)}N_{\alpha\beta,\rho}^{(k)} - \frac{1}{10}S_{33\alpha\beta}^{(k)}M_{\alpha\beta,\rho}^{(k)} - \frac{2}{5}S_{\rho 3r3}^{(k)}V_r^{(k)}$$

$$- \frac{t_k^3}{120}S_{3333}^{(k)}(\sigma_{r3,r\rho}^{+(k)} - \sigma_{r3,r\rho}^{-(k)}) + \frac{t_k}{3}S_{\rho 3r3}^{(k)}(\sigma_{r3}^{+(k)} - \sigma_{r3}^{-(k)})$$

$$- \frac{t_k^3}{840}S_{3333}^{(k)}(\sigma_{r3,r\rho}^{+(k)} + \sigma_{r3,r\rho}^{-(k)}) + \frac{t_k}{5}S_{\rho 3r3}^{(k)}(\sigma_{r3}^{+(k)} + \sigma_{r3}^{-(k)})$$

$$- \frac{3}{280}t_k^2 S_{3333}^{(k)}(\sigma_{33,\rho}^{+(k)} - \sigma_{33,\rho}^{-(k)}) - \frac{t_k^2}{24}S_{3333}^{(k)}(\sigma_{33,\rho}^{+(k)} + \sigma_{33,\rho}^{-(k)}) \tag{151}$$

$$u_\rho^{(k)}(-\frac{t_k}{2}) = \bar{v}_\rho^{(k)} - \frac{t_k}{2}\bar{\phi}_\rho^{(k)} - \frac{t_k}{12}S_{33\alpha\beta}^{(k)}N_{\alpha\beta,\rho}^{(k)} + \frac{1}{10}S_{33\alpha\beta}^{(k)}M_{\alpha\beta,\rho}^{(k)} + \frac{2}{5}S_{\rho 3r3}^{(k)}V_r^{(k)}$$

$$- \frac{t_k^3}{120}S_{3333}^{(k)}(\sigma_{r3,r\rho}^{+(k)} - \sigma_{r3,r\rho}^{-(k)}) + \frac{t_k}{3}S_{\rho 3r3}^{(k)}(\sigma_{r3}^{+(k)} - \sigma_{r3}^{-(k)})$$

$$+ \frac{t_k^3}{840}S_{3333}^{(k)}(\sigma_{r3,r\rho}^{+(k)} + \sigma_{r3,r\rho}^{-(k)}) - \frac{t_k}{5}S_{\rho 3r3}^{(k)}(\sigma_{r3}^{+(k)} + \sigma_{r3}^{-(k)})$$

$$+ \frac{3}{280}t_k^2 S_{3333}^{(k)}(\sigma_{33,\rho}^{+(k)} - \sigma_{33,\rho}^{-(k)}) - \frac{t_k^2}{24}S_{3333}^{(k)}(\sigma_{33,\rho}^{+(k)} + \sigma_{33,\rho}^{-(k)}) \tag{152}$$

$$u_3^k(\pm\frac{t_k}{2}) = \bar{v}_3^{(k)} \pm \frac{1}{2}S_{\alpha\beta 33}^{(k)}N_{\alpha\beta}^{(k)} + \frac{6}{5t_k}S_{33\alpha\beta}^{(k)}M_{\alpha\beta}^{(k)} \pm \frac{t_k}{4}S_{3333}^{(k)}(\sigma_{33}^{+(k)} + \sigma_{33}^{-(k)})$$

$$+ \frac{17}{140}t_k S_{3333}^{(k)}(\sigma_{33}^{+(k)} - \sigma_{33}^{-(k)})$$

$$+ t_k^2 S_{3333}^{(k)}[\frac{3}{280}(\sigma_{r3,r}^{+(k)} + \sigma_{r3,r}^{-(k)}) \pm \frac{1}{24}(\sigma_{r3,r}^{+(k)} - \sigma_{r3,r}^{-(k)})] \tag{153}$$

where

$$\bar{v}_3^{(k)} = \frac{3}{4}(\bar{u}_3^{(k)} - \hat{u}_3^{(k)}) \tag{154}$$

4.4.2 Constitutive Equations for Generalized Displacements

The eight constitutive equations for the kth layer are given by (60), (61), and (71). Substituting (48) and (49) into (60), (61), we have for kth layer, with (149) and (150)

$$\frac{1}{2}\Gamma_2\bar{v}_r^{(k)}=\frac{1}{2}\bar{u}_{(\mu,\rho)}^{(k)}=\frac{1}{t_k}S_{\mu\rho\alpha\beta}^{(k)}N_{\alpha\beta}^{(k)}+\frac{1}{t_k}S_{\mu\rho33}^{(k)}[\frac{t_k}{2}(\sigma_{33}^{+(k)}+\sigma_{33}^{-(k)})+\frac{t_k^2}{12}(\sigma_{r3,r}^{+(k)}-\sigma_{r3,r}^{-(k)})] \tag{155}$$

$$\frac{1}{2}\Gamma_2\bar{\phi}_r^{(k)}=\frac{3}{t_k}u_{(\mu,\rho)}^{*(k)}=\frac{12}{t_k^3}S_{\mu\rho\alpha\beta}^{(k)}M_{\alpha\beta}^{(k)}+\frac{12}{t_k^3}S_{\mu\rho33}^{(k)}[\frac{t_k^2}{10}(\sigma_{33}^{+(k)}-\sigma_{33}^{-(k)})+\frac{t_k^3}{120}(\sigma_{r3,r}^{+(k)}+\sigma_{r3,r}^{-(k)})] \tag{156}$$

(71) can be rewritten as

$$\bar{v}_{3,\rho}^{(k)}+\bar{\phi}_\rho^{(k)} = \frac{3}{4}(\bar{u}_{3,\rho}^{(k)}-\hat{u}_{3,\rho}^{(k)})+\frac{3}{t_k}u_\rho^{*(k)} = \frac{24}{5t_k}S_{\rho3r3}^{(k)}V_r^{(k)} - \frac{2}{5}S_{\rho3r3}^{(k)}(\sigma_{r3}^{+(k)} + \sigma_{r3}^{-(k)}) \tag{157}$$

Combining (113) through (115) and (155) through (157) the equations for the kth layer, are

$$[A]^{(k)}\{u\}^{(k)} + [B]^{(k)}\{\sigma\}^{-(k)} + [C]^{(k)}\{\sigma\}^{+(k)} = 0 \qquad k=1\ ,\ 2\ ,\ 3\ ,\ .\ .\ .\ N \tag{158}$$

where $[A]^{(k)}$, $[B]^{(k)}$, $[C]^{(k)}$ are operator matrices and $\{u\}^{(k)}$, $\{\sigma\}^{+(k)}$, $\{\sigma\}^{-(k)}$ are sets of field variables, respectively. Explicitly,

$$
[A]^{(k)} = \begin{bmatrix}
0 & 0 & 0 & \frac{1}{2}\Gamma_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}\Gamma_1 & -\delta_{\gamma\alpha} \\
0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial r} \\
-\frac{1}{2}\Gamma_2 & 0 & 0 & \frac{1}{t_k}S^{(k)}_{\mu\rho\alpha\beta} & 0 & 0 \\
0 & -\frac{1}{2}\Gamma_2 & 0 & 0 & \frac{12}{t_k^3}S^{(k)}_{\mu\rho\alpha\beta} & 0 \\
0 & -\delta_{r\rho} & -\frac{\partial}{\partial\rho} & 0 & 0 & \frac{24}{5t_k}S^{(k)}_{\rho 3 r 3}
\end{bmatrix} \tag{159}
$$

$$
[B]^{(k)} = \begin{bmatrix}
-\delta_{\gamma\alpha} & 0 \\
\frac{t_k}{2} & 0 \\
0 & -1 \\
-\frac{t_k}{12}S^{(k)}_{\mu\rho 33}\frac{\partial}{\partial r} & \frac{1}{2}S^{(k)}_{\mu\rho 33} \\
\frac{1}{10}S^{(k)}_{\mu\rho 33}\frac{\partial}{\partial r} & -\frac{6}{5t_k}S^{(k)}_{\mu\rho 33} \\
-\frac{2}{5}S^{(k)}_{\rho 3 r 3} & 0
\end{bmatrix} \tag{160}
$$

$$[C]^{(k)} = \begin{bmatrix} \delta_{\gamma\alpha} & 0 \\ \frac{t_k}{2} & 0 \\ 0 & 1 \\ \frac{t_k}{12} S^{(k)}_{\mu\rho 33} \frac{\partial}{\partial\gamma} & \frac{1}{2} S^{(k)}_{\mu\rho 33} \\ \frac{1}{10} S^{(k)}_{\mu\rho 33} \frac{\partial}{\partial\gamma} & \frac{6}{5t_k} S^{(k)}_{\mu\rho 33} \\ -\frac{2}{5} S^{(k)}_{\rho 3\gamma 3} & 0 \end{bmatrix} \tag{161}$$

$$\{\sigma\}^{\pm(k)} = \begin{Bmatrix} \sigma^{\pm(k)}_{\gamma 3} \\ \sigma^{\pm(k)}_{33} \end{Bmatrix} \qquad k = 1, 2, 3, \ldots, N \tag{162}$$

$$\{u\}^{(k)} = \begin{Bmatrix} \bar{v}^{(k)}_{\gamma} \\ \bar{\phi}^{(k)}_{\gamma} \\ \bar{v}^{(k)}_{3} \\ N^{(k)}_{\alpha\beta} \\ M^{(k)}_{\alpha\beta} \\ V^{(k)}_{\gamma} \end{Bmatrix} \tag{163}$$

The operator matrix $[A]^{(k)}$ is self-adjoint in the sense of (111).

Substituting (151) through (153) into (81) leads to, for k=1, 2, . . . N-1,

$$[\Lambda]^{(k)}\{\sigma\}^{-(k-1)}+[\bar{B}]^{(k)}\{u\}^{(k)}+[\Xi]^{(k)}\{\sigma\}^{-(k)}+[\bar{C}]^{(k+1)}\{u\}^{(k+1)}+[\bar{\Lambda}]^{(k+1)}[\sigma]^{-(k+1)} = 0 \tag{164}$$

where

$$[\overline{B}]^{(k)} = \begin{pmatrix} -\delta_{\gamma\rho} & \frac{t_k}{2} & 0 & \frac{t_k}{12}S^{(k)}_{\alpha\beta33}\frac{\partial}{\partial\rho} & -\frac{1}{10}S^{(k)}_{\alpha\beta33}\frac{\partial}{\partial\rho} & -\frac{2}{5}S^{(k)}_{\rho3\gamma3} \\ 0 & 0 & -1 & \frac{1}{2}S^{(k)}_{\alpha\beta33} & -\frac{6}{5t_k}S^{(k)}_{\alpha\beta33} & 0 \end{pmatrix} \tag{165}$$

$$[\overline{C}]^{(k)} = \begin{pmatrix} \delta_{\gamma\rho} & \frac{t_k}{2} & 0 & -\frac{t_k}{12}S^{(k)}_{\alpha\beta33}\frac{\partial}{\partial\rho} & -\frac{1}{10}S^{(k)}_{\alpha\beta33}\frac{\partial}{\partial\rho} & -\frac{2}{5}S^{(k)}_{\rho3\gamma3} \\ 0 & 0 & 1 & \frac{1}{2}S^{(k)}_{\alpha\beta33} & \frac{6}{5t_k}S^{(k)}_{\alpha\beta33} & 0 \end{pmatrix} \tag{166}$$

$$[\Xi]^{(k)} = \begin{pmatrix} \Xi^{(k)}_{11} & \Xi^{(k)}_{12} \\ \Xi^{(k)}_{21} & \Xi^{(k)}_{22} \end{pmatrix} \tag{167}$$

with

$$\Xi^{(k)}_{11} = (-\frac{1}{120}-\frac{1}{840})S^{(k)}_{3333}t_k^3\frac{\partial^2}{\partial\rho\partial\gamma} + (\frac{1}{3} + \frac{1}{5})t_k S^{(k)}_{\rho3\gamma3}$$
$$+ (-\frac{1}{120} - \frac{1}{840})S^{(k+1)}_{3333}t^3_{k+1}\frac{\partial^2}{\partial\rho\partial\gamma} + (\frac{1}{3} + \frac{1}{5})t_{k+1}S^{(k+1)}_{\rho3\gamma3}$$

$$\Xi^{(k)}_{12} = (\frac{3}{280} + \frac{1}{24})S^{(k)}_{3333}t_k^2\frac{\partial}{\partial\rho} - (\frac{3}{280} + \frac{1}{24})S^{(k+1)}_{3333}t^2_{k+1}\frac{\partial}{\partial\rho}$$

$$\Xi^{(k)}_{21} = -(\frac{3}{280} + \frac{1}{24})S^{(k)}_{3333}t_k^2\frac{\partial}{\partial\gamma} + (\frac{1}{24} + \frac{3}{280})S^{(k+1)}_{3333}t^2_{k+1}\frac{\partial}{\partial\gamma}$$

$$\Xi^{(k)}_{22} = (\frac{1}{4} + \frac{17}{140})S^{(k)}_{3333}t_k + (\frac{1}{4} + \frac{17}{140})S^{(k+1)}_{3333}t_{k+1}$$

and

$$[\Lambda]^{(k)} = \begin{pmatrix} \Lambda^{(k)}_{11} & \Lambda^{(k)}_{12} \\ \Lambda^{(k)}_{21} & \Lambda^{(k)}_{22} \end{pmatrix} \tag{168}$$

with

$$\Lambda^{(k)}_{11} = (\frac{1}{120}-\frac{1}{840})S^{(k)}_{3333}t_k^3\frac{\partial^2}{\partial\rho\partial\gamma} + (-\frac{1}{3}+\frac{1}{5})t_k S^{(k)}_{\rho3\gamma3}$$

$$\Lambda_{12}^{(k)} = \left(-\frac{3}{280}+\frac{1}{24}\right)S_{3333}^{(k)}t_k^2\frac{\partial}{\partial\rho}$$

$$\Lambda_{21}^{(k)} = \left(\frac{1}{24}-\frac{3}{280}\right)S_{3333}^{(k)}t_k^2\frac{\partial}{\partial\gamma}$$

$$\Lambda_{22}^{(k)} = \left(\frac{1}{4}-\frac{17}{140}\right)S_{3333}^{(k)}t_k$$

$$[\bar{\Lambda}]^{(k+1)} = \begin{bmatrix} \bar{\Lambda}_{11}^{(k+1)} & \bar{\Lambda}_{12}^{(k+1)} \\ \bar{\Lambda}_{21}^{(k+1)} & \bar{\Lambda}_{22}^{(k+1)} \end{bmatrix} \tag{169}$$

with

$$\bar{\Lambda}_{11}^{(k+1)} = \left(\frac{1}{120}-\frac{1}{840}\right)S_{3333}^{(k+1)}t_{k+1}^3\frac{\partial^2}{\partial\rho\partial\gamma}+\left(-\frac{1}{3}+\frac{1}{5}\right)t_{k+1}S_{\rho 3\gamma 3}^{(k+1)}$$

$$\bar{\Lambda}_{12}^{(k+1)} = \left(\frac{3}{280}-\frac{1}{24}\right)S_{3333}^{(k+1)}t_{k+1}^2\frac{\partial}{\partial\rho}$$

$$\bar{\Lambda}_{21}^{(k+1)} = \left(-\frac{1}{24}+\frac{3}{280}\right)S_{3333}^{(k+1)}t_{k+1}^2\frac{\partial}{\partial\gamma}$$

$$\bar{\Lambda}_{22}^{(k+1)} = \left(\frac{1}{4}-\frac{17}{140}\right)S_{3333}^{(k+1)}t_{k+1}$$

The two equations in (164) restate (81) using (151) through (153).

The quantities $\{\sigma\}^{+(1)}$ and $\{\sigma\}^{-(N)}$ are given for a problem and appear as forcing functions defined as:

$$\{\sigma\}^{+(1)} = \begin{Bmatrix} \hat{\sigma}_{\gamma 3}^{(0)} \\ \hat{\sigma}_{33}^{(0)} \end{Bmatrix} \qquad \{\sigma\}^{-(N)} = \begin{Bmatrix} \hat{\sigma}_{\gamma 3}^{(N)} \\ \hat{\sigma}_{33}^{(N)} \end{Bmatrix} \tag{170}$$

Define, $\{Q\}^{(1)}$ and $\{Q\}^{(N-1)}$ as

$$[Q]^{(1)} = [\Lambda]^{(1)}\ \{\sigma\}^{(0)}$$

$$
= \begin{Bmatrix} \left(\frac{1}{120}-\frac{1}{840}\right)S^{(1)}_{3333}t_1^3\hat{\sigma}^{(0)}_{r3,r\rho}+\left(-\frac{1}{3}+\frac{1}{5}\right)t_1S^{(1)}_{\rho3r3}\hat{\sigma}^{(0)}_{r3}+\left(-\frac{3}{280}+\frac{1}{24}\right)S^{(1)}_{3333}t_1^2\hat{\sigma}^{(0)}_{33,\rho} \\ \left(\frac{1}{24}-\frac{3}{280}\right)S^{(1)}_{3333}t_1^2\hat{\sigma}^{(0)}_{r3,r}+\left(\frac{1}{4}-\frac{17}{140}\right)S^{(1)}_{3333}t_1\hat{\sigma}^{(0)}_{33} \end{Bmatrix} \tag{171}
$$

and

$$
[Q]^{(N-1)} = [\overline{\Lambda}]^{(N)}\{\sigma\}^{-(N)}
$$

$$
= \begin{Bmatrix} \left(\frac{1}{120}-\frac{1}{840}\right)S^{(N)}_{3333}t_N^3\hat{\sigma}^{(N)}_{r3,r\rho}+\left(-\frac{1}{3}+\frac{1}{5}\right)t_NS^{(N)}_{\rho3r3}\hat{\sigma}^{(N)}_{r3}+\left(\frac{3}{280}-\frac{1}{24}\right)S^{(N)}_{3333}t_N^2\hat{\sigma}^{(N)}_{33,\rho} \\ \left(-\frac{1}{24}+\frac{3}{280}\right)S^{(N)}_{3333}t_N^2\hat{\sigma}^{(N)}_{r3,r}+S^{(N)}_{3333}t_N\left(\frac{1}{4}-\frac{17}{140}\right)\hat{\sigma}^{(N)}_{33} \end{Bmatrix} \tag{172}
$$

where superscript (0) denotes the top of the 1st layer and (N) denotes the bottom of the Nth layer. Combining (158), (164), (171), and (172), the complete set of field equations for the laminate is:

$$
[X]\ \{Y\} = \{Z\} \tag{173}
$$

where

$$
[X]=\left[\begin{array}{ccccccccccccc}
[A]^{(1)} & [B]^{(1)} & 0 & 0 & 0 & 0 & 0 & & & & & & \\
[\overline{B}]^{(1)} & [\Xi]^{(1)} & [\overline{C}]^{(2)} & [\overline{\Lambda}]^{(2)} & 0 & 0 & 0 & & & & & & \\
0 & [C]^{(2)} & [A]^{(2)} & [B]^{(2)} & 0 & 0 & 0 & & & & & & \\
0 & [\Lambda]^{(2)} & [\overline{B}]^{(2)} & [\Xi]^{(2)} & [\overline{C}]^{(3)} & [\overline{\Lambda}]^{(3)} & 0 & & & & & & \\
0 & 0 & 0 & [C]^{(3)} & [A]^{(3)} & [B]^{(3)} & 0 & & & & & & \\
0 & 0 & 0 & [\Lambda]^{(3)} & [\overline{B}]^{(3)} & [\Xi]^{(3)} & [\overline{C}]^{(4)} & [\overline{\Lambda}]^{(4)} & & & & & \\
 & & & 0 & 0 & [C]^{(4)} & [A]^{(4)} & [B]^{(4)} & & & 0 & 0 & 0 \\
 & & & 0 & 0 & [\Lambda]^{(4)} & [B]^{(4)} & [\Xi]^{(4)} & . & . & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 & . & . & . & [\overline{C}]^{(N-1)} & [\overline{\Lambda}]^{(N-1)} & 0 \\
 & & & & & & & . & . & [C]^{(N-1)} & [A]^{(N-1)} & [B]^{(N-1)} & 0 \\
 & & & & & & & & 0 & [\Lambda]^{(N-1)} & [\overline{B}]^{(N-1)} & [\Xi]^{(N-1)} & [\overline{C}]^{(N)} \\
 & & & & & & & & 0 & 0 & 0 & [C]^{(N)} & [A]^{(N)}
\end{array}\right]
$$

$$
\{Y\} = \begin{Bmatrix} \{u\}^{(1)} \\ \{\sigma\}^{(1)} \\ \{u\}^{(2)} \\ \{\sigma\}^{-(2)} \\ \{u\}^{(3)} \\ \{\sigma\}^{(3)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \{u\}^{(N-1)} \\ \{\sigma\}^{-(N-1)} \\ \{u\}^{(N)} \end{Bmatrix}
\qquad
\{Z\} = \begin{Bmatrix} -\{P\}^{(1)} \\ -\{Q\}^{(1)} \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\{Q\}^{(N-1)} \\ -\{P\}^{(N)} \end{Bmatrix}
$$

$$
\{P\}^{(1)} = [C]^{(1)} \begin{Bmatrix} \hat{\sigma}_{\gamma 3}^{(0)} \\ \hat{\sigma}_{33}^{(0)} \end{Bmatrix}
$$

$$
= \begin{Bmatrix} \hat{\sigma}^{(0)}_{\gamma 3} \\ \\ \frac{t_1}{2}\hat{\sigma}^{(0)}_{\gamma 3} \\ \\ \hat{\sigma}^{(0)}_{33} \\ \\ \frac{t_1}{12} S^{(1)}_{\mu\rho 33}\hat{\sigma}^{(0)}_{\gamma 3,\gamma} + \frac{1}{2} S^{(1)}_{\mu\rho 33}\hat{\sigma}^{(0)}_{33} \\ \\ \frac{1}{10} S^{(1)}_{\mu\rho 33}\hat{\sigma}^{(0)}_{\gamma 3,\gamma} + \frac{6}{5t_1} S^{(1)}_{\mu\rho 33}\hat{\sigma}^{(0)}_{33} \\ \\ -\frac{2}{5} S^{(1)}_{\rho 3\gamma 3}\hat{\sigma}^{(0)}_{\gamma 3} \end{Bmatrix} \tag{174}
$$

$$
\{P\}^{(N)} = [B]^{(N)} \begin{Bmatrix} \hat{\sigma}^{(N)}_{\gamma 3} \\ \\ \hat{\sigma}^{(N)}_{33} \end{Bmatrix}
$$

$$
= \begin{Bmatrix} -\hat{\sigma}^{(N)}_{\gamma 3} \\ \\ \frac{t_N}{2}\hat{\sigma}^{(N)}_{\gamma 3} \\ \\ -\hat{\sigma}^{(N)}_{33} \\ \\ -\frac{t_N}{12} S^{(N)}_{\mu\rho 33}\hat{\sigma}^{(N)}_{\gamma 3,\gamma} + \frac{1}{2} S^{(N)}_{\mu\rho 33}\hat{\sigma}^{(N)}_{33} \\ \\ \frac{1}{10} S^{(N)}_{\mu\rho 33}\hat{\sigma}^{(N)}_{\gamma 3,\gamma} - \frac{6}{5t_N} S^{(N)}_{\mu\rho 33}\hat{\sigma}^{(N)}_{33} \\ \\ -\frac{2}{5} S^{(N)}_{\rho 3\gamma 3}\hat{\sigma}^{(N)}_{\gamma 3} \end{Bmatrix} \tag{175}
$$

4.4.3 Adjointness of the Field equations

For the operator matrix [X] to be self-adjoint in the sense of (111) a sufficient condition is that $[A]^{(k)}$, $[\Xi]^{(k)}$ be self-adjoint and that $[\overline{C}]^{(k)}$, $[C]^{(k)}$; $[\Lambda]^{(k)}$, $[\overline{\Lambda}]^{(k)}$; $[B]^{(k)}$, $[\overline{B}]^{(k)}$ constitute adjoint pairs. Using the symbol $<\ ,\ >_R$ to denote inner products i.e.,

$$<f\ ,\ g>_R = \int_R f\ g\ dR$$

where f, g are functions defined over R, we establish the following relationships for the field variables associated with the kth layer:

4.4.3.1 Operator matrix $[A]^{(k)}$

Considering the operators $A_{14}^{(k)}$ and $A_{41}^{(k)}$ of the operator matrix $[A]^{(k)}$, taking inner product of $\tilde{v}_\gamma^{(k)}$, an arbitrary function in the domain of Γ_2, with $\frac{1}{2}\Gamma_1 N_{\alpha\beta}^{(k)}$, application of Green's first theorem [Kreyszig, 1979] gives

$$\begin{aligned}<\tilde{v}_\gamma^{(k)}\ ,\ A_{14}^{(k)}N_{\alpha\beta}^{(k)}>_{R^{(k)}} &= <\tilde{v}_\gamma^{(k)}\ ,\ \frac{1}{2}\Gamma_1 N_{\alpha\beta}^{(k)}>_{R^{(k)}} \\ &= -<N_{\mu\rho}^{(k)}\ ,\ \frac{1}{2}\Gamma_2\tilde{v}_\gamma^{(k)}>_{R^{(k)}} + <N_{\alpha\beta}^{(k)}\eta_\beta\ ,\ \tilde{v}_\alpha^{(k)}>_{R^{(k)}} \\ &= <N_{\mu\rho}^{(k)}\ ,\ A_{41}^{(k)}\tilde{v}_\gamma^{(k)}>_{R^{(k)}} + <N_{\alpha\beta}^{(k)}\eta_\beta\ ,\ \tilde{v}_\alpha^{(k)}>_{S^{(k)}}\end{aligned} \tag{176}$$

Here $R^{(k)}$ is the configuration of the kth layer and $S^{(k)}$ is its boundary. Similarly, considering the operators $A_{25}^{(k)}$, $A_{52}^{(k)}$ and taking inner product of $\tilde{\phi}_\gamma^{(k)}$, an arbitrary function in the domain of Γ_2, with $\frac{1}{2}\Gamma_1 M_{\alpha\beta}^{(k)}$, Green first theorem [Kreyszig, 1979] gives

$$\begin{aligned}<\tilde{\phi}_\gamma^{(k)}\ ,\ A_{25}^{(k)}M_{\alpha\beta}^{(k)}>_{R^{(k)}} &= <\tilde{\phi}_\gamma^{(k)}\ ,\ \frac{1}{2}\Gamma_1 M_{\alpha\beta}^{(k)}>_{R^{(k)}} \\ &= -<M_{\mu\rho}^{(k)}\ ,\ \frac{1}{2}\Gamma_2\tilde{\phi}_\gamma^{(k)}>_{R^{(k)}} + <M_{\alpha\beta}^{(k)}\eta_\beta\ ,\ \tilde{\phi}_\alpha^{(k)}>_{S^{(k)}}\end{aligned}$$

$$= <M^{(k)}_{\mu\rho}\ ,\ A^{(k)}_{52}\bar{\Phi}^{(k)}_{\gamma}>_{R^{(k)}} + <M^{(k)}_{\alpha\beta}\eta_{\beta}\ ,\ \bar{\Phi}^{(k)}_{\alpha}>_{S^{(k)}} \tag{177}$$

For operators $A^{(k)}_{36}$ and $A^{(k)}_{63}$ Green's first theorem [Kreyszig, 1979] gives, for an arbitrary function of $\tilde{v}^{(k)}_{3}$ in the domain of $R^{(k)}$,

$$<\tilde{v}^{(k)}_{3}\ ,\ A^{(k)}_{36}V^{(k)}_{\alpha}>_{R^{(k)}} = <\tilde{v}^{(k)}_{3}\ ,\ V^{(k)}_{\alpha,\alpha}>_{R^{(k)}}$$

$$= -<V^{(k)}_{\alpha}\ ,\ \tilde{v}^{(k)}_{3,\alpha}>_{R^{(k)}} + <V^{(k)}_{\alpha}\eta_{\alpha}\ ,\ \tilde{v}^{(k)}_{3}>_{S^{(k)}}$$

$$= <V^{(k)}_{\alpha}\ ,\ A^{(k)}_{63}\tilde{v}^{(k)}_{3}>_{R^{(k)}} + <V^{(k)}_{\alpha}\eta_{\alpha}\ ,\ \tilde{v}^{(k)}_{3}>_{S^{(k)}} \tag{178}$$

Operators $A^{(k)}_{44}$, $A^{(k)}_{55}$, $A^{(k)}_{66}$ are tensors which are symmetric in the sense of (97). For arbitrary functions $\bar{N}^{(k)}_{\mu\rho}$, $\bar{M}^{(k)}_{\mu\rho}$ and $\bar{V}^{(k)}_{\rho}$ which are square integrable over $R^{(k)}$, we have

$$<\bar{N}^{(k)}_{\mu\rho}\ ,\ A^{(k)}_{44}N^{(k)}_{\alpha\beta}>_{R^{(k)}} = <\bar{N}^{(k)}_{\mu\rho}\ ,\ \frac{S^{(k)}_{\mu\rho\alpha\beta}}{t_k}N^{(k)}_{\alpha\beta}>_{R^{(k)}} = <N^{(k)}_{\alpha\beta}\ ,\ \frac{S^{(k)}_{\mu\rho\alpha\beta}}{t_k}\bar{N}^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$= <N^{(k)}_{\alpha\beta}\ ,\ A^{(k)}_{44}\bar{N}^{(k)}_{\mu\rho}>_{R^{(k)}} \tag{179}$$

$$<\bar{M}^{(k)}_{\mu\rho}\ ,\ A^{(k)}_{55}M^{(k)}_{\alpha\beta}>_{R^{(k)}} = <\bar{M}^{(k)}_{\mu\rho}\ ,\ \frac{12S^{(k)}_{\mu\rho\alpha\beta}}{t^{3}_{k}}M^{(k)}_{\alpha\beta}>_{R^{(k)}} = <M^{(k)}_{\alpha\beta}\ ,\ \frac{12S^{(k)}_{\mu\rho\alpha\beta}}{t^{3}_{k}}\bar{M}^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$= <M^{(k)}_{\alpha\beta}\ ,\ A^{(k)}_{55}\bar{M}^{(k)}_{\mu\rho}>_{R^{(k)}} \tag{180}$$

$$<\bar{V}^{(k)}_{\rho}\ ,\ A^{(k)}_{66}V^{(k)}_{\gamma}>_{R^{(k)}} = <\bar{V}^{(k)}_{\rho}\ ,\ \frac{24S^{(k)}_{\rho3\gamma3}}{5t_k}V^{(k)}_{\gamma}>_{R^{(k)}} = <V^{(k)}_{\gamma}\ ,\ \frac{24S^{(k)}_{\rho3\gamma3}}{5t_k}\bar{V}^{(k)}_{\rho}>_{R^{(k)}}$$

$$= <V^{(k)}_{\gamma}\ ,\ A^{(k)}_{66}\bar{V}^{(k)}_{\rho}>_{R^{(k)}} \tag{181}$$

The pair $A^{(k)}_{26}$, $A^{(k)}_{62}$ consists of linear algebraic operators which are transpose of each other. This fact along with (176) through (181) satisfies the requirement of self-adjointness of operator matrix, $[A]^{(k)}$.

4.4.3.2 Operator matrix $[\Xi]^{(k)}$

Considering the operator matrix $[\Xi]^{(k)}$, the diagonal operators $\Xi_{22}^{(k)}$ is symmetric in the sense of (97) because, for $\tilde{\sigma}_{33}^{-(k)}$, $\sigma_{33}^{-(k)}$ defined over $R^{(k)}$:

$$<\tilde{\sigma}_{33}^{-(k)}, \Xi_{22}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}} = <\sigma_{33}^{-(k)}, \Xi_{22}^{(k)}\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}} \tag{182}$$

For the operator $\Xi_{11}^{(k)}$, taking inner product of $\tilde{\sigma}_{\rho 3}^{-(k)}$, defined over $\Xi_{11}^{(k)}$, with $\Xi_{11}^{(k)}\sigma_{\gamma 3}^{-(k)}$, the Green's second theorem [Kreyszig, 1979] gives

$$\begin{aligned}
<\tilde{\sigma}_{\rho 3}^{-(k)}, \Xi_{11}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} &= <\tilde{\sigma}_{\rho 3}^{-(k)}, (-\frac{1}{120}-\frac{1}{840})(S_{3333}^{(k)}t_k^3+S_{3333}^{(k+1)}t_{k+1}^3)\sigma_{\gamma 3,\rho\gamma}^{-(k)} \\
&\quad + (\frac{1}{3}+\frac{1}{5})(t_k S_{\rho 3\gamma 3}^{(k)}+t_{k+1}S_{\rho 3\gamma 3}^{(k+1)})\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} \\
&= <\sigma_{\gamma 3}^{-(k)}, (-\frac{1}{120}-\frac{1}{840})(S_{3333}^{(k)}t_k^3+S_{3333}^{(k+1)}t_{k+1}^3)\tilde{\sigma}_{\rho 3,\rho\gamma}^{-(k)} \\
&\quad + (\frac{1}{3}+\frac{1}{5})(t_k S_{\rho 3\gamma 3}^{(k)}+t_{k+1}S_{\rho 3\gamma 3}^{(k+1)})\tilde{\sigma}_{\gamma 3}^{-(k)}>_{R^{(k)}} \\
&\quad + <\tilde{\sigma}_{\gamma 3}^{-(k)}\eta_\gamma, (-\frac{1}{120}-\frac{1}{840})[S_{3333}^{(k)}t_k^3+S_{3333}^{(k+1)}t_{k+1}^3]\sigma_{\rho 3,\rho}^{-(k)}>_{S^{(k)}} \\
&\quad - <\sigma_{\gamma 3}^{-(k)}\eta_\gamma, (-\frac{1}{120}-\frac{1}{840})[S_{3333}^{(k)}t_k^3+S_{3333}^{(k+1)}t_{k+1}^3]\tilde{\sigma}_{\rho 3,\rho}^{-(k)}>_{S^{(k)}} \\
&= <\sigma_{\gamma 3}^{-(k)}, \Xi_{11}^{(k)}\tilde{\sigma}_{\rho 3}^{-(k)}>_{R^{(k)}} \\
&\quad + <\tilde{\sigma}_{\gamma 3}^{-(k)}\eta_\gamma, (-\frac{1}{120}-\frac{1}{840})[S_{3333}^{(k)}t_k^3+S_{3333}^{(k+1)}t_{k+1}^3]\sigma_{\rho 3,\rho}^{-(k)}>_{S^{(k)}} \\
&\quad - <\sigma_{\gamma 3}^{-(k)}\eta_\gamma, (-\frac{1}{120}-\frac{1}{840})[S_{3333}^{(k)}t_k^3+S_{3333}^{(k+1)}t_{k+1}^3]\tilde{\sigma}_{\rho 3,\rho}^{-(k)}>_{S^{(k)}}
\end{aligned} \tag{183}$$

For the off-diagonal pair $\Xi_{12}^{(k)}$ and $\Xi_{21}^{(k)}$ setting up an inner product of $\Xi_{12}\sigma_{33}^{-(k)}$ and an arbitrary $\tilde{\sigma}_{\rho 3}^{-(k)}$ in the domain of $\Xi_{21}^{(k)}$, gives:

$$\begin{aligned}
<\tilde{\sigma}_{\rho 3}^{-(k)}, \Xi_{12}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}} &= <\tilde{\sigma}_{\rho 3}^{-(k)}, (\frac{3}{280}+\frac{1}{24})[S_{3333}^{(k)}t_k^2-S_{3333}^{(k+1)}t_{k+1}^2]\sigma_{33,\rho}^{-(k)}>_{R^{(k)}} \\
&= -<\sigma_{33}^{-(k)}, (\frac{3}{280}+\frac{1}{24})[S_{3333}^{(k)}t_k^2-S_{3333}^{(k+1)}t_{k+1}^2]\tilde{\sigma}_{\rho 3,\rho}^{-(k)}>_{R^{(k)}} \\
&\quad + <\tilde{\sigma}_{\rho 3}^{(k)}\eta_\rho, (\frac{3}{280}+\frac{1}{24})[S_{3333}^{(k)}t_k^2-S_{3333}^{(k+1)}t_{k+1}^2]\sigma_{33}^{(k)}>_{S^{(k)}}
\end{aligned}$$

$$= <\sigma_{33}^{-(k)}, \Xi_{21}^{(k)}\bar{\sigma}_{\rho 3}^{-(k)}>_{R^{(k)}} + <(\frac{3}{280}+\frac{1}{24})[S_{3333}^{(k)}t_k^2 - S_{3333}^{(k+1)}t_{k+1}^2]\sigma_{33}^{-(k)}, \bar{\sigma}_{\rho 3}^{-(k)}\eta_\rho>_{S^{(k)}} \quad (184)$$

Equations (182) through (184) ensure the self-adjointness of operator matrix $[\Xi]^{(k)}$.

4.4.3.3 Operator matrices $[C]^{(k)}$ and $[\bar{C}]^{(k)}$

Considering the adjoint operators $C_{41}^{(k)}$, $\bar{C}_{14}^{(k)}$; $C_{51}^{(k)}$, $\bar{C}_{15}^{(k)}$ of matrices $[C]^{(k)}$ and $[\bar{C}]^{(k)}$, taking inner product of $\bar{N}_{\alpha\beta}^{(k)}$, an arbitrary function in the domain of $\bar{C}_{14}^{(k)}$, with $C_{41}^{(k)}\sigma_{r3}^{+(k)}$, Green's first theorem [Kreyszig, 1979] gives

$$<\bar{N}_{\mu\rho}^{(k)}, C_{41}^{(k)}\sigma_{r3}^{+(k)}>_{R^{(k)}} = <\bar{N}_{\mu\rho}^{(k)}, \frac{t_k}{12}S_{\mu\rho 33}^{(k)}\sigma_{r3,r}^{+(k)}>_{R^{(k)}}$$

$$= -<\sigma_{r3}^{+(k)}, \frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\bar{N}_{\alpha\beta,r}^{(k)}>_{R^{(k)}} + <\sigma_{r3}^{+(k)}\eta_r, \frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\bar{N}_{\alpha\beta}^{(k)}>_{S^{(k)}}$$

$$= <\sigma_{r3}^{+(k)}, \bar{C}_{14}^{(k)}\bar{N}_{\alpha\beta}^{(k)}>_{R^{(k)}} + <\sigma_{r3}^{+(k)}\eta_r, \frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\bar{N}_{\alpha\beta}^{(k)}>_{S^{(k)}} \quad (185)$$

Similarly, for the inner product of an arbitrary function $\bar{M}_{\alpha\beta}^{(k)}$, in the domain of $\bar{C}_{15}^{(k)}$, with $C_{51}^{(k)}\sigma_{r3}^{+(k)}$, Green's first theorem [Kreyszig, 1979] gives:

$$<\bar{M}_{\mu\rho}^{(k)}, C_{51}^{(k)}\sigma_{r3}^{+(k)}>_{R^{(k)}} = <\bar{M}_{\mu\rho}^{(k)}, \frac{1}{10}S_{\mu\rho}^{(k)}\sigma_{r3,r}^{+(k)}>_{R^{(k)}}$$

$$= -<\sigma_{r3}^{+(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\bar{M}_{\alpha\beta,r}^{(k)}>_{R^{(k)}} + <\sigma_{r3}^{+(k)}\eta_r, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\bar{M}_{\alpha\beta}^{(k)}>_{S^{(k)}}$$

$$= <\sigma_{r3}^{+(k)}, \bar{C}_{15}^{(k)}\bar{M}_{\alpha\beta}^{(k)}>_{R^{(k)}} + <\sigma_{r3}^{+(k)}\eta_r, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\bar{M}_{\alpha\beta}^{(k)}>_{S^{(k)}} \quad (186)$$

Each of the six pairs of algebraic operators viz. $C_{11}^{(k)}$, $\bar{C}_{11}^{(k)}$; $C_{21}^{(k)}$, $\bar{C}_{12}^{(k)}$; $C_{32}^{(k)}$, $\bar{C}_{23}^{(k)}$; $C_{42}^{(k)}$, $\bar{C}_{24}^{(k)}$; $C_{52}^{(k)}$, $\bar{C}_{52}^{(k)}$; $C_{61}^{(k)}$, $\bar{C}_{16}^{(k)}$ consists of operators which are transpose of each other. This fact along with (185) and (186) ensures the adjointness of the pair $[C]^{(k)}$ and $[\bar{C}]^{(k)}$.

4.4.3.4 Operator matrices $[\Lambda]^{(k)}$ and $[\overline{\Lambda}]^{(k)}$

Considering the operators $\overline{\Lambda}_{11}^{(k)}$, and $\Lambda_{11}^{(k)}$ of the operator matrices $[\overline{\Lambda}]^{(k)}$ and $[\Lambda]^{(k)}$, application of Green's second theorem [Kreyszig, 1979] to the inner product of $\tilde{\sigma}_{r3}^{+(k)}$, defined over, $\Lambda_{11}^{(k)}$ with $\overline{\Lambda}_{11}^{(k)}\sigma_{r3}^{-(k)}$ implies

$$<\tilde{\sigma}_{\rho3}^{+(k)},\overline{\Lambda}_{11}^{(k)}\sigma_{r3}^{-(k)}>_{R^{(k)}}=<\tilde{\sigma}_{\rho3}^{+(k)},(\frac{1}{120}-\frac{1}{840})S_{3333}^{(k)}t_k^3\sigma_{r3,r\rho}^{-(k)}+(-\frac{1}{3}+\frac{1}{5})t_kS_{\rho3r3}^{(k)}\sigma_{r3}^{-(k)}>_{R^{(k)}}$$

$$=<\sigma_{r3}^{-(k)},(\frac{1}{120}-\frac{1}{840})S_{3333}^{(k)}t_k^3\tilde{\sigma}_{\rho3,\rho r}^{+(k)}+(-\frac{1}{3}+\frac{1}{5})t_kS_{\rho3r3}^{(k)}\tilde{\sigma}_{\rho3}^{+(k)}>_{R^{(k)}}$$

$$+<\tilde{\sigma}_{r3}^{+(k)}\eta_r,(\frac{1}{120}-\frac{1}{840})t_k^3S_{3333}^{(k)}\tilde{\sigma}_{\rho3,\rho}^{-(k)}>_{S^{(k)}}$$

$$-<\sigma_{r3}^{-(k)}\eta_r,(\frac{1}{120}-\frac{1}{840})t_k^3S_{3333}^{(k)}\tilde{\sigma}_{\rho3,\rho}^{+(k)}>_{S^{(k)}}$$

$$=<\sigma_{r3}^{-(k)},\Lambda_{11}^{(k)}\tilde{\sigma}_{r3}^{+(k)}>_{R^{(k)}}$$

$$+<\tilde{\sigma}_{r3}^{+(k)}\eta_r,(\frac{1}{120}-\frac{1}{840})t_k^3S_{3333}^{(k)}\sigma_{\rho3,\rho}^{-(k)}>_{S^{(k)}}$$

$$-<\sigma_{r3}^{-(k)}\eta_r,(\frac{1}{120}-\frac{1}{840})t_k^3S_{3333}^{(k)}\tilde{\sigma}_{\rho3,\rho}^{+(k)}>_{S^{(k)}} \tag{187}$$

Application of Green's first theorem [Kreyszig, 1979] to the inner product of $\tilde{\sigma}_{\rho3}^{+(k)}$, an arbitrary function in the domain of $\Lambda_{21}^{(k)}$, with $\overline{\Lambda}_{12}^{(k)}\sigma_{33}^{-(k)}$ gives

$$<\tilde{\sigma}_{\rho3}^{+(k)},\overline{\Lambda}_{12}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}=<\tilde{\sigma}_{\rho3}^{+(k)},(\frac{3}{280}-\frac{1}{24})S_{3333}^{(k)}t_k^2\sigma_{33,\rho}^{-(k)}>_{R^{(k)}}$$

$$=-<\sigma_{33}^{-(k)},(\frac{3}{280}-\frac{1}{24})S_{3333}^{(k)}t_k^2\tilde{\sigma}_{\rho3,\rho}^{+(k)}>_{R^{(k)}}+<\sigma_{33}^{-(k)},(\frac{3}{280}-\frac{1}{24})t_k^2S_{3333}^{(k)}\tilde{\sigma}_{\rho3}^{+(k)}\eta_\rho>_{S^{(k)}}$$

$$=<\sigma_{33}^{-(k)},\Lambda_{21}^{(k)}\tilde{\sigma}_{r3}^{+(k)}>_{R^{(k)}}+<\sigma_{33}^{-(k)},(\frac{3}{280}-\frac{1}{24})t_k^2S_{3333}^{(k)}\tilde{\sigma}_{r3}^{+(k)}\eta_r>_{S^{(k)}} \tag{188}$$

Similarly, for $\tilde{\sigma}_{33}^{+(k)}$, an arbitrary function in the domain of $\Lambda_{12}^{(k)}$, and $\overline{\Lambda}_{21}^{(k)}\sigma_{r3}^{-(k)}$, application of Green's first theorem [Kreyszig, 1979] gives

$$<\tilde{\sigma}_{33}^{+(k)},\overline{\Lambda}_{21}^{(k)}\sigma_{r3}^{-(k)}>_{R^{(k)}}=<\tilde{\sigma}_{33}^{+(k)},(-\frac{1}{24}+\frac{3}{280})S_{3333}^{(k)}t_k^2\sigma_{r3,r}^{-(k)}>_{R^{(k)}}$$

$$=-<\sigma_{\rho3}^{-(k)},(-\frac{1}{24}+\frac{3}{280})S_{3333}^{(k)}t_k^2\tilde{\sigma}_{33,\rho}^{+(k)}>_{R^{(k)}}$$

$$+ \ <(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\sigma_{\gamma 3}^{-(k)}\eta_\gamma \ , \ \tilde{\sigma}_{33}^{+(k)}>_{S^{(k)}}$$

$$= <\sigma_{\rho 3}^{-(k)} \ , \ \Lambda_{12}^{(k)}\tilde{\sigma}_{33}^{+(k)}>_{R^{(k)}}+<(\frac{3}{280}-\frac{1}{24})t_k^2 S_{3333}^{(k)}\sigma_{\gamma 3}^{-(k)}\eta_\gamma \ , \ \tilde{\sigma}_{33}^{+(k)}>_{S^{(k)}} \qquad (189)$$

Operators $\overline{\Lambda}_{22}^{(k)}$ and $\Lambda_{22}^{(k)}$ are identical scalars and for $\tilde{\sigma}_{33}^{+(k)}$, and $\sigma_{33}^{-(k)}$, arbitrary functions defined over of $R^{(k)}$, implies

$$<\tilde{\sigma}_{33}^{+(k)} \ , \ \overline{\Lambda}_{22}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}=<\tilde{\sigma}_{33}^{+(k)} \ , \ (\frac{1}{4}-\frac{17}{140})S_{3333}^{(k)}t_k^2\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$= <\sigma_{33}^{-(k)} \ , \ (\frac{1}{4}-\frac{17}{140})S_{3333}^{(k)}t_k^2\tilde{\sigma}_{33}^{+(k)}>_{R^{(k)}}=<\sigma_{33}^{-(k)} \ , \ \Lambda_{22}^{(k)}\tilde{\sigma}_{33}^{+(k)}>_{R^{(k)}} \qquad (190)$$

Equations (187) through (190) ensure the adjointness of the pair consisting of $[\Lambda]^{(k)}$ and $[\overline{\Lambda}]^{(k)}$.

4.4.3.5 Operator matrices $[B]^{(k)}$ and $[\overline{B}]^{(k)}$

Considering the pair of operators $B_{41}^{(k)}$ and $\overline{B}_{14}^{(k)}$ belonging respectively to the operator matrices $[B]^{(k)}$ and $[\overline{B}]^{(k)}$, the inner product of an arbitrary function $\overline{N}_{\alpha\beta}^{(k)}$, in the domain of $\overline{B}_{14}$, with $B_{41}\sigma_{\gamma 3}^{-(k)}$, the Green's first theorem [Kreyszig, 1979] gives

$$<\overline{N}_{\alpha\beta}^{(k)} \ , \ B_{41}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} = <\overline{N}_{\alpha\beta}^{(k)} \ , \ -\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k)}>_{R^{(k)}}$$

$$= -<\sigma_{\gamma 3}^{-(k)} \ , \ -\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\overline{N}_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}}+<\sigma_{\gamma 3}^{-(k)}\eta_\gamma \ , \ -\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\overline{N}_{\alpha\beta}^{(k)}>_{S^{(k)}}$$

$$= <\sigma_{\gamma 3}^{-(k)} \ , \ \overline{B}_{14}^{(k)}\overline{N}_{\alpha\beta}^{(k)}>_{R^{(k)}}+<\sigma_{\gamma 3}^{-(k)}\eta_\gamma \ , \ -\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\overline{N}_{\alpha\beta}^{(k)}>_{S^{(k)}} \qquad (191)$$

For the pair of operators $B_{51}^{(k)}$ and $\overline{B}_{15}^{(k)}$, the inner product of an arbitrary function $\overline{M}_{\alpha\beta}^{(k)}$, in the domain of $\overline{B}_{15}^{(k)}$, with $B_{51}^{(k)}\sigma_{\gamma 3}^{-(k)}$ gives, upon application of Green's first theorem [Kreyszig, 1979].

$$<\overline{M}_{\mu\rho}^{(k)} \ , \ B_{51}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}}=<\overline{M}_{\mu\rho}^{(k)} \ , \ \frac{1}{10}S_{\mu\rho 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k)}>_{R^{(k)}}$$

$$= -<\sigma_{\rho 3}^{-(k)} \ , \ \frac{1}{10}S_{\alpha\beta 33}^{(k)}\overline{M}_{\alpha\beta,\rho}^{(k)}>_{R^{(k)}}+<\sigma_{\rho 3}^{-(k)}\eta_\gamma \ , \ \frac{1}{10}S_{\alpha\beta 33}^{(k)}\overline{M}_{\alpha\beta}^{(k)}>_{S^{(k)}}$$

$$= <\sigma_{\gamma 3}^{-(k)}, \overline{B}_{15}^{(k)} \overline{M}_{\alpha\beta}^{(k)}>_{R^{(k)}} + <\sigma_{\gamma 3}^{-(k)} \eta_\gamma , \frac{1}{10} S_{\alpha\beta 33}^{(k)} \overline{M}_{\alpha\beta}^{(k)}>_{S^{(k)}} \tag{192}$$

The elements in the six pairs of symmetric operators viz. $B_{11}^{(k)}$, $\overline{B}_{11}^{(k)}$; $B_{21}^{(k)}$, $\overline{B}_{12}^{(k)}$; $B_{32}^{(k)}$, $\overline{B}_{23}^{(k)}$; $B_{42}^{(k)}$, $\overline{B}_{24}^{(k)}$; $B_{52}^{(k)}$, $\overline{B}_{25}^{(k)}$; $B_{61}^{(k)}$, $\overline{B}_{16}^{(k)}$, consist of algebraic operators such that each element is transpose of the other. This property along with (191) and (192) ensures the adjointness of the pair of operator matrices $[B]^{(k)}$ and $[\overline{B}]^{(k)}$.

4.5 Consistent Boundary Operators

The general mixed boundary-value problem of linear elastostatics consists in finding a state i.e., a set of displacement, strain, and stress fields, which satisfies the governing field equations in a given spatial domain and meets the specified boundary conditions. A variational form of the boundary-value problem exists if a function over the space of admissible states can be defined such that its Gateaux differential along arbitrary paths vanishes only at the solution state. The set of admissible states is the collection of all possible {Y} in the domain of the operator matrix [X]. Closure of this collection includes the exact solution. A variational formulation for coupled boundary value problems was proposed by Sandhu and Salaam [1975] for the case in which the the operator matrix [X] is self-adjoint with respect to the bilinear mapping used and the boundary operator is consistent with [X] in the sense of (110). The adjointness of [X] has been shown in previous section. In the following section the consistent boundary operators associated with the adjoint pairs of non-zero operators in [X] with respect to the bilinear mapping used in the sense of (111) are identified.

Following (111), considering non-zero operators in [X] and taking inner product of a typical $\{\tilde{u}\}^{(k)}$ with the corresponding set of equations, we have

$$<\{\tilde{u}\}^{(k)}, [C]^{(k)}\{\sigma\}^{-(k-1)} + [A]^{(k)}\{u\}^{(k)} + [B]^{(k)}\{\sigma\}^{-(k)}>_{R^{(k)}}$$

$$= <\{\sigma\}^{-(k-1)}, [\overline{C}]^{(k)}\{\tilde{u}\}^{(k)}>_{R^{(k)}} + <\{u\}^{(k)}, [A]^{(k)}\{\tilde{u}\}^{(k)}>_{R^{(k)}} + <\{\sigma\}^{-(k)}, [\overline{B}]^{(k)}\{\tilde{u}\}^{(k)}>_{R^{(k)}}$$

$$+ <\{\sigma\}^{-(k-1)}, [\tilde{F}]^{(k)}\{\tilde{u}\}^{(k)}>_{S^{(k)}} + <\{u\}^{(k)}, [\tilde{D}]^{(k)}\{\tilde{u}\}^{(k)}>_{S_u^{(k)}} + <\{\sigma\}^{-(k)}, [\tilde{E}]^{(k)}\{\tilde{u}\}^{(k)}>_{S^{(k)}}$$

$$- <\{\tilde{u}\}^{(k)}, [F]^{(k)}\{\sigma\}^{-(k-1)} + [D]^{(k)}\{u\}^{(k)} + [E]^{(k)}\{\sigma\}^{-(k)}>_{S_u^{(k)}} \tag{193}$$

Setting

$$\{\tilde{u}\} = \begin{Bmatrix} \tilde{v}_\alpha^{(k)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

the left hand side of (193) is identified as

$$<\tilde{v}_\alpha^{(k)}, \sigma_{\alpha 3}^{-(k-1)} + N_{\alpha\beta,\beta}^{(k)} - \sigma_{\alpha 3}^{-(k)}>_{R^{(k)}} \tag{194}$$

Using (176)

$$<\tilde{v}_\alpha^{(k)}, \sigma_{\alpha 3}^{-(k-1)} + N_{\alpha\beta,\beta}^{(k)} - \sigma_{\alpha 3}^{-(k)}>_{R^{(k)}}$$

$$= <\sigma_{\alpha 3}^{-(k-1)}, \tilde{v}_\alpha^{(k)}>_{R^{(k)}} + <\sigma_{\alpha 3}^{-(k)}, -\tilde{v}_\alpha^{(k)}>_{R^{(k)}} + <N_{\alpha\beta}^{(k)}, -\tilde{v}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}$$

$$+ <N_{\alpha\beta}^{(k)}, \tilde{v}_\alpha^{(k)}\eta_\beta>_{S^{(k)}} \tag{195}$$

To obtain the consistent form (111), we rewrite the boundary term as

$$<N_{\alpha\beta}^{(k)}, \tilde{v}_\alpha^{(k)}\eta_\beta>_{S^{(k)}} = <N_{\alpha\beta}^{(k)}, \tilde{v}_\alpha^{(k)}\eta_\beta>_{S_2^{(k)}} - <\tilde{v}_\alpha^{(k)}, -N_{\alpha\beta}^{(k)}\eta_\beta>_{S_1^{(k)}} \tag{196}$$

where $\bar{S}_1^{(k)} \bigcup \bar{S}_2^{(k)} = \bar{S}^{(k)}$ and $S_1^{(k)} \bigcap S_2^{(k)} = 0$.

Proceeding similarly with

$$\{\tilde{u}\} = \begin{Bmatrix} 0 \\ \tilde{\phi}_\alpha^{(k)} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

we have

$$<\phi_{\alpha}^{(k)}, \frac{t_k}{2}\sigma_{\alpha 3}^{-(k-1)}+M_{\alpha\beta,\beta}^{(k)}-V_{\alpha}^{(k)}+\frac{t_k}{2}\sigma_{\alpha 3}^{-(k)}>_{R^{(k)}}$$

$$= <\sigma_{\alpha 3}^{-(k-1)}, \frac{t_k}{2}\tilde{\phi}_{\alpha}^{(k)}>_{R^{(k)}}+<\sigma_{\alpha 3}^{-(k)}, \frac{t_k}{2}\tilde{\phi}_{\alpha}^{(k)}>_{R^{(k)}}+<M_{\alpha\beta}^{(k)}, -\tilde{\phi}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}$$

$$+ <V_{\alpha}^{(k)}, -\tilde{\phi}_{\alpha}^{(k)}>_{R^{(k)}}+<M_{\alpha\beta}^{(k)}\eta_{\beta}, \tilde{\phi}_{\alpha}^{(k)}>_{S^{(k)}} \qquad (197)$$

As before, the form of (111) is realized by writing the boundary as

$$<M_{\alpha\beta}^{(k)}\eta_{\beta}, \tilde{\phi}_{\alpha}^{(k)}>_{S^{(k)}} = <M_{\alpha\beta}^{(k)}\eta_{\beta}, \tilde{\phi}_{\alpha}^{(k)}>_{S_4^{(k)}}-<\tilde{\phi}_{\alpha}^{(k)}, -\eta_{\beta}M_{\alpha\beta}^{(k)}>_{S_3^{(k)}} \qquad (198)$$

where $\bar{S}_3^{(k)} \bigcup \bar{S}_4^{(k)}=\bar{S}^{(k)}$ and $S_3^{(k)} \bigcap S_4^{(k)}=0$.

$$\{\tilde{u}\} = \begin{Bmatrix} 0 \\ 0 \\ \tilde{v}_3^{(k)} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

yields

$$<\tilde{v}_3^{(k)}, \sigma_{33}^{-(k-1)}+V_{\alpha,\alpha}^{(k)}-\sigma_{33}^{-(k)}>_{R^{(k)}}=<\sigma_{33}^{-(k-1)}, \tilde{v}_3^{(k)}>_{R^{(k)}}+<\sigma_{33}^{-(k)}, -\tilde{v}_3^{(k)}>_{R^{(k)}}$$

$$+ <V_{\alpha}^{(k)}, -\tilde{v}_{3,\alpha}^{(k)}>_{R^{(k)}}+<V_{\alpha}^{(k)}\eta_{\alpha}, \tilde{v}_3^{(k)}>_{S^{(k)}} \qquad (199)$$

The form of (111) is realized by writing the boundary terms as

$$<V_{\alpha}^{(k)}\eta_{\alpha}, \tilde{v}_3^{(k)}>_{S^{(k)}} = <V_{\alpha}^{(k)}, \eta_{\alpha}\tilde{v}_3^{(k)}>_{S_6^{(k)}}-<\tilde{v}_3^{(k)}, -\eta_{\alpha}V_{\alpha}^{(k)}>_{S_5^{(k)}} \qquad (200)$$

where $\bar{S}_5^{(k)} \bigcup \bar{S}_6^{(k)}=\bar{S}^{(k)}$ and $S_5^{(k)} \bigcap S_6^{(k)}=0$.

For

$$\{\tilde{u}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \tilde{N}_{\alpha\beta}^{(k)} \\ 0 \\ 0 \end{Bmatrix}$$

we have

$$<\bar{N}^{(k)}_{\alpha\beta}\ ,\ \frac{t_k}{12}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k-1)}_{\gamma3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k-1)}_{33}-\bar{v}^{(k)}_{(\alpha,\beta)}+\frac{1}{t_k}S^{(k)}_{\alpha\beta\mu\rho}N^{(k)}_{\mu\rho}$$

$$-\ \frac{t_k}{12}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k)}_{\gamma3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$=\ <\bar{\sigma}^{(k-1)}_{\gamma3}\ ,\ -\frac{t_k}{12}S^{(k)}_{\alpha\beta33}\bar{N}^{(k)}_{\alpha\beta,\gamma}>_{R^{(k)}}+<\bar{\sigma}^{(k-1)}_{33}\ ,\ \frac{1}{2}S^{(k)}_{\alpha\beta33}\bar{N}^{(k)}_{\alpha\beta}>_{R^{(k)}}$$

$$+\ <\bar{v}^{(k)}_{\alpha}\ ,\ \bar{N}^{(k)}_{\alpha\beta,\beta}>_{R^{(k)}}$$

$$+\ <N^{(k)}_{\alpha\beta}\ ,\ \frac{1}{t_k}S^{(k)}_{\alpha\beta\mu\rho}\bar{N}^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$+\ <\bar{\sigma}^{(k)}_{\gamma3}\ ,\ \frac{t_k}{12}S^{(k)}_{\alpha\beta33}\bar{N}^{(k)}_{\alpha\beta,\gamma}>_{R^{(k)}}+<\bar{\sigma}^{(k)}_{33}\ ,\ \frac{1}{2}S^{(k)}_{\alpha\beta33}\bar{N}^{(k)}_{\alpha\beta}>_{R^{(k)}}$$

$$+\ <\bar{N}^{(k)}_{\alpha\beta}\ ,\ -\eta_\beta\bar{v}^{(k)}_{\alpha}>_{S^{(k)}}+<\bar{N}^{(k)}_{\alpha\beta}\ ,\ -\frac{t_k}{12}S^{(k)}_{\alpha\beta33}\eta_\gamma\bar{\sigma}^{(k)}_{\gamma3}>_{S^{(k)}}$$

$$+\ <\bar{N}^{(k)}_{\alpha\beta}\ ,\ \frac{t_k}{12}S^{(k)}_{\alpha\beta33}\eta_\gamma\bar{\sigma}^{(k-1)}_{\gamma3}>_{S^{(k)}} \tag{201}$$

To obtain the form of (111), we rewrite the boundary term as

$$<\bar{N}^{(k)}_{\alpha\beta}\ ,\ -\eta_\beta\bar{v}^{(k)}_{\alpha}>_{S^{(k)}}=<\bar{v}^{(k)}_{\alpha}\ ,\ -\eta_\beta\bar{N}^{(k)}_{\alpha\beta}>_{S^{(k)}_1}-<\bar{N}^{(k)}_{\alpha\beta}\ ,\ \eta_\beta\bar{v}^{(k)}_{\alpha}>_{S^{(k)}_2} \tag{202}$$

For

$$\{\tilde{u}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \bar{M}^{(k)}_{\alpha\beta} \\ 0 \end{Bmatrix}$$

we have

$$<\bar{M}^{(k)}_{\alpha\beta}\ ,\ \frac{1}{10}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k-1)}_{\gamma3,\gamma}+\frac{6}{5t_k}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k-1)}_{33}-\bar{\phi}^{(k)}_{(\alpha,\beta)}+\frac{12}{t_k^3}S^{(k)}_{\alpha\beta\mu\rho}M^{(k)}_{\mu\rho}$$

$$+\ \frac{1}{10}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k)}_{\gamma3,\gamma}-\frac{6}{5t_k}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$= <\sigma_{\gamma 3}^{-(k-1)}, -\frac{1}{10}S_{\alpha\beta 33}^{(k)}\bar{M}_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}} + <\sigma_{33}^{-(k-1)}, \frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\bar{M}_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+ <\bar{\phi}_{\alpha}^{(k)}, \bar{M}_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}$$

$$+ <M_{\alpha\beta}^{(k)}, \frac{12}{t_k^3}S_{\alpha\beta\mu\rho}^{(k)}\bar{M}_{\mu\rho}^{(k)}>_{R^{(k)}}$$

$$+ <\sigma_{\gamma 3}^{-(k)}, -\frac{1}{10}S_{\alpha\beta 33}^{(k)}\bar{M}_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}} + <\sigma_{33}^{-(k)}, -\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\bar{M}_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+ <\bar{M}_{\alpha\beta}^{(k)}, -\eta_{\beta}\bar{\phi}_{\alpha}^{(k)}>_{S^{(k)}} + <\bar{M}_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_{\gamma}\sigma_{\gamma 3}^{-(k)}>_{S^{(k)}}$$

$$+ <\bar{M}_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_{\gamma}\sigma_{\gamma 3}^{-(k-1)}>_{S^{(k)}} \tag{203}$$

To obtain the form of (111), we rewrite the boundary terms as

$$<\bar{M}_{\alpha\beta}^{(k)}, -\eta_{\beta}\bar{\phi}_{\alpha}^{(k)}>_{S^{(k)}} = <\bar{\phi}_{\alpha}^{(k)}, -\eta_{\beta}\bar{M}_{\alpha\beta}^{(k)}>_{S_3^{(k)}} - <\bar{M}_{\alpha\beta}^{(k)}, \eta_{\beta}\bar{\phi}_{\alpha}^{(k)}>_{S^{(k)}} \tag{204}$$

For

$$\{\tilde{u}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \bar{V}_{\rho}^{(k)} \end{Bmatrix}$$

we have

$$<\bar{V}_{\rho}^{(k)}, -\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\sigma_{\gamma 3}^{-(k-1)} - \bar{\phi}_{\rho}^{(k)} - \bar{v}_{3,\rho}^{(k)} + \frac{24}{5t_k}S_{\rho 3\gamma 3}^{(k)}V_{\gamma}^{(k)} - \frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$= <\sigma_{\gamma 3}^{-(k-1)}, -\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\bar{V}_{\rho}^{(k)}>_{R^{(k)}} + <\bar{\phi}_{\rho}^{(k)}, -\bar{V}_{\rho}^{(k)}>_{R^{(k)}} + <\bar{v}_{3}^{(k)}, V_{\alpha,\alpha}^{(k)}>_{R^{(k)}}$$

$$+ <V_{\gamma}^{(k)}, \frac{24}{5t_k}S_{\rho 3\gamma 3}^{(k)}\bar{V}_{\rho}^{(k)}>_{R^{(k)}} + <\sigma_{\gamma 3}^{-(k)}, -\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\bar{V}_{\rho}^{(k)}>_{R^{(k)}}$$

$$+ <\bar{v}_{3}^{(k)}, -\eta_{\alpha}\bar{V}_{\alpha}^{(k)}>_{S^{(k)}} \tag{205}$$

To obtain the form of (111), we rewrite the boundary terms as

$$< \bar{v}_3^{(k)}, -\eta_\alpha \bar{V}_\alpha^{(k)} >_{S^{(k)}} = < \bar{v}_3^{(k)}, -\eta_\alpha \bar{V}_\alpha^{(k)} >_{S_5^{(k)}} - < \bar{V}_\alpha^{(k)}, \eta_\alpha \bar{v}_3^{(k)} >_{S_6^{(k)}} \tag{206}$$

Taking inner product of a typical $\{\tilde{\sigma}\}^{-(k)}$ with the corresponding set of equations in (173), we have

$$\begin{aligned}
&< \{\tilde{\sigma}\}^{-(k)}, [\Lambda]^{(k)}\{\sigma\}^{-(k-1)} + [B]^{(k)}\{u\}^{(k)} + [\Xi]^{(k)}\{\sigma\}^{-(k)} + [\bar{C}]^{(k+1)}\{u\}^{(k+1)} + [\bar{\Lambda}]^{(k+1)}\{\sigma\}^{-(k+1)} >_{R^{(k)}} \\
&= < \{\sigma\}^{-(k-1)}, [\bar{\Lambda}]^{(k)}\{\tilde{\sigma}\}^{-(k)} >_{R^{(k)}} + < \{u\}^{(k)}, [B]^{(k)}\{\tilde{\sigma}\}^{-(k)} >_{R^{(k)}} \\
&+ < \{\sigma\}^{-(k)}, [\Xi]^{(k)}\{\tilde{\sigma}\}^{-(k)} >_{R^{(k)}} + < \{u\}^{(k+1)}, [C]^{(k+1)}\{\tilde{\sigma}\}^{-(k)} >_{R^{(k)}} \\
&+ < \{\sigma\}^{-(k+1)}, [\Lambda]^{(k+1)}\{\tilde{\sigma}\}^{-(k)} >_{R^{(k)}} + < \{\sigma\}^{-(k-1)}, [\bar{\theta}]^{(k)}\{\tilde{\sigma}\}^{-(k)} >_{S^{(k)}} \\
&+ < \{\sigma\}^{-(k)}, [\psi]^{(k)}\{\tilde{\sigma}\}^{-(k)} >_{S^{(k)}} + < \{\sigma\}^{-(k+1)}, [\theta]^{(k+1)}\{\tilde{\sigma}\}^{-(k)} >_{S^{(k)}} \\
&- < \{\tilde{\sigma}\}^{-(k)}, [\theta]^{(k)}\{\sigma\}^{-(k-1)} + [E]^{(k)}\{u\}^{(k)} + [\psi]^{(k)}\{\sigma\}^{-(k)} + [\bar{F}]^{(k+1)}\{u\}^{(k+1)} + [\bar{\theta}]^{(k+1)}\{\sigma\}^{-(k+1)} >_{S^{(k)}}
\end{aligned} \tag{207}$$

Setting

$$\{\tilde{\sigma}\} = \begin{pmatrix} \tilde{\sigma}_{\gamma 3}^{-(k)} \\ 0 \end{pmatrix}$$

the left hand side of (207) becomes

$$\begin{aligned}
&< \tilde{\sigma}_{\gamma 3}^{-(k)}, \Lambda_{11}^{(k)}\sigma_{\rho 3}^{-(k-1)} + \Lambda_{12}^{(k)}\sigma_{33}^{-(k-1)} \\
&- \bar{v}_\gamma^{(k)} + \frac{t_k}{2}\bar{\phi}_\gamma^{(k)} + \frac{t_k}{12}S_{\alpha\beta 33}^{(k)}N_{\alpha\beta,\gamma}^{(k)} - \frac{1}{10}S_{\alpha\beta 33}^{(k)}M_{\alpha\beta,\gamma}^{(k)} - \frac{2}{5}S_{\rho 3\gamma 3}^{(k)}V_\rho^{(k)} \\
&+ \Xi_{11}^{(k)}\sigma_{\rho 3}^{-(k)} + \Xi_{12}^{(k)}\sigma_{33}^{-(k)} \\
&+ \bar{v}_\gamma^{(k+1)} + \frac{t_{k+1}}{2}\bar{\phi}_\gamma^{(k+1)} - \frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta,\gamma}^{(k+1)} - \frac{1}{10}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta,\gamma}^{(k+1)} - \frac{2}{5}S_{\rho 3\gamma 3}^{(k+1)}V_\rho^{(k+1)} \\
&+ \bar{\Lambda}_{11}^{(k+1)}\sigma_{\rho 3}^{-(k+1)} + \bar{\Lambda}_{12}^{(k+1)}\sigma_{33}^{-(k+1)} >_{R^{(k)}}
\end{aligned}$$

This can be rewritten using (182) through (192) as

$$< \sigma_{\gamma 3}^{-(k-1)}, \bar{\Lambda}_{11}^{(k)}\tilde{\sigma}_{\rho 3}^{-(k)} >_{R^{(k)}} + < \sigma_{33}^{-(k-1)}, \bar{\Lambda}_{21}^{(k)}\tilde{\sigma}_{\rho 3}^{-(k)} >_{R^{(k)}}$$

$$+ \ <\bar{v}_{\gamma}^{(k)} \ , \ -\tilde{\sigma}_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\bar{\phi}_{\gamma}^{(k)} \ , \ \frac{t_k}{2}\tilde{\sigma}_{\gamma 3}^{(k)}>_{R^{(k)}}$$

$$+ \ <N_{\alpha\beta}^{(k)} \ , \ -\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\tilde{\sigma}_{\gamma 3,\gamma}^{-(k)}>_{R^{(k)}}$$

$$+ \ <M_{\alpha\beta}^{(k)} \ , \ \frac{1}{10}S_{\alpha\beta 33}^{(k)}\tilde{\sigma}_{\gamma 3,\gamma}^{-(k)}>_{R^{(k)}}$$

$$+ \ <V_{\rho}^{(k)} \ , \ -\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\tilde{\sigma}_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{\gamma 3}^{-(k)} \ , \ \Xi_{11}^{(k)}\tilde{\sigma}_{\rho 3}^{-(k)}>_{R^{(k)}}+<\sigma_{33}^{-(k)} \ , \ \Xi_{21}^{(k)}\tilde{\sigma}_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\bar{v}_{\gamma}^{(k+1)} \ , \ \tilde{\sigma}_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\bar{\phi}_{\gamma}^{(k+1)} \ , \ \frac{t_{k+1}}{2}\tilde{\sigma}_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <N_{\alpha\beta}^{(k+1)} \ , \ \frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\tilde{\sigma}_{\gamma 3,\gamma}^{-(k)}>_{R^{(k)}}$$

$$+ \ <M_{\alpha\beta}^{(k+1)} \ , \ \frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\tilde{\sigma}_{\gamma 3,\gamma}^{-(k)}>_{R^{(k)}}$$

$$+ \ <V_{\rho}^{(k+1)} \ , \ -\frac{2}{5}S_{\rho 3\gamma 3}^{(k+1)}\tilde{\sigma}_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{\gamma 3}^{-(k+1)} \ , \ \Lambda_{11}^{(k+1)}\tilde{\sigma}_{\rho 3}^{-(k)}>_{R^{(k)}}+<\sigma_{33}^{-(k+1)} \ , \ \Lambda_{21}^{(k+1)}\tilde{\sigma}_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\tilde{\sigma}_{\gamma 3}\eta_{\gamma}^{-(k)} \ , \ -(\frac{t_{k+1}}{12}N_{\alpha\beta}^{(k+1)}+\frac{1}{10}M_{\alpha\beta}^{(k+1)})S_{\alpha\beta 33}^{(k+1)}+(\frac{t_k}{12}N_{\alpha\beta}^{(k)}-\frac{1}{10}M_{\alpha\beta}^{(k)})S_{\alpha\beta 33}^{(k)}>_{S^{(k)}}$$

$$+ \ (-\frac{1}{120}-\frac{1}{840})(S_{3333}^{(k)}t_k^3+S_{3333}^{(k+1)}t_{k+1}^3)[<\tilde{\sigma}_{\gamma 3}^{-(k)}\eta_{\gamma} \ , \ \sigma_{\rho 3,\rho}^{-(k)}>_{S^{(k)}}-<\sigma_{\gamma 3}^{-(k)}\eta_{\gamma} \ , \ \tilde{\sigma}_{\rho 3,\rho}^{-(k)}>_{S^{(k)}}]$$

$$+ \ <(\frac{1}{24}+\frac{3}{280})(S_{3333}^{(k)}t_k^3-S_{3333}^{(k+1)}t_{k+1}^3)\tilde{\sigma}_{\gamma 3}^{-(k)} \ , \ \sigma_{33}^{-(k)}\eta_{\gamma}>_{S^{(k)}}$$

$$+ \ (\frac{1}{120}-\frac{1}{840})S_{3333}^{(k+1)}t_{k+1}^3[<\tilde{\sigma}_{\gamma 3}^{-(k)}\eta_{\gamma} \ , \ \sigma_{\rho 3,\rho}^{-(k+1)}>_{S^{(k)}}-<\sigma_{\gamma 3}^{-(k+1)}\eta_{\gamma} \ , \ \tilde{\sigma}_{\rho 3,\rho}^{-(k)}>_{S^{(k)}}]$$

$$+ \ (\frac{1}{120}-\frac{1}{840})S_{3333}^{(k)}t_k^3[<\tilde{\sigma}_{\gamma 3}^{-(k)}\eta_{\gamma} \ , \ \sigma_{\rho 3,\rho}^{-(k-1)}>_{S^{(k)}}-<\sigma_{\gamma 3}^{-(k-1)}\eta_{\gamma} \ , \ \tilde{\sigma}_{\rho 3,\rho}^{-(k)}>_{S^{(k)}}]$$

$$+ \ <\tilde{\sigma}_{\gamma 3}^{-(k)}\eta_{gama} \ , -(\frac{3}{280}-\frac{1}{24})t_k^2 S_{3333}^{(k)}\sigma_{33}^{-(k-1)}>_{S^{(k)}}$$

$$+ \ <\tilde{\sigma}_{\gamma 3}^{-(k)}\eta_{\gamma} \ , -(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\sigma_{33}^{-(k+1)}>_{S^{(k)}}$$

For

$$\{\tilde{\sigma}\} = \begin{Bmatrix} 0 \\ \tilde{\sigma}_{33}^{-(k)} \end{Bmatrix}$$

we have

$$<\tilde{\sigma}_{33}^{-(k)} \ , \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)}$$

$$- \ \bar{v}_3^{(k)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}N_{\alpha\beta}^{(k)}-\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}M_{\alpha\beta}^{(k)}$$

$$+ \ \Xi_{21}^{(k)}\sigma_{\gamma 3}^{-(k)}+\Xi_{22}^{(k)}\sigma_{33}^{-(k)}$$

$$+ \ \bar{v}_3^{(k+1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta}^{(k+1)}+\frac{6}{5t_{k+1}}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta}^{(k+1)}>_{R^{(k)}}$$

$$+ \ \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k)}}$$

Using (184) and (189) through (191), this can be rewritten as

$$<\sigma_{\gamma 3}^{-(k-1)} \ , \bar{\Lambda}_{12}^{(k)}\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}+<\sigma_{33}^{-(k-1)} \ , \bar{\Lambda}_{22}^{(k)}\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\bar{v}_3^{(k)} \ , -\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ <N_{\alpha\beta}^{(k)} \ , \frac{1}{2}S_{\alpha\beta 33}^{(k)}\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ <M_{\alpha\beta}^{(k)} \ , -\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{\gamma 3}^{-(k)} \ , \Xi_{12}^{(k)}\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}+<\sigma_{33}^{-(k)} \ , \Xi_{22}^{(k)}\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\bar{v}_3^{(k+1)} \ , \tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ <N_{\alpha\beta}^{(k+1)} \ , \frac{1}{2}S_{\alpha\beta 33}^{(k+1)}\tilde{\sigma}_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ < M^{(k+1)}_{\alpha\beta} \ , \ \frac{6}{5t_{k+1}} S^{(k+1)}_{\alpha\beta 33} \tilde{\sigma}^{-(k)}_{33} >_{R^{(k)}}$$

$$+ \ < \sigma^{-(k+1)}_{\gamma 3} \ , \ \Lambda^{(k+1)}_{12} \tilde{\sigma}^{-(k)}_{33} >_{R^{(k)}} + < \sigma^{-(k+1)}_{33} \ , \ \Lambda^{(k+1)}_{22} \tilde{\sigma}^{-(k)}_{33} >_{R^{(k)}}$$

$$+ \ < \tilde{\sigma}^{-(k)}_{33} \ , \ -(\frac{1}{24} + \frac{3}{280})(t_k^2 S^{(k)}_{3333} - t_{k+1} S^{(k+1)}_{3333}) \sigma^{-(k)}_{\gamma 3} \eta_\gamma >_{S^{(k)}}$$

$$+ \ < \tilde{\sigma}^{-(k)}_{33} \ , \ -(-\frac{3}{280} + \frac{1}{24}) t^2_{k+1} S^{(k+1)}_{3333} \sigma^{-(k+1)}_{\gamma 3} \eta_\gamma >_{S^{(k)}}$$

$$+ \ < \tilde{\sigma}^{-(k)}_{33} \ , \ -(\frac{3}{280} - \frac{1}{24}) t^2_k S^{(k)}_{3333} \sigma^{-(k-1)}_{\gamma 3} \eta_\gamma >_{S^{(k)}}$$

The consistent boundary operators for N layers can be written collectively as follows:

$$[W] \ \{Y\} = \{G\} \tag{208}$$

where

$$
[W]=\left|\begin{array}{ccccccccccccc}
[D]^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
[E]^{(1)} & [\psi]^{(1)} & [F]^{(2)} & [\bar{\theta}]^{(2)} & 0 & 0 & 0 & & & & & & \\
0 & 0 & [D]^{(2)} & 0 & 0 & 0 & 0 & & & & & & \\
0 & [\theta]^{(2)} & [E]^{(2)} & [\psi]^{(2)} & [F]^{(3)} & [\bar{\theta}]^{(3)} & 0 & & & & & & \\
0 & 0 & 0 & 0 & [D]^{(3)} & 0 & 0 & & & & & & \\
0 & 0 & 0 & [\theta]^{(3)} & [E]^{(3)} & [\psi]^{(3)} & [F]^{(4)} & [\bar{\theta}]^{(4)} & & & & & \\
 & & & 0 & 0 & 0 & [D]^{(4)} & 0 & & & 0 & 0 & 0 \\
 & & & 0 & 0 & [\theta]^{(4)} & [E]^{(4)} & [\psi]^{(4)} & . & . & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 & . & . & . & [F]^{(N-1)} & [\bar{\theta}]^{(N-1)} & 0 \\
 & & & & & & & . & . & 0 & [D]^{(N-1)} & 0 & 0 \\
 & & & & & & & & 0 & [\theta]^{(N-1)} & [E]^{(N-1)} & [\psi]^{(N-1)} & [F]^{(N)} \\
 & & & & & & & & 0 & 0 & 0 & 0 & [D]^{(N)}
\end{array}\right|
$$

$$
\{Y\} = \begin{Bmatrix} \{u\}^{(1)} \\ \{\sigma\}^{-(1)} \\ \{u\}^{(2)} \\ \{\sigma\}^{-(2)} \\ \{u\}^{(3)} \\ \{\sigma\}^{-(3)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \{u\}^{(N-1)} \\ \{\sigma\}^{-(N-1)} \\ \{u\}^{(N)} \end{Bmatrix} \qquad \{G\} = \begin{Bmatrix} \{g_u\}^{(1)} \\ \{g_\sigma\}^{(1)} \\ \{g_u\}^{(2)} \\ \{g_\sigma\}^{(2)} \\ \{g_u\}^{(3)} \\ \{g_\sigma\}^{(3)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \{g_u\}^{(N-1)} \\ \{g_\sigma\}^{(N-1)} \\ \{g_u\}^{(N)} \end{Bmatrix}
$$

$$
\{\tilde{u}\}^{(k)} = \begin{Bmatrix} \tilde{V}_\gamma^{(k)} \\ \tilde{\Phi}_\gamma^{(k)} \\ \tilde{V}_3^{(k)} \\ \bar{N}_{\alpha\beta}^{(k)} \\ \bar{M}_{\alpha\beta}^{(k)} \\ \bar{V}_\gamma^{(k)} \end{Bmatrix} \tag{209}
$$

$$[D]^{(k)} = \begin{bmatrix} 0 & 0 & 0 & -\eta_\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\eta_\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta_\alpha \\ \eta_\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_\alpha & 0 & 0 & 0 \end{bmatrix} \tag{210}$$

$$[E]^{(k)} = \begin{bmatrix} 0 & 0 & 0 & -\frac{t_k}{12}S^{(k)}_{\alpha\beta 33}\eta_\gamma & \frac{1}{10}S^{(k)}_{\alpha\beta 33}\eta_\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{211}$$

$$[F]^{(k)} = \begin{bmatrix} 0 & 0 & 0 & \frac{t_k}{12}S^{(k)}_{\alpha\beta 33}\eta_\gamma & \frac{1}{10}S^{(k)}_{\alpha\beta 33}\eta_\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{212}$$

$$\{\tilde{\sigma}\}^{(k)} = \begin{Bmatrix} \bar{\sigma}^{(k)}_{\gamma 3} \\ \bar{\sigma}^{(k)}_{33} \end{Bmatrix} \tag{213}$$

$$[\theta]^{(k)} = \begin{bmatrix} -\left(\frac{1}{120}-\frac{1}{840}\right)t_k^3 S^{(k)}_{3333}\eta_\rho \frac{\partial}{\partial\gamma} & \left(-\frac{1}{24}+\frac{3}{280}\right)t_k^2 S^{(k)}_{3333}\eta_\gamma \\ 0 & 0 \end{bmatrix} \tag{214}$$

$$[\bar{\theta}]^{(k)} = \begin{bmatrix} -\left(\frac{1}{120}-\frac{1}{840}\right)t_k^3 S^{(k)}_{3333}\eta_\rho \frac{\partial}{\partial\gamma} & \left(\frac{1}{24}-\frac{3}{280}\right)t_k^2 S^{(k)}_{3333}\eta_\gamma \\ 0 & 0 \end{bmatrix} \tag{215}$$

$$[\psi]^{(k)} = \begin{bmatrix} \left(\frac{1}{120}+\frac{1}{840}\right)\left[t_k^3 S^{(k)}_{3333}+t_{k+1}^3 S^{(k+1)}_{3333}\right]\eta_\rho \frac{\partial}{\partial\gamma} & \left(\frac{1}{24}+\frac{3}{280}\right)\left[S^{(k+1)}_{3333}t_{k+1}^2 - S^{(k)}_{3333}t_k^2\right]\eta_\rho \\ 0 & 0 \end{bmatrix} \tag{216}$$

The vector [G] consists of values of the variables specified at various points on the boundary. Explicitly, typical equations in (208) are:

$$[\theta]^{(k)}\{\sigma\}^{-(k-1)}+[E]^{(k)}\{u\}^{(k)}+[\psi]^{(k)}\{\sigma\}^{(k)}+[F]^{(k)}\{u\}^{(k+1)}+[\bar{\theta}]^{(k+1)}\{\sigma\}^{(k+1)}=\{g_\sigma\}^{(k)} \quad \text{on } S^{(k)} \tag{217}$$

and

$$[D]^{(k)}\{u\}^{(k)}=\{g_u\}^{(k)} \quad \text{on } S_u^{(k)} \tag{218}$$

4.5.1 Prescribed Boundary Conditions

From (208), the couplings of the field variables in the boundary terms are realized. For existence of a variational formulation for the boundary value problem in the form proposed by Sandhu [1975], the boundary conditions must be specified in the consistent form represented by (217) and (218). The boundary terms will occur in the governing function (103) in the form:

$$<\{u\}^{(k)}, [D]^{(k)}\{u\}^{(k)}-2\{g_u\}^{(k)}>_{S_u^{(k)}}+<\{\sigma\}^{-(k)}, [\theta]^{(k)}\{\sigma\}^{-(k-1)}+[E]^{(k)}\{u\}^{(k)}$$

$$+ [\psi]^{(k)}\{\sigma\}^{-(k)}+[F]^{(k)}\{u\}^{(k+1)}+[\bar{\theta}]^{(k+1)}\{\sigma\}^{-(k+1)}-2\{g_\sigma\}^{(k)}>_{S^{(k)}} \tag{219}$$

$\{g_u\}^{(k)}$, $\{g_\sigma\}^{(k)}$ are values specified on appropriate portions of the boundary of the kth layer. Explicitly, (219), is

$$<\bar{v}_\alpha^{(k)}, -\eta_\beta N_{\alpha\beta}^{(k)}-2g_1^{(k)}>_{S_1^{(k)}}$$

$$+ <\bar{\Phi}_\alpha^{(k)}, -\eta_\beta M_{\alpha\beta}^{(k)}-2g_3^{(k)}>_{S_3^{(k)}}$$

$$+ <\bar{v}_3^{(k)}, -\eta_\alpha V_\alpha^{(k)}-2g_5^{(k)}>_{S_5^{(k)}}$$

$$+ <N_{\alpha\beta}^{(k)}, \eta_\beta \bar{v}_\alpha^{(k)}-2g_2^{(k)}>_{S_2^{(k)}}$$

$$+ <M_{\alpha\beta}^{(k)}, \eta_\beta \bar{\Phi}_\alpha^{(k)}-2g_4^{(k)}>_{S_4^{(k)}}$$

$$+ <V_\alpha^{(k)}, \eta_\alpha v_3^{(k)}-2g_6^{(k)}>_{S_6^{(k)}}$$

$$+ <\sigma_{\gamma 3}^{-(k)}, [-(\frac{1}{120}-\frac{1}{840})h_k^3 S_{3333}^{(k)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k-1)}+[(-\frac{1}{24}+\frac{3}{280})h_k^2 S_{3333}^{(k)}\eta_\gamma]\sigma_{33}^{-(k-1)}$$

$$+\ [-\frac{t_k}{12}S^{(k)}_{\alpha\beta 33}\eta_\gamma]N^{(k)}_{\alpha\beta}+[\frac{1}{10}S^{(k)}_{\alpha\beta 33}\eta_\gamma]M^{(k)}_{\alpha\beta}$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_k^3S^{(k)}_{3333}+t^3_{k+1}S^{(k+1)}_{3333})\eta_\gamma]\bar{\sigma}^{(k)}_{\rho 3,\rho}$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t^2_{k+1}S^{(k+1)}_{3333}-t_k^2S^{(k)}_{3333})\eta_\gamma]\bar{\sigma}^{(k)}_{33}+[\frac{t_{k+1}}{12}S^{(k+1)}_{\alpha\beta 33}\eta_\gamma]N^{(k+1)}_{\alpha\beta}$$

$$+\ [\frac{1}{10}S^{(k+1)}_{\alpha\beta 33}\eta_\gamma]M^{(k+1)}_{\alpha\beta}+[-(\frac{1}{120}-\frac{1}{840})t^3_{k+1}S^{(k+1)}_{3333}\eta_\gamma]\bar{\sigma}^{(k+1)}_{\rho 3,\rho}$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t^2_{k+1}S^{(k+1)}_{3333}\eta_\gamma]\bar{\sigma}^{(k+1)}_{33}-2g^{(k)}_\sigma>_{S^{(k)}} \tag{220}$$

Thus, the specified boundary conditions are:

$$\left.\begin{array}{ll}
-N^{(k)}_{\alpha\beta}\eta_\beta=g^{(k)}_1 & \text{on } S^{(k)}_1 \\
-M^{(k)}_{\alpha\beta}\eta_\beta=g^{(k)}_3 & \text{on } S^{(k)}_3 \\
-V^{(k)}_{\alpha}\eta_\alpha=g^{(k)}_5 & \text{on } S^{(k)}_5 \\
\bar{v}^{(k)}_{\alpha}\eta_\beta=g^{(k)}_2 & \text{on } S^{(k)}_2 \\
\bar{\phi}^{(k)}_{\alpha}\eta_\beta=g^{(k)}_4 & \text{on } S^{(k)}_4 \\
\bar{v}^{(k)}_{3}\eta_\alpha=g^{(k)}_6 & \text{on } S^{(k)}_6
\end{array}\right| \tag{221}$$

and the continuity conditions are:

$$[-(\frac{1}{120}-\frac{1}{840})t_k^3S^{(k)}_{3333}\eta_\gamma]\bar{\sigma}^{(k-1)}_{\rho 3,\rho}+[(-\frac{1}{24}+\frac{3}{280})t_k^2S^{(k)}_{3333}\eta_\gamma]\bar{\sigma}^{(k-1)}_{33}$$

$$+\ [-\frac{t_k}{12}S^{(k)}_{\alpha\beta 33}\eta_\gamma]N^{(k)}_{\alpha\beta}+[\frac{1}{10}S^{(k)}_{\alpha\beta 33}\eta_\gamma]M^{(k)}_{\alpha\beta}$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_k^3S^{(k)}_{3333}+t^3_{k+1}S^{(k+1)}_{3333})\eta_\gamma]\bar{\sigma}^{(k)}_{\rho 3,\rho}$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t^2_{k+1}S^{(k+1)}_{3333}-t_k^2S^{(k)}_{3333})\eta_\gamma]\bar{\sigma}^{(k)}_{33}+[\frac{t_{k+1}}{12}S^{(k+1)}_{\alpha\beta 33}\eta_\gamma]N^{(k+1)}_{\alpha\beta}$$

$$+\ [\frac{1}{10}S^{(k+1)}_{\alpha\beta 33}\eta_\gamma]M^{(k+1)}_{\alpha\beta}+[-(\frac{1}{120}-\frac{1}{840})t^3_{k+1}S^{(k+1)}_{3333}\eta_\gamma]\bar{\sigma}^{(k+1)}_{\rho 3,\rho}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^{2}S_{3333}^{(k+1)}\eta_{\gamma}]\sigma_{33}^{-(k+1)}=g_{\sigma}^{(k)} \quad \text{on } S^{(k)} \tag{222}$$

Jump discontinuities in the field variables can be introduced as conditions at internal boundaries. Indeed, as finite element bases generally have limited smoothness across interelement boundaries, even if there are no discontinuities in the physical problem, the smoothness condition of the physical problem needs to be introduced as a set of homogeneous discontinuity conditions. Similar to the format for the conditions on the boundary, these are:

$$\left.\begin{array}{ll}
-(N_{\alpha\beta}^{(k)})'\eta_{\beta}=g_{1}'^{(k)} & \text{on } S_{1i}^{(k)} \\
-(M_{\alpha\beta}^{(k)})'\eta_{\beta}=g_{3}'^{(k)} & \text{on } S_{3i}^{(k)} \\
-(V_{\alpha}^{(k)})'\eta_{\alpha}=g_{5}'^{(k)} & \text{on } S_{5i}^{(k)} \\
(\bar{v}_{\alpha}^{(k)})'\eta_{\beta}=g_{2}'^{(k)} & \text{on } S_{2i}^{(k)} \\
(\bar{\Phi}_{\alpha}^{(k)})'\eta_{\beta}=g_{4}'^{(k)} & \text{on } S_{4i}^{(k)} \\
(\bar{v}_{3}^{(k)})'\eta_{\alpha}=g_{6}'^{(k)} & \text{on } S_{6i}^{(k)}
\end{array}\right| \tag{223}$$

$$[-(\frac{1}{120}-\frac{1}{840})t_{k}^{3}S_{3333}^{(k)}\eta_{\gamma}](\sigma_{\rho3,\rho}^{-(k-1)})'+[(-\frac{1}{24}+\frac{3}{280})t_{k}^{2}S_{3333}^{(k)}\eta_{\gamma}](\sigma_{33}^{-(k-1)})'$$

$$+ [-\frac{t_k}{12}S_{\alpha\beta33}^{(k)}\eta_{\gamma}](N_{\alpha\beta}^{(k)})'+[\frac{1}{10}S_{\alpha\beta33}^{(k)}\eta_{\gamma}](M_{\alpha\beta}^{(k)})'$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t_{k}^{3}S_{3333}^{(k)}+t_{k+1}^{3}S_{3333}^{(k+1)})\eta_{\rho}](\sigma_{\rho3,\rho}^{-(k)})'$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^{2}S_{3333}^{(k+1)}-t_{k}^{2}S_{3333}^{(k)})\eta_{\gamma}](\sigma_{33}^{-(k)})'+[\frac{t_{k+1}}{12}S_{\alpha\beta33}^{(k+1)}\eta_{\gamma}](N_{\alpha\beta}^{(k+1)})'$$

$$+ [\frac{1}{10}S_{\alpha\beta33}^{(k+1)}\eta_{\gamma}](M_{\alpha\beta}^{(k+1)})'+[-(\frac{1}{120}-\frac{1}{840})t_{k+1}^{3}S_{3333}^{(k+1)}\eta_{\gamma}](\sigma_{\rho3,\rho}^{-(k+1)})'$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^{2}S_{3333}^{(k+1)}\eta_{\gamma}](\sigma_{33}^{-(k+1)})'=g_{\sigma}'^{(k)} \quad \text{on } S_{i}^{(k)} \tag{224}$$

The homogeneous discontinuity condition, i.e. vanishing $g_i'^{(k)}$, represents the internal continuity constraints for the physical problem. In allowing for jump discontinuities, generalizing Sandhu's [1975] assumption for the problem of elastostatics, one could require

$$S_{1i}^{(k)} \bigcap S_{2i}^{(k)} = 0 \tag{225}$$

$$S_{3i}^{(k)} \bigcap S_{4i}^{(k)} = 0 \tag{226}$$

$$S_{5i}^{(k)} \bigcap S_{6i}^{(k)} = 0 \tag{227}$$

where $S_{\alpha i}^{(k)}$, $\alpha = 1, 2, . . , 6$ are surfaces imbedded in the region $R^{(k)}$. However, Bufler [1979] showed that Sandhu's assumption is unnecessary. The relationship between $S_i^{(k)}$, $i = 1, 2, . . 6$ which are subsets of $S^{(k)}$ the boundary of $R^{(k)}$ have already been determined. $\{S_i^{(k)}, S_{i+1}^{(k)}, \ i = 1, 3, 5\}$ constitute pairs of complementary subsets of $S^{(k)}$, $k = 1, 2, .., .., N$.

4.6 Variational Principle

4.6.1 Variational Formulation for Linear Coupled Self-Adjoint Problem

For a boundary value problem with n independent field variables, defined by (112), (104) and (105), the governing function is [Sandhu, 1976]

$$\Omega = \sum_{i=1}^{n} [<u_i , \sum_{j=1}^{n} A_{ij} u_j - 2f_i>_R + <u_i , \sum_{j=1}^{n} C_{ij} u_j - 2g_i>_{S_i}$$

$$+ <u_i , \sum_{j=1}^{n} (C_{ij} u_j)' - 2g_i'>_{S_i}] \tag{228}$$

4.6.2 Governing Function for the Laminated Plate

The set of equations (173) along with the boundary conditions, (221) and the continuity condition (222), including the jump discontinuity conditions in (228) gives

$$\Omega(u\,,\sigma)=2\langle\{u\}^{(1)}\,,\{P\}^{(1)}\rangle_{R^{(1)}}+2\langle\{u\}^{(N)}\,,\{P\}^{(N)}\rangle_{R^{(1)}}$$

$$+\sum_{k=2}^{N}\langle\{u\}^{(k)}\,,[C]^{(k)}\{\sigma\}^{-(k-1)}\rangle_{R^{(k)}}+\sum_{k=1}^{N}\langle\{u\}^{(k)}\,,[A]^{(k)}\{u\}^{(k)}\rangle_{R^{(k)}}$$

$$+\sum_{k=1}^{N-1}\langle\{u\}^{(k)}\,,[B]^{(k)}\{\sigma\}^{-(k)}\rangle_{R^{(k)}}$$

$$+\sum_{k=1}^{N-1}\langle\{\sigma\}^{-(k)}\,,[\overline{B}]^{(k)}\{u\}^{(k)}\rangle_{R^{(k)}}+\sum_{k=1}^{N-1}\langle\{\sigma\}^{-(k)}\,,[\Xi]^{(k)}\{\sigma\}^{-(k)}\rangle_{R^{(k)}}$$

$$+\sum_{k=1}^{N-1}\langle\{\sigma\}^{-(k)}\,,[\overline{C}]^{(k+1)}\{u\}^{(k+1)}\rangle_{R^{(k+1)}}+\sum_{k=2}^{N-1}\langle\{\sigma\}^{-(k)}\,,[\Lambda]^{(k)}\{\sigma\}^{-(k-1)}\rangle_{R^{(k)}}$$

$$+\sum_{k=1}^{N-2}\langle\{\sigma\}^{-(k)}\,,[\overline{\Lambda}]^{(k+1)}\{\sigma\}^{-(k+1)}\rangle_{R^{(k+1)}}$$

$$+2\langle\{\sigma\}^{-(N-1)}\,,[Q]^{(N-1)}\rangle_{R^{(N)}}+2\langle\{\sigma\}^{-(1)}\,,[Q]^{(1)}\rangle_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\langle\{u\}^{(k)}\,,[D]^{(k)}\{u\}^{(k)}-2\{g_u\}^{(k)}\rangle_{S_u^{(k)}}$$

$$+\langle\{\sigma\}^{-(1)},[E]^{(1)}\{u\}^{(1)}+[\psi]^{(1)}\{\sigma\}^{-(1)}+[F]^{(1)}\{u\}^{(2)}+[\overline{\theta}]^{(2)}\{\sigma\}^{-(2)}-2\{g_\sigma\}^{(1)}\rangle_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\langle\{\sigma\}^{-(k)},[\theta]^{(k)}\{\sigma\}^{-(k-1)}+[E]^{(k)}\{u\}^{(k)}+[\psi]^{(k)}\{\sigma\}^{-(k)}$$

$$+[F]^{(k+1)}\{u\}^{(k+1)}+[\overline{\theta}]^{(k+1)}\{\sigma\}^{-(k+1)}-2\{g_\sigma\}^{(k)}\rangle_{S^{(k)}}$$

$$+\langle\{\sigma\}^{-(N-1)},[\theta]^{(N-1)}\{\sigma\}^{-(N-2)}+[E]^{(N-1)}\{u\}^{(N-1)}+[\psi]^{(N-1)}\{\sigma\}^{-(N-1)}+[F]^{(N)}\{u\}^{(N)}$$

$$-2\{g_\sigma\}^{(N-1)}\rangle_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\langle\{u\}^{(k)}\,,[D]^{(k)}(\{u\}^{(k)})'-2\{g'_u\}^{(k)}\rangle_{S_{ui}^{(k)}}$$

$$+<\{\sigma\}^{-(1)}, [E]^{(1)}(\{u\}^{(1)})'+[\psi]^{(1)}(\{\sigma\}^{-(1)})'+[F]^{(1)}(\{u\}^{(2)})'+[\theta]^{(2)}(\{\sigma\}^{-(2)})'$$

$$-2\{g'_\sigma\}^{(1)}>_{S_i^{(1)}}$$

$$+\sum_{k=2}^{N-2}<\{\sigma\}^{-(k)}, [\theta]^{(k)}(\{\sigma\}^{-(k-1)})'+[E]^{(k)}(\{u\}^{(k)})'+[\psi]^{(k)}(\{\sigma\}^{-(k)})'$$

$$+[F]^{(k+1)}(\{u\}^{(k+1)})'+[\theta]^{(k+1)}(\{\sigma\}^{-(k+1)})'-2\{g'_\sigma\}^{(k)}>_{S_i^{(k)}}$$

$$+<\{\sigma\}^{-(N-1)}, [\theta]^{(N-1)}(\{\sigma\}^{-(N-2)})'+[E]^{(N-1)}(\{u\}^{(N-1)})'+[\psi]^{(N-1)}(\{\sigma\}^{-(N-1)})'$$

$$+[F]^{(N)}(\{u\}^{(N)})'-2\{g'_\sigma\}^{(N-1)}>_{S_i^{(N-1)}} \qquad (229)$$

where $R^{(k)}$ is the two-dimensional region of the kth lamina and $S_u^{(k)}$ symbolically represents appropriate portions of the boundary of $R^{(k)}$. $\{u\}^{(k)}$, $\{\sigma\}^{-(k)}$ denote the set $\{u\}^{(k)}=\{\bar{v}_\alpha^{(k)}, \bar{\phi}_\alpha^{(k)}, \bar{v}_3^{(k)}, N_{\alpha\beta}^{(k)}, M_{\alpha\beta}^{(k)}, V_\alpha^{(k)}\}$ and $\{\sigma\}^{-(k)}=\{\sigma_{\gamma 3}^{-(k)}, \sigma_{33}^{-(k)}\}$, $k = 1, 2, .., N$ respectively. $S_{ui}^{(k)}$ represent appropriates subsets of internal boundaries in the region. Substitution of (159), (160), (161), (165), (166), (167), (168), (169), (171), (172), (210) through (212), and (214) through (216) into (229) gives the explicit form of the functional, including the jump discontinuities, as

$$\Omega(u\ ,\sigma)=2<\bar{v}_\gamma^{(1)}\ ,\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{\phi}_\gamma^{(1)}\ ,\frac{t_1}{2}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{v}_3^{(1)}\ ,\sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+\ 2<N_{\alpha\beta}^{(1)}\ ,\frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{1}{2}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+\ 2<M_{\alpha\beta}^{(1)}\ ,\frac{1}{10}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{6}{5t_1}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}+\ 2<V_\rho^{(1)}\ ,-\frac{2}{5}S_{\rho 3\gamma 3}^{(1)}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}$$

$$+\ 2<\bar{v}_\gamma^{(N)}\ ,-\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{\phi}_\gamma^{(N)}\ ,\frac{t_N}{2}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{v}_3^{(N)}\ ,-\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<N_{\alpha\beta}^{(N)}\ ,-\frac{t_N}{12}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}+\frac{1}{2}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<M_{\alpha\beta}^{(N)}\ ,\frac{1}{10}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}-\frac{6}{5t_N}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}+2<V_\rho^{(N)}\ ,-\frac{2}{5}S_{\rho 3\gamma 3}^{(N)}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}\{<\bar{v}_{\gamma}^{(k)},\ \sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{\phi}_{\gamma}^{(k)},\ \frac{t_k}{2}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)},\ \sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)},\ \frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k-1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\ \frac{1}{10}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k-1)}+\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <V_{\rho}^{(k)},\ -\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{<\bar{v}_{\alpha}^{(k)},\ N_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}+<\bar{\phi}_{\alpha}^{(k)},\ M_{\alpha\beta,\beta}^{(k)}-V_{\alpha}^{(k)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)},\ V_{\alpha,\alpha}^{(k)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)},\ -\bar{v}_{(\alpha,\beta)}^{(k)}+\frac{1}{t_k}S_{\alpha\beta\mu\rho}^{(k)}N_{\mu\rho}^{(k)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\ -\bar{\phi}_{(\alpha,\beta)}^{(k)}+\frac{12}{t_k^3}S_{\alpha\beta\mu\rho}^{(k)}M_{\mu\rho}^{(k)}>_{R^{(k)}}+<V_{\gamma}^{(k)},\ -\bar{\phi}_{\gamma}^{(k)}-\bar{v}_{3,\gamma}^{(k)}+\frac{24}{5t_k}S_{\rho 3\gamma 3}^{(k)}V_{\rho}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\bar{v}_{\gamma}^{(k)},\ -\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}}+<\bar{\phi}_{\gamma}^{(k)},\ \frac{t_k}{2}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)},\ -\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)},\ -\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\ \frac{1}{10}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k)}-\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+\ <V_{\rho}^{(k)},\ -\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\sigma_{\gamma 3}^{-(k)},\ -\bar{v}_{\gamma}^{(k)}+\frac{t_k}{2}\bar{\phi}_{\gamma}^{(k)}+\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}N_{\alpha\beta,\gamma}^{(k)}-\frac{1}{10}S_{\alpha\beta 33}^{(k)}M_{\alpha\beta,\gamma}^{(k)}-\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}V_{\rho}^{(k)}>_{R^{(k)}}$$

$$+\ <\sigma_{\gamma 3}^{-(k)},\ \Xi_{11}^{(k)}\sigma_{\gamma 3}^{-(k)}+\Xi_{12}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+<\sigma_{\gamma 3}^{-(k)},\ \bar{v}_{\gamma}^{(k+1)}+\frac{t_{k+1}}{2}\bar{\phi}_{\gamma}^{(k+1)}-\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta,\gamma}^{(k+1)}-\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta,\gamma}^{(k+1)}-\frac{2}{5}S_{\rho 3\gamma 3}^{(k+1)}V_{\rho}^{(k+1)}>_{R^{(k+1)}}$$

$$+\ <\sigma_{33}^{-(k)},\ -\bar{v}_{3}^{(k)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}N_{\alpha\beta}^{(k)}-\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}M_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+ \ <\bar{\sigma}_{33}^{(k)}\ ,\ \Xi_{21}^{(k)}\bar{\sigma}_{\gamma 3}^{(k)}+\Xi_{22}^{(k)}\bar{\sigma}_{33}^{(k)}>_{R^{(k)}}$$

$$+ \ <\bar{\sigma}_{33}^{(k)}\ ,\ \bar{v}_{3}^{(k+1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta}^{(k+1)}+\frac{6}{5t_{k+1}}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta}^{(k+1)}>_{R^{(k+1)}}\}$$

$$+\sum_{k=2}^{N-1}\{<\bar{\sigma}_{\gamma 3}^{(k)}\ ,\ \Lambda_{11}^{(k)}\bar{\sigma}_{\gamma 3}^{(k-1)}+\Lambda_{12}^{(k)}\bar{\sigma}_{33}^{(k-1)}>+<\bar{\sigma}_{33}^{(k)}\ ,\ \Lambda_{21}^{(k)}\bar{\sigma}_{\gamma 3}^{(k-1)}+\Lambda_{22}^{(k)}\bar{\sigma}_{33}^{(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{<\bar{\sigma}_{\gamma 3}^{(k)}\ ,\ \bar{\Lambda}_{11}^{(k+1)}\bar{\sigma}_{\gamma 3}^{(k+1)}+\bar{\Lambda}_{12}^{(k+1)}\bar{\sigma}_{33}^{(k+1)}>_{R^{(k+1)}}$$

$$+ \ <\bar{\sigma}_{33}^{(k)}\ ,\ \bar{\Lambda}_{21}^{(k+1)}\bar{\sigma}_{\gamma 3}^{(k+1)}+\bar{\Lambda}_{22}^{(k+1)}\bar{\sigma}_{33}^{(k+1)}>_{R^{(k+1)}}\}$$

$$+ \ 2<\{\sigma\}^{-(N-1)}\ ,\ [\bar{\Lambda}]^{(N)}\{\sigma\}^{(N)}>_{R^{(N)}}$$

$$+ \ 2<\{\sigma\}^{-(1)}\ ,\ [\bar{\Lambda}]^{(1)}\{\sigma\}^{(0)}>_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}_{\alpha}^{(k)}\ ,\ -\eta_{\beta}N_{\alpha\beta}^{(k)}-2g_{1}^{(k)}>_{S_{1}^{(k)}}$$

$$+ \ <\bar{\phi}_{\alpha}^{(k)}\ ,\ -\eta_{\beta}M_{\alpha\beta}^{(k)}-2g_{3}^{(k)}>_{S_{3}^{(k)}}$$

$$+ \ <\bar{v}_{3}^{(k)}\ ,\ -\eta_{\alpha}V_{\alpha}^{(k)}-2g_{5}^{(k)}>_{S_{5}^{(k)}}$$

$$+ \ <N_{\alpha\beta}^{(k)}\ ,\ \eta_{\beta}\bar{v}_{\alpha}^{(k)}-2g_{2}^{(k)}>_{S_{2}^{(k)}}$$

$$+ \ <M_{\alpha\beta}^{(k)}\ ,\ \eta_{\beta}\bar{\phi}_{\alpha}^{(k)}-2g_{4}^{(k)}>_{S_{4}^{(k)}}$$

$$+ \ <V_{\alpha}^{(k)}\ ,\ \eta_{\alpha}v_{3}^{(k)}-2g_{6}^{(k)}>_{S_{6}^{(k)}}\ \}$$

$$+ \ <\bar{\sigma}_{\gamma 3}^{(1)}\ ,\ [-\frac{t_{1}}{12}S_{\alpha\beta 33}^{(1)}\eta_{\gamma}]N_{\alpha\beta}^{(1)}+[\frac{1}{10}S_{\alpha\beta 33}^{(1)}\eta_{\gamma}]M_{\alpha\beta}^{(1)}$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_{1}^{3}S_{3333}^{(1)}+t_{2}^{3}S_{3333}^{(2)})\eta_{\gamma}]\bar{\sigma}_{\rho 3,\rho}^{(1)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{2}^{2}S_{3333}^{(2)}-t_{1}^{2}S_{3333}^{(1)})\eta_{\gamma}]\bar{\sigma}_{33}^{(1)}+[\frac{t_{2}}{12}S_{\alpha\beta 33}^{(2)}\eta_{\gamma}]N_{\alpha\beta}^{(2)}$$

$$+ [\frac{1}{10}S^{(2)}_{\alpha\beta33}\eta_\gamma]M^{(2)}_{\alpha\beta}+[-(\frac{1}{120}-\frac{1}{840})t_2^3S^{(2)}_{3333}\eta_\gamma]\sigma^{-(2)}_{\rho3,\rho}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_2^2S^{(2)}_{3333}\eta_\gamma]\sigma^{-(2)}_{33}-2g^{(1)}_\sigma>_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{ <\sigma^{-(k)}_{\gamma3}, [-(\frac{1}{120}-\frac{1}{840})t_k^3S^{(k)}_{3333}\eta_\gamma]\sigma^{-(k-1)}_{\rho3,\rho}$$

$$+ [(-\frac{1}{24}+\frac{3}{280})t_k^2S^{(k)}_{3333}\eta_\gamma]\sigma^{-(k-1)}_{33}+[-\frac{t_k}{12}S^{(k)}_{\alpha\beta33}\eta_\gamma]N^{(k)}_{\alpha\beta}+[\frac{1}{10}S^{(k)}_{\alpha\beta33}\eta_\gamma]M^{(k)}_{\alpha\beta}$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t_k^3S^{(k)}_{3333}+t^3_{k+1}S^{(k+1)}_{3333})\eta_\gamma]\sigma^{-(k)}_{\rho3,\rho}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t^2_{k+1}S^{(k+1)}_{3333}-t_k^2S^{(k)}_{3333})\eta_\gamma]\sigma^{-(k)}_{33}+[\frac{t_{k+1}}{12}S^{(k+1)}_{\alpha\beta33}\eta_\gamma]N^{(k+1)}_{\alpha\beta}$$

$$+ [\frac{1}{10}S^{(k+1)}_{\alpha\beta33}\eta_\gamma]M^{(k+1)}_{\alpha\beta}+[-(\frac{1}{120}-\frac{1}{840})t^3_{k+1}S^{(k+1)}_{3333}\eta_\gamma]\sigma^{-(k+1)}_{\rho3,\rho}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t^2_{k+1}S^{(k+1)}_{3333}\eta_\gamma]\sigma^{-(k+1)}_{33}-2g^{(k)}_\sigma>_{S^{(k)}}\}$$

$$+ <\sigma^{-(N-1)}_{\gamma3}, [-(\frac{1}{120}-\frac{1}{840})t^3_{N-1}S^{(N-1)}_{3333}\eta_\gamma]\sigma^{-(N-2)}_{\rho3,\rho}$$

$$+ [(-\frac{1}{24}+\frac{3}{280})t^2_{N-1}S^{(N-1)}_{3333}\eta_\gamma]\sigma^{-(N-2)}_{33}$$

$$+ [-\frac{t_{N-1}}{12}S^{(N-1)}_{\alpha\beta33}\eta_\gamma]N^{(N-1)}_{\alpha\beta}+[\frac{1}{10}S^{(N-1)}_{\alpha\beta33}\eta_\gamma]M^{(N-1)}_{\alpha\beta}$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t^3_{N-1}S^{(N-1)}_{3333}+t_N^3S^{(N)}_{3333})\eta_\gamma]\sigma^{-(N-1)}_{\rho3,\rho}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_N^2S^{(N)}_{3333}-t^2_{N-1}S^{(N-1)}_{3333})\eta_\gamma]\sigma^{-(N-1)}_{33}+[\frac{t_N}{12}S^{(N)}_{\alpha\beta33}\eta_\gamma]N^{(N)}_{\alpha\beta}$$

$$+ [\frac{1}{10}S^{(N)}_{\alpha\beta33}\eta_\gamma]M^{(N)}_{\alpha\beta}-2g^{(N-1)}_\sigma>_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}^{(k)}_\alpha, (N^{(k)}_{\alpha\beta})'-2g'^{(k)}_1>_{S^{(k)}_{1i}}$$

$$+ <\bar{\phi}^{(k)}_\alpha, -\eta_\beta(M^{(k)}_{\alpha\beta})'-2g'^{(k)}_3>_{S^{(k)}_{3i}}$$

$$+ \ <\bar{v}_3^{(k)}, -\eta_\alpha(V_\alpha^{(k)})' - 2g_5'^{(k)}>_{S_{5i}^{(k)}}$$

$$+ \ <N_{\alpha\beta}^{(k)}, \eta_\beta(\bar{v}_\alpha^{(k)})' - 2g_2'^{(k)}>_{S_{2i}^{(k)}}$$

$$+ \ <M_{\alpha\beta}^{(k)}, \eta_\beta(\bar{\phi}_\alpha^{(k)})' - 2g_4'^{(k)}>_{S_{4i}^{(k)}}$$

$$+ \ <V_\alpha^{(k)}, \eta_\alpha(\bar{v}_3^{(k)})' - 2g_6'^{(k)}>_{S_{6i}^{(k)}}\}$$

$$+ \ <\bar{\sigma}_{\gamma 3}^{(1)}, [-\frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\eta_\gamma](N_{\alpha\beta}^{(1)})' + [\frac{1}{10}S_{\alpha\beta 33}^{(1)}\eta_\gamma](M_{\alpha\beta}^{(1)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_1^3S_{3333}^{(1)}+t_2^3S_{3333}^{(2)})\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(1)})'$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_2^2S_{3333}^{(2)}-t_1^2S_{3333}^{(1)})\eta_\gamma](\bar{\sigma}_{33}^{(1)})' + [\frac{t_2}{12}S_{\alpha\beta 33}^{(2)}\eta_\gamma](N_{\alpha\beta}^{(2)})'$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(2)}\eta_\gamma](M_{\alpha\beta}^{(2)})' + [-(\frac{1}{120}-\frac{1}{840})t_2^3S_{3333}^{(2)}\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(2)})'$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_2^2S_{3333}^{(2)}\eta_\gamma](\bar{\sigma}_{33}^{(2)})' - 2g_\sigma'^{(1)}>_{S_i^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{ \ <\bar{\sigma}_{\gamma 3}^{(k)}, [-(\frac{1}{120}-\frac{1}{840})t_k^3S_{3333}^{(k)}\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(k-1)})'$$

$$+ \ [(-\frac{1}{24}+\frac{3}{280})t_k^2S_{3333}^{(k)}\eta_\gamma](\bar{\sigma}_{33}^{(k-1)})'$$

$$+ \ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_\gamma](N_{\alpha\beta}^{(k)})' + [\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma](M_{\alpha\beta}^{(k)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_k^3S_{3333}^{(k)}+t_{k+1}^3S_{3333}^{(k+1)})\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(k)})'$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2S_{3333}^{(k+1)}-t_k^2S_{3333}^{(k)})\eta_\gamma](\bar{\sigma}_{33}^{(k)})' + [\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma](N_{\alpha\beta}^{(k+1)})'$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma](M_{\alpha\beta}^{(k+1)})' + [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3S_{3333}^{(k+1)}\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(k+1)})'$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2S_{3333}^{(k+1)}\eta_\gamma](\bar{\sigma}_{33}^{(k+1)})' - 2g_\sigma'^{(k)}>_{S_i^{(k)}} \}$$

$$+ \ <\bar{\sigma}^{(N-1)}_{\gamma 3}, [-(\frac{1}{120}-\frac{1}{840})t^3_{N-1}S^{(N-1)}_{3333}\eta_\gamma][\bar{\sigma}^{(N-2)}_{\rho 3,\rho})'$$

$$+ \ [(-\frac{1}{24}+\frac{3}{280})t^2_{N-1}S^{(N-1)}_{3333}\eta_\gamma][\bar{\sigma}^{(N-2)}_{33})'$$

$$+ \ [-\frac{t_{N-1}}{12}S^{(N-1)}_{\alpha\beta 33}\eta_\gamma][N^{(N-1)}_{\alpha\beta})'+[\frac{1}{10}S^{(N-1)}_{\alpha\beta 33}\eta_\gamma][M^{(N-1)}_{\alpha\beta})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t^3_{N-1}S^{(N-1)}_{3333}+t^3_N S^{(N)}_{3333})\eta_\gamma][\bar{\sigma}^{(N-1)}_{\rho 3,\rho})'$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t^2_N S^{(N)}_{3333}-t^2_{N-1}S^{(N-1)}_{3333})\eta_\gamma][\bar{\sigma}^{(N-1)}_{33})'+[\frac{t_N}{12}S^{(N)}_{\alpha\beta 33}\eta_\gamma][N^{(N)}_{\alpha\beta})'$$

$$+ \ [\frac{1}{10}S^{(N)}_{\alpha\beta 33}\eta_\gamma][M^{(N)}_{\alpha\beta})'-2g'^{(N-1)}_\sigma>_{S^{(N-1)}_i} \tag{230}$$

We shall show that the Gateaux differential of this function vanishes if and only if the field equations as well as the boundary conditions of the problem are satisfied.

4.6.3 The Set of Admissible States

The governing function defined in the previous section is defined over the linear vector space U of the ordered set of functions $u=\{\bar{V}^{(k)}_\alpha, \bar{\Phi}^{(k)}_\alpha, \bar{V}^{(k)}_3, N^{(k)}_{\alpha\beta}, M^{(k)}_{\alpha\beta}, V^{(k)}_\alpha$ and $\sigma^{(j)}_{\gamma 3}, \sigma^{(j)}_{33}$, k, j = 1, 2, ., ., N-1 } so that each function belongs to the intersection of the domains of the set of operators which act on it. The domain of definition M of any operator A is the set V such that for any $u, v \in V$, the inner products $<u, Av>$ and $<v, Au>$ exist. Thus if we denote the space of function of which derivatives up to order q are continuous by C^q, an admissible state is the set of $u=\{\bar{V}^{(k)}_\alpha, \bar{\Phi}^{(k)}_\alpha, \bar{V}^{(k)}_3, N^{(k)}_{\alpha\beta}, M^{(k)}_{\alpha\beta}, V^{(k)}_\alpha$, and $\sigma^{(j)}_{\gamma 3}, \sigma^{(j)}_{33}$, k, j = 1, 2, ., ., N-1 } such that $<\bar{Y}, WY>_{R(k)}$ exists. To ensure this, it is only necessary that

$$\bar{\sigma}^{(k)}_{\gamma 3} \in C^2 \tag{231}$$

$$\bar{\sigma}^{(k)}_{33} \in C^1 \tag{232}$$

$$V^{(k)}_\alpha \in C^1 \tag{233}$$

$$M_{\alpha\beta}^{(k)} \in C^1 \tag{234}$$

$$N_{\alpha\beta}^{(k)} \in C^1 \tag{235}$$

$$\bar{v}_{\alpha}^{(k)} \in C^1 \tag{236}$$

$$\bar{\phi}_{\alpha}^{(k)} \in C^1 \tag{237}$$

$$\bar{v}_{3}^{(k)} \in C^1 \tag{238}$$

However, to ensure that all the differential equations can be satisfied by an element in the set of admissible states, it is necessary that various field variables have appropriate smoothness. The equilibrium equations (113) through (115) require that:

1. $\sigma_{\gamma 3}^{(k)} \in C^0$
2. $\sigma_{33}^{(k)} \in C^0$
3. $V_{\alpha}^{(k)} \in C^1$, at least one order of continuous differentiability higher than that of $\sigma_{33}^{(k)}$
4. $M_{\alpha\beta}^{(k)} \in C^2$, at least one order of continuous differentiability higher than that of $\sigma_{\gamma 3}^{(k)}$, $\sigma_{33}^{(k)}$ and one order higher than that of $V_{\alpha}^{(k)}$
5. $N_{\alpha\beta}^{(k)} \in C^1$, at least one order of continuous differentiability higher than that of $\sigma_{\gamma 3}^{(k)}$

From constitutive relations, (155) through (157), for $\sigma_{33}^{(k)} \in C^0$

1. $\sigma_{\gamma 3}^{(k)} \in C^1$, one order of continuous differentiability higher that that of $\sigma_{33}^{(k)}$. Hence, $N_{\alpha\beta}^{(k)} \in C^2$ from (5) above.
2. $\bar{v}_{\gamma}^{(k)} \in C^3$, one order of continuous differentiability higher than that of $N_{\alpha\beta}^{(k)}$ and at least the same as that of $\sigma_{\gamma 3}^{(k)}$
3. $\bar{\phi}_{\gamma}^{(k)} \in C^3$, one order of continuous differentiability higher than that of $M_{\alpha\beta}^{(k)}$ and at least the same as that of $\sigma_{\gamma 3}^{(k)}$

4. $\bar{v}_3^{(k)} \epsilon C^4$, one order of continuous differentiability higher than that of $V_\alpha^{(k)}$ and $\bar{\phi}_\rho^{(k)}$ and at least one order higher than that of $\sigma_{\gamma 3}^{-(k)}$

The continuity equations, (164), require that:

$$V_\alpha^{(k)}, \bar{v}_\alpha^{(k)}, \bar{\phi}_\alpha, \bar{v}_3 \epsilon C^0 \tag{239}$$

$$M_{\alpha\beta}^{(k)}, N_{\alpha\beta}^{(k)}, \sigma_{33}^{-(k)} \epsilon C^1 \tag{240}$$

and

$$\sigma_{\gamma 3}^{-(k)} \epsilon C^2 \tag{241}$$

Combining the requirements of continuity, equilibrium, and constitutive equations, we have the following restrictions upon the field variables so that the differential equations can be simultaneously meaningful:

$$\bar{v}_\alpha^{(k)} \epsilon C^3 \tag{242}$$

$$\bar{\phi}_\alpha^{(k)} \epsilon C^3 \tag{243}$$

$$\bar{v}_3^{(k)} \epsilon C^4 \tag{244}$$

$$N_{\alpha\beta}^{(k)} \epsilon C^3 \tag{245}$$

$$M_{\alpha\beta}^{(k)} \epsilon C^3 \tag{246}$$

$$V_\alpha^{(k)} \epsilon C^2 \tag{247}$$

$$\sigma_{\gamma 3}^{-(k)} \epsilon C^2 \tag{248}$$

$$\sigma_{33}^{-(k)} \epsilon C^1 \tag{249}$$

We note here that the domain of the differential operators defined above is contained in the domain over which the governing function (230) is defined. In the above discussion C^0 may be regarded as the space of piecewise continuous functions over the domain $R^{(k)}$. This is important for finite element approximations.

4.6.4 Proof of the Variational Theorem

It is important that even if there are no internal jump discontinuities, i.e., if $g'_1, \ldots, g'_6$ vanish, the jump terms in the functional $\Omega(u, \sigma)$ be introduced in the formulation [Sandhu and Salaam 1975]. If these terms are excluded, and approximants which have interelement discontinuities are used, a built-in source of error in the approximation may exist.

The adjointness relations hold only if the functions have appropriate smoothness properties over $R^{(k)}$. If there are internal discontinuities, the relations have to be restated. The functions may be sufficiently smooth for the divergence theorem to be applicable over subregions or elements but not over the entire region. Thus if a partition of $R^{(k)}$ into subregions or elements is available such that all $S^{(k)}_{ui}$ are included in the intersection of the closure of the elements, application of the divergence theorem to the sum of the inner products over n elements will lead to equations of the type

$$\sum_{e=1}^{n} \langle u , Av \rangle_{R_e^{(k)}} = \sum_{e=1}^{n} \langle v , Au \rangle_{R_e^{(k)}} + \sum_{e=1}^{n} \langle v , Cu \rangle_{S_e^{(k)}}$$

where n is total number of elements; $R_e^{(k)}$ is the region occupied by the eth element; C is a boundary operator; $S_e^{(k)}$ is the boundary of the $R_e^{(k)}$. Clearly, summation over all the elements leads to

$$\sum_{e=1}^{n} \langle u , Cv \rangle_{S_e^{(k)}} = \langle u , Cv \rangle_{S^{(k)}} + \int_{S_i^{(k)}} (u\,Cv)' dS_i^{(k)} \tag{250}$$

where $S^{(k)}$ is the boundary of the region $R^{(k)}$ or of its finite element approximation; $S_i^{(k)}$ is the union of interelement boundaries and the superscripted prime denotes a jump in the quantity.

To evaluate Gateaux differential of $\Omega(u, \sigma)$ along arbitrary paths in the space of admissible states following the definition given in (98) through (100), it is necessary and sufficient to consider paths in the space of admissible states of each field variable separately e.g. in (98) the v could be arbitrary in the set of admissible states of one variable and have zero components in the spaces of admissible states of the remaining variables. Thus considering paths exclusively in the space of $\bar{v}_\alpha^{(k)}$, denoting an arbitrary path by $\bar{x}_\alpha^{(k)}$, the Gateaux differential of Ω with respect to $\bar{v}_\alpha^{(k)}$ along path $\bar{x}_\alpha^{(k)}$ is

$$\Delta_{\bar{x}_\alpha^{(k)}}\Omega = <\bar{x}_\alpha^{(k)}, \bar{\sigma}_{\alpha 3}^{(k-1)} + N_{\alpha\beta,\beta}^{(k)} - \bar{\sigma}_{\alpha 3}^{(k)} - \bar{\sigma}_{\alpha 3}^{(k)} + \bar{\sigma}_{\alpha 3}^{(k-1)} >_{R^{(k)}}$$

$$+ <N_{\alpha\beta}^{(k)}, -\bar{x}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}$$

$$+ <\bar{x}_\alpha^{(k)}, -\eta_\beta N_{\alpha\beta}^{(k)} - 2g_1^{(k)}>_{S_1^{(k)}}$$

$$+ <N_{\alpha\beta}^{(k)}, \eta_\beta \bar{x}_\alpha^{(k)}>_{S_2^{(k)}}$$

$$+ <\bar{x}_\alpha^{(k)}, -\eta_\beta (N_{\alpha\beta}^{(k)})' - 2g_1'^{(k)}>_{S_{1i}^{(k)}}$$

$$+ <N_{\alpha\beta}^{(k)}, \eta_\beta (\bar{x}_\alpha^{(k)})'>_{S_{2i}^{(k)}}$$

Noting that

$$<N_{\alpha\beta}^{(k)}, -\bar{x}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}} = <N_{\alpha\beta,\beta}^{(k)}, \bar{x}_\alpha^{(k)}>_{R^{(k)}} - <N_{\alpha\beta}^{(k)}, \bar{x}_\alpha^{(k)}\eta_\beta>_{S^{(k)}} - \int_{S_i^{(k)}} (N_{\alpha\beta}^{(k)}\bar{x}_\alpha^{(k)}\eta_\beta)' dS_i^{(k)}$$

and that $\bar{S}_1^{(k)} \bigcup \bar{S}_2^{(k)} = \bar{S}^{(k)}$ and $S_1^{(k)} \bigcap S_2^{(k)} = 0$

$$\Delta_{\bar{x}_\alpha^{(k)}}\Omega = 2<\bar{x}_\alpha^{(k)}, N_{\alpha\beta,\beta}^{(k)} - \bar{\sigma}_{\alpha 3}^{(k)} + \bar{\sigma}_{\alpha 3}^{(k-1)}>_{R^{(k)}} + 2<\bar{x}_\alpha^{(k)}, -\eta_\beta N_{\alpha\beta}^{(k)} - g_1^{(k)}>_{S_1^{(k)}}$$

$$+ 2<\bar{x}_\alpha^{(k)}, -\eta_\beta (N_{\alpha\beta}^{(k)}) - g_1'^{(k)}>_{S_{1i}^{(k)}} \tag{251}$$

Noting that inner product is a nondegenerate bilinear mapping, i.e.,

$$<u, v>_R = 0 \text{ for all } v \iff u = 0 \quad \text{on } R$$

and that arbitrary $\bar{x}_\alpha^{(k)}$ could be selected such that $\bar{x}_\alpha^{(k)}$ vanish on any two of $R^{(k)}$, $S_1^{(k)}$ and $S_{1i}^{(k)}$, vanishing of $\Delta_{\bar{x}_\alpha^{(k)}}\Omega$ for all $\bar{x}_\alpha^{(k)}$ implies

$$N^{(k)}_{\alpha\beta,\beta}-\sigma^{-(k)}_{\alpha 3}+\sigma^{-(k-1)}_{\alpha 3} = 0 \quad \text{on } R^{(k)}$$

$$-\eta_\beta N^{(k)}_{\alpha\beta} = g^{(k)}_1 \quad \text{on } S^{(k)}_1$$

and

$$-\eta_\beta (N^{(k)}_{\alpha\beta})' = g'^{(k)}_1 \quad \text{on } S^{(k)}_{1i}$$

Similarly, Gateaux differential of Ω with respect to $\overline{\phi}^{(k)}_\alpha$ along path $\overline{y}^{(k)}_\alpha$ is

$$\Delta_{\overline{y}^{(k)}_\alpha}\Omega = <\overline{y}^{(k)}_\alpha \,,\, \frac{t_k}{2}\sigma^{-(k-1)}_{\alpha 3}+M^{(k)}_{\alpha\beta,\beta}-V^{(k)}_\alpha+\frac{t_k}{2}\sigma^{-(k)}_{\alpha 3}+\frac{t_k}{2}\sigma^{-(k)}_{\alpha 3}+\frac{t_k}{2}\sigma^{-(k-1)}_{\alpha 3}>_{R^{(k)}}$$

$$+ <M^{(k)}_{\alpha\beta} \,,\, -\overline{y}^{(k)}_{(\alpha,\beta)}>_{R^{(k)}}$$

$$+ <V^{(k)}_\alpha \,,\, -\overline{y}^{(k)}_\alpha>_{R^{(k)}}$$

$$+ <\overline{y}^{(k)}_\alpha \,,\, -\eta_\beta M^{(k)}_{\alpha\beta}-2g^{(k)}_3>_{S^{(k)}_3}$$

$$+ <M^{(k)}_{\alpha\beta} \,,\, \eta_\beta \overline{y}^{(k)}_\alpha>_{S^{(k)}_4}$$

$$+ <\overline{y}^{(k)}_\alpha \,,\, -\eta_\beta (M^{(k)}_{\alpha\beta})'-2g'^{(k)}_3>_{S^{(k)}_{3i}}$$

$$+ <M^{(k)}_{\alpha\beta} \,,\, \eta_\beta (\overline{y}^{(k)}_\alpha)'>_{S^{(k)}_{4i}}$$

Noting that

$$<M^{(k)}_{\alpha\beta} \,,\, -\overline{y}^{(k)}_{(\alpha,\beta)}>_{R^{(k)}} = <M^{(k)}_{\alpha\beta,\beta} \,,\, \overline{y}^{(k)}_\alpha>_{R^{(k)}} - <M^{(k)}_{\alpha\beta} \,,\, \overline{y}^{(k)}_\alpha \eta_\beta>_{S^{(k)}} - \int_{S^{(k)}_i} (M^{(k)}_{\alpha\beta}\, \overline{y}^{(k)}_\alpha \eta_\beta)' dS^{(k)}_i$$

and $\overline{S}^{(k)}_3 \bigcup \overline{S}^{(k)}_4 = \overline{S}^{(k)}$; $S^{(k)}_3 \bigcap S^{(k)}_4 = 0$

$$\Delta_{\overline{y}^{(k)}_\alpha}\Omega = 2<\overline{y}^{(k)}_\alpha \,,\, M^{(k)}_{\alpha\beta,\beta}-V^{(k)}_\alpha+\frac{t_k}{2}(\sigma^{-(k)}_{\alpha 3}+\sigma^{-(k-1)}_{\alpha 3})>_{R^{(k)}}+2<\overline{y}^{(k)}_\alpha \,,\, -\eta_\beta M^{(k)}_{\alpha\beta}-g^{(k)}_3>_{S^{(k)}_3}$$

$$+ 2<\overline{y}^{(k)}_\alpha \,,\, -\eta_\beta (M^{(k)}_{\alpha\beta})'-g'^{(k)}_3>_{S^{(k)}_{3i}}$$

Vanishing of $\Delta_{\overline{y}^{(k)}_\alpha}\Omega$ for arbitrary $\overline{y}^{(k)}_\alpha$ implies

$$M^{(k)}_{\alpha\beta,\beta} - V^{(k)}_{\alpha} + \frac{t_k}{2}(\sigma^{-(k)}_{\alpha 3} + \sigma^{-(k-1)}_{\alpha 3}) = 0 \quad \text{on } R^{(k)}$$

$$-\eta_\beta M^{(k)}_{\alpha\beta} = g^{(k)}_3 \text{ on } S^{(k)}_3$$

and

$$-\eta_\beta (M^{(k)}_{\alpha\beta})' = g'^{(k)}_3 \text{ on } S^{(k)}_{3i}$$

Gateaux differential of Ω with respect to $\bar{v}^{(k)}_3$ along path $\bar{z}^{(k)}_3$ is

$$\Delta_{\bar{z}^{(k)}_3}\Omega = <\bar{z}^{(k)}_3, \sigma^{-(k-1)}_{33} + V^{(k)}_{\alpha,\alpha} - \sigma^{-(k)}_{33} - \sigma^{-(k)}_{33} + \sigma^{-(k-1)}_{33}>_{R^{(k)}}$$

$$+ <V^{(k)}_{\alpha}, -\bar{x}^{(k)}_{3,\alpha}>_{R^{(k)}}$$

$$+ <\bar{z}^{(k)}_3, -\eta_\alpha V^{(k)}_\alpha - 2g^{(k)}_5>_{S^{(k)}_5}$$

$$+ <V^{(k)}_\alpha, \eta_\alpha \bar{z}^{(k)}_3>_{S^{(k)}_6}$$

$$+ <\bar{z}^{(k)}_3, -\eta_\alpha (V^{(k)}_\alpha)' - 2g'^{(k)}_5>_{S^{(k)}_{5i}}$$

$$+ <V^{(k)}_\alpha, \eta_\alpha (\bar{z}^{(k)}_3)>_{S^{(k)}_{6i}}$$

Noting that

$$<V^{(k)}_\alpha, -\bar{x}^{(k)}_{3,\alpha}>_{R^{(k)}} = <V^{(k)}_{\alpha,\alpha}, \bar{z}^{(k)}_3>_{R^{(k)}} - <V^{(k)}_\alpha, \bar{z}^{(k)}_3 \eta_\alpha>_{S^{(k)}} - \int_{S^{(k)}_i} (V^{(k)}_\alpha \bar{z}^{(k)}_3 \eta_\alpha)' dS^{(k)}_i$$

and $\bar{S}^{(k)}_5 \bigcup \bar{S}^{(k)}_6 = \bar{S}^{(k)}$; $S^{(k)}_5 \bigcap S^{(k)}_6 = 0$

$$\Delta_{\bar{z}^{(k)}_3}\Omega = 2<\bar{z}^{(k)}_3, V^{(k)}_{\alpha,\alpha} + \sigma^{-(k-1)}_{33} - \sigma^{-(k)}_{33}>_{R^{(k)}} + 2<\bar{z}^{(k)}_3, -\eta_\alpha V^{(k)}_\alpha - g^{(k)}_5>_{S^{(k)}_5}$$

$$+ 2<\bar{z}^{(k)}_3, -\eta_\alpha (V^{(k)}_\alpha)' - g'^{(k)}_5>_{S^{(k)}_{5i}} \tag{252}$$

Vanishing of $\Delta_{\bar{z}^{(k)}_3}\Omega$ for arbitrary $\bar{z}^{(k)}_3$ implies

$$V^{(k)}_{\alpha,\alpha} + \sigma^{-(k-1)}_{33} - \sigma^{-(k)}_{33} = 0 \text{ on } R^{(k)}$$

$$-\eta_\alpha V^{(k)}_\alpha = g^{(k)}_5 \text{ on } S^{(k)}_5$$

and

$$-\eta_\alpha(V_\alpha^{(k)}) = g_5'^{(k)} \text{ on } S_{5i}^{(k)}$$

Gateaux differential of Ω with respect to $N_{\alpha\beta}^{(k)}$ along path $n_{\alpha\beta}^{(k)}$ is

$$\Delta_{n_{\alpha\beta}^{(k)}}\Omega = <n_{\alpha\beta}^{(k)}, \frac{t_k}{12}S_{\alpha\beta33}^{(k)}\bar{\sigma}_{\gamma3,\gamma}^{(k-1)} + \frac{1}{2}S_{\alpha\beta33}^{(k)}\bar{\sigma}_{33}^{(k)} - \bar{v}_{(\alpha,\beta)}^{(k)} + \frac{2}{t_k}S_{\alpha\beta\mu\rho}^{(k)}N_{\mu\rho}^{(k)}$$

$$- \frac{t_k}{12}S_{\alpha\beta33}^{(k)}\bar{\sigma}_{\gamma3,\gamma}^{(k)} + \frac{1}{2}S_{\alpha\beta33}^{(k)}\bar{\sigma}_{33}^{(k)} + \frac{1}{2}S_{\alpha\beta33}^{(k)}\bar{\sigma}_{33}^{(k-1)} + \frac{1}{2}S_{\alpha\beta33}^{(k)}\bar{\sigma}_{33}^{(k-1)}>_{R^{(k)}}$$

$$+ <\bar{v}_\alpha^{(k)}, n_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}$$

$$+ <\bar{\sigma}_{\gamma3}^{(k)}, \frac{t_k}{12}S_{\alpha\beta33}^{(k)}n_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}}$$

$$+ <\bar{\sigma}_{\gamma3}^{(k-1)}, -\frac{t_k}{12}S_{\alpha\beta33}^{(k)}n_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}}$$

$$+ <n_{\alpha\beta}^{(k)}, \eta_\beta\bar{v}_\alpha^{(k)} - 2g_2^{(k)}>_{S_2^{(k)}}$$

$$+ <\bar{v}_\alpha^{(k)}, -\eta_\beta n_{\alpha\beta}^{(k)}>_{S_1^{(k)}}$$

$$+ <n_{\alpha\beta}^{(k)}, -\frac{t_k}{12}S_{\alpha\beta33}^{(k)}\eta_\gamma\bar{\sigma}_{\gamma3}^{(k)} + \frac{t_k}{12}S_{\alpha\beta33}^{(k)}\eta_\gamma\bar{\sigma}_{\gamma3}^{(k-1)}>_{S^{(k)}}$$

$$+ <n_{\alpha\beta}^{(k)}, \eta_\beta(\bar{v}_\alpha^{(k)})' - 2g_2'^{(k)}>_{S_{2i}^{(k)}}$$

$$+ <\bar{v}_\alpha^{(k)}, -\eta_\beta(n_{\alpha\beta}^{(k)})'>_{S_{1i}^{(k)}}$$

$$+ <n_{\alpha\beta}^{(k)}, -\frac{t_k}{12}S_{\alpha\beta33}^{(k)}\eta_\gamma(\bar{\sigma}_{\gamma3}^{(k)})' + \frac{t_k}{12}S_{\alpha\beta33}^{(k)}\eta_\gamma(\bar{\sigma}_{\gamma3}^{(k-1)})'>_{S^{(k)}}$$

Noting that

$$<\bar{v}_{\alpha\beta}^{(k)}, n_{(\alpha,\beta)}^{(k)}>_{R^{(k)}} = -<n_{\alpha\beta}^{(k)}, \bar{v}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}} + <n_{\alpha\beta}^{(k)}\eta_\beta, \bar{v}_\alpha^{(k)}>_{S^{(k)}} + \int_{S_i^{(k)}}(n_{\alpha\beta}^{(k)}\eta_\beta\bar{v}_\alpha^{(k)})'dS_i^{(k)}$$

$$<\bar{\sigma}_{\gamma3}^{(k)}, \frac{t_k}{12}S_{\alpha\beta33}^{(k)}n_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}} = -<n_{\alpha\beta}^{(k)}, \frac{t_k}{12}S_{\alpha\beta33}^{(k)}\bar{\sigma}_{\gamma3,\gamma}^{(k)}>_{R^{(k)}}$$

$$+ <n_{\alpha\beta}^{(k)}, \frac{t_k}{12}S_{\alpha\beta33}^{(k)}\eta_\gamma\bar{\sigma}_{\gamma3}^{(k)}>_{S^{(k)}}$$

$$+\int_{S_i^{(k)}} (n_{\alpha\beta}^{(k)} \frac{t_k}{12} S_{\alpha\beta 33}^{(k)} \eta_\gamma \sigma_{\gamma 3}^{-(k)})' dS_i^{(k)}$$

$$<\sigma_{\gamma 3}^{-(k-1)}, -\frac{t_k}{12} S_{\alpha\beta 33}^{(k)} n_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}} = <n_{\alpha\beta}^{(k)}, \frac{t_k}{12} S_{\alpha\beta 33}^{(k)} \sigma_{\gamma 3,\gamma}^{-(k-1)}>_{R^{(k)}}$$

$$- <n_{\alpha\beta}^{(k)}, \frac{t_k}{12} S_{\alpha\beta 33}^{(k)} \eta_\gamma \sigma_{\gamma 3}^{-(k-1)}>_{S^{(k)}}$$

$$- \int_{S_i^{(k)}} (n_{\alpha\beta}^{(k)} \frac{t_k}{12} S_{\alpha\beta 33}^{(k)} \eta_\gamma \sigma_{\gamma 3}^{-(k-1)})' dS_i^{(k)}$$

and $\bar{S}_1^{(k)} \bigcup \bar{S}_2^{(k)} = \bar{S}^{(k)}$; $S_1^{(k)} \bigcap S_2^{(k)} = 0$

$$\Delta_{n_{\alpha\beta}^{(k)}} \Omega = 2 <n_{\alpha\beta}, \frac{t_k}{12} S_{\alpha\beta 33}^{(k)} \sigma_{\gamma 3,\gamma}^{-(k-1)} - \frac{t_k}{12} S_{\alpha\beta 33}^{(k)} \sigma_{\gamma 3,\gamma}^{-(k)} - \bar{v}_{(\alpha,\beta)}^{(k)}$$

$$+ \frac{1}{t_k} S_{\alpha\beta\mu\rho}^{(k)} N_{\mu\rho}^{(k)} + \frac{1}{2} S_{\alpha\beta 33}^{(k)} (\sigma_{33}^{-(k-1)} + \sigma_{33}^{-(k)})>_{R^{(k)}}$$

$$+ 2 <n_{\alpha\beta}^{(k)}, \eta_\beta \bar{v}_\alpha^{(k)} - g_2^{(k)}>_{S_2^{(k)}}$$

$$+ 2 <n_{\alpha\beta}^{(k)}, \eta_\beta (\bar{v}_\alpha^{(k)})' - g_2'^{(k)}>_{S_{2i}^{(k)}} \tag{253}$$

Vanishing of $\Delta_{n_{\alpha\beta}^{(k)}} \Omega$ for arbitrary $n_{\alpha\beta}^{(k)}$ implies

$$\frac{t_k}{12} S_{\alpha\beta 33}^{(k)} (\sigma_{\gamma 3,\gamma}^{-(k-1)} - \sigma_{\gamma 3,\gamma}^{-(k)}) - \bar{v}_{(\alpha,\beta)}^{(k)} + \frac{1}{t_k} S_{\alpha\beta\mu\rho}^{(k)} N_{\mu\rho}^{(k)} + \frac{1}{2} S_{\alpha\beta 33}^{(k)} (\sigma_{33}^{-(k-1)} + \sigma_{33}^{-(k)}) = 0 \quad \text{on} \quad R^{(k)}$$

$$\eta_\beta \bar{v}_\alpha^{(k)} = g_2^{(k)} \text{ on } S_2^{(k)}$$

and

$$\eta_\beta (\bar{v}_\alpha^{(k)})' = g_2'^{(k)} \text{ on } S_{2i}^{(k)}$$

Gateaux differential of Ω with respect to $M_{\alpha\beta}^{(k)}$ along path $m_{\alpha\beta}^{(k)}$ is

$$\Delta_{m_{\alpha\beta}^{(k)}} \Omega = <m_{\alpha\beta}^{(k)}, \frac{1}{10} S_{\alpha\beta 33}^{(k)} \sigma_{\gamma 3,\gamma}^{-(k-1)} + \frac{6}{5t_k} S_{\alpha\beta 33}^{(k)} \sigma_{33}^{-(k-1)} - \bar{\phi}_{(\alpha,\beta)}^{(k)} + \frac{24}{t_k^3} S_{\alpha\beta\mu\rho}^{(k)} M_{\mu\rho}^{(k)}$$

$$+ \frac{1}{10} S_{\alpha\beta 33}^{(k)} \sigma_{\gamma 3,\gamma}^{-(k)} - \frac{6}{5t_k} S_{\alpha\beta 33}^{(k)} \sigma_{33}^{-(k)} - \frac{6}{5t_k} S_{\alpha\beta 33}^{(k)} \sigma_{33}^{-(k)} + \frac{6}{5t_k} S_{\alpha\beta 33}^{(k)} \sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+ \ <\bar{\phi}_{\alpha}^{(k)}, m_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{\gamma 3}^{-(k)}, -\frac{1}{10}S_{\alpha\beta 33}^{(k)}m_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{\gamma 3}^{-(k-1)}, -\frac{1}{10}S_{\alpha\beta 33}^{(k)}m_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}}$$

$$+ \ <m_{\alpha\beta}^{(k)}, \eta_\beta\bar{\phi}_{\alpha}^{(k)}-2g_4^{(k)}>_{S_4^{(k)}}$$

$$+ \ <\bar{\phi}_{\alpha}^{(k)}, -\eta_\beta m_{\alpha\beta}^{(k)}>_{S_3^{(k)}}$$

$$+ \ <m_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma\sigma_{\gamma 3}^{-(k)}+\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma\sigma_{\gamma 3}^{-(k-1)}>_{S^{(k)}}$$

$$+ \ <m_{\alpha\beta}^{(k)}, \eta_\beta(\bar{\phi}_{\alpha}^{(k)})'-2g_4'^{(k)}>_{S_{4i}^{(k)}}$$

$$+ \ <\bar{\phi}_{\alpha}^{(k)}, -\eta_\beta(m_{\alpha\beta}^{(k)})'>_{S_{3i}^{(k)}}$$

$$+ \ <m_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma(\sigma_{\gamma 3}^{-(k)})'+\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma(\sigma_{\gamma 3}^{-(k-1)})>_{Si^{(k)}}$$

Noting that

$$<\bar{\phi}_{\alpha\beta}^{(k)}, m_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}=-<m_{\alpha\beta}^{(k)}, \bar{\phi}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}+<m_{\alpha\beta}^{(k)}, \eta_\beta\bar{\phi}_{\alpha}^{(k)}>_{S^{(k)}}+\int_{S_i^{(k)}}(m_{\alpha\beta}^{(k)}\eta_\beta\bar{\phi}_{\alpha}^{(k)})'dS_i^{(k)}$$

$$\begin{aligned}<\sigma_{\gamma 3}^{-(k)}, -\frac{1}{10}S_{\alpha\beta 33}^{(k)}m_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}}&=<m_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k)}>_{R^{(k)}}\\&- <m_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma\sigma_{\gamma 3}^{-(k)}>_{S^{(k)}}\\&- \int_{S_i^{(k)}}(m_{\alpha\beta}^{(k)}\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma\sigma_{\gamma 3}^{-(k)})'dS_i^{(k)}\end{aligned}$$

$$\begin{aligned}<\sigma_{\gamma 3}^{-(k-1)}, -\frac{1}{10}S_{\alpha\beta 33}^{(k)}m_{\alpha\beta,\gamma}^{(k)}>_{R^{(k)}}&=<m_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k-1)}>_{R^{(k)}}\\&- <m_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma\sigma_{\gamma 3}^{-(k-1)}>_{S^{(k)}}\\&- \int_{S_i^{(k)}}(m_{\alpha\beta}^{(k)}\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma\sigma_{\gamma 3}^{-(k-1)})'dS_i^{(k)}\end{aligned}$$

and $\bar{S}_3^{(k)} \bigcup \bar{S}_4^{(k)} = \bar{S}^{(k)}$; $S_3^{(k)} \bigcap S_4^{(k)} = 0$

$$\Delta_{m_{\alpha\beta}^{(k)}}\Omega = 2 < m_{\alpha\beta} , \frac{1}{10} S^{(k)}_{\alpha\beta 33} \sigma^{-(k-1)}_{\gamma 3,\gamma} + \frac{1}{10} S^{(k)}_{\alpha\beta 33} \sigma^{-(k)}_{\gamma 3,\gamma} - \bar{\phi}^{(k)}_{(\alpha,\beta)}$$

$$+ \frac{12}{t_k^3} S^{(k)}_{\alpha\beta\mu\rho} M^{(k)}_{\mu\rho} + \frac{6}{5t_k} S^{(k)}_{\alpha\beta 33} (\sigma^{-(k-1)}_{33} - \sigma^{-(k)}_{33}) >_{R^{(k)}}$$

$$+ 2 < m^{(k)}_{\alpha\beta} , \eta_\beta \bar{\phi}^{(k)}_\alpha - g_4^{(k)} >_{S_4^{(k)}}$$

$$+ 2 < m^{(k)}_{\alpha\beta} , \eta_\beta (\bar{\phi}^{(k)}_\alpha) - g'^{(k)}_4 >_{S_{4i}^{(k)}} \tag{254}$$

Vanishing of $\Delta_{m_{\alpha\beta}^{(k)}}\Omega$ for arbitrary $m^{(k)}_{\alpha\beta}$ implies

$$\frac{1}{10} S^{(k)}_{\alpha\beta 33} (\sigma^{-(k-1)}_{\gamma 3,\gamma} \sigma^{-(k)}_{\gamma 3,\gamma}) - \bar{\phi}^{(k)}_{(\alpha,\beta)} + \frac{12}{t_k^3} S^{(k)}_{\alpha\beta\mu\rho} M^{(k)}_{\mu\rho} + \frac{6}{5t_k} S^{(k)}_{\alpha\beta 33} (\sigma^{-(k-1)}_{33} - \sigma^{-(k)}_{33}) = 0 \text{ on } R^{(k)}$$

$$\eta_\beta \bar{\phi}^{(k)}_\alpha = g_4^{(k)} \quad \text{on } S_4^{(k)}$$

and

$$\eta_\beta (\bar{\phi}^{(k)}_\alpha)' = g'^{(k)}_4 \quad \text{on } S_{4i}^{(k)}$$

Gateaux differential of Ω with respect to $V^{(k)}_\alpha$ along path $q^{(k)}_\rho$

$$\Delta_{q_\rho^{(k)}}\Omega = < q^{(k)}_\rho , -\frac{2}{5} S^{(k)}_{\rho 3\gamma 3} \sigma^{-(k-1)}_{\gamma 3} - \bar{\phi}^{(k)}_\rho - \bar{\phi}^{(k)}_\rho - \bar{v}^{(k)}_{3,\rho} + \frac{48}{5t_k} S^{(k)}_{\rho 3\gamma 3} V^{(k)}_\gamma$$

$$- \frac{2}{5} S^{(k)}_{\rho 3\gamma 3} \sigma^{-(k)}_{\gamma 3} - \frac{2}{5} S^{(k)}_{\rho 3\gamma 3} \sigma^{-(k)}_{\gamma 3} - \frac{2}{5} S^{(k)}_{\rho 3\gamma 3} \sigma^{-(k-1)}_{\gamma 3} >_{R^{(k)}}$$

$$+ < \bar{v}^{(k)}_3 , q^{(k)}_{\alpha,\alpha} >_{R^{(k)}}$$

$$+ < q^{(k)}_\rho , \eta_\rho \bar{v}^{(k)}_3 - 2g^{(k)}_6 >_{S_6^{(k)}}$$

$$+ < \bar{v}^{(k)}_3 , -\eta_\alpha q^{(k)}_\alpha >_{S_5^{(k)}}$$

$$+ < q^{(k)}_\rho , \eta_\rho (\bar{v}^{(k)}_3)' - 2g'^{(k)}_6 >_{S_{6i}^{(k)}}$$

$$+ < \bar{v}^{(k)}_3 , -\eta_\alpha (q^{(k)}_\alpha) >_{S_{5i}^{(k)}}$$

Noting that

$$<\bar{v}_3^{(k)}, q_{\alpha,\alpha}^{(k)}>_{R^{(k)}} = -<\bar{v}_{3,\alpha}^{(k)}, q_\alpha^{(k)}>_{R^{(k)}} + <\bar{v}_3^{(k)}, q_\alpha^{(k)}\eta_\alpha>_{S^{(k)}} + \int_{S_i^{(k)}} (\bar{v}_3^{(k)} q_\alpha^{(k)} \eta_\alpha)' dS_i^{(k)}$$

and $\bar{S}_5^{(k)} \bigcup \bar{S}_6^{(k)} = \bar{S}^{(k)}$; $S_5^{(k)} \bigcap S_6^{(k)} = 0$

$$\Delta_{q_\rho^{(k)}}\Omega = 2<q_\rho^{(k)}, -\frac{2}{5}S_{\rho3\gamma3}^{(k)}(\sigma_{\gamma3}^{-(k-1)} + \sigma_{\gamma3}^{-(k)}) - \bar{\phi}_\rho^{(k)} - \bar{v}_{3,\rho}^{(k)} + \frac{24}{5t_k}S_{\rho3\gamma3}^{(k)}V_\gamma^{(k)}>_{R^{(k)}}$$

$$+ 2<q_\rho^{(k)}, \eta_\rho\bar{v}_3^{(k)} - g_6^{(k)}>_{S_6^{(k)}}$$

$$+ 2<q_\rho^{(k)}, \eta_\rho(\bar{v}_3^{(k)})' - g_6'^{(k)}>_{S_{6i}^{(k)}} \tag{255}$$

Vanishing of $\Delta_{q_\rho^{(k)}}\Omega$ for arbitrary q_ρ implies

$$-\frac{2}{5}S_{\rho3\gamma3}^{(k)}(\sigma_{\gamma3}^{-(k-1)} + \sigma_{\gamma3}^{-(k)}) - \bar{\phi}_\rho^{(k)} - \bar{v}_{3,\rho}^{(k)} + \frac{24}{5t_k}S_{\rho3\gamma3}^{(k)}V_\gamma^{(k)} = 0 \text{ on } R^{(k)}$$

$$\eta_\rho\bar{v}_3^{(k)} = g_6^{(k)} \text{ on } S_6^{(k)}$$

and

$$\eta_\rho(\bar{v}_3^{(k)})' = g_{6i}^{(k)} \text{ on } S_{6i}^{(k)}$$

Gateaux differential of Ω with respect to $\sigma_{\gamma3}^{-(k)}$ along path $\tau_{\gamma3}^{-(k)}$ is

$$\Delta_{\tau_{\gamma3}^{-(k)}}\Omega = <\tau_{\gamma3}^{-(k)}, \bar{v}_\gamma^{(k+1)} + \frac{t_{k+1}}{2}\bar{\phi}_\gamma^{(k+1)} - \frac{2}{5}S_{\rho3\gamma3}^{(k+1)}V_\rho^{(k+1)} - \bar{v}_\gamma^{(k)} + \frac{t_k}{2}\bar{\phi}_\gamma^{(k)} - \frac{2}{5}S_{\rho3\gamma3}^{(k)}V_\rho^{(k)}$$

$$- \bar{v}_\gamma^{(k)} + \frac{t_k}{2}\bar{\phi}_\gamma^{(k)} + \frac{t_k}{12}S_{\alpha\beta33}^{(k)}N_{\alpha\beta,\gamma}^{(k)} - \frac{1}{10}S_{\alpha\beta33}^{(k)}M_{\alpha\beta,\gamma}^{(k)} - \frac{2}{5}S_{\rho3\gamma3}^{(k)}V_\rho^{(k)}$$

$$+ \Xi_{11}^{(k)}\sigma_{\gamma3}^{-(k)} + \Xi_{12}^{(k)}\sigma_{33}^{-(k)} + \bar{v}_\gamma^{(k+1)} + \frac{t_{k+1}}{2}\bar{\phi}_\gamma^{(k+1)} - \frac{t_{k+1}}{12}S_{\alpha\beta33}^{(k+1)}N_{\alpha\beta,\gamma}^{(k+1)}$$

$$- \frac{1}{10}S_{\alpha\beta33}^{(K+1)}M_{\alpha\beta,\gamma}^{(k+1)} - \frac{2}{5}S_{\rho3\gamma3}^{(k+1)}V_\rho^{(k+1)} + \Lambda_{11}^{(k)}\sigma_{\gamma3}^{-(k-1)} + \Lambda_{12}^{(k)}\sigma_{33}^{-(k-1)}$$

$$+ \bar{\Lambda}_{11}^{(k+1)}\sigma_{\gamma3}^{-(k+1)} + \bar{\Lambda}_{12}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k)}}$$

$$+ <\tau_{\gamma3,\gamma}^{-(k)}, \frac{t_{k+1}}{12}S_{\alpha\beta33}^{(k+1)}N_{\alpha\beta}^{(k+1)} + \frac{1}{10}S_{\alpha\beta33}^{(k+1)}M_{\alpha\beta}^{(k+1)}$$

$$- \frac{t_k}{12}S_{\alpha\beta33}^{(k)}N_{\alpha\beta}^{(k)} + \frac{1}{10}S_{\alpha\beta33}^{(k)}M_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{\gamma 3}^{-(k)}\ ,\ \Xi_{11}^{(k)}\tau_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \Xi_{21}^{(k)}\tau_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{\gamma 3}^{-(k+1)}\ ,\ \Lambda_{11}^{(k+1)}\tau_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{33}^{-(k+1)}\ ,\ \Lambda_{21}^{(k+1)}\tau_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{\gamma 3}^{-(k-1)}\ ,\ \overline{\Lambda}_{11}^{(k)}\tau_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{33}^{-(k-1)}\ ,\ \overline{\Lambda}_{21}^{(k)}\tau_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\tau_{\gamma 3}^{-(k)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k-1)}+[(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\eta_\gamma]\sigma_{33}^{-(k-1)}$$

$$+ \ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_\gamma]N_{\alpha\beta}^{(k)}+[\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma]M_{\alpha\beta}^{(k)}$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)}-t_k^2 S_{3333}^{(k)})\eta_\gamma]\sigma_{33}^{-(k)}+[\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma]N_{\alpha\beta}^{(k+1)}$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma]M_{\alpha\beta}^{(k+1)}+[-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k+1)}$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{33}^{-(k+1)}-2g_\sigma^{(k)}>_{S^{(k)}}$$

$$+ \ <\tau_{\rho 3,\rho}^{-(k)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{\gamma 3}^{-(k+1)}$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma]\sigma_{\gamma 3}^{-(k)}+[-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma]\sigma_{\gamma 3}^{-(k-1)}>_{S^{(k)}}$$

$$+ \ <\tau_{\gamma 3}^{-(k)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k-1)})'+[(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\eta_\gamma](\sigma_{33}^{-(k-1)})'$$

$$+ \ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_\gamma](N_{\alpha\beta}^{(k)})'+[\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma](M_{\alpha\beta}^{(k)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k)})'$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)} - t_k^2 S_{3333}^{(k)})\eta_\gamma](\sigma_{33}^{-(k)})' + [\frac{t_{k+1}}{12} S_{\alpha\beta 33}^{(k+1)}\eta_\gamma](N_{\alpha\beta}^{(k+1)})'$$

$$+ \ [\frac{1}{10} S_{\alpha\beta 33}^{(k+1)}\eta_\gamma](M_{\alpha\beta}^{(k+1)})' + [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k+1)})'$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma](\sigma_{33}^{-(k+1)})' - 2g_\sigma'^{(k)} >_{S_i^{(k)}}$$

$$+ \ <\tau_{\rho 3,\rho}^{-(k)} , [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma](\sigma_{\gamma 3}^{-(k+1)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)} + t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma](\sigma_{\gamma 3}^{-(k)})'$$

$$+ \ [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma](\sigma_{\gamma 3}^{-(k-1)})' >_{S_i^{(k)}}$$

Using the divergence theorem:

$$<\tau_{\gamma 3,\gamma}^{-(k)} , (\frac{t_{k+1}}{12}N_{\alpha\beta}^{(k+1)} + \frac{1}{10}M_{\alpha\beta}^{(k+1)})S_{\alpha\beta 33}^{(k+1)} >_{R^{(k)}}$$

$$= \ <\tau_{\gamma 3}^{-(k)} , -(\frac{t_{k+1}}{12}N_{\alpha\beta,\gamma}^{(k+1)} + \frac{1}{10}M_{\alpha\beta,\gamma}^{(k+1)})S_{\alpha\beta 33}^{(k+1)} >_{R^{(k)}}$$

$$+ \ <\tau_{\gamma 3}\eta_\gamma^{-(k)} , (\frac{t_{k+1}}{12}N_{\alpha\beta}^{(k+1)} + \frac{1}{10}M_{\alpha\beta}^{(k+1)})S_{\alpha\beta 33}^{(k+1)} >_{S^{(k)}}$$

$$+ \int_{S_i^{(k)}} (\tau_{\gamma 3}\eta_\gamma^{-(k)} (\frac{t_1}{12}N_{\alpha\beta}^{(k+1)} + \frac{1}{10}M_{\alpha\beta}^{(k+1)})S_{\alpha\beta 33}^{(k+1)})' dS_i^{(k)}$$

$$<\tau_{\gamma 3,\gamma}^{-(k)} , (-\frac{t_k}{12}N_{\alpha\beta}^{(k)} + \frac{1}{10}M_{\alpha\beta}^{(k)})S_{\alpha\beta 33}^{(k)} >_{R^{(k)}}$$

$$= \ <\tau_{\gamma 3}^{-(k)} , -(-\frac{t_k}{12}N_{\alpha\beta,\gamma}^{(k)} + \frac{1}{10}M_{\alpha\beta,\gamma}^{(k)})S_{\alpha\beta 33}^{(k)} >_{R^{(k)}}$$

$$+ \ <\tau_{\gamma 3}\eta_\gamma^{-(k)} , (-\frac{t_k}{12}N_{\alpha\beta}^{(k)} + \frac{1}{10}M_{\alpha\beta}^{(k)})S_{\alpha\beta 33}^{(k)} >_{S^{(k)}}$$

$$+ \int_{S_i^{(k)}} (\tau_{\gamma 3}\eta_\gamma^{-(k)}(-\frac{t_k}{12}N_{\alpha\beta}^{(k)} + \frac{1}{10}M_{\alpha\beta}^{(k)})' S_{\alpha\beta 33}^{(k)} >_{S^{(k)}}$$

$$<\sigma_{\gamma 3}^{-(k)} , \Xi_{11}^{(k)}\tau_{\gamma 3}^{-(k)} >_{R^{(k)}} \ = \ <\tau_{\gamma 3}^{-(k)} , \Xi_{11}^{(k)}\sigma_{\gamma 3}^{-(k)} >_{R^{(k)}}$$

$$+\left(-\frac{1}{120}-\frac{1}{840}\right)\left(S^{(k)}_{3333}t^3_k+S^{(k+1)}_{3333}t^3_{k+1}\right)\left[<\bar{\sigma}^{(k)}_{\gamma 3}\eta_\gamma\ ,\ \bar{\tau}^{(k)}_{\rho 3,\rho}>_{S^{(k)}}-<\bar{\tau}^{(k)}_{\gamma 3}\eta_\gamma\ ,\ \bar{\sigma}^{(k)}_{\rho 3,\rho}>_{S^{(k)}}\right]$$

$$+\left(-\frac{1}{120}-\frac{1}{840}\right)\left(S^{(k)}_{3333}t^3_k+S^{(k+1)}_{3333}t^3_{k+1}\right)\left[\int_{S^{(k)}_i}(\bar{\sigma}^{(k)}_{\gamma 3}\eta_\gamma\ \bar{\tau}^{(k)}_{\rho 3,\rho})'dS^{(k)}_i\right.$$

$$\left.-\int_{S^{(k)}_i}(\bar{\tau}^{(k)}_{\gamma 3}\eta_\gamma\ \bar{\sigma}^{(k)}_{\rho 3,\rho})'dS^{(k)}_i\right]$$

$$<\bar{\sigma}^{(k)}_{33}\ ,\ \Xi^{(k)}_{21}\bar{\tau}^{(k)}_{\gamma 3}>_{R^{(k)}}\ =\ <\bar{\tau}^{(k)}_{\gamma 3}\ ,\ \bar{\Xi}^{(k)}_{12}\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$-\ <\bar{\tau}^{(k)}_{\gamma 3}\eta_\gamma\ ,\ \left(\frac{1}{24}+\frac{3}{280}\right)\left(S^{(k)}_{3333}t^3_k-S^{(k+1)}_{3333}t^3_{k+1}\right)\bar{\sigma}^{(k)}_{33}>_{S^{(k)}}$$

$$-\ \int_{S^{(k)}_i}\left(\frac{1}{24}+\frac{3}{280}\right)\left(S^{(k)}_{3333}t^3_k-S^{(k+1)}_{3333}t^3_{k+1}\right)(\eta_\gamma\bar{\tau}^{(k)}_{\gamma 3}\bar{\sigma}^{(k)}_{33})'>_{S^{(k)}_i}$$

$$<\bar{\sigma}^{(k+1)}_{\gamma 3}\ ,\ \Lambda^{(k+1)}_{11}\bar{\tau}^{(k)}_{\gamma 3}>_{R^{(k)}}\ =\ <\bar{\tau}^{(k)}_{\gamma 3}\ ,\ \bar{\Lambda}^{(k+1)}_{11}\bar{\sigma}^{(k+1)}_{\gamma 3}>_{R^{(k)}}$$

$$+\left(\frac{1}{120}-\frac{1}{840}\right)S^{(k+1)}_{3333}t^3_{k+1}\left[<\bar{\sigma}^{(k+1)}_{\gamma 3}\eta_\gamma\ ,\ \bar{\tau}^{(k)}_{\rho 3,\rho}>_{S^{(k)}}-<\bar{\tau}^{(k)}_{\gamma 3}\eta_\gamma\ ,\ \bar{\sigma}^{(k+1)}_{\rho 3,\rho}>_{S^{(k)}}\right]$$

$$+\left(\frac{1}{120}-\frac{1}{840}\right)S^{(k+1)}_{3333}t^3_{k+1}\left[\int_{S^{(k)}_i}(\bar{\sigma}^{(k+1)}_{\gamma 3}\eta_\gamma\bar{\tau}^{(k)}_{\rho 3,\rho})'dS^{(k)}_i-\int_{S^{(k)}_i}(\bar{\tau}^{(k)}_{\gamma 3}\eta_\gamma\ \bar{\sigma}^{(k+1)}_{\rho 3,\rho})'dS^{(k)}_i\right]$$

$$<\bar{\sigma}^{(k-1)}_{\gamma 3}\ ,\ \bar{\Lambda}^{(k)}_{11}\bar{\tau}^{(k)}_{\gamma 3}>_{R^{(k)}}\ =\ <\bar{\tau}^{(k)}_{\gamma 3}\ ,\ \Lambda^{(k)}_{11}\bar{\sigma}^{(k-1)}_{\gamma 3}>_{R^{(k)}}$$

$$+\left(\frac{1}{120}-\frac{1}{840}\right)S^{(k)}_{3333}t^3_k\left[<\bar{\sigma}^{(k-1)}_{\gamma 3}\eta_\gamma\ ,\ \bar{\tau}^{(k)}_{\rho 3,\rho}>_{S^{(k)}}-<\bar{\tau}^{(k)}_{\gamma 3}\eta_\gamma\ ,\ \bar{\sigma}^{(k-1)}_{\rho 3,\rho}>_{S^{(k)}}\right]$$

$$+\left(\frac{1}{120}-\frac{1}{840}\right)S^{(k)}_{3333}t^3_k\left[\int_{S^{(k)}_i}(\bar{\sigma}^{(k-1)}_{\gamma 3}\eta_\gamma\ \bar{\tau}^{(k)}_{\rho 3,\rho})'dS^{(k)}_i-\int_{S^{(k)}_i}(\bar{\tau}^{(k)}_{\gamma 3}\eta_\gamma\ \bar{\sigma}^{(k-1)}_{\rho 3,\rho})'dS^{(k)}_i\right]$$

$$<\bar{\sigma}^{(k-1)}_{33}\ ,\ \bar{\Lambda}^{(k)}_{21}\bar{\tau}^{(k)}_{\gamma 3}>_{R^{(k)}}\ =\ <\bar{\tau}^{(k)}_{\gamma 3}\ ,\ \Lambda^{(k)}_{12}\bar{\sigma}^{(k-1)}_{33}>_{R^{(k)}}$$

$$+\ <\bar{\tau}^{(k)}_{\gamma 3}\eta_\gamma\ ,\ \left(\frac{3}{280}-\frac{1}{24}\right)t^2_kS^{(k)}_{3333}\bar{\sigma}^{(k-1)}_{33}>_{S^{(k)}}$$

$$+\int_{S^{(k)}_i}\left(\left(\frac{3}{280}-\frac{1}{24}\right)t^2_kS^{(k)}_{3333}\eta_\gamma\bar{\tau}^{(k)}_{\gamma 3}\bar{\sigma}^{(k-1)}_{33}\right)'dS^{(k)}_i$$

$$<\bar{\sigma}^{(k+1)}_{33}\ ,\ \Lambda^{(k+1)}_{21}\bar{\tau}^{(k)}_{\gamma 3}>_{R^{(k)}}\ =\ <\bar{\tau}^{(k)}_{\gamma 3}\ ,\ \bar{\Lambda}^{(k+1)}_{12}\bar{\sigma}^{(k+1)}_{33}>_{R^{(k)}}$$

$$+ \ <\tau_{\gamma 3}^{-(k)}\eta_\gamma \ , (\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\sigma_{33}^{-(k+1)}>_{S^{(k)}}$$

$$+ \int_{S_1^{(k)}} ((\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma\tau_{\gamma 3}^{-(k)}\sigma_{33}^{-(k+1)})' >_{S_1^{(k)}}$$

Substituting these relations into the expression for the Gateaux differential we have

$$\Delta_{\tau_{\gamma 3}^{-(k)}}\Omega = 2<\tau_{\gamma 3}^{-(k)} \ , \Lambda_{11}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{12}^{(k)}\sigma_{33}^{-(k-1)}$$

$$- \ \bar{V}_\gamma^{(k)}+\frac{t_k}{2}\bar{\Phi}_\gamma^{(k)}+S_{\alpha\beta 33}^{(k)}(\frac{t_k}{12}N_{\alpha\beta,\gamma}^{(k)}-\frac{1}{10}M_{\alpha\beta,\gamma}^{(k)})-\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}V_\rho^{(k)}$$

$$+ \ \Xi_{11}^{(k)}\sigma_{\gamma 3}^{-(k)}+\Xi_{12}^{(k)}\sigma_{33}^{-(k)}$$

$$+ \ \bar{V}_\gamma^{(k+1)}+\frac{t_{k+1}}{2}\bar{\Phi}_\gamma^{(k+1)}+S_{\alpha\beta 33}^{(k+1)}(-\frac{t_{k+1}}{12}N_{\alpha\beta,\gamma}^{(k+1)}-\frac{1}{10}M_{\alpha\beta,\gamma}^{(k+1)})-\frac{2}{5}S_{\rho 3\gamma 3}^{(k+1)}V_\rho^{(k+1)}$$

$$+ \ \bar{\Lambda}_{11}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{12}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k)}}$$

$$+ \ 2<\tau_{\gamma 3}^{-(k)} \ , [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k-1)}+[(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\eta_\gamma]\sigma_{33}^{-(k-1)}$$

$$+ \ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_\gamma]N_{\alpha\beta}^{(k)}+[\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma]M_{\alpha\beta}^{(k)}$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)}-t_k^2 S_{3333}^{(k)})\eta_\gamma]\sigma_{33}^{-(k)}+[\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma]N_{\alpha\beta}^{(k+1)}$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma]M_{\alpha\beta}^{(k+1)}+[-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k+1)}$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{33}^{-(k+1)}-g_\sigma^{(k)}>_{S^{(k)}}$$

$$+ \ 2<\tau_{\gamma 3}^{-(k)} \ , [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k-1)})'+[(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\eta_\gamma](\sigma_{33}^{-(k-1)})'$$

$$+ \ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_\gamma](N_{\alpha\beta}^{(k)})'+[\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma](M_{\alpha\beta}^{(k)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k)})'$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^{2}S_{3333}^{(k+1)}-t_{k}^{2}S_{3333}^{(k)})\eta_{\gamma}](\sigma_{33}^{-(k)})'+[\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\eta_{\gamma}](N_{\alpha\beta}^{(k+1)})'$$

$$+ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_{\gamma}](M_{\alpha\beta}^{(k+1)})'+[-(\frac{1}{120}-\frac{1}{840})t_{k+1}^{3}S_{3333}^{(k+1)}\eta_{\gamma}](\sigma_{\rho 3,\rho}^{-(k+1)})'$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^{2}S_{3333}^{(k+1)}\eta_{\gamma}](\sigma_{33}^{-(k+1)})'-g_{\sigma}^{\prime(k)}>_{S_{i}^{(k)}} \qquad (256)$$

Vanishing of $\Delta_{\tau_{\gamma 3}^{-(k)}}\Omega$ for arbitrary $\tau_{\gamma 3}^{-(k)}$ implies

$$\Lambda_{11}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{12}^{(k)}\sigma_{33}^{-(k-1)}$$

$$- \bar{v}_{\gamma}^{(k)}+\frac{t_k}{2}\bar{\phi}_{\gamma}^{(k)}+S_{\alpha\beta 33}^{(k)}(\frac{t_k}{12}N_{\alpha\beta,\gamma}^{(k)}-\frac{1}{10}M_{\alpha\beta,\gamma}^{(k)})-\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}V_{\rho}^{(k)}$$

$$+ \Xi_{11}^{(k)}\sigma_{\gamma 3}^{-(k)}+\Xi_{12}^{(k)}\sigma_{33}^{-(k)}$$

$$+ \bar{v}_{\gamma}^{(k+1)}+\frac{t_{k+1}}{2}\bar{\phi}_{\gamma}^{(k+1)}+S_{\alpha\beta 33}^{(k+1)}(-\frac{t_{k+1}}{12}N_{\alpha\beta,\gamma}^{(k+1)}-\frac{1}{10}M_{\alpha\beta,\gamma}^{(k+1)})-\frac{2}{5}S_{\rho 3\gamma 3}^{(k+1)}V_{\rho}^{(k+1)}$$

$$+ \bar{\Lambda}_{11}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{12}^{(k+1)}\sigma_{33}^{-(k+1)} = 0 \text{ on } R^{(k)}$$

$$[-(\frac{1}{120}-\frac{1}{840})t_{k}^{3}S_{3333}^{(k)}\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(k-1)}+[(-\frac{1}{24}+\frac{3}{280})t_{k}^{2}S_{3333}^{(k)}\eta_{\gamma}]\sigma_{33}^{-(k-1)}$$

$$+ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_{\gamma}]N_{\alpha\beta}^{(k)}+[\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_{\gamma}]M_{\alpha\beta}^{(k)}$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t_{k}^{3}S_{3333}^{(k)}+t_{k+1}^{3}S_{3333}^{(k+1)})\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(k)}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^{2}S_{3333}^{(k+1)}-t_{k}^{2}S_{3333}^{(k)})\eta_{\gamma}]\sigma_{33}^{-(k)}+[\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\eta_{\gamma}]N_{\alpha\beta}^{(k+1)}$$

$$+ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_{\gamma}]M_{\alpha\beta}^{(k+1)}+[-(\frac{1}{120}-\frac{1}{840})t_{k+1}^{3}S_{3333}^{(k+1)}\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(k+1)}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^{2}S_{3333}^{(k+1)}\eta_{\gamma}]\sigma_{33}^{-(k+1)} = g_{\sigma}^{(k)} \text{ on } S^{(k)}$$

and

$$[-(\frac{1}{120}-\frac{1}{840})t_{k}^{3}S_{3333}^{(k)}\eta_{\gamma}](\sigma_{\rho 3,\rho}^{-(k-1)})'+[(-\frac{1}{24}+\frac{3}{280})t_{k}^{2}S_{3333}^{(k)}\eta_{\gamma}](\sigma_{33}^{-(k-1)})'$$

$$+ [-\frac{t_k}{12}S^{(k)}_{\alpha\beta33}\eta_\gamma](N^{(k)}_{\alpha\beta})' + [\frac{1}{10}S^{(k)}_{\alpha\beta33}\eta_\gamma](M^{(k)}_{\alpha\beta})'$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S^{(k)}_{3333}+t^3_{k+1}S^{(k+1)}_{3333})\eta_\gamma](\sigma^{-(k)}_{\rho3,\rho})'$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t^2_{k+1}S^{(k+1)}_{3333}-t_k^2 S^{(k)}_{3333})\eta_\gamma](\sigma^{-(k)}_{33})' + [\frac{t_{k+1}}{12}S^{(k+1)}_{\alpha\beta33}\eta_\gamma](N^{(k+1)}_{\alpha\beta})'$$

$$+ [\frac{1}{10}S^{(k+1)}_{\alpha\beta33}\eta_\gamma](M^{(k+1)}_{\alpha\beta})' + [-(\frac{1}{120}-\frac{1}{840})t^3_{k+1}S^{(k+1)}_{3333}\eta_\gamma](\sigma^{-(k+1)}_{\rho3,\rho})'$$

$$+ [(\frac{1}{24}-\frac{3}{280})t^2_{k+1}S^{(k+1)}_{3333}\eta_\gamma](\sigma^{-(k+1)}_{33})' = g'^{(k)}_{\sigma} \quad \text{on } S^{(k)}_i$$

Gateaux differential of Ω with respect to $\sigma^{-(k)}_{33}$ along path $\tau^{-(k)}_{33}$ is

$$\Delta_{\tau^{-(k)}_{33}}\Omega = <\tau^{-(k)}_{33}\ ,\ \bar{v}^{(k+1)}_3 + \frac{1}{2}S^{(k+1)}_{\alpha\beta33}N^{(k+1)}_{\alpha\beta} + \frac{6}{5t_{k+1}}S^{(k+1)}_{\alpha\beta33}M^{(k+1)}_{\alpha\beta} + \frac{1}{2}S^{(k)}_{\alpha\beta33}N^{(k)}_{\alpha\beta} - \frac{6}{5t_k}S^{(k)}_{\alpha\beta33}M^{(k)}_{\alpha\beta}$$

$$- \bar{v}^{(k)}_3 - \bar{v}^{(k)}_3 + \bar{v}^{(k+1)}_3 + \frac{1}{2}S^{(k)}_{\alpha\beta33}N^{(k)}_{\alpha\beta} - \frac{6}{5t_k}S^{(k)}_{\alpha\beta33}M^{(k)}_{\alpha\beta}$$

$$+ \frac{1}{2}S^{(k+1)}_{\alpha\beta33}N^{(k+1)}_{\alpha\beta} + \frac{6}{5t_{k+1}}S^{(k+1)}_{\alpha\beta33}M^{(k+1)}_{\alpha\beta} >_{R^{(k)}}$$

$$+ <\tau^{-(k)}_{33}\ ,\ \Xi^{(k)}_{21}\sigma^{-(k)}_{\gamma3} + \Xi^{(k)}_{22}\sigma^{-(k)}_{33} + \Lambda^{(k)}_{21}\sigma^{-(k-1)}_{\gamma3} + \Lambda^{(k)}_{22}\sigma^{-(k-1)}_{33}$$

$$+ \bar{\Lambda}^{(k+1)}_{21}\sigma^{-(k+1)}_{\gamma3} + \bar{\Lambda}^{(k+1)}_{22}\sigma^{-(k+1)}_{33} >_{R^{(k)}}$$

$$+ <\sigma^{-(k)}_{\gamma3}\ ,\ \Xi^{(k)}_{12}\tau^{-(k)}_{33}>_{R^{(k)}} + <\sigma^{-(k)}_{33}\ ,\ \Xi^{(k)}_{22}\tau^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <\sigma^{-(k+1)}_{\gamma3}\ ,\ \Lambda^{(k+1)}_{12}\tau^{-(k)}_{33}>_{R^{(k)}} + <\sigma^{-(k+1)}_{33}\ ,\ \Lambda^{(k+1)}_{22}\tau^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <\sigma^{-(k-1)}_{\gamma3}\ ,\ \bar{\Lambda}^{(k)}_{12}\tau^{-(k)}_{33}>_{R^{(k)}} + <\sigma^{-(k-1)}_{33}\ ,\ \bar{\Lambda}^{(k)}_{22}\tau^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <\tau^{-(k)}_{33}\ ,\ [(-\frac{1}{24}+\frac{3}{280})t^2_{k+1}S^{(k+1)}_{3333}\eta_\gamma]\sigma^{-(k+1)}_{\gamma3}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t^2_{k+1}S^{(k+1)}_{3333}-t_k^2 S^{(k)}_{3333})\eta_\gamma]\sigma^{-(k)}_{\gamma3}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_k^2 S^{(k)}_{3333}\eta_\gamma]\sigma^{-(k-1)}_{\gamma3} >_{S^{(k)}}$$

$$+ \ <\tau_{33}^{-(k)}, [(-\frac{1}{24}+\frac{3}{280})t_{k+1}^{2}S_{3333}^{(k+1)}\eta_\gamma](\sigma_{\gamma 3}^{-(k+1)})'$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^{2}S_{3333}^{(k+1)}-t_k^2S_{3333}^{(k)})\eta_\gamma](\sigma_{\gamma 3}^{-(k)})'$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_k^2S_{3333}^{(k)}\eta_\gamma](\sigma_{\gamma 3}^{-(k-1)})' >_{S_i^{(k)}}$$

Using the relationships

$$<\sigma_{\gamma 3}^{-(k)}, \Xi_{12}^{(k)}\tau_{33}^{-(k)}>_{R^{(k)}} = <\tau_{33}^{-(k)}, \Xi_{21}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\tau_{33}^{-(k)}, (\frac{1}{24}+\frac{3}{280})(t_k^2S_{3333}^{(k)}-t_{k+1}S_{3333}^{(k+1)})\sigma_{\gamma 3}^{-(k)}\eta_\gamma>_{S^{(k)}}$$

$$+ \int_{S_i^{(k)}} ((\frac{1}{24}+\frac{3}{280})(t_k^2S_{3333}^{(k)}-t_{k+1}S_{3333}^{(k+1)})\eta_\gamma\,\tau_{33}^{-(k)}\sigma_{\gamma 3}^{-(k)})'dS_i^{(k)}$$

$$<\sigma_{\gamma 3}^{-(k+1)}, \Lambda_{12}^{(k+1)}\tau_{33}^{-(k)}>_{R^{(k)}} = <\tau_{33}^{-(k)}, \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}>_{R^{(k)}}$$

$$+ \ <\tau_{33}^{-(k)}, (-\frac{3}{280}+\frac{1}{24})t_{k+1}^{2}S_{3333}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}\eta_\gamma>_{S^{(k)}}$$

$$+ \int_{S_i^{(k)}} ((-\frac{3}{280}+\frac{1}{24})t_{k+1}^{2}S_{3333}^{(k+1)}\eta_\gamma\tau_{33}^{-(k)}\sigma_{\gamma 3}^{-(k+1)})' >_{S_i^{(k)}}$$

$$<\sigma_{\gamma 3}^{-(k-1)}, \bar{\Lambda}_{12}^{(k)}\tau_{33}^{-(k)}>_{R^{(k)}} = <\tau_{33}^{-(k)}, \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}$$

$$+ \ <\tau_{33}^{-(k)}, (\frac{3}{280}-\frac{1}{24})t_k^2S_{3333}^{(k)}\sigma_{\gamma 3}^{-(k-1)}\eta_\gamma>_{S^{(k)}}$$

$$+ \int_{S_i^{(k)}} ((\frac{3}{280}-\frac{1}{24})t_k^2S_{3333}^{(k)}\eta_\gamma\tau_{33}^{-(k)}\sigma_{\gamma 3}^{-(k-1)})' >_{S_i^{(k)}}$$

$$<\sigma_{33}^{-(k)}, \Xi_{22}^{(k)}\tau_{33}^{-(k)}>_{R^{(k)}} = <\tau_{33}^{-(k)}, \Xi_{22}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$<\sigma_{33}^{-(k+1)}, \Lambda_{22}^{(k+1)}\tau_{33}^{-(k)}>_{R^{(k)}} = <\tau_{33}^{-(k)}, \bar{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

and

$$<\sigma_{33}^{-(k-1)}, \bar{\Lambda}_{22}^{(k+1)}\tau_{33}^{-(k)}>_{R^{(k)}} = <\tau_{33}^{-(k)}, \Lambda_{22}^{(k+1)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

we have

$$
\begin{aligned}
\Delta_{\tau_{33}^{(k)}}\Omega = 2 < \tau_{33}^{-(k)}\,,\ & \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)} \\
& - \bar{v}_3^{(k)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}N_{\alpha\beta}^{(k)}-\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}M_{\alpha\beta}^{(k)} \\
& + \Xi_{21}^{(k)}\sigma_{\gamma 3}^{-(k)}+\Xi_{22}^{(k)}\sigma_{33}^{-(k)} \\
& + \bar{v}_3^{(k+1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta}^{(k+1)}+\frac{6}{5t_{k+1}}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta}^{(k+1)} \\
& + \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k+1)} >_{R^{(k)}}
\end{aligned} \tag{257}
$$

Vanishing of $\Delta_{\tau_{33}^{(k)}}\Omega$ for arbitrary $\tau_{33}^{-(k)}$ implies

$$
\begin{aligned}
& \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)} \\
& \quad - \bar{v}_3^{(k)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}N_{\alpha\beta}^{(k)}-\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}M_{\alpha\beta}^{(k)} \\
& \quad + \Xi_{21}^{(k)}\sigma_{\gamma 3}^{-(k)}+\Xi_{22}^{(k)}\sigma_{33}^{-(k)} \\
& \quad + \bar{v}_3^{(k+1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta}^{(k+1)}+\frac{6}{5t_{k+1}}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta}^{(k+1)} \\
& \quad + \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k+1)} = 0 \text{ on } R^{(k)}
\end{aligned}
$$

Equation (230) represents the basic function governing the behavior of laminated composites. Use of this function is possible for both nonhomogeneous and homogeneous problems along with discontinuity conditions. This function is completely general, in the sense that it admits $\bar{v}_\gamma^{(k)}$, $\bar{\Phi}_\gamma^{(k)}$, $\bar{v}_3^{(k)}$, $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$, $V_\rho^{(k)}$, $\sigma_{i3}^{-(k)}$ as field variables and there is no requirement that admissible variables identically satisfy any of the field equations or the boundary conditions. We note that the function is defined on a space which includes the set of admissible states as a subset. The space is defined by (242) through (249). This directly represents an extension of the space of admissible states in which the approximate solutions are often sought.

4.7 Extended Variational Principles

The functions in (230) must belong to the domain of definition of each of $A_{ij}^{(k)}$ i, j = 1, . . . , n . This implies restrictions of smoothness on the choices of v_i in approximate solution schemes. Some of these restrictions can be relaxed using extended variational principles based on elimination of some of the operators. These extensions were first discussed by Prager [1967], and Pian [1969]. Sandhu [1975,1976] proposed a general scheme for these extensions using the self-adjoint property of the operator matrix. In the context of the finite element method, it is clear that the extension of the admissible space provides greater freedom in selection of approximating functions. To apply these ideas to the present problem, recall:

$$
\begin{aligned}
<\bar{v}_\alpha^{(k)}, N_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}} = & -<N_{\alpha\beta}^{(k)}, \bar{v}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}} \\
& + <N_{\alpha\beta}^{(k)}, \eta_\beta \bar{v}_\alpha^{(k)}>_{S_2^{(k)}} + <\bar{v}_\alpha^{(k)}, N_{\alpha\beta}^{(k)}\eta_\beta>_{S_1^{(k)}} \\
& + <[N_{\alpha\beta}^{(k)}]^{mean}, \eta_\beta(\bar{v}_\alpha^{(k)})'>_{S_i^{(k)}} + <[\bar{v}_\alpha^{(k)}]^{mean}, (N_{\alpha\beta}^{(k)})'\eta_\beta>_{S_i^{(k)}} \qquad (258)
\end{aligned}
$$

Here $[N_{\alpha\beta}^{(k)}]^{mean}$, $[\bar{v}_\alpha^{(k)}]^{mean}$ are the mean values across the internal surface $S_i^{(k)}$ of the quantities in the brackets.

$$
\begin{aligned}
<\bar{\phi}_\alpha^{(k)}, M_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}} = & -<M_{\alpha\beta}^{(k)}, \bar{\phi}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}} \\
& + <M_{\alpha\beta}^{(k)}, \eta_\beta \bar{\phi}_\alpha^{(k)}>_{S_4^{(k)}} + <\bar{\phi}_\alpha^{(k)}, M_{\alpha\beta}^{(k)}\eta_\beta>_{S_3^{(k)}} \\
& + <[M_{\alpha\beta}^{(k)}]^{mean}, \eta_\beta(\bar{\phi}_\alpha^{(k)})'>_{S_i^{(k)}} + <[\bar{\phi}_\alpha^{(k)}]^{mean}, (M_{\alpha\beta}^{(k)})'\eta_\beta>_{S_i^{(k)}} \qquad (259)
\end{aligned}
$$

where $[M_{\alpha\beta}^{(k)}]^{mean}$, $[\bar{\phi}_\alpha^{(k)}]^{mean}$ are the mean values across the internal surface $S_i^{(k)}$.

$$
\begin{aligned}
<\bar{v}_3^{(k)}, V_{\alpha,\alpha}^{(k)}>_{R^{(k)}} = & -<V_\alpha^{(k)}, \bar{v}_{3,\alpha}^{(k)}>_{R^{(k)}} \\
& + <V_\alpha^{(k)}, \eta_\alpha \bar{v}_3^{(k)}>_{S_6^{(k)}} + <\bar{v}_3^{(k)}, V_\alpha^{(k)}\eta_\alpha>_{S_5^{(k)}} \\
& + <[V_\alpha^{(k)}]^{mean}, \eta_\alpha(\bar{v}_3^{(k)})'>_{S_i^{(k)}} + <[\bar{v}_3^{(k)}]^{mean}, (V_\alpha^{(k)})'\eta_\alpha>_{S_i^{(k)}} \qquad (260)
\end{aligned}
$$

where $[V_\alpha^{(k)}]^{mean}$, $[\overline{V}_3^{(k)}]^{mean}$ are the mean values across the internal surface $S_i^{(k)}$. Equations (258) through (259) can be used to eliminate

1. $N^{(k)}_{\alpha\beta,\beta}$ or $\overline{v}^{(k)}_{(\alpha,\beta)}$

2. $M^{(k)}_{\alpha\beta,\beta}$ or $\overline{\phi}^{(k)}_{(\alpha,\beta)}$

3. $V^{(k)}_{\alpha,\alpha}$ or $\overline{v}^{(k)}_{3,\alpha}$

from the general variational formulation. For example, using (258) the set of admissible states for $N^{(k)}_{\alpha\beta}$ is extended from C^1 to C^0. Various combinations give rise to distinct extended formulations. To illustrate application of the general procedure to the analysis of laminated composites based on stress formulations, we shall only state six extensions leading to some useful formulations. Other formulations can be constructed on the lines indicated by Al-Ghothani [1986] in the context of a unified approach to the dynamics of bending and extension of moderately thick laminated composite plates. Elimination of $N^{(k)}_{\alpha\beta,\beta}$ from (230) gives

$$\Omega_1(u\ ,\sigma)=2<\overline{v}^{(1)}_\gamma\ ,\sigma^{(0)}_{\gamma3}>_{R^{(1)}}+2<\overline{\phi}^{(1)}_\gamma\ ,\frac{t_1}{2}\sigma^{(0)}_{\gamma3}>_{R^{(1)}}+2<\overline{v}^{(1)}_3\ ,\sigma^{(0)}_{33}>_{R^{(1)}}$$

$$+\ 2<N^{(1)}_{\alpha\beta}\ ,\frac{t_1}{12}S^{(1)}_{\alpha\beta33}\sigma^{(0)}_{\gamma3,\gamma}+\frac{1}{2}S^{(1)}_{\alpha\beta33}\sigma^{(0)}_{33}>_{R^{(1)}}$$

$$+\ 2<M^{(1)}_{\alpha\beta}\ ,\frac{1}{10}S^{(1)}_{\alpha\beta33}\sigma^{(0)}_{\gamma3,\gamma}+\frac{6}{5t_1}S^{(1)}_{\alpha\beta33}\sigma^{(0)}_{33}>_{R^{(1)}}\ +\ 2<V^{(1)}_\rho\ ,-\frac{2}{5}S^{(1)}_{\rho3\gamma3}\sigma^{(0)}_{\gamma3}>_{R^{(1)}}$$

$$+\ 2<\overline{v}^{(N)}_\gamma\ ,-\sigma^{(N)}_{\gamma3}>_{R^{(N)}}+2<\overline{\phi}^{(N)}_\gamma\ ,\frac{t_N}{2}\sigma^{(N)}_{\gamma3}>_{R^{(N)}}+2<\overline{v}^{(N)}_3\ ,-\sigma^{(N)}_{33}>_{R^{(N)}}$$

$$+\ 2<N^{(N)}_{\alpha\beta}\ ,-\frac{t_N}{12}S^{(N)}_{\alpha\beta33}\sigma^{(N)}_{\gamma3,\gamma}+\frac{1}{2}S^{(N)}_{\alpha\beta33}\sigma^{(N)}_{33}>_{R^{(N)}}$$

$$+\ 2<M^{(N)}_{\alpha\beta}\ ,\frac{1}{10}S^{(N)}_{\alpha\beta33}\sigma^{(N)}_{\gamma3,\gamma}-\frac{6}{5t_N}S^{(N)}_{\alpha\beta33}\sigma^{(N)}_{33}>_{R^{(N)}}+2<V^{(N)}_\rho\ ,-\frac{2}{5}S^{(N)}_{\rho3\gamma3}\sigma^{(N)}_{\gamma3}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}\{<\overline{v}^{(k)}_\gamma\ ,\sigma^{-(k-1)}_{\gamma3}>_{R^{(k)}}+<\overline{\phi}^{(k)}_\gamma\ ,\frac{t_k}{2}\sigma^{-(k-1)}_{\gamma3}>_{R^{(k)}}+<\overline{v}^{(k)}_3\ ,\sigma^{-(k-1)}_{33}>_{R^{(k)}}$$

$$+ <N^{(k)}_{\alpha\beta}, \frac{2t_k}{12}S^{(k)}_{\alpha\beta 33}\sigma^{-(k-1)}_{\gamma 3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta 33}\sigma^{-(k-1)}_{33}>_{R^{(k)}}$$

$$+ <M^{(k)}_{\alpha\beta}, \frac{1}{10}S^{(k)}_{\alpha\beta 33}\sigma^{-(k-1)}_{\gamma 3,\gamma}+\frac{6}{5t_k}S^{(k)}_{\alpha\beta 33}\sigma^{-(k-1)}_{33}>_{R^{(k)}}$$

$$+ <V^{(k)}_{\rho}, -\frac{2}{5}S^{(k)}_{\rho 3\gamma 3}\sigma^{-(k-1)}_{\gamma 3}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{<\overline{\phi}^{(k)}_{\alpha}, M^{(k)}_{\alpha\beta,\beta}-V^{(k)}_{\alpha}>_{R^{(k)}}+<\overline{v}^{(k)}_{3}, V^{(k)}_{\alpha,\alpha}>_{R^{(k)}}$$

$$- 2<N^{(k)}_{\alpha\beta}, \overline{v}^{(k)}_{(\alpha,\beta)}>_{R^{(k)}}+<N^{(k)}_{\alpha\beta}, \frac{1}{t_k}S^{(k)}_{\alpha\beta\mu\rho}N^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$+ <M^{(k)}_{\alpha\beta}, -\overline{\phi}^{(k)}_{(\alpha,\beta)}+\frac{12}{t_k^3}S^{(k)}_{\alpha 3\mu\rho}M^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$+ <V^{(k)}_{\gamma}, -\overline{\phi}^{(k)}_{\gamma}-\overline{v}^{(k)}_{3,\gamma}+\frac{24}{5t_k}S^{(k)}_{\rho 3\gamma 3}V^{(k)}_{\rho}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\overline{v}^{(k)}_{\gamma}, -\sigma^{-(k)}_{\gamma 3}>_{R^{(k)}}+<\overline{\phi}^{(k)}_{\gamma}, \frac{t_k}{2}\sigma^{-(k)}_{\gamma 3}>_{R^{(k)}}+<\overline{v}^{(k)}_{3}, -\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <N^{(k)}_{\alpha\beta}, -\frac{2t_k}{12}S^{(k)}_{\alpha\beta 33}\sigma^{-(k)}_{\gamma 3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta 33}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <M^{(k)}_{\alpha\beta}, \frac{1}{10}S^{(k)}_{\alpha\beta 33}\sigma^{-(k)}_{\gamma 3,\gamma}-\frac{6}{5t_k}S^{(k)}_{\alpha\beta 33}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <V^{(k)}_{\rho}, -\frac{2}{5}S^{(k)}_{\rho 3\gamma 3}\sigma^{-(k)}_{\gamma 3}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\sigma^{-(k)}_{\gamma 3}, -\overline{v}^{(k)}_{\gamma}+\frac{t_k}{2}\overline{\phi}^{(k)}_{\gamma}-\frac{1}{10}S^{(k)}_{\alpha\beta 33}M^{(k)}_{\alpha\beta,\gamma}-\frac{2}{5}S^{(k)}_{\rho 3\gamma 3}V^{(k)}_{\rho}>_{R^{(k)}}$$

$$+ <\sigma^{-(k)}_{\rho 3}, \Xi^{(k)}_{11}\sigma^{-(k)}_{\gamma 3}+\Xi^{(k)}_{12}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <\sigma^{-(k)}_{\gamma 3}, \overline{v}^{(k+1)}_{\gamma}+\frac{t_{k+1}}{2}\overline{\phi}^{(k+1)}_{\gamma}$$

$$- \frac{1}{10}S^{(k+1)}_{\alpha\beta 33}M^{(k+1)}_{\alpha\beta,\gamma}-\frac{2}{5}S^{(k+1)}_{\rho 3\gamma 3}V^{(k+1)}_{\rho}>_{R^{(k+1)}}$$

$$+ <\sigma^{-(k)}_{33}, -\overline{v}^{(k)}_{3}+\frac{1}{2}S^{(k)}_{\alpha\beta 33}N^{(k)}_{\alpha\beta}-\frac{6}{5t_k}S^{(k)}_{\alpha\beta 33}M^{(k)}_{\alpha\beta}>_{R^{(k)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \Xi_{21}^{(k)}\sigma_{\gamma 3}^{-(k)}+\Xi_{22}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \bar{v}_3^{(k+1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta}^{(k+1)}+\frac{6}{5t_{k+1}}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta}^{(k+1)}>_{R^{(k+1)}}\}$$

$$+\sum_{k=2}^{N-1}\{<\sigma_{\rho 3}^{-(k)}\ ,\ \Lambda_{11}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{12}^{(k)}\sigma_{33}^{-(k-1)}>+<\sigma_{33}^{-(k)}\ ,\ \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{<\sigma_{\gamma 3}^{-(k)}\ ,\ \bar{\Lambda}_{11}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{12}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}\}$$

$$+ \ 2<\{\sigma\}^{-(N-1)}\ ,\ [\bar{\Lambda}]^{(N)}\{\sigma\}^{(N)}>_{R^{(N)}}$$

$$+ \ 2<\{\sigma\}^{-(1)}\ ,\ [\Lambda]^{(1)}\{\sigma\}^{(0)}>_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}_\alpha^{(k)}\ ,\ -2g_1^{(k)}>_{S_1^{(k)}}$$

$$+ \ <\bar{\phi}_\alpha^{(k)}\ ,\ -\eta_\beta M_{\alpha\beta}^{(k)}-2g_3^{(k)}>_{S_3^{(k)}}$$

$$+ \ <\bar{v}_3^{(k)}\ ,\ -\eta_\alpha V_\alpha^{(k)}-2g_5^{(k)}>_{S_5^{(k)}}$$

$$+ \ 2<N_{\alpha\beta}^{(k)}\ ,\ \eta_\beta\bar{v}_\alpha^{(k)}-g_2^{(k)}>_{S_2^{(k)}}$$

$$+ \ <M_{\alpha\beta}^{(k)}\ ,\ \eta_\beta\bar{\phi}_\alpha^{(k)}-2g_4^{(k)}>_{S_4^{(k)}}$$

$$+ \ <V_\alpha^{(k)}\ ,\ \eta_\alpha\bar{v}_3^{(k)}-2g_6^{(k)}>_{S_6^{(k)}}\}$$

$$+ \ <\sigma_{\gamma 3}^{-(1)}\ ,\ [\frac{1}{10}S_{\alpha\beta 33}^{(1)}\eta_\gamma]M_{\alpha\beta}^{(1)}$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_1^3S_{3333}^{(1)}+t_2^3S_{3333}^{(2)})\eta_\gamma]\sigma_{\rho 3,\rho}^{-(1)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_2^2S_{3333}^{(2)}-t_1^2S_{3333}^{(1)})\eta_\gamma]\sigma_{33}^{-(1)}$$

$$+ [\frac{1}{10}S^{(2)}_{\alpha\beta33}\eta_\gamma]M^{(2)}_{\alpha\beta}+[-(\frac{1}{120}-\frac{1}{840})t_2^3S^{(2)}_{3333}\eta_\gamma]\sigma^{-(2)}_{\rho3,\rho}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_2^2S^{(2)}_{3333}\eta_\gamma]\sigma^{-(2)}_{33}-2g^{(1)}_\sigma>_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{ <\sigma^{-(k)}_{\gamma3}, [-(\frac{1}{120}-\frac{1}{840})t_k^3S^{(k)}_{3333}\eta_\gamma]\sigma^{-(k-1)}_{\rho3,\rho}+[(-\frac{1}{24}+\frac{3}{280})t_k^2S^{(k)}_{3333}\eta_\gamma]\sigma^{-(k-1)}_{33}$$

$$+ [\frac{1}{10}S^{(k)}_{\alpha\beta33}\eta_\gamma]M^{(k)}_{\alpha\beta}+[(\frac{1}{120}+\frac{1}{840})(t_k^3S^{(k)}_{3333}+t^3_{k+1}S^{(k+1)}_{3333})\eta_\gamma]\sigma^{-(k)}_{\rho3,\rho}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t^2_{k+1}S^{(k+1)}_{3333}-t_k^2S^{(k)}_{3333})\eta_\gamma]\sigma^{-(k)}_{33}$$

$$+ [\frac{1}{10}S^{(k+1)}_{\alpha\beta33}\eta_\gamma]M^{(k+1)}_{\alpha\beta}+[-(\frac{1}{120}-\frac{1}{840})t^3_{k+1}S^{(k+1)}_{3333}\eta_\gamma]\sigma^{-(k+1)}_{\rho3,\rho}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t^2_{k+1}S^{(k-1)}_{3333}\eta_\gamma]\sigma^{-(k+1)}_{33}-2g^{(k)}_\sigma>_{S^{(k)}} \}$$

$$+ <\sigma^{-(N-1)}_{\gamma3}, [-(\frac{1}{120}-\frac{1}{840})t^3_{N-1}S^{(N-1)}_{3333}\eta_\gamma]\sigma^{-(N-2)}_{\rho3,\rho}+[(-\frac{1}{24}+\frac{3}{280})t^2_{N-1}S^{(N-1)}_{3333}\eta_\gamma]\sigma^{-(N-2)}_{33}$$

$$+ [\frac{1}{10}S^{(N-1)}_{\alpha\beta33}\eta_\gamma]M^{(N-1)}_{\alpha\beta}$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t^3_{N-1}S^{(N-1)}_{3333}+t^3_NS^{(N)}_{3333})\eta_\gamma]\sigma^{-(N-1)}_{\rho3,\rho}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t^2_NS^{(N)}_{3333}-t^2_{N-1}S^{(N-1)}_{3333})\eta_\gamma]\sigma^{-(N-1)}_{33}$$

$$+ [\frac{1}{10}S^{(N)}_{\alpha\beta33}\eta_\gamma]M^{(N)}_{\alpha\beta}-2g^{(N-1)}_\sigma>_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}^{(k)}_o, -2g'^{(k)}_1>_{S^{(k)}_{1i}}$$

$$+ <\bar{\phi}^{(k)}_\alpha, -\eta_\beta(M^{(k)}_{\alpha\beta})'-2g'^{(k)}_3>_{S^{(k)}_{3i}}$$

$$+ <\bar{v}^{(k)}_3, -\eta_\alpha(V^{(k)}_\alpha)'-2g'^{(k)}_5>_{S^{(k)}_{5i}}$$

$$+ 2<N^{(k)}_{\alpha\beta}, \eta_\beta(\bar{v}^{(k)}_\alpha)'-g'^{(k)}_2>_{S^{(k)}_{2i}}$$

$$+ <M^{(k)}_{\alpha\beta}, \eta_\beta(\bar{\phi}^{(k)}_o)'-2g'^{(k)}_4>_{S^{(k)}_{4i}}$$

$$+ \ <V_\alpha^{(k)}\ ,\ \eta_\alpha(\bar{v}_3^{(k)})' - 2g_6'^{(k)}>_{S_{6i}^{(k)}}\}$$

$$+ \ <\bar{\sigma}_{\gamma 3}^{(1)}\ ,\ [\frac{1}{10}S_{\alpha\beta 33}^{(1)}\eta_\gamma](M_{\alpha\beta}^{(1)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_1^3 S_{3333}^{(1)} + t_2^3 S_{3333}^{(2)})\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(1)})'$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_2^2 S_{3333}^{(2)} - t_1^2 S_{3333}^{(1)})\eta_\gamma](\bar{\sigma}_{33}^{(1)})'$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(2)}\eta_\gamma](M_{\alpha\beta}^{(2)})' + [-(\frac{1}{120}-\frac{1}{840})t_2^3 S_{3333}^{(2)}\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(2)})'$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_2^2 S_{3333}^{(2)}\eta_\gamma](\bar{\sigma}_{33}^{(2)})' - 2g_\sigma'^{(1)}>_{S_i^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{\ <\bar{\sigma}_{\gamma 3}^{(k)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(k-1)})'$$

$$+ \ [(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\eta_\gamma](\bar{\sigma}_{33}^{(k-1)})'$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma](M_{\alpha\beta}^{(k)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)} + t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(k)})'$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)} - t_k^2 S_{3333}^{(k)})\eta_\gamma](\bar{\sigma}_{33}^{(k)})'$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma](M_{\alpha\beta}^{(k+1)})' + [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(k+1)})'$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma](\bar{\sigma}_{33}^{(k+1)})' - 2g_\sigma'^{(k)}>_{S_i^{(k)}}\ \}$$

$$+ \ <\bar{\sigma}_{\gamma 3}^{(N-1)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S_{3333}^{(N-1)}\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(N-2)})'$$

$$+ \ [(-\frac{1}{24}+\frac{3}{280})t_{N-1}^2 S_{3333}^{(N-1)}\eta_\gamma](\bar{\sigma}_{33}^{(N-2)})'$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(N-1)}\eta_\gamma](M_{\alpha\beta}^{(N-1)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^3 S_{3333}^{(N-1)} + t_N^3 S_{3333}^{(N)})\eta_\gamma](\bar{\sigma}_{\rho 3,\rho}^{(N-1)})'$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_N^2 S_{3333}^{(N)} - t_{N-1}^2 S_{3333}^{(N-1)})\eta_\gamma](\sigma_{33}^{-(N-1)})'$$

$$+ [\frac{1}{10} S_{\alpha\beta 33}^{(N)}\eta_\gamma](M_{\alpha\beta}^{(N)})' - 2g_\sigma^{\prime(N-1)} >_{S_i^{(N-1)}}$$

Clearly, Ω_1 is defined over an extension of the domain Ω insomuch as $N_{\alpha\beta}^{(k)}$ need only belong to C^0 and not to C^1. In finite element procedures this relaxation of continuity [Pian 1969] is quite important in allowing lower order interpolations or simpler approximating functions to be used. Elimination of $M_{\alpha\beta,\beta}^{(k)}$ from Ω_1 gives

$$\Omega_2(u\,,\sigma)=2<\bar{v}_\gamma^{(1)}\,,\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{\phi}_\gamma^{(1)}\,,\frac{t_1}{2}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{v}_3^{(1)}\,,\sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+\ 2<N_{\alpha\beta}^{(1)}\,,\frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{1}{2}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+\ 2<M_{\alpha\beta}^{(1)}\,,\frac{1}{10}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{6}{5t_1}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}+\ 2<V_\rho^{(1)}\,,-\frac{2}{5}S_{\rho 3\gamma 3}^{(1)}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}$$

$$+\ 2<\bar{v}_\gamma^{(N)}\,,-\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{\phi}_\gamma^{(N)}\,,\frac{t_N}{2}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{v}_3^{(N)}\,,-\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<N_{\alpha\beta}^{(N)}\,,-\frac{t_N}{12}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}+\frac{1}{2}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<M_{\alpha\beta}^{(N)}\,,\frac{1}{10}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}-\frac{6}{5t_N}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}+2<V_\rho^{(N)}\,,-\frac{2}{5}S_{\rho 3\gamma 3}^{(N)}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}\{<\bar{v}_\gamma^{(k)}\,,\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{\phi}_\gamma^{(k)}\,,\frac{t_k}{2}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{v}_3^{(k)}\,,\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)}\,,\frac{2t_k}{12}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k-1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)}\,,\frac{2}{10}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k-1)}+\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <V_\rho^{(k)}\,,-\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{<\bar{\phi}_\alpha^{(k)},\,-V_\alpha^{(k)}>_{R^{(k)}}+<\bar{v}_3^{(k)}\,,V_{\alpha,\alpha}^{(k)}>_{R^{(k)}}$$

$$- 2\langle N^{(k)}_{\alpha\beta} , \bar{v}^{(k)}_{(\alpha,\beta)}\rangle_{R^{(k)}}$$

$$+ \langle N^{(k)}_{\alpha\beta} , \frac{1}{t_k} S^{(k)}_{\alpha\beta\mu\rho} N^{(k)}_{\mu\rho}\rangle_{R^{(k)}}$$

$$- 2\langle M^{(k)}_{\alpha\beta} , \bar{\phi}^{(k)}_{(\alpha,\beta)}\rangle_{R^{(k)}}$$

$$+ \langle M^{(k)}_{\alpha\beta} , \frac{12}{t_k^3} S^{(k)}_{\alpha\beta\mu\rho} M^{(k)}_{\mu\rho}\rangle_{R^{(k)}} + \langle V^{(k)}_{\gamma} , -\bar{\phi}^{(k)}_{\gamma} - \bar{v}^{(k)}_{3,\gamma} + \frac{24}{5t_k} S^{(k)}_{\rho3\gamma3} V^{(k)}_{\rho}\rangle_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{\langle \bar{v}^{(k)}_{\gamma} , -\sigma^{-(k)}_{\gamma3}\rangle_{R^{(k)}} + \langle \bar{\phi}^{(k)}_{\gamma} , \frac{t_k}{2}\sigma^{-(k)}_{\gamma3}\rangle_{R^{(k)}} + \langle \bar{v}^{(k)}_{3} , -\sigma^{-(k)}_{33}\rangle_{R^{(k)}}$$

$$+ \langle N^{(k)}_{\alpha\beta} , -\frac{2t_k}{12} S^{(k)}_{\alpha\beta33}\sigma^{-(k)}_{\gamma3,\gamma} + \frac{1}{2} S^{(k)}_{\alpha\beta33}\sigma^{-(k)}_{33}\rangle_{R^{(k)}}$$

$$+ \langle M^{(k)}_{\alpha\beta} , \frac{2}{10} S^{(k)}_{\alpha\beta33}\sigma^{-(k)}_{\gamma3,\gamma} - \frac{6}{5t_k} S^{(k)}_{\alpha\beta33}\sigma^{-(k)}_{33}\rangle_{R^{(k)}}$$

$$+ \langle V^{(k)}_{\rho} , -\frac{2}{5} S^{(k)}_{\rho3\gamma3}\sigma^{-(k)}_{\gamma3}\rangle_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{\langle \sigma^{-(k)}_{\gamma3} , -\bar{v}^{(k)}_{\gamma} + \frac{t_k}{2}\bar{\phi}^{(k)}_{\gamma} - \frac{2}{5} S^{(k)}_{\rho3\gamma3} V^{(k)}_{\rho}\rangle_{R^{(k)}}$$

$$+ \langle \sigma^{-(k)}_{\rho3} , \Xi^{(k)}_{11}\sigma^{-(k)}_{\gamma3} + \Xi^{(k)}_{12}\sigma^{-(k)}_{33}\rangle_{R^{(k)}}$$

$$+ \langle \sigma^{-(k)}_{\gamma3} , \bar{v}^{(k+1)}_{\gamma} + \frac{t_{k+1}}{2}\bar{\phi}^{(k+1)}_{\gamma} - \frac{2}{5} S^{(k+1)}_{\rho3\gamma3} V^{(k+1)}_{\rho}\rangle_{R^{(k+1)}}$$

$$+ \langle \sigma^{-(k)}_{33} , -\bar{v}^{(k)}_{3} + \frac{1}{2} S^{(k)}_{\alpha\beta33} N^{(k)}_{\alpha\beta} - \frac{6}{5t_k} S^{(k)}_{\alpha\beta33} M^{(k)}_{\alpha\beta}\rangle_{R^{(k)}}$$

$$+ \langle \sigma^{-(k)}_{33} , \Xi^{(k)}_{21}\sigma^{-(k)}_{\gamma3} + \Xi^{(k)}_{22}\sigma^{-(k)}_{33}\rangle_{R^{(k)}}$$

$$+ \langle \sigma^{-(k)}_{33} , \bar{v}^{(k+1)}_{3} + \frac{1}{2} S^{(k+1)}_{\alpha\beta33} N^{(k+1)}_{\alpha\beta} + \frac{6}{5t_{k+1}} S^{(k+1)}_{\alpha\beta33} M^{(k+1)}_{\alpha\beta}\rangle_{R^{(k+1)}}\}$$

$$+\sum_{k=2}^{N-1}\{\langle \sigma^{-(k)}_{\rho3} , \Lambda^{(k)}_{11}\sigma^{-(k-1)}_{\gamma3} + \Lambda^{(k)}_{12}\sigma^{-(k-1)}_{33}\rangle + \langle \sigma^{-(k)}_{33} , \Lambda^{(k)}_{21}\sigma^{-(k-1)}_{\gamma3} + \Lambda^{(k)}_{22}\sigma^{-(k-1)}_{33}\rangle_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{\langle \sigma^{-(k)}_{\gamma3} , \bar{\Lambda}^{(k+1)}_{11}\sigma^{-(k+1)}_{\gamma3} + \bar{\Lambda}^{(k+1)}_{12}\sigma^{-(k+1)}_{33}\rangle_{R^{(k+1)}}$$

$$+ \ <\sigma_{33}^{-(k)} \ , \ \overline{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\overline{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}\}$$

$$+ \ 2<\{\sigma\}^{-(N-1)} \ , \ [\overline{\Lambda}]^{(N)}\{\sigma\}^{(N)}>_{R^{(N)}}$$

$$+ \ 2<\{\sigma\}^{(1)} \ , \ [\Lambda]^{(1)}\{\sigma\}^{(0)}>_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\{<\overline{v}_{\alpha}^{(k)} \ , \ -2g_{1}^{(k)}>_{S_{1}^{(k)}}$$

$$+ \ <\overline{\phi}_{\alpha}^{(k)} \ , \ -2g_{3}^{(k)}>_{S_{3}^{(k)}}$$

$$+ \ <\overline{v}_{3}^{(k)} \ , \ -\eta_{\alpha}V_{\alpha}^{(k)}-2g_{5}^{(k)}>_{S_{5}^{(k)}}$$

$$+ \ 2<N_{\alpha\beta}^{(k)} \ , \ \eta_{\beta}\overline{v}_{\alpha}^{(k)}-g_{2}^{(k)}>_{S_{2}^{(k)}}$$

$$+ \ 2<M_{\alpha\beta}^{(k)} \ , \ \eta_{\beta}\overline{\phi}_{\alpha}^{(k)}-g_{4}^{(k)}>_{S_{4}^{(k)}}$$

$$+ \ <V_{\alpha}^{(k)} \ , \ \eta_{\alpha}\overline{v}_{3}^{(k)}-2g_{6}^{(k)}>_{S_{6}^{(k)}}\}$$

$$+ \ <\sigma_{\gamma 3}^{-(1)} \ , \ [(\frac{1}{120}+\frac{1}{840})(t_{1}^{3}S_{3333}^{(1)}+t_{2}^{3}S_{3333}^{(2)})\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(1)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{2}^{2}S_{3333}^{(2)}-t_{1}^{2}S_{3333}^{(1)})\eta_{\gamma}]\sigma_{33}^{-(1)}$$

$$+ \ [-(\frac{1}{120}-\frac{1}{840})t_{2}^{3}S_{3333}^{(2)}\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(2)}$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_{2}^{2}S_{3333}^{(2)}\eta_{\gamma}]\sigma_{33}^{-(2)}-2g_{\sigma}^{(1)}>_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{ \ <\sigma_{\gamma 3}^{-(k)} \ , \ [-(\frac{1}{120}-\frac{1}{840})t_{k}^{3}S_{3333}^{(k)}\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(k-1)}+[(-\frac{1}{24}+\frac{3}{280})t_{k}^{2}S_{3333}^{(k)}\eta_{\gamma}]\sigma_{33}^{-(k-1)}$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_{k}^{3}S_{3333}^{(k)}+t_{k+1}^{3}S_{3333}^{(k+1)})\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(k)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^{2}S_{3333}^{(k+1)}-t_{k}^{2}S_{3333}^{(k)})\eta_{\gamma}]\sigma_{33}^{-(k)}$$

$$+ \ [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^{3}S_{3333}^{(k+1)}\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(k+1)}$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^{2}S_{3333}^{(k+1)}\eta_{\gamma}]\sigma_{33}^{-(k+1)}-2g_{\sigma}^{(k)}>_{S^{(k)}}\ \}$$

$$+\ <\sigma_{\gamma 3}^{-(N-1)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^{3}S_{3333}^{(N-1)}\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(N-2)}+[(-\frac{1}{24}+\frac{3}{280})t_{N-1}^{2}S_{3333}^{(N-1)}\eta_{\gamma}]\sigma_{33}^{-(N-2)}$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^{3}S_{3333}^{(N-1)}+t_{N}^{3}S_{3333}^{(N)})\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(N-1)}$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_{N}^{2}S_{3333}^{(N)}-t_{N-1}^{2}S_{3333}^{(N-1)})\eta_{\gamma}]\sigma_{33}^{-(N-1)}-2g_{\sigma}^{(N-1)}>_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}_{\alpha}^{(k)}\ ,\ -2g_{1}^{\prime(k)}>_{S_{1i}^{(k)}}$$

$$+\ <\bar{\phi}_{\alpha}^{(k)}\ ,\ -2g_{3}^{\prime(k)}>_{S_{3i}^{(k)}}$$

$$+\ <\bar{v}_{3}^{(k)}\ ,\ -\eta_{\alpha}(V_{\alpha}^{(k)})'-2g_{5}^{\prime(k)}>_{S_{5i}^{(k)}}$$

$$+\ 2<N_{\alpha\beta}^{(k)}\ ,\ \eta_{\beta}(\bar{v}_{\alpha}^{(k)})'-g_{2}^{\prime(k)}>_{S_{2i}^{(k)}}$$

$$+\ 2<M_{\alpha\beta}^{(k)}\ ,\ \eta_{\beta}(\bar{\phi}_{\alpha}^{(k)})'-g_{4}^{\prime(k)}>_{S_{4i}^{(k)}}$$

$$+\ <V_{\alpha}^{(k)}\ ,\ \eta_{\alpha}(\bar{v}_{3}^{(k)})'-2g_{6}^{\prime(k)}>_{S_{6i}^{(k)}}\}$$

$$+\ <\sigma_{\gamma 3}^{-(1)}\ ,\ [(\frac{1}{120}+\frac{1}{840})(t_{1}^{3}S_{3333}^{(1)}+t_{2}^{3}S_{3333}^{(2)})\eta_{\gamma}](\sigma_{\rho 3,\rho}^{-(1)})'$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_{2}^{2}S_{3333}^{(2)}-t_{1}^{2}S_{3333}^{(1)})\eta_{\gamma}](\sigma_{33}^{-(1)})'$$

$$+\ [-(\frac{1}{120}-\frac{1}{840})t_{2}^{3}S_{3333}^{(2)}\eta_{\gamma}](\sigma_{\rho 3,\rho}^{-(2)})'$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t_{2}^{2}S_{3333}^{(2)}\eta_{\gamma}](\sigma_{33}^{-(2)})'-2g_{\sigma}^{\prime(1)}>_{S_{i}^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{\ <\sigma_{\gamma 3}^{-(k)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_{k}^{3}S_{3333}^{(k)}\eta_{\gamma}](\sigma_{\rho 3,\rho}^{-(k-1)})'$$

$$+\ [(-\frac{1}{24}+\frac{3}{280})t_{k}^{2}S_{3333}^{(k)}\eta_{\gamma}](\sigma_{33}^{-(k-1)})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k)})'$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)}-t_k^2 S_{3333}^{(k)})\eta_\gamma](\sigma_{33}^{-(k)})'$$

$$+\ [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k+1)})'$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma](\sigma_{33}^{-(k+1)})'-2g_\sigma'^{(k)}>_{S_1^{(k)}}\}$$

$$+\ <\sigma_{\gamma 3}^{-(N-1)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S_{3333}^{(N-1)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-2)})'$$

$$+\ [(-\frac{1}{24}+\frac{3}{280})t_{N-1}^2 S_{3333}^{(N-1)}\eta_\gamma](\sigma_{33}^{-(N-2)})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^3 S_{3333}^{(N-1)}+t_N^3 S_{3333}^{(N)})\eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-1)})'$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_N^2 S_{3333}^{(N)}-t_{N-1}^2 S_{3333}^{(N-1)})\eta_\gamma](\sigma_{33}^{-(N-1)})'-2g_\sigma'^{(N-1)}>_{S_1^{(N-1)}}$$

The domain of Ω_2 is an extension of the domain of Ω_1 to include $M_{\alpha\beta}^{(k)}$ which $\in C^0$ but may not be continuously differentiable. Elimination of $V_{\alpha,\alpha}^{(k)}$ from Ω_2 gives

$$\Omega_3(u\ ,\ \sigma)=2<\bar{v}_\gamma^{(1)}\ ,\ \sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{\phi}_\gamma^{(1)}\ ,\ \frac{t_1}{2}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{v}_3^{(1)}\ ,\ \sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+\ 2<N_{\alpha\beta}^{(1)}\ ,\ \frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{1}{2}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+\ 2<M_{\alpha\beta}^{(1)}\ ,\ \frac{1}{10}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{6}{5t_1}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}\ +\ 2<V_\rho^{(1)}\ ,\ -\frac{2}{5}S_{\rho 3\gamma 3}^{(1)}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}$$

$$+\ 2<\bar{v}_\gamma^{(N)}\ ,\ -\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{\phi}_\gamma^{(N)}\ ,\ \frac{t_N}{2}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{v}_3^{(N)}\ ,\ -\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<N_{\alpha\beta}^{(N)}\ ,\ -\frac{t_N}{12}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}+\frac{1}{2}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<M_{\alpha\beta}^{(N)}\ ,\ \frac{1}{10}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}-\frac{6}{5t_N}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}+2<V_\rho^{(N)}\ ,\ -\frac{2}{5}S_{\rho 3\gamma 3}^{(N)}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}\{<\bar{v}_\gamma^{(k)}\ ,\ \sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{\phi}_\gamma^{(k)}\ ,\ \frac{t_k}{2}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{v}_3^{(k)}\ ,\ \sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+ <N^{(k)}_{\alpha\beta}\ ,\ \frac{2t_k}{12}S^{(k)}_{\alpha\beta33}\sigma^{-(k-1)}_{\gamma3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta33}\sigma^{-(k-1)}_{33}>_{R^{(k)}}$$

$$+ <M^{(k)}_{\alpha\beta}\ ,\ \frac{2}{10}S^{(k)}_{\alpha\beta33}\sigma^{-(k-1)}_{\gamma3,\gamma}+\frac{6}{5t_k}S^{(k)}_{\alpha\beta33}\sigma^{-(k-1)}_{33}>_{R^{(k)}}$$

$$+ <V^{(k)}_{\rho}\ ,\ -\frac{2}{5}S^{(k)}_{\rho3\gamma3}\sigma^{-(k-1)}_{\gamma3}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{<\bar{\Phi}^{(k)}_{\alpha},\ -V^{(k)}_{\alpha}>_{R^{(k)}}$$

$$- 2<N^{(k)}_{\alpha\beta}\ ,\ \bar{v}^{(k)}_{(\alpha,\beta)}>_{R^{(k)}}$$

$$+ <N^{(k)}_{\alpha\beta}\ ,\ \frac{1}{t_k}S^{(k)}_{\alpha\beta\mu\rho}N^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$- 2<M^{(k)}_{\alpha\beta}\ ,\ \bar{\Phi}^{(k)}_{(\alpha,\beta)}>_{R^{(k)}}$$

$$+ <M^{(k)}_{\alpha\beta}\ ,\ \frac{12}{t_k^3}S^{(k)}_{\alpha\beta\mu\rho}M^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$- 2<V^{(k)}_{\gamma}\ ,\ \bar{v}^{(k)}_{3,\gamma}>_{R^{(k)}}+<V^{(k)}_{\gamma}\ ,\ -\bar{\Phi}^{(k)}_{\gamma}+\frac{24}{5t_k}S^{(k)}_{\rho3\gamma3}V^{(k)}_{\rho}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\bar{v}^{-(k)}_{\gamma}\ ,\ -\sigma^{-(k)}_{\gamma3}>_{R^{(k)}}+<\bar{\Phi}^{(k)}_{\gamma}\ ,\ \frac{t_k}{2}\sigma^{-(k)}_{\gamma3}>_{R^{(k)}}+<\bar{v}^{-(k)}_{3}\ ,\ -\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <N^{(k)}_{\alpha\beta}\ ,\ -\frac{2t_k}{12}S^{(k)}_{\alpha\beta33}\sigma^{-(k)}_{\gamma3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta33}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <M^{(k)}_{\alpha\beta}\ ,\ \frac{2}{10}S^{(k)}_{\alpha\beta33}\sigma^{-(k)}_{\gamma3,\gamma}-\frac{6}{5t_k}S^{(k)}_{\alpha\beta33}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <V^{(k)}_{\rho}\ ,\ -\frac{2}{5}S^{(k)}_{\rho3\gamma3}\sigma^{-(k)}_{\gamma3}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\sigma^{-(k)}_{\gamma3}\ ,\ -\bar{v}^{(k)}_{\gamma}+\frac{t_k}{2}\bar{\Phi}^{(k)}_{\gamma}-\frac{2}{5}S^{(k)}_{\rho3\gamma3}V^{(k)}_{\rho}>_{R^{(k)}}$$

$$+ <\sigma^{-(k)}_{\rho3}\ ,\ \Xi^{(k)}_{11}\sigma^{-(k)}_{\gamma3}+\Xi^{(k)}_{12}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ <\sigma^{-(k)}_{\gamma3}\ ,\ \bar{v}^{(k+1)}_{\gamma}+\frac{t_{k+1}}{2}\bar{\Phi}^{(k+1)}_{\gamma}-\frac{2}{5}S^{(k+1)}_{\rho3\gamma3}V^{(k+1)}_{\rho}>_{R^{(k+1)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ -\bar{v}_3^{(k)}+\frac{1}{2}S_{\alpha\beta33}^{(k)}N_{\alpha\beta}^{(k)}-\frac{6}{5t_k}S_{\alpha\beta33}^{(k)}M_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \Xi_{21}^{(k)}\sigma_{\gamma3}^{-(k)}+\Xi_{22}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \bar{v}_3^{(k+1)}+\frac{1}{2}S_{\alpha\beta33}^{(k+1)}N_{\alpha\beta}^{(k+1)}+\frac{6}{5t_{k+1}}S_{\alpha\beta33}^{(k+1)}M_{\alpha\beta}^{(k+1)}>_{R^{(k+1)}}\}$$

$$+\sum_{k=2}^{N-1}\{<\sigma_{\rho3}^{-(k)}\ ,\ \Lambda_{11}^{(k)}\sigma_{\gamma3}^{-(k-1)}+\Lambda_{12}^{(k)}\sigma_{33}^{-(k-1)}>+<\sigma_{33}^{-(k)}\ ,\ \Lambda_{21}^{(k)}\sigma_{\gamma3}^{-(k-1)}+\Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{<\sigma_{\gamma3}^{-(k)}\ ,\ \bar{\Lambda}_{11}^{(k+1)}\sigma_{\gamma3}^{-(k+1)}+\bar{\Lambda}_{12}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma3}^{-(k+1)}+\bar{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}\}$$

$$+ \ 2<\{\sigma\}^{-(N-1)}\ ,\ [\bar{\Lambda}]^{(N)}\{\sigma\}^{(N)}>_{R^{(N)}}$$

$$+ \ 2<\{\sigma\}^{(1)}\ ,\ [\Lambda]^{(1)}\{\sigma\}^{(0)}>_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}_\alpha^{(k)}\ ,\ -2g_1^{(k)}>_{S_1^{(k)}}$$

$$+ \ <\bar{\phi}_\alpha^{(k)}\ ,\ -2g_3^{(k)}>_{S_3^{(k)}}$$

$$+ \ <\bar{v}_3^{(k)}\ ,\ -2g_5^{(k)}>_{S_5^{(k)}}$$

$$+ \ 2<N_{\alpha\beta}^{(k)}\ ,\ \eta_\beta\bar{v}_\alpha^{(k)}-g_2^{(k)}>_{S_2^{(k)}}$$

$$+ \ 2<M_{\alpha\beta}^{(k)}\ ,\ \eta_\beta\bar{\phi}_\alpha^{(k)}-g_4^{(k)}>_{S_4^{(k)}}$$

$$+ \ 2<V_\alpha^{(k)}\ ,\ \eta_\alpha\bar{v}_3^{(k)}-g_6^{(k)}>_{S_6^{(k)}}\}$$

$$+ \ <\sigma_{\gamma3}^{-(1)}\ ,\ [(\frac{1}{120}+\frac{1}{840})(t_1^3S_{3333}^{(1)}+t_2^3S_{3333}^{(2)})\eta_\gamma]\sigma_{\rho3,\rho}^{-(1)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_2^2S_{3333}^{(2)}-t_1^2S_{3333}^{(1)})\eta_\gamma]\sigma_{33}^{-(1)}$$

$$+\ [-(\frac{1}{120}-\frac{1}{840})t_2^3 S_{3333}^{(2)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(2)}$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t_2^2 S_{3333}^{(2)}\eta_\gamma]\sigma_{33}^{-(2)}-2g_\sigma^{(1)}>_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{\ <\sigma_{\gamma 3}^{-(k)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k-1)}+[(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\eta_\gamma]\sigma_{33}^{-(k-1)}$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k)}$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)}-t_k^2 S_{3333}^{(k)})\eta_\gamma]\sigma_{33}^{-(k)}$$

$$+\ [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k+1)}$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{33}^{-(k+1)}-2g_\sigma^{(k)}>_{S^{(k)}}\ \}$$

$$+\ <\sigma_{\gamma 3}^{-(N-1)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S_{3333}^{(N-1)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(N-2)}+[(-\frac{1}{24}+\frac{3}{280})t_{N-1}^2 S_{3333}^{(N-1)}\eta_\gamma]\sigma_{33}^{-(N-2)}$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^3 S_{3333}^{(N-1)}+t_N^3 S_{3333}^{(N)})\eta_\gamma]\sigma_{\rho 3,\rho}^{-(N-1)}$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_N^2 S_{3333}^{(N)}-t_{N-1}^2 S_{3333}^{(N-1)})\eta_\gamma]\sigma_{33}^{-(N-1)}-2g_\sigma^{(N-1)}>_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}_\alpha^{(k)}\ ,\ -2g_1'^{(k)}>_{S_{1i}^{(k)}}$$

$$+\ <\bar{\phi}_\alpha^{(k)}\ ,\ -2g_3'^{(k)}>_{S_{3i}^{(k)}}$$

$$+\ <\bar{v}_3^{(k)}\ ,\ -2g_5'^{(k)}>_{S_{5i}^{(k)}}$$

$$+\ 2<N_{\alpha\beta}^{(k)}\ ,\ \eta_\beta(\bar{v}_\alpha^{(k)})'-g_2'^{(k)}>_{S_{2i}^{(k)}}$$

$$+\ 2<M_{\alpha\beta}^{(k)}\ ,\ \eta_\beta(\bar{\phi}_\alpha^{(k)})'-g_4'^{(k)}>_{S_{4i}^{(k)}}$$

$$+\ 2<V_\alpha^{(k)}\ ,\ \eta_\alpha(\bar{v}_3^{(k)})'-g_6'^{(k)}>_{S_{6i}^{(k)}}\}$$

$$+\ <\sigma_{\gamma 3}^{\cdot(1)}\ ,\ [(\frac{1}{120}+\frac{1}{840})(t_1^3 S_{3333}^{(1)}+t_2^3 S_{3333}^{(2)})\eta_\gamma](\sigma_{\rho 3,\rho}^{-(1)})'$$

$$
\begin{aligned}
&+ [(\frac{1}{24}+\frac{3}{280})(t_2^2 S^{(2)}_{3333}-t_1^2 S^{(1)}_{3333})\eta_\gamma](\sigma^{-(1)}_{33})' \\
&+ [-(\frac{1}{120}-\frac{1}{840})t_2^3 S^{(2)}_{3333}\eta_\gamma](\sigma^{-(2)}_{\rho 3,\rho})' \\
&+ [(\frac{1}{24}-\frac{3}{280})t_2^2 S^{(2)}_{3333}\eta_\gamma](\sigma^{-(2)}_{33})'-2g'^{(1)}_{\sigma} >_{S_i^{(1)}} \\
+\sum_{k=2}^{N-2}\{ &<\sigma^{-(k)}_{\gamma 3}, [-(\frac{1}{120}-\frac{1}{840})t_k^3 S^{(k)}_{3333}\eta_\gamma](\sigma^{-(k-1)}_{\rho 3,\rho})' \\
&+ [(-\frac{1}{24}+\frac{3}{280})t_k^2 S^{(k)}_{3333}\eta_\gamma](\sigma^{-(k-1)}_{33})' \\
&+ [(\frac{1}{120}+\frac{1}{840})(t_k^3 S^{(k)}_{3333}+t_{k+1}^3 S^{(k+1)}_{3333})\eta_\gamma](\sigma^{-(k)}_{\rho 3,\rho})' \\
&+ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S^{(k+1)}_{3333}-t_k^2 S^{(k)}_{3333})\eta_\gamma](\sigma^{-(k)}_{33})' \\
&+ [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S^{(k+1)}_{3333}\eta_\gamma](\sigma^{-(k+1)}_{\rho 3,\rho})' \\
&+ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S^{(k+1)}_{3333}\eta_\gamma](\sigma^{-(k+1)}_{33})'-2g'^{(k)}_{\sigma} >_{S_i^{(k)}} \} \\
&+ <\sigma^{-(N-1)}_{\gamma 3}, [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S^{(N-1)}_{3333}\eta_\gamma](\sigma^{-(N-2)}_{\rho 3,\rho})' \\
&+ [(-\frac{1}{24}+\frac{3}{280})t_{N-1}^2 S^{(N-1)}_{3333}\eta_\gamma](\sigma^{-(N-2)}_{33})' \\
&+ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^3 S^{(N-1)}_{3333}+t_N^3 S^{(N)}_{3333})\eta_\gamma](\sigma^{-(N-1)}_{\rho 3,\rho})' \\
&+ [(\frac{1}{24}+\frac{3}{280})(t_N^2 S^{(N)}_{3333}-t_{N-1}^2 S^{(N-1)}_{3333})\eta_\gamma](\sigma^{-(N-1)}_{33})'-2g'^{(N-1)}_{\sigma} >_{S_i^{(N-1)}} \qquad (261)
\end{aligned}
$$

The domain of definition of Ω_3 is an extension of the domain of Ω_2 insomuch as $V^{(k)}_\alpha \in C^0$ and not necessarily $\in C^1$.

Alternatively, extended formulations which do not contain the derivatives of kinematic variables $\overline{v}^{(k)}_\alpha$, $\overline{\phi}^{(k)}_\alpha$, and $\overline{v}^{(k)}_3$ can be obtained. Elimination of $\overline{v}^{(k)}_{(\alpha,\beta)}$ from (230) gives:

$$J_1(u\ ,\ \sigma)=2<\bar{v}^{(1)}_{\gamma}\ ,\ \sigma^{(0)}_{\gamma 3}>_{R^{(1)}}+2<\bar{\phi}^{(1)}_{\gamma}\ ,\ \frac{t_1}{2}\sigma^{(0)}_{\gamma 3}>_{R^{(1)}}+2<\bar{v}^{(1)}_{3}\ ,\ \sigma^{(0)}_{33}>_{R^{(1)}}$$

$$+\ 2<N^{(1)}_{\alpha\beta}\ ,\ \frac{t_1}{12}S^{(1)}_{\alpha\beta 33}\sigma^{(0)}_{\gamma 3,\gamma}+\frac{1}{2}S^{(1)}_{\alpha\beta 33}\sigma^{(0)}_{33}>_{R^{(1)}}$$

$$+\ 2<M^{(1)}_{\alpha\beta}\ ,\ \frac{1}{10}S^{(1)}_{\alpha\beta 33}\sigma^{(0)}_{\gamma 3,\gamma}+\frac{6}{5t_1}S^{(1)}_{\alpha\beta 33}\sigma^{(0)}_{33}>_{R^{(1)}}\ +\ 2<V^{(1)}_{\rho}\ ,\ -\frac{2}{5}S^{(1)}_{\rho 3\gamma 3}\sigma^{(0)}_{\gamma 3}>_{R^{(1)}}$$

$$+\ 2<\bar{v}^{(N)}_{\gamma}\ ,\ -\sigma^{(N)}_{\gamma 3}>_{R^{(N)}}+2<\bar{\phi}^{(N)}_{\gamma}\ ,\ \frac{t_N}{2}\sigma^{(N)}_{\gamma 3}>_{R^{(N)}}+2<\bar{v}^{(N)}_{3}\ ,\ -\sigma^{(N)}_{33}>_{R^{(N)}}$$

$$+\ 2<N^{(N)}_{\alpha\beta}\ ,\ -\frac{t_N}{12}S^{(N)}_{\alpha\beta 33}\sigma^{(N)}_{\gamma 3,\gamma}+\frac{1}{2}S^{(N)}_{\alpha\beta 33}\sigma^{(N)}_{33}>_{R^{(N)}}$$

$$+\ 2<M^{(N)}_{\alpha\beta}\ ,\ \frac{1}{10}S^{(N)}_{\alpha\beta 33}\sigma^{(N)}_{\gamma 3,\gamma}-\frac{6}{5t_N}S^{(N)}_{\alpha\beta 33}\sigma^{(N)}_{33}>_{R^{(N)}}+2<V^{(N)}_{\rho}\ ,\ -\frac{2}{5}S^{(N)}_{\rho 3\gamma 3}\sigma^{(N)}_{\gamma 3}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}\{<\bar{v}^{(k)}_{\gamma}\ ,\ \bar{\sigma}^{(k-1)}_{\gamma 3}>_{R^{(k)}}+<\bar{\phi}^{(k)}_{\gamma}\ ,\ \frac{t_k}{2}\bar{\sigma}^{(k-1)}_{\gamma 3}>_{R^{(k)}}+<\bar{v}^{(k)}_{3}\ ,\ \bar{\sigma}^{(k-1)}_{33}>_{R^{(k)}}$$

$$+\ <N^{(k)}_{\alpha\beta}\ ,\ \frac{t_k}{12}S^{(k)}_{\alpha\beta 33}\bar{\sigma}^{(k-1)}_{\gamma 3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta 33}\bar{\sigma}^{(k-1)}_{33}>_{R^{(k)}}$$

$$+\ <M^{(k)}_{\alpha\beta}\ ,\ \frac{1}{10}S^{(k)}_{\alpha\beta 33}\bar{\sigma}^{(k-1)}_{\gamma 3,\gamma}+\frac{6}{5t_k}S^{(k)}_{\alpha\beta 33}\bar{\sigma}^{(k-1)}_{33}>_{R^{(k)}}$$

$$+\ <V^{(k)}_{\rho}\ ,\ -\frac{2}{5}S^{(k)}_{\rho 3\gamma 3}\bar{\sigma}^{(k-1)}_{\gamma 3}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{2<\bar{v}^{(k)}_{\alpha}\ ,\ N^{(k)}_{\alpha\beta,\beta}>_{R^{(k)}}+<\bar{\phi}^{(k)}_{\alpha},\ M^{(k)}_{\alpha\beta,\beta}-V^{(k)}_{\alpha}>_{R^{(k)}}+<\bar{v}^{(k)}_{3}\ ,\ V^{(k)}_{\alpha,\alpha}>_{R^{(k)}}$$

$$+\ <N^{(k)}_{\alpha\beta}\ ,\ \frac{1}{t_k}S^{(k)}_{\alpha\beta\mu\rho}N^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$+\ <M^{(k)}_{\alpha\beta}\ ,\ -\bar{\phi}^{(k)}_{(\alpha,\beta)}+\frac{12}{t_k^3}S^{(k)}_{\alpha\beta\mu\rho}M^{(k)}_{\mu\rho}>_{R^{(k)}}+<V^{(k)}_{\gamma}\ ,\ -\bar{\phi}^{(k)}_{\gamma}-\bar{v}^{(k)}_{3,\gamma}+\frac{24}{5t_k}S^{(k)}_{\rho 3gama3}V^{(k)}_{\rho}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\bar{v}^{(k)}_{\gamma}\ ,\ -\bar{\sigma}^{(k)}_{\gamma 3}>_{R^{(k)}}+<\bar{\phi}^{(k)}_{gama}\ ,\ \frac{t_k}{2}\bar{\sigma}^{(k)}_{\gamma 3}>_{R^{(k)}}+<\bar{v}^{(k)}_{3}\ ,\ -\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$+\ <N^{(k)}_{\alpha\beta}\ ,\ -\frac{t_k}{12}S^{(k)}_{\alpha\beta 33}\bar{\sigma}^{(k)}_{\gamma 3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta 33}\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$+ \ <M^{(k)}_{\alpha\beta} \ , \ \frac{1}{10}S^{(k)}_{\alpha\beta 33}\sigma^{-(k)}_{\gamma 3,\gamma} - \frac{6}{5t_k}S^{(k)}_{\alpha\beta 33}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ \ <V^{(k)}_{\rho} \ , \ -\frac{2}{5}S^{(k)}_{\rho 3\gamma 3}\sigma^{-(k)}_{\gamma 3}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\sigma^{-(k)}_{\gamma 3} \ , \ -\overline{v}^{(k)}_{\gamma} + \frac{t_k}{2}\overline{\phi}^{(k)}_{\gamma} + \frac{t_k}{12}S^{(k)}_{\alpha\beta 33}N^{(k)}_{\alpha\beta,\gamma} - \frac{1}{10}S^{(k)}_{\alpha\beta 33}M^{(k)}_{\alpha\beta,\gamma} - \frac{2}{5}S^{(k)}_{\rho 3\gamma 3}V^{(k)}_{\rho}>_{R^{(k)}}$$

$$+ \ <\sigma^{-(k)}_{\rho 3} \ , \ \Xi^{(k)}_{11}\sigma^{-(k)}_{\gamma 3} + \Xi^{(k)}_{12}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ \ <\sigma^{-(k)}_{\gamma 3} \ , \ \overline{v}^{(k+1)}_{\gamma} + \frac{t_{k+1}}{2}\overline{\phi}^{(k+1)}_{\gamma} - \frac{t_{k+1}}{12}S^{(k+1)}_{\alpha\beta 33}N^{(k+1)}_{\alpha\beta,\gamma}$$

$$- \ \frac{1}{10}S^{(k+1)}_{\alpha\beta 33}M^{(k+1)}_{\alpha\beta,\gamma} - \frac{2}{5}S^{(k+1)}_{\rho 3\gamma 3}V^{(k+1)}_{\rho}>_{R^{(k+1)}}$$

$$+ \ <\sigma^{-(k)}_{33} \ , \ -\overline{v}^{(k)}_{3} + \frac{1}{2}S^{(k)}_{\alpha\beta 33}N^{(k)}_{\alpha\beta} - \frac{6}{5t_k}S^{(k)}_{\alpha\beta 33}M^{(k)}_{\alpha\beta}>_{R^{(k)}}$$

$$+ \ <\sigma^{-(k)}_{33} \ , \ \Xi^{(k)}_{21}\sigma^{-(k)}_{\gamma 3} + \Xi^{(k)}_{22}\sigma^{-(k)}_{33}>_{R^{(k)}}$$

$$+ \ <\sigma^{-(k)}_{33} \ , \ \overline{v}^{(k+1)}_{3} + \frac{1}{2}S^{(k+1)}_{\alpha\beta 33}N^{(k+1)}_{\alpha\beta} + \frac{6}{5t_{k+1}}S^{(k+1)}_{\alpha\beta 33}M^{(k+1)}_{\alpha\beta}>_{R^{(k+1)}}\}$$

$$+\sum_{k=2}^{N-1}\{<\sigma^{-(k)}_{\rho 3} \ , \ \Lambda^{(k)}_{11}\sigma^{-(k-1)}_{\gamma 3} + \Lambda^{(k)}_{12}\sigma^{-(k-1)}_{33}> - <\sigma^{-(k)}_{33} \ , \ \Lambda^{(k)}_{21}\sigma^{-(k-1)}_{\gamma 3} + \Lambda^{(k)}_{22}\sigma^{-(k-1)}_{33}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{<\sigma^{-(k)}_{\gamma 3} \ , \ \overline{\Lambda}^{(k+1)}_{11}\sigma^{-(k+1)}_{\gamma 3} + \overline{\Lambda}^{(k+1)}_{12}\sigma^{-(k+1)}_{33}>_{R^{(k+1)}}$$

$$+ \ <\sigma^{-(k)}_{33} \ , \ \overline{\Lambda}^{(k+1)}_{21}\sigma^{-(k+1)}_{\gamma 3} + \overline{\Lambda}^{(k+1)}_{22}\sigma^{-(k+1)}_{33}>_{R^{(k+1)}}\}$$

$$+ \ 2<\{\sigma\}^{-(N-1)} \ , \ [\overline{\Lambda}]^{(N)}\{\sigma\}^{(N)}>_{R^{(N)}}$$

$$+ \ 2<\{\sigma\}^{(1)} \ , \ [\Lambda]^{(1)}\{\sigma\}^{(0)}>_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\{2<\overline{v}^{(k)}_{\alpha} \ , \ -\eta_{\beta}N^{(k)}_{\alpha\beta} - g^{(k)}_{1}>_{S^{(k)}_{1}}$$

$$+ \ <\overline{\phi}^{(k)}_{\alpha} \ , \ -\eta_{\beta}M^{(k)}_{\alpha\beta} - 2g^{(k)}_{3}>_{S^{(k)}_{3}}$$

$$+ \ <\bar{v}_3^{(k)}, -\eta_\alpha V_\alpha^{(k)} - 2g_5^{(k)}>_{S_5^{(k)}}$$

$$- \ 2<N_{\alpha\beta}^{(k)}, g_2^{(k)}>_{S_2^{(k)}}$$

$$+ \ <M_{\alpha\beta}^{(k)}, \eta_\beta \bar{\phi}_\alpha^{(k)} - 2g_4^{(k)}>_{S_4^{(k)}}$$

$$+ \ <V_\alpha^{(k)}, \eta_\alpha \bar{v}_3^{(k)} - 2g_6^{(k)}>_{S_6^{(k)}}\}$$

$$+ \ <\bar{\sigma}_{\gamma 3}^{(1)}, [-\frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\eta_\gamma]N_{\alpha\beta}^{(1)} + [\frac{1}{10}S_{\alpha\beta 33}^{(1)}\eta_\gamma]M_{\alpha\beta}^{(1)}$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_1^3 S_{3333}^{(1)} + t_2^3 S_{3333}^{(2)})\eta_\gamma]\bar{\sigma}_{\rho 3,\rho}^{(1)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_2^2 S_{3333}^{(2)} - t_1^2 S_{3333}^{(1)})\eta_\gamma]\bar{\sigma}_{33}^{(1)} + [\frac{t_2}{12}S_{\alpha\beta 33}^{(2)}\eta_\gamma]N_{\alpha\beta}^{(2)}$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(2)}\eta_\gamma]M_{\alpha\beta}^{(2)} + [-(\frac{1}{120}-\frac{1}{840})t_2^3 S_{3333}^{(2)}\eta_\gamma]\bar{\sigma}_{\rho 3,\rho}^{(2)}$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_2^2 S_{3333}^{(2)}\eta_\gamma]\bar{\sigma}_{33}^{(2)} - 2g_\sigma^{(1)}>_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{ \ <\bar{\sigma}_{\gamma 3}^{(k)}, [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma]\bar{\sigma}_{\rho 3,\rho}^{(k-1)} + [(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\eta_\gamma]\bar{\sigma}_{33}^{(k-1)}$$

$$+ \ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_\gamma]N_{\alpha\beta}^{(k)} + [\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma]M_{\alpha\beta}^{(k)} + [(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)} + t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma]\bar{\sigma}_{\rho 3,\rho}^{(k)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)} - t_k^2 S_{3333}^{(k)})\eta_\gamma]\bar{\sigma}_{33}^{(k)} + [\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma]N_{\alpha\beta}^{(k+1)}$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma]M_{\alpha\beta}^{(k+1)} + [-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma]\bar{\sigma}_{\rho 3,\rho}^{(k+1)}$$

$$+ \ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma]\bar{\sigma}_{33}^{(k+1)} - 2g_\sigma^{(k)}>_{S^{(k)}} \}$$

$$+ \ <\bar{\sigma}_{\gamma 3}^{(N-1)}, [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S_{3333}^{(N-1)}\eta_\gamma]\bar{\sigma}_{\rho 3,\rho}^{(N-2)} + [(-\frac{1}{24}+\frac{3}{280})t_{N-1}^2 S_{3333}^{(N-1)}\eta_\gamma]\bar{\sigma}_{33}^{(N-2)}$$

$$+ \ [-\frac{t_{N-1}}{12}S_{\alpha\beta 33}^{(N-1)}\eta_\gamma]N_{\alpha\beta}^{(N-1)} + [\frac{1}{10}S_{\alpha\beta 33}^{(N-1)}\eta_\gamma]M_{\alpha\beta}^{(N-1)}$$

$$+\ \Big[\Big(\frac{1}{120}+\frac{1}{840}\Big)\Big(t_{N-1}^{3}S_{3333}^{(N-1)}+t_{N}^{3}S_{3333}^{(N)}\Big)\eta_{\gamma}\Big]\bar{\sigma}_{\rho 3,\rho}^{(N-1)}$$

$$+\ \Big[\Big(\frac{1}{24}+\frac{3}{280}\Big)\Big(t_{N}^{2}S_{3333}^{(N)}-t_{N-1}^{2}S_{3333}^{(N-1)}\Big)\eta_{\gamma}\Big]\bar{\sigma}_{33}^{(N-1)}+\Big[\frac{t_{N}}{12}S_{\alpha\beta 33}^{(N)}\eta_{\gamma}\Big]N_{\alpha\beta}^{(N)}$$

$$+\ \Big[\frac{1}{10}S_{\alpha\beta 33}^{(N)}\eta_{\gamma}\Big]M_{\alpha\beta}^{(N)}-2g_{\sigma}^{(N-1)}>_{S^{(N\text{-}1)}}$$

$$+\sum_{k=1}^{N}\Big\{2<\bar{v}_{\alpha}^{(k)}\ ,\ -\eta_{\beta}(N_{\alpha\beta}^{(k)})'-g_{1}^{\prime(k)}>_{S_{1i}^{(k)}}$$

$$+\ <\bar{\phi}_{\alpha}^{(k)}\ ,\ -\eta_{\beta}(M_{\alpha\beta}^{(k)})'-2g_{3}^{\prime(k)}>_{S_{3i}^{(k)}}$$

$$+\ <\bar{v}_{3}^{(k)}\ ,\ -\eta_{\alpha}(V_{\alpha}^{(k)})'-2g_{5}^{\prime(k)}>_{S_{5i}^{(k)}}$$

$$-\ 2<N_{\alpha\beta}^{(k)}\ ,\ g_{2}^{\prime(k)}>_{S_{2i}^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)}\ ,\ \eta_{\beta}(\bar{\phi}_{\alpha}^{(k)})'-2g_{4}^{\prime(k)}>_{S_{4i}^{(k)}}$$

$$+\ <V_{\alpha}^{(k)}\ ,\ \eta_{\alpha}(\bar{v}_{3}^{(k)})'-2g_{6}^{\prime(k)}>_{S_{6i}^{(k)}}\Big\}$$

$$+\ <\bar{\sigma}_{\gamma 3}^{(1)}\ ,\ \Big[-\frac{t_{1}}{12}S_{\alpha\beta 33}^{(1)}\eta_{\gamma}\Big](N_{\alpha\beta}^{(1)})'+\Big[\frac{1}{10}S_{\alpha\beta 33}^{(1)}\eta_{\gamma}\Big](M_{\alpha\beta}^{(1)})'$$

$$+\ \Big[\Big(\frac{1}{120}+\frac{1}{840}\Big)\Big(t_{1}^{3}S_{3333}^{(1)}+t_{2}^{3}S_{3333}^{(2)}\Big)\eta_{\gamma}\Big](\bar{\sigma}_{\rho 3,\rho}^{(1)})'$$

$$+\ \Big[\Big(\frac{1}{24}+\frac{3}{280}\Big)\Big(t_{2}^{2}S_{3333}^{(2)}-t_{1}^{2}S_{3333}^{(1)}\Big)\eta_{\gamma}\Big](\bar{\sigma}_{33}^{(1)})'+\Big[\frac{t_{2}}{12}S_{\alpha\beta 33}^{(2)}\eta_{\gamma}\Big](N_{\alpha\beta}^{(2)})'$$

$$+\ \Big[\frac{1}{10}S_{\alpha\beta 33}^{(2)}\eta_{\gamma}\Big](M_{\alpha\beta}^{(2)})'+\Big[-\Big(\frac{1}{120}-\frac{1}{840}\Big)t_{2}^{3}S_{3333}^{(2)}\eta_{\gamma}\Big](\bar{\sigma}_{\rho 3,\rho}^{(2)})'$$

$$+\ \Big[\Big(\frac{1}{24}-\frac{3}{280}\Big)t_{2}^{2}S_{3333}^{(2)}\eta_{\gamma}\Big](\bar{\sigma}_{33}^{(2)})'-2g_{\sigma}^{\prime(1)}>_{S_{1}^{(1)}}$$

$$+\sum_{k=2}^{N-2}\Big\{\ <\bar{\sigma}_{\gamma 3}^{(k)}\ ,\ \Big[-\Big(\frac{1}{120}-\frac{1}{840}\Big)t_{k}^{3}S_{3333}^{(k)}\eta_{\gamma}\Big](\bar{\sigma}_{\rho 3,\rho}^{(k-1)})'$$

$$+\ \Big[\Big(-\frac{1}{24}+\frac{3}{280}\Big)t_{k}^{2}S_{3333}^{(k)}\eta_{\gamma}\Big](\bar{\sigma}_{33}^{(k-1)})'$$

$$+\ [-\frac{t_k}{12}S^{(k)}_{\alpha\beta33}\eta_\gamma][N^{(k)}_{\alpha\beta})'+[\frac{1}{10}S^{(k)}_{\alpha\beta33}\eta_\gamma][M^{(k)}_{\alpha\beta})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t^3_kS^{(k)}_{3333}+t^3_{k+1}S^{(k+1)}_{3333})\eta_\gamma][\sigma^{-(k)}_{\rho3,\rho})'$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t^2_{k+1}S^{(k+1)}_{3333}-t^2_kS^{(k)}_{3333})\eta_\gamma][\sigma^{-(k)}_{33})'+[\frac{t_{k+1}}{12}S^{(k+1)}_{\alpha\beta33}\eta_\gamma][N^{(k+1)}_{\alpha\beta})'$$

$$+\ [\frac{1}{10}S^{(k+1)}_{\alpha\beta33}\eta_\gamma][M^{(k+1)}_{\alpha\beta})'+[-(\frac{1}{120}-\frac{1}{840})t^3_{k+1}S^{(k+1)}_{3333}\eta_\gamma][\sigma^{-(k+1)}_{\rho3,\rho})'$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t^2_{k+1}S^{(k+1)}_{3333}\eta_\gamma][\sigma^{-(k+1)}_{33})'-2g'^{(k)}_{\sigma}>_{S^{(k)}_1}\}$$

$$+\ <\sigma^{-(N-1)}_{\gamma3}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t^3_{N-1}S^{(N-1)}_{3333}\eta_\gamma][\sigma^{-(N-2)}_{\rho3,\rho})'$$

$$+\ [(-\frac{1}{24}+\frac{3}{280})t^2_{N-1}S^{(N-1)}_{3333}\eta_\gamma][\sigma^{-(N-2)}_{33})'$$

$$+\ [-\frac{t_{N-1}}{12}S^{(N-1)}_{\alpha\beta33}\eta_\gamma][N^{(N-1)}_{\alpha\beta})'+[\frac{1}{10}S^{(N-1)}_{\alpha\beta33}\eta_\gamma][M^{(N-1)}_{\alpha\beta})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t^3_{N-1}S^{(N-1)}_{3333}+t^3_NS^{(N)}_{3333})\eta_\gamma][\sigma^{-(N-1)}_{\rho3,\rho})'$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t^2_NS^{(N)}_{3333}-t^2_{N-1}S^{(N-1)}_{3333})\eta_\gamma][\sigma^{-(N-1)}_{33})'+[\frac{t_N}{12}S^{(N)}_{\alpha\beta33}\eta_\gamma][N^{(N)}_{\alpha\beta})'$$

$$+\ [\frac{1}{10}S^{(N)}_{\alpha\beta33}\eta_\gamma][M^{(N)}_{\alpha\beta})'-2g'^{(N-1)}_{\sigma}>_{S^{(N-1)}_1}$$

The domain of J_1 is an extension of the domain of Ω requiring $\overline{v}^{(k)}_\alpha \in C^0$ and not necessarily $\in C^1$. Elimination $\overline{\phi}^{(k)}_{(\alpha,\beta)}$ from J_1 gives:

$$J_2(u\ ,\ \sigma)=2<\overline{v}^{(1)}_\gamma\ ,\ \sigma^{(0)}_{\gamma3}>_{R^{(1)}}+2<\overline{\phi}^{(1)}_\gamma\ ,\ \frac{t_1}{2}\sigma^{(0)}_{\gamma3}>_{R^{(1)}}+2<\overline{v}^{(1)}_3\ ,\ \sigma^{(0)}_{33}>_{R^{(1)}}$$

$$+\ 2<N^{(1)}_{\alpha\beta}\ ,\ \frac{t_1}{12}S^{(1)}_{\alpha\beta33}\sigma^{(0)}_{\gamma3,\gamma}+\frac{1}{2}S^{(1)}_{\alpha\beta33}\sigma^{(0)}_{33}>_{R^{(1)}}$$

$$+\ 2<M^{(1)}_{\alpha\beta}\ ,\ \frac{1}{10}S^{(1)}_{\alpha\beta33}\sigma^{(0)}_{\gamma3,\gamma}+\frac{6}{5t_1}S^{(1)}_{\alpha\beta33}\sigma^{(0)}_{33}>_{R^{(1)}}\ +\ 2<V^{(1)}_\rho\ ,\ -\frac{2}{5}S^{(1)}_{\rho3\gamma3}\sigma^{(0)}_{\gamma3}>_{R^{(1)}}$$

$$+\ 2<\overline{v}^{(N)}_\gamma\ ,\ -\sigma^{(N)}_{\gamma3}>_{R^{(N)}}+2<\overline{\phi}^{(N)}_\gamma\ ,\ \frac{t_N}{2}\sigma^{(N)}_{\gamma3}>_{R^{(N)}}+2<\overline{v}^{(N)}_3\ ,\ -\sigma^{(N)}_{33}>_{R^{(N)}}$$

$$+\ 2<N_{\alpha\beta}^{(N)}\ ,\ -\frac{t_N}{12}S_{\alpha\beta33}^{(N)}\sigma_{\gamma3,\gamma}^{(N)}+\frac{1}{2}S_{\alpha\beta33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<M_{\alpha\beta}^{(N)}\ ,\ \frac{1}{10}S_{\alpha\beta33}^{(N)}\sigma_{\gamma3,\gamma}^{(N)}-\frac{6}{5t_N}S_{\alpha\beta33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}+2<V_{\rho}^{(N)}\ ,\ -\frac{2}{5}S_{\rho3\gamma3}^{(N)}\sigma_{\gamma3}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}\{<\bar{v}_{\gamma}^{(k)}\ ,\ \sigma_{\gamma3}^{-(k-1)}>_{R^{(k)}}+<\bar{\phi}_{\gamma}^{(k)}\ ,\ \frac{t_k}{2}\sigma_{\gamma3}^{-(k-1)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)}\ ,\ \sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)}\ ,\ \frac{t_k}{12}S_{\alpha\beta33}^{(k)}\sigma_{\gamma3,\gamma}^{-(k-1)}+\frac{1}{2}S_{\alpha\beta33}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)}\ ,\ \frac{1}{10}S_{\alpha\beta33}^{(k)}\sigma_{\gamma3,\gamma}^{-(k-1)}+\frac{6}{5t_k}S_{\alpha\beta33}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <V_{\rho}^{(k)}\ ,\ -\frac{2}{5}S_{\rho3\gamma3}^{(k)}\sigma_{\gamma3}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{2<\bar{v}_{\alpha}^{(k)}\ ,\ N_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}+2<\bar{\phi}_{\alpha}^{(k)}\ ,\ M_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}+<\bar{\phi}_{\alpha}^{(k)},\ -V_{\alpha}^{(k)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)}\ ,\ V_{\alpha,\alpha}^{(k)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)}\ ,\ \frac{1}{t_k}S_{\alpha\beta\mu\rho}^{(k)}N_{\mu\rho}^{(k)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)}\ ,\ \frac{12}{t_k^3}S_{\alpha\beta\mu\rho}^{(k)}M_{\mu\rho}^{(k)}>_{R^{(k)}}+<V_{\gamma}^{(k)}\ ,\ -\bar{\phi}_{\gamma}^{(k)}-\bar{v}_{3,\gamma}^{(k)}+\frac{24}{5t_k}S_{\rho3\gamma3}^{(k)}V_{\rho}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\bar{v}_{\gamma}^{(k)}\ ,\ -\sigma_{\gamma3}^{-(k)}>_{R^{(k)}}+<\bar{\phi}_{\gamma}^{(k)}\ ,\ \frac{t_k}{2}\sigma_{\gamma3}^{-(k)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)}\ ,\ -\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)}\ ,\ -\frac{t_k}{12}S_{\alpha\beta33}^{(k)}\sigma_{\gamma3,\gamma}^{-(k)}+\frac{1}{2}S_{\alpha\beta33}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)}\ ,\ \frac{1}{10}S_{\alpha\beta33}^{(k)}\sigma_{\gamma3,\gamma}^{-(k)}-\frac{6}{5t_k}S_{\alpha\beta33}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+\ <V_{\rho}^{(k)}\ ,\ -\frac{2}{5}S_{\rho3\gamma3}^{(k)}\sigma_{\gamma3}^{-(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\sigma_{\gamma3}^{-(k)}\ ,\ -\bar{v}_{\gamma}^{(k)}+\frac{t_k}{2}\bar{\phi}_{\gamma}^{(k)}+\frac{t_k}{12}S_{\alpha\beta33}^{(k)}N_{\alpha\beta,\gamma}^{(k)}-\frac{1}{10}S_{\alpha\beta33}^{(k)}M_{\alpha\beta,\gamma}^{(k)}-\frac{2}{5}S_{\rho3\gamma3}^{(k)}V_{\rho}^{(k)}>_{R^{(k)}}$$

$$+\ <\sigma_{\rho3}^{-(k)}\ ,\ \Xi_{11}^{(k)}\sigma_{\gamma3}^{-(k)}+\Xi_{12}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+ <\bar{\sigma}_{\gamma 3}^{(k)}, \bar{v}_{\gamma}^{(k+1)} + \frac{t_{k+1}}{2}\bar{\Phi}_{\gamma}^{(k+1)} - \frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta,\gamma}^{(k+1)}$$

$$- \frac{1}{10}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta,\gamma}^{(k+1)} - \frac{2}{5}S_{\rho 3\gamma 3}^{(k+1)}V_{\rho}^{(k+1)} >_{R^{(k+1)}}$$

$$+ <\bar{\sigma}_{33}^{(k)}, -\bar{v}_{3}^{(k)} + \frac{1}{2}S_{\alpha\beta 33}^{(k)}N_{\alpha\beta}^{(k)} - \frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}M_{\alpha\beta}^{(k)} >_{R^{(k)}}$$

$$+ <\bar{\sigma}_{33}^{(k)}, \Xi_{21}^{(k)}\bar{\sigma}_{\gamma 3}^{(k)} + \Xi_{22}^{(k)}\bar{\sigma}_{33}^{(k)} >_{R^{(k)}}$$

$$+ <\bar{\sigma}_{33}^{(k)}, \bar{v}_{3}^{(k+1)} + \frac{1}{2}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta}^{(k+1)} + \frac{6}{5t_{k+1}}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta}^{(k+1)} >_{R^{(k+1)}}\}$$

$$+\sum_{k=2}^{N-1}\{<\bar{\sigma}_{\rho 3}^{(k)}, \Lambda_{11}^{(k)}\bar{\sigma}_{\gamma 3}^{(k-1)} + \Lambda_{12}^{(k)}\bar{\sigma}_{33}^{(k-1)}> + <\bar{\sigma}_{33}^{(k)}, \Lambda_{21}^{(k)}\bar{\sigma}_{\gamma 3}^{(k-1)} + \Lambda_{22}^{(k)}\bar{\sigma}_{33}^{(k-1)} >_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{<\bar{\sigma}_{\gamma 3}^{(k)}, \bar{\Lambda}_{11}^{(k+1)}\bar{\sigma}_{\gamma 3}^{(k+1)} + \bar{\Lambda}_{12}^{(k+1)}\bar{\sigma}_{33}^{(k+1)} >_{R^{(k+1)}}$$

$$+ <\bar{\sigma}_{33}^{(k)}, \bar{\Lambda}_{21}^{(k+1)}\bar{\sigma}_{\gamma 3}^{(k+1)} + \bar{\Lambda}_{22}^{(k+1)}\bar{\sigma}_{33}^{(k+1)} >_{R^{(k+1)}}\}$$

$$+ 2<\{\sigma\}^{-(N-1)}, [\bar{\Lambda}]^{(N)}\{\sigma\}^{(N)} >_{R^{(N)}}$$

$$+ 2<\{\sigma\}^{(1)}, [\Lambda]^{(1)}\{\sigma\}^{(0)} >_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\{2<\bar{v}_{\alpha}^{(k)}, -\eta_{\beta}N_{\alpha\beta}^{(k)} - g_{1}^{(k)} >_{S_1^{(k)}}$$

$$+ 2<\bar{\Phi}_{\alpha}^{(k)}, -\eta_{\beta}M_{\alpha\beta}^{(k)} - g_{3}^{(k)} >_{S_3^{(k)}}$$

$$+ <\bar{v}_{3}^{(k)}, -\eta_{\alpha}V_{\alpha}^{(k)} - 2g_{5}^{(k)} >_{S_5^{(k)}}$$

$$- 2<N_{\alpha\beta}^{(k)}, g_{2}^{(k)} >_{S_2^{(k)}}$$

$$- 2<M_{\alpha\beta}^{(k)}, g_{4}^{(k)} >_{S_4^{(k)}}$$

$$+ <V_{\alpha}^{(k)}, \eta_{\alpha}\bar{v}_{3}^{(k)} - 2g_{6}^{(k)} >_{S_6^{(k)}}\}$$

$$+ <\sigma^{-(1)}_{\gamma 3}, [-\frac{t_1}{12}S^{(1)}_{\alpha\beta 33}\eta_\gamma]N^{(1)}_{\alpha\beta}+[\frac{1}{10}S^{(1)}_{\alpha\beta 33}\eta_\gamma]M^{(1)}_{\alpha\beta}$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t_1^3S^{(1)}_{3333}+t_2^3S^{(2)}_{3333})\eta_\gamma]\sigma^{-(1)}_{\rho 3,\rho}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_2^2S^{(2)}_{3333}-t_1^2S^{(1)}_{3333})\eta_\gamma]\sigma^{-(1)}_{33}+[\frac{t_2}{12}S^{(2)}_{\alpha\beta 33}\eta_\gamma]N^{(2)}_{\alpha\beta}$$

$$+ [\frac{1}{10}S^{(2)}_{\alpha\beta 33}\eta_\gamma]M^{(2)}_{\alpha\beta}+[-(\frac{1}{120}-\frac{1}{840})t_2^3S^{(2)}_{3333}\eta_\gamma]\sigma^{-(2)}_{\rho 3,\rho}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_2^2S^{(2)}_{3333}\eta_\gamma]\sigma^{-(2)}_{33}-2g^{(1)}_\sigma>_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{ <\sigma^{-(k)}_{\gamma 3}, [-(\frac{1}{120}-\frac{1}{840})t_k^3S^{(k)}_{3333}\eta_\gamma]\sigma^{-(k-1)}_{\rho 3,\rho}+[(-\frac{1}{24}+\frac{3}{280})t_k^2S^{(k)}_{3333}\eta_\gamma]\sigma^{-(k-1)}_{33}$$

$$+ [-\frac{t_k}{12}S^{(k)}_{\alpha\beta 33}\eta_\gamma]N^{(k)}_{\alpha\beta}+[\frac{1}{10}S^{(k)}_{\alpha\beta 33}\eta_\gamma]M^{(k)}_{\alpha\beta}+[(\frac{1}{120}+\frac{1}{840})(t_k^3S^{(k)}_{3333}+t_{k+1}^3S^{(k+1)}_{3333})\eta_\gamma]\sigma^{-(k)}_{\rho 3,\rho}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2S^{(k+1)}_{3333}-t_k^2S^{(k)}_{3333})\eta_\gamma]\sigma^{-(k)}_{33}+[\frac{t_{k+1}}{12}S^{(k+1)}_{\alpha\beta 33}\eta_\gamma]N^{(k+1)}_{\alpha\beta}$$

$$+ [\frac{1}{10}S^{(k+1)}_{\alpha\beta 33}\eta_\gamma]M^{(k+1)}_{\alpha\beta}+[-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3S^{(k+1)}_{3333}\eta_\gamma]\sigma^{-(k+1)}_{\rho 3,\rho}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2S^{(k+1)}_{3333}\eta_\gamma]\sigma^{-(k+1)}_{33}-2g^{(k)}_\sigma>_{S^{(k)}} \}$$

$$+ <\sigma^{-(N-1)}_{\gamma 3}, [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3S^{(N-1)}_{3333}\eta_\gamma]\sigma^{-(N-2)}_{\rho 3,\rho}+[(-\frac{1}{24}+\frac{3}{280})t_{N-1}^2S^{(N-1)}_{3333}\eta_\gamma]\sigma^{-(N-2)}_{33}$$

$$+ [-\frac{t_{N-1}}{12}S^{(N-1)}_{\alpha\beta 33}\eta_\gamma]N^{(N-1)}_{\alpha\beta}+[\frac{1}{10}S^{(N-1)}_{\alpha\beta 33}\eta_\gamma]M^{(N-1)}_{\alpha\beta}$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^3S^{(N-1)}_{3333}+t_N^3S^{(N)}_{3333})\eta_\gamma]\sigma^{-(N-1)}_{\rho 3,\rho}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_N^2S^{(N)}_{3333}-t_{N-1}^2S^{(N-1)}_{3333})\eta_\gamma]\sigma^{-(N-1)}_{33}+[\frac{t_N}{12}S^{(N)}_{\alpha\beta 33}\eta_\gamma]N^{(N)}_{\alpha\beta}$$

$$+ [\frac{1}{10}S^{(N)}_{\alpha\beta 33}\eta_\gamma]M^{(N)}_{\alpha\beta}-2g^{(N-1)}_\sigma>_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\{2<\bar{v}^{(k)}_\alpha, -\eta_\beta(N^{(k)}_{\alpha\beta})'-g'^{(k)}_1>_{S^{(k)}_{1i}}$$

$$+\ 2<\bar{\phi}_{\alpha}^{(k)}\ ,\ -\eta_{\beta}(M_{\alpha\beta}^{(k)})'-g_{3}'^{(k)}>_{S_{3i}^{(k)}}$$

$$+\ <\bar{v}_{3}^{(k)}\ ,\ -\eta_{\alpha}(V_{\alpha}^{(k)})'-2g_{5}'^{(k)}>_{S_{5i}^{(k)}}$$

$$-\ 2<N_{\alpha\beta}^{(k)}\ ,\ g_{2}'^{(k)}>_{S_{2i}^{(k)}}$$

$$-\ 2<M_{\alpha\beta}^{(k)}\ ,\ g_{4}'^{(k)}>_{S_{4i}^{(k)}}$$

$$+\ <V_{\alpha}^{(k)}\ ,\ \eta_{\alpha}(\bar{v}_{3}^{(k)})'-2g_{6}'^{(k)}>_{S_{6i}^{(k)}}\}$$

$$+\ <\bar{\sigma}_{\gamma 3}^{(1)}\ ,\ [-\frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\eta_{\gamma}](N_{\alpha\beta}^{(1)})'+[\frac{1}{10}S_{\alpha\beta 33}^{(1)}\eta_{\gamma}](M_{\alpha\beta}^{(1)})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_1^3S_{3333}^{(1)}+t_2^3S_{3333}^{(2)})\eta_{\gamma}](\bar{\sigma}_{\rho 3,\rho}^{(1)})'$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_2^2S_{3333}^{(2)}-t_1^2S_{3333}^{(1)})\eta_{\gamma}](\bar{\sigma}_{33}^{(1)})'+[\frac{t_2}{12}S_{\alpha\beta 33}^{(2)}\eta_{\gamma}](N_{\alpha\beta}^{(2)})'$$

$$+\ [\frac{1}{10}S_{\alpha\beta 33}^{(2)}\eta_{\gamma}](M_{\alpha\beta}^{(2)})'+[-(\frac{1}{120}-\frac{1}{840})t_2^3S_{3333}^{(2)}\eta_{\gamma}](\bar{\sigma}_{\rho 3,\rho}^{(2)})'$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t_2^2S_{3333}^{(2)}\eta_{\gamma}](\bar{\sigma}_{33}^{(2)})'-2g_{\sigma}'^{(1)}>_{S_{i}^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{\ <\bar{\sigma}_{\gamma 3}^{(k)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_k^3S_{3333}^{(k)}\eta_{\gamma}](\bar{\sigma}_{\rho 3,\rho}^{(k-1)})'$$

$$+\ [(-\frac{1}{24}+\frac{3}{280})t_k^2S_{3333}^{(k)}\eta_{\gamma}](\bar{\sigma}_{33}^{(k-1)})'$$

$$+\ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_{\gamma}](N_{\alpha\beta}^{(k)})'+[\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_{\gamma}](M_{\alpha\beta}^{(k)})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_k^3S_{3333}^{(k)}+t_{k+1}^3S_{3333}^{(k+1)})\eta_{\gamma}](\bar{\sigma}_{\rho 3,\rho}^{(k)})'$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2S_{3333}^{(k+1)}-t_k^2S_{3333}^{(k)})\eta_{\gamma}](\bar{\sigma}_{33}^{(k)})'+[\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\eta_{\gamma}](N_{\alpha\beta}^{(k+1)})'$$

$$+\ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_{\gamma}](M_{\alpha\beta}^{(k+1)})'+[-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3S_{3333}^{(k+1)}\eta_{\gamma}](\bar{\sigma}_{\rho 3,\rho}^{(k+1)})'$$

$$+\ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^{2}S_{3333}^{(k+1)}\eta_{\gamma}](\sigma_{33}^{-(k+1)})'-2g_{\sigma}^{\prime(k)}>_{S_1^{(k)}}\}$$

$$+\ <\sigma_{\gamma 3}^{-(N-1)}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^{3}S_{3333}^{(N-1)}\eta_{\gamma}](\sigma_{\rho 3,\rho}^{-(N-2)})'$$

$$+\ [(-\frac{1}{24}+\frac{3}{280})t_{N-1}^{2}S_{3333}^{(N-1)}\eta_{\gamma}](\sigma_{33}^{-(N-2)})'$$

$$+\ [-\frac{t_{N-1}}{12}S_{\alpha\beta 33}^{(N-1)}\eta_{\gamma}](N_{\alpha\beta}^{(N-1)})'+[\frac{1}{10}S_{\alpha\beta 33}^{(N-1)}\eta_{\gamma}](M_{\alpha\beta}^{(N-1)})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^{3}S_{3333}^{(N-1)}+t_{N}^{3}S_{3333}^{(N)})\eta_{\gamma}](\sigma_{\rho 3,\rho}^{-(N-1)})'$$

$$+\ [(\frac{1}{24}+\frac{3}{280})(t_{N}^{2}S_{3333}^{(N)}-t_{N-1}^{2}S_{3333}^{(N-1)})\eta_{\gamma}](\sigma_{33}^{-(N-1)})'+[\frac{t_N}{12}S_{\alpha\beta 33}^{(N)}\eta_{\gamma}](N_{\alpha\beta}^{(N)})'$$

$$+\ [\frac{1}{10}S_{\alpha\beta 33}^{(N)}\eta_{\gamma}](M_{\alpha\beta}^{(N)})'-2g_{\sigma}^{\prime(N-1)}>_{S_1^{(N-1)}}$$

The domain of definition of J_2 is an extension of the domain of J_1 to admit $\bar{\phi}_{\alpha}^{(k)}\in C^{0}$ instead of requiring $\bar{\phi}_{\alpha}^{(k)}\in C^{1}$. Elimination of $\bar{v}_{3,\alpha}^{(k)}$ from J_2 gives:

$$J_3(u\ ,\ \sigma)=2<\bar{v}_{\gamma}^{(1)}\ ,\ \sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{\phi}_{\gamma}^{(1)}\ ,\ \frac{t_1}{2}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{v}_{3}^{(1)}\ ,\ \sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+\ 2<N_{\alpha\beta}^{(1)}\ ,\ \frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{1}{2}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+\ 2<M_{\alpha\beta}^{(1)}\ ,\ \frac{1}{10}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{6}{5t_1}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}\ +\ 2<V_{\rho}^{(1)}\ ,\ -\frac{2}{5}S_{\rho 3\gamma 3}^{(1)}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}$$

$$+\ 2<\bar{v}_{\gamma}^{(N)}\ ,\ -\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{\phi}_{\gamma}^{(N)}\ ,\ \frac{t_N}{2}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{v}_{3}^{(N)}\ ,\ -\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<N_{\alpha\beta}^{(N)}\ ,\ -\frac{t_N}{12}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}+\frac{1}{2}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<M_{\alpha\beta}^{(N)}\ ,\ \frac{1}{10}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}-\frac{6}{5t_N}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}+2<V_{\rho}^{(N)}\ ,\ -\frac{2}{5}S_{\rho 3\gamma 3}^{(N)}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}\{<\bar{v}_{\gamma}^{(k)}\ ,\ \sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{\phi}_{\gamma}^{(k)}\ ,\ \frac{t_k}{2}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)}\ ,\ \sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+ \ <N^{(k)}_{\alpha\beta}\ ,\ \frac{t_k}{12}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k-1)}_{\gamma3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k-1)}_{33}>_{R^{(k)}}$$

$$+ \ <M^{(k)}_{\alpha\beta}\ ,\ \frac{1}{10}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k-1)}_{\gamma3,\gamma}+\frac{6}{5t_k}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k-1)}_{33}>_{R^{(k)}}$$

$$+ \ <V^{(k)}_{\rho}\ ,\ -\frac{2}{5}S^{(k)}_{\rho3\gamma3}\bar{\sigma}^{(k-1)}_{\gamma3}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{2<\bar{v}^{(k)}_{\alpha}\ ,\ N^{(k)}_{\alpha\beta,\beta}>_{R^{(k)}}+2<\bar{\phi}^{(k)}_{\alpha}\ ,\ M^{(k)}_{\alpha\beta,\beta}>_{R^{(k)}}+<\bar{\phi}^{(k)}_{\alpha},\ -V^{(k)}_{\alpha}>_{R^{(k)}}+2<\bar{v}^{(k)}_{3}\ ,\ V^{(k)}_{\alpha,\alpha}>_{R^{(k)}}$$

$$+ \ <N^{(k)}_{\alpha\beta}\ ,\ \frac{1}{t_k}S^{(k)}_{\alpha\beta\mu\rho}N^{(k)}_{\mu\rho}>_{R^{(k)}}$$

$$+ \ <M^{(k)}_{\alpha\beta}\ ,\ \frac{12}{t_k^3}S^{(k)}_{\alpha\beta\mu\rho}M^{(k)}_{\mu\rho}>_{R^{(k)}}+<V^{(k)}_{\gamma}\ ,\ -\bar{\phi}^{(k)}_{\gamma}+\frac{24}{5t_k}S^{(k)}_{\rho3\gamma3}V^{(k)}_{\rho}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\bar{v}^{(k)}_{\gamma}\ ,\ -\bar{\sigma}^{(k)}_{\gamma3}>_{R^{(k)}}+<\bar{\phi}^{(k)}_{\gamma}\ ,\ \frac{t_k}{2}\bar{\sigma}^{(k)}_{\gamma3}>_{R^{(k)}}+<\bar{v}^{(k)}_{3}\ ,\ -\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$+ \ <N^{(k)}_{\alpha\beta}\ ,\ -\frac{t_k}{12}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k)}_{\gamma3,\gamma}+\frac{1}{2}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$+ \ <M^{(k)}_{\alpha\beta}\ ,\ \frac{1}{10}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k)}_{\gamma3,\gamma}-\frac{6}{5t_k}S^{(k)}_{\alpha\beta33}\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$+ \ <V^{(k)}_{\rho}\ ,\ -\frac{2}{5}S^{(k)}_{\rho3\gamma3}\bar{\sigma}^{(k)}_{\gamma3}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\bar{\sigma}^{(k)}_{\gamma3}\ ,\ -\bar{v}^{(k)}_{\gamma}+\frac{t_k}{2}\bar{\phi}^{(k)}_{\gamma}+\frac{t_k}{12}S^{(k)}_{\alpha\beta33}N^{(k)}_{\alpha\beta,\gamma}-\frac{1}{10}S^{(k)}_{\alpha\beta33}M^{(k)}_{\alpha\beta,\gamma}-\frac{2}{5}S^{(k)}_{\rho3\gamma3}V^{(k)}_{\rho}>_{R^{(k)}}$$

$$+ \ <\bar{\sigma}^{(k)}_{\rho3}\ ,\ \Xi^{(k)}_{11}\bar{\sigma}^{(k)}_{\gamma3}+\Xi^{(k)}_{12}\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$+ \ <\bar{\sigma}^{(k)}_{\gamma3}\ ,\ \bar{v}^{(k+1)}_{\gamma}+\frac{t_{k+1}}{2}\bar{\phi}^{(k+1)}_{\gamma}-\frac{t_{k+1}}{12}S^{(k+1)}_{\alpha\beta33}N^{(k+1)}_{\alpha\beta,\gamma}$$

$$-\ \frac{1}{10}S^{(k+1)}_{\alpha\beta33}M^{(k+1)}_{\alpha\beta,\gamma}-\frac{2}{5}S^{(k+1)}_{\rho3\gamma3}V^{(k+1)}_{\rho}>_{R^{(k+1)}}$$

$$+ \ <\bar{\sigma}^{(k)}_{33}\ ,\ -\bar{v}^{(k)}_{3}+\frac{1}{2}S^{(k)}_{\alpha\beta33}N^{(k)}_{\alpha\beta}-\frac{6}{5t_k}S^{(k)}_{\alpha\beta33}M^{(k)}_{\alpha\beta}>_{R^{(k)}}$$

$$+ \ <\bar{\sigma}^{(k)}_{33}\ ,\ \Xi^{(k)}_{21}\bar{\sigma}^{(k)}_{\gamma3}+\Xi^{(k)}_{22}\bar{\sigma}^{(k)}_{33}>_{R^{(k)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \bar{v}_3^{(k+1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k+1)}N_{\alpha\beta}^{(k+1)}+\frac{6}{5t_{k+1}}S_{\alpha\beta 33}^{(k+1)}M_{\alpha\beta}^{(k+1)}>_{R^{(k+1)}}\}$$

$$+\sum_{k=2}^{N-1}\{<\sigma_{\rho 3}^{-(k)}\ ,\ \Lambda_{11}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{12}^{(k)}\sigma_{33}^{-(k-1)}>+<\sigma_{33}^{-(k)}\ ,\ \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}+\Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{<\sigma_{\gamma 3}^{-(k)}\ ,\ \bar{\Lambda}_{11}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{12}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}$$

$$+ \ <\sigma_{33}^{-(k)}\ ,\ \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}+\bar{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}\}$$

$$+ \ 2<\{\sigma\}^{-(N-1)}\ ,\ [\bar{\Lambda}]^{(N)}\{\sigma\}^{(N)}>_{R^{(N)}}$$

$$+ \ 2<\{\sigma\}^{(1)}\ ,\ [\Lambda]^{(1)}\{\sigma\}^{(0)}>_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\{2<\bar{v}_{\alpha}^{(k)}\ ,\ -\eta_{\beta}N_{\alpha\beta}^{(k)}-g_1^{(k)}>_{S_1^{(k)}}$$

$$+ \ 2<\bar{\phi}_{\alpha}^{(k)}\ ,\ -\eta_{\beta}M_{\alpha\beta}^{(k)}-g_3^{(k)}>_{S_3^{(k)}}$$

$$+ \ 2<\bar{v}_3^{(k)}\ ,\ -\eta_{\alpha}V_{\alpha}^{(k)}-g_5^{(k)}>_{S_5^{(k)}}$$

$$- \ 2<N_{\alpha\beta}^{(k)}\ ,\ g_2^{(k)}>_{S_2^{(k)}}$$

$$- \ 2<M_{\alpha\beta}^{(k)}\ ,\ g_4^{(k)}>_{S_4^{(k)}}$$

$$- \ 2<V_{\alpha}^{(k)}\ ,\ g_6^{(k)}>_{S_6^{(k)}}\}$$

$$+ \ <\sigma_{\gamma 3}^{-(1)}\ ,\ [-\frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\eta_{\gamma}]N_{\alpha\beta}^{(1)}+[\frac{1}{10}S_{\alpha\beta 33}^{(1)}\eta_{\gamma}]M_{\alpha\beta}^{(1)}$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})(t_1^3S_{3333}^{(1)}+t_2^3S_{3333}^{(2)})\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(1)}$$

$$+ \ [(\frac{1}{24}+\frac{3}{280})(t_2^2S_{3333}^{(2)}-t_1^2S_{3333}^{(1)})\eta_{\gamma}]\sigma_{33}^{-(1)}+[\frac{t_2}{12}S_{\alpha\beta 33}^{(2)}\eta_{\gamma}]N_{\alpha\beta}^{(2)}$$

$$+ \ [\frac{1}{10}S_{\alpha\beta 33}^{(2)}\eta_{\gamma}]M_{\alpha\beta}^{(2)}+[-(\frac{1}{120}-\frac{1}{840})t_2^3S_{3333}^{(2)}\eta_{\gamma}]\sigma_{\rho 3,\rho}^{-(2)}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_2^2 S_{3333}^{(2)}\eta_\gamma]\sigma_{33}^{-(2)}-2g_\sigma^{(1)}>_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{ <\sigma_{\gamma 3}^{-(k)}, [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k-1)}+[(-\frac{1}{24}+\frac{3}{280})t_k^2 S_{3333}^{(k)}\eta_\gamma]\sigma_{33}^{-(k-1)}$$

$$+ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\eta_\gamma]N_{\alpha\beta}^{(k)}+[\frac{1}{10}S_{\alpha\beta 33}^{(k)}\eta_\gamma]M_{\alpha\beta}^{(k)}+[(\frac{1}{120}+\frac{1}{840})(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k)}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)}-t_k^2 S_{3333}^{(k)})\eta_\gamma]\sigma_{33}^{-(k)}+[\frac{t_{k+1}}{12}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma]N_{\alpha\beta}^{(k+1)}$$

$$+ [\frac{1}{10}S_{\alpha\beta 33}^{(k+1)}\eta_\gamma]M_{\alpha\beta}^{(k+1)}+[-(\frac{1}{120}-\frac{1}{840})t_{k+1}^3 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k+1)}$$

$$+ [(\frac{1}{24}-\frac{3}{280})t_{k+1}^2 S_{3333}^{(k+1)}\eta_\gamma]\sigma_{33}^{-(k+1)}-2g_\sigma^{(k)}>_{S^{(k)}} \}$$

$$+ <\sigma_{\gamma 3}^{-(N-1)}, [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S_{3333}^{(N-1)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(N-2)}+[(-\frac{1}{24}+\frac{3}{280})t_{N-1}^2 S_{3333}^{(N-1)}\eta_\gamma]\sigma_{33}^{-(N-2)}$$

$$+ [-\frac{t_{N-1}}{12}S_{\alpha\beta 33}^{(N-1)}\eta_\gamma]N_{\alpha\beta}^{(N-1)}+[\frac{1}{10}S_{\alpha\beta 33}^{(N-1)}\eta_\gamma]M_{\alpha\beta}^{(N-1)}$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^3 S_{3333}^{(N-1)}+t_N^3 S_{3333}^{(N)})\eta_\gamma]\sigma_{\rho 3,\rho}^{-(N-1)}$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_N^2 S_{3333}^{(N)}-t_{N-1}^2 S_{3333}^{(N-1)})\eta_\gamma]\sigma_{33}^{-(N-1)}+[\frac{t_N}{12}S_{\alpha\beta 33}^{(N)}\eta_\gamma]N_{\alpha\beta}^{(N)}$$

$$+ [\frac{1}{10}S_{\alpha\beta 33}^{(N)}\eta_\gamma]M_{\alpha\beta}^{(N)}-2g_\sigma^{(N-1)}>_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\{2<\bar{v}_\alpha^{(k)}, -\eta_\beta(N_{\alpha\beta}^{(k)})'-g_1^{\prime(k)}>_{S_{1i}^{(k)}}$$

$$+ 2<\bar{\phi}_\alpha^{(k)}, -\eta_\beta(M_{\alpha\beta}^{(k)})'-g_3^{\prime(k)}>_{S_{3i}^{(k)}}$$

$$+ 2<\bar{v}_3^{(k)}, -\eta_\alpha(V_\alpha^{(k)})'-g_5^{\prime(k)}>_{S_{5i}^{(k)}}$$

$$- 2<N_{\alpha\beta}^{(k)}, g_2^{\prime(k)}>_{S_{2i}^{(k)}}$$

$$- 2<M_{\alpha\beta}^{(k)}, g_4^{\prime(k)}>_{S_{4i}^{(k)}}$$

$$- 2 < V_o^{(k)} , g_6^{\prime(k)} >_{S_{6i}^{(k)}} \}$$

$$+ \ < \sigma_{\gamma 3}^{-(1)} , [-\frac{t_1}{12} S_{\alpha\beta 33}^{(1)} \eta_\gamma](N_{\alpha\beta}^{(1)})' + [\frac{1}{10} S_{\alpha\beta 33}^{(1)} \eta_\gamma](M_{\alpha\beta}^{(1)})'$$

$$+ \ [(\frac{1}{120} + \frac{1}{840})(t_1^3 S_{3333}^{(1)} + t_2^3 S_{3333}^{(2)}) \eta_\gamma](\sigma_{\rho 3,\rho}^{-(1)})'$$

$$+ \ [(\frac{1}{24} + \frac{3}{280})(t_2^2 S_{3333}^{(2)} - t_1^2 S_{3333}^{(1)}) \eta_\gamma](\sigma_{33}^{-(1)})' + [\frac{t_2}{12} S_{\alpha\beta 33}^{(2)} \eta_\gamma](N_{\alpha\beta}^{(2)})'$$

$$+ \ [\frac{1}{10} S_{\alpha\beta 33}^{(2)} \eta_\gamma](M_{\alpha\beta}^{(2)})' + [-(\frac{1}{120} - \frac{1}{840}) t_2^3 S_{3333}^{(2)} \eta_\gamma](\sigma_{\rho 3,\rho}^{-(2)})'$$

$$+ \ [(\frac{1}{24} - \frac{3}{280}) t_2^2 S_{3333}^{(2)} \eta_\gamma](\sigma_{33}^{-(2)})' - 2 g_\sigma^{\prime(1)} >_{S_i^{(1)}}$$

$$+ \sum_{k=2}^{N-2} \{ \ < \sigma_{\gamma 3}^{-(k)} , [-(\frac{1}{120} - \frac{1}{840}) t_k^3 S_{3333}^{(k)} \eta_\gamma](\sigma_{\rho 3,\rho}^{-(k-1)})'$$

$$+ \ [(-\frac{1}{24} + \frac{3}{280}) t_k^2 S_{3333}^{(k)} \eta_\gamma](\sigma_{33}^{-(k-1)})'$$

$$+ \ [-\frac{t_k}{12} S_{\alpha\beta 33}^{(k)} \eta_\gamma](N_{\alpha\beta}^{(k)})' + [\frac{1}{10} S_{\alpha\beta 33}^{(k)} \eta_\gamma](M_{\alpha\beta}^{(k)})'$$

$$+ \ [(\frac{1}{120} + \frac{1}{840})(t_k^3 S_{3333}^{(k)} + t_{k+1}^3 S_{3333}^{(k+1)}) \eta_\gamma](\sigma_{\rho 3,\rho}^{-(k)})'$$

$$+ \ [(\frac{1}{24} + \frac{3}{280})(t_{k+1}^2 S_{3333}^{(k+1)} - t_k^2 S_{3333}^{(k)}) \eta_\gamma](\sigma_{33}^{-(k)})' + [\frac{t_{k+1}}{12} S_{\alpha\beta 33}^{(k+1)} \eta_\gamma](N_{\alpha\beta}^{(k+1)})'$$

$$+ \ [\frac{1}{10} S_{\alpha\beta 33}^{(k+1)} \eta_\gamma](M_{\alpha\beta}^{(k+1)})' + [-(\frac{1}{120} - \frac{1}{840}) t_{k+1}^3 S_{3333}^{(k+1)} \eta_\gamma](\sigma_{\rho 3,\rho}^{-(k+1)})'$$

$$+ \ [(\frac{1}{24} - \frac{3}{280}) t_{k+1}^2 S_{3333}^{(k+1)} \eta_\gamma](\sigma_{33}^{-(k+1)})' - 2 g_\sigma^{\prime(k)} >_{S_i^{(k)}} \}$$

$$+ \ < \sigma_{\gamma 3}^{-(N-1)} , [-(\frac{1}{120} - \frac{1}{840}) t_{N-1}^3 S_{3333}^{(N-1)} \eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-2)})'$$

$$+ \ [(-\frac{1}{24} + \frac{3}{280}) t_{N-1}^2 S_{3333}^{(N-1)} \eta_\gamma](\sigma_{33}^{-(N-2)})'$$

$$+ \ [-\frac{t_{N-1}}{12} S_{\alpha\beta 33}^{(N-1)} \eta_\gamma](N_{\alpha\beta}^{(N-1)})' + [\frac{1}{10} S_{\alpha\beta 33}^{(N-1)} \eta_\gamma](M_{\alpha\beta}^{(N-1)})'$$

$$+ [(\frac{1}{120}+\frac{1}{840})(t_{N-1}^{3}S_{3333}^{(N-1)}+t_{N}^{3}S_{3333}^{(N)})\eta_{\gamma}](\bar{\sigma}_{\rho 3,\rho}^{(N-1)})'$$

$$+ [(\frac{1}{24}+\frac{3}{280})(t_{N}^{2}S_{3333}^{(N)}-t_{N-1}^{2}S_{3333}^{(N-1)})\eta_{\gamma}](\bar{\sigma}_{33}^{(N-1)})'+[\frac{t_N}{12}S_{\alpha\beta 33}^{(N)}\eta_{\gamma}](N_{\alpha\beta}^{(N)})'$$

$$+ [\frac{1}{10}S_{\alpha\beta 33}^{(N)}\eta_{\gamma}](M_{\alpha\beta}^{(N)})'-2g_{\sigma}^{\prime(N-1)}>_{S_i^{(N-1)}}$$

The domain of definition of J_3 is an extension of the domain of J_2 to which $\bar{v}_3^{(k)}\epsilon C^0$ instead of $\bar{v}_3^{(k)}\epsilon C^1$.

4.7.1 Some Specializations

Forcing some of the field equations and/or boundary conditions to be satisfied identically, the number of field variables is reduced and some interesting specializations of the extended variational principles are realized. Combining (184), for layers k=1 to N-1; each of (187) and (189), for layers k=2 to N-1 and (188) and (190), for layers k=1 to N-2, the following self-adjointness relationships arise.

$$\sum_{k=1}^{N-1}<\bar{\sigma}_{\gamma 3}^{(k)}, \Xi_{12}^{(k)}\bar{\sigma}_{33}^{(k)}>_{R^{(k)}}=\sum_{k=1}^{N-1}<\bar{\sigma}_{33}^{(k)}, \Xi_{21}^{(k)}\bar{\sigma}_{\gamma 3}^{(k)}>_{R^{(k)}}$$

$$+ \sum_{k=1}^{N-1}<\bar{\sigma}_{\rho 3}^{(k)}, (\frac{3}{280}+\frac{1}{24})[S_{3333}^{(k)}t_k^2-S_{3333}^{(k+1)}t_{k+1}^2]\eta_{\rho}\bar{\sigma}_{33}^{(k)}>_{S^{(k)}}$$

$$+ \sum_{k=1}^{N-1}\int_{S_i^{(k)}}(\bar{\sigma}_{\rho 3}^{(k)}(\frac{3}{280}+\frac{1}{24})[S_{3333}^{(k)}t_k^2-S_{3333}^{(k+1)}t_{k+1}^2]\eta_{\rho}\bar{\sigma}_{33}^{(k)})'dS_i^{(k)} \qquad (262)$$

$$\sum_{k=2}^{N-1}<\bar{\sigma}_{\gamma 3}^{(k-1)}, \bar{\Lambda}_{11}^{(k)}\bar{\sigma}_{\rho 3}^{(k)}>_{R^{(k)}}=\sum_{k=2}^{N-1}<\bar{\sigma}_{\rho 3}^{(k)}, \Lambda_{11}^{(k)}\bar{\sigma}_{\gamma 3}^{(k-1)}>_{R^{(k)}}$$

$$+ \sum_{k=1}^{N-2}<\bar{\sigma}_{\gamma 3}^{(k)}, (\frac{1}{120}-\frac{1}{840})t_{k+1}^{3}S_{3333}^{(k+1)}\eta_{\gamma}\bar{\sigma}_{\rho 3,\rho}^{(k+1)}>_{S^{(k+1)}}$$

$$- \sum_{k=2}^{N-1}<\bar{\sigma}_{\gamma 3}^{(k)}, (\frac{1}{120}-\frac{1}{840})t_{k}^{3}S_{3333}^{(k)}\eta_{\gamma}\bar{\sigma}_{\rho 3,\rho}^{(k-1)}>_{S^{(k-1)}}$$

$$+\sum_{k=1}^{N-2}\int_{S_i^{(k+1)}}(\sigma_{\gamma 3}^{-(k)}(\frac{1}{120}-\frac{1}{840})t_{k+1}^{3}S_{3333}^{(k+1)}\eta_{\gamma}\sigma_{\rho 3,\rho}^{(k+1)})'dS_i^{(k+1)}$$

$$-\sum_{k=2}^{N-1}\int_{S_i^{(k-1)}}(\sigma_{\gamma 3}^{-(k)}(\frac{1}{120}-\frac{1}{840})t_{k}^{3}S_{3333}^{(k)}\eta_{\gamma}\sigma_{\rho 3,\rho}^{-(k-1)})'dS_i^{(k-1)} \tag{263}$$

$$\sum_{k=1}^{N-2}<\sigma_{\gamma 3}^{-(k)}, \overline{\Lambda}_{12}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}=\sum_{k=2}^{N-1}<\sigma_{33}^{-(k)}, \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}$$

$$+\sum_{k=2}^{N-1}<\sigma_{\gamma 3}^{-(k-1)}, (\frac{3}{280}-\frac{1}{24})t_k^2S_{3333}^{(k)}\eta_\gamma\sigma_{33}^{-(k)}>_{S^{(k)}}$$

$$+\sum_{k=2}^{N-1}\int_{S_i^{(k)}}(\sigma_{\gamma 3}^{-(k-1)}(\frac{3}{280}-\frac{1}{24})t_k^2S_{3333}^{(k)}\eta_\gamma\sigma_{33}^{-(k)})'dS_i^{(k)} \tag{264}$$

$$\sum_{k=2}^{N-1}<\sigma_{\gamma 3}^{-(k)}, \Lambda_{12}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}=\sum_{k=2}^{N-1}<\sigma_{33}^{-(k-1)}, \overline{\Lambda}_{21}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}}$$

$$+\sum_{k=2}^{N-1}<\sigma_{\gamma 3}^{-(k)}, -(\frac{3}{280}-\frac{1}{24})t_k^2S_{3333}^{(k)}\eta_\gamma\sigma_{33}^{-(k-1)}>_{S^{(k)}}$$

$$-\sum_{k=2}^{N-1}\int_{S_i^{(k)}}(\sigma_{\gamma 3}^{-(k)}(\frac{3}{280}-\frac{1}{24})t_k^2S_{3333}^{(k)}\eta_\gamma\sigma_{33}^{-(k-1)})'dS_i^{(k)} \tag{265}$$

$$\sum_{k=1}^{N-2}<\sigma_{33}^{-(k)}, \overline{\Lambda}_{22}^{(k+1)}\sigma_{33}^{-(k+1)}>_{R^{(k+1)}}=\sum_{k=2}^{N-1}<\sigma_{33}^{-(k)}, \Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}} \tag{266}$$

Substituting (262) through (266) into (261) to eliminate $\Xi_{12}^{(k)}$, $\overline{\Lambda}_{11}^{(k)}$, $\overline{\Lambda}_{12}^{(k+1)}$, $\Lambda_{12}^{(k)}$ and $\overline{\Lambda}_{22}^{(k+1)}$, $\Omega_3(u, \sigma)$ becomes

$$\Omega_4(u, \sigma)=2<\overline{v}_\gamma^{(1)}, \sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\overline{\phi}_\gamma^{(1)}, \frac{t_1}{2}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\overline{v}_3^{(1)}, \sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+ 2<N_{\alpha\beta}^{(1)}, \frac{t_1}{12}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{1}{2}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}}$$

$$+ 2<M_{\alpha\beta}^{(1)}, \frac{1}{10}S_{\alpha\beta 33}^{(1)}\sigma_{\gamma 3,\gamma}^{(0)}+\frac{6}{5t_1}S_{\alpha\beta 33}^{(1)}\sigma_{33}^{(0)}>_{R^{(1)}} + 2<V_\rho^{(1)}, -\frac{2}{5}S_{\rho 3\gamma 3}^{(1)}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}$$

$$+\ 2<\bar{v}_{\gamma}^{(N)},-\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{\phi}_{\gamma}^{(N)},\frac{t_N}{2}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{v}_{3}^{(N)},-\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<N_{\alpha\beta}^{(N)},-\frac{t_N}{12}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}+\frac{1}{2}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\ 2<M_{\alpha\beta}^{(N)},\frac{1}{10}S_{\alpha\beta 33}^{(N)}\sigma_{\gamma 3,\gamma}^{(N)}-\frac{6}{5t_N}S_{\alpha\beta 33}^{(N)}\sigma_{33}^{(N)}>_{R^{(N)}}+2<V_{\rho}^{(N)},-\frac{2}{5}S_{\rho 3\gamma 3}^{(N)}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}2\{<\bar{v}_{\gamma}^{(k)},\bar{\sigma}_{\gamma 3}^{(k-1)}>_{R^{(k)}}+<\bar{\phi}_{gama}^{(k)},\frac{t_k}{2}\bar{\sigma}_{\gamma 3}^{(k-1)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)},\bar{\sigma}_{33}^{(k-1)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)},\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{\gamma 3,\gamma}^{(k-1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{33}^{(k-1)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\frac{1}{10}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{\gamma 3,\gamma}^{(k-1)}+\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{33}^{(k-1)}>_{R^{(k)}}$$

$$+\ <V_{\rho}^{(k)},-\frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\bar{\sigma}_{\gamma 3}^{(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{2<N_{\alpha\beta}^{(k)},-\bar{v}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)},\frac{1}{t_k}S_{\alpha\beta\mu\rho}^{(k)}N_{\mu\rho}^{(k)}>_{R^{(k)}}$$

$$+\ 2<M_{\alpha\beta}^{(k)},-\bar{\phi}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\frac{12}{t_k^3}S_{\alpha\beta\mu\rho}^{(k)}M_{\mu\rho}^{(k)}>_{R^{(k)}}$$

$$+\ 2<V_{\gamma}^{(k)},-(\bar{v}_{3,\gamma}^{(k)}+\bar{\phi}_{\gamma}^{(k)})>_{R^{(k)}}$$

$$+\ <V_{\gamma}^{(k)},\frac{24}{5t_k}S_{\rho 3\gamma 3}^{(k)}V_{\rho}^{(k)}>_{R^{(k)}}$$

$$+\sum_{k=1}^{N-1}2\{<\bar{v}_{\gamma}^{(k)},-\bar{\sigma}_{\gamma 3}^{(k)}>_{R^{(k)}}+<\bar{\phi}_{\gamma}^{(k)},\frac{t_k}{2}\bar{\sigma}_{\gamma 3}^{(k)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)},-\bar{\sigma}_{33}^{(k)}>_{R^{(k)}}$$

$$+\ <N_{\alpha\beta}^{(k)},-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{\gamma 3,\gamma}^{(k)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{33}^{(k)}>_{R^{(k)}}$$

$$+ \langle M_{\alpha\beta}^{(k)} , \frac{1}{10} S_{\alpha\beta 33}^{(k)} \bar{\sigma}_{\gamma 3,\gamma}^{(k)} - \frac{6}{5t_k} S_{\alpha\beta 33}^{(k)} \bar{\sigma}_{33}^{(k)} \rangle_{R^{(k)}}$$

$$+ \langle V_{\rho}^{(k)} , -\frac{2}{5} S_{\rho 3\gamma 3}^{(k)} \bar{\sigma}_{\gamma 3}^{(k)} \rangle_{R^{(k)}} \}$$

$$+\sum_{k=1}^{N-1} \{ \langle \bar{\sigma}_{\rho 3}^{(k)} , \Xi_{11}^{(k)} \bar{\sigma}_{\gamma 3}^{(k)} \rangle_{R^{(k)}} + 2 \langle \bar{\sigma}_{33}^{(k)} , \Xi_{21}^{(k)} \bar{\sigma}_{\gamma 3}^{(k)} \rangle_{R^{(k)}} + \langle \bar{\sigma}_{33}^{(k)} , \Xi_{22}^{(k)} \bar{\sigma}_{33}^{(k)} \rangle_{R^{(k)}} \}$$

$$+\sum_{k=2}^{N-1} \{ 2 \langle \bar{\sigma}_{\rho 3}^{(k)} , \Lambda_{11}^{(k)} \bar{\sigma}_{\gamma 3}^{(k-1)} \rangle_{R^{(k)}} + 2 \langle \bar{\sigma}_{33}^{(k)} , \Lambda_{21}^{(k)} \bar{\sigma}_{\gamma 3}^{(k-1)} \rangle_{R^{(k)}} + 2 \langle \bar{\sigma}_{33}^{(k)} , \Lambda_{22}^{(k)} \bar{\sigma}_{33}^{(k-1)} \rangle_{R^{(k)}} \}$$

$$+\sum_{k=1}^{N-2} \{ 2 \langle \bar{\sigma}_{33}^{(k)} , \bar{\Lambda}_{21}^{(k+1)} \bar{\sigma}_{\gamma 3}^{(k+1)} \rangle_{R^{(k+1)}} \}$$

$$+ 2 \langle [\sigma]^{(N-1)} , [\bar{\Lambda}]^{(N)} [\sigma]^{(N)} \rangle_{R^{(N)}}$$

$$+ 2 \langle [\sigma]^{-(1)} , [\Lambda]^{(1)} [\sigma]^{(0)} \rangle_{R^{(1)}}$$

$$+\sum_{k=1}^{N} \{ \langle \bar{v}_{\alpha}^{(k)} , -2g_1^{(k)} \rangle_{S_1^{(k)}}$$

$$+ \langle \bar{\phi}_{\alpha}^{(k)} , -2g_3^{(k)} \rangle_{S_3^{(k)}}$$

$$+ \langle \bar{v}_3^{(k)} , -2g_5^{(k)} \rangle_{S_5^{(k)}}$$

$$+ 2 \langle N_{\alpha\beta}^{(k)} , \eta_\beta \bar{v}_{\alpha}^{(k)} - g_2^{(k)} \rangle_{S_2^{(k)}}$$

$$+ 2 \langle M_{\alpha\beta}^{(k)} , \eta_\beta \bar{\phi}_{\alpha}^{(k)} - g_4^{(k)} \rangle_{S_4^{(k)}}$$

$$+ 2 \langle V_{\alpha}^{(k)} , \eta_\alpha \bar{v}_3^{(k)} - g_6^{(k)} \rangle_{S_6^{(k)}} \}$$

$$+ 2 \langle \bar{\sigma}_{\gamma 3}^{(1)} , [(\frac{1}{120} + \frac{1}{840}) \frac{(t_1^3 S_{3333}^{(1)} + t_2^3 S_{3333}^{(2)})}{2} \eta_\gamma] \bar{\sigma}_{\rho 3,\rho}^{(1)} - g_\sigma^{(1)} \rangle_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2} \{ 2 \langle \bar{\sigma}_{\gamma 3}^{(k)} , [-(\frac{1}{120} - \frac{1}{840}) t_k^3 S_{3333}^{(k)} \eta_\gamma] \bar{\sigma}_{\rho 3,\rho}^{(k-1)}$$

$$+ [(\frac{1}{120}+\frac{1}{840})\frac{(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})}{2}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(k)} - g_\sigma^{(k)} >_{S^{(k)}} \}$$

$$+ 2<\sigma_{\gamma 3}^{-(N-1)}, [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S_{3333}^{(N-1)}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(N-2)}$$

$$+ [(\frac{1}{120}+\frac{1}{840})\frac{(t_{N-1}^3 S_{3333}^{(N-1)}+t_N^3 S_{3333}^{(N)})}{2}\eta_\gamma]\sigma_{\rho 3,\rho}^{-(N-1)} - g_\sigma^{(N-1)} >_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\{ <\bar{v}_\alpha^{(k)}, -2g_1^{\prime(k)}>_{S_{1i}^{(k)}}$$

$$+ <\bar{\phi}_\alpha^{(k)}, -2g_3^{\prime(k)}>_{S_{3i}^{(k)}}$$

$$+ <\bar{v}_3^{(k)}, -2g_5^{\prime(k)}>_{S_{5i}^{(k)}}$$

$$+ 2<N_{\alpha\beta}^{(k)}, \eta_\beta(\bar{v}_\alpha^{(k)})' - g_2^{\prime(k)}>_{S_{2i}^{(k)}}$$

$$+ 2<M_{\alpha\beta}^{(k)}, \eta_\beta(\bar{\phi}_\alpha^{(k)})' - g_4^{\prime(k)}>_{S_{4i}^{(k)}}$$

$$+ 2<V_\alpha^{(k)}, \eta_\alpha(\bar{v}_3^{(k)})' - g_6^{\prime(k)}>_{S_{6i}^{(k)}}\}$$

$$+ 2<\sigma_{\gamma 3}^{-(1)}, [(\frac{1}{120}+\frac{1}{840})\frac{(t_1^3 S_{3333}^{(1)}+t_2^3 S_{3333}^{(2)})}{2}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(1)})' - g_\sigma^{\prime(1)}>_{S_i^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{ 2<\sigma_{\gamma 3}^{-(k)}, [-(\frac{1}{120}-\frac{1}{840})t_k^3 S_{3333}^{(k)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k-1)})'$$

$$+ [(\frac{1}{120}+\frac{1}{840})\frac{(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})}{2}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k)})' - g_\sigma^{\prime(k)}>_{S_i^{(k)}} \}$$

$$+ 2<\sigma_{\gamma 3}^{-(N-1)}, [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S_{3333}^{(N-1)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-2)})'$$

$$+ [(\frac{1}{120}+\frac{1}{840})\frac{(t_{N-1}^3 S_{3333}^{(N-1)}+t_N^3 S_{3333}^{(N)})}{2}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-1)})' - g_\sigma^{\prime(N-1)}>_{S_i^{(N-1)}} \qquad (267)$$

For the boundary value problem considered, if the set of admissible states is restricted to one that identically satisfies the constitutive equations, (155) through (157), the functional Ω_4 is specialized to

$$\Omega_S(\bar{v}_\alpha^{(k)}, \bar{\phi}_\alpha^{(k)}, \bar{v}_3^{(k)}, \sigma_{i3}^{-(k)}) = 2<\bar{v}_\gamma^{(1)}, \sigma_{\gamma 3}^{(0)}>_{R^{(1)}} + 2<\bar{\phi}_\gamma^{(1)}, \frac{t_1}{2}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}} + 2<\bar{v}_3^{(1)}, \sigma_{33}^{(0)}>_{R^{(1)}}$$

$$-2<\bar{v}_\gamma^{(N)}, \sigma_{\gamma 3}^{(N)}>_{R^{(N)}} + 2<\bar{\phi}_\gamma^{(N)}, \frac{t_N}{2}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}} + 2<\bar{v}_3^{(N)}, -\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N} 2\{<\bar{v}_\gamma^{(k)}, \sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}} + <\bar{\phi}_\gamma^{(k)}, \frac{t_k}{2}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}} + <\bar{v}_3^{(k)}, \sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$-\sum_{k=1}^{N}\{<N_{\alpha\beta}^{(k)}, \frac{1}{t_k}S_{\alpha\beta\mu\rho}^{(k)}N_{\mu\rho}^{(k)}>_{R^{(k)}}$$

$$+ <M_{\alpha\beta}^{(k)}, \frac{12}{t_k^3}S_{\alpha\beta\mu\rho}^{(k)}M_{\mu\rho}^{(k)}>_{R^{(k)}}$$

$$+ <V_\rho^{(k)}, \frac{24}{5t_k}S_{\rho 3\gamma 3}^{(k)}V_\rho^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1} 2\{<\bar{v}_\gamma^{(k)}, -\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} + <\bar{\phi}_\gamma^{(k)}, \frac{t_k}{2}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} + <\bar{v}_3^{(k)}, -\sigma_{33}^{-(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\sigma_{\rho 3}^{-(k)}, \Xi_{11}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} + 2<\sigma_{33}^{-(k)}, \Xi_{21}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} + <\sigma_{33}^{-(k)}, \Xi_{22}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=2}^{N-1}\{2<\sigma_{\rho 3}^{-(k)}, \Lambda_{11}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}} + 2<\sigma_{33}^{-(k)}, \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}} + 2<\sigma_{33}^{-(k)}, \Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{2<\sigma_{33}^{-(k)}, \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{-(k+1)}>_{R^{(k+1)}}\}$$

$$+ 2<[\sigma]^{-(N-1)}, [\bar{\Lambda}]^{(N)}[\sigma]^{(N)}>_{R^{(N)}}$$

$$+ 2<[\sigma]^{-(1)}, [\Lambda]^{(1)}[\sigma]^{(0)}>_{R^{(1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}_\alpha^{(k)}, -2g_1^{(k)}>_{S_1^{(k)}}$$

$$+ <\bar{\phi}_\alpha^{(k)}, -2g_3^{(k)}>_{S_3^{(k)}}$$

$$+ <\bar{v}_3^{(k)}, -2g_5^{(k)}>_{S_5^{(k)}}$$

$$+\ 2<N^{(k)}_{\alpha\beta}\ ,\ \eta_\beta \bar{v}^{(k)}_\alpha - g^{(k)}_2>_{S^{(k)}_2}$$

$$+\ 2<M^{(k)}_{\alpha\beta}\ ,\ \eta_\beta \bar{\phi}^{(k)}_\alpha - g^{(k)}_4>_{S^{(k)}_4}$$

$$+\ 2<V^{(k)}_{\alpha}\ ,\ \eta_\alpha \bar{v}^{(k)}_3 - g^{(k)}_6>_{S^{(k)}_6}\}$$

$$+\ 2<\sigma^{-(1)}_{\gamma 3}\ ,\ [(\frac{1}{120}+\frac{1}{840})\frac{(t_1^3 S^{(1)}_{3333}+t_2^3 S^{(2)}_{3333})}{2}\eta_\gamma]\sigma^{-(1)}_{\rho 3,\rho} - g^{(1)}_\sigma>_{S^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{\ 2<\sigma^{-(k)}_{\gamma 3}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_k^3 S^{(k)}_{3333}\eta_\gamma]\sigma^{-(k-1)}_{\rho 3,\rho}$$

$$+\ [(\frac{1}{120}+\frac{1}{840})\frac{(t_k^3 S^{(k)}_{3333}+t_{k+1}^3 S^{(k+1)}_{3333})}{2}\eta_\gamma]\sigma^{-(k)}_{\rho 3,\rho} - g^{(k)}_\sigma>_{S^{(k)}}\ \}$$

$$+\ 2<\sigma^{-(N-1)}_{\gamma 3}\ ,\ [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S^{(N-1)}_{3333}\eta_\gamma]\sigma^{-(N-2)}_{\rho 3,\rho}$$

$$+\ [(\frac{1}{120}+\frac{1}{840})\frac{(t_{N-1}^3 S^{(N-1)}_{3333}+t_N^3 S^{(N)}_{3333})}{2}\eta_\gamma]\sigma^{-(N-1)}_{\rho 3,\rho} - g^{(N-1)}_\sigma>_{S^{(N-1)}}$$

$$+\sum_{k=1}^{N}\{<\bar{v}^{(k)}_\alpha\ ,\ -2g'^{(k)}_1>_{S^{(k)}_{1i}}$$

$$+\ <\bar{\phi}^{(k)}_\alpha\ ,\ -2g'^{(k)}_3>_{S^{(k)}_{3i}}$$

$$+\ <\bar{v}^{(k)}_3\ ,\ -2g'^{(k)}_5>_{S^{(k)}_{5i}}$$

$$+\ 2<N^{(k)}_{\alpha\beta}\ ,\ \eta_\beta (\bar{v}^{(k)}_\alpha)' - g'^{(k)}_2>_{S^{(k)}_{2i}}$$

$$+\ 2<M^{(k)}_{\alpha\beta}\ ,\ \eta_\beta (\bar{\phi}^{(k)}_\alpha)' - g'^{(k)}_4>_{S^{(k)}_{4i}}$$

$$+\ 2<V^{(k)}_{\alpha}\ ,\ \eta_\alpha (\bar{v}^{(k)}_3)' - g'^{(k)}_6>_{S^{(k)}_{6i}}\}$$

$$+\ 2<\sigma^{-(1)}_{\gamma 3}\ ,\ [(\frac{1}{120}+\frac{1}{840})\frac{(t_1^3 S^{(1)}_{3333}+t_2^3 S^{(2)}_{3333})}{2}\eta_\gamma](\sigma^{-(1)}_{\rho 3,\rho})' - g'^{(1)}_\sigma>_{S^{(1)}_i}$$

$$+\sum_{k=2}^{N-2}\{\ 2<\bar{\sigma}_{\gamma 3}^{(k)}\ ,[-(\frac{1}{120}-\frac{1}{840})t_k^3S_{3333}^{(k)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k-1)})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})\frac{(t_k^3S_{3333}^{(k)}+t_{k+1}^3S_{3333}^{(k+1)})}{2}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k)})'-g_\sigma^{\prime(k)}>_{S_i^{(k)}}\ \}$$

$$+\ 2<\bar{\sigma}_{\gamma 3}^{(N-1)}\ ,[-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3S_{3333}^{(N-1)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-2)})'$$

$$+\ [(\frac{1}{120}+\frac{1}{840})\frac{(t_{N-1}^3S_{3333}^{(N-1)}+t_N^3S_{3333}^{(N)})}{2}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-1)})'-g_\sigma^{\prime(N-1)}>_{S_i^{(N-1)}} \tag{268}$$

Here $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$, $V_\alpha^{(k)}$ are not independent field variables but defined completely by $\bar{v}_\alpha^{(k)}$, $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$ and $\bar{\sigma}_{i3}^{(k)}$ through the constitutive relations (155) through (157). Even if the physical problem has no discontinuities i.e., $g_i^{\prime(k)}$ vanish, the discontinuity terms must be included. These vanish if $\bar{v}_\alpha^{(k)}$, $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$ and $\bar{\sigma}_{\rho 3,\rho}^{(k)}$ are restricted to being continuous across all internal boundaries. Satisfying the displacement boundary conditions (i.e. the last three conditions in (221) and in (223)) identically, the traction-free boundary conditions and assuming no physical discontinuities (i.e., g'_1, g'_3, g'_5 and g'_α) vanishing, $\Omega_5(u\ ,\sigma)$ leads to

$$\Omega_6(u\ ,\sigma)=2<\bar{v}_\gamma^{(1)}\ ,\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{\phi}_\gamma^{(1)}\ ,\frac{t_1}{2}\sigma_{\gamma 3}^{(0)}>_{R^{(1)}}+2<\bar{v}_3^{(1)}\ ,\sigma_{33}^{(0)}>_{R^{(1)}}$$

$$-2<\bar{v}_\gamma^{(N)}\ ,\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{\phi}_\gamma^{(N)}\ ,\frac{t_N}{2}\sigma_{\gamma 3}^{(N)}>_{R^{(N)}}+2<\bar{v}_3^{(N)}\ ,-\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N}2\{<\bar{v}_\gamma^{(k)}\ ,\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{\phi}_\gamma^{(k)}\ ,\frac{t_k}{2}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}+<\bar{v}_3^{(k)}\ ,\sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{<N_{\alpha\beta}^{(k)}\ ,\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k-1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)}\ ,\frac{1}{10}S_{\alpha\beta 33}^{(k)}\sigma_{\gamma 3,\gamma}^{-(k-1)}+\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$+\ <V_\rho^{(k)}\ ,-\frac{2}{5}S_{\rho 3gamma3}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}\}$$

$$-\sum_{k=1}^{N}\{<N_{\alpha\beta}^{(k)}, \bar{v}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}+<M_{\alpha\beta}^{(k)}, \bar{\phi}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}}+<V_{\gamma}^{(k)}, \bar{v}_{3,\gamma}^{(k)}+\bar{\phi}_{\gamma}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{<N_{\alpha\beta}^{(k)}, -\frac{t_k}{12}S_{\alpha\beta33}^{(k)}\sigma_{\gamma3,\gamma}^{-(k)}+\frac{1}{2}S_{\alpha\beta33}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+ <M_{\alpha\beta}^{(k)}, \frac{1}{10}S_{\alpha\beta33}^{(k)}\sigma_{\gamma3,\gamma}^{-(k)}-\frac{6}{5t_k}S_{\alpha\beta33}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$+ <V_{\rho}^{(k)}, -\frac{2}{5}S_{\rho3\gamma3}^{(k)}\sigma_{\gamma3}^{-(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}2\{<\bar{v}_{\gamma}^{(k)}, -\sigma_{\gamma3}^{-(k)}>_{R^{(k)}}+<\bar{\phi}_{\gamma}^{(k)}, \frac{t_k}{2}\sigma_{\gamma3}^{-(k)}>_{R^{(k)}}+<\bar{v}_{3}^{(k)}, -\sigma_{33}^{-(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\sigma_{\rho3}^{-(k)}, \Xi_{11}^{(k)}\sigma_{\gamma3}^{-(k)}>_{R^{(k)}}+2<\sigma_{33}^{-(k)}, \Xi_{21}^{(k)}\sigma_{\gamma3}^{-(k)}>_{R^{(k)}}+<\sigma_{33}^{-(k)}, \Xi_{22}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=2}^{N-1}\{2<\sigma_{\rho3}^{-(k)}, \Lambda_{11}^{(k)}\sigma_{\gamma3}^{-(k-1)}>_{R^{(k)}}+2<\sigma_{33}^{-(k)}, \Lambda_{21}^{(k)}\sigma_{\gamma3}^{-(k-1)}>_{R^{(k)}}+2<\sigma_{33}^{-(k)}, \Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-2}\{2<\sigma_{33}^{-(k)}, \bar{\Lambda}_{21}^{(k+1)}\sigma_{\gamma3}^{-(k+1)}>_{R^{(k+1)}}\}$$

$$+ 2<[\sigma]^{-(N-1)}, [\bar{\Lambda}]^{(N)}[\sigma]^{(N)}>_{R^{(N)}}$$

$$+ 2<[\sigma]^{-(1)}, [\Lambda]^{(1)}[\sigma]^{(0)}>_{R^{(1)}}$$

$$-2\sum_{k=1}^{N}\{<\bar{v}_{\alpha}^{(k)}, g_1^{(k)}>_{S_1^{(k)}}$$

$$+ <\bar{\phi}_{\alpha}^{(k)}, g_3^{(k)}>_{S_3^{(k)}}$$

$$+ <\bar{v}_{3}^{(k)}, g_5^{(k)}>_{S_5^{(k)}}\}$$

$$+ 2<\sigma_{\gamma3}^{-(1)}, [(\frac{1}{120}+\frac{1}{840})\frac{(t_1^3S_{3333}^{(1)}+t_2^3S_{3333}^{(2)})}{2}\eta_\gamma](\sigma_{\rho3,\rho}^{-(1)})'>_{S_1^{(1)}}$$

$$+\sum_{k=2}^{N\ 2}\{2<\sigma_{\gamma3}^{-(k)}, [-(\frac{1}{120}-\frac{1}{840})t_k^3S_{3333}^{(k)}\eta_\gamma](\sigma_{\rho3,\rho}^{-(k-1)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})\frac{(t_k^3 S_{3333}^{(k)}+t_{k+1}^3 S_{3333}^{(k+1)})}{2}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(k)})' >_{S_i^{(k)}} \}$$

$$+ \ 2<\sigma_{\gamma 3}^{-(N-1)}, [-(\frac{1}{120}-\frac{1}{840})t_{N-1}^3 S_{3333}^{(N-1)}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-2)})'$$

$$+ \ [(\frac{1}{120}+\frac{1}{840})\frac{(t_{N-1}^3 S_{3333}^{(N-1)}+t_N^3 S_{3333}^{(N)})}{2}\eta_\gamma](\sigma_{\rho 3,\rho}^{-(N-1)})' >_{S_i^{(N-1)}} \tag{269}$$

Because the mechanical quantities, $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$ and $V_\alpha^{(k)}$ in Ω_6 are completely defined by the constitutive relations (155) through (157), the independent field variables are $\overline{v}_\alpha^{(k)}$, $\overline{\phi}_\alpha^{(k)}$, $\overline{v}_3^{(k)}$ and $\sigma_{i3}^{-(k)}$. The number of independent field variables, therefore, is 8N+3 (N= number of layers) for this specialization. This is less than the 13N used by Pagano [1978] for obtaining numerical solutions. For other cases, where any of the force resultants may not satisfy the constitutive relations, these would have to be included as field variables. Thus for the most general case, there would be 16N+3 field variables.

The specialization defined by Ω_6, having the least number of field variables, was used to develop a finite element solution procedure described in the next section.

SECTION V

FINITE ELEMENT FORMULATION

5.1 Finite Element Discretization

In the finite element method, the region R is replaced by a collection of m disjoint open subregions called elements $\{R_e, e = 1, \ldots, m\}$ such that in a sequence of refinements

$$\bar{R} = \lim_{m \to \infty} \bigcup_{e=1}^{m} \bar{R}_e \tag{270}$$

and the subregions have the property that

$$R_e \bigcap R_f = 0 \qquad \text{if } e \neq f \tag{271}$$

These elements are connected at a finite number, N, of nodal points. We assume that a finite element representation of R is available such that $\{(S_{1i}, S_{2i}), (S_{3i}, S_{4i})$ and $(S_{5i}, S_{6i})\}$ are contained in the union of intersection of element boundaries.

Over an element, let approximation to the unknown field variables be, in matrix form, as follows

$$\bar{v}_\alpha = [\psi]_e^T \{a_v\}_e \tag{272}$$

where $[\psi]_e^T$, the set of base functions, is a row vector and $\{a_v\}_e$ is a column vector of coefficients. Evaluating the function, and its derivatives up to a certain order, at nodal points yields

$$\{\bar{v}_\alpha\}_e = [\psi]_e^{iT} \{a_v\}_e \tag{273}$$

where $\{\bar{v}_\alpha\}_e$ is the vector of nodal point values of the function and its derivatives up to the order selected, and $[\psi]_e^{iT}$ is the matrix of base functions and their appropriate derivatives, if required, evaluated at each nodal point. The rows and columns of $[\psi]_e^{iT}$ must be linearly independent. If square, the matrix is invertible. Hence, we can write

$$\{a_\nu\}_e = [[\psi]_e^{iT}]^{-1} \{\bar{v}_\alpha\}_e$$

$$= [A]^{-1} \{\bar{v}_\alpha\}_e \tag{274}$$

where $[A] = [\psi]_e^{iT}$

Substitution of (274) into (272) leads to

$$\bar{v}_\alpha = [\psi]_e^T [A]^{-1} \{\bar{v}_\alpha\}_e$$

$$= [\psi_\nu]_e^T \{\bar{v}_\alpha\}_e \tag{275}$$

where

$$[\psi_\nu]_e^T = [\psi]_e^T [A]^{-1} \tag{276}$$

For the kth layer, we have

$$\bar{v}_\alpha^{(k)} = [\psi_\nu]_e^T \{\bar{v}_\alpha\}_e^{(k)} \tag{277}$$

where $[\psi_\nu]_e^T$ can now be regraded as a set of interpolating functions relating nodal point values of a function and its derivatives up to a preselected order, to the value of the function $\bar{v}_\alpha^{(k)}$ at an arbitrary point within the element e. Similarly the other field variables are approximated, in the finite element procedure, as

$$\left.\begin{aligned}
\bar{\Phi}_\alpha^{(k)} &= [\psi_\phi]_e^T \{\bar{\Phi}_\alpha\}_e^{(k)} \\
\bar{v}_3^{(k)} &= [\psi_w]_e^T \{\bar{v}_3\}_e^{(k)} \\
\bar{\sigma}_{\gamma 3}^{(k)} &= [\psi_{\gamma 3}]_e^T \{\sigma_{\gamma 3}\}_e^{-(k)} \\
\bar{\sigma}_{33}^{(k)} &= [\psi_{33}]_e^T \{\sigma_{33}\}_e^{-(k)}
\end{aligned}\right\} \tag{278}$$

5.1.1 Finite Element Formulation

In the present investigation, the specialized functional $\Omega_6(u, \sigma)$ was used as the basis for setting up an approximate numerical solution scheme. $\Omega_6(u, \sigma)$ is completely defined by the kinematic variables, $\bar{v}_\alpha^{(k)}$, $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$, and the stress variables, $\bar{\sigma}_{i3}^{(k)}$. In deriving the governing function $\Omega_6(u, \sigma)$ the following assumptions were made:

1. Constitutive equations are identically satisfied, i.e., $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$, $V_\alpha^{(k)}$ are not field variables but completely defined by $\bar{v}_\alpha^{(k)}$, $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$ and the stress variables $\bar{\sigma}_{i3}^{(k)}$.
2. Displacement boundary conditions are exactly satisfied, i.e., $\bar{v}_\alpha^{(k)}$, $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$ are restricted to the set exactly satisfying the last three equations of the set (221).
3. Stress boundary conditions are satisfied at the free edges of the plate.
4. No jump discontinuities in the force quantities $N_{\alpha\beta}^{(k)}\eta_\beta$, $M_{\alpha\beta}^{(K)}\eta_\beta$, $V_\alpha^{(k)}\eta_\alpha$ exist in the interior of the plate.
5. $\bar{v}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$, $\bar{\phi}_\alpha^{(k)}$ satisfy the last three of the discontinuity equations (223) identically i.e., if the physical problem does not have kinematic discontinuities, $\bar{v}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$, $\bar{\phi}_\alpha^{(k)}$ are continuous in the interior of the plate.

To explicitly write $\Omega_6(u, \sigma)$ in terms of the free field variables, it is necessary to use the constitutive relations (155) through (157) to eliminate $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$ and $V_\rho^{(k)}$. Upon appropriate rearrangement, (155) through (157) yield:

$$N_{\alpha\beta}^{(k)}=[C_{\mu\rho\alpha\beta}^{-(k)}]\{t_k\bar{v}_{(\mu,\rho)}^{(k)} - \frac{t_k}{2}S_{\alpha\beta33}^{(k)}(\bar{\sigma}_{33}^{(k-1)} + \bar{\sigma}_{33}^{(k)}) + \frac{t_k^2}{12}S_{\alpha\beta33}^{(k)}(\bar{\sigma}_{\gamma3,\gamma}^{(k)} - \bar{\sigma}_{\gamma3,\gamma}^{(k-1)})\} \tag{279}$$

$$M_{\alpha\beta}^{(k)}=[C_{\mu\rho\alpha\beta}^{-(k)}]\{\frac{t_k^3}{12}\bar{\phi}_{(\mu,\rho)}^{(k)}-\frac{t_k^2}{10}S_{\alpha\beta33}^{(k)}(\bar{\sigma}_{33}^{(k-1)}-\bar{\sigma}_{33}^{(k)})-\frac{t_k^3}{120}S_{\alpha\beta33}^{(k)}(\bar{\sigma}_{\gamma3,\gamma}^{(k-1)}+\bar{\sigma}_{\gamma3,\gamma}^{(k)})\} \tag{280}$$

$$V_\rho^{(k)} = \frac{5t_k}{24}C_{\gamma3\rho3}^{-(k)}(\bar{v}_{3,\gamma}^{(k)}+\bar{\phi}_\gamma^{(k)})+\frac{t_k}{12}(\bar{\sigma}_{\rho3}^{(k-1)}+\bar{\sigma}_{\rho3}^{(k)}) \tag{281}$$

where $[C_{\mu\rho\alpha\beta}] = [S_{\mu\rho\alpha\beta}]^{1}$ and $[C_{\gamma 3\rho 3}] = [S_{\gamma 3\rho 3}]^{1}$. To recover continuous $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$, $V_{\alpha}^{(k)}$, it is necessary that $\bar{v}_{\alpha}^{(k)}$, $\bar{\phi}_{\alpha}$, $\bar{v}_{3}^{(k)}$, $\bar{\sigma}_{i3}^{(k)}$ be continuously differentiable. Otherwise, $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$, $V_{\rho}^{(k)}$ will be discontinuous at inter-element boundaries. However, this requirement of differentiability makes the finite element scheme unwieldy and is not necessary for convergence of sequence of solutions, ordered by mesh refinement, to the correct value. For this reason, in the finite element scheme discussed later in this section, simpler interpolation schemes yielding discontinuous force resultants at inter-element boundaries was adopted. Substituting (277) and (278) into (279) through (281), $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$, and $V_{\rho}^{(k)}$ can be expressed in terms of shape functions and generalized coordinates associated with $\bar{v}_{\alpha}^{(k)}$, $\bar{\phi}_{\alpha}^{(k)}$, $\bar{v}_{3}^{(k)}$ and $\bar{\sigma}_{i3}^{(k)}$. The governing functional $\Omega_6(u\ ,\sigma)$ after discretizing the spatial domain R into m disjoint elements has the form:

$$\Omega_6(u\ ,\sigma)=\sum_{e=1}^{m}\Omega_e(u\ ,\sigma) \tag{282}$$

where

$$\begin{aligned}
\Omega_e(u\ ,\sigma)=&\sum_{k=1}^{N}\{\int_{R_e} N_{\alpha\beta}^{(k)}\ [\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{\gamma 3,\gamma}^{(k-1)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{33}^{(k-1)}]dR_e \\
&+\int_{R_e} M_{\alpha\beta}^{(k)}\ [\frac{1}{10}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{\gamma 3,\gamma}^{(k-1)}+\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{33}^{(k-1)}]dR_e \\
&-\int_{R_e} V_{\rho}^{(k)}\ \frac{2}{5}S_{\rho 3\gamma 3}^{(k)}\bar{\sigma}_{\gamma 3}^{(k-1)}dR_e\} \\
-&\sum_{k=1}^{N}\{\int_{R_e} N_{\alpha\beta}^{(k)}\ \bar{v}_{(\alpha,\beta)}^{(k)}dR_e+\int_{R_e} M_{\alpha\beta}^{(k)}\ \bar{\phi}_{(\alpha,\beta)}^{(k)}dR_e+\int_{R_e} V_{\gamma}^{(k)}\ (\bar{v}_{3,r}^{(k)}+\bar{\phi}_{\gamma}^{(k)})dR_e\} \\
+&\sum_{k=1}^{N}\{\int_{R_e} N_{\alpha\beta}^{(k)}\ [-\frac{t_k}{12}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{\gamma 3,\gamma}^{(k)}+\frac{1}{2}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{33}^{(k)}]dR_e \\
&+\int_{R_e} M_{\alpha\beta}^{(k)}\ [\frac{1}{10}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{\gamma 3,\gamma}^{(k)}-\frac{6}{5t_k}S_{\alpha\beta 33}^{(k)}\bar{\sigma}_{33}^{(k)}]dR_e
\end{aligned}$$

$$- \int_{R_e} V_\rho^{(k)} \; \frac{2}{5} S_{\rho 3 \gamma 3}^{(k)} \sigma_{\gamma 3}^{-(k)} dR_e \}$$

$$+ \sum_{k=1}^{N} \{ \, 2\int_{R_e} \bar{v}_\gamma^{(k)} \; \sigma_{\gamma 3}^{-(k-1)} dR_e \; + \; 2\int_{R_e} \bar{\phi}_\gamma^{(k)} \; \frac{t_k}{2} \sigma_{\gamma 3}^{-(k-1)} dR_e \; + \; 2\int_{R_e} \bar{v}_3^{(k)} \; \sigma_{33}^{-(k-1)} dR_e \, \}$$

$$+ \sum_{k=1}^{N} \{ \, -2\int_{R_e} \bar{v}_\gamma^{(k)} \; \sigma_{\gamma 3}^{-(k)} dR_e \; + \; 2\int_{R_e} \bar{\phi}_\gamma^{(k)} \; \frac{t_k}{2} \sigma_{\gamma 3}^{-(k)} dR_e \; - \; 2\int_{R_e} \bar{v}_3^{(k)} \; \sigma_{33}^{-(k)} dR_e \, \}$$

$$+ \sum_{k=1}^{N-1} \{ \int_{R_e} \sigma_{\rho 3}^{-(k)} \; \Xi_{11}^{(k)} \sigma_{\gamma 3}^{-(k)} dR_e + 2\int_{R_e} \sigma_{33}^{-(k)} \; \Xi_{21}^{(k)} \sigma_{\gamma 3}^{-(k)} dR_e + \int_{R_e} \sigma_{33}^{-(k)} \; \Xi_{22}^{(k)} \sigma_{33}^{-(k)} dR_e \}$$

$$+ \sum_{k=2}^{N-1} \{ 2\int_{R_e} \sigma_{\rho 3}^{-(k)} \; \Lambda_{11}^{(k)} \sigma_{\gamma 3}^{-(k-1)} dR_e + 2\int_{R_e} \sigma_{33}^{-(k)} \; \Lambda_{21}^{(k)} \sigma_{\gamma 3}^{-(k-1)} dR_e$$

$$+ \; 2\int_{R_e} \sigma_{33}^{-(k-1)} \; \bar{\Lambda}_{21}^{(k)} \sigma_{\gamma 3}^{-(k)} dR_e + 2\int_{R_e} \sigma_{33}^{-(k)} \; \Lambda_{22}^{(k)} \sigma_{33}^{-(k-1)} dR_e \}$$

$$+ \; 2\int_{R_e} \sigma_{\rho 3}^{-(N-1)} \; \bar{\Lambda}_{11}^{(N)} \sigma_{\gamma 3}^{-(N)} dR_e + 2\int_{R_e} \sigma_{\gamma 3}^{-(N-1)} \; \bar{\Lambda}_{12}^{(N)} \sigma_{33}^{-(N)} dR_e$$

$$+ \; 2\int_{R_e} \sigma_{33}^{-(N-1)} \; \bar{\Lambda}_{21}^{(N)} \sigma_{\gamma 3}^{-(N)} dR_e + 2\int_{R_e} \sigma_{33}^{-(N-1)} \; \bar{\Lambda}_{22}^{(N)} \sigma_{33}^{-(N)} dR_e$$

$$+ \; 2\int_{R_e} \sigma_{\rho 3}^{-(1)} \; \Lambda_{11}^{(1)} \sigma_{\gamma 3}^{(0)} dR_e + 2\int_{R_e} \sigma_{\gamma 3}^{-(1)} \; \Lambda_{12}^{(1)} \sigma_{33}^{(0)} dR_e$$

$$+ \; 2\int_{R_e} \sigma_{33}^{-(1)} \; \Lambda_{21}^{(1)} \sigma_{\gamma 3}^{(0)} dR_e + 2\int_{R_e} \sigma_{33}^{-(1)} \; \Lambda_{22}^{(1)} \sigma_{33}^{(0)} dR_e$$

$$- \sum_{k=1}^{N} \{ 2\int_{S_1 \cap S_e} \bar{v}_\alpha^{(k)} \; g_1^{(k)} dS_{1e} + 2\int_{S_3 \cap S_e} \bar{\phi}_\alpha^{(k)} \; g_3^{(k)} dS_{3e}$$

$$+ \; 2\int_{S_5 \cap S_e} \bar{v}_3^{(k)} \; g_5^{(k)} dS_{5e} \, \}$$

$$+ \; (\frac{1}{120} + \frac{1}{840}) \int_{S_i} (\sigma_{\gamma 3}^{-(1)} \eta_\gamma t_1^3 S_{3333}^{(1)} \sigma_{\rho 3,\rho}^{-(1)})' dS_i$$

$$+ \; (\frac{1}{120} + \frac{1}{840}) \int_{S_i} (\sigma_{\gamma 3}^{-(1)} \eta_\gamma t_2^3 S_{3333}^{(2)} \sigma_{\rho 3,\rho}^{-(1)})' dS_i$$

$$+ \sum_{k=2}^{N\;2} \{ -2(\frac{1}{120} - \frac{1}{840}) \int_{S_i} (\sigma_{\Gamma 3}^{-(k)} \eta_\gamma t_k^3 S_{3333}^{(k)} \sigma_{\rho 3,\rho}^{-(k\;1)})' dS_i$$

$$+\left(\frac{1}{120}+\frac{1}{840}\right)\int_{S_i}(\sigma_{\Gamma 3}^{-(k)}\eta_\gamma t_k^3 S_{3333}^{(k)}\sigma_{\rho 3,\rho}^{-(k)})' dS_i$$

$$+\left(\frac{1}{120}+\frac{1}{840}\right)\int_{S_i}(\sigma_{\gamma 3}^{-(k)}\eta_\gamma t_{k+1}^3 S_{3333}^{(k+1)}\sigma_{\rho 3,\rho}^{-(k)})' dS_i \}$$

$$-2\left(\frac{1}{120}-\frac{1}{840}\right)\int_{S_i}(\sigma_{\gamma 3}^{-(N-1)}\eta_\gamma t_{N-1}^3 S_{3333}^{(N-1)}\sigma_{\rho 3,\rho}^{-(N-2)})' dS_i$$

$$+\left(\frac{1}{120}+\frac{1}{840}\right)\int_{S_i}(\sigma_{\gamma 3}^{-(N-1)}\eta_\gamma t_{N-1}^3 S_{3333}^{(N-1)}\sigma_{\rho 3,\rho}^{-(N-1)})' dS_i$$

$$+\left(\frac{1}{120}+\frac{1}{840}\right)\int_{S_i}(\sigma_{\gamma 3}^{-(N-1)}\eta_\gamma t_N^3 S_{3333}^{(N)}\sigma_{\rho 3,\rho}^{-(N-1)})' dS_i \tag{283}$$

Substituting (277) through (281) into (283), the spatially discretized governing functional is obtained.

$$- \{\sigma_{33}\}_e^{-(k)T}\frac{K_{33\gamma 3}^{(k)}}{24}\{\sigma_{\gamma 3}\}_e^{-(k-1)}+\{\sigma_{\gamma 3}\}_e^{-(k)T}\frac{K_{\gamma 3\gamma 3}^{(k)}}{144}\{\sigma_{\gamma 3}\}_e^{-(k-1)}$$

$$- \{\sigma_{\gamma 3}\}_e^{-(k-1)T}\frac{K_{\gamma 3\gamma 3}^{(k)}}{144}\{\sigma_{\gamma 3}\}_e^{-(k-1)}+\{\overline{v}_\alpha\}_e^{(k)T}K_{v33}^{(k)}\{\sigma_{33}\}_e^{-(k-1)}$$

$$- \{\sigma_{33}\}_e^{-(k-1)T}\frac{K_{3333}^{(k)}}{4}\{\sigma_{33}\}_e^{-(k-1)}-\{\sigma_{33}\}_e^{-(k)T}\frac{K_{3333}^{(k)}}{4}\{\sigma_{33}\}_e^{-(k-1)}$$

$$+ \{\sigma_{\gamma 3}\}_e^{-(k)T}\frac{K_{\gamma 333}^{(k)}}{24}\{\sigma_{33}\}_e^{-(k-1)}-\{\sigma_{\gamma 3}\}_e^{-(k-1)T}\frac{K_{\gamma 333}^{(k)}}{24}\{\sigma_{33}\}_e^{-(k-1)}$$

$$+ \{\overline{\phi}_\alpha\}_e^{(k)T}K_{D\phi\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k-1)}-\{\sigma_{33}\}_e^{-(k-1)T}\frac{K_{33\gamma 3}^{(k)}}{100}\{\sigma_{\gamma 3}\}_e^{-(k-1)}$$

$$+ \{\sigma_{33}\}_e^{-(k)T}\frac{K_{33\gamma 3}^{(k)}}{100}\{\sigma_{\gamma 3}\}_e^{-(k-1)}-\{\sigma_{\gamma 3}\}_e^{-(k-1)T}\frac{K_{\gamma 3\gamma 3}^{(k)}}{1200}\{\sigma_{\gamma 3}\}_e^{-(k-1)}$$

$$- \{\sigma_{\gamma 3}\}_e^{-(k)T}\frac{K_{\gamma 3\gamma 3}^{(k)}}{1200}\{\sigma_{\gamma 3}\}_e^{-(k-1)}$$

$$+ \{\overline{\phi}_\alpha\}_e^{(k)T}K_{D\phi 33}^{(k)}\{\sigma_{33}\}_e^{-(k-1)}-\{\sigma_{33}\}_e^{-(k-1)T}\frac{3}{25}K_{3333}^{(k)}\{\sigma_{33}\}_e^{-(k-1)}$$

$$+ \{\sigma_{33}\}_e^{-(k)^T} \frac{3}{25} K_{3333}^{(k)} \{\sigma_{33}\}_e^{-(k-1)} - \{\sigma_{\gamma 3}\}^{-(k-1)} \frac{K_{\gamma 333}^{(k)}}{100} \{\sigma_{33}\}_e^{-(k-1)}$$

$$- \{\sigma_{\gamma 3}\}_e^{-(k)^T} \frac{K_{\gamma 333}^{(k)}}{100} \{\sigma_{33}\}_e^{-(k-1)} - \{\bar{v}_3\}_e^{(k)^T} K_{w\gamma 3}^{(k)} \{\sigma_{\gamma 3}\}_e^{-(k-1)}$$

$$- \{\bar{\phi}_\alpha\}_e^{(k)^T} K_{\phi\gamma 3}^{(k)} \{\sigma_{\gamma 3}\}_e^{-(k-1)} - \{\sigma_{\rho 3}\}_e^{-(k-1)^T} K_{\rho 3\gamma 3}^{(k)} \{\sigma_{\gamma 3}\}_e^{-(k-1)} - \{\sigma_{\rho 3}\}_e^{-(k)^T} K_{\rho 3\gamma 3}^{(k)} \{\sigma_{\gamma 3}\}_e^{-(k-1)} \quad]$$

$$-\sum_{k=1}^{N} [\quad \{\bar{v}_\alpha\}_e^{(k)^T} K_{vv}^{(k)} \{\bar{v}_\alpha\}_e^{(k)} - \{\sigma_{33}\}_e^{-(k-1)^T} K_{33v}^{(k)} \{\bar{v}_\alpha\}_e^{(k)}$$

$$- \{\sigma_{33}\}_e^{-(k)^T} K_{33v}^{(k)} \{\bar{v}_\alpha\}_e^{(k)} + \{\sigma_{\gamma 3}\}_e^{-(k)^T} K_{\gamma 3v}^{(k)} \{\bar{v}_\alpha\}_e^{(k)}$$

$$- \{\sigma_{\gamma 3}\}_e^{-(k-1)^T} K_{\gamma 3v}^{(k)} \{\bar{v}_\alpha\}_e^{(k)}$$

$$+ \{\bar{\phi}_\alpha\}_e^{(k)^T} K_{D\phi\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)} - \{\sigma_{33}\}_e^{-(k-1)^T} K_{33D\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)}$$

$$+ \{\sigma_{33}\}_e^{-(k)^T} K_{33D\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)} - \{\sigma_{\gamma 3}\}_e^{-(k-1)^T} K_{\gamma 3D\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)}$$

$$- \{\sigma_{\gamma 3}\}_e^{-(k)^T} K_{\gamma 3D\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)}$$

$$+ \{\bar{v}_3\}_e^{(k)^T} K_{ww}^{(k)} \{\bar{v}_3\}_e^{(k)} + \{\bar{v}_3\}_e^{(k)^T} K_{w\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)}$$

$$+ \{\sigma_{\gamma 3}\}_e^{-(k-1)^T} R_{\gamma 3w}^{(k)} \{\bar{v}_3\}_e^{(k)} + \{\sigma_{\gamma 3}\}_e^{-(k)^T} R_{\gamma 3w}^{(k)} \{\bar{v}_3\}_e^{(k)}$$

$$+ \{\bar{\phi}_\alpha\}_e^{(k)^T} K_{\phi w}^{(k)} \{\bar{v}_3\}_e^{(k)} + \{\bar{\phi}_\alpha\}_e^{(k)^T} K_{\phi\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)}$$

$$+ \{\sigma_{\gamma 3}\}_e^{-(k-1)^T} R_{\gamma 3\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)} + \{\sigma_{\gamma 3}\}_e^{-(k)^T} R_{\gamma 3\phi}^{(k)} \{\bar{\phi}_\alpha\}_e^{(k)} \quad]$$

$$+\sum_{k=1}^{N} [\quad -\{\bar{v}_\alpha\}_e^{(k)^T} K_{v\gamma 3}^{(k)} \{\sigma_{\gamma 3}\}_e^{-(k)} + \{\sigma_{33}\}_e^{-(k-1)^T} \frac{K_{33\gamma 3}^{(k)}}{24} \{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$+ \{\sigma_{33}\}_e^{-(k)^T} \frac{K_{33\gamma 3}^{(k)}}{24} \{\sigma_{\gamma 3}\}^{-(k)} - \{\sigma_{\gamma 3}\}_e^{-(k)^T} \frac{K_{\gamma 3\gamma 3}^{(k)}}{144} \{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$+ \{\sigma_{\gamma 3}\}_e^{-(k-1)^T} \frac{K_{\gamma 3\gamma 3}^{(k)}}{144} \{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$+ \{\bar{v}_\alpha\}_e^{(k)^T} K_{v33}^{(k)} \{\sigma_{33}\}_e^{-(k)} - \{\sigma_{33}\}_e^{-(k-1)^T} \frac{K_{3333}^{(k)}}{4} \{\sigma_{33}\}_e^{-(k)}$$

$$- \{\sigma_{33}\}_e^{-(k)^T} \frac{K_{3333}^{(k)}}{4}\{\sigma_{33}\}_e^{-(k)} + \{\sigma_{\gamma 3}\}_e^{-(k)^T} \frac{K_{\gamma 333}^{(k)}}{24}\{\sigma_{33}\}_e^{-(k)}$$

$$- \{\sigma_{\gamma 3}\}_e^{-(k-1)^T} \frac{K_{\gamma 333}^{(k)}}{24}\{\sigma_{33}\}_e^{-(k)}$$

$$+ \{\bar{\phi}_\alpha\}_e^{(k)^T} K_{D\phi\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k)} - \{\sigma_{33}\}_e^{-(k-1)^T} \frac{K_{33\gamma 3}^{(k)}}{100}\{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$+ \{\sigma_{33}\}_e^{-(k)^T} \frac{K_{33\gamma 3}^{(k)}}{100}\{\sigma_{\gamma 3}\}_e^{-(k)} - \{\sigma_{\gamma 3}\}_e^{-(k-1)^T} \frac{K_{\gamma 3\gamma 3}^{(k)}}{1200}\{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$- \{\sigma_{\gamma 3}\}_e^{-(k)^T} \frac{K_{\gamma 3\gamma 3}^{(k)}}{1200}\{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$- \{\bar{\phi}_\alpha\}_e^{(k)^T} K_{D\phi 33}^{(k)}\{\sigma_{33}\}_e^{-(k)} + \{\sigma_{33}\}_e^{-(k-1)^T} \frac{3}{25} K_{3333}^{(k)}\{\sigma_{33}\}_e^{-(k)}$$

$$- \{\sigma_{33}\}_e^{-(k)^T} \frac{3}{25} K_{3333}^{(k)}\{\sigma_{33}\}_e^{-(k)} + \{\sigma_{\gamma 3}\}_e^{-(k-1)^T} \frac{K_{\gamma 333}^{(k)}}{100}\{\sigma_{33}\}_e^{-(k)}$$

$$+ \{\sigma_{\gamma 3}\}_e^{-(k)^T} \frac{K_{\gamma 333}^{(k)}}{100}\{\sigma_{33}\}_e^{-(k)}$$

$$- \{\bar{v}_3\}_e^{(k)^T} K_{w\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k)} - \{\bar{\phi}_\alpha\}_e^{(k)^T} K_{\phi\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$- \{\sigma_{\rho 3}\}_e^{-(k-1)} K_{\rho 3\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k)} - \{\sigma_{\rho 3}\}_e^{-(k)^T} K_{\rho 3\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k)} \;]$$

$$+\sum_{k=1}^{N-1}[\; -\{\sigma_{\rho 3}\}_e^{-(k)^T}(\frac{1}{120}+\frac{1}{840})D_{\rho 3\gamma 3D}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k)} + \{\sigma_{\rho 3}\}_e^{-(k)^T}(\frac{1}{3}+\frac{1}{5})D_{\rho 3\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$- \{\sigma_{\rho 3}\}_e^{-(k)^T}(\frac{1}{120}+\frac{1}{840})D_{\rho 3\gamma 3D}^{(k+1)}\{\sigma_{\gamma 3}\}_e^{-(k)} + \{\sigma_{\rho 3}\}_e^{-(k)^T}(\frac{1}{3}+\frac{1}{5})D_{\rho 3\gamma 3}^{(k+1)}\{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$- 2\{\sigma_{33}\}_e^{-(k)^T}(\frac{1}{24}+\frac{3}{280})D_{33\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k)} + 2\{\sigma_{33}\}_e^{-(k)^T}(\frac{1}{24}+\frac{3}{280})D_{33\gamma 3}^{(k+1)}\{\sigma_{\gamma 3}\}_e^{-(k)}$$

$$+ \{\sigma_{33}\}_e^{-(k)^T}(\frac{1}{4}+\frac{17}{140})D_{3333}^{(k)}\{\sigma_{33}\}_e^{-(k)} + \{\sigma_{33}\}_e^{-(k)^T}(\frac{1}{4}+\frac{17}{140})D_{3333}^{(k+1)}\{\sigma_{33}\}_e^{-(k)} \;]$$

$$+\sum_{k=2}^{N-1}[\; 2\{\sigma_{\rho 3}\}_e^{-(k)^T}(\frac{1}{120}-\frac{1}{840})D_{\rho 3\gamma 3D}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k-1)} - 2\{\sigma_{\rho 3}\}_e^{-(k)^T}(\frac{1}{3}-\frac{1}{5})D_{\rho 3\gamma 3}^{(k)}\{\sigma_{\gamma 3}\}_e^{-(k-1)}$$

$$+\ 2\{\sigma_{33}\}_e^{-(k)^T}(\frac{1}{24}-\frac{3}{280})D_{33\gamma3}^{(k)}\{\sigma_{\gamma3}\}_e^{-(k-1)}-2\{\sigma_{33}\}_e^{-(k-1)^T}(\frac{1}{24}-\frac{3}{280})D_{33\gamma3}^{(k)}\{\sigma_{\gamma3}\}_e^{-(k)}$$

$$+\ 2\{\sigma_{33}\}_e^{-(k)^T}(\frac{1}{4}-\frac{17}{140})D_{3333}^{(k)}\{\sigma_{33}\}_e^{-(k-1)}\ ]$$

$$+\sum_{k=1}^{N}\{\ 2\{\bar{v}_\alpha\}_e^{(k)^T}R_{v\gamma3}\{\sigma_{\gamma3}\}_e^{-(k-1)}+2\{\bar{\phi}_\alpha\}_e^{(k)^T}6R_{\phi\gamma3}^{(k)}\{\sigma_{\gamma3}\}_e^{-(k-1)}+2\{\bar{v}_3\}_e^{(k)^T}R_{w33}\{\sigma_{33}\}_e^{-(k-1)}\ \}$$

$$+\sum_{k=1}^{N}\{\ -2\{\bar{v}_\alpha\}_e^{(k)^T}R_{v\gamma3}\{\sigma_{\gamma3}\}_e^{-(k)}+2\{\bar{\phi}_\alpha\}_e^{(k)^T}6R_{\phi\gamma3}^{(k)}\{\sigma_{\gamma3}\}_e^{-(k)}-2\{\bar{v}_3\}_e^{(k)^T}R_{w33}\{\sigma_{33}\}_e^{-(k)}\ \}$$

$$+\ 2\{\sigma_{\rho3}\}_e^{-(N-1)^T}(\frac{1}{120}-\frac{1}{840})D_{\rho3\gamma3D}^{(N)}\{\sigma_{\gamma3}\}_e^{-(N)}-2\{\sigma_{\rho3}\}_e^{-(N-1)^T}(\frac{1}{3}-\frac{1}{5})D_{\rho3\gamma3}^{(N)}\{\sigma_{\gamma3}\}_e^{-(N)}$$

$$+\ 2\{\sigma_{\gamma3}\}_e^{-(N-1)^T}(\frac{3}{280}-\frac{1}{24})D_{\gamma333D}^{(N)}\{\sigma_{33}\}_e^{-(N)}$$

$$+\ 2\{\sigma_{33}\}_e^{-(N-1)^T}(\frac{3}{280}-\frac{1}{24})D_{33\gamma3}^{(N)}\{\sigma_{\gamma3}\}_e^{-(N)}+2\{\sigma_{33}\}_e^{-(N-1)^T}(\frac{1}{4}-\frac{17}{140})D_{3333}^{(N)}\{\sigma_{33}\}_e^{-(N)}$$

$$+\ 2\{\sigma_{\rho3}\}_e^{-(1)^T}(\frac{1}{120}-\frac{1}{840})D_{\rho3\gamma3D}^{(1)}\{\sigma_{\gamma3}\}_e^{(0)}-2\{\sigma_{\rho3}\}_e^{-(1)^T}(\frac{1}{3}-\frac{1}{5})D_{\rho3\gamma3}^{(1)}\{\sigma_{\gamma3}\}_e^{(0)}$$

$$+\ 2\{\sigma_{\gamma3}\}_e^{-(1)^T}(\frac{1}{24}-\frac{3}{280})D_{\gamma333D}^{(1)}\{\sigma_{33}\}_e^{(0)}+2\{\sigma_{33}\}_e^{-(1)^T}(\frac{1}{24}-\frac{3}{280})D_{33\gamma3}^{(1)}\{\sigma_{\gamma3}\}_e^{(0)}$$

$$+\ 2\{\sigma_{33}\}_e^{-(1)^T}(\frac{1}{4}-\frac{17}{140})D_{3333}^{(1)}\{\sigma_{33}\}_e^{(0)}\ ]$$

$$-\sum_{k=1}^{N}[\ 2\{\bar{v}_\alpha\}_e^{(k)^T}R_{vn}^{(k)}+2\{\bar{\phi}_\alpha\}_e^{(k)^T}R_{\phi n}^{(k)}+2\{\bar{v}_3\}_e^{(k)^T}R_{wn}^{(k)}\ ]$$

$$+\ \{\sigma_{\gamma3}\}_e^{-(1)^T}(\frac{1}{120}+\frac{1}{840})K_{\gamma3\rho3J}^{(1)}\{\sigma_{\rho3}\}_e^{-(1)}$$

$$+\ \{\sigma_{\gamma3}\}_e^{-(1)^T}(\frac{1}{120}+\frac{1}{840})K_{\gamma3\rho3J}^{(2)}\{\sigma_{\rho3}\}_e^{-(1)}$$

$$+\sum_{k=2}^{N-2}\ \{-2\{\sigma_{\gamma3}\}_e^{-(k)^T}(\frac{1}{120}-\frac{1}{840})K_{\gamma3\rho3J}^{(k)}\{\sigma_{\rho3}\}_e^{-(k-1)}$$

$$+\ \{\sigma_{\gamma3}\}_e^{-(k)^T}(\frac{1}{120}+\frac{1}{840})K_{\gamma3\rho3J}^{(k)}\{\sigma_{\rho3}\}_e^{-(k)}$$

$$+ \{\sigma_{\gamma 3}\}_e^{-(k)^T}(\frac{1}{120}+\frac{1}{840})K^{(k+1)}_{\gamma 3\rho 3J}\{\sigma_{\rho 3}\}_e^{-(k)} \}$$

$$- 2\{\sigma_{\gamma 3}\}_e^{-(N-1)^T}(\frac{1}{120}-\frac{1}{840})K^{(N-1)}_{\gamma 3\rho 3J}\{\sigma_{\rho 3}\}_e^{-(N-2)}$$

$$+ \{\sigma_{\gamma 3}\}_e^{-(N-1)^T}(\frac{1}{120}+\frac{1}{840})K^{(N-1)}_{\gamma 3\rho 3J}\{\sigma_{\rho 3}\}_e^{-(N-1)}$$

$$+ \{\sigma_{\gamma 3}\}_e^{-(N-1)^T}(\frac{1}{120}+\frac{1}{840})K^{(N)}_{\gamma 3\rho 3J}\{\sigma_{\rho 3}\}_e^{-(N-1)} \tag{284}$$

where

$$K^{(k)}_{v\gamma 3} = \int_{R_e} [D\psi_v]_e \frac{t_k^2}{12} C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\alpha\beta 33} [\psi_{\gamma 3,\gamma}]_e^T dR_e \tag{285}$$

$$K^{(k)}_{\gamma 3v} = \int_{R_e} [\psi_{\gamma 3,\gamma}]_e \frac{t_k^2}{12} C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\mu\rho 33} [D\psi_v]_e^T dR_e \tag{286}$$

$$K^{(k)}_{v33} = \int_{R_e} [D\psi_v]_e \frac{t_k}{2} C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\alpha\beta 33} [\psi_{33}]_e^T dR_e \tag{287}$$

$$K^{(k)}_{33v} = \int_{R_e} [\psi_{33}]_e \frac{t_k}{2} C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\mu\rho 33} [D\psi_v]_e^T dR_e \tag{288}$$

$$K^{(k)}_{33\gamma 3} = \int_{R_e} [\psi_{33}]_e t_k^2 C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\mu\rho 33} S^{(k)}_{\alpha\beta 33} [\psi_{\gamma 3,\gamma}]_e^T dR_e \tag{289}$$

$$K^{(k)}_{3333} = \int_{R_e} [\psi_{33}]_e t_k C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\mu\rho 33} S^{(k)}_{\alpha\beta 33} [\psi_{33}]_e^T dR_e \tag{290}$$

$$K^{(k)}_{\gamma 3\gamma 3} = \int_{R_e} [\psi_{\gamma 3,\gamma}]_e t_k^3 C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\mu\rho 33} S^{(k)}_{\alpha\beta 33} [\psi_{\gamma 3,\gamma}]_e^T dR_e \tag{291}$$

$$K^{(k)}_{\gamma 333} = \int_{R_e} [\psi_{\gamma 3,\gamma}]_e t_k^2 C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\mu\rho 33} S^{(k)}_{\alpha\beta 33} [\psi_{33}]_e^T dR_e \tag{292}$$

$$K^{(k)}_{D\phi\gamma 3} = \int_{R_e} [D\psi_\phi]_e \frac{t_k^3}{120} C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\alpha\beta 33} [\psi_{\gamma 3,\gamma}]_e^T dR_e \tag{293}$$

$$K^{(k)}_{\gamma 3D\phi} = \int_{R_e} [\psi_{\gamma 3,\gamma}]_e \frac{t_k^3}{120} C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\mu\rho 33} [D\psi_\phi]_e^T dR_e \tag{294}$$

$$K^{(k)}_{D\phi 33} = \int_{R_e} [D\psi_\phi]_e \frac{t_k^2}{10} C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\alpha\beta 33} [\psi_{33}]_e^T dR_e \tag{295}$$

$$K^{(k)}_{33D\phi} = \int_{R_e} [\psi_{33}]_e \frac{t_k^2}{10} C^{(k)}_{\mu\rho\alpha\beta} S^{(k)}_{\mu\rho 33} [D\psi_\phi]_e^T dR_e \tag{296}$$

$$K^{(k)}_{w\gamma 3} = \int_{R_e} [\psi_{w,\rho}]_e \frac{t_k}{12} C^{(k)}_{\rho 3\alpha 3} S^{(k)}_{\alpha 3\gamma 3} [\psi_{\gamma 3}]_e^T dR_e \tag{297}$$

$$K^{(k)}_{\phi\gamma 3} = \int_{R_e} [\psi_\phi]_e \frac{t_k}{12} C^{(k)}_{\rho 3\alpha 3} S^{(k)}_{\alpha 3\gamma 3} [\psi_{\gamma 3}]_e^T dR_e \tag{298}$$

$$K^{(k)}_{\rho 3\gamma 3} = \int_{R_e} [\psi_{\rho 3}]_e \frac{t_k}{30} S^{(k)}_{\rho 3\gamma 3} [\psi_{\gamma 3}]_e^T dR_e \tag{299}$$

$$K^{(k)}_{vv} = \int_{R_e} [D\psi_v]_e t_k C^{(k)}_{\mu\rho\alpha\beta} [D\psi_v]_e^T dR_e \tag{300}$$

$$K^{(k)}_{D\phi\phi} = \int_{R_e} [D\psi_\phi]_e \frac{t_k^3}{12} C^{(k)}_{\mu\rho\alpha\beta} [D\psi_\phi]_e^T dR_e \tag{301}$$

$$K^{(k)}_{ww} = \int_{R_e} [\psi_{w,\gamma}]_e \frac{5t_k}{24} C^{(k)}_{\rho 3\gamma 3} [\psi_{w,\rho}]_e^T dR_e \tag{302}$$

$$K^{(k)}_{\phi\phi} = \int_{R_e} [\psi_\phi]_e \frac{5t_k}{24} C^{(k)}_{\rho 3\gamma 3} [\psi_\phi]_e^T dR_e \tag{303}$$

$$K^{(k)}_{w\phi} = \int_{R_e} [\psi_{w,\gamma}]_e \frac{5t_k}{24} C^{(k)}_{\rho 3\gamma 3} [\psi_\phi]_e^T dR_e \tag{304}$$

$$K^{(k)}_{\phi w} = \int_{R_e} [\psi_\phi]_e \frac{5t_k}{24} C^{(k)}_{\rho 3\gamma 3} [\psi_{w,\rho}]_e^T dR_e \tag{305}$$

$$K^{(k)}_{\gamma 3\rho 3J} = \int_{S_{I_e}} [\psi_{\gamma 3}]_e \eta_\gamma t_k^3 S^{(k)}_{3333} [\psi_{\rho 3,\rho}]_e^T dS_{I_e} \tag{306}$$

$$D^{(k)}_{\rho 3\gamma 3D} = \int_{R_e} [\psi_{\rho 3}]_e S^{(k)}_{3333} t_k^3 [\psi_{\gamma 3,\gamma\rho}]_e^T dR_e \tag{307}$$

$$D^{(k)}_{\rho 3\gamma 3} = \int_{R_e} [\psi_{\rho 3}]_e t_k S^{(k)}_{\rho 3\gamma 3} [\psi_{\gamma 3}]_e^T dR_e \tag{308}$$

$$D^{(k)}_{3333} = \int_{R_e} [\psi_{33}]_e t_k S^{(k)}_{3333} [\psi_{33}]_e^T dR_e \tag{309}$$

$$D^{(k)}_{\gamma 333} = \int_{R_e} [\psi_{\gamma 3,\gamma}]_e t_k^2 S^{(k)}_{3333} [\psi_{33}]_e^T dR_e \tag{310}$$

$$D^{(k)}_{33\gamma 3} = \int_{R_e} [\psi_{33}]_e t_k^2 S^{(k)}_{3333} [\psi_{\gamma 3,\gamma}]_e^T dR_e \tag{311}$$

$$D^{(k)}_{\gamma 333D} = \int_{R_e} [\psi_{\gamma 3}]_e t_k^2 S^{(k)}_{3333} [\psi_{33,\gamma}]_e^T dR_e \tag{312}$$

$$D^{(k)}_{33\gamma 3D} = \int_{R_e} [\psi_{33,\gamma}]_e t_k^2 S^{(k)}_{3333} [\psi_{\gamma 3}]_e^T dR_e \tag{313}$$

$$R_{v\gamma 3} = \int_{R_e} [\psi_v]_e \, [\psi_{\gamma 3}]_e^T dR_e \tag{314}$$

$$R^{(k)}_{w\gamma 3} = \int_{R_e} [\psi_{w,\gamma}]_e \frac{t_k}{12} [\psi_{\gamma 3}]_e^T dR_e \tag{315}$$

$$R^{(k)}_{\gamma 3w} = \int_{R_e} [\psi_{\gamma 3}]_e \frac{t_k}{12} [\psi_{w,\gamma}]_e^T dR_e \tag{316}$$

$$R_{w33} = \int_{R_e} [\psi_w]_e [\psi_{33}]_e^T dR_e \tag{317}$$

$$R^{(k)}_{\phi\gamma 3} = \int_{R_e} [\psi_\phi]_e \frac{t_k}{12} [\psi_{\gamma 3}]_e^T dR_e \tag{318}$$

$$R^{(k)}_{\gamma 3\phi} = \int_{R_e} [\psi_{\gamma 3}]_e \frac{t_k}{12} [\psi_\phi]_e^T dR_e \tag{319}$$

$$R^{(k)}_{vn} = \int_{S_1 \cap S_e} [\psi_v]_e \; g_1^{(k)} dS_{1e} \tag{320}$$

$$R^{(k)}_{\phi n} = \int_{S_3 \cap S_e} [\psi_\phi]_e \; g_3^{(k)} dS_{3e} \tag{321}$$

$$R^{(k)}_{wn} = \int_{S_5 \cap S_e} [\psi_w]_e \; g_5^{(k)} dS_{5e} \tag{322}$$

The spatially discretized functional (284) can expressed in matrix form as

$$\Omega_e = -\{U\}_e^T [K]_e \{U\}_e + 2 \{U\}_e^T \{F\}_e \tag{323}$$

where

$$
[K]_e = \begin{bmatrix}
[\eta]^{(1)} & [\gamma]^{(1)} & [\pi]^{(1)} & 0 & 0 & 0 & 0 & 0 & & & & \\
[\gamma]^{(1)^T} & [\alpha]^{(1)} & [\beta]^{(1)} & 0 & 0 & 0 & 0 & 0 & & & & \\
[\pi]^{(1)^T} & [\beta]^{(1)^T} & [\lambda]^{(1)} & [\gamma]^{(2)} & [\zeta]^{(2)} & 0 & 0 & 0 & & & & \\
0 & 0 & [\gamma]^{2^T} & [\alpha]^{(2)} & [\beta]^{(2)} & 0 & 0 & 0 & & & & \\
0 & 0 & [\zeta]^{2^T} & [\beta]^{(2)^T} & [\lambda]^{(2)} & [\gamma]^{(3)} & [\zeta]^{(3)} & 0 & & & & \\
0 & 0 & 0 & 0 & [\gamma]^{(3)^T} & [\alpha]^{(3)} & [\beta]^{(3)} & . & & & & \\
 & & & 0 & [\zeta]^{(3)^T} & [\beta]^{(3)^T} & [\lambda]^{(3)} & . & & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & [\gamma]^{(4)^T} & . & [\gamma]^{(N-1)} & [\zeta]^{(N-1)} & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 & . & [\alpha]^{(N-1)} & [\beta]^{(N-1)} & 0 & 0 \\
 & & & & & & & . & [\beta]^{(N-1)^T} & [\lambda]^{(N-1)} & [\gamma]^{(N)} & [\pi]^{(N)} \\
 & & & & & & & & . & [\gamma]^{(N)^T} & [\alpha]^{(N)} & [\beta]^{(N)} \\
 & & & & & & & & 0 & [\pi]^{(N)^T} & [\beta]^{(N)^T} & [\eta]^{(N)}
\end{bmatrix}
$$

$$
\{U\}_e = \begin{pmatrix} \{\sigma\}^{(0)} \\ \{u\}^{(1)} \\ \{\sigma\}^{-(1)} \\ \{u\}^{(2)} \\ \{\sigma\}^{-(2)} \\ \{u\}^{(3)} \\ \{\sigma\}^{\ (3)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \{u\}^{(N-1)} \\ \{\sigma\}^{-(N-1)} \\ \{u\}^{(N)} \\ \{\sigma\}^{-(N)} \end{pmatrix}
\qquad
\{F\}_e = \begin{pmatrix} \{0\} \\ \{R\}^{(1)} \\ \{0\} \\ \{R\}^{(2)} \\ \{0\} \\ \{R\}^{(3)} \\ \{0\} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \{R\}^{(N-1)} \\ \{0\} \\ \{R\}^{(N)} \\ \{0\} \end{pmatrix}
$$

and

$$\{u\}^{(k)} = \begin{Bmatrix} \bar{V}_\gamma^{(k)} \\ \bar{\phi}_\gamma^{(k)} \\ \bar{V}_3^{(k)} \end{Bmatrix} \quad k = 1, 2, \ . \ . \ N$$

$$\{\sigma\}^{-(k)} = \begin{Bmatrix} \sigma_{\gamma 3}^{-(k)} \\ \sigma_{33}^{-(k)} \end{Bmatrix} \quad k = 1, 2, \ . \ . \ . \ N\text{-}1$$

$$\{\sigma\}^{(0)} = \begin{Bmatrix} \sigma_{\gamma 3}^{(0)} \\ \sigma_{33}^{(0)} \end{Bmatrix}$$

$$\{\sigma\}^{-(N)} = \begin{Bmatrix} \sigma_{\gamma 3}^{(N)} \\ \sigma_{33}^{(N)} \end{Bmatrix}$$

$$\{R\}^{(k)} = \begin{Bmatrix} -R_{vn}^{(k)} \\ -R_{\phi n}^{(k)} \\ -R_{wn}^{(k)} \end{Bmatrix} \qquad k = 1, 2, 3, \ . \ . \ . \quad N \tag{324}$$

Superposed bar on a quantity in the following denotes average of the quantity itself and its transpose. Here, elements of the $[K]_e$ are explicitly,

$$[\eta]^{(1)} = \begin{bmatrix} (\frac{1}{144}+\frac{1}{1200})K_{\gamma 3\gamma 3}^{(1)}+K_{\rho 3\gamma 3}^{(1)} & (\frac{1}{24}+\frac{1}{100})K_{\gamma 333}^{(1)} \\ (\frac{1}{24}+\frac{1}{100})K_{33\gamma 3}^{(1)} & (\frac{1}{4}+\frac{3}{25})K_{3333}^{(1)} \end{bmatrix} \tag{325}$$

$$[\pi]^{(1)} = \begin{bmatrix} \pi_{11}^{(1)} & \pi_{12}^{(1)} \\ \pi_{21}^{(1)} & \pi_{22}^{(1)} \end{bmatrix} \tag{326}$$

$$\pi_{11}^{(1)} = -(\frac{1}{144}-\frac{1}{1200})K_{\gamma 3\gamma 3}^{(1)}+K_{\rho 3\gamma 3}^{(1)}-(\frac{1}{120}-\frac{1}{840})\bar{D}_{\rho 3\gamma 3D}^{(1)}+(\frac{1}{3}-\frac{1}{5})D_{\rho 3\gamma 3}^{(1)}$$

$$\pi_{12}^{(1)} = (\frac{1}{24}-\frac{1}{100})K_{\gamma 333}^{(1)}-(\frac{1}{24}-\frac{3}{280})D_{\gamma 333}^{(1)}$$

$$\pi_{21}^{(1)} = -(\frac{1}{24}-\frac{1}{100})K_{33\gamma 3}^{(1)}-(\frac{1}{24}-\frac{3}{280})D_{33\gamma 3D}^{(1)}$$

$$\pi_{22}^{(1)} = (\frac{1}{4}-\frac{3}{25})K_{3333}^{(1)}-(\frac{1}{4}-\frac{17}{140})D_{3333}^{(1)}$$

$$[\eta]^{(N)} = \begin{bmatrix} (\frac{1}{144}+\frac{1}{1200})K_{\gamma 3\gamma 3}^{(N)}+K_{\rho 3\gamma 3}^{(N)} & -(\frac{1}{24}+\frac{1}{100})K_{\gamma 333}^{(N)} \\ -(\frac{1}{24}+\frac{1}{100})K_{33\gamma 3}^{(N)} & (\frac{1}{4}+\frac{3}{25})K_{3333}^{(N)} \end{bmatrix} \tag{327}$$

$$[\pi]^{(N)} = \begin{bmatrix} \pi_{11}^{(N)} & \pi_{12}^{(N)} \\ \pi_{21}^{(N)} & \pi_{22}^{(N)} \end{bmatrix} \tag{328}$$

$$\pi_{11}^{(N)} = -(\frac{1}{144}-\frac{1}{1200})K_{\gamma 3\gamma 3}^{(N)}+K_{\rho 3\gamma 3}^{(N)}-(\frac{1}{120}-\frac{1}{840})D_{\rho 3\gamma 3D}^{(N)}+(\frac{1}{3}-\frac{1}{5})D_{\rho 3\gamma 3}^{(N)}$$

$$\pi_{12}^{(N)} = (\frac{1}{24}-\frac{1}{100})K_{\gamma 333}^{(N)}+(\frac{1}{24}-\frac{3}{280})D_{\gamma 333D}^{(N)}$$

$$\pi_{21}^{(N)} = -(\frac{1}{24}-\frac{1}{100})K_{33\gamma 3}^{(N)}+(\frac{1}{24}-\frac{3}{280})D_{33\gamma 3}^{(N)}$$

$$\pi_{22}^{(N)} = (\frac{1}{4}-\frac{3}{25})K_{3333}^{(N)}-(\frac{1}{4}-\frac{17}{140})D_{3333}^{(N)}$$

$$[\alpha]^{(k)} = \begin{bmatrix} K_{vv}^{(k)} & 0 & 0 \\ 0 & (K_{D\phi\phi}^{(k)}+K_{\phi\phi}^{(k)}) & K_{\phi w}^{(k)} \\ 0 & K_{w\phi}^{(k)} & K_{ww}^{(k)} \end{bmatrix} \quad k = 1, 2, ., \ldots N \tag{329}$$

$$[\beta]^{(k)} = \begin{bmatrix} K_{v\gamma 3}^{(k)}+R_{v\gamma 3} & -K_{v33}^{(k)} \\ -(K_{D\phi\gamma 3}^{(k)}+5K_{\phi\gamma 3}^{(k)}) & K_{D\phi 33}^{(k)} \\ K_{w\gamma 3}^{(k)} & R_{w33} \end{bmatrix} \quad k = 1, 2, ., \ldots N \tag{330}$$

$$[\gamma]^{(k)} = \begin{bmatrix} -(K^{(k)}_{\gamma 3v}+R_{\gamma 3v}) & -(K^{(k)}_{\gamma 3D\phi}+5K^{(k)}_{\gamma 3\phi}) & K^{(k)}_{\gamma 3w} \\ -K^{(k)}_{33v} & -K^{(k)}_{33D\phi} & -R_{33w} \end{bmatrix} \quad k = 1, \; 2, \ldots \; N \tag{331}$$

$$[\lambda]^{(k)} = \begin{bmatrix} \lambda^{(k)}_{11} & \lambda^{(k)}_{12} \\ \lambda^{(k)}_{12} & \lambda^{(k)}_{22} \end{bmatrix} \quad k = 1, \; 2, \; 3, \ldots \; N\text{-}1 \tag{332}$$

$$[\zeta]^{(k)} = \begin{bmatrix} \zeta^{(k)}_{11} & \zeta^{(k)}_{12} \\ \zeta^{(k)}_{12} & \zeta^{(k)}_{22} \end{bmatrix} \quad k = 2, \; 3, \ldots \; N\text{-}1 \tag{333}$$

where

$$\lambda^{(k)}_{11} = (\frac{1}{144}+\frac{1}{1200})(K^{(k)}_{gamnma3\gamma 3}+K^{(k+1)}_{\gamma 3\gamma 3})+(K^{(k)}_{\rho 3\gamma 3}+K^{(k+1)}_{\rho 3\gamma 3})$$

$$+ (\frac{1}{120}+\frac{1}{840})(\overline{D}^{(k)}_{\rho 3\gamma 3D}+\overline{D}^{(k+1)}_{\rho 3\gamma 3D})-(\frac{1}{3}+\frac{1}{5})(D^{(k)}_{\rho 3\gamma 3}+D^{(k+1)}_{\rho 3\gamma 3})$$

$$- (\frac{1}{120}+\frac{1}{840})(\overline{K}^{(k)}_{\gamma 3\rho 3J}+\overline{K}^{(k+1)}_{\gamma 3\rho 3J})$$

$$\lambda^{(k)}_{12} = (\frac{1}{24}+\frac{1}{100})(K^{(k+1)}_{\gamma 333}-K^{(k)}_{\gamma 333})+(\frac{1}{24}+\frac{3}{280})(D^{(k)}_{\gamma 333}-D^{(k+1)}_{\gamma 333})$$

$$\lambda^{(k)}_{21} = (\frac{1}{24}+\frac{1}{100})(K^{(k+1)}_{33\gamma 3}-K^{(k)}_{33\gamma 3})+(\frac{1}{24}+\frac{3}{280})(D^{(k)}_{33\gamma 3}-D^{(k+1)}_{33\gamma 3})$$

$$\lambda^{(k)}_{22} = (\frac{1}{4}+\frac{3}{25})(K^{(k+1)}_{3333}+K^{(k)}_{3333})-(\frac{1}{4}+\frac{17}{140})(D^{(k+1)}_{3333}+D^{(k)}_{3333})$$

$$\zeta^{(k)}_{11} = -(\frac{1}{144}-\frac{1}{1200})K^{(k)}_{\gamma 3\gamma 3}+K^{(k)}_{\rho 3\gamma 3}-(\frac{1}{120}-\frac{1}{840})\overline{D}^{(k)}_{\rho 3\gamma 3D}+(\frac{1}{3}-\frac{1}{5})D^{(k)}_{\rho 3\gamma 3}$$

$$+ (\frac{1}{120}-\frac{1}{840})\overline{K}^{(k)}_{\gamma 3\rho 3J}$$

$$\zeta^{(k)}_{12} = (\frac{1}{24}-\frac{1}{100})K^{(k)}_{\gamma 333}-(\frac{1}{24}-\frac{3}{280})D^{(k)}_{\gamma 333}$$

$$\zeta^{(k)}_{21} = -(\frac{1}{24}-\frac{1}{100})K^{(k)}_{33\gamma 3}+(\frac{1}{24}-\frac{3}{280})D^{(k)}_{33\gamma 3}$$

$$\zeta^{(k)}_{22} = (\frac{1}{4}-\frac{3}{25})K^{(k)}_{3333}-(\frac{1}{4}-\frac{17}{140})D^{(k)}_{3333}$$

The spatially discretized governing function for the global system is

$$\Omega_6 = \sum_{e=1}^{m} \Omega_e = -\{U\}^T[K]\{U\}+2\{U\}^T\{F\} \tag{334}$$

where $\{U\}$ is the vector of values of the field variables at the global nodal points, $\{F\}$ is the set of corresponding load vectors and $[K]$ is the system matrix corresponding to $[K]_e$ for an element. Vanishing of the differential of Ω_6 in (334) with respect to $\{U\}$ gives the set of equations

$$[K]\{U\} = \{F\} \tag{335}$$

where

$$[K] = \sum_{e=1}^{m} [K]_e \tag{336}$$

and

$$\{F\} = \sum_{e=1}^{m} [F]_e \tag{337}$$

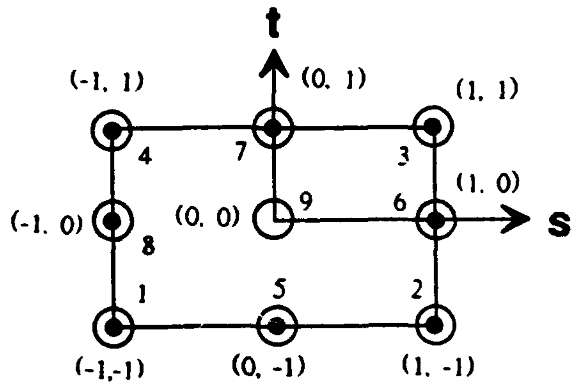
5.2 Selection of the Interpolation Scheme

5.2.1 General Considerations

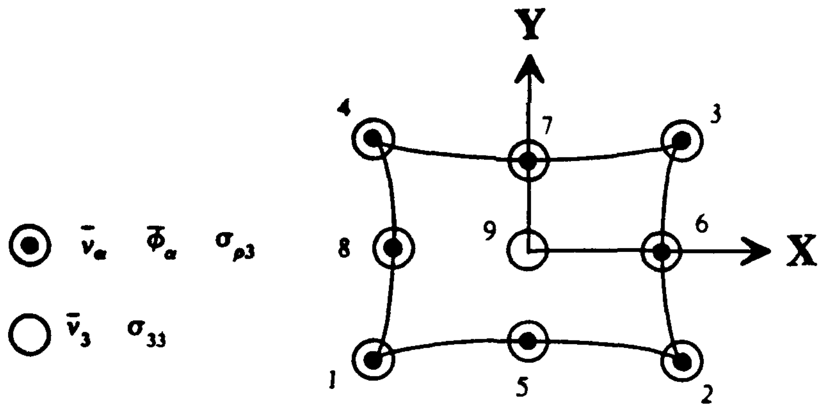
The basic requirement of selections of element to be used in the numerical solution procedure is that over each element the first order derivatives of $\bar{v}_\alpha^{(k)}$, $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$, $\bar{\sigma}_{33}^{(k)}$ and the second order derivatives for $\bar{\sigma}_{\gamma 3}^{(k)}$ include, at least, constant values. Also, $\bar{v}_\alpha^{(k)}$, $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$, $\bar{\sigma}_{33}^{(k)}$, and $\bar{\sigma}_{\gamma 3}^{(k)}$ should be continuous across interelement boundaries. This would require, for triangular elements, $\bar{v}_\alpha^{(k)}$ $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$ and $\bar{\sigma}_{33}^{(k)}$ to be linear or higher order. $\bar{\sigma}_{\gamma 3}^{(k)}$ needs to be at least quadratic. For rectangular elements, bilinear interpolants for $\bar{v}_\alpha^{(k)}$ $\bar{\phi}_\alpha^{(k)}$, $\bar{v}_3^{(k)}$ and $\bar{\sigma}_{33}^{(k)}$ would be necessary and higher order interpolants would be required for $\bar{\sigma}_{\gamma 3}^{(k)}$.

The Heterosis element introduced by Hughes [1978] is a higher order element satisfying all of the above requirements. This element has been found [Hughes 1978] to be very good for isotropic plates. It has been used for laminated plates by Hong [1988] and no comparative studies of its effectiveness in comparison with new possibilities are available. For the present application, this element was used.

In this section, the interpolation functions of the Heterosis element and the differential operator matrices involving in the (285) through (322) used to implement the finite element analysis are summarized. The finite element computer program incorporated the Heterosis plate bending element [Hughes 1978] without using reduced/selective integration technique. This element is a variant of isoparametric finite element and, therefore, element matrix can be formed following the usual procedure for isoparametric element formulation. However, the Heterosis element differs from other isoparametric elements in using different interpolation schemes for lateral displacement, $\overline{v}_3^{(k)}$, and transverse normal stress, $\tilde{\sigma}_{33}^{(k)}$. In-plane kinematic field variables $\overline{v}_\alpha^{(k)}$, $\overline{\phi}_\alpha^{(k)}$ and transverse shear stresses $\tilde{\sigma}_{\gamma 3}^{(k)}$ are approximated by nine-node Lagrange interpolation functions while the lateral displacement, $\overline{v}_3^{(k)}$ and transverse normal stress $\tilde{\sigma}_{33}^{(k)}$, are approximated by quadratic functions for eight-node isoparametric element. Figure 1 shows the geometry and the nodal points of the Heterosis element.



(a) Local



(b) Global

Figure 1: (a) Local, (b) Global Coordinate Systems of Heterosis Element

5.2.2 Interpolation function for the Heterosis Element

Interpolations functions of eight-node isoparametric element and nine-node Lagrange element in terms of natural/local coordinates (s,t) and their derivatives with respect to s and t are as follows:

$$N = \frac{1}{4}\begin{bmatrix} (1-s)(1-t)(-1-s-t) \\ (1+s)(1-t)(-1+s-t) \\ (1+s)(1+t)(-1+s+t) \\ (1-s)(1+t)(-1-s+t) \\ 2(1-s^2)(1-t) \\ 2(1+s)(1-t^2) \\ 2(1-s^2)(1+t) \\ 2(1-s)(1-t^2) \end{bmatrix} \quad \frac{\partial N}{\partial s} = \frac{1}{4}\begin{bmatrix} (1-t)(2s+t) \\ (1-t)(2s-t) \\ (1+t)(2s+t) \\ (1+t)(2s-t) \\ -4s(1-t) \\ 2(1-t^2) \\ -4s(1+t) \\ -2(1-t^2) \end{bmatrix} \quad \frac{\partial N}{\partial t} = \frac{1}{4}\begin{bmatrix} (1-s)(2t+s) \\ (1+s)(2t-s) \\ (1+s)(2t+s) \\ (1-s)(2t-s) \\ -2(1-s^2) \\ -4t(1+s) \\ 2(1-s^2) \\ -4t(1-s) \end{bmatrix} \tag{338}$$

$$L = \frac{1}{4}\begin{bmatrix} st(1-s)(1-t) \\ st(1+s)(t-1) \\ st(1+s)(1+t) \\ st(s-1)(t+1) \\ 2t(1-s^2)(t-1) \\ 2s(s+1)(1-t^2) \\ 2t(1-s^2)(t+1) \\ 2s(s-1)(1-t^2) \\ 4(1-s^2)(1-t^2) \end{bmatrix} \quad \frac{\partial L}{\partial s} = \frac{1}{4}\begin{bmatrix} t(2s-1)(t-1) \\ t(2s+1)(t-1) \\ t(2s+1)(t+1) \\ t(2s-1)(t+1) \\ 4st(1-t) \\ 2(2s+1)(1-t^2) \\ -4st(t+1) \\ 2(2s-1)(1-t^2) \\ 8s(t^2-1) \end{bmatrix} \quad \frac{\partial L}{\partial t} = \frac{1}{4}\begin{bmatrix} s(2t-1)(s-1) \\ s(2t-1)(s+1) \\ s(2t+1)(s+1) \\ s(2t+1)(s-1) \\ 2(2t-1)(1-s^2) \\ -4st(s+1) \\ 2(2t+1)(1-s^2) \\ 4st(1-s) \\ 8t(s^2-1) \end{bmatrix} \tag{339}$$

$$\frac{\partial L^2}{\partial s \partial t} = \frac{1}{4}\begin{bmatrix} (2s-1)(2t-1) \\ (2s+1)(2t-1) \\ (2s+1)(2t+1) \\ (2s-1)(2t+1) \\ 4s(1-2t) \\ -4t(2s+1) \\ -4s(2t+1) \\ -4t(2s-1) \\ 16st \end{bmatrix} \quad \frac{\partial L^2}{\partial s^2} = \frac{1}{4}\begin{bmatrix} 2t(t-1) \\ 2t(t-1) \\ 2t(t+1) \\ 2t(t+1) \\ 4t(1-t) \\ 4(1-t^2) \\ -4t(t+1) \\ 4(1-t^2) \\ 8(t^2-1) \end{bmatrix} \quad \frac{\partial L^2}{\partial t \partial t} = \frac{1}{4}\begin{bmatrix} 2s(s-1) \\ 2s(s+1) \\ 2s(s+1) \\ 2s(s-1) \\ 4(1-s^2) \\ -4s(s+1) \\ 4(1-s^2) \\ 4s(1-s) \\ 8(s^2-1) \end{bmatrix} \tag{340}$$

Here, **N** and **L** denote interpolation functions for eight-node isoparametric and the nine-node Lagrange elements, respectively.

5.3 Computer Implementation

Since the field variables are interpolated over an element in natural coordinates (s,t), it is necessary to set up the relation of the global coordinates and natural (local) coordinates for evaluation of the element matrices defined in (285) through (322). We consider a mapping of global coordinate system (x_1, x_2) to local coordinate system (s,t). We assume that this mapping is one-to-one and onto. By chain rule, the derivative in each coordinate system is related by

$$\begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} \tag{341}$$

where Jacobian matrix **J** and its inverse are defined as

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \quad \text{and} \quad \mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \end{bmatrix} \tag{342}$$

Here, $|\mathbf{J}|$ is the determinant of Jacobian matrix. Following the concept of isoparametric formulation, global coordinates are interpolated over an element as

$$x = \Psi^T \underline{x} \tag{343}$$

where Ψ is the vector of interpolation functions used for field variable. $\underline{x}$ is the vector of global coordinate values at nodal points.

Using (338) through (343), the differential operator matrices in (285) through (322), explicitly, are

$$\mathbf{H}_{v_\alpha^{(k)}} = \mathbf{H}_{\phi_\alpha^{(k)}} = \mathbf{H}_{\sigma_{\gamma 3}^{(k)}} = \begin{bmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{L} \end{bmatrix} \tag{344}$$

$$\mathbf{H}_{\sigma_{33}^{(k)}} = \mathbf{H}_w = \mathbf{N} \tag{345}$$

$$\{D\phi\}^T = \mathbf{T}_p^T = \frac{1}{|\mathbf{J}_L|}\begin{bmatrix} \underline{y}^T & 0 \\ 0 & \underline{-x}^T \\ \underline{-x}^T & \underline{y}^T \end{bmatrix}\mathbf{P} \tag{346}$$

$$\{\phi_{w,\gamma}\}^T = \mathbf{T}_s^T = \frac{1}{|\mathbf{J}_N|}\begin{bmatrix} \underline{y}^T \\ \underline{-x}^T \end{bmatrix}\mathbf{R} \tag{347}$$

$$\{\phi_{\gamma 3,\gamma}\}^T = \mathbf{T}_{pr}^T = \frac{1}{|\mathbf{J}_L|}(\underline{y}^T - \underline{x}^T)\mathbf{L} \tag{348}$$

$$\{\phi_{\gamma 3,\gamma\rho}\} = \mathbf{T}_{PRR} = \begin{bmatrix} \dfrac{\partial^2}{\partial x^2} & \dfrac{\partial^2}{\partial x \partial y} \\ \\ \dfrac{\partial^2}{\partial y \partial x} & \dfrac{\partial^2}{\partial y^2} \end{bmatrix}\mathbf{L} \tag{349}$$

where

$$|\mathbf{J}_L| = -\underline{x}^T \mathbf{P}\, \underline{y}$$

$$|\mathbf{J}_N| = -\underline{x}^T \mathbf{R}\, \underline{y}$$

$$\mathbf{P} = \mathbf{L},_t \mathbf{L},_s^T - \mathbf{L},_s \mathbf{L},_t^T$$

$$\mathbf{R} = \mathbf{N},_t \mathbf{N},_s^T - \mathbf{N},_s \mathbf{N},_t^T$$

$$\frac{\partial^2}{\partial x^2} = [\frac{A}{J}\underline{y}^T L,_t L,_t^T \underline{y} - \frac{B}{J}\underline{y}^T L,_s L,_t^T \underline{y} + \frac{1}{J^2}\underline{y}^T (L,_t L,_{ts}^T - L,_s L,_{tt}^T)\underline{y}]\frac{\partial}{\partial s}$$

$$+ [-\frac{A}{J}\underline{y}^T L,_t L,_s^T \underline{y} + \frac{B}{J}\underline{y}^T L,_s L,_s^T \underline{y} + \frac{1}{J^2}\underline{y}^T (L,_s L,_{st}^T - L,_t L,_{ss}^T)\underline{y}]\frac{\partial}{\partial t}$$

$$+ \frac{1}{J^2}\underline{y}^T L,_t L,_t^T \underline{y}\frac{\partial^2}{\partial s^2} + \frac{1}{J^2}\underline{y}^T L,_s L,_s^T \underline{y}\frac{\partial^2}{\partial t^2} - \frac{1}{J^2}\underline{y}^T (L,_s L,_t^T + L,_t L,_s^T)\underline{y}\frac{\partial^2}{\partial s \partial t}$$

and

$$A = -\frac{1}{J^2}\underline{x}^T [L,_{ss} L,_t^T + L,_s L,_{ts}^T - L,_{ts} L,_s^T - L,_t L,_{ss}^T]\underline{y}$$

$$B = -\frac{1}{J^2}\underline{x}^T [L,_{st} L,_t^T + L,_s L,_{tt}^T - L,_{tt} L,_s^T - L,_t L,_{st}^T]\underline{y}$$

$$\frac{\partial^2}{\partial y^2} = [\frac{A}{J}\underline{x}^T L_{,t} L_{,t}^T \underline{x} - \frac{B}{J}\underline{x}^T L_{,s} L_{,t}^T \underline{x} + \frac{1}{J^2}\underline{x}^T (L_{,t} L_{,ts}^T - L_{,s} L_{,tt}^T)\underline{x}]\frac{\partial}{\partial s}$$

$$+ [-\frac{A}{J}\underline{x}^T L_{,t} L_{,s}^T \underline{x} + \frac{B}{J}\underline{x}^T L_{,s} L_{,s}^T \underline{x} + \frac{1}{J^2}\underline{x}^T (L_{,s} L_{,st}^T - L_{,t} L_{,ss}^T)\underline{x}]\frac{\partial}{\partial t}$$

$$+ \frac{1}{J^2}\underline{x}^T L_{,t} L_{,t}^T \underline{x}\frac{\partial^2}{\partial s^2} + \frac{1}{J^2}\underline{x}^T L_{,s} L_{,s}^T \underline{x}\frac{\partial^2}{\partial t^2} - \frac{1}{J^2}\underline{x}^T (L_{,s} L_{,t}^T + L_{,t} L_{,s}^T)\underline{x}\frac{\partial^2}{\partial s \partial t}$$

$$\frac{\partial^2}{\partial x \partial y} = [-\frac{A}{J}\underline{y}^T L_{,t} L_{,t}^T \underline{x} + \frac{B}{J}\underline{y}^T L_{,s} L_{,t}^T \underline{x} - \frac{1}{J^2}\underline{y}^T (L_{,t} L_{,ts}^T - L_{,s} L_{,tt}^T)\underline{x}]\frac{\partial}{\partial s}$$

$$+ [\frac{A}{J}\underline{y}^T L_{,t} L_{,s}^T \underline{x} - \frac{B}{J}\underline{y}^T L_{,s} L_{,s}^T \underline{x} - \frac{1}{J^2}\underline{y}^T (L_{,s} L_{,st}^T - L_{,t} L_{,ss}^T)\underline{x}]\frac{\partial}{\partial t}$$

$$- \frac{1}{J^2}\underline{y}^T L_{,t} L_{,t}^T \underline{x}\frac{\partial^2}{\partial s^2} - \frac{1}{J^2}\underline{y}^T L_{,s} L_{,s}^T \underline{x}\frac{\partial^2}{\partial t^2} + \frac{1}{J^2}\underline{y}^T (L_{,s} L_{,t}^T + L_{,t} L_{,s}^T)\underline{x}\frac{\partial^2}{\partial s \partial t}$$

$$\frac{\partial^2}{\partial y \partial x} = [-\frac{A}{J}\underline{x}^T L_{,t} L_{,t}^T \underline{y} + \frac{B}{J}\underline{x}^T L_{,s} L_{,t}^T \underline{y} - \frac{1}{J^2}\underline{x}^T (L_{,t} L_{,ts}^T - L_{,s} L_{,tt}^T)\underline{y}]\frac{\partial}{\partial s}$$

$$+ [\frac{A}{J}\underline{x}^T L_{,t} L_{,s}^T \underline{y} - \frac{B}{J}\underline{x}^T L_{,s} L_{,s}^T \underline{y} - \frac{1}{J^2}\underline{x}^T (L_{,s} L_{,st}^T - L_{,t} L_{,ss}^T)\underline{y}]\frac{\partial}{\partial t}$$

$$- \frac{1}{J^2}\underline{x}^T L_{,t} L_{,t}^T \underline{y}\frac{\partial^2}{\partial s^2} - \frac{1}{J^2}\underline{x}^T L_{,s} L_{,s}^T \underline{y}\frac{\partial^2}{\partial t^2} + \frac{1}{J^2}\underline{x}^T (L_{,s} L_{,t}^T + L_{,t} L_{,s}^T)\underline{y}\frac{\partial^2}{\partial s \partial t}$$

In element matrices given in (285) through (322), integrands are functions of local coordinates (s,t). Therefore, the surface integration extends over the natural coordinate surface. Since, in general,

$$dR = |J|\, ds\, dt \tag{350}$$

integration in the two coordinate systems is related by

$$\int\int_A F(x_\alpha)\, dx\, dy = \int_{-1}^{1}\int_{-1}^{1} F(s,t)\, |J| ds\, dt \tag{351}$$

For numerical evaluation of the integrals, Gaussian quadrature implies:

$$\int_R F(x,y)\, dR = \sum_{i=1}^{m}\sum_{j=i}^{m} F(s_{ij},t_{ij})\, |J_{ij}|\, |W_{ij}| \tag{352}$$

where **m** is the number of Gaussian quadrature points and W_{ij} are the corresponding weighting values. Here, it should be mentioned that in the Heterosis element numerical integration was performed. In the matrices given in (285), through (322), the highest order of numerator of the integrands in (s-t) is order of nine and the highest order of denominator is order of six. To properly perform the numerical integration, it is necessary to use as many Gaussian quadrature points as possible until convergence is attained. It is quite expensive to perform the numerical integration when the number of Gaussian quadrature points is more than three. Therefore, three-point Gaussian quadrature was used in the examples described in the next section.

SECTION VI
EXAMPLES OF APPLICATION

6.1 Introduction

The finite element formulation of the modified Pagano's [1978] theory developed in the previous section was used to obtain solutions to displacement and stress fields in delamination coupons. Three examples were solved. The first two examples involved four-ply symmetric laminates. Both cross-ply and angle-ply coupons were considered. The purpose was to validate the finite element model by comparing the numerical solutions with those from Pagano's [1978] analysis. The third example consisted of studying stress distribution in a multi-ply laminate subjected to uniform stretch. A theoretical solution for this case was not available. The results were compared with Chang [1987] who used a continuous traction element for the problem of 3-D elasticity specialized to the coupon with a potential energy minimization procedure. A stacking sequence of $[(25.5/-25.5)_5/90]_s$ was used.

6.2 Delamination Coupons

6.2.1 Four-Ply Laminates

In this section, analysis of two long symmetric laminate strips made of graphite-epoxy materials, with fiber orientations of $[45/\text{-}45]_s$ and $[0/90]_s$ under uniform displacement in the longitudinal direction is described. The relation between laminate width and thickness was 2b=16h following Pagano [1978]. In the analysis each ply was idealized as a homogeneous, elastic orthotropic material. For comparison purpose, the material properties assumed, following Pagano's work [1978], were:

$$E_{11}=20\times10^{6}\text{psi}$$

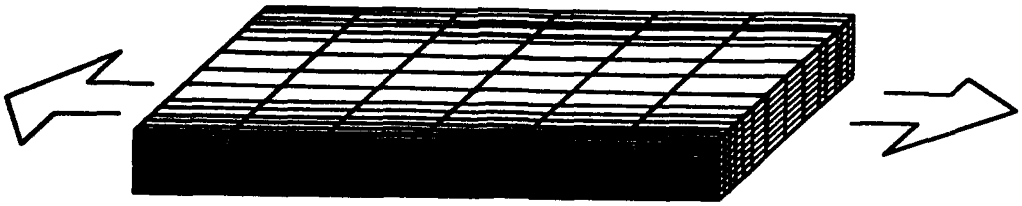
$$E_{22}=E_{33}=2.1\times10^{6}\text{psi}$$

$$G_{12}=G_{13}=G_{23}=0.85\times10^{6}\text{psi}$$

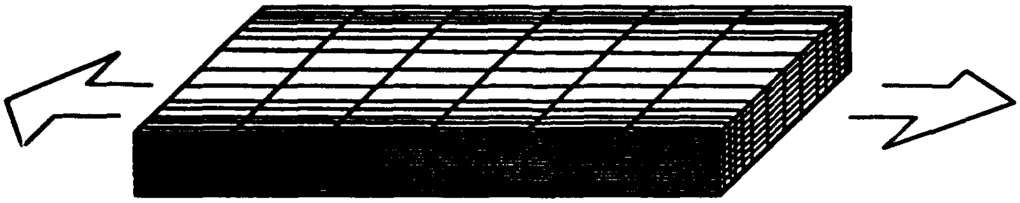
$$\nu_{12}=\nu_{13}=\nu_{23}=0.21$$

The subscripts 1, 2 and 3 denote the longitudinal, the transverse and the thickness directions respectively. The 6x14 and 6x18 finite element meshes as shown in Figure 2 were used to discretize a coupon. This corresponds to N=6 and N=4 respectively in Pagano's analysis [1978]. Numerical results based on the finite element model were compared with Pagano's [1978] analytical solution.

The value of N in the following figures corresponds to the number of sub-layers used in Pagano's theory. Thus, N=6 indicates that each physical layer of thickness h was modeled by three sub-layers each of thickness h/3, while N=2 denotes that each physical layer is treated as a unit as stated in Pagano [1978].



(a)



(b)

Figure 2: (a) 6x14 Element Mesh, (b) 6x18 Element Mesh

6.2.1.1 Cross-Ply Laminate

Distribution of σ_z along the width on the central plane of the $[0/90]_s$ laminate for N=6 shown in Figure 3, indicates a sharp rise near the free-edge boundary. Solutions obtained from the finite element model agree with Pagano's N=2 and N=6 solution over the entire width of the laminate.

Figure 4 and Figure 5 show the variations of σ_z along the interface between the 0° and 90° plies for N=2 and N=6. Due to the presence of the discontinuity in elastic properties, a singular stress behavior would be expected at the free edge.

Values of τ_{yz} along the interface between the 0/90 layers, calculated from the finite element model (Figure 6), showed that the refinement through the thickness is necessary to have satisfactory agreement with those calculated by Pagano's method.

Comparative results for the variation of transverse displacement along the top surface of the $[0/90]_s$ laminate are shown in Figure 7.

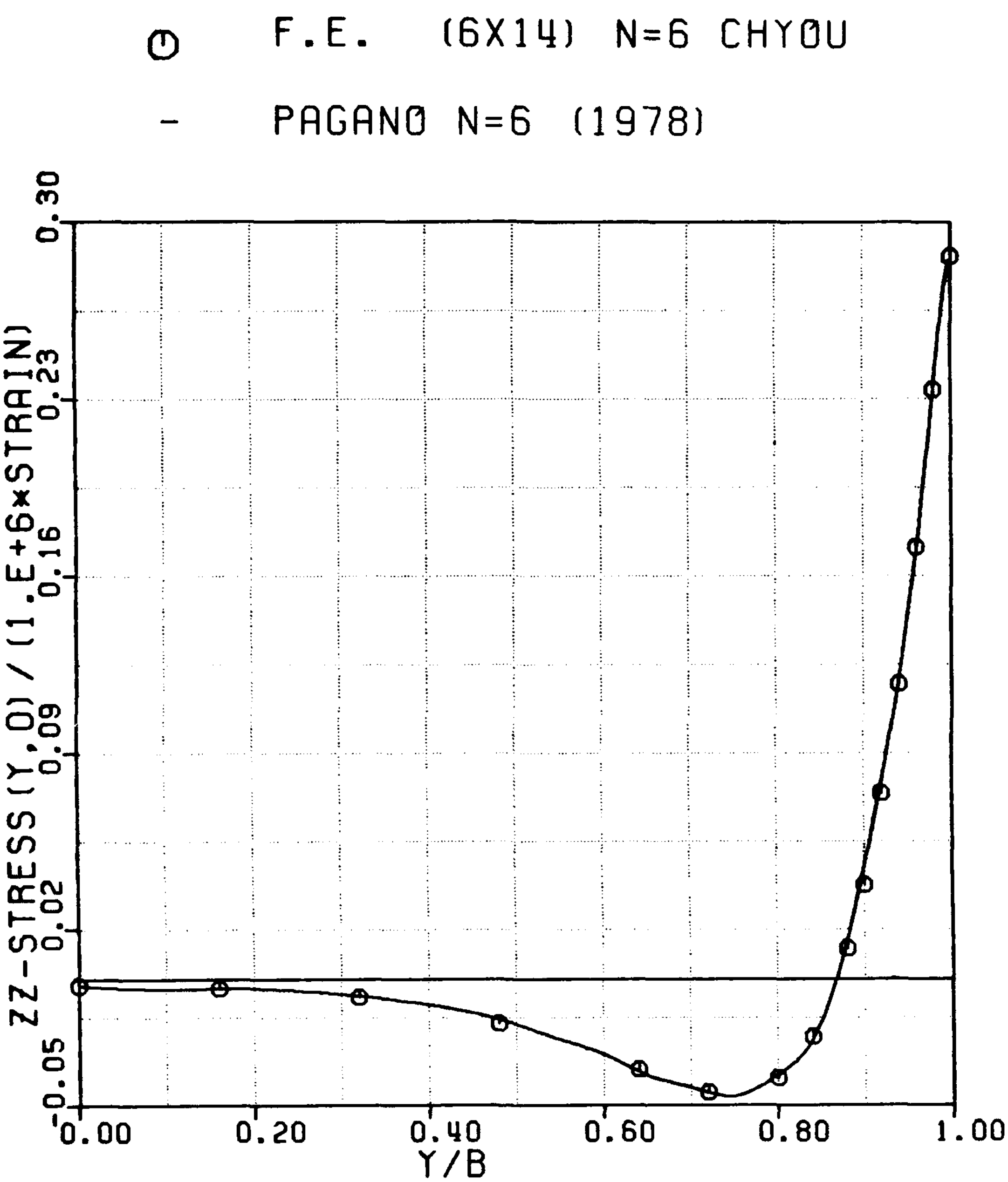


Figure 3: Distribution of Z-stress Along 90/90 Interface

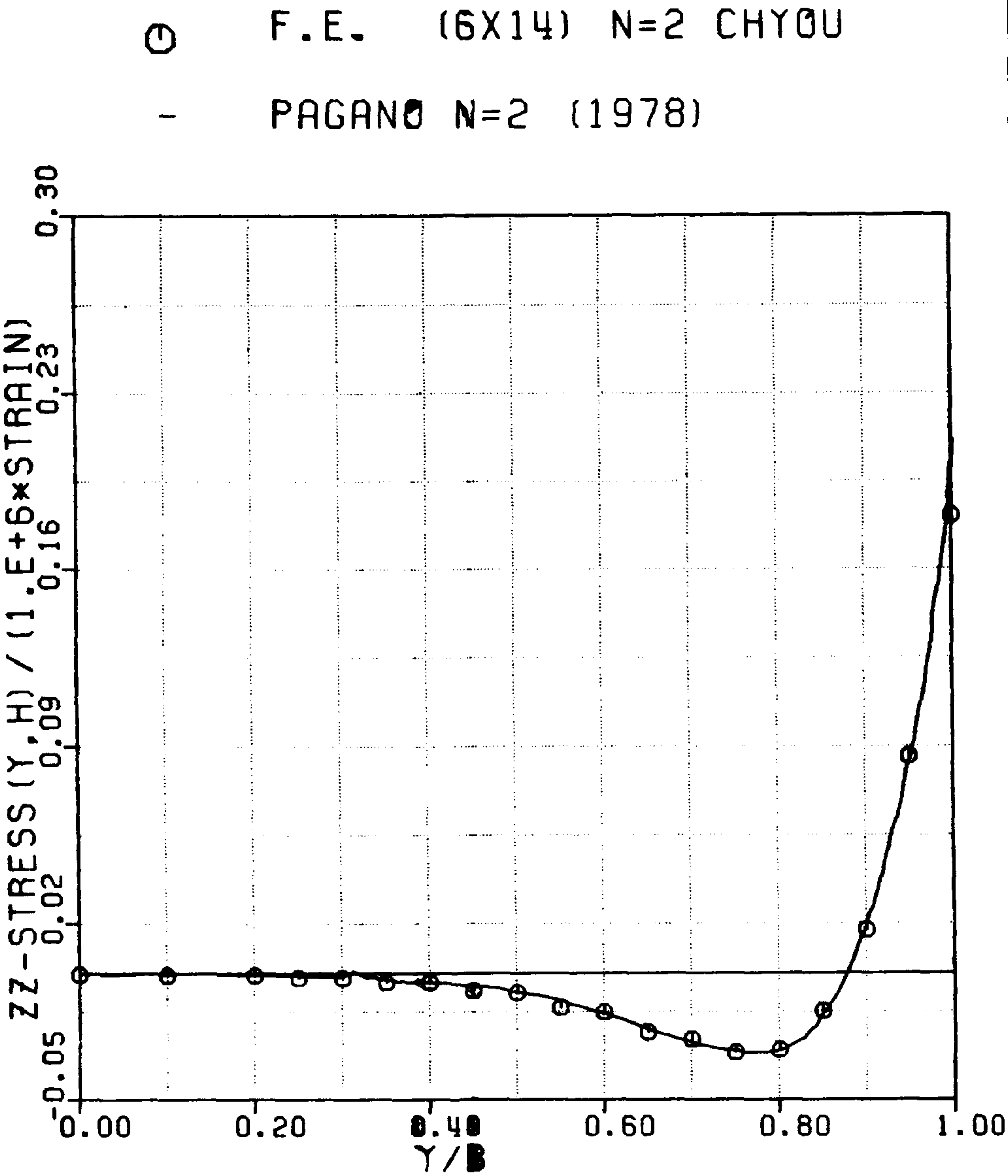


Figure 4: Distribution of Z-stress Along 0/90 Interface (N=2)

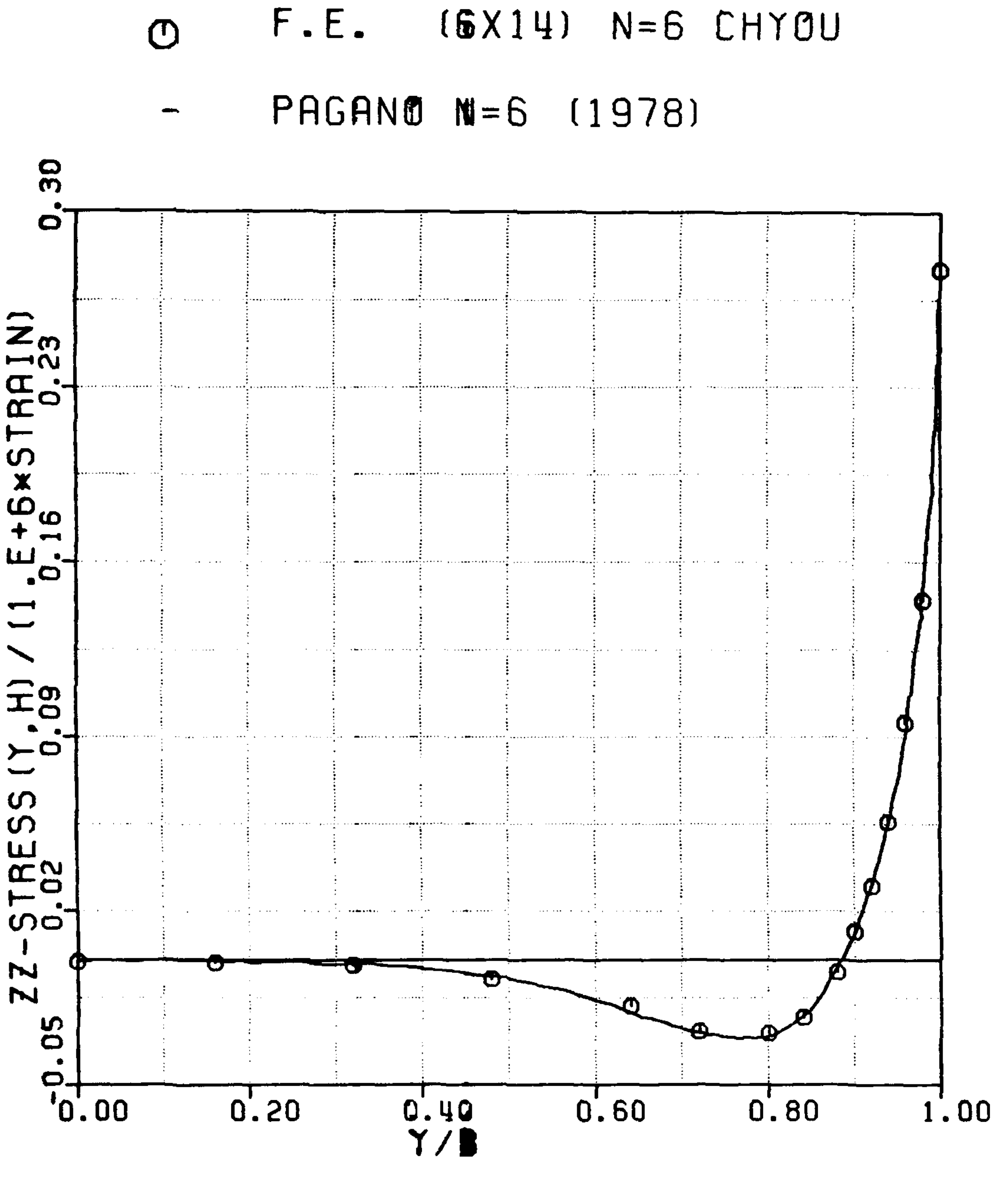


Figure 5: Distribution of Z-stress Along 0/90 Interface (N=6)

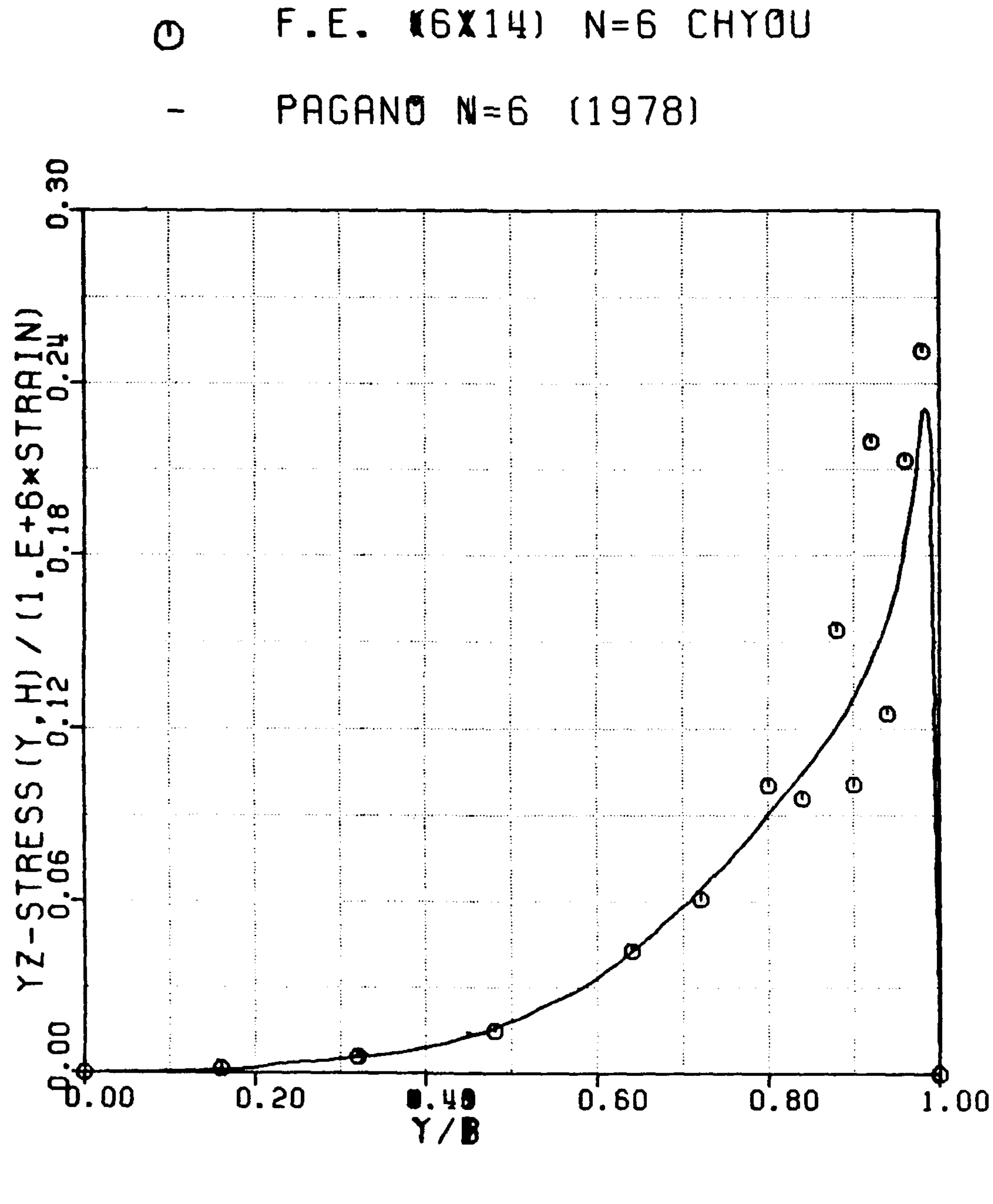


Figure 6: Distribution of YZ-stress Along the 0/90 Interface

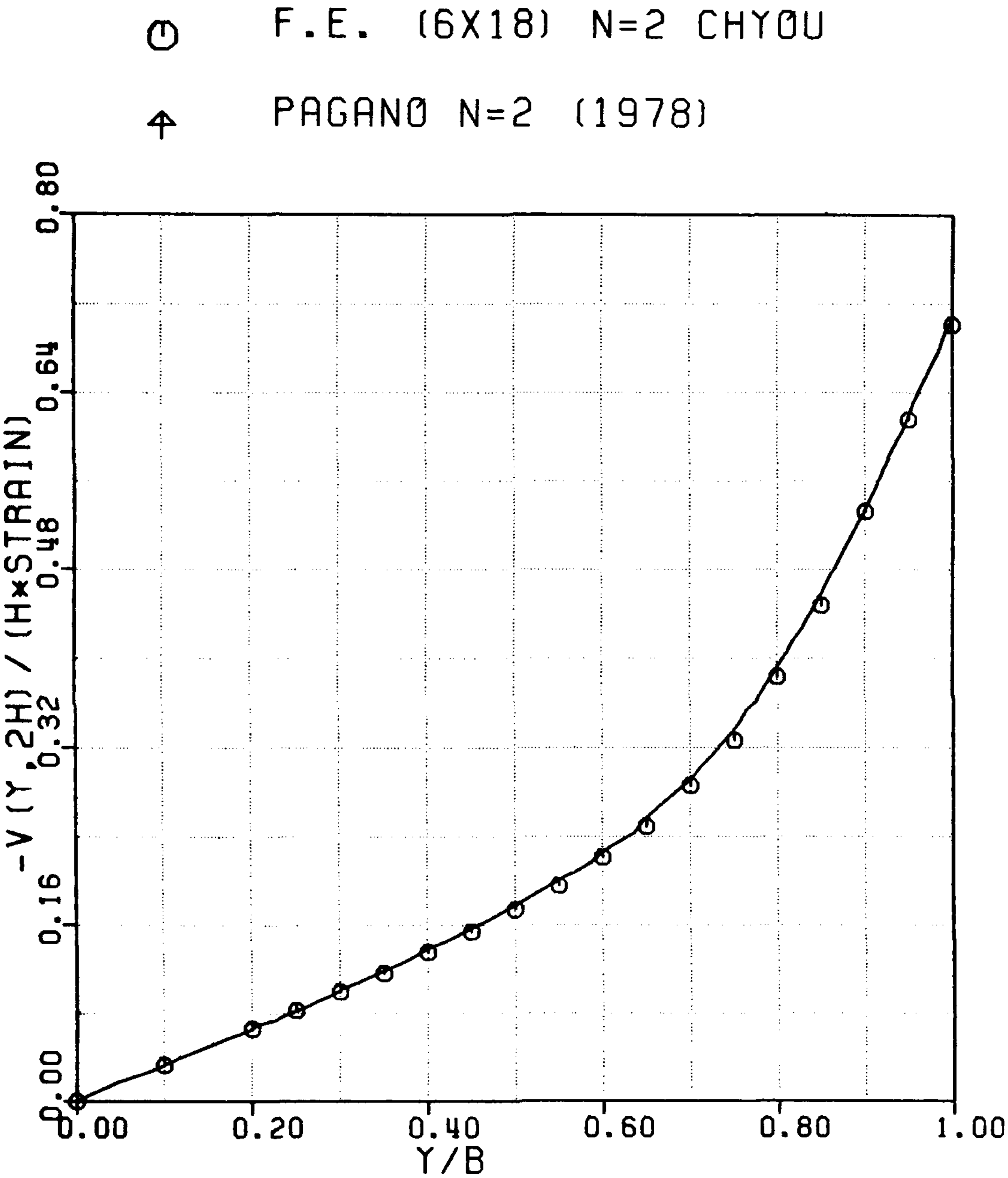


Figure 7: Transverse Displacement Across Top Surface

6.2.1.2 Angle-Ply Laminate

Figure 8 and Figure 9 show the distributions of σ_x along the width of the laminate a the center line of the top (45°) layer for N=2 and N=6. The distributions of τ_{xy} along the width of the coupon at the middle of the top (45 degree) layer for N=2 and N=6 are shown in Figure 10 and Figure 11. The results obtained using the finite element model agreed with Pagano's solutions for N=2 and N=6 across the entire width of the laminate.

A comparison of the shear stress (τ_{xz}) distributions along the interface of the 45/-45 layers for N=2 and N=6 (Figure 12 and Figure 13), indicated that the solutions of the finite element model had sharp rise toward the free-edge similar to Pagano's solutions with N=2 and N=6.

For the axial displacement distributions across the width of the top surface for N=2 and N=6, the finite element results compared well with Pagano's N=2 and N=6 solutions (Figure 14 and Figure 15).

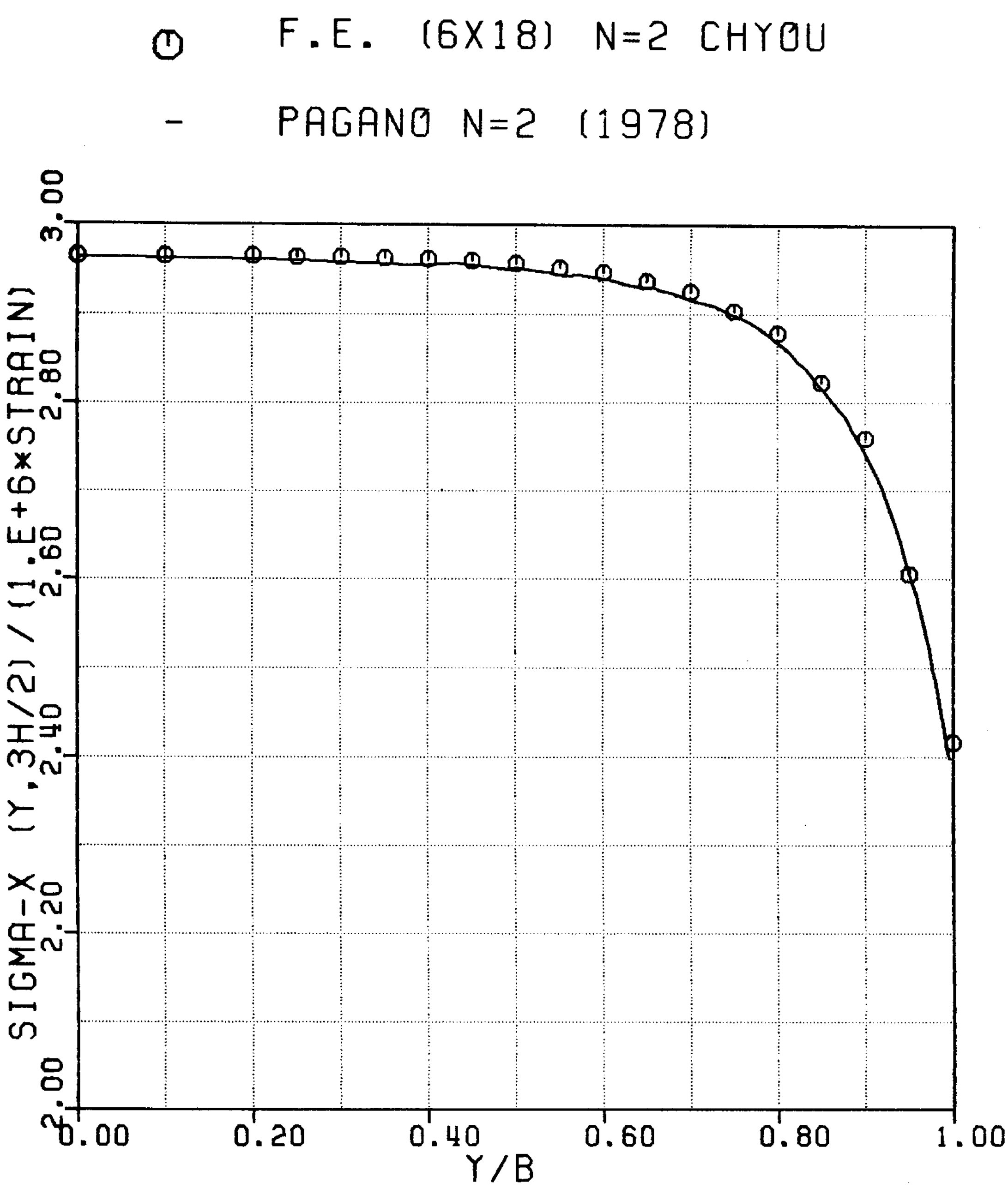


Figure 8: Distribution of X-stress Along Center of Top Layer (45 degree) for N=2

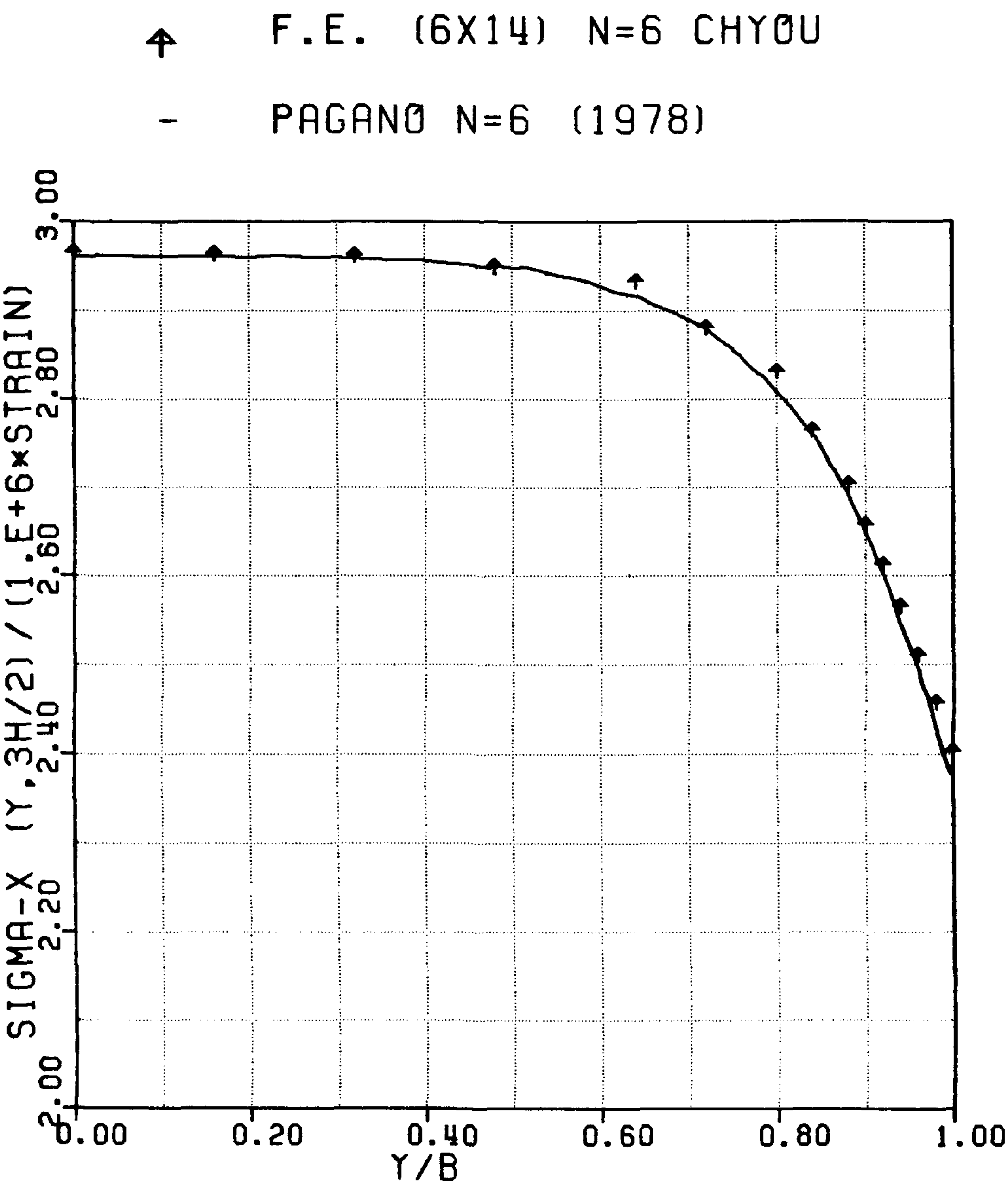


Figure 9: Distribution of X-stress Along Center of Top Layer (45 degree) for N=6

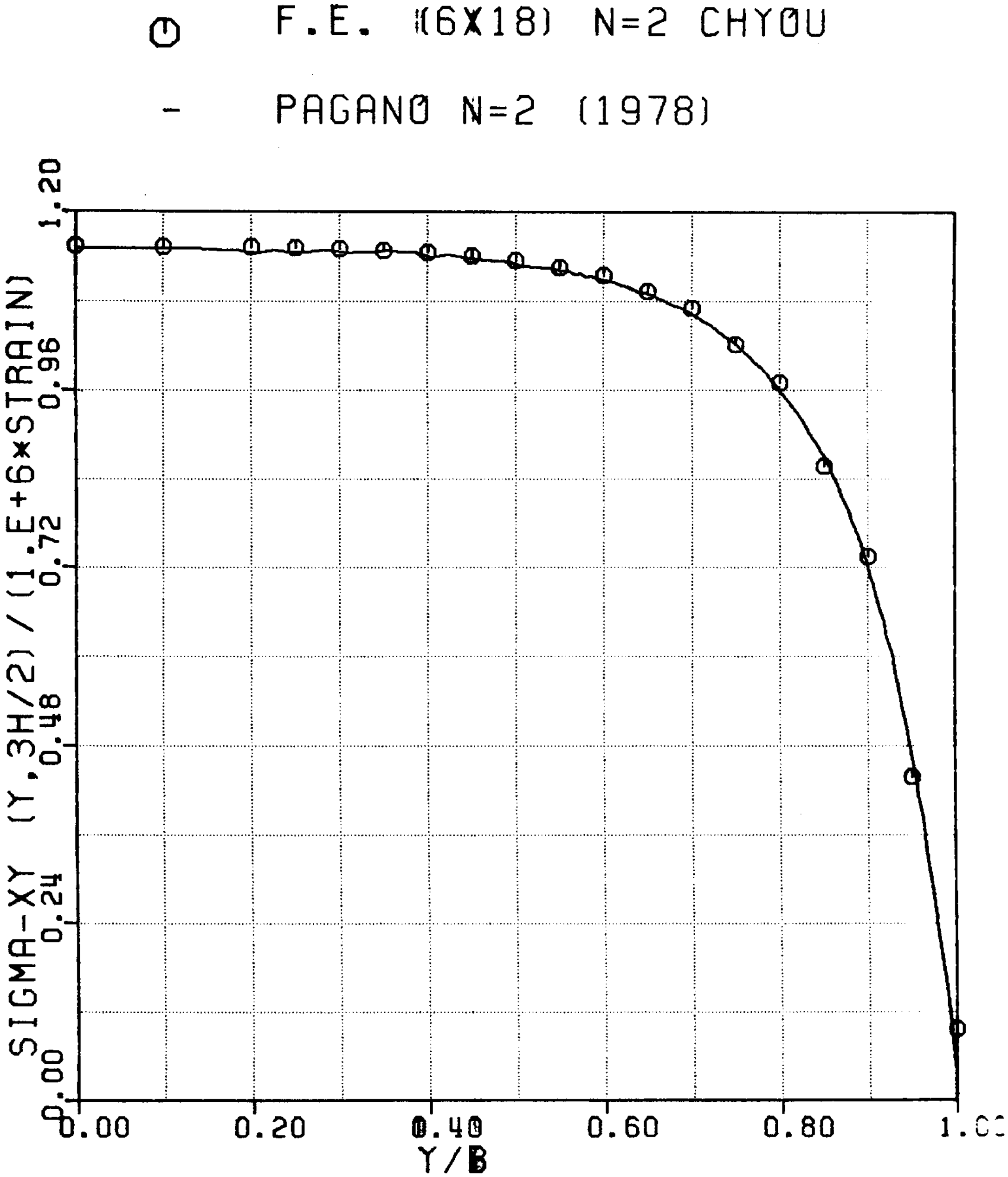


Figure 10: Distribution of XY-stress Along Center of Top Layer (45 degree) for N=2

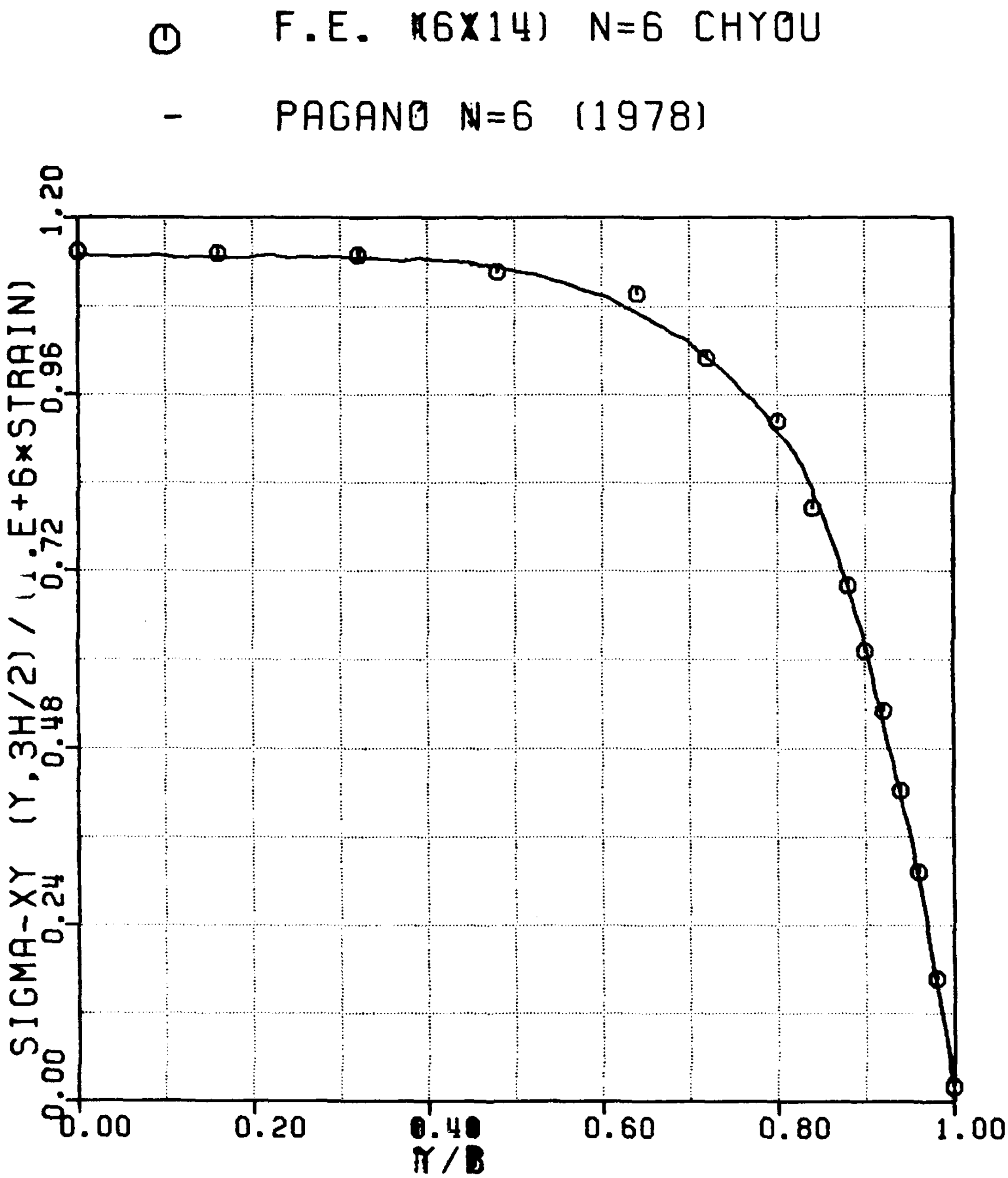


Figure 11: Distribution of XY-stress Along Center of Top Layer (45 degree) for N=6

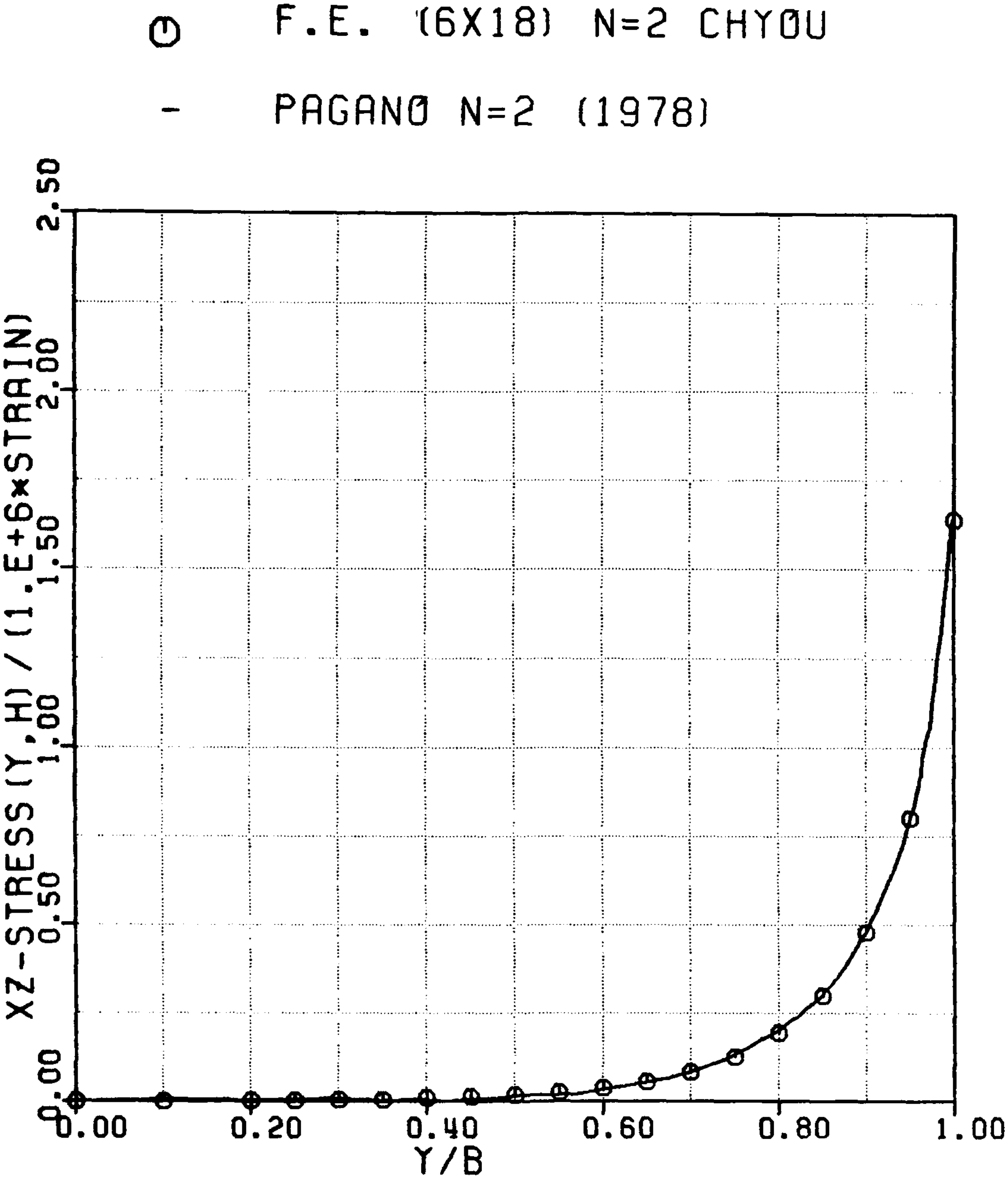


Figure 12: Distribution of XZ-stress Along 45/-45 Interface for N=2

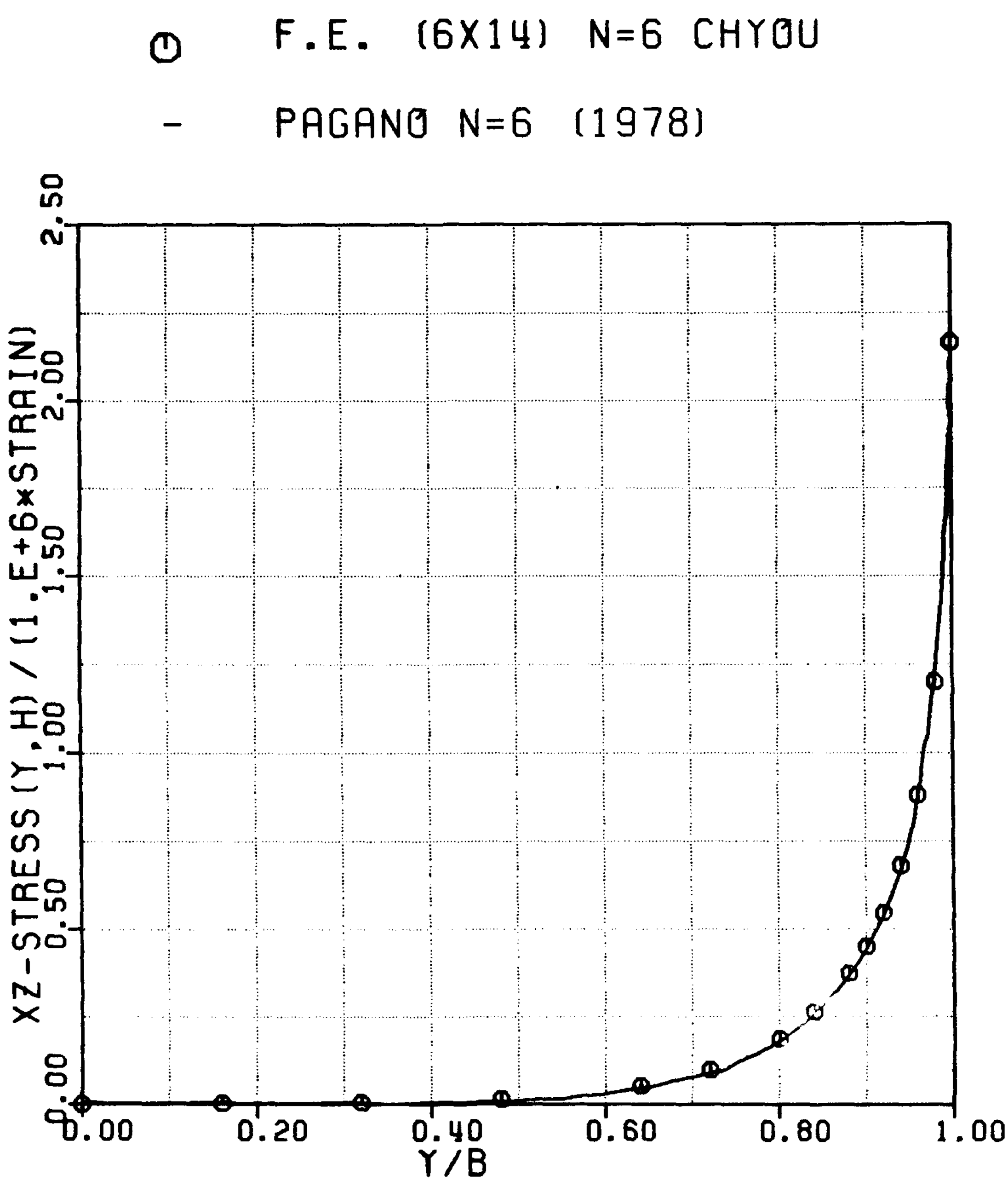


Figure 13: Distribution of XZ-stress Along 45/-45 Interface for N=6

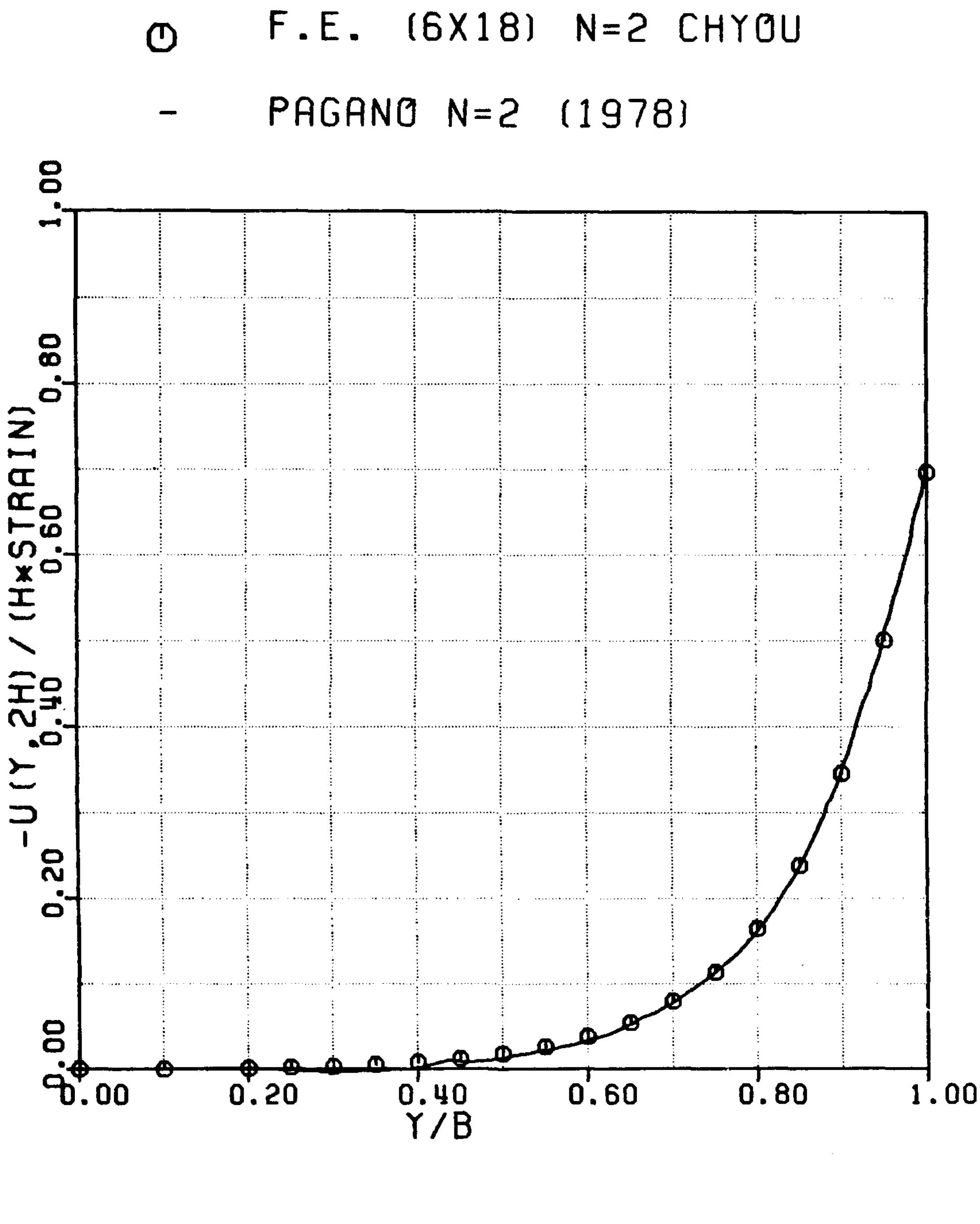


Figure 14: Axial Displacement Across Top Surface for N=2

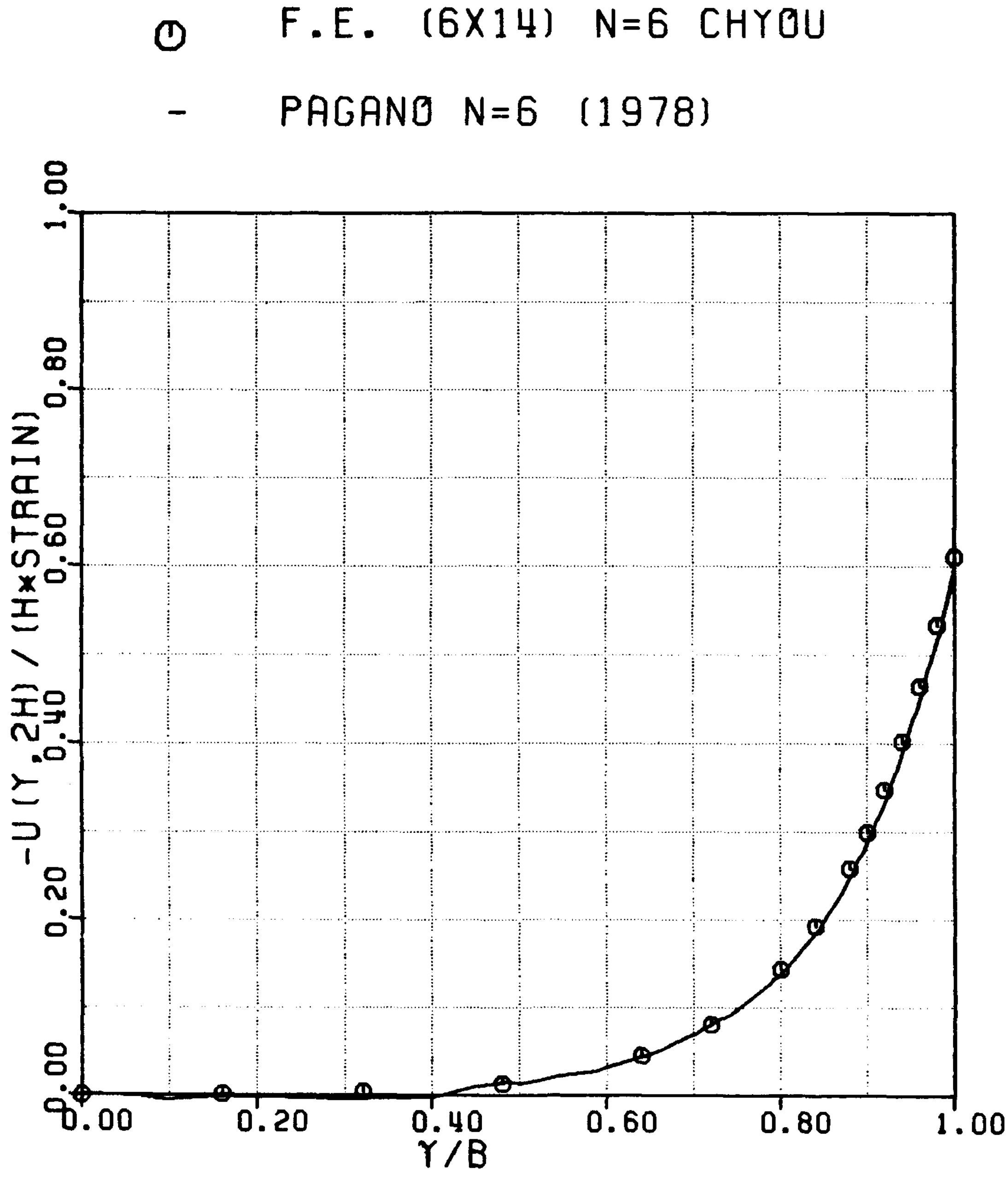


Figure 15: Axial Displacement Across Top Surface for N=6

6.2.2 A Multi-Ply Delamination Coupon

Analysis of the four-ply laminate specimens described in the previous section demonstrated the validity of using the proposed finite element procedures in solving delamination coupons. In this section, application of the finite element method to investigate the stress fields in the multi-ply laminate is described and the results are compared with those by Chang [1987]. The structure of the laminate with predetermined fiber orientations used in the present investigation was, following Sandhu and Sendeckyj [1987],

Stacking Sequence	Width	Ply thickness	Plies
$[(25.5/-25.5)_5/90]_s$	1.0 in	0.00505 in	22

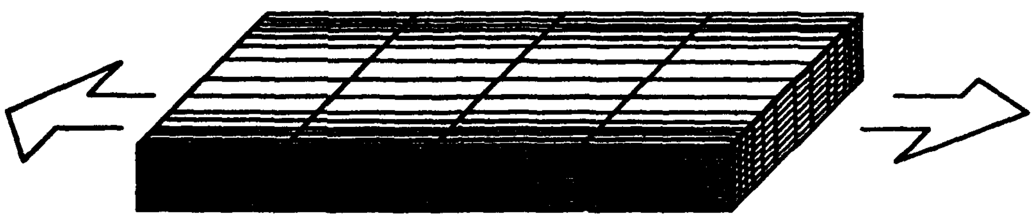
The material used in the study was AS4/3501-6, graphite-epoxy, and the elastic constants were [Sandhu and Sendeckj, 1987]

$$E_{11}=19.26\times10^6\text{psi}$$
$$E_{22}=1.32\times10^6\text{psi}$$
$$G_{12}=0.83\times10^6\text{psi}$$
$$\nu_{12}=0.35$$

6.2.2.1 Numerical Evaluation

A 64 element model shown in Figure 16 was used to discretize the delamination coupon. Interlaminar stress field within the delamination coupon for an applied longitudinal average unit strain was computed.

The distribution of σ_z along the midplane of the multi-ply laminate, shown in Figure 17 indicates that the finite element model predicts a sharp rise toward the free edge similar to Chang [1987].



64 Element Model

Figure 16: Finite Element Mesh for 22-ply Laminated Plate

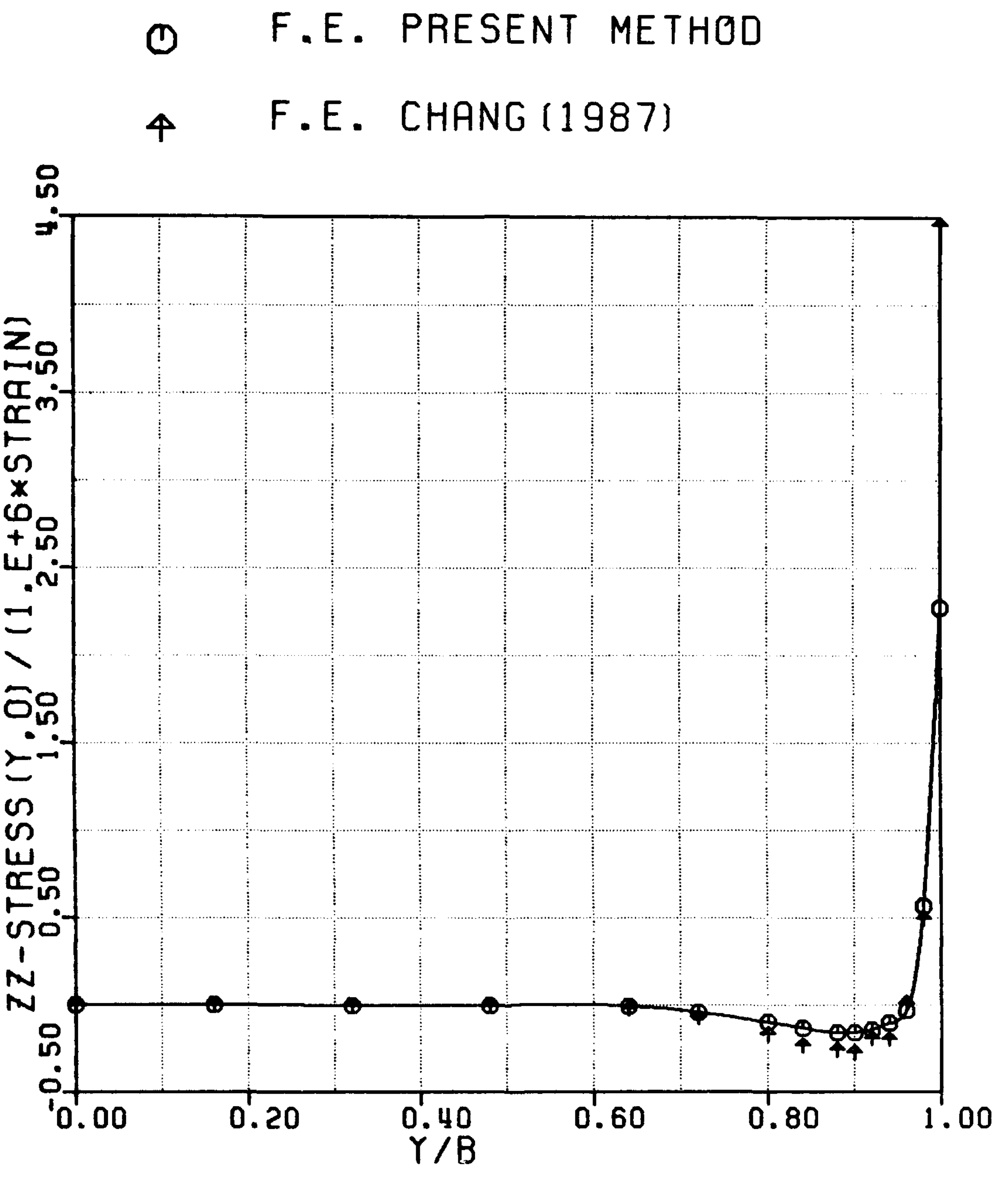


Figure 17: Distribution of Z-stress at Midplane

Figure 18 shows the through-the-thickness stress distributions of σ_z calculated from the present model and the continuous traction Q-23 element [Chang 1987] at the free edge of the laminate specimen. Figure 19 shows the through-the-thickness stress distributions of σ_z at the centroids of elements along the free-edge. It is observed that the stress field based on Q-23 element [Chang 1987] is not smooth at all compared with that by the present model. The slope discontinuity of σ_z at the interfaces in Chang's analysis indicates limitations of the displacement-based model in satisfying traction-free conditions. Table 4 gives a comparison of numerical results obtained by the present study and Chang [1987] at the free edge (through-the-thickness) of the 22-layer delamination coupon.

The stresses σ_{yz} at the interface of -25.5/90 calculated by the present model and Chang [1987] are shown in Figure 20. Figure 21 shows the effect of mesh refinement on the σ_{yz} distribution along the interface of -25.5/90 calculated by Chang's analysis [1987]. The oscillating pattern of σ_{yz} is present both in Chang's analysis and the present study. Chang [1987] found that refinement of mesh near the edge eliminated the oscillation. The same could be expected with the present model. However, due to limitations on core and auxiliary storage on the available computational facilities, refinement of meshes could not be implementated at this stage.

Table 4: A Comparison of Numerical Results (Through-the-thickness) at the Free Edge

Z/H	Chyou z-stress/(unit strain x 10^6)	Chang [1987] z-stress/(unit strain x 10^6)
0.0	0.00000	0.00000
0.5	-0.15750	-0.15460
1.0	-0.31500	1.89130
1.5	-0.25191	-0.42825
2.0	-0.18882	0.93902
2.5	-0.14466	-0.25979
3.0	-0.10050	1.66470
3.5	-0.08347	-0.27034
4.0	-0.06644	1.07290
4.5	-0.05523	-0.26235
5.0	-0.04402	1.65240
5.5	-0.02552	-0.25146
6.0	-0.00703	1.13480
6.5	0.03743	-0.25522
7.0	0.08188	1.94070
7.5	0.17704	-0.20345
8.0	0.27220	1.43390
8.5	0.48780	-0.17756
9.0	0.70340	3.05810
9.5	1.19430	1.82050
10.0	1.68520	-0.06381
10.5	1.97660	5.89320
11.0	2.26800	4.46050

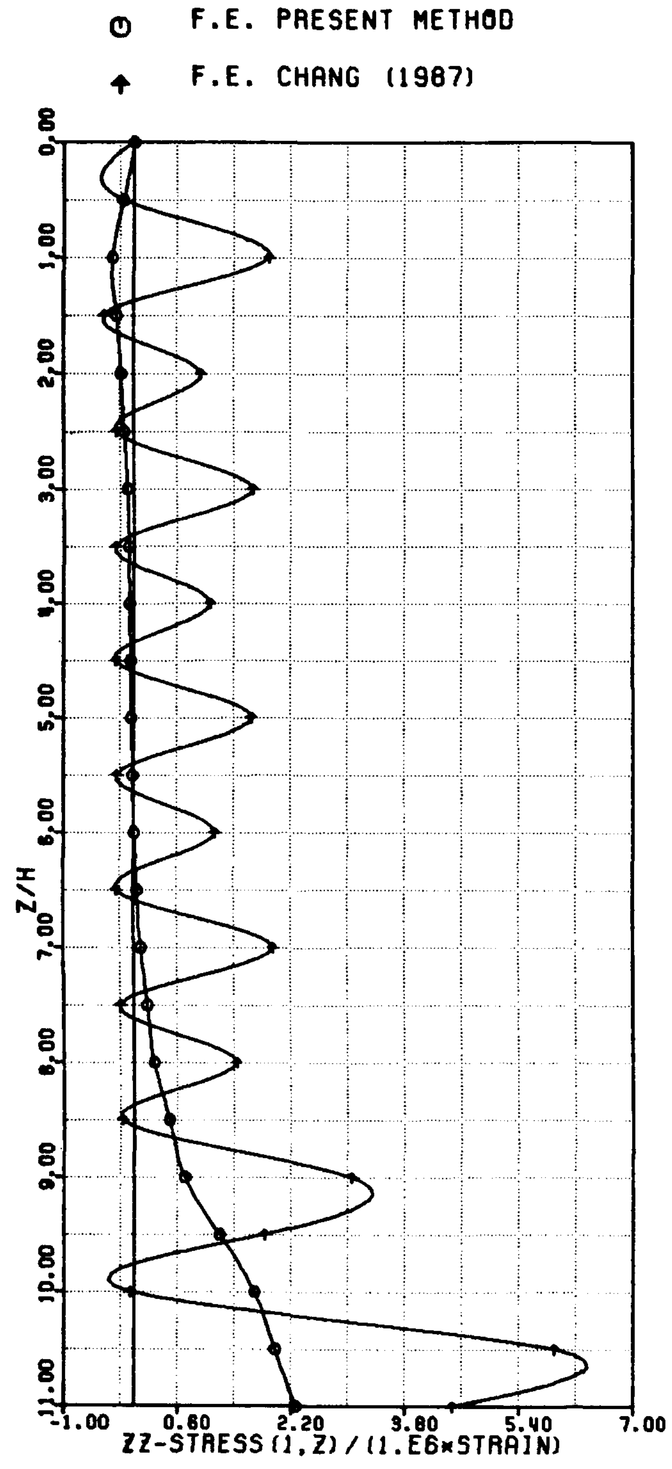


Figure 18: Through-the-thickness Z-stress Distribution Along the Free-Edge

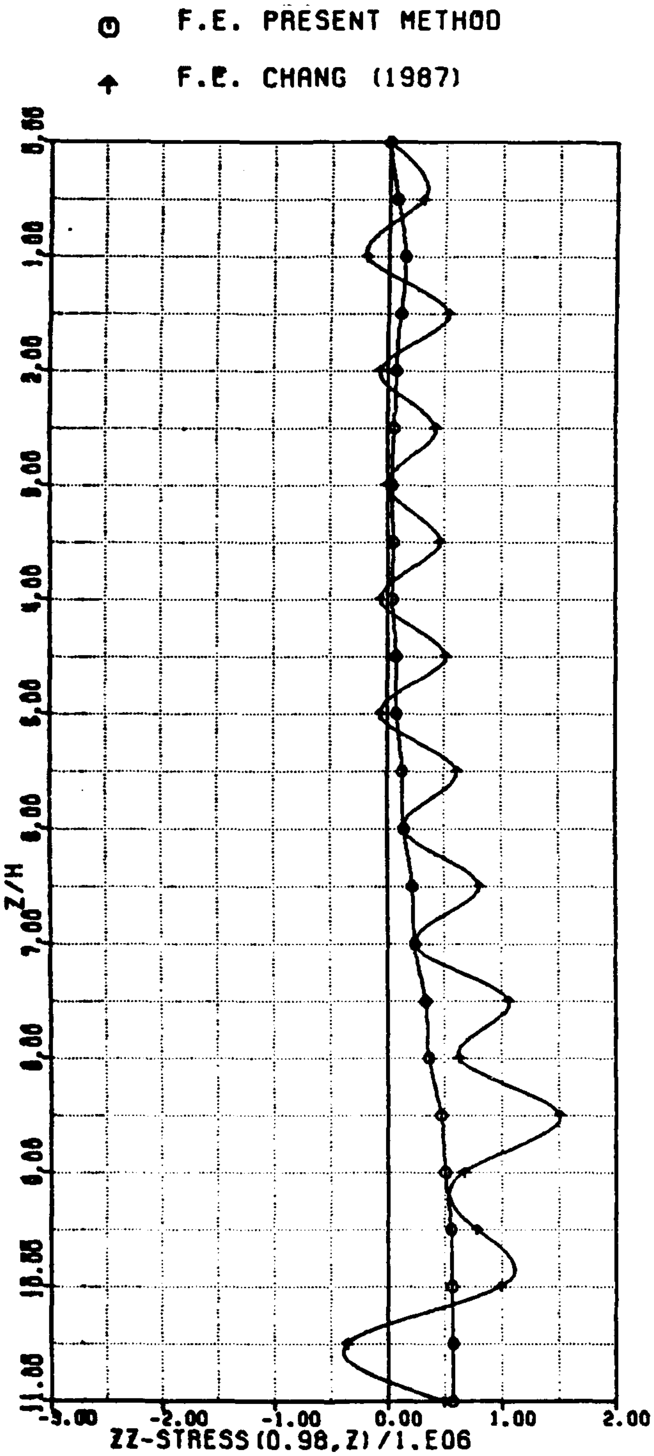


Figure 19: Through-the-thickness Z-stress Distribution at the Centroids of Elements Along the Free Edge

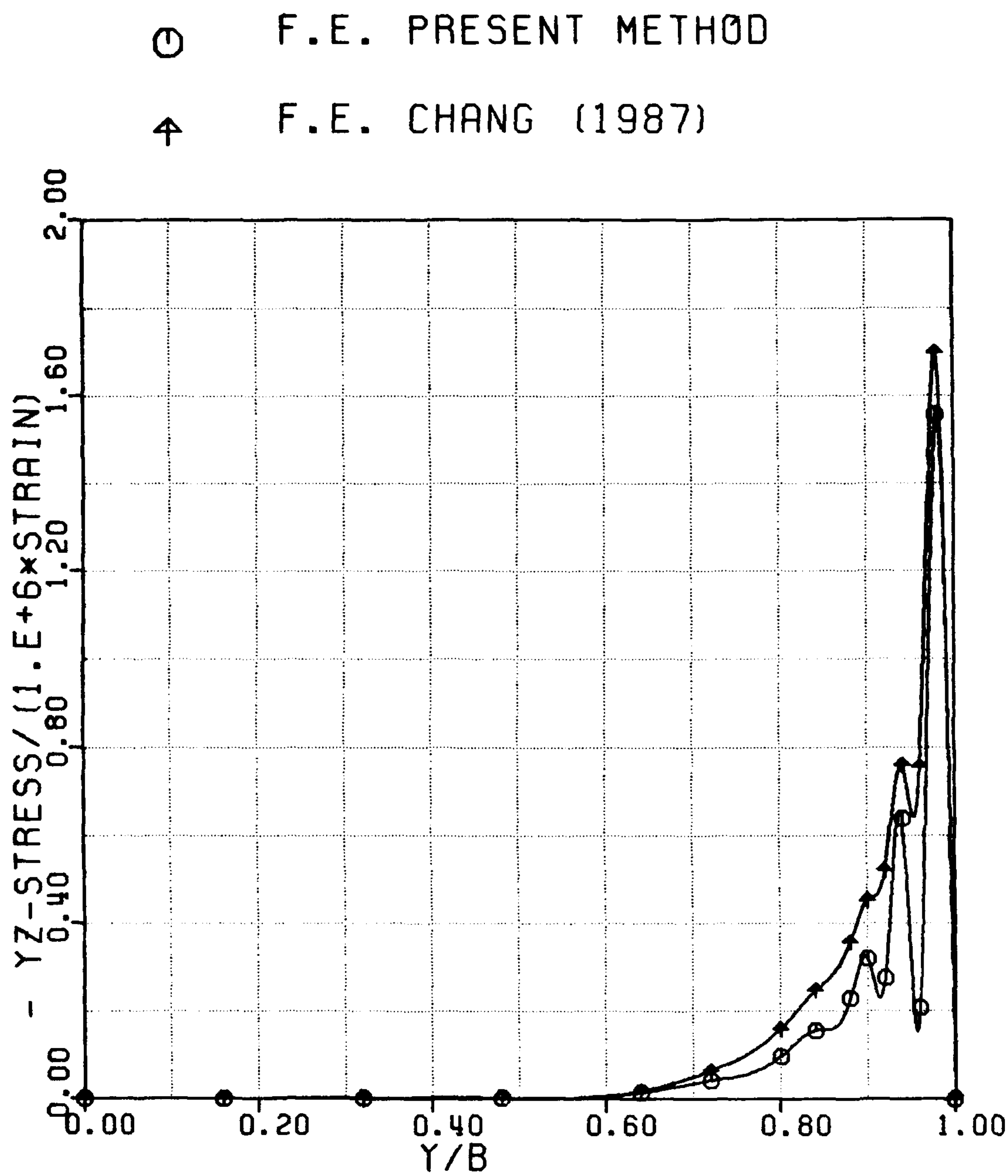


Figure 20: YZ-Stress Distribution at the Interface -25.5/90

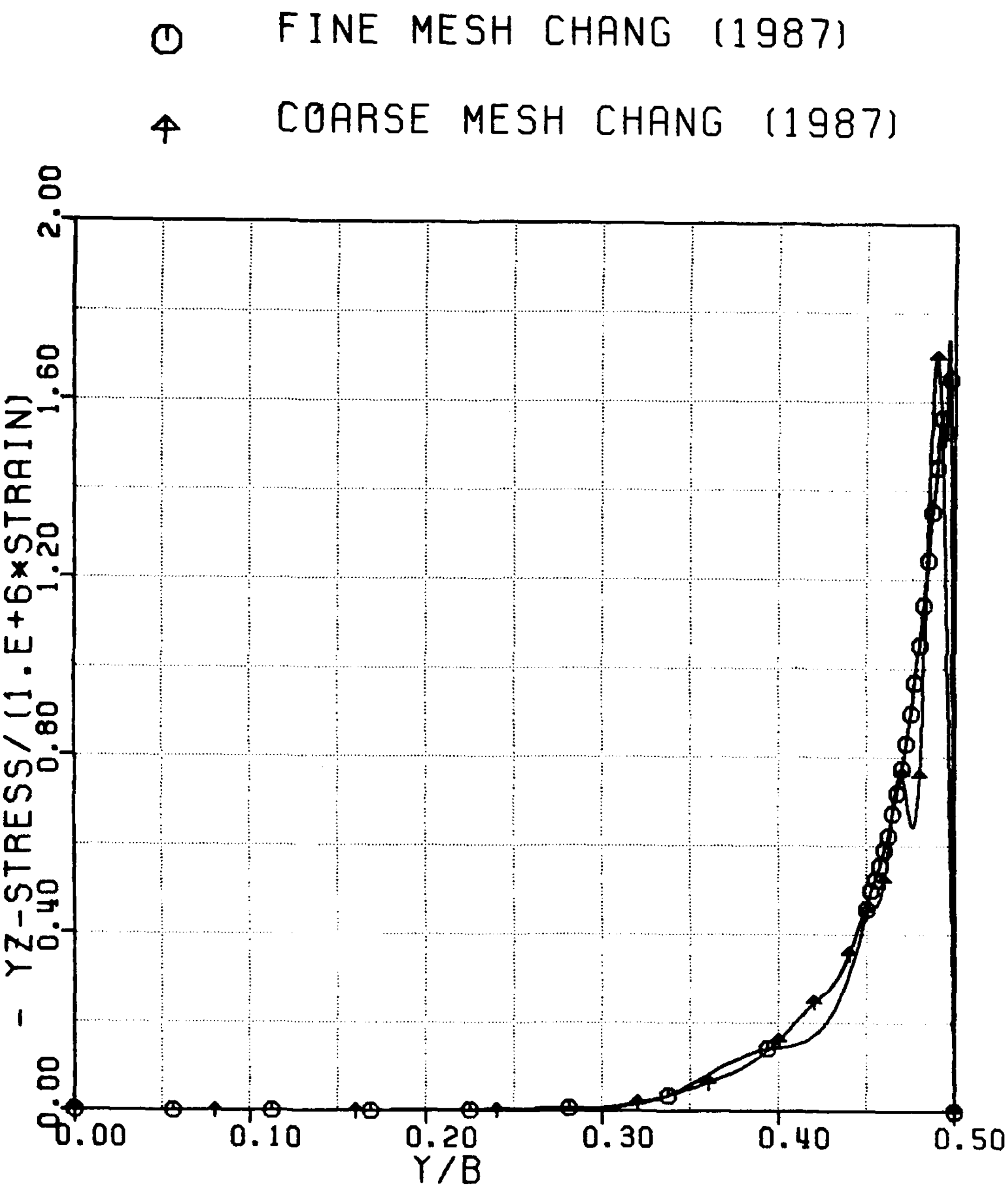


Figure 21: Effect of Mesh Refinement on YZ-Stress Distributions at the Interface -25.5/90

SECTION VII
DISCUSSION

The problem of analysis of composite laminates has been investigated. Pagano's theory has been examined carefully, rewritten in terms of a reduced number of field variables and stated in a self-adjoint form so that a general variational formulation could be developed. The resulting functional was specialized for implementation in a finite element model for stress and deformation analysis in laminated composite plates. The procedure was applied to study of stress fields in free-edge delamination specimens.

As originally stated, Pagano's [1978] theory used seven equilibrium equations, ten constitutive equations, and six interfacial continuity equations involving 23 field variables. An important feature of the present research was to rewrite the equations of Pagano's theory in terms of fewer field variables and to state the equations in a self-adjoint form. Physically, there are only five equations of equilibrium and, therefore, there can only be five corresponding displacement field variables. The quantities N_{33} and M_{33} introduced by Pagano could, therefore, be eliminated by writing explicit expressions for these in terms of stresses (termed equilibrium equations by Pagano). In this manner, the number of local mechanical variables reduced to eight requiring as many constitutive equations. The total number of field equations in Pagano's theory is thus reduced to 19 in five independent displacement field variables, eight mechanical quantities and six interfacial traction and displacement components. It is shown that the system of interfacial displacement continuity equations and field equations is self-adjoint in the sense of inner products.

Boundary operators consistent with the field operators for the problem have been identified following the procedure outlined by Sandhu and Salaam [1975] and Sandhu [1976]. For self-adjoint matrix of operators and consistent boundary operators a general variational principle was derivedallowing for possible internal discontinuities. Extensions to relax the requirement of differentiability of certain field variables have been developed, along with specializations to reduce the number of independent field variables.

For finite element implementation the interpolants can be restricted to satisfy some field equations and boundary conditions identically. One specialization in which the generalized displacements are continuous across boundaries of subregions or elements but generalized forces need not be continuously differentiable was used to develop a finite element program. This program was verified against Pagano's [1978] four-ply laminated free-edge delamination specimens, and also applied to a 22-ply specimen. The theory and the numerical procedure developed are quite general and applicable to laminated composite plates with arbitrary boundary conditions including internal boundaries e.g. holes. Discontinuities, e.g. delaminations, can be easily included by retaining corresponding terms in the governing functional. The general variational theory could form the basis of several alternative finite element schemes which could be used to define bounds to the solution.

As at present developed, the finite element computer program requires enormous amount of storage. This is largely due to the large number of algebraic equations with large bandwidth. Available equation solvers and storage strategies cannot handle the problem in an economical fashion. It appears necessary to develop efficient equation solving procedures taking advantage of the multi-banded nature of the system matrix. Global-local strategies to reduce the size of the problem and solve it in two or more steps should also be investigated. Different finite element interpolations could be implemented to obtain results in more economical fashion.

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