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On Impasse Points of Quasilinear Differential Algebraic Equations[1]

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1. Introduction.

An implicit differential algebraic equation (DAE) has the form $F(x, \dot{x}) = 0$ of a general ordinary differential equation (ODE) but involves a (sufficiently smooth) mapping $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $F = F(x,p)$, for which rank $D_pF(x,p) \equiv r < n$ is constant but not full in $F^{-1}(0)$. The terminology derives from the fact that, in practice, the simplest DAE problems are of the form

$$\begin{cases} g(x_1, x_2) = 0, \\ \dot{x}_2 = f(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^{n-r} \times \mathbb{R}^r, \end{cases}$$

where g and f map into $\mathbb{R}^{n-r}$ and $\mathbb{R}^r$, respectively, and thus exhibit clearly a differential and an algebraic part.

Because of the rank condition a DAE does not reduce locally to an explicit ODE via the implicit function theorem. Nevertheless, local reduction of a DAE to an explicit ODE on a *submanifold* of $\mathbb{R}^n$ was recently shown to be feasible under general assumptions ruling out certain geometric singularities ([RR91b]). The combination of this reduction procedure with standard ODE results yields an existence and uniqueness theory for "nonsingular" DAE's ([RR91a],[RR91b]) which is applicable to many problems.

While the above stresses strong analogies between DAE's and classical (that is, explicit) ODE's, important differences exist. For instance, a DAE $F(x, \dot{x}) = 0$ need not have a

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solution satisfying the initial condition $x(0) = x_0, \dot{x}(0) = p_0$ under the sole hypothesis that $F(x_0, p_0) = 0$; in fact, further compatibility conditions are usually required of (x_0, p_0) for such a solution to exist. Next, solutions of DAE's may exhibit features that solutions of explicit ODE's cannot possess. As an example with $n = 2$, consider the DAE

$$\begin{cases} x_1^2 + x_2 = 0, \\ \dot{x}_2 = 1, \end{cases} \tag{1.1}$$

with the initial condition $x(0) = (1, -1)$ (and hence $\dot{x}(0) = (-1/2, 1)$). This problem has the unique solution $x(t) = ((1-t)^{1/2}, t-1)$, which cannot be continued beyond $t = 1$ despite the fact that $x(1) = (0,0)$ exists and $\lim_{t \to 1^-} x(t) = x(1)$. If, instead, $x(t)$ were characterized as a solution of an explicit ODE $\dot{x} = h(x)$ with continuous h, it would follow from the initial condition $x(1) = (0,0)$ that $x(t)$ can be continued beyond $t = 1$.

The point $(0,0)$ where the solution $x(t) = ((1-t)^{1/2}, t-1)$ of (1.1) terminates in finite time $t = 1$ is an *impassse point*, as they are called in the engineering literature, especially in connection with nonlinear LRC circuits (see e.g. [C69], [CD89]).

We concentrate here on quasilinear equations

$$A(x)\dot{x} = G(x), \tag{1.2}$$

where $A = \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n)$ satisfies rank $A(x) \equiv r < n$ whenever $G(x) \in$ rge $A(x)$. This corresponds to the choice $F(x,p) = A(x)p - G(x)$ when F is linear in p. The purpose of this paper is to provide a mathematical characterization of the impasse points of such quasilinear DAE's which, to date, is lacking in general. However, the recent work by Chua and Deng [CD89] contains partial results for the special case when A is a constant projection.

Despite their rather special structure, quasilinear DAE's are relevant in a large number of physical problems. Obviously, the simple 2×2 example (1.1) fits into this framework. Moreover, nonautonomous problems $A(t,x)\dot{x} = G(t,x)$ can also be brought into the form (1.2) by adding the equation $\dot{t} = 1$.

Although impasse points have no analogue in the solutions of explicit ODE's, they are strongly reminiscent of "standard singular points" of *singular* ODE's. Singular ODE's differ from DAE's in that, when considered in the form (1.2), the matrix $A(x)$ fails to have full rank n only at exceptional points. As shown in [R89] the most frequently encountered type of singularity for a singular ODE is a standard singular point x_* where exactly *two* solutions([1]) either emanate or terminate. This is exactly the case in the simple example (1.1) where $t \mapsto (-(1-t)^{1/2}, t-1)$ is a solution, different from $x(t) = ((1-t)^{1/2}, t-1)$, that terminates at $(0,0)$ for $t = 1$. This strongly suggests that, in the vicinity of an impasse point, a DAE (1.2) can be reduced to a singular ODE with a standard singular point. In our presentation here, the possibility of performing such a reduction will serve as an equivalent mathematical definition for impasse points.

For convenience, the main results about singular points are summarized in Section 2. A reduction procedure for DAE's, which is essentially, but not completely, a specialization of the method in [RR91b] to the quasilinear case, is described in Section 3. The value of this procedure is to reduce the index of a DAE, and in particular to make an index 1 DAE into an implicit ODE that may or may not be singular. The short Section 4 summarizes the existence theory of [RR91b] for the index 1 quasilinear DAE's.

Impasse points are defined in Section 5. We have chosen to give a coordinate–free but rather abstract definition based upon the concept of intrinsic derivatives of a vector bundle morphism. This definition is equivalent to a simpler, coordinate dependent one, which is useful in most practical applications (Lemma 5.1).

In some simple but frequent cases, impasse points turn out to coincide with foldpoints of a specific manifold relative to some splitting of the ambient space. These matters are considered in Section 6 and relate to the results of [CD89].

Section 7 is devoted to two examples. The first example is a concrete one taken from electrical network theory. The second example shows that one–parameter stationary problems can be reformulated as DAE's and is meant to emphasize the fact that the study of DAE's, including impasse points and singularities (an aspect not touched upon here) encompasses all the issues usually investigated in the framework of bifurcation theory.

The theory can easily be extended to higher index problems (provided that the index is well defined). Here, the validity of our results for *nonconstant* $A(x)$ in (1.2) is essential,

([1])Modulo translations in time since (1.1) is autonomous.

even if $A(x)$ is constant in the given higher index problem. Some related comments can be found in Section 8.

2. Standard Singular Points of Singular ODE's: A Brief Review.

The material presented here is taken from [R89] but our notation corresponds to that used below in this paper. We begin by introducing the following terminology:

Definition 2.1. *Consider the implicit ODE*

$$A_1(\xi)\dot{\xi} = G_1(\xi), \tag{2.1}$$

where $A_1 : U^r \to \mathcal{L}(\mathbb{R}^r)$ and $G_1 : U^r \to \mathbb{R}^r$ are C^1 mappings defined on some open set $U^r \subset \mathbb{R}^r$. A point $\xi \in U^r$ is a regular point of (2.1) if rank $A_1(\xi) = r$ and a singular point if rank $A_1(\xi) < r$ but ξ is a limit point of regular points of (2.1).

Clearly, for any regular point $\xi_* \in U^r$ and given t_* the initial value problem

$$A_1(\xi)\dot{\xi} = G_1(\xi), \quad \xi(t_*) = \xi_* \tag{2.2}$$

has a unique solution in a neighborhood of ξ_*. But already simple examples show that for a singular point ξ_* the existence and number of solutions of (2.2) in the neighborhood of ξ_* depends strongly on the character of the singularity at that point. The most frequent and also simplest case is the following:

Definition 2.1. *A singular point $\xi \in U^r$ of (2.1) is a standard singular point if*

$$\dim\ker A_1(\xi) = 1 \quad (\text{ i.e. } \operatorname{rank} A_1(\xi) = r - 1), \tag{2.3}$$

$$G_1(\xi) \notin \operatorname{rge} A_1(\xi), \tag{2.4}$$

$$(DA_1(\xi)\eta)\eta \notin \operatorname{rge} A_1(\xi), \quad \forall \eta \in \ker A_1(\xi) \setminus \{0\}. \tag{2.5}$$

In particular, if (2.1) has a standard singular point then such points will exist for every problem obtained by a sufficiently small perturbation of (2.1). Moreover, the set of standard singular points of (2.1) is either empty or forms a hypersurface of U^r, and, at least in the C^∞ case, the standard singularities are the only singularities that satisfy the latter condition for generic choices of A_1 and G_1 (see [R89]).

Remark 2.1: The fact that standard singular points of (2.1) lie on a hypersurface, if they exist at all, has an important consequence regarding the dynamics. Indeed, if a solution of the initial value problem (2.2) for some regular point ξ_0 tends, at some later time, to a standard singular point, then the same will happen also with any trajectory passing through a point ξ_0' sufficiently close to ξ_0. In other words, standard singular points cannot be by-passed by simply perturbing the initial condition. □

If ξ_* is a standard singular point of (2.1), the condition (2.4) makes it impossible for any C^1 function $\xi : [t_*, t_* + T) \to \mathbb{R}^r$ with $\xi(t_*) = \xi_*$ to be a solution of (2.1) since the relation $A_1(\xi_*)\dot{\xi}(t_*) = G_1(\xi_*)$ cannot hold. This remark suggests the following definition.

Definition 2.3. *Let ξ_* be a standard singular point of (2.1). A solution of the initial value problem (2.2) for given $t_* \in \mathbb{R}$ is a continuous function $\xi : J \subset \mathbb{R}^1 \to U^r$ on either $J = [t_*, t_* + T)$ or $J = (t_* - T, t_*]$ for some $T > 0$ which is of class C^1 on $J^0 = J \setminus \{\xi_*\}$ and satisfies $\xi(t_*) = \xi_*$ and $A_1(\xi(t))\dot{\xi}(t) = G_1(\xi(t))$ for $t \in J^0$.*

Evidently the two conditions (2.4) and (2.5) are equivalent with

$$\alpha(\xi)(\eta, \tilde{\eta}) \equiv \langle G_1(\xi), \tilde{\eta}\rangle \; \langle (DA_1(\xi)\eta)\eta, \tilde{\eta}\rangle \; \neq \; 0,$$

$$\forall \eta \in \ker A_1(\xi) \setminus \{0\}, \; \tilde{\eta} \in \ker A_1(\xi)^T \setminus \{0\} \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural inner product of $\mathbb{R}^r$. Since the form $\alpha(\xi)(\eta, \tilde{\eta})$ is continuous and quadratic in η as well as $\tilde{\eta}$ it must have either a positive or negative value for all pairs of nonzero vectors $(\eta, \tilde{\eta})$ in $\ker A_1(\xi) \times \ker A_1(\xi)^T$ if only this holds for one such pair.

The main result ([R89, Theorem 5.1]) on the existence of solutions near standard singular points can then be phrased as follows:

Theorem 2.1. *Let ξ_* be a standard singular point of (2.1). Then, given $t_* \in \mathbb{R}$, the initial value problem (2.2) has exactly two solutions which are both defined on $J = [t_*, t_* + T)$ or on $J = (t_* - T, t_*]$ for some $T > 0$ if $\alpha(\xi_*)(\eta, \tilde{\eta}) \; > \; 0$ or $\alpha(\xi_*)(\eta, \tilde{\eta}) \; < \; 0$ for some pairs of nonzero vectors $(\eta, \tilde{\eta}) \; \in \; \ker A_1(\xi_*) \times \ker A_1(\xi_*)^T$, respectively. Furthermore, $\|\dot{\xi}(t)\|$ tends to ∞, as $t \in J \setminus \{t_*\}$ tends to t_*.*

It follows from Theorem 2.1 that a solution of (2.1) emanating from some regular point can reach a standard singular point ξ_* at some later time only if the quantity $\alpha(\xi_*)$ of (2.6) is negative. Standard singular points ξ_* with positive $\alpha(\xi_*)$ obviously cannot be reached in increasing time. Thus, it is appropriate to introduce the following terminology:

Definition 2.4. *The standard singular point ξ_* of (2.1) is accessible or inaccessible if $\alpha(\xi_*)(\eta, \tilde{\eta}) > 0$ or $\alpha(\xi_*)(\eta, \tilde{\eta}) < 0$, respectively, for some pair of nonzero vectors $\eta, \tilde{\eta}$ in* $\ker A_1(\xi_*) \times \ker A_1(\xi_*)^T$.

In [R89], accessible and inaccessible points are called attracting and repelling, respectively, but the terminology used here appears to be more appropriate. Of course, by reversing the evolution in time, inaccessible points become accessible and vice-versa.

It follows from Theorem 2.1 that accessible standard singular points are reached in *finite* time by trajectories emanating elsewhere in U^r. Since these trajectories cannot be continuously extended beyond these points, they represent "catastrophes" for the solutions of (2.1). This is further emphasized by the fact that, as mentioned in Remark 2.1, no small perturbation of the initial condition (and/or of A_1 or G_1) will affect the eventual encounter of such points. In this respect, note that the sign condition in (2.6) is unchanged if A_1, G_1 and ξ_* are replaced by sufficiently small approximations.

Although they cannot be reached, inaccessible standard singular points may also have drastic effects on the dynamics but not in the form of catastrophes (see again [R89]).

3. A Reduction Procedure for Differential-Algebraic Equations.

In this section, we consider DAE's of the form

$$A(x)\dot{x} = G(x), \quad x \in \mathbb{R}^n \tag{3.1}$$

with the following properties:

Assumption 3.1. *For some open subset $U^n \subset \mathbb{R}^n$ the mappings $A : U^n \to \mathcal{L}(\mathbb{R}^n)$ and $G : U^n \to \mathbb{R}^n$ are of class C^2 on U^n and for some fixed integer $0 \le r < n$, we have*

$$\{x \in U^n,\ G(x) \in \ rge\ A(x)\} \Rightarrow \ rank\ A(x) = r \tag{3.2}$$

and the mapping

$$(x,p) \in U^n \times \mathbb{R}^n \longmapsto A(x)p - G(x) \in \mathbb{R}^n, \tag{3.3}$$

is a submersion.

The submersion assumption for (3.3) implies that the mapping

$$(h,q) \in \mathbb{R}^n \times \mathbb{R}^n \longmapsto (DA(x)h)p + A(x)q - DG(x)h \in \mathbb{R}^n \tag{3.4}$$

is surjective for every $(x,p) \in U^n \times \mathbb{R}^n$. Hence the set

$$M = \{(x,p) \in U^n \times \mathbb{R}^n : A(x)p - G(x) = 0\} \tag{3.5}$$

is a closed n-dimensional C^2 submanifold of $U^n \times \mathbb{R}^n$.

Our aim is to show that (3.1) can be reduced to the similar form

$$A_1(\xi)\dot{\xi} = G_1(\xi), \tag{3.6}$$

where now A_1 and G_1 take values in $\mathcal{L}(\mathbb{R}^r)$ and $\mathbb{R}^r$, respectively. In essence, this reduction represents a simplification in the quasilinear case (3.1) of the theory developed in [RR91b] for general implicit DAE's $F(x,\dot{x}) = 0$. However, there is a special feature which will be of crucial importance for the later discussion. While the reduction of [RR91b] is valid only locally in the vicinity of a point $(x_*,p_*) \in F^{-1}(0)$, the reduction here is local only in the first variable. This is due to the linearity of (3.1) with respect to the derivative $\dot{x}$ and, roughly speaking, will allow us to analyze phenomena involving "infinite" p_* as must be done to discuss impasse points.

We consider the set

$$W = \{x \in U^n : G(x) \in \operatorname{rge} A(x)\}, \tag{3.7}$$

and note that $(x,p) \in M$ for some $p \in \mathbb{R}^n$ if and only if $x \in W$ and hence

$$W = \pi(M), \tag{3.8}$$

where $\pi : U^n \times \mathbb{R}^n \to U^n$ is the projection onto the first factor. Then we have:

Proposition 3.1. *The set W is an r-dimensional C^2 submanifold of U^n. In addition, if the set $\{x \in U^n : rankA(x) = r\}$ is closed in U^n then W is closed in U^n.*

Proof: By Assumption 3.1, rank $A(x) = r$ is independent of $x \in W$ which implies that rank $\pi_{|M}$ is constant and equal to r. Indeed, for $(x,p) \in M$, the mapping $T_{(x,p)}(\pi_{|M})$ is the restriction of π to $T_{(x,p)}M$ and, since $T_{(x,p)}M$ is the null-space of the mapping (3.4), we have

$$\ker T_{(x,p)}(\pi_{|M}) = \{(0,q) \in \mathbb{R}^n : A(x)q = 0\} = \{0\} \times \ker A(x).$$

Thus, for $(x,p) \in M$, and hence $x \in W$, we obtain $\dim \ker T_{(x,p)}(\pi_{|M}) = n-r$ and therefore rank $\pi_{|M} = r$ as desired.

In view of this result and the subimmersion theorem (see e.g. [D70], [AMR88]) it suffices to show that $\pi_{|M} : M \to W$ is open to prove that W is an r - dimensional C^2 submanifold of U^n. Let M_0 be any open subset of M. In order to prove that $\pi(M_0)$ is open in W let $x_0 \in \pi(M_0)$, so that there is some $p_0 \in \mathbb{R}^n$ with $(x_0, p_0) \in M_0$. We construct a continuous function $f : W_0 \to M$ on some open neighborhood $W_0 \subset W$ of x_0 such that $f(x_0) = (x_0, p_0)$ and $\pi \circ f = id_{W_0}$. The continuity of f and $(x_0, p_0) \in M_0$ then imply that $f^{-1}(M_0)$ is an open neighborhood of x_0 in W_0 and hence in W. On the other hand, $\pi \circ f = id_{W_0}$ implies $f^{-1}(M_0) \subset \pi(M_0)$. Thus it follows that $\pi(M_0)$ is a neighborhood of x_0 and therefore that $\pi(M_0)$ is open since x_0 was arbitrary.

Before constructing f let us observe that, in general, if

$$V = \{x \in U^n : \text{ rank } A(x) = r\} \tag{3.9}$$

and $x_0 \in V$ is a given point, then there exist an open neighborhood V_0 of x_0 in V and continuous functions $e_i : V_0 \to \mathbb{R}^n$ such that $\{e_1(x), \cdots, e_r(x)\}$ is an orthonormal basis of rge $A(x)$, $\forall x \in V_0$. Indeed, let $u_{01}, \cdots, u_{0r}$ be any basis of rge $A(x_0)$. Then there are linearly independent vectors $w_1, \cdots, w_r \in \mathbb{R}^n$ for which $u_{0i} = A(x_0)w_i, 1 \leq i \leq r$. A straightforward contradiction argument now shows that the vectors $u_i(x) = A(x)w_i$, $i = 1, \cdots, r$, remain linearly independent for all x sufficiently close to x_0 in V. Thus, for all x in some small open neighborhood V_0 of x_0 in V the vectors $u_1(x), \cdots, u_r(x)$ form a basis of rge $A(x)$ since rank $A(x) = r$. Now, by applying the Gram-Schmidt process to $u_1(x), \cdots, u_r(x)$, we obtain an orthonormal basis $\{e_1(x), \cdots, e_r(x)\}$. The functions $e_i : V_0 \to \mathbb{R}^n$ are continuous since the functions $x \mapsto u_i(x)$ are continuous and the Gram-Schmidt process involves only continuous operations. For future use we note that the orthogonal projections $P(x) \in \mathcal{L}(\mathbb{R}^n)$ onto rge $A(x)$ depend continuously upon $x \in W_0$ since $P(x)$ can be expressed as a sum of dyadic products of the vectors $e_1(x), \cdots, e_r(x)$.

Clearly, the above arguments can be repeated verbatim to obtain continuous functions $\tilde{e}_i : V_0 \to \mathbb{R}^n$, $i = 1, \cdots, r$, such that $\{\tilde{e}_1(x), \cdots \tilde{e}_r(x)\}$ is an orthonormal basis of rge $A(x)^T$ (after shrinking V_0 if necessary). In particular, the orthogonal projection $\tilde{P}(x) \in \mathcal{L}(\mathbb{R}^n)$ onto rge $A(x)^T$ depends continuously upon $x \in V_0$.

We return to the specific case when $x_0 \in W$. With the constructed neighborhood V_0 we obtain the open neighborhood $W_0 = W \cap V_0$ of x_0 in W. For $x \in W_0$, the equation $A(x)p = G(x)$, $p \in \{\ker A(x)\}^\perp =$ rge $A(x)^T$, has a unique solution $p = p(x)$. The components $p_i(x)$ of $p(x)$ in the basis $\tilde{e}_i(x)$, $1 \le i \le r$, satisfy the (invertible) linear system $\sum_{j=1}^{r} \alpha_{ij}(x) p_j(x) = \gamma_i(x)$, $1 \le i \le r$, where $\alpha_{ij}(x) = \langle \tilde{e}_i(x), A(x)\tilde{e}_j(x)\rangle$ and $\gamma_i(x) = \langle G(x), \tilde{e}_i(x)\rangle$ are continuous on W_0. Therefore the components p_i, and hence p itself, are continuous functions on W_0.

We construct the desired function f in the form $f(x) = (x, p(x) + q(x))$ where $q : W_0 \to \mathbb{R}^n$ is continuous and satisfies $q(x) \in \ker A(x)$ for $x \in W_0$, and $q(x_0) = p_0 - p(x_0) \in \ker A(x_0)$. Clearly then $f(W_0) \subset M$, $f(x_0) = (x_0, p_0)$ and $\pi \circ f = id_{W_0}$. In order to find q it suffices to note that, in view of the previous remarks $\tilde{Q} = I - \tilde{P} : W_0 \to \mathcal{L}(\mathbb{R}^n)$ is continuous and $\tilde{Q}(x)$ is the orthogonal projection onto $\ker A(x) = \{$ rge $A(x)^T\}^\perp$. Thus, $q(x) = \tilde{Q}(x)(p_0 - p(x_0))$ satisfies the desired conditions.

Fo the completion of the proof, we must show that W is closed in U^n whenever the set V of (3.9) is closed in U^n. Let $\{x_k\}_{k\ge1}$ be a sequence in W with $\lim_{k\to\infty} x_k = x_0 \in U^n$. Then we have $x_0 \in V$ since $W \subset V$ and V is closed in U^n. Let $V_0 \subset V$ be an open neighborhood of x_0 in V such that $P : V_0 \to \mathcal{L}(\mathbb{R}^n)$ is continuous where $P(x)$ denotes the orthogonal projection onto rge $A(x)$; the existence of such a neighborhood V_0 was shown above. Because of $x_k \in W$ we have $P(x_k)G(x_k) = G(x_k)$ for $k \ge 1$. In the limit as $k \to \infty$ it then follows that $P(x_0)G(x_0) = G(x_0)$ which means that $G(x_0) \in$ rge $A(x_0)$ and therefore $x_0 \in W$. □

Remark 3.1: Proposition 3.1 has no global analog for general implicit DAE's $F(x,\dot{x}) = 0$, although a local version exists (see [RR91b]). $\square$

We are now in a position to describe our reduction procedure. As a motivation, note that if $J \subset \mathbb{R}$ is an open interval and $x : J \to U^n$ a C^1 solution of (3.1) then necessarily $x(t) \in W$, $\forall t \in J$. Since W is a manifold this implies that $(x(t),\dot{x}(t)) \in TW$, $\forall t \in J$ where here and subsequently TW is viewed as a subset of $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Since $(x(t),\dot{x}(t)) \in M$ it follows that $(x(t),\dot{x}(t)) \in TW \cap M$, $\forall t \in J$. The reduction of (3.1) to the form (3.2) now is an easy consequence of the characterization of $TW \cap M$ given below.

Let $x_* \in W$. By Proposition 3.1 there exist open subsets $\mathcal{O} \subset W$ and $U^r \subset \mathbb{R}^r$ and a C^2 mapping $\varphi : U^r \to \mathbb{R}^r$ which is a diffeomorphism of U^r onto $\mathcal{O}$. The tangent bundle $T\mathcal{O}$ is the image of the mapping

$$(3.10) \qquad (\xi,\eta) \in U^r \times \mathbb{R}^r \longmapsto (\varphi(\xi), D\varphi(\xi)\eta).$$

From $T\mathcal{O} = \{(x,p) \in TW : x \in \mathcal{O}\}$ we obtain

$$(3.11) \qquad \{(x,p) \in TW \cap M : \ x \in \mathcal{O}\} = T\mathcal{O} \cap M,$$

and, using (3.11) and the fact that $T\mathcal{O}$ is the image of the mapping (3.10), that

$$(3.12) \qquad \{(x,p) \in TW \cap M,\ x \in \mathcal{O}\} \Leftrightarrow \begin{cases} x = \varphi(\xi),\ p = D\varphi(\xi)\eta, \\ A(\varphi(\xi))D\varphi(\xi)\eta - G(\varphi(\xi)) = 0, \end{cases}$$

for some pair $(\xi,\eta) \in U^r \times \mathbb{R}^r$.

By definition of φ and W, $G(\varphi(\xi)) \in$ rge $A(\varphi(\xi))$, $\forall \xi \in U^r$ which implies that

$$(3.13) \qquad A(\varphi(\xi))D\varphi(\xi)\eta - G(\varphi(\xi)) \in \text{ rge } A(\varphi(\xi)),\ \forall \xi \in U^r.$$

As we shall see now, (3.13) allows us to replace the equation in $\mathbb{R}^n$ appearing on the right-hand side of (3.12) by a similar equation in $\mathbb{R}^r$.

Let Z_* be a given complement of rge $A(x_*)$. Then Z_* is also a complement of rge $A(x)$ for all $x \in W$ sufficiently close to x_*. This can be shown, for instance, by using the fact that the orthogonal projection $P(x)$ onto rge $A(x)$ depends continuously on $x \in W$, as we saw in the proof of Proposition 3.1. Now suppose that there is a sequence $\{x_k\}_{k\geq 1}$ in W tending to x_* such that Z_* is not a complement of rge $A(x_k)$. Since rank $A(x_k) = r$ and $\dim Z_* = n - r$, there exists a unit vector $u_k \in$ rge $A(x_k) \cap Z_*$ and, by extracting a subsequence, we may assume that $\lim_{k\to\infty} u_k = u$. Clearly, u is a unit vector on Z_* and, since $P(x_k)u_k = u_k$ it follows that $P(x_*)u = u$; that is, $u \in$ rge $A(x_*)$ which is absurd.

Thus, by shrinking $\mathcal{O}$ and with it U^r, if necessary, we can ensure that Z_* is a complement of rge $A(\varphi(\xi))$, for $\xi \in U^r$. Equivalently, the projection operator P_* onto rge $A(x_*)$ relative to the decomposition

$$\mathbb{R}^n = \text{rge } A(x_*) \oplus Z_* \tag{3.14}$$

is a linear isomorphism of rge $A(\varphi(\xi))$ onto rge $A(x_*)$ for all $\xi \in U^r$. By (3.13) this implies that for any pair $(\xi, \eta) \in U^r \times \mathbb{R}^r$ we have $A(\varphi(\xi))D\varphi(\xi)\eta - G(\varphi(\xi)) = 0$ if and only if $P_*A(\varphi(\xi))D\varphi(\xi)\eta - P_*G(\varphi(\xi)) = 0$. Using this equivalence in (3.12), we infer that

$$\{(x,p) \in TW \cap M,\ x \in \mathcal{O}\} \Leftrightarrow \begin{cases} x = \varphi(\xi),\ p = D\varphi(\xi)\eta, \\ A_1(\xi)\eta - G_1(\xi) = 0, \end{cases} \tag{3.15}$$

for some pair $(\xi, \eta) \in U^r \times \mathbb{R}^r$, where we have set

$$A_1(\xi) = P_*A(\varphi(\xi))D\varphi(\xi) \tag{3.16}$$

$$G_1(\xi) = P_*G(\varphi(\xi)). \tag{3.17}$$

Note that by identifying rge $A(x_*)$ with $\mathbb{R}^r$ we see that the mappings A_1 and G_1 map into $\mathcal{L}(\mathbb{R}^r)$ and $\mathbb{R}^r$, respectively, and are of class C^1.

Suppose now that $J \subset \mathbb{R}$ is an open interval and $x : J \to U^n$ a C^1 solution of (3.1). As noted earlier, we have $x(J) \subset W$. If it so happens that $x(J) \subset \mathcal{O}$ so that $(x(t), \dot{x}(t)) \in TW \cap M$ and $x(t) \in \mathcal{O}$ for all $t \in J$, then it follows from (3.15) that the C^1 function $\xi : J \to U^r$ defined by $\xi(t) = \varphi^{-1} \circ x(t)$ satisfies

$$A_1(\xi(t))\dot{\xi}(t) = G_1(\xi(t)),\ \forall t \in J. \tag{3.18}$$

with A_1, G_1 given by (3.16) and (3.17), respectively. Conversely, for every C^1 solution $\xi : J \to U^r$ of (3.10), $x(t) = \varphi \circ \xi(t)$, is a C^1 solution of (3.1). Henceforth, Equation (3.18) will be referred to as the *reduction* of (3.1) near x_*.

4. Existence Theory for Nonsingular DAE's of Index 1.

The results presented in this section are particular cases of the general theory in [RR91b] but the main existence and uniqueness Theorem 4.1 can here be given an independent and much simplified treatment by taking advantage of the special quasilinear form (2.1) of the DAE and by confining attention to index 1 problems as defined below. We retain the hypotheses and notation of Section 3.

Definition 4.1. *The quasilinear system (3.1) (satisfying Assumption 3.1) is a nonsingular DAE of index* 1 *if*

$$\{x \in U^n,\ G(x) \in \text{ rge } A(x)_{|T_x W}\}(^2) \Rightarrow \text{rank } A(x)_{|T_x W} = \text{ rank } A(x)(= r). \tag{4.1}$$

For nonsingular DAE's of index 1 the local existence and uniqueness theory is very similar to that for ODE's:

Theorem 4.1. *Let (3.1) be a nonsingular DAE of index* 1*. Then, for any given $x_* \in W_1 = \pi(TW \cap M) \subset W$ and $t_* \in \mathbb{R}$, there exists an open interval J containing t_* and a unique C^1 solution $x : J \to U^n$ of*

$$A(x)\dot{x} = G(x), \quad x(t_*) = x_*. \tag{4.2}$$

Moreover, no C^1 solution of (4.2) exists for $x_* \notin W_1$.

Proof: In Section 3 it was already noted that when $x : J \to U^n$ is a C^2 solution of (3.1) then $(x(t), \dot{x}(t)) \in TW \cap M$, $\forall t \in J$, and hence $x(J) \subset W_1 = \pi(TW \cap M)$. In particular, (4.2) cannot have a C^1 solution if $x_* \notin W_1$.

Suppose now that $x_* \in W_1 \subset W$ so that $(x_*, p_*) \in TW \cap M$ for some $p_* \in \mathbb{R}^n$. Then $p_* \in T_{x_*}W$ and $A(x_*)p_* = G(x_*)$ implies that $G(x_*) \in \text{ rge } A(x_*)_{|T_{x_*}W}$ and therefore, by

(2) Hence $x \in W$; see (3.7).

(4.1), that rank $A(x_*)_{|T_{x_*}W} = r$. Equivalently, this shows that

$$A(x_*) \in GL(\ T_{x_*}W,\ \mathrm{rge}\ A(x_*)\ ). \tag{4.3}$$

The reduction procedure of the previous section ensured the existence of a C^2 diffeomorphism φ of an open subset U^r of $\mathbb{R}^r$ onto an open neighborhood $\mathcal{O}$ of x_* in W such that on $\mathcal{O}$ (4.2) is equivalent to

$$A_1(\xi)\dot{\xi} = G_1(\xi), \quad \xi(t_*) = \xi_*, \tag{4.4}$$

where $\xi_* = \varphi^{-1}(x_*)$, $A_1(\xi) = P_*A(\varphi(\xi))D\varphi(\xi)$, $G_1(\xi) = P_*G(\varphi(\xi))$ and P_* denotes a projection operator onto rge $A(x_*)$.

For $\xi = \xi_*$ we have $A_1(\xi_*) = A(x_*)D\varphi(\xi_*) \in GL(\mathbb{R}^r, \mathrm{rge}\ A(x_*))$ by (4.3) and $D\varphi(\xi_*) \in GL(\mathbb{R}^r, T_{x_*}W)$. Thus, in the vicinity of ξ_*, (4.4) is equivalent to the explicit ODE $\dot{\xi} = A_1(\xi)^{-1}G_1(\xi), \quad \xi(t_*) = \xi_*$. Hence the classical ODE theory guarantees that on some open interval J containing t_* there exists a unique C^1 solution $\xi : J \to U^r$ of (4.4). Clearly then $x = \varphi \circ \xi$ is the unique C^1 solution of (4.2). □

Remark 4.1: It is easily checked from the above proof that the solution x in Theorem 4.1 is actually of class C^2. □

The Definition 4.1 of a nonsingular DAE of index 1 does not rule out the existence of points $x_* \in W$ where rank $A(x_*)_{|T_{x_*}W} < r$. Such points do not belong to the set $W_1 = \pi(TW \cap M)$ and hence, by Theorem 4.1, no C^1 solution to the corresponding initial value problem (4.2) exists. Nevertheless, on some interval $J = [t_* - T, t_*]$ or $J = [t_*, t_* + T]$ with $T > 0$, continuous functions may well exist which are differentiable on $J \setminus \{t_*\}$ and satisfy there $A(x(t))\dot{x}(t) = G(x(t))$. The impasse points discussed in the next section allow for solutions of this type.

5. Impasse Points of Nonsingular DAE's with Index 1.

First, we recall the definition and basic properties of the intrinsic derivative of a vector bundle morphism (see e.g. [AGV85], [GG73]). Let X be a manifold and E, F vector bundles with base X. A vector bundle morphism $\rho : E \to F$ can always be viewed as a section $\rho : X \to$ Hom (E, F) of the vector bundle Hom (E, F) with base X. Thus, for $x \in X$, we have $\rho(x) = (x, B(x))$ where $B(x) \in \mathcal{L}(E_x, F_x)$ and, as usual, E_x and F_x denote the fibers of E and F, respectively, above $x \in X$. If ρ is differentiable, $T_x\rho$ is a linear mapping $T_x\rho : T_xX \to T_{\rho(x)}$ Hom (E, F). As $T_{\rho(x)}$ Hom $(E, F) \simeq T_xX \times \mathcal{L}(E_x, F_x)$ and ρ is a section, we have $(T_x\rho)h = (h, \tilde{B}(x)h) \in T_xX \times \mathcal{L}(E_x, F_x)$, $\forall h \in T_xX$, where then $\tilde{B}(x) \in \mathcal{L}(T_xX, \mathcal{L}(E_x, F_x))$.

The intrinsic derivative of ρ at $x \in X$ is defined as the mapping (loc. cit.)

$$(5.1) \qquad h \in T_xX \longmapsto \sigma(x)[\tilde{B}(x)h]_{|\ker B(x)} \in \mathcal{L}(\ker B(x),\ \text{coker } B(x)),$$

where $\sigma(x) : F_x \to$ coker $B(x) = F_x/\text{rge } B(x)$ is the canonical projection.

When X is an open subset of $\mathbb{R}^r$ and $E = X \times \mathbb{R}^n$, $F = X \times \mathbb{R}^m$ then B is a mapping $B : X \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $\tilde{B}(x)$ is just the derivative $DB(x)$. Hence the intrinsic derivative (5.1) becomes

$$(5.2) \qquad h \in \mathbb{R}^r \longmapsto \sigma(x)[DB(x)h]_{|\ker B(x)} \in \mathcal{L}(\ker B(x),\ \text{coker } B(x)),$$

where $\sigma(x) : \mathbb{R}^m \to$ coker $B(x) = \mathbb{R}^m/\text{rge } B(x)$ is the canonical projection. It is well known that, in general, the calculation of the intrinsic derivative (5.1) reduces to the special case (5.2) via local trivializations of the vector bundles E and F (loc. cit.).

In our setting, a useful choice is given by $X = W$, $E = TW$, $F = R_W(A)$ where

$$(5.3) \qquad R_W(A) = \{(x, q) \in W \times \mathbb{R}^n;\ q \in \text{ rge } A(x)\}.$$

As rank $A(x) = r$ is independent of $x \in W$, it is indeed easily seen that $R_W(A)$ is a vector bundle with base W and r-dimensional fiber rge $A(x)$, $x \in W$. We set

$$(5.4) \qquad A_W : (x, p) \in TW \longmapsto A_W(x, p) = (x, A(x)p) \in R_W(A),$$

where, alternatively, A_W may be viewed as the section

$$(5.4') \qquad A_W : x \in W \longmapsto (x, A(x)_{|T_xW}) \in \text{ Hom } (TW, R_W(A)).$$

Because of the embedding $\text{rge } A(x) \subset \mathbb{R}^n$, $\text{coker } A(x)_{|T_xW} = \text{rge } A(x)/ \text{ rge } A(x)_{|T_xW}$ may be identified with

$$\text{coker } A(x)_{|T_xW} = \text{rge } A(x) \cap (\text{rge } A(x)_{|T_xW})^{\perp}, \tag{5.5}$$

so that $\sigma(x): \text{ rge } A(x) \to \text{ coker } A(x)_{|T_xW}$ corresponds to the restriction to $\text{rge } A(x)$ of the orthogonal projection onto $\text{coker } A(x)_{|T_xW}$.

Definition 5.1. *Let (3.1) be a nonsingular DAE with index* 1. *The point $x_* \in W$ is an impasse point of (3.1) if the two conditions*

$$\dim\ker A(x_*)_{|T_{x_*}W} = 1 \textit{ (i.e. rank } A(x_*)_{|T_{x_*}W} = r - 1), \tag{5.6}$$

$${}^iDA_W(x_*)_{|\ker A(x_*)_{|T_{x_*}W}} \neq 0. \tag{5.7}$$

hold, where ${}^iDA_W(x_*)$ denotes the intrinsic derivative of the bundle morphism A_W at x_*.

The impasse point x_* is said to be accessible or inaccessible if

$$\alpha(\xi)(e, \tilde{e}) \equiv \langle G(x_*), \tilde{e}\rangle\langle({}^iDA_W(x_*)e)e, \tilde{e}\rangle \tag{5.8}$$

is < 0 or > 0, respectively, for some pair $(e, \tilde{e})$ of nonzero vectors in $\ker A(x_*)_{|T_{x_*}W} \times$ coker $A(x_*)_{|T_{x_*}W}$ where $\langle\cdot,\cdot\rangle$ denotes the usual inner product of $\mathbb{R}^n$.

A few comments are in order. First, as in the case of (2.6) it follows that the form (5.8) must have either a positive or negative value for all pairs of nonzero vectors $(e, \tilde{e})$ in $\ker A(x_*)_{|T_{x_*}W} \times \text{ coker } A(x_*)_{|T_{x_*}W}$ if only this holds for one such pair of vectors. Secondly, it follows from (5.6) that ${}^iDA_W(x_*)$ is a linear mapping from $T_{x_*}W$ into the *one-dimensional* space $\mathcal{L}(\ker A(x_*)_{|T_{x_*}W}, \text{ coker } A(x_*)_{|T_{x_*}W})$. Hence, condition (5.7) is equivalent to the assumption that

$${}^iDA_W(x_*)e \neq 0, \tag{5.9}$$

for some, or equivalently every, nonzero vector $e \in \ker A(x_*)_{|T_{x_*}W}$. In turn, (5.9) holds if and only if for some, or equivalently every, nonzero vector $\tilde{e} \in$ coker $A(x_*)_{|T_{x_*}W}$ we have (using the identification in (5.5))

$$\langle ({}^{i}DA_W(x_*)e)e, \tilde{e}\rangle \neq 0. \tag{5.10}$$

On the other hand, because of $x_* \in W$ it follows that $G(x_*) \in$ rge $A(x_*)$ and therefore by (5.6) and the hypothesis that the DAE (3.1) is nonsingular (Definition 4.1) that $\langle G(x_*), \tilde{e}\rangle \neq 0$ for $\tilde{e} \in$ coker $A(x_*)_{|T_{x_*}W} \setminus \{0\}$. Together with (5.10) this shows that the left-hand side of (5.8) is nonzero. Moreover, it is obvious that its sign is independent of the nonzero vectors $e, \tilde{e}$.

Remark 5.1: It should be emphasized that the intrinsic derivative ${}^{i}DA_W$ *cannot* be replaced by the much simpler DA in (5.7) or (5.8), although the conditions obtained through the substitution formally make sense but are nevertheless irrelevant. To corroborate the above statement, it suffices to note that when ${}^{i}DA_W$ is replaced by DA in (5.7) or (5.8), these modified conditions never hold when $A(x)$ is independent of x, whereas constancy of $A(x)$ is not an obstacle to the existence of impasse points as we will see in the next section.

The following lemma relates Definition 5.1 to standard singular points of ODE's (see Section 2) via the reduction procedure of Section 3.

Lemma 5.1. *Let (3.1) be a nonsingular DAE with index* 1. *The point $x_* \in W$ is an accessible or inaccessible impasse point of (3.1) if and only if $\xi_* = \varphi^{-1}(x_*)$ is an accessible or inaccessible standard singular point, respectively, of the reduction (3.18) of (3.1) locally near x_*.*

Note: From this characterization, it follows that a *nonsingular* DAE may reduce to a *singular* ODE. Of course, this is not due to any ambiguity in the choice of terminology but, rather, to the fact that the reduction procedure may, of necessity, give rise to singularities.

Proof: The proof follows from the calculation of the intrinsic derivative ${}^{i}DA_W(x_*)$ using appropriate trivializations of the bundles TW and $R_W(A)$ near x_*. A trivialization of TW near x_* is given by the inverse of the mapping $(\xi, \eta) \in U^r \times \mathbb{R}^r \longmapsto (\varphi(\xi), D\varphi(\xi)\eta)$ where U^r is an open subset of $\mathbb{R}^r$ and φ a diffeomorphism of U^r onto an open neighborhood

$\mathcal{O}$ of x_* in W. On the other hand, with a projection operator P_* onto rge $A(x_*)$, the mapping $(x, q) \in R_W(A) \longmapsto (\varphi^{-1}(x),\ P_* q) \in U^r \times$ rge $A(x_*)$ provides a trivialization of $R_W(A)$ near x_* after shrinking U^r if necessary. Indeed, as noticed in the proof of Proposition 3.1, we have $P_* \in GL(\text{rge } A(x), \text{ rge } A(x_*))$ for $x \in W$ sufficiently close to x_*.

The local expression of the bundle morphism A_W in (5.4) becomes

$$(5.11) \qquad (\xi, \eta) \in U^r \times \mathbb{R}^r \longmapsto (\xi, P_* A(\varphi(\xi)) D\varphi(\xi)\eta) \in U^r \times \text{ rge } A(x_*) \simeq U^r \times \mathbb{R}^r.$$

As mentioned at the beginning of this section, the corresponding local expression of ${}^i DA_W(x_*)$ is the intrinsic derivative at $\xi_* = \varphi^{-1}(x_*)$ of the morphism (5.11) of trivial bundles over an open subset of $\mathbb{R}^r$. Thus, in line with (5.2), it is given by $h \in \mathbb{R}^r \longmapsto \sigma(\xi_*)[DA_1(\xi_*)h]_{|\ker A_1(\xi_*)}$ where

$$(5.12) \qquad A_1(\xi) = P_* A(\varphi(\xi)) D\varphi(\xi)$$

and $\sigma(\xi_*) : \mathbb{R}^r \to \mathbb{R}^r/\text{rge } A_1(\xi_*)$ is the canonical projection.

Conditions (5.6) and (5.7) amount to

$$(5.13) \qquad \dim \ker A_1(\xi_*) = 1 \quad (\text{i.e. rank } A_1(\xi_*) = r - 1)$$

and to $h \in \ker A_1(\xi_*) \longmapsto \sigma(\xi_*)[DA_1(\xi_*)h]_{|\ker A_1(\xi_*)} \neq 0$, respectively. But, in view of (5.13), it is clear that the latter relation reads

$$(5.14) \qquad (DA_1(\xi_*)\eta)\eta \notin \text{ rge } A_1(\xi_*),\ \forall \eta \in \ker A_1(\xi_*) \setminus \{0\}.$$

Finally, with

$$(5.15) \qquad G_1(\xi) = P_* G(\varphi(\xi)),\ \forall \xi \in U^r,$$

we have $G_1(\xi_*) = P_*G(x_*) = G(x_*)$ since $P_* = I$ on rge $A(x_*)$ and $x_* \in W$. As $\text{rge}A(x_*)_{|T_{x_*}W} =$ rge $A_1(\xi_*)$ is obvious from (5.12) and $D\varphi(\xi_*) \in GL(\mathbb{R}^r, T_{x_*}W)$, it follows that

$$G(x_*) \notin \text{ rge } A(x_*)_{|T_{x_*}W} \Leftrightarrow G_1(\xi_*) \notin \text{ rge } A_1(\xi_*). \tag{5.16}$$

Suppose now that (3.1) is a nonsingular DAE and $x_* \in W$ is an impasse point of (3.1), so that (5.13) and (5.14) hold, and $G(x_*) \notin$ rge $A(x_*)_{|T_{x_*}W}$ by (5.6) and the nonsingularity of (3.1). Then it follows from (5.16) that $G_1(\xi_*) \notin$ rge $A_1(\xi_*)$, and hence that $\xi_* = \varphi^{-1}(x_*)$ is a standard singular point of the reduction (3.18) of (3.1) near x_* since A_1 in (5.12) and G_1 in (5.15) are exactly as in that reduction. Conversely, if ξ_* is a standard singular point of the reduction of (3.1) near x_*, then (5.13) and (5.14) hold (and also $G(x_*) \notin$ rge $A(x_*)_{|T_{x_*}W}$ by (5.16)), and hence (5.6) and (5.7) also hold. This means that x_* is an impasse point of (3.1).

If x_* is an impasse point of (3.1); that is, if $\xi_* = \varphi^{-1}(x_*)$ is a standard singular point of the reduction (3.18) of (3.1) near x_*, it is straightforward to check that the criteria for accessibility and inaccessibility in Definitions 5.1 and 2.4 are equivalent. □

Remark 5.2: From the above proof we see that, by nonsingularity of the DAE (3.1), the condition $G_1(\xi_*) \notin$ rge $A_1(\xi_*)$ is always satisfied when $A_1(\xi_*)$ is singular and hence need not be checked again. In other words, to prove that ξ_* is a standard singular point of the reduction of the nonsingular DAE (3.1) (with index 1) near x_*, it suffices to show that conditions (2.3) and (2.5) of Definition 2.1 hold with $\xi = \xi_*$. □

As a by-product, Lemma 5.1 yields an equivalent and useful definition for impasse points of nonsingular DAE's. This definition is useful, for it rarely happens that intrinsic derivatives can be calculated without making explicit use of trivializations and local coordinates. Definition 5.1 is mostly theoretical, but is has the advantage of being intrinsic: that is, independent of any reduction procedure.

Motivated by Lemma 5.1 and the definition of solutions of singular ODE's in the vicinity of standard singular points (see Definition 2.3), we introduce the concept of a solution of a nonsingular DAE with index 1 near an impasse point as follows:

Definition 5.2. *Let the DAE (3.1) be nonsingular with index* 1 *and* x_* *an impasse point of (3.1). For given* $t_* \in \mathbb{R}$, *a solution of the initial value problem*

$$(5.17) \qquad \begin{cases} A(x)\dot{x} = G(x), \\ x(t_*) = x_*, \end{cases}$$

is a continuous function $x : J \to \mathbb{R}^n$ *on an interval* $J = [t_*, t_* + T)$ *or* $J = (t_* - T, t_*]$ *for some* $T > 0$ *which is of class* C^1 *on* $J^0 = J \setminus \{t_*\}$ *and satisfies* $A(x(t))\dot{x}(t) = G(x(t))$ *for* $t \in J^0$.

It is straightforward to check that the reduction procedure of Section 3 transforms the solutions of (5.17) in the above sense into solutions of

$$\begin{cases} A_1(\xi)\dot{\xi} = G_1(\xi), \\ \xi(t_*) = \xi_*, \end{cases}$$

in the sense of Definition 2.2, where $A_1(\xi)\dot{\xi} = G_1(\xi)$ is the reduction of (3.1) near x_*, and vice-versa. Thus, combining Theorem 2.1 and Lemma 5.1, we find at once:

Theorem 5.1. *Let the DAE (3.1) be nonsingular with index* 1 *and let* x_* *be an accessible (resp. inaccessible) impasse point of (3.1). Then, given* $t_* \in \mathbb{R}$, *the initial value problem (5.17) has exactly two solutions in the sense of Definition 5.2, both defined either on* $J = [t_*, t_* + T)$ *(resp.* $J = (t_* - T, t_*]$*) for some* $T > 0$. *Moreover,* $\lim_{t \to t_*} \|\dot{x}(t)\| = \infty$.

Remark 5.3: Naturally, the solutions found in Theorem 5.1 are C^1 solutions of (5.17) on $J^0 = (t_*, t_* + T)$ or $J^0 = (t_* - T, t_*)$ and hence are automatically C^2 on this interval (see Remark 4.1). Moreover, we have $x(J^0) \subset W_1 = \pi(TW \cap M)$ by Theorem 4.1. This shows that impasse points of (3.1) lie in the closure of W_1 in W. In fact, they lie on the boundary of W_1 in W since $x_* \in W_1$ is impossible for an impasse point by the nonsingularity of (3.1). □

The behavior of the solutions of (5.17) in the vicinity of an impasse point x_*, described in Theorem 5.1, justifies the terminology "impasse point", at least in the accessible case, since such points are reached in finite time by trajectories emanating elsewhere in W

which cannot be continuously extended beyond that impasse point. A justification of the terminology in the inaccessible case can be seen in the fact that inaccessible impasse points become accessible by simply reversing the evolution in time.

Because C^1 solutions of a (not necessarily nonsingular) DAE (3.1) lie in $W_1 = \pi(W \cap TM)$, their closure relative to the open set U^n must lie in W when W is closed in U^n, as, for example when the set $\{x \in U^n : \text{ rank } A(x) = r\}$ is closed in U^n (Proposition 3.1). This is the case in many practical applications but not the only mathematically sound possibility. When W is not closed in U^n, it becomes possible, a priori at least, for points $x_* \notin W$ to be reached in finite time by C^1 trajectories that cannot be continuously extended beyond x_*. Such points could also be called impasse points but they are not covered by our theory. Likewise, there may be points $x_* \in W$ corresponding to higher singularities of the reduction; that is, with $\dim\ker A_1(\xi_*) > 1$, at which C^1 trajectories stop. Examples of this kind are easily obtained by considering systems of DAE's in independent variables all having at (say) 0 an accessible impasse point in the sense of Definition 5.1. This is to say that our treatment of impasse points of DAE's is not exhaustive and that the impasse points of this paper, definitely the most frequently encountered ones, should perhaps be called "standard impasse points" since other types may exist in physical problems.

6. Impasse Points and Foldpoints.

A simple class of DAE's (3.1) has the form

$$\text{(6.1)} \qquad \begin{cases} g(x_1, x_2) = 0 \in \mathbb{R}^{n-r}, \\ \dot{x}_2 = f(x_1, x_2) \in \mathbb{R}^r, \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^{n-r} \times \mathbb{R}^r$ and $f : U^n \to \mathbb{R}^r, g : U^n \to \mathbb{R}^{n-r}$ are C^2 mappings. (The differentiability assumption is retained here for consistency with the previous setting although it would suffice to assume only that $f \in C^1$).

As we shall see, for (6.1) the impasse points of the previous section coincide essentially with the simple foldpoints of $g^{-1}(0)$ relative to the splitting $\mathbb{R}^n = \mathbb{R}^{n-r} \dotplus \mathbb{R}^r$. This result relates our work to various discussions of impasse points found in the literature where the form (6.1) is assumed, notably [CD89] (see also [M91]).

Theorem 6.1. *(i) The DAE (6.1) satisfies the general hypothesis of Section 3 if and only if the mapping g is a submersion in U^n.*

(ii) If (i) holds then (6.1) is nonsingular with index 1 if and only if

$$\{x \in g^{-1}(0),\ D_2 g(x) f(x) \in \text{ rge } D_1 g(x)\} \Rightarrow D_1 g(x) \in GL(\mathbb{R}^{n-r}), \tag{6.2}$$

where D_α denotes partial differentiation with respect to x_α, $\alpha = 1, 2$.

(iii) If both (i) and (ii) hold then $x_* \in U^n$ is an impasse point of (6.1) if and only if $x_* \in g^{-1}(0)$ and

$$\dim \ker D_1 g(x_*) = 1, \tag{6.3}$$

$$\langle D_1^2 g(x_*)(u)^2, \tilde{u} \rangle \neq 0, \tag{6.4}$$

for some, or equivalently all, pairs of nonzero vectors $(u, \tilde{u})$ in $\ker D_1 g(x_*) \times \ker D_1 g(x_*)^T$, where $\langle \cdot, \cdot \rangle$ denotes the natural inner product of $\mathbb{R}^{n-r}$. Moreover, the impasse point x_* is accessible or inaccessible if and only if

$$\alpha(x_*)(u, \tilde{u}) \equiv \langle D_2 g(x_*) f(x_*), \tilde{u} \rangle \langle D_1^2 g(x_*)(u, u), \tilde{u} \rangle \tag{6.5}$$

is < 0 or > 0, respectively, for some, or equivalently every, pair $(u, \tilde{u})$ of nonzero vectors in $\ker D_1 g(x_*) \times \ker D_1 g(x_*)^T$.

Proof: The DAE (6.1) is the special case of (3.1) in which

$$A(x) = \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix}, \quad I_r = \text{ identity of } \mathbb{R}^r, \tag{6.6}$$

is independent of x and

$$G(x) = \begin{pmatrix} g(x) \\ f(x) \end{pmatrix} \in \mathbb{R}^{n-r} \times \mathbb{R}^r. \tag{6.7}$$

For the proof of (i) we write

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^{n-r} \times \mathbb{R}^r, \quad F(x,p) = A(x)p - G(x) = \begin{pmatrix} g(x) \\ p_2 - f(x) \end{pmatrix}.$$

Then for $(h,q) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$DF(x,p)(h,q) = \begin{pmatrix} Dg(x) \\ q_2 - Df(x)h \end{pmatrix}.$$

which makes it obvious that F is a submersion on $U^n \times \mathbb{R}^n$ if and only if g is a submersion on U^n. In this case, $M = F^{-1}(0) = \{(x,p) \in U^n \times \mathbb{R}^n;\ x \in g^{-1}(0), p_2 = f(x)\}$ has the projection $W = \pi(M) = g^{-1}(0)$. Since $A(x)$ is independent of x and has rank r the condition rank $A(x) = r$, $\forall x \in W$, holds trivially.

For (ii) assume that g is a submersion on U^n. Because of $W = g^{-1}(0)$ we have

(6.8) $$x \in W \ \Rightarrow\ T_xW\{h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathbb{R}^{n-r} \times \mathbb{R}^r :\ D_1g(x)h_1 + D_2g(x)h_2 = 0\}$$

and from (6.6) and (6.8) we find that

(6.9) $$\ker A(x)_{|T_xW} = \ker D_1g(x) \times \{0\}$$

and

(6.10) $$\text{rge } A(x)_{|T_xW} = \{\begin{pmatrix} 0 \\ h_2 \end{pmatrix} \in \mathbb{R}^{n-r} \times \mathbb{R}^r : \exists h_1 \in \mathbb{R}^{n-r}, D_1g(x)h_1 + D_2g(x)h_2 = 0\}.$$

Thus, it follows from (6.7) and (6.10) that $x \in W = g^{-1}(0)$ and $G(x) \in$ rge $A(x)_{|T_xW}$ if and only if $x \in g^{-1}(0)$ and $D_2g(x)f(x) \in$ rge $D_1f(x)$. Also, since T_xW is r-dimensional, rank $A(x)_{|T_xW} = r$ if and only if $A(x)_{|T_xW}$ is one-to-one; that is, by (6.9) if $\ker D_1g(x) = \{0\}$. In turn, this amounts to $D_1g(x) \in GL(\mathbb{R}^{n-r})$, and hence the condition (6.2) is indeed equivalent to the DAE (6.1) being nonsingular with index 1 (see Definition 4.1).

Finally, for the proof of (iii) let $x_* \in U^n$ be an impasse point of (6.1). Then, by definition, $x_* \in W = g^{-1}(0)$. To prove that conditions (6.3) and (6.4) characterize x_* as an impasse point, we use Lemma 5.1 to show that (6.3) and (6.4) are equivalent to the condition that $\xi_* = \varphi^{-1}(x_*)$ is a standard singular point of the reduction of (6.1) near x_* (under the standing assumption that (6.1) is nonsingular with index 1).

With $\varphi_1 : U^r \to W = g^{-1}(0)$ being a C^2 diffeomorphism of an open subset U^r of $\mathbb{R}^r$ onto an open neighborhood $\mathcal{O}$ of x_* in W, the method of Section 3 with A and G given by (6.6) and (6.7), respectively, shows that the reduction of (6.1) near x_* has the form

$$A_1(\xi)\dot{\xi} = G_1(\xi), \tag{6.11}$$

with

$$A_1(\xi) = P_* D\varphi(\xi) \tag{6.12}$$

where P_* is the projection onto $\mathbb{R}^r$ corresponding to the splitting $\mathbb{R}^n = \mathbb{R}^{n-r} \times \mathbb{R}^r$ and

$$G_1(\xi) = f(\varphi(\xi))(= P_* G(\varphi(\xi))). \tag{6.13}$$

With $\xi_* = \varphi^{-1}(x_*)$ we have $D\varphi(\xi_*) \in GL(\mathbb{R}^r, T_{x_*}W)$. Thus, by (6.8) and (6.12) it follows that

$$\ker A_1(\xi_*) = D\varphi(\xi_*)^{-1}(\ker D_1 g(x_*)) \tag{6.14}$$

and therefore that $\dim\ker A_1(\xi_*) = 1$ if and only if $\dim\ker D_1 g(x_*) = 1$; that is, if (6.3) holds. Now, we show that (6.4) is equivalent to the condition

$$(DA_1(\xi_*)\eta)\eta \neq \text{ rge } A_1(\xi_*), \tag{6.15}$$

where $\eta \in \ker A_1(\xi_*) \setminus \{0\}$ and thus, by (6.14), $\eta = D\varphi(\xi_*)^{-1}u, u \in \ker D_1 g(x_*) \setminus \{0\}$. Because of $\dim\ker A_1(\xi_*) = 1$, (6.15) amounts to

$$\langle (DA_1(\xi_*)\eta)\eta \, , \tilde{\eta}\rangle \; \neq 0, \tag{6.16}$$

where $\tilde{\eta}$ is any nonzero vector in $(\text{rge } A_1(\xi_*))^\perp$. We claim that any such vector has the form $\tilde{\eta} = D_2 g(x_*)^T \tilde{u}$ with $\tilde{u} \in \ker D_1 g(x_*) \setminus \{0\}$. To see this, note first that $Dg(x_*)^T \in \mathcal{L}(\mathbb{R}^{n-r}, \mathbb{R}^n)$ is one-to-one since $Dg(x_*) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n-r})$ is onto. Thus, for $\tilde{u} \in \ker D_1 g(x_*)^T \setminus \{0\}$, we have $D_2 g(x_*)^T \tilde{u} \neq 0$. Next, from (6.8) and $D\varphi(\xi_*) \in GL(\mathbb{R}^r, T_{x_*} W)$ we deduce that

$$\text{rge } A_1(\xi_*) = \{h_2 \in \mathbb{R}^r : \exists h_1 \in \mathbb{R}^{n-r}, D_1 g(x_*) h_1 + D_2 g(x_*) h_2 = 0\}. \tag{6.17}$$

It follows at once that $D_2 g(x_*)^T \tilde{u} \in (\text{rge } A_1(\xi_*))^\perp$ for all $\tilde{u} \in \ker D_1 g(x_*)^T \setminus \{0\}$. This proves the claim since, evidently, $D_2 g(x_*)^T \tilde{u} \neq 0$ and $(\text{rge } A_1(\xi_*))^\perp$ is one-dimensional.

From (6.12) we deduce that (6.16) may be written as

$$\langle P_* D^2 \varphi(\xi_*)(\eta, \eta), (D_2 g(x_*))^T \tilde{u} \rangle \neq 0, \tag{6.18}$$

for $\eta = D\varphi(\xi_*)^{-1} u$ and $(u, \tilde{u})$ an arbitrary pair of nonzero vectors in $\ker D_1 g(x_*) \times \ker D_1 g(x_*)^T$. To calculate the left-hand side of (6.18) we use the relation $g \circ \varphi \equiv 0$ following from the definition of φ. Differentiating twice at $\xi = \xi_*$ we obtain $D^2 g(x_*)(D\varphi(\xi_*)\eta)^2 + Dg(x_*) D^2 \varphi(x_*)(\eta)^2 = 0$. Taking now the inner product with $\tilde{u}$ and using $\eta = D\varphi(\xi_*)^{-1} u$ we get $\langle D^2 \varphi(\xi_*)(\eta, \eta), (Dg(x_*))^T \tilde{u} \rangle = \langle D^2 g(x_*)(u, u), \tilde{u} \rangle$. But the left-hand side is just $\langle P_* D\varphi(\xi_*)(\eta, \eta)\ (D_2 g(x_*))^T \tilde{u} \rangle$ because $(D_1 g(x_*))^T \tilde{u} = 0$ and the right-hand side is $-\langle D_1^2 g(x_*)(u, u), \tilde{u} \rangle$ because cf $u \in \ker D_1 g(x_*) \subset \mathbb{R}^{n-r}$. Thus,

$$\langle P_* D^2 \varphi(\xi_*)(\eta, \eta), D_2 g(x_*)^T \tilde{u} \rangle = -\langle D_1^2 g(x_*)(u, u), \tilde{u} \rangle, \tag{6.19}$$

which shows that (6.4) is equivalent to (6.18) and hence to (6.15).

By Remark 5.2 the equivalence of (6.3) with $\dim \ker A_1(\xi_*) = 1$ and of (6.4) with (6.15) suffices to prove that, together, (6.3) and (6.4) are equivalent to x_* being an impasse point of (6.1).

For the accessibility or inaccessibility criterion note that by the previous calculation $\langle (DA_1(\xi_*)\eta)\eta, \tilde{\eta} \rangle = -\langle D_1^2 g(x_*)(u, u), \tilde{u} \rangle$ and that by (6.18) and $\tilde{\eta} = D_2 g(x_*)^T \tilde{u}$ we have

$\langle G_1(\xi_*), \tilde{\eta}\rangle = \langle f(x_*), D_2 g(x_*)^T \tilde{u}\rangle$. From these relations and the criterion (2.6) for accessibility or inaccessibility of ξ_* we obtain the criterion (6.5) for the accessibility or inaccessibility criterion of x_*. $\square$

Under the sole assumption that g is a submersion in U^n the conditions (6.3) and (6.4) characterize $x_* \in g^{-1}(0)$ as a simple foldpoint of $g^{-1}(0)$ relative to the splitting $\mathbb{R}^n = \mathbb{R}^{n-r} \times \mathbb{R}^r$; that is, as a turning point in the case when $r = n-1$ and hence g is scalar. The accessibility or inaccessibility criterion (6.5) has no geometric interpretation and translates a purely dynamic property.

In [CD89], Chua and Deng characterize an impasse point of (6.1) as a point $x_* = (x_{*1}, x_{*2})$ such that the mapping $(x_1, \lambda) \longmapsto g(x_1, x_{*2} + \lambda f(x_*))$ has a turning point at $(x_{1*}, 0)$. Now, a simple (or generic) turning point $(x_{1*}, 0)$ of this mapping is characterized by the three conditions: $\dim \ker D_1 g(x_*) = 1$, $D_2 g(x_*) f(x_*) \notin \operatorname{rge} D_1 g(x_*)$ and $\langle D_1^2 g(x_*)(u)^2, \tilde{u}\rangle \neq 0$ for every pair of nonzero vectors $(u, \tilde{u}) \in \ker D_1 g(x_*) \times \ker D_1 g(x_*)^T$. In our exposition, the first and third of these conditions are exactly (6.3) and (6.4), respectively, and the second one is contained in (6.2) and (6.3). Thus, for the special case of (6.1) our definition of impasse points is essentially the same as that of [CD89].

7. Examples.

7.1 A nonlinear LRC network. Consider a nonlinear LRC network, consisting of a resistor branch (1), a capacitor branch (2) and an inductor branch (3) in parallel. Denote by I_i the currents in the i^{th}-branch, and by V the (common) voltage drop in each branch. Let $C(V)$ and $L(I_3)$ represent the capacity and the inductivity, respectively, where $C : \mathbb{R} \to (0, \infty)$ and $L : \mathbb{R} \to (0, \infty)$ are smooth enough functions (see e.g. Smale [S72] or Takens [T76]). From Kirchhoff's law, we have $I_1 + I_2 + I_3 = 0$, while the time-evolution of the voltage drop and currents in the capacitor and inductor branches is given by $I_2 = C(V)\dot{V}$, $V = L(I_3)\dot{I}_3$. Finally, in the resistor branch (1), I_1 and V are related through an equation of the form $g(I_1, V) = 0$, where $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth enough function.

For convenience, we shall use the variables $x_i = I_i$, $1 \leq i \leq 3$, $x_4 = V$. In this notation, the equations governing the evolution of the voltage drop and currents of the above network have the form

$$A(x)\dot{x} = G(x) \tag{7.1}$$

where $x = (x_1, x_2, x_3, x_4)^T$ and

$$A(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & L(x_3) & 0 \\ 0 & 0 & 0 & C(x_4) \end{pmatrix}, \quad G(x) = \begin{pmatrix} x_1 + x_2 + x_3 \\ g(x_1, x_4) \\ x_4 \\ x_2 \end{pmatrix}. \tag{7.2}$$

Clearly, rank $A(x) = 2$, $\forall x \in \mathbb{R}^4$, since L and C assume strictly positive values by hypothesis. Under the mild condition that $(\partial g/\partial x_4)(x_1, x_4) \neq 0$ for $x \in \mathbb{R}^4$, the reduction procedure of Section 3 can be applied. In fact, here the manifold W is given by $W = \{x \in \mathbb{R}^4 : x_1 + x_2 + x_3 = 0,\ g(x_1, x_4) = 0\}$, and hence can be parametrized by $\xi = (x_1, x_2)$, i.e. $W = \{(x_1, x_2, -x_1 - x_2, \theta(x_1)) : (x_1, x_2) \in \mathbb{R}^2\}$, where $\theta : \mathbb{R} \to \mathbb{R}$ is characterized by $g(x_1, \theta(x_1)) = 0$. We leave it to the reader to check that the DAE (7.1) is nonsingular (Definition 4.1) in some open subset $U^4 \subset \mathbb{R}^4$ if and only if

$$\{x \in W \cap U^4, \frac{d\theta}{dx_1}(x_1) = 0\} \Rightarrow x_2 \neq 0, \tag{7.3}$$

and that the reduced system reads

$$A_1(x_1, x_2) \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = G_1(x_1, x_2), \tag{7.4}$$

where

$$A_1(x_1, x_2) = \begin{pmatrix} L(-x_1 - x_2) & L(-x_1 - x_2) \\ 0 & C(\theta(x_1))\frac{d\theta}{dx_1}(x_1) \end{pmatrix}, \quad G(x_1, x_2) = \begin{pmatrix} -\theta(x_1) \\ x_2 \end{pmatrix}.$$

From Lemma 5.1, $x = (x_1, x_2, -x_1 - x_2, \theta(x_1)) \in W \cap U^4$ is an impasse point of (7.1) if and only if (x_1, x_2) is a standard singular point of (7.4): that is, $\frac{d\theta}{dx_1}(x_1) = 0$, $\frac{d^2\theta}{dx_1^2}(x_1) \neq 0$.

Under the assumption (7.3) this is straightforward to verify; see also Remark 5.2. Moreover, accessibility or inaccessibility of the impasse point depends upon $x_2 \frac{d^2\theta}{dx_1^2}(x_1) < 0$ or > 0, respectively. For instance if $g(x_1, x_4) = x_4 - x_1^2 - \gamma$, $\gamma \in \mathbb{R}$ (an example chosen in Takens [T76]), then $\theta(x_1) = \gamma + x_1^2$ and the impasse points of (7.1) are points of the form $(0, x_2, -x_2, \gamma)$, $x_2 \neq 0$ (accessible if $x_2 < 0$, inaccessible if $x_2 > 0$). The DAE (7.1) is singular in any open subset U^4 containing the point $(0,0,0,\gamma)$, for (7.2) fails to hold and, in fact, this point is not an impasse point but a "funnel" if $\gamma \neq 0$ (see [T76]; see also [RR92] for further comments and numerical results).

Remark 7.1: The choice of $A(x)$ and $G(x)$ in (7.2) is not canonical. For instance, (7.1) is unchanged if, instead of (7.2), we choose

$$A(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, G(x) = \begin{pmatrix} x_1 + x_2 + x_3 \\ g(x_1, x_4) \\ x_4/L(x_3) \\ x_2/C(x_4) \end{pmatrix}.$$

With the above choice, the problem fits into the framework of Section 6 and hence the impasse points of (7.1) appear as foldpoints of the mapping

$$(x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_1 + x_2 + x_3 \\ g(x_1, x_4) \end{pmatrix},$$

relative to the splitting $\mathbb{R}^4 = \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}^2_{(x_3,x_4)}$. $\square$

7.2 One–parameter stationary problems. We shall show that one–parameter stationary problems, in which no evolution of any kind is involved, can be reformulated as a quasilinear DAE, usually of index ≤ 1, and that their turning points coincide with the impasse points of this paper. In fact, all the singularities, bifurcations and others, that may appear in a stationary one–parameter problem may also be characterized as singularities of the DAE formulation. Thus, the analysis of DAE's, their impasse points and their singularities encompasses the entire study of one–parameter stationary problems, their turning points and singularities.

Let us begin with the equation

$$f(\lambda, x) = 0, \tag{7.5}$$

where $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth enough function. If solutions are sought in the form $x = x(\lambda)$, a C^1 function of the parameter λ (which is legitimate as long as the derivative $D_x f(\lambda, x)$ is invertible) then $x(\lambda)$ is a solution of $D_\lambda f(\lambda, x) + D_x f(\lambda, x)\dot{x} = 0$, where the dot represents $d/d\lambda$. Introducing the auxiliary equation $\dot{\lambda} = 1$, this equation can be rewritten as

$$A(\lambda, x)\begin{pmatrix} \dot{\lambda} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where

$$A(\lambda, x) = \begin{pmatrix} 1 & 0 \\ D_\lambda f(\lambda, x) & D_x f(\lambda, x) \end{pmatrix}.$$

Evidently, $A(\lambda, x)$ is singular if and only if $D_x f(\lambda, x)$ is singular, and a straightforward verification shows that (λ, x) is a standard singular point of (7.6) if and only if

(i) $\dim \ker D_x f(\lambda, x) = 1$,

(ii) $D_\lambda f(\lambda, x) \notin \operatorname{rge} D_x f(\lambda, x)$,

(iii) $D_x^2 f(\lambda, x)h^2 \notin \operatorname{rge} D_x f(\lambda, x)$, $\forall h \in \ker D_x f(\lambda, x) - \{0\}$.

As is well-known, conditions (i) - (iii) above characterize the solution (x, λ) of (7.5) as a simple turning point of (7.5). Since if f is constant along the trajectories of (7.5), standard singular points of (7.6) along trajectories emanating at (λ_0, x_0) with $f(\lambda_0, x_0) = 0$ coincide with the simple turning points of (7.5). More generally, equivalence between singularity of $A(\lambda, x)$ and singularity of $D_x f(\lambda, x)$ shows that all the singularities of (7.5) appear as singularities of the quasilinear singular ODE (7.6).

Quasilinear DAE's rather than ODE's are needed to reformulate constrained problems, i.e. problems of the form

$$\begin{cases} g(x) = 0 \in \mathbb{R}^{n-r}, \\ f(\lambda, x) = 0 \in \mathbb{R}^r. \end{cases} \tag{7.7}$$

In (7.7), the equation $g(x) = 0$ may be viewed as a condition characterizing the admissible "state" variables x. Using the same procedure as before, i.e. differentiating the identity $f(\lambda, x(\lambda)) = 0$, we see that (7.7) can be reformulated as the problem of solving the DAE

$$A(\lambda, x)\begin{pmatrix} \dot{\lambda} \\ \dot{x} \end{pmatrix} = G(\lambda, x), \tag{7.8}$$

where $A(\lambda, x)$ is the $(n+1) \times (n+1)$ matrix

$$A(\lambda, x) = \begin{pmatrix} 1 & 0 \\ D_\lambda f(\lambda, x) & D_x f(\lambda, x) \\ 0 & 0 \end{pmatrix}$$

and $G(\lambda, x) \in \mathbb{R} \times \mathbb{R}^r \times \mathbb{R}^{n-r} \cong \mathbb{R}^{n+1}$ is given by $G(\lambda, x) = (1, 0, g(x))^T$ (independent of λ).

It is easily seen that under the general conditions ensuring that (7.8) is a nonsingular DAE, the impasse points of (7.8) along trajectories emanating at a point (λ_0, x_0) where $g(x_0) = 0$, and $f(\lambda_0, x_0) = 0$, coincide with the simple turning points of the equation $f(\lambda, \varphi(\xi)) = 0$, where $\varphi : \mathbb{R}^r \to \mathbb{R}^n$ is a local parametrization of the manifold $g^{-1}(0)$ (Note that $g^{-1}(0)$ is a manifold because (7.8) is a nonsingular DAE).

8. Higher Index Problems.

It was shown in Section 3 that under Assumption 3.1, the n-dimensional quasilinear DAE in $U^n \subset \mathbb{R}^n$

$$A(x)\dot{x} = G(x) \tag{8.1}$$

can be reduced locally to the r-dimensional form

$$A_1(\xi)\dot{\xi} = G_1(\xi), \tag{8.2}$$

where $r \leq n$ is the (constant) rank of $A(x)$. Recall that (8.1) was called a nonsingular DAE with index 1 if the conditions of Definition 4.1 are fulfilled. If so, (8.2) reduces to the ODE $\dot{\xi} = A_1(\xi)^{-1}G_1(\xi)$ (see the proof of Theorem 4.1).

Now, it may also happen that (8.2) is a nonsingular DAE with index 1. In this case. (8.1) will be called a nonsingular DAE with index 2. From the reduction procedure of Section 3, it appears that if ξ_* is an impasse point of (8.2), the images $x(t) = \varphi(\xi(t))$ of the two solutions of (8.2) emanating or terminating at ξ_* (in the notation of Section 3) solve (8.1) and both emanate or terminate at $x_* = \varphi(\xi_*)$. Also, no other solution of (8.1) passing through x_* exists. Thus, x_* possesses all the properties characterizing impasse points of quasilinear DAE's of index 1 and hence should be referred to as an impasse point of the quasilinear DAE (8.1) of index 2. It is straightforward (but somewhat cumbersome) to check that, as expected, this definition is independent of the specific parametrization φ used to reduce (8.1) to the form (8.2). It is equally clear how higher index (quasilinear) DAE's and their impasse points can be defined. Naturally, these definitions implicitly require sufficient smoothness of A and G in (8.1). Note that even if $A(x)$ in (8.1) is independent of x, $A_1(\xi)$ in (8.2) will not be constant in general. Thus, for higher index problems, the simplified framework of Section 6 cannot be used.

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This paper presents a mathematical characterization of the impasse points of quasi-linear DAE's $A(x)\dot{x} = G(x)$, where then $A(x)$ is a nxn matrix having constant but not full rank $r<n$ in the domain of interest. We show that a reduction procedure permits to reduce the DAE to a quasilinear ODE $A_1(\xi)\xi' = G_1(\xi)$ in R^r, and that impasse points correspond to special singularities of the reduced ODE. Impasse points of higher index DAE's can also be defined; the only difference is that the above reduction procedure takes more than one step.

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