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Diffusion of the Self Magnetic Fields of an Electron Beam through a Resistive Toroidal Chamber

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13. ABSTRACT (Maximum 200 words) We have studied the diffusion of the self magnetic field of a relativistic electron beam through a resistive toroidal chamber. In contrast with previous calculations, the solution is not limited near the minor axis of the torus. Under the assumptions of small aspect ratio, thin conducting wall and to zero order in toroidal corrections, it has been found that there are three characteristic times with which the magnetic field leaks out of the resistive torus. The computed fields have been used to determine the beam centroid orbit during trapping that follows the beam injection in the NRL modified betatron accelerator. The predicted orbit is in very good agreement with the results of the experiment.				
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DIFFUSION OF THE SELF MAGNETIC FIELDS OF AN ELECTRON BEAM THROUGH A RESISTIVE TOROIDAL CHAMBER

I. Introduction

Extensive work has been done in the past on the diffusion of external magnetic fields into hollow circular cylindrical conductors of infinite length¹⁻⁵ and also on the diffusion of the self magnetic field of beams out of hollow cylinders.⁶⁻⁹ The beam self magnetic field diffusion studies have also considered the effect of the diffusing fields on the beam dynamics and have furnished interesting results on the beam stability⁶⁻⁸ and beam trapping.⁹ However, these studies are linear and the expressions for the fields are valid only near the axis of the cylinder.

Following the installation of strong focusing windings in the NRL modified betatron accelerator it is routinely observed that for several combinations of injection parameters the injected beam consistently spirals from the injection position to the magnetic minor axis and is trapped.^{10,11} Attempts to explain this interesting phenomenon using the existing linear resistive model have been unsuccessful.¹¹ The decay rate predicted by the linear theory⁹ for the parameters of the experiment is at least 10-20 times longer than that observed in the experiment, even when wake field effects are taken into account.¹²

In contrast to the analysis^{9,12} that assumes the beam to be near the minor axis, the beam in the experiment during injection is at least temporarily near the wall. In addition, the geometry of the NRL device is toroidal¹⁰ and not cylindrical and, therefore, there are additional characteristic times⁵ that may modify the diffusion process.

The present work extends the results of the linear theory. The expressions for the diffusing fields are valid not only near the axis but almost over the entire cross-section of the chamber and toroidal effects

are included to lowest order. The results have been obtained under the following assumptions. First, it is assumed that the toroidal vacuum chamber has a small aspect ratio, i.e., the results are valid provided that the ratio of the minor to the major radius is much less than unity and the radial distances of the observation point from the minor axis is considerably smaller than the major radius of the torus. Second, since the results are confined in the vicinity of the toroidal chamber, propagation effects do not play any role, i.e., the displacement current is omitted in Maxwell's equations. Third, in order to obtain tractable analytical results, it is assumed that the conducting wall is thin, i.e., its thickness is much smaller than the minor radius of the torus. In this case, the analytical results are not valid very near the conducting wall. In the limit when the ratio of the wall thickness to the minor radius of the torus tends to zero, i.e., for a toroidal conducting shell, the analytical results are exact everywhere inside of the toroidal vessel. Finally, the analytical results on the beam dynamics are further simplified under the assumption that the current ring moves slowly in comparison to the fastest of the characteristic times that dictate the diffusion process.

Under the assumptions mentioned above, it is found that there are three characteristic times with which the magnetic field leaks out of a resistive torus, when a current ring turns on at $t = 0$ inside the torus. The shortest is the "plane" or "fast" diffusion time $\tau_{FD} = \mu_0 \sigma (b-a)^2 / \pi^2$, where σ is the wall conductivity, a and b are the inner and outer minor radii of the conducting wall and μ_0 is the permeability of the vacuum. The terms associated with τ_{FD} are responsible for the electric field to be zero at $t=0$ outside the torus since no leakage has occurred as yet. The "cylinder" diffusion time $\tau_D = \mu_0 \sigma (b-a) a / 2$ together with τ_{FD} determines the speed with which the fields penetrate the wall chamber so that the images

of these fields gradually disappear. Finally, the "loop" diffusion time $\tau_{oo} = 2\tau_D [\ln (8r_o/a) - 2]$, where r_o is the major radius of the torus, determines the speed with which the fields diffuse into the hole of the doughnut and is responsible for the gradual disappearance of the wall current. It turns out that the radial component of the self magnetic field, which is responsible for the beam trapping in the MBA, is independent of τ_{oo} , and, therefore, the "loop" diffusion time does not play any role in the resistive trapping of the beam. The expressions of the fields predicted by the present work have been used to compute the beam centroid orbit and several other trapping parameters measured in recent detailed beam trapping studies ¹³ in the NRL modified betatron accelerator. The shape of the computed orbits is very similar to those observed in the experiment.

In Section II, the diffusion problem for a current ring inside a toroidal conductor is formulated. In Section III, the vector potential for a current ring in the absence of the conductor is derived. This is the particular solution of the problem. In Section IV, the homogeneous solution inside and outside the torus (but not inside the conductor) is derived in toroidal geometry. The toroidal geometry removes the ambiguity on the value of certain constants associated with the logarithmic dependence of the solutions. In Section V, the initial conditions are established that determine the time-dependent arbitrary parameters in the homogeneous solutions. In Section VI, the solution inside the conductor is derived and the boundary conditions are applied. The vector potential is computed in the three regions inside and outside the torus and inside the conductor. In Section VII, approximate results are obtained under the assumption of a thin conducting wall. Section VIII contains exact analytic results for the shell model and Section IX provides a summary of the main results and lists the most important conclusions of the present study.

II. Formulation of the Diffusion Problem

The configuration and system of coordinates is shown in Fig. 1. The toroidal chamber has a major radius r_0 , an inner and outer minor radius a and b , respectively, and conductivity σ . In the presence of an external driver, namely, a current ring which is axisymmetric, time-dependent and is located inside the torus, the magnetic vector potential in that region is determined by the equation

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A}^{\text{in}} = -\mu_0 \vec{J}, \quad (1)$$

where the vector potential $\vec{A}$ has only one nonzero component A_θ^{in} that depends only on the cylindrical components (r, z) and on time. The current density $\vec{J}$ of the current ring has only one nonzero component J_θ which is equal to $J_\theta = I_c / \pi r_c^2$ inside the ring and zero outside it. Here, I_c is the ring current and r_c is its minor radius.

The magnetic field inside the conductor is determined from the vector potential $\vec{A}^{\text{con}}$ that is described by the diffusion equation

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A}^{\text{con}} = -\mu_0 \sigma \frac{\partial \vec{A}^{\text{con}}}{\partial t}, \quad (2)$$

where, again, $\vec{A}^{\text{con}}$ has only one component A_θ^{con} , which depends on (r, z) and on time. Finally, the vector potential outside the torus is determined by the homogeneous equation

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A}^{\text{out}} = 0 \quad (3)$$

where the component A_θ^{out} of $\vec{A}^{\text{out}}$ depends on (r, z) and on time. The magnetic and electric field components, in each region, are given by

$$B_r = - \frac{\partial A_\theta}{\partial z}, \quad (4a)$$

$$B_z = \frac{1}{r} \frac{\partial r A_\theta}{\partial r}, \quad (4b)$$

$$E_\theta = - \frac{\partial A_\theta}{\partial t}, \quad (4c)$$

where A_θ is one of the components defined above, depending on the region of interest. Notice that Eq. (2) is identical to Ampere's law combined with Ohm's law inside the conductor.

It is convenient to express the vector potential and the fields in terms of the local cylindrical coordinates (ρ, ϕ) which are related to the global cylindrical coordinates (r, z) by:

$$r = r_0 + \rho \cos \phi, \quad (5a)$$

$$z = \rho \sin \phi. \quad (5b)$$

Then, in the region inside the conductor, Eq. (2) reduces to

$$\begin{aligned} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial A_\theta^{\text{con}}}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 A_\theta^{\text{con}}}{\partial \phi^2} + \frac{\cos \phi \frac{\partial A_\theta^{\text{con}}}{\partial \rho} - \sin \phi \frac{1}{\rho} \frac{\partial A_\theta^{\text{con}}}{\partial \phi}}{r_0 + \rho \cos \phi} \\ - \frac{A_\theta^{\text{con}}}{(r_0 + \rho \cos \phi)^2} = + \mu_0 \sigma \frac{\partial A_\theta^{\text{con}}}{\partial t}, \end{aligned} \quad (6)$$

after taking account of the fact that A_θ^{con} is independent of the toroidal angle θ . Also, in the local coordinate system Eqs. (4a), (4b) are replaced by

$$B_\rho = - \frac{1}{\rho} \frac{\partial A_\theta}{\partial \phi} + \frac{\sin \phi}{r_0 + \rho \cos \phi} A_\theta, \quad (7a)$$

$$B_{\phi} = \frac{\partial A_{\theta}}{\partial \rho} + \frac{\cos \phi}{r_0 + \rho \cos \phi} A_{\theta}, \quad (7b)$$

while Eq. (4c) remains the same.

At the surface of the toroidal conductor, i.e., at $\rho = a$ and $\rho = b$, the tangential components of the electric and magnetic fields are continuous. Therefore, in the local coordinate system, the boundary conditions are:

$$E_{\theta}^{\text{in}}(\rho = a, \phi, t) = E_{\theta}^{\text{con}}(\rho = a, \phi, t), \quad (8a)$$

$$E_{\theta}^{\text{out}}(\rho = b, \phi, t) = E_{\theta}^{\text{con}}(\rho = b, \phi, t), \quad (8b)$$

$$B_{\phi}^{\text{in}}(\rho = a, \phi, t) = B_{\phi}^{\text{con}}(\rho = a, \phi, t), \quad (8c)$$

$$B_{\phi}^{\text{out}}(\rho = b, \phi, t) = B_{\phi}^{\text{con}}(\rho = b, \phi, t). \quad (8d)$$

Since the vector potential is zero at $t = 0$, the first two boundary conditions can also be expressed as

$$A_{\theta}^{\text{in}}(\rho = a, \phi, t) = A_{\theta}^{\text{con}}(\rho = a, \phi, t), \quad (9a)$$

$$A_{\theta}^{\text{out}}(\rho = b, \phi, t) = A_{\theta}^{\text{con}}(\rho = b, \phi, t). \quad (9b)$$

Thus, the diffusion fields in the three regions inside and outside the torus and inside the conductor are determined by the solutions of Eqs. (1), (3) and (6) with the boundary conditions given by Eqs. (9a), (9b), (8c) and (8d) on the inner and outer surface of the toroidal conductor.

III: Vector Potential of a Current Ring

In this section, an approximate expression for the vector potential of a current ring external driver is obtained. This is the particular solution of Eq. (1). For that purpose, the toroidal coordinate system associated with the current ring is used. Toroidal coordinates are most appropriate for a toroidal conductor and their significance will become evident in the next section when the arbitrariness as to the value of certain constants in the solution is removed.

The global cylindrical coordinates (r, z) are related to the ring toroidal coordinates (η', ξ') by:

$$r = b_c \frac{\sinh \eta'}{\cosh \eta' - \cos \xi'}, \quad (10a)$$

$$z = b_c \frac{\sin \xi'}{\cosh \eta' - \cos \xi'}, \quad (10b)$$

where b_c is a constant. These relations can be easily inverted, namely:

$$e^{-2\eta'} = \frac{(r - b_c)^2 + z^2}{(r + b_c)^2 + z^2}, \quad (11a)$$

$$e^{-\eta'} \cos \xi' = \frac{1}{2} \left(1 - \frac{b_c}{r}\right) + \frac{1}{2} \left(1 + \frac{b_c}{r}\right) e^{-2\eta'}, \quad (11b)$$

$$e^{-\eta'} \sin \xi' = \frac{z}{2r} (1 - e^{-2\eta'}). \quad (11c)$$

According to Eq. (11a), when η' is fixed, the coordinates (r, z) describe a circle whose radius is $b_c / \sinh \eta'$. If for $\eta' = \eta'_c$ this circle coincides with the current ring surface whose major radius is R_c , then it is straightforward to show that $b_c = [R_c^2 - r_c^2]^{1/2}$. The points (η', ξ') outside

the ring are determined by the inequality $\eta' < \eta'_c$, while the points inside the ring are determined by $\eta' > \eta'_c$.

In the absence of the toroidal conductor, the solution of Eq. (1) for a current ring with constant current density $J_\theta = I_c / \pi r_c^2$ is equal to

$$A_\theta^{\text{ext}} = \frac{\mu_0 I_c}{4\pi^2 r_c^2} \int_V \frac{\cos(\theta' - \theta'')}{|\vec{x}' - \vec{x}''|} d^3x'', \quad (12)$$

where $\vec{x}'$ and $\vec{x}''$ are the observation point and a point inside the ring, respectively,, and the integration is over the volume V of the ring. The Green's function of $|\vec{x}' - \vec{x}''|^{-1}$, in toroidal geometry is equal to ¹⁴

$$\begin{aligned} \frac{1}{|\vec{x}' - \vec{x}''|} &= \frac{1}{\pi b_c} (\cosh \eta' - \cos \xi')^{1/2} (\cosh \eta'' - \cos \xi'')^{1/2} \\ &\star \sum_{m,n=0}^{\infty} \varepsilon_n \varepsilon_m (-1)^n \frac{\Gamma(m-n+\frac{1}{2})}{\Gamma(m+n+\frac{1}{2})} \cos n(\theta' - \theta'') \cos m(\xi' - \xi'') \\ &\star \left\{ \begin{array}{ll} P_{m-1/2}^n (\cosh \eta') Q_{m-1/2}^n (\cosh \eta''); & \eta'' > \eta' \\ P_{m-1/2}^n (\cosh \eta'') Q_{m-1/2}^n (\cosh \eta'); & \eta'' < \eta' \end{array} \right\}, \quad (13) \end{aligned}$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ when $m = 1, 2, 3, \dots$, $\Gamma(x)$ is the gamma function, and $P_{m-1/2}^n (\cosh \eta)$, $Q_{m-1/2}^n (\cosh \eta)$ are the associated Legendre functions of the first and second kind, respectively. Without giving the details of the calculation, substitution of Eq. (13) into Eq. (12) leads to the following expression of the vector potential of the current ring:

$$A_\theta^{\text{ext}} = b_c (\cosh \eta' - \cos \xi')^{1/2} \sum_{m=0}^{\infty} \varepsilon_m a_m^{\text{ext}} Q_{m-1/2}^1 (\cosh \eta') \cos m \xi'$$

$$+ b_c (\cosh \eta' - \cos \xi')^{1/2} \sum_{m=0}^{\infty} \epsilon_m c_m^{\text{ext}} p_{m-1/2}^1 (\cosh \eta') \cos m \xi', \quad (14a)$$

inside the current ring, i.e., when $\eta' > \eta'_c$, and

$$A_{\theta}^{\text{ext}} = b_c (\cosh \eta' - \cos \xi')^{1/2} \sum_{m=0}^{\infty} \epsilon_m b_m^{\text{ext}} p_{m-1/2}^1 (\cosh \eta') \cos m \xi', \quad (14b)$$

outside the current ring, i.e., when $\eta' < \eta'_c$, where

$$a_m^{\text{ext}} = - \frac{4\sqrt{2}\mu_o I_c b_c}{3\pi^2 r_c^2} \frac{1}{m^2 - \frac{1}{4}} \int_{\eta'_c}^{\eta'} p_{m-1/2}^1 (\cosh \eta'') Q_{m-1/2}^2 (\cosh \eta'') \frac{d\eta''}{\sinh \eta''}, \quad (15a)$$

$$c_m^{\text{ext}} = - \frac{4\sqrt{2}\mu_o I_c b_c}{3\pi^2 r_c^2} \frac{1}{m^2 - \frac{1}{4}} \int_{\eta'}^{\infty} Q_{m-1/2}^1 (\cosh \eta'') Q_{m-1/2}^2 (\cosh \eta'') \frac{d\eta''}{\sinh \eta''} \quad (15b)$$

$$b_m^{\text{ext}} = - \frac{4\sqrt{2}\mu_o I_c b_c}{3\pi^2 r_c^2} \frac{1}{m^2 - \frac{1}{4}} \int_{\eta'_c}^{\infty} Q_{m-1/2}^1 (\cosh \eta'') Q_{m-1/2}^2 (\cosh \eta'') \frac{d\eta''}{\sinh \eta''}, \quad (15c)$$

and $\eta'_c = \ln [(R_c + r_c + b_c)/(R_c + r_c - b_c)]$. In the derivation of Eqs. (14a), (14b) from Eq. (12), use was made of the identity¹⁴

$$\int_0^{\pi} \frac{\cos m \xi''}{(\cosh \eta'' - \cos \xi'')^{5/2}} d\xi'' = \frac{4\sqrt{2}}{3 \sinh^2 \eta''} Q_{m-1/2}^2 (\cosh \eta''). \quad (16)$$

The toroidal functions $P_m^n - 1/2 (\cosh \eta)$ and $Q_m^n - 1/2 (\cosh \eta)$ appearing in Eqs. (13), (14a), (14b), (15a), (15b) and (15c) are given by the following exact expressions: ¹⁵

$$\begin{aligned}
 P_m^n - 1/2 (\cosh \eta) &= \frac{2^n \Gamma(m)}{\pi^{1/2} \Gamma(m - n + \frac{1}{2})} (\sinh \eta)^n e^{(m - n - \frac{1}{2})\eta} \\
 &+ (1 - \delta_{m0}) \sum_{s=0}^{m-1} \frac{(n + \frac{1}{2})_s (n - m + \frac{1}{2})_s}{s! (1 - m)_s} e^{-2s\eta} \\
 &+ \frac{(-1)^n 2^{n+1} \Gamma(m+n+\frac{1}{2})}{\pi^{3/2} \Gamma(m+1)} (\sinh \eta)^n e^{-(m+n+\frac{1}{2})\eta} \\
 &+ \sum_{s=0}^{\infty} \frac{(n+\frac{1}{2})_s (m+n+\frac{1}{2})_s}{s! (m+1)_s} e^{-2s\eta} [\ln(4e^\eta) + u_s + u_{m+s} - v_{n+s} - v_{m+n+s}],
 \end{aligned} \tag{17a}$$

$$\begin{aligned}
 Q_m^n - 1/2 (\cosh \eta) &= \frac{(-1)^n 2^n \pi^{1/2} \Gamma(m+n+\frac{1}{2})}{\Gamma(m+1)} (\sinh \eta)^n \\
 &- (m+n+\frac{1}{2})\eta \sum_{s=0}^{\infty} \frac{(n+\frac{1}{2})_s (m+n+\frac{1}{2})_s}{s! (m+1)_s} e^{-2s\eta},
 \end{aligned} \tag{17b}$$

where

$$(a)_s = a(a+1)(a+2)\dots(a+s-1), \quad (a)_0 = 1, \tag{18a}$$

$$u_n = \frac{1}{2} \sum_{k=1}^n \frac{1}{k}, \quad u_0 = 0, \tag{18b}$$

$$v_n = \sum_{k=1}^n \frac{1}{2k-1}, \quad v_0 = 0. \quad (18c)$$

For $m=0$, the first term in Eq. (17a) is omitted, since $\delta_{mn} = 1$ for $m = n$, and $\delta_{mn} = 0$ for $m \neq n$. The expressions above are appropriate for the region inside as well as outside the torus but on its vicinity.

Up to this point, the results given above are exact. In the following, an approximate expression of $\Lambda_\theta^{\text{ext}}$ will be obtained from Eqs. (14a) and (14b) under the assumption of a small aspect ratio r_c/R_c of the current ring by keeping terms up to order $e^{-\eta'_c}$. Notice that, when $r_c/R_c \ll 1$, then to first order in the aspect ratio, or to first order in the toroidal corrections, we have the approximate relations: $b_c = R_c$, $e^{-\eta'_c} = r_c/2R_c$ and $e^{-\eta'} = \rho'/2R_c$. Here, (ρ', ϕ') are the local cylindrical coordinates with respect to the ring position, i.e.,

$$r = R_c + \rho' \cos \phi', \quad (19a)$$

$$z = \rho' \sin \phi'. \quad (19b)$$

Making use of the identity

$$(\cosh \eta' - \cos \xi')^{1/2} = \sum_{m=0}^{\infty} \epsilon_m D_m(\eta') \cos m \xi', \quad (20)$$

where

$$D_0(\eta') = \frac{\sqrt{2}}{2\pi} e^{\eta'} (1 + e^{-2\eta'}) \left[Q_{-1/2}(\cosh \eta') - \frac{2 e^{-\eta'}}{1 + e^{-2\eta'}} Q_{1/2}(\cosh \eta') \right], \quad (21a)$$

$$D_m(\eta') = \frac{\sqrt{2}}{2\pi} e^{\eta'} (1+e^{-2\eta'}) \left[Q_{m-1/2}(\cosh \eta') - \frac{e^{-\eta'}}{1+e^{-2\eta'}} \left(Q_{m-1/2}(\cosh \eta') + Q_{m+1/2}(\cosh \eta') \right) \right], \quad (21b)$$

and also of Eq. (17b), a straightforward calculation leads to the approximate expression, to first order in toroidal corrections,

$$b_c (\cosh \eta' - \cos \xi')^{1/2} \sum_{m=0}^{\infty} \epsilon_m a_m^{\text{ext}} Q_{m-1/2}^1(\cosh \eta') \cos m \xi' \\ = - \frac{\pi}{2\sqrt{2}} b_c \left[a_0^{\text{ext}} - (a_0^{\text{ext}} - 3 a_1^{\text{ext}}) e^{-\eta'} \cos \xi' \right]. \quad (22)$$

The number of terms kept on the right hand side of Eq. (22) was determined by the fact that the quantities $b_c a_m^{\text{ext}}$ are of zero order in toroidal corrections, as is indicated by Eqs. (15a), (17a) and (17b). Similarly, the quantities $b_c b_m^{\text{ext}}$ and $b_c c_m^{\text{ext}}$ are of order $(b_c/r_c)^2 e^{-2(m+1)\eta'}$ or $(b_c/r_c)^2 e^{-2(m+1)\eta'_c}$ and, consequently, we have the approximate expression

$$b_c (\cosh \eta' - \cos \xi')^{1/2} \sum_{m=0}^{\infty} \epsilon_m b_m^{\text{ext}} P_{m-1/2}^1(\cosh \eta') \cos m \xi' \\ = - \frac{\sqrt{2}}{2\pi} b_c \left[b_0^{\text{ext}} (\ln (4e^{\eta'}) - 2) - b_0^{\text{ext}} (\ln (4e^{\eta'}) - 2) e^{-\eta'} \cos \xi' - 2b_1^{\text{ext}} e^{\eta'} \cos \xi' \right], \quad (23)$$

to first order in toroidal corrections. The same relation above holds for the quantities $b_c c_m^{\text{ext}}$.

From Eqs. (14a), (14b), (19a), (19b), (22) and (23), it follows that in the local coordinate system (ρ', ϕ') of the current ring and to first order in r_c/R_c , we have inside the ring, i.e., when $\rho' \leq r_c$:

$$\begin{aligned}
A_{\theta}^{\text{ext}} \approx & -\frac{\pi}{2\sqrt{2}} R_c \left[a_0^{\text{ext}} - \frac{1}{2} (a_0^{\text{ext}} - 3 a_1^{\text{ext}}) \frac{\rho' \cos \phi'}{R_c} \right] \\
& - \frac{\sqrt{2}}{2\pi} R_c \left[c_0^{\text{ext}} \left(\ln \frac{8R_c}{\rho'} - 2 \right) + \frac{c_0^{\text{ext}}}{2} \left(\frac{\rho' \cos \phi'}{R_c} - \frac{r_c^2 \cos \phi'}{R_c \rho'} \right) \right. \\
& \left. - \frac{c_0^{\text{ext}}}{2} \frac{\rho' \cos \phi'}{R_c} \left(\ln \frac{8R_c}{\rho'} - 2 \right) - 4c_1^{\text{ext}} \frac{R_c \cos \phi'}{\rho'} \right], \quad (24a)
\end{aligned}$$

while outside the ring, i.e., when $\rho' \geq r_c$, we have:

$$\begin{aligned}
A_{\theta}^{\text{ext}} \approx & -\frac{\sqrt{2}}{2\pi} R_c \left[b_0^{\text{ext}} \left(\ln \frac{8R_c}{\rho'} - 2 \right) + \frac{b_0^{\text{ext}}}{2} \left(\frac{\rho' \cos \phi'}{R_c} - \frac{r_c^2 \cos \phi'}{R_c \rho'} \right) \right. \\
& \left. - \frac{b_0^{\text{ext}}}{2} \frac{\rho' \cos \phi'}{R_c} \left(\ln \frac{8R_c}{\rho'} - 2 \right) - 4b_1^{\text{ext}} \frac{R_c \cos \phi'}{\rho'} \right]. \quad (24b)
\end{aligned}$$

Next, we need the approximate expressions, to first order in r_c/R_c , for the quantities $R_c a_m^{\text{ext}}$, $R_c c_m^{\text{ext}}$ and $R_c b_m^{\text{ext}}$, when $m = 0, 1$. From Eqs. (15a), (15b), (15c), (17a), and (17b), it is easy to show that

$$R_c a_0^{\text{ext}} \approx -\frac{\sqrt{2} \mu_0 I_c}{\pi^2} \left[\ln \frac{8R_c}{r_c} - \frac{3}{2} - \left(\frac{\rho'}{r_c} \right)^2 \left(\ln \frac{8R_c}{\rho'} - \frac{3}{2} \right) \right], \quad (25a)$$

$$R_c a_1^{\text{ext}} \approx \frac{5 \sqrt{2} \mu_0 I_c}{6\pi^2} \left[\left(\frac{\rho'}{r_c} \right)^2 - 1 \right], \quad (25b)$$

$$R_c c_0^{\text{ext}} \approx -\frac{\sqrt{2}}{2} \mu_0 I_c \left(\frac{\rho'}{r_c} \right)^2, \quad (25c)$$

$$R_c c_1^{\text{ext}} \approx \frac{5 \sqrt{2} \mu_0 I_c}{64} \frac{\rho'^4}{R_c^2 r_c^2}, \quad (25d)$$

$$R_c b_o^{\text{ext}} = - \frac{\sqrt{2}}{2} \mu_o I_c, \quad (25e)$$

$$R_c b_1^{\text{ext}} = \frac{5 \sqrt{2} \mu_o I_c}{64} \left(\frac{r_c}{R_c} \right)^2. \quad (25f)$$

Substitution of Eqs. (25a) through (25f) into Eqs. (24a) and (24b) leads to the following expressions for A_θ^{ext} to first order in r_c/R_c in the local coordinate system of the ring:

$$\begin{aligned} A_\theta^{\text{ext}} = & \frac{\mu_o}{2\pi} I_c \left[\ln \frac{8R_c}{r_c} - \frac{3}{2} - \frac{1}{2} \left(\frac{\rho'}{r_c} \right)^2 \right] \\ & - \frac{\mu_o}{4\pi} I_c \frac{\rho' \cos \phi'}{R_c} \left[\ln \frac{8R_c}{r_c} - 3 - \frac{1}{4} \left(\frac{\rho'}{r_c} \right)^2 \right], \end{aligned} \quad (26a)$$

when $\rho' \leq r_c$, and

$$\begin{aligned} A_\theta^{\text{ext}} = & \frac{\mu_o}{2\pi} I_c \left[\ln \frac{8R_c}{\rho'} - 2 \right] \\ & - \frac{\mu_o}{4\pi} I_c \frac{\rho' \cos \phi'}{R_c} \left[\ln \frac{8R_c}{\rho'} - 3 - \frac{1}{4} \left(\frac{r_c}{\rho'} \right)^2 \right], \end{aligned} \quad (26b)$$

when $\rho' \geq r_c$. Let (Δ, α) be the ring position in the local cylindrical coordinates of the toroidal conductor (cf. Eqs. (5a), (5b)). Then, in the relation above, R_c , ρ' and $\rho' \cos \phi'$ are replaced by

$$R_c = r_o + \Delta \cos \alpha, \quad (27a)$$

$$\rho' = \left[\rho^2 + \Delta^2 - 2 \rho \Delta \cos (\phi - \alpha) \right]^{1/2}, \quad (27b)$$

$$\rho' \cos \phi' = \rho \cos \phi - \Delta \cos \alpha. \quad (27c)$$

Equations (26a) and (26b) are useful in the diffusion problem of a current ring inside a toroidal conducting shell with first order toroidal corrections. This will be reported elsewhere. Here, only the zero order solution to the diffusion problem is considered. In this case, the vector potential of the current ring, in the absence of the conductor, is equal to

$$A_{\theta 0}^{\text{ext}} = \frac{\mu_0}{2\pi} I_c \left[\ln \frac{8r_0}{r_c} - \frac{3}{2} - \frac{1}{2} \left(\frac{\rho'}{r_c} \right)^2 \right], \quad (28a)$$

inside the ring, and

$$A_{\theta 0}^{\text{ext}} = \frac{\mu_0}{2\pi} I_c \left[\ln \frac{8r_0}{\rho'} - 2 \right], \quad (28b)$$

outside the ring. Equation (28b) can also be written as:

$$\begin{aligned} A_{\theta 0}^{\text{ext}} &= \frac{\mu_0}{2\pi} I_c \left(\ln \frac{8r_0}{\rho} - 2 \right) \\ &- \frac{\mu_0}{4\pi} I_c \ln \left[1 + \left(\frac{\Delta}{\rho} \right)^2 - 2 \frac{\Delta}{\rho} \cos (\phi - \alpha) \right]. \end{aligned} \quad (28c)$$

This expression of the ring vector potential is used in the application of the boundary conditions at the inner surface of the toroidal conductor.

IV. Homogeneous Solution of the Vector Potential

Inside the torus, the most general solution of Eq. (1) is

$$A_{\theta}^{\text{in}} = A_{\theta}^{\text{ext}} + A_{\theta}^{\text{in}}, \quad (29)$$

where A_{θ}^{in} is the homogeneous solution of Eq. (1). The exact homogeneous solution can be expressed most appropriately in toroidal coordinates with respect to the toroidal conductor rather than the current ring. For that purpose, we replace ξ', η', b_c by $\xi_1, \eta_1, b_1 = [r_0^2 - a^2]^{1/2}$, in Eqs. (10a), (10b), (11a), (11b), and (11c). When $\eta_1 = \eta_{1c}$, where $\eta_{1c} = \ln[(r_0 + a + b_1)/(r_0 + a - b_1)]$, the coordinates (r, z) describe a circle which coincides with the inner surface of the toroidal conductor. In terms of the toroidal coordinates (η_1, ξ_1) , the exact homogeneous solution of Eq. (1) inside the torus is

$$A_{\theta}^{\text{in}} = b_1 (\cosh \eta_1 - \cos \xi_1)^{1/2} \sum_{m=0}^{\infty} \epsilon_m 0_m^{-1/2} (\cosh \eta_1) \\ * [\tilde{a}_m^{(c)} \cos m \xi_1 + \tilde{a}_m^{(s)} \sin m \xi_1]. \quad (30)$$

For small aspect ratio a/r_0 , and, to lowest order in this ratio, we have the approximate relations:

$$b_1 = r_0, \quad e^{-\eta_{1c}} = a/2r_0 \quad \text{and} \quad e^{-(\eta_1 + i\xi_1)} = \rho e^{-i\phi/2r_0},$$

where (ρ, ϕ) are the local cylindrical coordinates with respect to the toroidal conductor (cf. Eqs. (5a), (5b)). Moreover, if it is assumed that each of the coefficients $b_1 \tilde{a}_m^{(c)}$, $b_1 \tilde{a}_m^{(s)}$ is of order $(r_0/a)^m$, then it follows from Eqs. (17b) and (30) that the homogeneous solution of Eq. (1), to zero order in toroidal corrections, is equal to

$$A_{\theta 0}^{\text{in}} = -b_1 \sum_{m=0}^{\infty} \epsilon_m \frac{\pi^{1/2} \Gamma(m + \frac{3}{2})}{2^{1/2} \Gamma(m + 1)} \left(\frac{\rho}{2r_0}\right)^m * \left[\tilde{a}_m^{(c)} \cos m\phi + \tilde{a}_m^{(s)} \sin m\phi \right]. \quad (31)$$

By redefining the coefficients $\tilde{a}_m^{(c)}$, $\tilde{a}_m^{(s)}$, in terms of the zero order coefficients a_0 , $a_m^{(c)}$, $a_m^{(s)}$, we conclude from Eqs. (29) and (31) that the most general solution inside the torus (inside and outside the current ring) and to zero order in toroidal corrections is

$$A_{\theta 0}^{\text{in}} = A_{\theta 0}^{\text{ext}} + a_0 + \sum_{m=1}^{\infty} \left(\frac{\rho}{a}\right)^m \left[a_m^{(c)} \cos m\phi + a_m^{(s)} \sin m\phi \right] \quad (32)$$

The undefined coefficients a_0 , $a_m^{(c)}$, $a_m^{(s)}$ will be determined from the boundary conditions.

The homogeneous zero order solution $A_{\theta 0}^{\text{in}}$ satisfies also the zero order homogeneous equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial A_{\theta 0}}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 A_{\theta 0}}{\partial \phi^2} = 0, \quad (33)$$

which follows from Eq. (1) by expressing it in the local coordinates of the toroidal conductor (cf. Eq. (6)) and neglecting the terms with toroidal corrections.

For the vector potential outside the torus, we define the toroidal coordinates (η_0, ξ_0) in a similar fashion, i.e., we replace ξ' , η' and b_c by ξ_0 , η_0 , $b_0 = [r_0^2 - b^2]^{1/2}$, in Eqs. (10a), (10b), (11a), (11b) and (11c). When $\eta_0 = \eta_{0c}$, where $\eta_{0c} = \ln[(r_0 + b + b_0)/(r_0 + b - b_0)]$, the coordinates (r, z) describe a circle which coincides with the outer surface of the toroidal conductor. In terms of the toroidal coordinates (η_0, ξ_0) , the exact solution of Eq. (3) outside the torus is

$$A_{\theta}^{\text{out}} = b_0 (\cosh \eta_0 - \cos \xi_0)^{1/2} \sum_{m=0}^{\infty} \epsilon_m P_m^1 - 1/2 (\cosh \eta_0) \\ * [\bar{b}_m^{(c)} \cos m \xi_0 + \bar{b}_m^{(s)} \sin m \xi_0]. \quad (34)$$

As before, for small aspect ratio b/r_0 , we have the approximate relations:

$b_0 \approx r_0$, $e^{-\eta_{0c}} \approx b/2r_0$ and $e^{\eta_0 + i\xi_0} \approx 2r_0 e^{i\phi/\rho}$. If it is assumed that each of the coefficients $b_0 \bar{b}_m^{(c)}$, $b_0 \bar{b}_m^{(s)}$ is of order $(b/r_0)^m$, then it follows from Eqs. (17a) and (34) that the solution of Eq. (3), to zero order in toroidal corrections, is equal to

$$A_{\theta 0}^{\text{out}} = \left(\ln \frac{8r_0}{\rho} - 2 \right) b_0 + \sum_{m=1}^{\infty} \left(\frac{b}{\rho} \right)^m [\bar{b}_m^{(c)} \cos m\phi + \bar{b}_m^{(s)} \sin m\phi]. \quad (35)$$

Here, the coefficients $\bar{b}_m^{(c)}$, $\bar{b}_m^{(s)}$ have been redefined in terms of the zero order coefficients b_0 , $b_m^{(c)}$, $b_m^{(s)}$. These undefined coefficients will be determined from the boundary conditions.

It is apparent that Eq. (35) satisfies the zero order homogeneous equation (33), but the solution of Eq. (33) does not provide all the information included in Eq. (35). The most general solution of Eq. (33) which is independent of the toroidal angle ϕ is equal to $C_0 + C_1 \ln \rho$ where C_0 and C_1 are arbitrary constants. But Eq. (35) indicates that these two constants are related to each other and their dependence on each other is established only by solving the problem in toroidal geometry rather than making some ad hoc assumption. For example, if we assumed that, at infinite time, the vector potential outside the conductor is equal to that of the current ring in the absence of the conductor, we would probably obtain the correct relationship between C_0 and C_1 , but this assumption

would be imposed on the solution of the diffusion problem rather than coming out naturally as a result from the solution.

V. Initial Conditions of the Vector Potential

When I_c , Δ and α are time dependent quantities, the coefficients a_0 , $a_m^{(c)}$, $a_m^{(s)}$, b_0 , $b_m^{(c)}$, and $b_m^{(s)}$ in Eqs. (32) and (35) are also time dependent. Since outside the torus the vector potential is zero at $t = 0$, $b_0(t)$, $b_m^{(c)}(t)$ and $b_m^{(s)}(t)$ are also zero at $t = 0$. In addition, the vector potential is zero at $t = 0$ inside the conductor. From the continuity of the vector potential at the inner surface of the conductor and from Eq. (32), it follows that the coefficients $a_0(t)$, $a_m^{(c)}(t)$, $a_m^{(s)}(t)$ are not zero at $t = 0$. Since the image fields constitute a zero order homogeneous solution inside the torus, it is convenient to redefine the as yet undetermined coefficients

$a_m^{(c)}(t)$, $a_m^{(s)}(t)$ by subtracting the image solution from them, so that they are zero at $t = 0$. As to the coefficient $a_0(t)$, in order that it becomes zero at $t = 0$, it is convenient to redefine it by replacing it with $a_0(t) - (\mu_0/2\pi)I_c(t) (\ln 8r_0/a - 2)$. Then, Eq. (32) should be replaced by

$$\begin{aligned}
 A_{\theta 0}^{in}(\rho, \phi, t) = & \frac{\mu_0}{4\pi} I_c(t) \left[2 \left(\ln \frac{a}{r_c} + \frac{1}{2} \right) \right. \\
 & - \frac{\rho^2 + \Delta^2(t) - 2\rho \Delta(t) \cos(\phi - \alpha(t))}{r_c^2} \\
 & + \ln \left(1 + \left(\frac{\rho \Delta(t)}{a^2} \right)^2 - 2 \frac{\rho \Delta(t)}{a^2} \cos(\phi - \alpha(t)) \right) \\
 & + a_0(t) + \sum_{m=1}^{\infty} \left(\frac{\rho}{a} \right)^m \left[a_m^{(c)}(t) \cos m\phi + a_m^{(s)}(t) \sin m\phi \right],
 \end{aligned} \tag{36a}$$

inside the ring, and

$$\begin{aligned}
A_{\theta 0}^{\text{in}}(\rho, \phi, t) = & \frac{\mu_0}{4\pi} I_c(t) \left[2 \ln \frac{a}{\rho} \right. \\
& - \ln \left(1 + \left(\frac{\Delta(t)}{\rho} \right)^2 - 2 \frac{\Delta(t)}{\rho} \cos(\phi - \alpha(t)) \right) \\
& + \ln \left(1 + \left(\frac{\rho \Delta(t)}{a^2} \right)^2 - 2 \frac{\rho \Delta(t)}{a^2} \cos(\phi - \alpha(t)) \right) \Big] \\
& + a_0(t) + \sum_{n=1}^{\infty} \left(\frac{\rho}{a} \right)^n \left[a_n^{(c)}(t) \cos n\phi + a_n^{(s)}(t) \sin n\phi \right],
\end{aligned} \tag{36b}$$

outside the current ring but inside the torus. The vector potential from the image has a logarithmic singularity at the image position $(a^2/\Delta(t), \alpha(t))$, which lies outside the inner surface of the conductor. Therefore, inside the torus, it is a zero order homogeneous solution of Eq. (33).

When $\Delta(t)/\rho < 1$, Eq. (36b) is equivalent to

$$\begin{aligned}
A_{\theta 0}^{\text{in}}(\rho, \phi, t) = & \frac{\mu_0}{2\pi} I_c(t) \left[\ln \frac{a}{\rho} \right. \\
& - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\Delta(t)}{a} \right)^n \left(\left(\frac{\rho}{a} \right)^n - \left(\frac{a}{\rho} \right)^n \right) \cos n(\phi - \alpha(t)) \Big] \\
& + a_0(t) + \sum_{n=1}^{\infty} \left(\frac{\rho}{a} \right)^n \left[a_n^{(c)}(t) \cos n\phi + a_n^{(s)}(t) \sin n\phi \right], \tag{37}
\end{aligned}$$

where $a_0(t)$, $a_n^{(c)}(t)$, $a_n^{(s)}(t)$ are zero at $t = 0$. The zero initialization of these coefficients will lead to simple expressions when the boundary conditions will be applied in the next Section.

VI. Diffusion Fields Inside and Outside the Toroidal Conductor

In order to compute the fields, the zero order diffusion equation must be solved inside the conductor and the boundary conditions be applied on its inner and outer surface. The zero order diffusion equation is obtained from Eq. (6) by omitting the toroidal corrections, namely,

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial A_{\theta 0}^{\text{con}}}{\partial \rho} + \frac{1}{2} \frac{\partial^2 A_{\theta 0}^{\text{con}}}{\partial \phi^2} - \mu_0 \sigma \frac{\partial A_{\theta 0}^{\text{con}}}{\partial t} = 0 \quad (38)$$

In general, let

$$\hat{f}(p) = \int_0^\infty f(t) e^{-pt} dt, \quad (39)$$

be the Laplace transform of $f(t)$. Then, in the Laplace transform domain, Eq. (38) becomes

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial \hat{A}_{\theta 0}^{\text{con}}}{\partial \rho} + \frac{1}{2} \frac{\partial^2 \hat{A}_{\theta 0}^{\text{con}}}{\partial \phi^2} - \mu_0 \sigma p \hat{A}_{\theta 0}^{\text{con}} = 0, \quad (40)$$

where $A_{\theta 0}^{\text{con}}(\rho, \phi, t)$ was assumed to be zero at $t = 0$. The most general solution of Eq. (40) is

$$\begin{aligned} \hat{A}_{\theta 0}^{\text{con}}(\rho, \phi, p) &= \hat{c}_0(p) I_0(\lambda \rho) \\ &+ \sum_{m=1}^{\infty} I_m(\lambda \rho) \left[\hat{c}_m^{(c)}(p) \cos m\phi + \hat{c}_m^{(s)}(p) \sin m\phi \right] \\ &+ \hat{d}_0(p) K_0(\lambda \rho) + \sum_{m=1}^{\infty} K_m(\lambda \rho) \left[\hat{d}_m^{(c)}(p) \cos m\phi + \hat{d}_m^{(s)}(p) \sin m\phi \right], \quad (41) \end{aligned}$$

where

$$\lambda = [\mu_0 \sigma p]^{1/2}, \quad (42)$$

and $I_m(x)$, $K_m(x)$ are the modified Bessel functions of order m .

The boundary conditions, to zero order in toroidal corrections, follow from Eqs. (7a), (7b), (8c), (8d), (9a), (9b) and (39). They are given by

$$\hat{A}_{\theta 0}^{\text{in}}(a, \phi, p) = \hat{A}_{\theta 0}^{\text{con}}(a, \phi, p), \quad (43a)$$

$$\left. \frac{\partial \hat{A}_{\theta 0}^{\text{in}}(\rho, \phi, p)}{\partial \rho} \right|_{\rho=a} = \left. \frac{\partial \hat{A}_{\theta 0}^{\text{con}}(\rho, \phi, p)}{\partial \rho} \right|_{\rho=a}, \quad (43b)$$

and the same two relations at $\rho=b$. Defining $\hat{F}_m^{(c)}(p)$ and $\hat{F}_m^{(s)}(p)$ to be the Laplace transforms of $(\Delta(t)/a)^m \cos m \alpha(t)$ and $(\Delta(t)/a)^m \sin m \alpha(t)$, respectively, the boundary conditions at $\rho=a$ and $\rho=b$, using Eqs. (35), (37) and (41), lead to the following algebraic system of equations:

$$\hat{a}_0(p) = \hat{c}_0(p)I_0(\lambda a) + \hat{d}_0(p)K_0(\lambda a), \quad (44a)$$

$$- \frac{\mu_0}{2\pi} \hat{i}_c(p) = \lambda a [\hat{c}_0(p)I'_0(\lambda a) + \hat{d}_0(p)K'_0(\lambda a)], \quad (44b)$$

$$\left(\ell n \frac{8r_0}{b} - 2 \right) \hat{b}_0(p) = \hat{c}_0(p)I_0(\lambda b) + \hat{d}_0(p)K_0(\lambda b), \quad (44c)$$

$$- \hat{b}_0(p) = \lambda b [\hat{c}_0(p)I'_0(\lambda b) + \hat{d}_0(p)K'_0(\lambda b)], \quad (44d)$$

when $m=0$, and

$$\hat{a}_m^{(i)}(p) = \hat{c}_m^{(i)}(p)I_m(\lambda a) + \hat{d}_m^{(i)}(p)K_m(\lambda a), \quad (45a)$$

$$- \frac{\mu_0}{\pi} \hat{F}_m^{(i)}(p) + m \hat{a}_m^{(i)}(p) - \lambda a \left[\hat{c}_m^{(i)}(p) I'_m(\lambda a) + \hat{d}_m^{(i)}(p) K'_m(\lambda a) \right] \quad (45b)$$

$$\hat{b}_m^{(i)}(p) = \hat{c}_m^{(i)}(p) I_m(\lambda b) + \hat{d}_m^{(i)}(p) K_m(\lambda b) \quad (45c)$$

$$- m \hat{b}_m^{(i)}(p) = \lambda b \left[\hat{c}_m^{(i)}(p) I'_m(\lambda b) + \hat{d}_m^{(i)}(p) K'_m(\lambda b) \right], \quad (45d)$$

when $m = 1, 2, \dots$, and $i = c, s$. Also, $I'_m(x)$, $K'_m(x)$ are the derivatives of $I_m(x)$, $K_m(x)$. The solution of the first algebraic system of Eqs. (44a) - (44d) is

$$\hat{a}_0(p) = \frac{\mu_0}{2\pi} \hat{I}_c(p) \frac{\tilde{g}_0(\lambda)}{\tilde{f}_0(\lambda)}, \quad (46a)$$

$$\hat{b}_0(p) = \frac{\mu_0}{2\pi} \hat{I}_c(p) \frac{1}{\tilde{f}_0(\lambda)}, \quad (46b)$$

$$\hat{c}_0(p) = \hat{b}_0(p) \left[\left(\ln \frac{8r_0}{b} - 2 \right) \lambda b K_1(\lambda b) - K_0(\lambda b) \right], \quad (46c)$$

$$\hat{d}_0(p) = \hat{b}_0(p) \left[\left(\ln \frac{8r_0}{b} - 2 \right) \lambda b I_1(\lambda b) + I_0(\lambda b) \right], \quad (46d)$$

where

$$\begin{aligned} \tilde{f}_0(\lambda) &= \left(\ln \frac{8r_0}{b} - 2 \right) \lambda a \lambda b \left[K_1(\lambda a) I_1(\lambda b) - I_1(\lambda a) K_1(\lambda b) \right] \\ &+ \lambda a \left[K_1(\lambda a) I_0(\lambda b) + I_1(\lambda a) K_0(\lambda b) \right], \end{aligned} \quad (47a)$$

$$\tilde{g}_0(\lambda) = \left(\ln \frac{8r_0}{b} - 2 \right) \lambda b \left[K_0(\lambda a) I_1(\lambda b) + I_0(\lambda a) K_1(\lambda b) \right]$$

$$+ K_0(\lambda a) I_0(\lambda b) - I_0(\lambda a) K_0(\lambda b). \quad (47b)$$

Similarly, the solution of the second algebraic system of Eqs. (45a) - (45d) is

$$\hat{a}_m^{(i)}(p) = \frac{\mu_0}{2\pi} \hat{F}_m^{(i)}(p) \frac{\tilde{g}_m(\lambda)}{\tilde{f}_m(\lambda)}, \quad (48a)$$

$$\hat{b}_m^{(i)}(p) = \frac{\mu_0}{2\pi} \hat{F}_m^{(i)}(p) \frac{1}{\tilde{f}_m(\lambda)}, \quad (48b)$$

$$\hat{c}_m^{(i)}(p) = \hat{b}_m^{(i)}(p) \lambda b K_{m-1}(\lambda b), \quad (48c)$$

$$\hat{d}_m^{(i)}(p) = \hat{b}_m^{(i)}(p) \lambda b I_{m-1}(\lambda b), \quad (48d)$$

where

$$\tilde{f}_m(\lambda) = -\frac{\lambda a \lambda b}{2} [I_{m+1}(\lambda a) K_{m-1}(\lambda b) - K_{m+1}(\lambda a) I_{m-1}(\lambda b)], \quad (49a)$$

$$\tilde{g}_m(\lambda) = \lambda b [I_m(\lambda a) K_{m-1}(\lambda b) + K_m(\lambda a) I_{m-1}(\lambda b)], \quad (49b)$$

and $m = 1, 2, 3, \dots, i = c, s$.

The inverse Laplace transforms of the coefficients given above are determined from the sum of the residues at the poles of these coefficients. The poles are computed at the zeroes of $\tilde{f}_m(\lambda_{mk}) = 0$, where $m = 0, 1, 2, \dots, k = 0, 1, 2, \dots$. Since all the zeroes occur for imaginary values of λ , we define the real quantities α_{mk} by means of the relation $\lambda_{mk} = i\alpha_{mk}$. Then a pole occurs at $p_{mk} = -\alpha_{mk}^2/\mu_0\sigma$ (cf. Eq. (42)) and the inverse Laplace transform $h_m(t)$ of $\tilde{g}_m(\lambda)/\tilde{f}_m(\lambda)$ is

$$h_m(t) = \sum_{k=0}^{\infty} \frac{\tilde{g}_m(i\alpha_{mk})}{\left. \frac{d\lambda}{dp} \right|_{\lambda=i\alpha_{mk}}} \frac{\tilde{f}'_m(i\alpha_{mk})}{e^{-\frac{\alpha_{mk}^2}{\mu_0 \sigma} t}} \Theta(t), \quad (50)$$

where $\tilde{f}'_m(z)$ is the derivative of $\tilde{f}_m(z)$ and $\Theta(t) = 1$, when $t > 0$, while $\Theta(t) = 0$, when $t < 0$. Using the convolution theorem in the Laplace transform domain, we obtain the following expressions for the coefficients inside and outside the torus:

$$a_o(t) = \sum_{k=0}^{\infty} A_{ok} U_{ok}(t), \quad (51a)$$

$$b_o(t) = \sum_{k=0}^{\infty} B_{ok} U_{ok}(t), \quad (51b)$$

$$a_m^{(i)}(t) = \sum_{k=0}^{\infty} A_{mk} U_{mk}^{(i)}(t), \quad (51c)$$

$$b_m^{(i)}(t) = \sum_{k=0}^{\infty} B_{mk} U_{mk}^{(i)}(t), \quad (51d)$$

where $m = 1, 2, \dots$, $i = c, s$, and

$$U_{ok}(t) = \frac{1}{\tau_{ok}} e^{-t/\tau_{ok}} \int_0^t e^{t'/\tau_{ok}} \frac{\mu_0}{2\pi} I_c(t') dt', \quad (52a)$$

$$U_{mk}^{(c)}(t) = \frac{1}{\tau_{mk}} e^{-t/\tau_{mk}} \int_0^t e^{t'/\tau_{mk}} \frac{\mu_0}{2\pi} I_c(t') \left(\frac{\Delta(t')}{a} \right)^m \cos m\alpha(t') dt', \quad (52b)$$

$$U_{mk}^{(s)}(t) = \frac{1}{\tau_{mk}} e^{-t/\tau_{mk}} \int_0^t e^{t'/\tau_{mk}} \frac{\mu_0}{2\pi} I_c(t') \left(\frac{\Delta(t')}{a} \right)^m \sin m\alpha(t') dt', \quad (52c)$$

Also,

$$\frac{1}{\tau_{mk}} = \frac{\alpha_{mk}^2}{\mu_0 \sigma}, \quad (53a)$$

$$A_{mk} = - \frac{2g_m(\alpha_{mk})}{\alpha_{mk} f'_m(\alpha_{mk})}, \quad (53b)$$

$$B_{mk} = - \frac{2}{\alpha_{mk} f'_m(\alpha_{mk})}, \quad (53c)$$

where $m = 0, 1, 2, \dots$ $k = 0, 1, 2, \dots$ Finally, the functions $f_m(\alpha)$, $g_m(\alpha)$ and the derivative $f'_m(\alpha)$ of $f_m(\alpha)$ are as follows:

$$f_0(\alpha) = \tilde{f}_0(i\alpha) \quad (54a)$$

$$= \frac{\pi}{2} x_0 [J_1(x_0)Y_0(x_1) - Y_1(x_0)J_0(x_1)]$$

$$- \frac{\pi}{2} \left(\ln \frac{8r_0}{b} - 2 \right) x_0 x_1 [J_1(x_0)Y_1(x_1) - Y_1(x_0)J_1(x_1)],$$

$$f_m(\alpha) = \tilde{f}_m(i\alpha) = \frac{\pi}{2} \frac{x_0 x_1}{2} [J_{m+1}(x_0)Y_{m-1}(x_1) - Y_{m+1}(x_0)J_{m-1}(x_1)], \quad (54b)$$

$$g_0(\alpha) = \tilde{g}_0(i\alpha) = \frac{\pi}{2} [J_0(x_0)Y_0(x_1) - Y_0(x_0)J_0(x_1)] \quad (55a)$$

$$- \frac{\pi}{2} \left(\ln \frac{8r_0}{b} - 2 \right) x_1 [J_0(x_0)Y_1(x_1) - Y_0(x_0)J_1(x_1)],$$

$$g_m(\alpha) = \tilde{g}_m(i\alpha) = \frac{\pi}{2} x_1 [J_m(x_0)Y_{m-1}(x_1) - Y_m(x_0)J_{m-1}(x_1)], \quad (55b)$$

$$\alpha f'_0(\alpha) = i\alpha f'_0(i\alpha) \quad (56a)$$

$$\begin{aligned} &= f_0(\alpha) + \frac{\pi}{2} x_0^2 [J_0(x_0)Y_0(x_1) - Y_0(x_0)J_0(x_1)] \\ &+ \frac{\pi}{2} \left(\ln \frac{8r_0}{b} - 2 \right) x_0 x_1 [J_1(x_0)Y_1(x_1) - Y_1(x_0)J_1(x_1)] \\ &- \frac{\pi}{2} \left[\left(\ln \frac{8r_0}{b} - 2 \right) x_1^2 + 1 \right] x_0 [J_1(x_0)Y_0(x_1) - Y_1(x_0)J_0(x_1)] \\ &- \frac{\pi}{2} \left(\ln \frac{8r_0}{b} - 2 \right) x_0^2 x_1 [J_0(x_0)Y_1(x_1) - Y_0(x_0)J_1(x_1)], \end{aligned}$$

$$\alpha f'_m(\alpha) = i\alpha f'_m(i\alpha) \quad (56b)$$

$$\begin{aligned} &= \frac{\pi}{2} \frac{x_0^2 x_1}{2} [J_m(x_0)Y_{m-1}(x_1) - Y_m(x_0)J_{m-1}(x_1)] \\ &- \frac{\pi}{2} \frac{x_0 x_1^2}{2} [J_{m+1}(x_0)Y_m(x_1) - Y_{m+1}(x_0)J_m(x_1)], \end{aligned}$$

where $x_0 = \alpha a$, $x_1 = \alpha b$, $m = 1, 2, \dots$ and $J_m(x)$, $Y_m(x)$ are the Bessel functions of order m . Notice that α_{mk} are the zeroes of $f_m(\alpha_{mk}) = 0$, for $m = 0, 1, 2, \dots$, $k = 0, 1, 2, \dots$

From Eqs. (36a), (36b), (35), (51a)-(51d), we conclude that the zero order vector potential inside the ring is equal to

$$\begin{aligned} A_{\theta 0}^{in}(\rho, \phi, t) &= \frac{\mu_0}{4\pi} I_c(t) \left[2 \left(\ln \frac{a}{r_c} + \frac{1}{2} \right) \right. \\ &\left. - \frac{\rho^2 + \Delta^2(t) - 2\rho\Delta(t)\cos(\phi - \alpha(t))}{r_c^2} \right] \end{aligned} \quad (57a)$$

$$\begin{aligned}
& + \ln \left[1 + \left(\frac{\rho \Delta(t)}{a^2} \right)^2 - 2 \frac{\rho \Delta(t)}{a^2} \cos (\phi - \alpha(t)) \right] \Bigg] \\
& + \sum_{k=0}^{\infty} A_{ok} U_{ok}(t) + \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} A_{mk} \left(\frac{\rho}{a} \right)^m \left[U_{mk}^{(c)}(t) \cos m\phi + U_{mk}^{(s)}(t) \sin m\phi \right].
\end{aligned}$$

Outside the ring and inside the torus (i.e., $\rho \leq a$), it is equal to

$$\begin{aligned}
A_{\Theta 0}^{\text{in}}(\rho, \phi, t) &= \frac{\mu_0}{4\pi} I_c(t) \left[2 \ln \frac{a}{\rho} \right. \\
&- \ln \left(1 + \left(\frac{\Delta(t)}{\rho} \right)^2 - 2 \frac{\Delta(t)}{\rho} \cos (\phi - \alpha(t)) \right) \\
&+ \ln \left(1 + \left(\frac{\rho \Delta(t)}{a^2} \right)^2 - 2 \frac{\rho \Delta(t)}{a^2} \cos (\phi - \alpha(t)) \right) \Bigg] \\
&+ \sum_{k=0}^{\infty} A_{ok} U_{ok}(t) + \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} A_{mk} \left(\frac{\rho}{a} \right)^m \left[U_{mk}^{(c)}(t) \cos m\phi + U_{mk}^{(s)}(t) \sin m\phi \right]
\end{aligned} \tag{57b}$$

and outside the torus (i.e., $\rho \geq b$) it is equal to

$$\begin{aligned}
A_{\Theta 0}^{\text{out}}(\rho, \phi, t) &= \left(\ln \frac{8r_0}{\rho} - 2 \right) \sum_{k=0}^{\infty} B_{ok} U_{ok}(t) \\
&+ \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} B_{mk} \left(\frac{\rho}{b} \right)^m \left[U_{mk}^{(c)}(t) \cos m\phi + U_{mk}^{(s)}(t) \sin m\phi \right].
\end{aligned} \tag{58}$$

From Eqs. (52a), (52b) and (52c) it is easy to show that the time dependent coefficients $U_{ok}(t)$, $U_{mk}^{(c)}(t)$, $U_{mk}^{(s)}(t)$ satisfy the first order differential equations

$$\dot{U}_{ok}(t) + \frac{1}{\tau_{ok}} U_{ok}(t) = \frac{1}{\tau_{ok}} \frac{\mu_0}{2\pi} I_c(t), \tag{59a}$$

$$\dot{U}_{mk}^{(c)}(t) + \frac{1}{\tau_{mk}} U_{mk}^{(c)}(t) = \frac{1}{\tau_{mk}} \frac{\mu_0}{2\pi} I_c(t) \left(\frac{\Delta(t)}{a} \right)^m \cos m\alpha(t) \quad (59b)$$

$$\dot{U}_{mk}^{(s)}(t) + \frac{1}{\tau_{mk}} U_{mk}^{(s)}(t) = \frac{1}{\tau_{mk}} \frac{\mu_0}{2\pi} I_c(t) \left(\frac{\Delta(t)}{a} \right)^m \sin m\alpha(t) \quad (59c)$$

with initial conditions $U_{ok}(0) = U_{mk}^{(c)}(0) = U_{mk}^{(s)}(0) = 0$, for $m = 1, 2, \dots, k = 0, 1, 2, \dots$. These differential equations are very useful when the current ring moves and its equations of motion depend on the diffusion fields, i.e., when the ring dynamics is coupled to the diffusion fields. Then, the state vector of the system consists not only of the position and velocity of the ring, but also of the diffusion coefficients $U_{ok}(t)$, $U_{mk}^{(c)}(t)$, $U_{mk}^{(s)}(t)$, for $m = 1, 2, \dots, k = 0, 1, 2, \dots$, and its time derivative is determined by the ring dynamics, as well as by Eqs. (59a)-(59c). Notice that it is much easier to solve in the computer a set of coupled first order differential equations rather than a set of coupled first order differential equations and the convolution integrals given by Eqs. (52a)-(52c).

Next, four exact identities will be established for the time independent coefficients A_{mk} , B_{mk} . For a motionless, step function ring current, i.e., when $I_c(t) = I_0 \theta(t)$, Eqs. (51a)-(51d) and Eqs. (52a)-(52c) give

$$a_0(t) = \frac{\mu_0}{2\pi} I_0 \sum_{k=0}^{\infty} A_{ok} \left(1 - e^{-t/\tau_{ok}} \right), \quad (60a)$$

$$b_0(t) = \frac{\mu_0}{2\pi} I_0 \sum_{k=0}^{\infty} B_{ok} \left(1 - e^{-t/\tau_{ok}} \right), \quad (60b)$$

$$a_m^{(c)}(t) + i a_m^{(s)}(t) = \frac{\mu_0}{2\pi} I_0 \left(\frac{\Delta}{a}\right)^m e^{im\alpha} \sum_{k=0}^{\infty} A_{mk} \left(1 - e^{-t/\tau_{mk}}\right), \quad (60c)$$

$$b_m^{(c)}(t) + i b_m^{(s)}(t) = \frac{\mu_0}{2\pi} I_0 \left(\frac{\Delta}{a}\right)^m e^{im\alpha} \sum_{k=0}^{\infty} B_{mk} \left(1 - e^{-t/\tau_{mk}}\right), \quad (60d)$$

The Laplace transform of $I_c(t)$ is $\hat{I}_c(p) = I_0/p$, and we have from Eqs. (46a), (46b), (48a), (48b):

$$\hat{a}_0(p) = \frac{\mu_0}{2\pi} I_0 \frac{1}{p} \frac{\tilde{g}_0(\lambda)}{\tilde{f}_0(\lambda)}, \quad (61a)$$

$$\hat{b}_0(p) = \frac{\mu_0}{2\pi} I_0 \frac{1}{p} \frac{1}{\tilde{f}_0(\lambda)}, \quad (61b)$$

$$\hat{a}_m^{(c)}(p) + i \hat{a}_m^{(s)}(p) = \frac{\mu_0}{2\pi} I_c \left(\frac{\Delta}{a}\right)^m e^{im\alpha} \frac{1}{p} \frac{\tilde{g}_m(\lambda)}{\tilde{f}_m(\lambda)}, \quad (61c)$$

$$\hat{b}_m^{(c)}(p) + i \hat{b}_m^{(s)}(p) = \frac{\mu_0}{2\pi} I_0 \left(\frac{\Delta}{a}\right)^m e^{im\alpha} \frac{1}{p} \frac{1}{\tilde{f}_m(\lambda)}, \quad (61d)$$

From the well-known theorem of Laplace transforms it follows that $\lim_{t \rightarrow \infty} f(t) = \lim_{p \rightarrow 0} pf(p)$. Eqs. (47a), (47b), (49a), (49b), give

$$\lim_{\lambda \rightarrow 0} \tilde{f}_0(\lambda) = 1, \quad (62a)$$

$$\lim_{\lambda \rightarrow 0} \tilde{g}_0(\lambda) = \ln \frac{8r_0}{a} - 2, \quad (62b)$$

$$\lim_{\lambda \rightarrow 0} \tilde{f}_m(\lambda) = m \left(\frac{b}{a}\right)^m, \quad (62c)$$

$$\lim_{\lambda \rightarrow 0} \tilde{g}_m(\lambda) = \left(\frac{b}{a}\right)^m. \quad (62d)$$

Application of the theorem just mentioned in Eqs. (60a)-(60d) and Eqs. (61a)-(61d), in conjunction with Eqs. (62a)-(62d) leads to the following identities:

$$\sum_{k=0}^{\infty} A_{ok} = \ln \frac{8r_0}{a} - 2, \quad (63a)$$

$$\sum_{k=0}^{\infty} B_{ok} = 1, \quad (63b)$$

$$\sum_{k=0}^{\infty} A_{mk} = \frac{1}{m}, \quad (63c)$$

$$\sum_{k=0}^{\infty} B_{mk} = \frac{1}{m} \left(\frac{a}{b}\right)^m. \quad (63d)$$

These identities have been verified by the computer for a thin as well as a thick conducting wall. Substitution of these identities into Eqs. (60a)-(60d) when $t \rightarrow \infty$, leads to

$$\lim_{t \rightarrow \infty} a_0(t) = \frac{\mu_0}{2\pi} I_0 \left(\ln \frac{8r_0}{a} - 2 \right), \quad (64a)$$

$$\lim_{t \rightarrow \infty} b_0(t) = \frac{\mu_0}{2\pi} I_0, \quad (64b)$$

$$\lim_{t \rightarrow \infty} \left[a_m^{(c)}(t) + i a_m^{(s)}(t) \right] = \frac{\mu_0}{2\pi} I_0 \frac{1}{m} \left(\frac{\Delta}{a}\right)^m e^{im\alpha}, \quad (64c)$$

$$\lim_{t \rightarrow \infty} \left[b_m^{(c)}(t) + i b_m^{(s)}(t) \right] = \frac{\mu_0}{2\pi} I_0 \frac{1}{m} \left(\frac{\Delta}{b}\right)^m e^{im\alpha}. \quad (64d)$$

With the help of these identities and Eqs. (35), (36a) and (36b), it is easy to show that, as $t \rightarrow \infty$, the zero order vector potential inside and outside the toroidal conductor (as well as inside the ring) becomes equal to that of the current ring in the absence of the conductor, i.e., it becomes equal to $A_{\theta 0}^{\text{ext}}$, due to the diffusion process. This conclusion demonstrates the importance of the identities (63a)-(63d).

The magnetic and electric fields inside and outside the torus can be computed from Eqs. (57a), (57b) and (58). Thus, the self-magnetic and self-electric fields, i.e., the fields of the ring at its centroid $\rho = \Delta(t)$, $\phi = \alpha(t)$, are equal to

$$B_{\rho}^{\text{self}}(t) = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{m}{a} A_{mk} \left(\frac{\Delta(t)}{a} \right)^{m-1} \quad (65a)$$

$$* \left[-U_{mk}^{(s)}(t) \cos m\alpha(t) + U_{mk}^{(c)}(t) \sin m\alpha(t) \right],$$

$$B_{\phi}^{\text{self}}(t) = -\frac{\mu_0}{2\pi} I_c(t) \frac{\Delta(t)}{a^2} \frac{1}{1 - \left(\frac{\Delta(t)}{a} \right)^2} \quad (65b)$$

$$+ \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{m}{a} A_{mk} \left(\frac{\Delta(t)}{a} \right)^{m-1}$$

$$* \left[U_{mk}^{(c)}(t) \cos m\alpha(t) + U_{mk}^{(s)}(t) \sin m\alpha(t) \right],$$

$$E_{\theta 0}^{\text{self}}(t) = -\frac{\mu_0}{2\pi} \dot{I}_c(t) \left[\ln \frac{a}{r_c} + \frac{1}{2} + \ln \left(1 - \left(\frac{\Delta(t)}{a} \right)^2 \right) \right]$$

$$+ \frac{\mu_0}{2\pi} I_c(t) \frac{\frac{x_c(t)\dot{x}_c(t) + z_c(t)\dot{z}_c(t)}{a^2}}{1 - \left(\frac{\Delta(t)}{a} \right)^2}$$

$$- \sum_{k=0}^{\infty} \frac{A_{ok}}{\tau_{ok}} \left[\frac{\mu_o}{2\pi} I_c(t) - U_{ok}(t) \right] \quad (65c)$$

$$- \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{A_{mk}}{\tau_{mk}} \left(\frac{\Delta(t)}{a} \right)^m \left[\frac{\mu_o}{2\pi} I_c(t) \left(\frac{\Delta(t)}{a} \right)^m \right. \\ \left. - U_{mk}^{(c)}(t) \cos m\alpha(t) - U_{mk}^{(s)}(t) \sin m\alpha(t) \right],$$

where $x_c(t) = \Delta(t) \cos \alpha(t)$ and $z_c(t) = \Delta(t) \sin \alpha(t)$.

The last three relations, and in particular the B_{op}^{self} component were useful in providing an analytical model for the beam trapping that occurs after injection in the modified betatron accelerator.

A typical example of the effect that the diffusion process has on the ring dynamics, immediately after injection is shown in Fig. 2. The numerical integration of the ring equations of motion coupled with the diffusion fields was done for the parameters listed in Table I. Figure 2(a) shows the projection of the centroid orbit on the r-z plane that moves with the same toroidal angular velocity as the ring centroid. There is a slow (bounce) motion and, due to the presence of the stellarator windings (i.e., strong focusing), there is also an intermediate motion. Both of these modes are indicated in Fig. 2(a). Since there are six field periods of the stellarator field in the range $0 \leq \theta \leq 2\pi$, the electrons perform six oscillations during one revolution around the major axis. To take into account the intermediate motion that has been neglected in the diffusion model presented in this paper, the resistivity in the code is computed

using the skin depth that corresponds to the intermediate frequency and not the actual thickness of the wall. The dots in Fig. 2(b) show the positions the beam crosses the r-z plane at $\theta = 0^\circ$. The time difference between two dots is equal to the period around the major axis, i.e., ~ 27 nsec, and therefore the speed of the ring on the r-z plane can be inferred from the relative position of the dots. Fig. 2(c) provides the relativistic factor γ vs. time and the reduction in γ is obvious due to the energy lost on the resistive wall and to establish the electromagnetic field outside the torus. Another example is given in reference 13, which refers to the beam trapping in the modified betatron accelerator and is in good agreement with the experimental results.

VII. Approximate Results for a Thin Conducting Wall

The results presented in the previous Sections are approximate in the sense that they include only zero order toroidal corrections, i.e., are valid only for small aspect ratio vessels. Otherwise, they are exact. In this Section, the additional assumption is made that the conducting wall is thin. This assumption allows us to compute approximate expressions of the zeroes α_{mk} of $f_m(\alpha_{mk}) = 0$ (cf. Eqs. (54a), (54b)) and of the vector potential and the fields.

When the conducting wall is thin, i.e., $(b-a)/a \ll 1$, both $x_0 = \alpha a$ and $x_1 = \alpha b$, where α is a zero of $f_m(\alpha) = 0$, are very large numbers and the asymptotic expansions of the Bessel functions can be used in Eqs. (54a), (54b). This is valid only up to some maximum value of m . Keeping terms up to order $1/z$, the asymptotic expansions of $J_n(z)$, $Y_n(z)$ are: ¹⁶

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}} \left[\cos X_n - \frac{4n^2 - 1}{8z} \sin X_n \right], \quad (66a)$$

$$Y_n(z) \sim \sqrt{\frac{2}{\pi z}} \left[\sin X_n + \frac{4n^2 - 1}{8z} \cos X_n \right], \quad (66b)$$

where $X_n = z - (n + 1/2) \pi/2$. We substitute these expansions into Eqs. (54a), (54b). Then, the zeroes of $f_0(\alpha_{ok}) = 0$ are determined from

$$\tan(x_1 - x_0) = \frac{1}{\left(\ln \frac{8r_0}{b} - 2 \right) x_0}, \quad (67a)$$

while the zeroes of $f_m(\alpha_{mk}) = 0$ are determined from

$$\tan(x_1 - x_0) = \frac{2m}{x_0}. \quad (67b)$$

A more accurate expression for the zeroes of $f_m(\alpha_{mk}) = 0$, correct to order $(m/x_0)^2$, is derived in the Appendix, and is given by

$$\tan(x_1 - x_0) \approx \frac{\frac{2m}{x_0} - \frac{3}{2} \frac{m^2 - \frac{1}{4}}{x_0^2} + \frac{1}{2} \left(\frac{b-a}{a}\right)^2 x_0}{1 - \left(m - \frac{1}{2}\right) \frac{b-a}{a} + \left(m - \frac{3}{2}\right) \left(m + \frac{1}{2}\right) \left(\frac{b-a}{a}\right)^2}. \quad (67c)$$

If $x_0 \gg 1$, we see from Eq. (67a) that $|x_1 - x_0| \ll 1$. Therefore, $\tan(x_1 - x_0) \approx x_1 - x_0 \approx (b-a)\alpha$, and one of the zeroes is

$$\alpha_{00} = \left[\frac{1}{\left(\ln \frac{8r_0}{b} - 2 \right) a(b-a)} \right]^{1/2}, \quad (68a)$$

while the others are given by

$$\alpha_{ok} = \frac{k\pi}{b-a}, \quad (68b)$$

where $k = 1, 2, \dots$. The small additive correction term $1/[k\pi(\ln(8r_0/b) - 2)a]$ has been omitted in Eq. (68b). The zeroes of Eq. (67b) can be obtained in a similar fashion, except when m is as large or larger than x_0 . Let $m = 1, 2, \dots, M$, where $M = \text{Int}[a/4(b-a)]$, and $\text{Int}(x)$ is the integral part of x . Then one set of zeroes of Eq. (67b) is approximately given by

$$\alpha_{m0} = \left[\frac{2m}{a(b-a)} \right]^{1/2}, \quad (69a)$$

with an error of only a few percent when $(b-a)/a \leq 10^{-2}$. For m as specified above and $k = 1, 2, \dots, K$, where $K = \text{Int}[a/\pi(b-a)]$, the rest of the zeroes are given by

$$\alpha_{mk} = \frac{k\pi}{b-a}. \quad (69b)$$

The small correction term $[2m + (k\pi)^2/2]/(k\pi a)$ has been omitted in Eq. (69b). For values of m and k larger than M and K , respectively, the zeroes should be computed numerically from Eq. (54b). Notice that the presence of the terms $(\rho/a)^m$ and $(b/\rho)^m$ in the series expansions of the vector potential and the fields indicates that the large values of m become important when these quantities are computed close to the conducting wall, where ρ/a and b/ρ become almost equal to 1 and more m -terms must be included in the sums to converge within a prescribed accuracy. An estimate of the minor radius ρ_i within which the vector potential and the fields are sufficiently accurate is determined by $\rho_i/a = [(b - a)/a]^{1/M}$. Since $(b - a)/a \ll 1$, all the terms associated with $(\rho/a)^m$, for $m = M + 1, M + 2, \dots$ in the series expansions of the vector potential and the fields, have a negligible contribution, provided $\rho \leq \rho_i$. A similar argument can be made for the vector potential and the fields outside the torus. Their accuracy is within a few percent when $\rho \geq \rho_o$, where $\rho_o/a = [a/(b - a)]^{1/M}$. Within the distances $d_i = a - \rho_i$ and $d_o = \rho_o - b$ from the inner and outer conducting walls the zeroes (and, therefore, the vector potential and the fields) cannot be computed analytically in terms of a simple expression. In this case they should be computed numerically from Eq. (54b) and then use the analytic expressions for the vector potential and the fields. As an example, when $(b - a)/a = 10^{-3}$, then $M = 250$, $K = 318$, $d_i/a = 0.027$, $d_o/a = 0.028$, but when $(b - a)/a = 10^{-2}$, then $M = 25$, $K = 31$, $d_i/a = 0.17$, $d_o/a = 0.20$. In the following, the various quantities will be computed to order $(b - a)/a$, with the understanding that they are not accurate close to the conducting wall. But in the limit when the ratio $(b - a)/a$ tends to zero (but $\sigma(b - a)/a$ remains finite), i.e., when the toroidal conductor becomes a toroidal conducting shell, the distances d_i, d_o are zero and the results become exact, to zero order in toroidal corrections, everywhere inside as well as outside and in the vicinity of the torus.

Under the assumptions stated above, the time constants τ_{mk} in Eq. (53a) become

$$\tau_{00} = 2 \tau_D \left(\ln \frac{8r_0}{b} - 2 \right), \quad (70a)$$

$$\tau_{ok} = \frac{\tau_{FD}}{k^2}, \quad (70b)$$

$$\tau_{m0} = \frac{\tau_D}{m}, \quad (70c)$$

$$\tau_{mk} = \frac{\tau_{FD}}{k^2}, \quad (70d)$$

where

$$\tau_D = \frac{\mu_0 \sigma (b - a) a}{2}, \quad (71a)$$

$$\tau_{FD} = \frac{\mu_0 \sigma (b - a)^2}{\pi^2}, \quad (71b)$$

and $m = 1, 2, \dots, k = 1, 2, \dots$. Therefore, there are three characteristic time constants associated with the diffusion process. The "loop" diffusion time τ_{00} is the slowest and determines the speed with which the external field of the ring diffuses into the hole of the doughnut. This time is present because of the toroidal geometry of the conductor. The "cylinder" diffusion time τ_D and the "fast" diffusion time τ_{FD} are associated with the diffusion process in a cylinder and determine the speed with which the field of the ring penetrates the conducting wall. Notice that, in the limit of a toroidal conducting shell, any terms associated with the fast diffusion time τ_{FD} diffuse instantaneously at $t = 0$ outside the torus.

This explains the origin of the electric field that is immediately established at $t = 0$ outside the torus for the shell model (cf. statement after Eq. (90) in the next section). On the other hand, we know from Eq. (58), that the magnetic field does not diffuse instantaneously at $t = 0$ outside the torus.

Under the same assumptions stated above and to lowest order in $(b - a)/a$, we have the following approximate relations

$$A_{00} \approx \ln \frac{8r_0}{a} - 2 - \frac{1}{3} \frac{b - a}{a}, \quad (72a)$$

$$A_{0k} \approx \frac{2}{(k\pi)^2} \frac{b - a}{a}, \quad (72b)$$

$$A_{m0} \approx \frac{1}{m} - \frac{2}{3} \frac{b - a}{a}, \quad (72c)$$

$$A_{mk} \approx \frac{4}{(k\pi)^2} \frac{b - a}{a}, \quad (72d)$$

$$B_{00} \approx 1 + \frac{1}{6 \left(\ln \frac{8r_0}{b} - 2 \right)} \frac{b - a}{a}, \quad (73a)$$

$$B_{0k} \approx \frac{2 (-1)^k}{\left(\ln \frac{8r_0}{b} - 2 \right) (k\pi)^2} \frac{b - a}{a}, \quad (73b)$$

$$B_{m0} \approx \frac{1}{m} + \frac{4}{3} \frac{b - a}{a}, \quad (73c)$$

$$B_{mk} \approx \frac{4 (-1)^k}{(k\pi)^2} \frac{b - a}{a}, \quad (73d)$$

for $m = 1, 2, \dots$, $k = 1, 2, \dots$. The relations (73b), (73d) were derived directly from Eqs. (56a), (56b), while the relations (72b), (72d) were derived with the help of Eqs. (73b), (73d) and the relations $g_0(\alpha_{0k}) = (-1)^k (\ln 8r_0/a - 2)$, $g_m(\alpha_{mk}) = (-1)^k$, for $m = 1, 2, \dots$, $k = 1, 2, \dots$

Finally, A_{oo} , A_{mo} , B_{oo} , B_{mo} were derived using the identities (63a) - (63d).

As mentioned above, the terms associated with the fast diffusion time τ_{FD} vary in time on a much faster time scale than the terms associated with the times τ_{oo} and τ_D . If the ring current $I_c(t)$ and its position ($\Delta(t)$, $\alpha(t)$) vary slowly within a few e-folds of τ_{FD} , then the part of the vector potential (or the fields) which is associated with τ_{FD} can be simplified considerably.

First, let us consider the self-magnetic field. Substitution of Eqs. (70b), (70d) and Eqs. (72a) - (72d) into Eqs. (65a), (65b) leads to the relations

$$B_{o\rho}^{self}(t) = \frac{1}{a} \sum_{m=1}^{\infty} \left(1 - \frac{2m}{3} \frac{b-a}{a}\right) \left(\frac{\Delta(t)}{a}\right)^{m-1} \quad (74a)$$

$$* \left[-U_{mo}^{(s)}(t) \cos m\alpha(t) + U_{mo}^{(c)} \sin m\alpha(t) \right]$$

$$+ \frac{4}{\pi^2} \frac{b-a}{a^2} \sum_{k=1}^{\infty} \frac{1}{\tau_{FD}} \int_0^t h_s^{self}(t, t-t') e^{-\frac{t'}{\tau_{FD}} k^2} dt',$$

$$B_{o\phi}^{self}(t) = -\frac{\mu_o}{2\pi} I_c(t) \frac{\Delta(t)}{a^2} \frac{1}{1 - \left(\frac{\Delta(t)}{a}\right)^2}$$

$$+ \frac{1}{a} \sum_{m=1}^{\infty} \left(1 - \frac{2m}{3} \frac{b-a}{a}\right) \left(\frac{\Delta(t)}{a}\right)^{m-1} \quad (74b)$$

$$* \left[U_{mo}^{(c)}(t) \cos m\alpha(t) + U_{mo}^{(s)}(t) \sin m\alpha(t) \right]$$

$$+ \frac{4}{\pi^2} \frac{b-a}{a^2} \sum_{k=1}^{\infty} \frac{1}{\tau_{PD}} \int_0^t h_c^{\text{self}}(t, t-t') e^{-\frac{t'}{\tau_{PD}}} k^2 dt',$$

where

$$h_c^{\text{self}}(t, t') = \frac{\mu_0}{2\pi} I_c(t') \frac{\Delta(t')}{a} \quad (75a)$$

$$* \frac{\left[1 + \left(\frac{\Delta(t)\Delta(t')}{a^2}\right)^2\right] \cos(\alpha(t) - \alpha(t')) - 2 \frac{\Delta(t)\Delta(t')}{a^2}}{\left[1 + \left(\frac{\Delta(t)\Delta(t')}{a^2}\right)^2 - 2 \frac{\Delta(t)\Delta(t')}{a^2} \cos(\alpha(t) - \alpha(t'))\right]^2},$$

$$h_s^{\text{self}}(t, t') = \frac{\mu_0}{2\pi} I_c(t') \frac{\Delta(t')}{a} \quad (75b)$$

$$* \frac{\left[1 - \left(\frac{\Delta(t)\Delta(t')}{a^2}\right)^2\right] \sin(\alpha(t) - \alpha(t'))}{\left[1 + \left(\frac{\Delta(t)\Delta(t')}{a^2}\right)^2 - 2 \frac{\Delta(t)\Delta(t')}{a^2} \cos(\alpha(t) - \alpha(t'))\right]^2}.$$

If $I_c(t - t')$, $\Delta(t - t')$, $\alpha(t - t')$ vary only slightly as t' varies within a few e-folds of τ_{PD} , they can be replaced by $I_c(t)$, $\Delta(t)$ and $\alpha(t)$, except in the sine in Eq. (75b) we should set $\alpha(t) - \alpha(t - t') \approx \alpha'(t)t'$ to get the lowest order contribution. Here $\alpha'(t)$ is the derivative of $\alpha(t)$.

Equations (74a) and (74b) then become

$$B_{op}^{\text{self}}(t) \approx \frac{1}{a} \sum_{m=1}^{\infty} \left(1 - \frac{2m}{3} \frac{b-a}{a}\right) \left(\frac{\Delta(t)}{a}\right)^{m-1}$$

$$* \left[-U_{mo}^{(s)}(t) \cos m\alpha(t) + U_{mo}^{(c)}(t) \sin m\alpha(t) \right]$$

$$\begin{aligned}
& + \frac{b-a}{a} \frac{2\mu_0}{\pi^3} I_c(t) \alpha'(t) \tau_{FD} \frac{\Delta(t)}{a^2} \frac{1 + \left(\frac{\Delta(t)}{a}\right)^2}{\left[1 - \left(\frac{\Delta(t)}{a}\right)^2\right]^3} \\
& * \sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - \frac{t}{\tau_{FD}} e^{-\frac{t}{\tau_{FD}} k^2} \right), \tag{76a}
\end{aligned}$$

$$\begin{aligned}
B_{o\phi}^{\text{self}}(t) &= - \frac{\mu_0}{2\pi} I_c(t) \frac{\Delta(t)}{a^2} \frac{1}{1 - \left(\frac{\Delta(t)}{a}\right)^2} \\
& + \frac{1}{a} \sum_{m=1}^{\infty} \left(1 - \frac{2m}{3} \frac{b-a}{a} \right) \left(\frac{\Delta(t)}{a}\right)^{m-1} \\
& * \left[U_{mo}^{(c)}(t) \cos m\alpha(t) + U_{mo}^{(s)}(t) \sin m\alpha(t) \right] \\
& + \frac{b-a}{a} \frac{2\mu_0}{\pi^3} I_c(t) \frac{\Delta(t)}{a^2} \frac{1}{\left[1 - \left(\frac{\Delta(t)}{a}\right)^2\right]^2} \\
& * \sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - e^{-\frac{t}{\tau_{FD}} k^2} \right). \tag{76b}
\end{aligned}$$

After a few e-folds of τ_{FD} , the sum over k becomes equal to $\pi^2/6$ and there is a residual contribution in the self-magnetic field from the fast diffusing terms. Since $(b-a)/a \ll 1$, Eqs. (76a), (76b) indicate that this contribution is small unless the ring is close to the conducting wall. But in that region, these relations are no longer valid and, therefore, they provide only a hint as to the significance of the fast diffusing terms when the ring is close to the conducting wall.

From Eqs. (65c), (70b) (70d) and Eqs. (72a) - (72d), and to lowest order in $(b - a)/a$, the electric field at the ring centroid is given by

$$\begin{aligned}
 E_{\theta 0}^{\text{self}}(t) = & -\frac{\mu_0}{2\pi} \dot{I}_c(t) \left[\ln \frac{a}{r_c} + \frac{1}{2} + \ln \left(1 - \left(\frac{\Delta(t)}{a} \right)^2 \right) \right] \\
 & + \frac{\mu_0}{2\pi} I_c(t) \frac{\frac{x_c(t)\dot{x}_c(t) + z_c(t)\dot{z}_c(t)}{a^2}}{1 - \left(\frac{\Delta(t)}{a} \right)^2} \\
 & + \frac{1}{2\tau_D} \left[U_{00}(t) + 2 \sum_{m=1}^{\infty} \left(\frac{\Delta(t)}{a} \right)^m \left[U_{m0}^{(c)}(t) \cos m\alpha(t) + U_{m0}^{(s)}(t) \sin m\alpha(t) \right] \right] \\
 & - \frac{1}{\tau_D} \left[\frac{\mu_0}{4\pi} I_c(t) + h_e^{\text{self}}(t, t) \right] \\
 & - \frac{1}{\tau_D} \frac{\mu_0}{4\pi} 2 \sum_{k=1}^{\infty} \left[I_c(t) - \frac{k^2}{\tau_{FD}} \int_0^t I_c(t-t') e^{-\frac{t'}{\tau_{FD}} k^2} dt' \right] \\
 & - \frac{1}{\tau_D} 2 \sum_{k=1}^{\infty} \left[h_e^{\text{self}}(t, t) - \frac{k^2}{\tau_{FD}} \int_0^t h_e^{\text{self}}(t, t-t') e^{-\frac{t'}{\tau_{FD}} k^2} dt' \right],
 \end{aligned} \tag{77}$$

where

$$\begin{aligned}
 h_e^{\text{self}}(t, t') = & \frac{\mu_0}{2\pi} I_c(t') \frac{\Delta(t)\Delta(t')}{a^2} \\
 & * \frac{\cos(\alpha(t) - \alpha(t')) - \frac{\Delta(t)\Delta(t')}{a^2}}{1 + \left(\frac{\Delta(t)\Delta(t')}{a^2} \right)^2 - 2 \frac{\Delta(t)\Delta(t')}{a^2} \cos(\alpha(t) - \alpha(t'))}.
 \end{aligned} \tag{78}$$

When $I_c(t)$, $\Delta(t)$, $\alpha(t)$ vary slowly, in a similar fashion as for the self-magnetic field, we obtain the simplified equation for the self-electric field

$$\begin{aligned}
 E_{\theta 0}^{\text{self}}(t) = & -\frac{\mu_0}{2\pi} \dot{I}_c(t) \left[\ln \frac{a}{r_c} + \frac{1}{2} + \ln \left(1 - \left(\frac{\Delta(t)}{a} \right)^2 \right) \right] \\
 & + \frac{\mu_0}{2\pi} I_c(t) \frac{\frac{x_c(t)\dot{x}_c(t) + z_c(t)\dot{z}_c(t)}{a^2}}{1 - \left(\frac{\Delta(t)}{a} \right)^2} \\
 & + \frac{1}{2\tau_D} \left[U_{00}(t) + 2 \sum_{m=1}^{\infty} \left(\frac{\Delta(t)}{a} \right)^m \left[U_{m0}^{(c)}(t) \cos m\alpha(t) + U_{m0}^{(s)}(t) \sin m\alpha(t) \right] \right] \\
 & - \frac{\mu_0}{4\pi} \frac{I_c(t)}{\tau_D} \theta_3(0, e^{-t/\tau_{FD}}) \left[1 + 2 \frac{\left(\frac{\Delta(t)}{a} \right)^2}{1 - \left(\frac{\Delta(t)}{a} \right)^2} \right], \quad (79)
 \end{aligned}$$

where $\theta_3(z, q)$ is the theta function of order 3,¹⁶ i.e.,

$$\theta_3(z, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos 2kz. \quad (80)$$

Notice that $\theta_3(0, q)$ can be expressed in terms of the complete elliptic integral of the first kind $K(m)$, i.e., $\theta_3(0, q) = [2K(m)/\pi]^{1/2}$. Here, m is related with the nome q by the relation $q = \exp[-\pi K(1-m)/K(m)]$ and when q tends to 1, then m tends also to 1. When $t/\tau_{FD} \ll 1$, the nome q is very close to 1, and in this case $K(1-m) \approx K(0) = \pi/2$, so that

$[2K(m)/\pi]^{1/2} \approx [\pi/\ln(1/q)]^{1/2}$. Therefore, $\theta_3(0, e^{-t/\tau_{FD}}) \approx [\pi/(t/\tau_{FD})]^{1/2}$, i.e., the self-electric field is proportional to $I_c(t)[\tau_{FD}/t]^{1/2}$ when $t/\tau_{FD} \ll 1$. If the ring current is a step function of time, i.e., $I_c(t) = I_0 \theta(t)$, then the self-electric field is infinite at $t = 0$ (actually, it is infinite everywhere inside the torus). This result is not surprising if we

take into account that the vector potential rises in time instantaneously at $t = 0$, because the ring current does so. When $t/\tau_{FD} \gg 1$, we have $\theta_3(0, e^{-t/\tau_{FD}}) \approx 1$ and the fast diffusing part in $E_{\theta 0}^{self}$ provides a residual contribution.

Finally, consider the electric field outside the torus, i.e., when $\rho \geq b$. From Eqs. (4c), (58), (59a)-(59c), (70d) and Eqs. (72a)-(72d) we see that, to lowest order in $(b-a)/a$, it is given by

$$\begin{aligned}
 E_{\theta 0}^{out}(\rho, \phi, t) = & \frac{1}{2\tau_D} \left[\frac{\ln \frac{8r_0}{\rho} - 2}{\ln \frac{8r_0}{b} - 2} U_{00}(t) \right. \\
 & \left. + 2 \sum_{m=1}^{\infty} \left(\frac{b}{\rho} \right)^m \left(U_{m0}^{(c)}(t) \cos m\phi + U_{m0}^{(s)}(t) \sin m\phi \right) \right] \\
 & - \frac{1}{\tau_D} \left[\frac{\ln \frac{8r_0}{\rho} - 2}{\ln \frac{8r_0}{b} - 2} \frac{\mu_0}{4\pi} I_c(t) + h_0(\rho, \phi, t) \right] \\
 & - \frac{1}{\tau_D} \frac{\ln \frac{8r_0}{\rho} - 2}{\ln \frac{8r_0}{b} - 2} \frac{\mu_0}{4\pi} 2 \sum_{k=1}^{\infty} (-1)^k \left[I_c(t) - \frac{k^2}{\tau_{FD}} \int_0^t I_c(t-t') e^{-\frac{t'}{\tau_{FD}} k^2} dt' \right] \\
 & - \frac{1}{\tau_D} 2 \sum_{k=1}^{\infty} (-1)^k \left[h_0(\rho, \phi, t) - \frac{k^2}{\tau_{FD}} \int_0^t h_0(\rho, \phi, t-t') e^{-\frac{t'}{\tau_{FD}} k^2} dt' \right],
 \end{aligned} \tag{81}$$

where

$$h_0(\rho, \phi, t) = \frac{\mu_0}{2\pi} I_c(t) \frac{b\Delta(t)}{\rho a} \tag{82}$$

$$* \frac{\cos(\phi - \alpha(t)) - \frac{b\Delta(t)}{\rho a}}{1 + \left(\frac{b\Delta(t)}{\rho a}\right)^2 - 2 \frac{b\Delta(t)}{\rho a} \cos(\phi - \alpha(t))}.$$

When $I_c(t)$, $\Delta(t)$, $\alpha(t)$ vary slowly, Eq. (81) simplifies to

$$E_{\theta 0}^{\text{out}}(\rho, \phi, t) = \frac{1}{2\tau_D} \left[\frac{\ln \frac{8r_0}{\rho} - 2}{\ln \frac{8r_0}{b} - 2} U_{00}(t) \right. \quad (83)$$

$$+ 2 \sum_{m=1}^{\infty} \left(\frac{b}{\rho} \right)^m \left(U_{m0}^{(c)}(t) \cos m\phi + U_{m0}^{(s)}(t) \sin m\phi \right) \Bigg]$$

$$- \theta_4 \left(0, e^{-t/\tau_{FD}} \right) \frac{1}{\tau_D} \left[\frac{\ln \frac{8r_0}{\rho} - 2}{\ln \frac{8r_0}{b} - 2} \frac{\mu_0}{4\pi} I_c(t) + h_0(\rho, \phi, t) \right]$$

where $\theta_4(z, q)$ is the theta function of order 4,¹⁶ i.e.,

$$\theta_4(z, q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos 2kz. \quad (84)$$

Notice that $\theta_4(0, q) = [(1-m)^{1/2} 2 K(m)/\pi]^{1/2}$. When $t/\tau_{FD} \ll 1$, the nome q is very close to 1 and in this case, $K(m) \approx (1/2) \ln[16/(1-m)]$ and $2 K(m)/\pi \approx \pi/\ln(1/q)$. Therefore, we have

$$\theta_4 \left(0, e^{-t/\tau_{FD}} \right) \approx 2 \left[\frac{\pi}{t/\tau_{FD}} e^{-\frac{\pi^2}{2(t/\tau_{FD})}} \right]^{1/2}, \quad (85)$$

i.e., when the ring current is a step function of time, the electric field is zero at $t = 0$ outside the torus. On the other hand, when

$t/\tau_{FD} \gg 1$, then $\theta_4(0, e^{-t/\tau_{FD}}) \approx 1$ and the electric field reduces to that computed from the shell model. It appears, therefore, that the fast diffusing part in $E_{\theta 0}^{out}$ contributes the exact amount needed for $E_{\theta 0}^{out}$ to vanish at $t = 0$, but very quickly $E_{\theta 0}^{out}$ increases to the value predicted by the shell model when the conducting wall is thin.

To calculate the wall current density $J_{\theta 0}^{wall}(\rho, \phi, t) = \sigma E_{\theta 0}^{con}(\rho, \phi, t)$, the electric field inside the conductor is needed. From the continuity of the electric field at the inner and outer surfaces of the conductor and when $t/\tau_{FD} \ll 1$, $E_{\theta 0}^{con}$ varies from a very large value at the inner surface to a very small value at the outer surface. However, when $t/\tau_{FD} \gg 1$, but $t/\tau_D \ll 1$, and in the special case of a thin conducting wall, it is easy to show from Eqs. (57b), (72a), (72c) and (83) that the electric fields at the inner and outer surface are approximately equal to each other. In the extreme case of the shell model, they become exactly equal to each other, and the surface wall current density is equal to $J_{\theta s}^{wall}(\phi, t) = \sigma(b/a)E_{\theta 0}^{out}(a, \phi, t)$, where $E_{\theta 0}^{out}(a, \phi, t)$ is given by Eq. (83). The surface wall current $I_s^{wall}(t)$ is computed by integrating $J_{\theta s}^{wall}(\phi, t)$ over the poloidal angle ϕ .

VIII. Two Applications for the Shell Model

When $(b-a)/a$ tends to zero but $\sigma(b-a)/a$ remains finite, i.e., for the shell model, Eqs. (57a), (57b) and (58) simplify considerably, i.e.,

$$A_{\theta 0}^{\text{in}}(\rho, \phi, t) = \frac{\mu_0}{4\pi} I_c(t) \left[2 \left(\ln \frac{a}{r_c} + \frac{1}{2} \right) - \left(\frac{\rho'}{r_c} \right)^2 \right. \\ \left. + \ln \left(1 + \left(\frac{\rho \Delta(t)}{a^2} \right)^2 - 2 \frac{\rho \Delta(t)}{a^2} \cos(\phi - \alpha(t)) \right) \right] \quad (86a)$$

$$+ \left(\ln \frac{8r_0}{a} - 2 \right) U_{00}(t) + \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{a} \right)^m \left(U_{m0}^{(c)}(t) \cos m\phi + U_{m0}^{(s)}(t) \sin m\phi \right),$$

inside the ring ($\rho' \leq r_c$),

$$A_{\theta 0}^{\text{in}}(\rho, \phi, t) = \frac{\mu_0}{4\pi} I_c(t) \left[2 \ln \frac{a}{\rho'} \right. \quad (86b)$$

$$\left. + \ln \left(1 + \left(\frac{\rho \Delta(t)}{a^2} \right)^2 - 2 \frac{\rho \Delta(t)}{a^2} \cos(\phi - \alpha(t)) \right) \right]$$

$$+ \left(\ln \frac{8r_0}{a} - 2 \right) U_{00}(t) + \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{a} \right)^m \left(U_{m0}^{(c)}(t) \cos m\phi + U_{m0}^{(s)}(t) \sin m\phi \right)$$

outside the ring but inside the torus ($\rho \leq a$), and

$$A_{\theta 0}^{\text{out}}(\rho, \phi, t) = \left(\ln \frac{8r_0}{\rho} - 2 \right) U_{00}(t) \quad (87)$$

$$+ \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{a}{\rho} \right)^m \left(U_{m0}^{(c)}(t) \cos m\phi + U_{m0}^{(s)}(t) \sin m\phi \right),$$

outside the torus ($\rho \geq a$), where ρ' is defined by Eq. (27b).

First, consider the case of a motionless ring, i.e., Δ and α are constant, and its current is a step function of time, i.e., $I_c(t) = I_0 \Theta(t)$. From Eqs. (52a)-(52c) we have:

$$U_{00}(t) = \frac{\mu_0}{2\pi} I_0 \left(1 - e^{-t/\tau_{00}}\right), \quad (88a)$$

$$U_{m0}^{(c)}(t) + i U_{m0}^{(s)}(t) = \frac{\mu_0}{2\pi} I_0 \left(\frac{\Delta}{a}\right)^m e^{im\alpha} \left(1 - e^{-\frac{m}{\tau_D} t}\right), \quad (88b)$$

and the vector potential becomes:

$$\begin{aligned} A_{\Theta 0}^{in}(\rho, \phi, t) = & \frac{\mu_0}{4\pi} I_0 \left[2 \left(\ln \frac{8r_0}{r_c} - \frac{3}{2} \right) - \left(\frac{\rho'}{r_c} \right)^2 \right. \\ & \left. - 2 \left(\ln \frac{8r_0}{a} - 2 \right) e^{-t/\tau_{00}} + \ln \left(1 + \left(\frac{\rho \Delta}{a^2} e^{-t/\tau_D} \right)^2 - 2 \frac{\rho \Delta}{a^2} e^{-t/\tau_D} \cos(\phi - \alpha) \right) \right] \end{aligned} \quad (89a)$$

inside the ring ($\rho' \leq r_c$),

$$\begin{aligned} A_{\Theta 0}^{in}(\rho, \phi, t) = & \frac{\mu_0}{4\pi} I_0 \left[2 \left(\ln \frac{8r_0}{\rho'} - 2 \right) \right. \\ & \left. - 2 \left(\ln \frac{8r_0}{a} - 2 \right) e^{-t/\tau_{00}} + \ln \left(1 + \left(\frac{\rho \Delta}{a^2} e^{-t/\tau_D} \right)^2 - 2 \frac{\rho \Delta}{a^2} e^{-t/\tau_D} \cos(\phi - \alpha) \right) \right] \end{aligned} \quad (89b)$$

outside the ring but inside the torus ($\rho \leq a$), and

$$\begin{aligned} A_{\Theta 0}^{out}(\rho, \phi, t) = & \frac{\mu_0}{4\pi} I_0 \left[2 \left(\ln \frac{8r_0}{\rho'} - 2 \right) \right. \\ & \left. - 2 \left(\ln \frac{8r_0}{\rho} - 2 \right) e^{-t/\tau_{00}} + \ln \left(1 + \left(\frac{\Delta}{\rho} e^{-t/\tau_D} \right)^2 - 2 \frac{\Delta}{\rho} e^{-t/\tau_D} \cos(\phi - \alpha) \right) \right] \end{aligned} \quad (90)$$

outside the torus ($\rho \geq a$). Notice that although $A_{\theta 0}^{\text{out}}(\rho, \phi, t)$ is zero at $t = 0$, its partial time derivative is not zero at $t = 0$, because, as explained in the previous section, the fast diffusing terms which render the electric field zero at $t = 0$ outside the torus, are missing in the shell model. Also, notice that at $t=0$ the vector potential is the sum of the external vector potential of the ring and its image, but for $t \gg \tau_{00}$ only the vector potential of the ring remains present.

As a second application, consider the case in which the ring moves on a circle, i.e., Δ is constant, $\alpha = \omega t$, and its current is a step function of time, i.e., $I_c(t) = I_0 \Theta(t)$. From Eqs. (52a)-(52c) we have:

$$U_{00}(t) = \frac{\mu_0}{2\pi} I_0 \left(1 - e^{-t/\tau_{00}} \right), \quad (91a)$$

$$U_{m0}^{(c)}(t) = \frac{\mu_0}{2\pi} I_0 \left(\frac{\Delta}{a} \right)^m \frac{\cos m\omega t + \omega \tau_D \sin m\omega t - e^{-m \frac{t}{\tau_D}}}{1 + (\omega \tau_D)^2}, \quad (91b)$$

$$U_{m0}^{(s)}(t) = \frac{\mu_0}{2\pi} I_0 \left(\frac{\Delta}{a} \right)^m \frac{\sin m\omega t - \omega \tau_D \cos m\omega t + \omega \tau_D e^{-m \frac{t}{\tau_D}}}{1 + (\omega \tau_D)^2}, \quad (91c)$$

and the vector potential becomes:

$$\begin{aligned} A_{\theta 0}^{\text{in}}(\rho, \phi, t) = & \frac{\mu_0}{4\pi} I_0 \left[2 \left(\ell n \frac{8r_0}{r_c} - \frac{3}{2} \right) - \left(\frac{\rho'}{r_c} \right)^2 \right. \\ & - 2 \left(\ell n \frac{8r_0}{a} - 2 \right) e^{-t/\tau_{00}} + \ell n \left(1 + \left(\frac{\rho \Delta}{a} \right)^2 - 2 \frac{\rho \Delta}{a} \cos(\phi - \omega t) \right) \\ & \left. - \frac{1}{1 + (\omega \tau_D)^2} \left(\ell n \left(1 + \left(\frac{\rho \Delta}{a} \right)^2 - 2 \frac{\rho \Delta}{a} \cos(\phi - \omega t) \right) \right) \right] \end{aligned} \quad (92a)$$

$$\begin{aligned}
& - \ln \left(1 + \left(\frac{\rho \Delta}{a} e^{-t/\tau_D} \right)^2 - 2 \frac{\rho \Delta}{a} e^{-t/\tau_D} \cos(\phi - \omega t) \right) \\
& - \frac{2\omega\tau_D}{1 + (\omega\tau_D)^2} \left(\text{Arctan} \frac{\frac{\rho \Delta}{a} \sin(\phi - \omega t)}{1 - \frac{\rho \Delta}{a} \cos(\phi - \omega t)} - \text{Arctan} \frac{\frac{\rho \Delta}{a} e^{-t/\tau_D} \sin \phi}{1 - \frac{\rho \Delta}{a} e^{-t/\tau_D} \cos \phi} \right) \Bigg]
\end{aligned}$$

inside the ring ($\rho' \leq r_c$),

$$\begin{aligned}
A_{\theta 0}^{\text{in}}(\rho, \phi, t) &= \frac{\mu_0}{4\pi} I_0 \left[2 \left(\ln \frac{8r_0}{\rho'} - 2 \right) \right. \\
& - 2 \left(\ln \frac{8r_0}{a} - 2 \right) e^{-t/\tau_{00}} + \ln \left(1 + \left(\frac{\rho \Delta}{a} \right)^2 - 2 \frac{\rho \Delta}{a} \cos(\phi - \omega t) \right) \\
& - \frac{1}{1 + (\omega\tau_D)^2} \left(\ln \left(1 + \left(\frac{\rho \Delta}{a} \right)^2 - 2 \frac{\rho \Delta}{a} \cos(\phi - \omega t) \right) \right. \\
& \left. \left. - \ln \left(1 + \left(\frac{\rho \Delta}{a} e^{-t/\tau_D} \right)^2 - 2 \frac{\rho \Delta}{a} e^{-t/\tau_D} \cos(\phi - \omega t) \right) \right) \right. \\
& \left. - \frac{2\omega\tau_D}{1 + (\omega\tau_D)^2} \left(\text{Arctan} \frac{\frac{\rho \Delta}{a} \sin(\phi - \omega t)}{1 - \frac{\rho \Delta}{a} \cos(\phi - \omega t)} - \text{Arctan} \frac{\frac{\rho \Delta}{a} e^{-t/\tau_D} \sin \phi}{1 - \frac{\rho \Delta}{a} e^{-t/\tau_D} \cos \phi} \right) \right],
\end{aligned} \tag{92b}$$

outside the ring, but inside the torus ($\rho \leq a$), and

$$\begin{aligned}
A_{\theta 0}^{\text{out}}(\rho, \phi, t) &= \frac{\mu_0}{4\pi} I_0 \left[2 \left(\ln \frac{8r_0}{\rho} - 2 \right) \right. \\
& - 2 \left(\ln \frac{8r_0}{\rho} - 2 \right) e^{-t/\tau_{00}} + \ln \left(1 + \left(\frac{\Delta}{\rho} \right)^2 - 2 \frac{\Delta}{\rho} \cos(\phi - \omega t) \right)
\end{aligned} \tag{93}$$

$$\begin{aligned}
& - \frac{1}{1 + (\omega\tau_D)^2} \left(\ln \left(1 + \left(\frac{\Delta}{\rho} \right)^2 - 2 \frac{\Delta}{\rho} \cos(\phi - \omega t) \right) \right. \\
& \left. - \ln \left(1 + \left(\frac{\Delta}{\rho} e^{-t/\tau_D} \right)^2 - 2 \frac{\Delta}{\rho} e^{-t/\tau_D} \cos(\phi - \omega t) \right) \right) \\
& - \frac{2\omega\tau_D}{1 + (\omega\tau_D)^2} \left(\text{Arctan} \frac{\frac{\Delta}{\rho} \sin(\phi - \omega t)}{1 - \frac{\Delta}{\rho} \cos(\phi - \omega t)} - \text{Arctan} \frac{\frac{\Delta}{\rho} e^{-t/\tau_D} \sin\phi}{1 - \frac{\Delta}{\rho} e^{-t/\tau_D} \cos\phi} \right) \Bigg],
\end{aligned}$$

outside the torus ($\rho \geq a$). There are two extreme cases of interest:

i) when $\omega\tau_D \ll 1$; then the vector potential is the same as that of a motionless ring. ii) when $\omega\tau_D \gg 1$; then there is diffusion of the terms associated with the loop time τ_{00} , but the image fields do not dissipate to zero, but they follow in phase the circular motion of the ring.

IX. Conclusion

The diffusion of the self magnetic field of a beam inside a toroidal conductor is governed by three different diffusion times. The loop time τ_{oo} is responsible for the diffusion of the fields into the hole of the doughnut and, after a few e-folds of the fast diffusion time, the time behavior of the wall current is completely determined by τ_{oo} . The "cylinder" diffusion time τ_D is responsible for the dissipation of the image fields which are present initially, but they vanish after a few e-folds of τ_D , if the ring current does not vary with time. Finally, the fast diffusion time is responsible for the electric field outside the conductor to be zero initially, but it acquires approximately the value associated with the shell model after a few e-folds of τ_{FD} .

After a few e-folds of the loop time and if the ring current does not vary with time, the vector potential becomes equal to that in the absence of the conductor. In addition, to zero order in the toroidal corrections, the radial component of the self-magnetic field, which is responsible for the beam trapping, is independent of the loop time. Therefore, the time scale of the trapping mechanism should be independent of τ_{oo} . Reliable results close to the conducting wall can be obtained only by numerical computation of the poles and by including a very large number of terms in the series expansions of the vector potential and the fields. But in the extreme case of the shell model, the results are exact everywhere inside as well as outside and in the vicinity of the torus. This model provides quite an accurate description of the diffusion process for a toroidal conductor with a thin wall, except during the first few e-folds of τ_{FD} , since the effect of the fast diffusing terms is not included. Due to the simplicity of the shell model, it is rather easy to compute the first order toroidal corrections. These results will be reported in a future publication.

Appendix

An expression for $f_m(\alpha)$, given by Eq. (54b), will be derived here, correct to second order in m/x_0 . Using the multiplication theorem for the Bessel functions, we obtain the relation¹⁶

$$\begin{aligned} \frac{2}{\pi} \frac{1}{x_0 x_1} f_m(\alpha) &= J_{m+1}(x_0) Y_{m-1}(x_1) - Y_{m+1}(x_0) J_{m-1}(x_1) \quad (A1) \\ &= \left(\frac{b}{a}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{b^2 - a^2}{2a^2} x_0\right)^k [J_{m+1}(x_0) Y_{m-1+k}(x_0) - Y_{m+1}(x_0) J_{m-1+k}(x_0)]. \end{aligned}$$

With the help of the identities

$$J_\ell(x_0) Y_{\ell+1}(x_0) - Y_\ell(x_0) J_{\ell+1}(x_0) = -\frac{2}{\pi x_0}, \quad (A2a)$$

$$Z_{\ell-1}(x_0) + Z_{\ell+1}(x_0) = \frac{2\ell}{x_0} Z_\ell(x_0), \quad (A2b)$$

where $Z_\ell(x_0)$ is either $J_\ell(x_0)$ or $Y_\ell(x_0)$, we can show that

$$J_{m+1}(x_0) Y_{m-1}(x_0) - Y_{m+1}(x_0) J_{m-1}(x_0) = \frac{2m}{x_0} \frac{2}{\pi x_0}, \quad (A3a)$$

$$J_{m+1}(x_0) Y_m(x_0) - Y_{m+1}(x_0) J_m(x_0) = \frac{2}{\pi x_0}, \quad (A3b)$$

$$J_{m+1}(x_0) Y_{m+2}(x_0) - Y_{m+1}(x_0) J_{m+2}(x_0) = -\frac{2}{\pi x_0}, \quad (A3c)$$

and for $\ell = 1, 2, \dots$ we can also show that

$$\begin{aligned} &J_{m+1}(x_0) Y_{m+4\ell-1}(x_0) - Y_{m+1}(x_0) J_{m+4\ell-1}(x_0) \\ &= \frac{2}{\pi x_0} \left[-\frac{2(2\ell-1)(m+2\ell)}{x_0} + O\left(\frac{1}{x_0^3}\right) \right], \end{aligned} \quad (A4a)$$

$$J_{m+1}(x_0) Y_{m+4\ell}(x_0) - Y_{m+1}(x_0) J_{m+4\ell}(x_0)$$

$$= \frac{2}{\pi x_0} \left[1 - \frac{4\ell(2\ell-1)(m+2\ell)(m+2\ell+1)}{x_0^2} + o\left(\frac{1}{x_0^4}\right) \right], \quad (A4b)$$

$$\begin{aligned} & J_{m+1}(x_0) Y_{m+4\ell+1}(x_0) - Y_{m+1}(x_0) J_{m+4\ell+1}(x_0) \\ &= \frac{2}{\pi x_0} \left[\frac{4\ell(m+2\ell+1)}{x_0} + o\left(\frac{1}{x_0^3}\right) \right], \end{aligned} \quad (A4c)$$

$$\begin{aligned} & J_{m+1}(x_0) Y_{m+4\ell+2}(x_0) - Y_{m+1}(x_0) J_{m+4\ell+2}(x_0) \\ &= \frac{2}{\pi x_0} \left[1 + \frac{4\ell(2\ell+1)(m+2\ell+1)(m+2\ell+2)}{x_0^2} + o\left(\frac{1}{x_0^4}\right) \right]. \end{aligned} \quad (A4d)$$

Therefore, we conclude that

$$\begin{aligned} & \left(\frac{a}{b}\right)^m \frac{1}{x_0} f_m(\alpha) = \frac{2m}{x_0} - \sin \left(\frac{b^2 - a^2}{2a^2} x_0 \right) \\ & - \frac{2}{x_0} \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(m+k+2)}{(2k+4)!} \left(\frac{b^2 - a^2}{2a^2} x_0 \right)^{2k+4} \\ & + \frac{4}{x_0^2} \sum_{\ell=1}^{\infty} \frac{\ell(2\ell-1)(m+2\ell)(m+2\ell+1)}{(4\ell+1)!} \left(\frac{b^2 - a^2}{2a^2} x_0 \right)^{4\ell+1} \\ & - \frac{4}{x_0^2} \sum_{\ell=1}^{\infty} \frac{\ell(4\ell+1)(m+2\ell+1)(m+2\ell+2)}{(4\ell+3)!} \left(\frac{b^2 - a^2}{2a^2} x_0 \right)^{4\ell+1}. \end{aligned} \quad (A5)$$

where terms of order $1/x_0^3$ and higher have been omitted. A straightforward

and lengthy calculation leads to the relations

$$\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(m+k+2)}{(2k+4)!} z^{2k+4} \quad (A6a)$$

$$= m - \frac{z}{2} \left(m - \frac{1}{2} \right) \sin z - \left(m + \frac{z^2}{4} \right) \cos z,$$

$$\sum_{\ell=1}^{\infty} \frac{\ell(2\ell-1)(m+2\ell)(m+2\ell+1)}{(4\ell+1)!} z^{4\ell+1} \quad (A6b)$$

$$- \sum_{\ell=1}^{\infty} \frac{\ell(2\ell+1)(m+2\ell+1)(m+2\ell+2)}{(4\ell+3)!} z^{4\ell+3}$$

$$= \frac{1}{8} \left[\frac{z^4}{4} - \left(m - \frac{3}{2} \right) \left(m + \frac{1}{2} \right) z^2 + \frac{3}{4} \right] \sin z$$

$$- \left(m + \frac{1}{2} \right) \frac{z}{8} \left[z^2 + 3 \left(m - \frac{1}{2} \right) \right] \cos z.$$

Substituting Eqs. (A6a), (A6b) into Eq. (A5), we conclude that

$$\left(\frac{a}{b} \right)^m \frac{1}{x_1} f_m(\alpha) = - \left[1 - \left(m - \frac{1}{2} \right) \left(\frac{b^2 - a^2}{2a^2} \right) \right] \quad (A7)$$

$$- \frac{1}{2x_0^2} \left[\frac{9}{4} - \left(m - \frac{3}{2} \right) \left(m + \frac{1}{2} \right) \left(\frac{b^2 - a^2}{2a^2} x_0 \right)^2 + \frac{1}{4} \left(\frac{b^2 - a^2}{2a^2} x_0 \right)^4 \right] + 0 \left(\left(\frac{m}{x_0} \right)^3 \right)$$

$$* \sin \left(\frac{b^2 - a^2}{2a^2} x_0 \right)$$

$$+ \left[\frac{2}{x_0} \left(m + \frac{1}{4} \left(\frac{b^2 - a^2}{2a^2} x_0 \right)^2 \right) - \frac{1}{2x_0^2} \left(m + \frac{1}{2} \right) \left(3 \left(m - \frac{1}{2} \right) + \left(\frac{b^2 - a^2}{2a^2} x_0 \right)^2 \right) \right]$$

$$+ 0 \left(\left(\frac{m}{x_0} \right)^3 \right) \cos \left(\frac{b^2 - a^2}{2a^2} x_0 \right)$$

We see that the zeroes of $f_m(\alpha)$ are given by the relation

$$\tan \left(\frac{b^2 - a^2}{2a^2} x_0 \right) = A/B, \quad (A8)$$

where

$$A = \frac{2m}{x_0} + \frac{1}{2} \left(\frac{b^2 - a^2}{2a^2} \right)^2 x_0 - \frac{1}{2} \left(m + \frac{1}{2} \right) \left(\frac{3(m - \frac{1}{2})}{x_0^2} + \left(\frac{b^2 - a^2}{2a^2} \right)^2 \right)$$

$$B = 1 - \left(m - \frac{1}{2} \right) \left(\frac{b^2 - a^2}{2a^2} \right) - \frac{1}{2} \frac{9}{4x_0^2} - \left(m - \frac{3}{2} \right) \left(m + \frac{1}{2} \right) \left(\frac{b^2 - a^2}{2a^2} \right)^2 + \frac{1}{4} \left(\frac{b^2 - a^2}{2a^2} \right)^4 x_0^2,$$

correct to order $(m/x_0)^2$. If $(b - a)/a \ll 1$, and $x_0 \gg 1$, the relation above simplifies to

$$\tan (x_1 - x_0) \approx \frac{\frac{2m}{x_0} - \frac{3}{2} \frac{m^2 - \frac{1}{4}}{x_0^2} + \frac{1}{2} \left(\frac{b - a}{a} \right)^2 x_0}{1 - \left(m - \frac{1}{2} \right) \frac{b - a}{a} + \left(m - \frac{3}{2} \right) \left(m + \frac{1}{2} \right) \left(\frac{b - a}{a} \right)^2}. \quad (A9)$$

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Table I. Parameters of the run shown in Fig. 2

Torus major radius r_0	100 cm
Torus inner minor radius a	15.2 cm
Torus outer minor radius b	15.217 cm
Strong focusing radius ρ_0	23.4 cm
Strong focusing current I_{SF}	24 kA
Strong focusing Periodicity m	6
Vertical magnetic field B_{z0}	26 Gauss
Toroidal magnetic field $B_{\theta 0}$	4000 Gauss
Beam relativistic factor γ	1.69714
Beam minor radius r_c	3 mm
Beam current I_c	0.5 kA
Wall resistivity	8.84 m Ω cm
Intermediate frequency ω_v	$1.8 \times 10^9 \text{ sec}^{-1}$

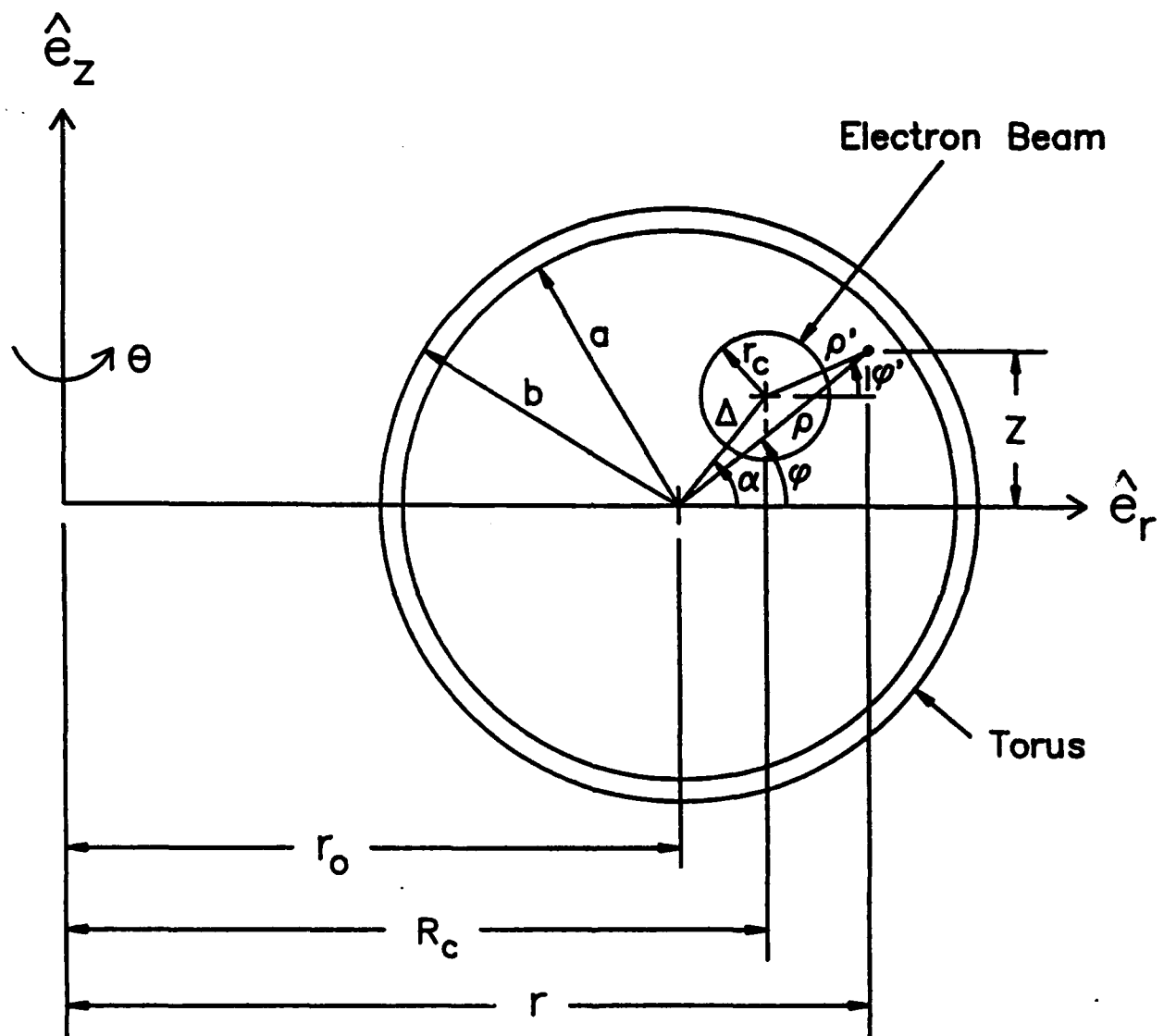


Fig. 1 System of coordinates

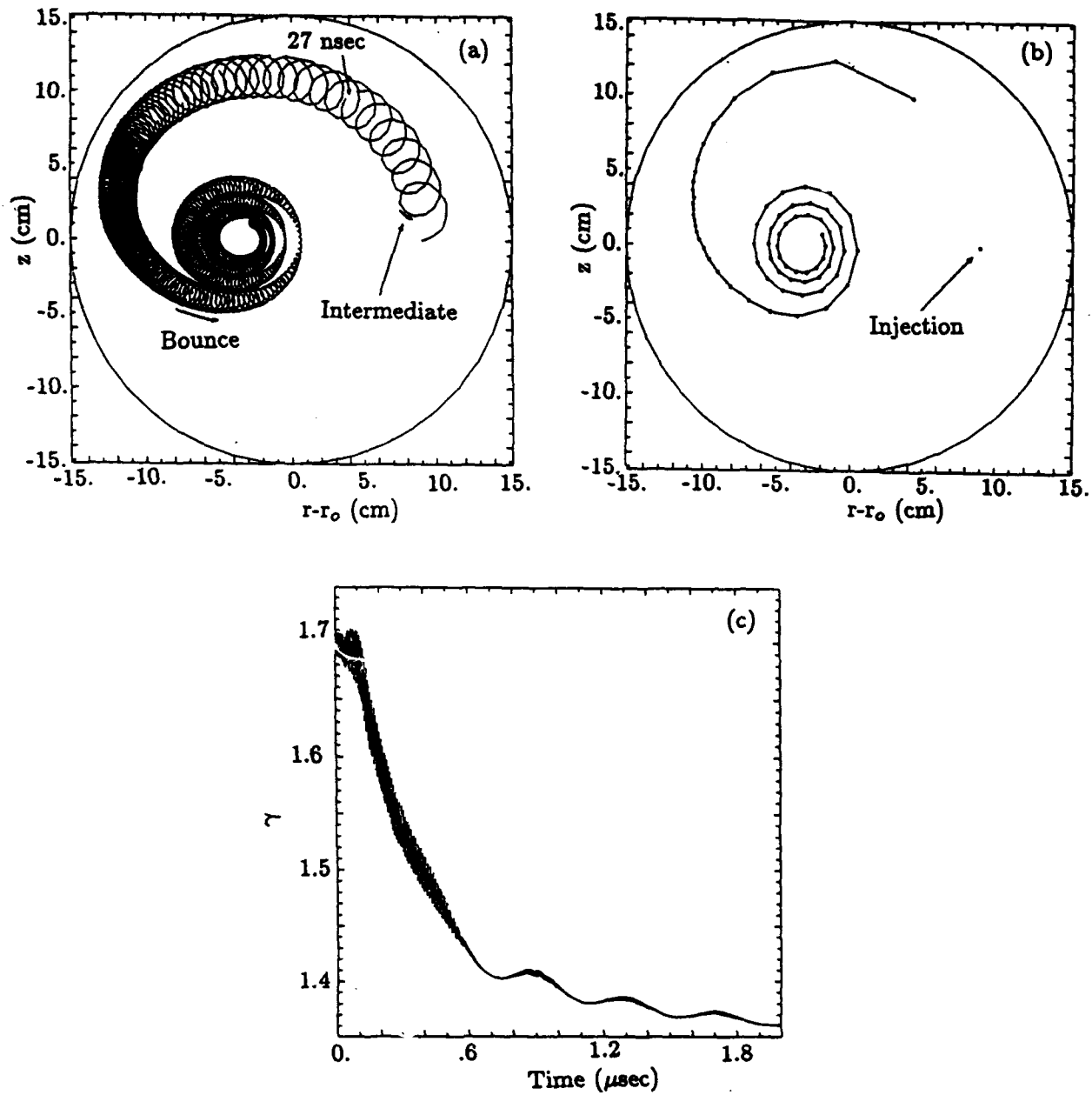


Fig. 2 Beam centroid orbit [(a) and (b)] and relativistic factor vs. time [(c)] from the numerical integration of the ring equations of motion coupled with the diffusion fields.