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13. ABSTRACT *(Maximum 200 words)*

In this report, we will focus on minimizing the cost due to inspection and repair of hydraulic structures. The optimization is based on symmetric properties of the underlying physical deterioration process: i.e. the damages per time-unit are exchangeable and the probabilities of preventive repair and failure are obtained by conditioning on the average amount of deterioration with regard to a finite or an infinite time-horizon. By introducing a prior for the average deterioration per time-unit we can account for uncertainty in the decision problems. Advantages of our Bayesian approach are that we base our mathematical models on an observable quantity, namely the damage, and, surprisingly, that our results are just sums of products which can be easily evaluated. Two examples from the field of hydraulic engineering are studied: determining a preventive repair interval when a safety norm is given and a cost-optimal periodic inspection rate if there is a possibility for a preventive repair during each inspection.

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INSPECTION & REPAIR DECISIONS FOR HYDRAULIC STRUCTURES UNDER SYMMETRIC DETERIORATION

Interim Technical Report

by

Ir. Jan M. van Noortwijk

June 1992

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Inspection and Repair Decisions for Hydraulic Structures under Symmetric Deterioration*

Jan M. van Noortwijk†

June 18, 1992

Abstract

In this report we will focus on minimising the cost due to inspection and repair of hydraulic structures. The optimisation is based on symmetric properties of the underlying physical deterioration process: i.e. the damages per time-unit are exchangeable and the probabilities of preventive repair and failure are obtained by conditioning on the average amount of deterioration with regard to a finite or an infinite time-horizon. By introducing a prior for the average deterioration per time-unit we can account for uncertainty in the decision problems. Advantages of our Bayesian approach are that we base our mathematical models on an observable quantity, namely the damage, and, surprisingly, that our results are just sums of products which can be easily evaluated. Two examples from the field of hydraulic engineering are studied: determining a preventive repair interval when a safety norm is given and a cost-optimal periodic inspection rate if there is a possibility for a preventive repair during each inspection.

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1 Introduction

This report will discuss the subject of preventive maintenance of hydraulic structures. The problem emerges from the field of structural engineering, although the same approach can be applied within the field of mechanical and electrical engineering.

Preventive maintenance of hydraulic structures consists mainly of two parts: inspection and repair. A repair can either be a preventive repair (before failure) or a corrective repair (after failure). Inspection is often needed, because it is not possible to observe continuously the physical condition of a structure, e.g. the height of a dyke or the strength of a bridge. For an extensive problem description about maintenance of hydraulic structures see van Noortwijk [11].

Inspection and repair entail substantial costs. Money can be saved by weighing inspection cost on one hand with repair and failure cost on the other. A high inspection rate will result in high inspection cost and low repair and failure cost. No inspection at all will surely lead to failure and high cost. An optimal or near-optimal maintenance policy lies somewhere in between. In practice the failure cost is often much higher than the cost of preventive repair, where failure cost includes both cost of corrective repair and other public or private losses due to failure.

To lower the expected sum of inspection, preventive repair and failure cost a wide variety of mathematical models has been built. For surveys see Barlow & Proschan [4], Pierskalla & Voelker [12], Sherif & Smith [13] and Sherif [14]. In most cases inspection and repair are modelled in a probabilistic way. Quite popular are maintenance models based on lifetime distributions assuming that the deterioration law of a structure is Markovian. A problem in practice is to get data for either a lifetime distribution or the transition probabilities of a Markov chain (where a transition is a change of the structure's physical condition). Unfortunately, only a few applications can be found.

Instead of assuming a lifetime distribution or a Markov chain, we will follow a different approach. As in structural engineering we will define a failure as the event at which a structure's resistance drops below the stress. We will focus on stochastic resistance and a constant deterministic stress.

The resistance will be modelled with damages appearing per time-unit. The advantage of this approach is that damage is observable, while a lifetime or a transition probability matrix is not. Furthermore we do not make the usual assumption of mutual independent damages, but base our proposition on the physics and the decision problem.

Our Bayesian approach is applicable for decision problems, where the joint probability density of the amounts of damage per time-unit is invariant under permutation and depends on the average amount of damage, while considering a finite or infinite time-horizon. These properties are known as exchangeability and l_1 symmetry respectively. They will be discussed in section 2 and were first applied to decision problems by Mendel [9, 10].

An advantage of the discretisation in time-units and the application of exchangeability and l_1 symmetry is that the calculation of the probabilities of preventive repair and failure becomes tractable. All probabilities can be expressed in sums and products. Moreover, the Bayesian modelling is based on the underlying physics. In general a classical Bayesian provides a prior for every uncertain parameter in a probabilistic problem formulation, without accounting for ways to determine these priors or even to make sure whether the parameter is observable in practice.

Concerning to the inspections we will focus on a periodic inspection policy, although Barlow, Hunter & Proschan [2] proved that a constant inspection rate is not optimal when the lifetime distribution is not exponential. Our failure distribution based on accumulated damages will not be exponential. In practice a periodic inspection policy is often chosen, because it is easier to plan manpower. In the future it is interesting to look at a resistance dependent inspection policy.

Two examples will be studied in section 3.

First, we focus on determining a preventive repair interval (without needed or possible inspection), while the failure probability should not exceed a certain safety norm.

Second, we study an inspection model which is an extension of the basic model of Barlow, Hunter & Proschan [2]. It includes periodic inspection, while at an inspection a preventive repair only will be executed if the resistance is below a preventive repair level. If an inspection is scheduled too late the structure can pass into the failed state. A large amount of money is involved when this happens. Passing the preventive repair level can only be noted through inspection, whereas a failure will be noted immediately, like an inundation of an area protected with flood defences against the sea. The time-horizon can be finite and the decision variable is the inspection rate.

All technical proofs can be found in the Appendix.

Notation

c_F	Cost of failure.
c_I	Cost of one inspection.
c_P	Cost of preventive repair.
$C(k)$	Expected sum of inspection, preventive repair and failure cost.
D_l	Amount of damage in the lth time-unit.
$k\Delta$	Length of inspection interval.
$N\Delta$	Time-horizon.
$P_{norm}(n)$	Safety probability norm.
r	Resistance at time zero.
R	Resistance or strength.
R_n	Resistance in time-unit n.
$R - S < 0$	Failure.
s	Constant stress or failure level.
S	Stress, load or action effect.
Δ	Length of time-unit.
θ	Average amount of damage per time-unit.
ρ	Preventive repair level.

2 Assumptions and definitions

In the field of mechanical and electrical engineering one often considers lifetime distributions to model the appearance of failure in a system, e.g. a motor or switch is working or not, whereas in the area of structural engineering failure is decomposed to comparing a structure's *resistance* or *strength* R with its *stress*, *load* or *action effect* S (see e.g. Ang & Tang [1] and [6]). A structure is said to *fail* if its resistance is below the stress, i.e. if the so-called *performance function* $R - S$ is below zero. In general both resistance and stress are unknown functions of time and moreover not necessarily independent. In this paper we will restrict ourselves with stochastic resistance and deterministic stress. This means that a structure will fail if its resistance R is below a constant failure level s.

In time the strength of a structure will degrade due to deterioration. Since deterioration mostly is observed according to a particular time-unit, we subdivide the time-axis $(0, \infty)$ into time-units $((n-1)\Delta, n\Delta]$, with $n = 1, 2, 3, \ldots$ and time-unit length Δ. Often a structure is planned to function for a finite time-interval, say $(0, N\Delta]$. If the structure has met its needs it will be replaced at $N\Delta$. In general this replacement will be an improvement. With the passage of time new techniques come available and can result in a higher original resistance and a lower deterioration rate. In our terminology an improvement differs from the maintenance action preventive repair.

Suppose that in time-unit n the structure's resistance suffers a stochastic degradation D_n, $D_n \geq 0$, and assume the resistance at time zero equals r, where $r > s$. Consequently, the resistance in time-unit n can be written as

$$R_n = r - \sum_{l=1}^{n} D_l, \quad n = 1, \ldots, N. \tag{1}$$

See also figure 1.

We study a physical deterioration process, where the expected resistance is linear decreasing with time. Or in other words: the expected amounts of damage per time-unit are equal to each other. Examples within the field of hydraulic engineering are according to Vrijling, Klatter & Kuiper [15] settlement[1] and the behaviour of a dune under the continuous action of waves and wind. Other examples are corrosion of underwater pipelines and the scour-depth near underwater footings of a bridge (Tang [personal communication]).

Instead of supposing independence we make weaker assumptions: i.e. exchangeability and l_1 symmetry.

[1] When a civil structure is built on clay some water present in the clay will be pressed out with passage of time, causing sinking of the structure.

The vector $\mathbf{D}_N$ of N uncertain amounts of damages,

$$\mathbf{D}_N = (D_1, \ldots, D_N), \tag{2}$$

is assumed to be *exchangeable*. $\mathbf{D}_N$ is exchangeable if the probability density is invariant under all $N!$ permutations of the coordinates, i.e. if

$$\Pr\{D_1 = \delta_1, \ldots, D_N = \delta_N\} = \Pr\{D_1 = \delta_{\pi 1}, \ldots, D_N = \delta_{\pi N}\}, \tag{3}$$

where $\pi \in N!$ is any permutation of $1, \ldots, N$. The variables of an infinite sequence $\mathbf{D}_\infty$, where $\mathbf{D}_\infty = \lim_{N\to\infty} \mathbf{D}_N$, are exchangeable if $D_1, \ldots, D_n$ are exchangeable for each n (see Mendel [10]).

If the coordinates of $\mathbf{D}_N$ are not exchangeable Mendel [9] provides a way to transform them in such a way that they are exchangeable. Also if the expected accumulated degradation is not linear in time, but exponential for example, it would be possible to apply exchangeability. In a subsequent paper we will discuss also nonlinear expected deterioration.

In all examples it is assumed that the decision-maker has subjective information about the average amount of damage determined with regard to the time-horizon $(0, N\Delta]$. Consequently we condition the probability of $\delta_1, \ldots, \delta_N$ on the average amount of deterioration over $(0, N\Delta]$, i.e. on $\frac{1}{N}\sum_{l=1}^{N} D_l = \theta$. Given a constant value of the sum the probability density of the underlying amounts of damages is furthermore assumed to be uniform or

$$\begin{aligned} &\Pr\left\{D_1 = \delta_1, \ldots, D_N = \delta_N \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} = \\ &= \Pr\left\{D_1 = \tilde{\delta}_1, \ldots, D_N = \tilde{\delta}_N \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\}, \end{aligned} \tag{4}$$

for $(\delta_1, \ldots, \delta_N), (\tilde{\delta}_1, \ldots, \tilde{\delta}_N) \in \mathbb{R}_+^N$. $D_1, \ldots, D_N$ are now said to be l_1 *symmetric*[2] or l_1 *isotropic* (see Mendel [10] for details). This leads to a distribution uniform on simplices.

The likelihood of realisations $\delta_1, \ldots, \delta_n$, $1 \leq n \leq N$, given that $\mathbf{D}_N$ is l_1 symmetric[3], can be obtained by integrating the uniform distribution out on the simplex

$$\sum_{l=1}^{N} D_l = N\theta \tag{5}$$

[2] l_p symmetric stochastic variables $D_1, \ldots, D_N$ have a uniform distribution on $\sum_{l=1}^{N} D_l^p = N\theta$ (see Mendel [10]).

[3] l_1 symmetry implies exchangeability, on the other hand, an exchangeable measure which is l_1 symmetric in two coordinates is l_1 symmetric.

over the $(n+1)$th through the Nth amount of damage. This can be achieved by application of the Dirichlet integral and results in

$$l_N(\delta_1,\cdots,\delta_n|\theta) = \frac{(N-1)_n}{(N\theta)^n}\left[1-\frac{\sum_{l=1}^n \delta_l}{N\theta}\right]^{N-n-1} I_{[0,N\theta]}\left(\textstyle\sum_{l=1}^n \delta_l\right), \tag{6}$$

where $(N-1)_n$ is the usual shorthand for $(N-1)\cdots(N-1-n+1)$. $I_{[a,b]}(x)=1$ for $x\in[a,b]$ and equals to zero elsewhere. For a proof of (6) see Mendel [10].

An interesting case arises when taking the limit for N to infinity. Then we obtain a product of n exponentials,

$$\begin{aligned}\lim_{N\to\infty} l_N(\delta_1,\cdots,\delta_n|\theta) &= l_\infty(\delta_1,\cdots,\delta_n|\theta) = \\ &= \frac{1}{\theta^n}\exp\left\{-\frac{\sum_{l=1}^n \delta_l}{\theta}\right\} I_{[0,\infty)}\left(\textstyle\sum_{l=1}^n \delta_l\right).\end{aligned} \tag{7}$$

This result coincides with the classical Bayesian exponential model, where the coordinates of $\mathbf{D}_n$ are mutually independent and identically exponentially distributed random variables with mean θ.

The probability density of $\delta_1,\ldots,\delta_n$ can be found by formulating a finite de Finetti-type representation

$$p(\delta_1,\ldots,\delta_n) = \int_{\theta=0}^{\infty} l_N(\delta_1,\cdots,\delta_n|\theta)\pi(\theta)\,d\theta, \tag{8}$$

where $\pi(\theta)$ is a prior density, which is uniquely determined by the probability density of $\delta_1,\ldots,\delta_n$ (see de Finetti [7] and Barlow & Mendel [3]). Mendel [10] provides a way to obtain a prior for the infinite case. He gets the inverted gamma distribution

$$\pi(\theta) = \frac{\mu^r}{\Gamma(r)}\theta^{-(r+1)}\exp\left\{-\frac{\mu}{\theta}\right\} I_{(0,\infty)}(\theta), \tag{9}$$

$\mu, r > 0$, where the parameters μ and r may be based on data and/or subjective opinions of experts. An advantage of this prior is that the calculations lead to closed-form results in both the finite and infinite case.

By using the prior (9) in the representation (8) we get, after applying the binomial formula, for finite N

$$\begin{aligned}&p(\delta_1,\ldots,\delta_n) = \\ &= \frac{(N-1)_n}{N^n}\sum_{i=0}^{N-n-1}\binom{N-n-1}{i}\left[-\frac{1}{N}\sum_{l=1}^{n}\delta_l\right]^i \mathrm{E}\left\{\theta^{-i-n}\right\} \\ &= \frac{(N-1)_n}{N^n}\sum_{i=0}^{N-n-1}\binom{N-n-1}{i}\left[-\frac{1}{N}\sum_{l=1}^{n}\delta_l\right]^i \frac{\Gamma(i+n+r)}{\mu^{i+n}\Gamma(r)}\end{aligned} \tag{10}$$

and for taking the limit of N to infinity

$$\lim_{N\to\infty} p(\delta_1,\ldots,\delta_n) = \frac{\Gamma(n+r)}{\Gamma(r)} \frac{\mu^r}{[\mu+\sum_{l=1}^{n}\delta_l]^{n+r}} \tag{11}$$

With (6), (9) and (10), for finite N, and with (7), (9) and (11), for infinite N, we can derive the posterior density of the expected amount of damage per time-unit by updating our prior with available data $\delta_1,\ldots,\delta_m$ after applying Bayes' theorem:

$$\pi(\theta|\delta_1,\cdots,\delta_m) = \frac{l_N(\delta_1,\cdots,\delta_m|\theta)\pi(\theta)}{p(\delta_1,\ldots,\delta_m)}. \tag{12}$$

Note that for the infinite case the posterior is again an inverted gamma distribution with parameters $\mu+\sum_{l=1}^{m}\delta_l$ and $r+m$. That is why the inverted gamma distribution is said to be *natural conjugate* to our likelihood. For calculations in detail see Cooke, Misiewicz & Mendel [5]. They also deal with censored data. We get this kind of data in the case of *imperfect inspection*, i.e. the actual resistance will be determined with uncertainty.

3 Decision problems from hydraulic engineering

In this section we will apply the theory developed in the former section to two examples taken from the field of hydraulic engineering. Subsequently the decision variables per act will be: the preventive repair interval with no inspection (§ 3.1) and the inspection rate with given preventive repair level (§ 3.2).

3.1 Preventive repair interval with no inspection

In some cases a decision maker needs to know when he has to carry out a preventive repair, while there is no need or no possibility for inspecting the structure. The expected preventive repair time is fully determined by a certain safety norm, laid down by the government. An example is the minimal permitted probability of inundation of a region protected by a dyke ring in the Netherlands (see [6]). Assuming that the cost of preventive repair is constant in the sense that it is fully determined by having the maintenance people and the right materials at your disposal and not by the actual amount of degradation (provided that it has not been failed), so a dyke manager wants to postpone preventive maintenance as long as possible.

Suppose $P_{norm}(n)$ is the safety probability norm provided for each time-interval $(0, n\Delta]$, $1 \leq n \leq N$, where $P_{norm}(n)$ is strictly nondecreasing in n. The probability of failure should not exceed $P_{norm}(n)$ or with (1)

$$\Pr\{R_n < s\} = \Pr\left\{\sum_{l=1}^{n} D_l > r - s\right\} < P_{norm}(n), \; n = 1, \ldots, N. \tag{13}$$

Hence a preventive repair should be carried out just before that n, where the probability of failure exceeds the safety norm.

The probability of failure in time-interval n can be calculated with (7) as follows:

$$\begin{aligned}
&\Pr\left\{\sum_{l=1}^{n} D_l > r - s\right\} = \\
&= 1 - \int_{\theta=0}^{\infty} \Pr\left\{\sum_{l=1}^{n} D_l \leq r - s \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} \pi(\theta)\, d\theta = \\
&= 1 - \int_{\theta=0}^{\infty} \int_{\delta_1=0}^{r-s} \cdots \int_{\delta_n=0}^{r-s-\sum_{l=1}^{n-1}\delta_l} l_N(\delta_1, \cdots, \delta_n|\theta)\pi(\theta)\, d\delta_n \cdots d\delta_1 d\theta,
\end{aligned} \tag{14}$$

with $r - s \leq N\theta$ and $\pi(\theta)$ the inverted gamma distribution.

With theorem 3 of the Appendix it follows that, for the finite case,

$$\Pr\left\{\sum_{l=1}^{n} D_l \leq r - s \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$

$$= 1 - \sum_{l=1}^{n} \binom{N-1}{l-1} \left[1 - \frac{r-s}{N\theta}\right]^{N-l} \left[\frac{r-s}{N\theta}\right]^{l-1}, \tag{15}$$

$n = 1, \ldots, N, \ldots, r - s \leq N\theta$, and with theorem 8 of the Appendix

$$\lim_{N \to \infty} \Pr\left\{ \sum_{l=1}^{n} D_l \leq r - s \,\middle|\, \frac{1}{N} \sum_{l=1}^{N} D_l = \theta \right\} =$$
$$= 1 - \sum_{l=1}^{n} \frac{1}{(l-1)!} \left[\frac{r-s}{\theta}\right]^{l-1} \exp\left\{-\frac{r-s}{\theta}\right\}, \tag{16}$$

$n = 1, 2, \ldots$, for $N \to \infty$.

(15) is a beta distribution in $(r-s)/N\theta$ with parameters n and $N-n$ (see theorem 3 of the Appendix). The latter is the gamma distribution in $(r-s)/\theta$ with parameters n and 1, which is exactly the distribution of the sum of n mutually independent and identically exponentially distributed variables with mean θ.

The mean of $\sum_{l=1}^{n} D_l$, the amount of damage in time-interval $(0, n\Delta]$, equals $n\theta$ in both the finite and infinite case. As we see the expected resistance is indeed linear decreasing in time. The variance of $\sum_{l=1}^{n} D_l$ in case of the beta distribution (finite time-horizon) is $\frac{N-n}{N+1} n\theta^2$, while the variance equals $n\theta^2$ for the gamma distribution (infinite time-horizon). The variance for finite N is zero if $n = N$, which is obviously correct since we are conditioning on $\frac{1}{N} \sum_{l=1}^{N} D_l = \theta$.

For finite N the discrete lifetime probability density function, given θ, is a binomial distribution (see theorem 2 of the Appendix),

$$\Pr\left\{ \sum_{l=1}^{n-1} D_l \leq x \cap \sum_{l=1}^{n} D_l > x \,\middle|\, \frac{1}{N} \sum_{l=1}^{N} D_l = \theta \right\} =$$
$$= \binom{N-1}{n-1} \left[1 - \frac{r-s}{N\theta}\right]^{N-n} \left[\frac{r-s}{N\theta}\right]^{n-1}, \tag{17}$$

for $n = 1, \ldots, N$, $r - s \leq N\theta$ and with parameters $(r-s)/N\theta$ and N. For infinite N this lifetime probability density approaches a Poisson distribution (see theorem 9 of the Appendix):

$$\lim_{N \to \infty} \Pr\left\{ \sum_{l=1}^{n-1} D_l \leq r - s \cap \sum_{l=1}^{n} D_l > r - s \,\middle|\, \frac{1}{N} \sum_{l=1}^{N} D_l = \theta \right\} =$$
$$= \frac{1}{(n-1)!} \left[\frac{r-s}{\theta}\right]^{n-1} \exp\left\{-\frac{r-s}{\theta}\right\}, \tag{18}$$

for $n = 1, 2, \ldots$ and parameter $(r-s)/\theta$. The mean life is $1 + \frac{N-1}{N}(r-s)/\theta$ and $1 + (r-s)/\theta$ respectively.

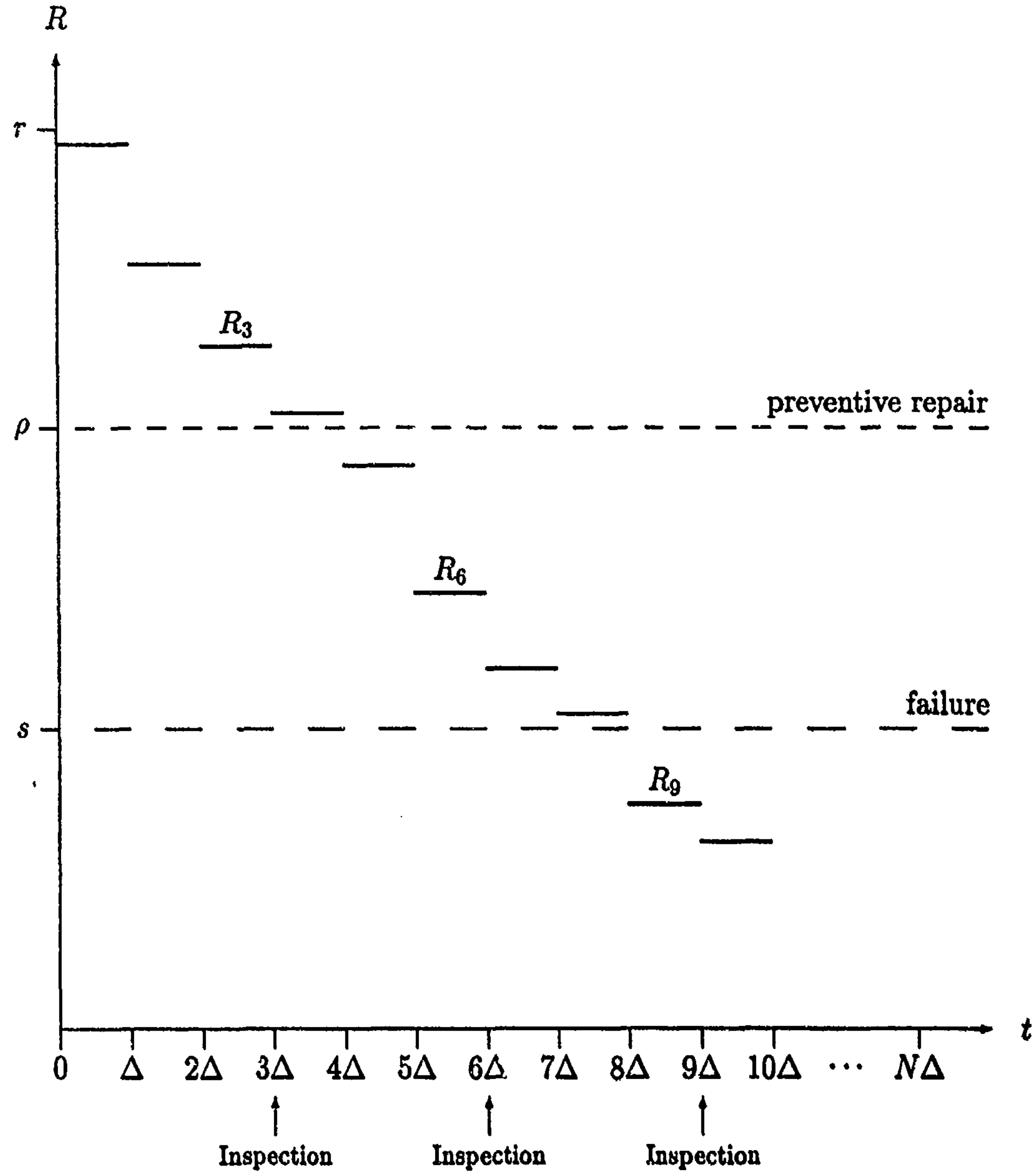

Figure 1: The (discretised) resistance R_n in time-unit n with inspection interval length $k = 3$. At the first inspection (t=3Δ) $R_3 \geq \rho$, hence no preventive repair takes place. At the second inspection $s \leq R_6 < \rho$, so there is no failure, but a preventive repair is needed. The total cost is $2c_I + c_P$. Suppose $k = 9$ then there would have been a failure in the first inspection interval with cost c_F.

3.2 Inspection rate with given preventive repair level

In structural engineering corrective repairs and the collateral losses are too expensive and consequently need to be avoided by well planned inspections and preventive repairs. In this example inspection is needed to know whether we need to carry out a preventive repair. Object of study is to obtain that inspection rate leading to minimal expected cost. A high inspection rate will avoid a failure (almost surely), but entails high inspection cost. No inspection at all leads to failure cost, so the optimum inspection rate lies in between these two extremes.

Inspection will be executed with a constant rate: with an inspection interval of k times Δ, where k can be chosen from $1, \ldots, N$. Hence the inspection times form the set

$$jk\Delta,\ j = 1(1)K, \tag{19}$$

where $K = \lfloor N/k \rfloor$ is the greatest integer less than or equal to N/k. Furthermore, we are dealing with *perfect inspection*, i.e. the actual resistance R can be determined without any uncertainty. Inspection takes negligible time and does not degrade the structure. Each inspection costs c_I.

During an inspection we now have the possibility to execute a preventive repair. In case of no failure there are two possible actions depending on the inspected condition: no repair or preventive repair. We will assume that a preventive repair takes place if the structure's resistance is below a certain preventive repair level, denoted by ρ, where $s < \rho < r$, given by the decision maker. A preventive repair will be executed at the end of the jth inspection interval, i.e. at time $jk\Delta$, if

$$R_{jk} < \rho. \tag{20}$$

There will be no action if R_{jk} is greater than or equal to ρ. See also figure 1. Future research will determine the preventive repair level ρ for a finite and infinite time-horizon.

If the preventive repair level is reached the structure need not yet be in the failed condition. However with a low inspection rate a maintenance action can be too late: a disastrous failure can happen. A failure will be noted immediately, e.g. inundation, while exceedence of the preventive repair level can only be known through inspection.

Both the failure cost c_F and preventive repair cost c_P are constant, where $c_P < c_F$. Note that if c_P would have been greater than or equal to c_F then the optimal value of k would have been N. The cost function to minimise the inspection, preventive repair and failure cost is given in (21).

$$
\begin{aligned}
C(k) \;=\; & \sum_{j=1}^{K}\left[jc_I + c_P\right]\Pr\left\{R_{(j-1)k} \geq \rho \cap s \leq R_{jk} < \rho\right\} + \\
& + \sum_{j=1}^{K}\left[jc_I + c_F\right]\Pr\left\{R_{(j-1)k} \geq \rho \cap R_{jk} < s\right\} + \\
& + \left[Kc_I + c_F\right]\Pr\left\{R_{Kk} \geq \rho \cap R_N < s\right\}, \qquad (21)
\end{aligned}
$$

where

$$
\begin{aligned}
& \Pr\left\{R_{Kk} \geq \rho \cap R_N < s\right\} = \\
& = \sum_{n=Kk+1}^{N} \Pr\left\{R_{Kk} \geq \rho \cap R_{n-1} \geq s \cap R_n < s\right\}. \qquad (22)
\end{aligned}
$$

Note that the model is an extension of the basic model of Barlow, Hunter & Proschan [2] and is also studied by Gijsbers [8]. The latter assumes a normal distribution for $R(t)$ (in case of a continuous model) and does not give analytical results. Under these assumptions the civil structure actually has a probability to improve!

Now we are concerned with calculating the probabilities in (21) and (22). For technical details we refer to the Appendix.

By theorem 7 of the Appendix it follows that the probability of preventive repair at the jth inspection, given the structure has not been preventively repaired at the former inspection (i.e. at time $(j-1)\Delta$), conditional on θ, equals

$$
\begin{aligned}
& \Pr\left\{R_{(j-1)k} \geq \rho \cap s \leq R_{jk} < \rho \,\Big|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} = \\
& = \binom{N-1}{jk-1}\left\{\left[\frac{r-s}{N\theta}\right]^{jk-1} - \sum_{i=1}^{(j-1)k}\binom{jk-1}{i-1}\left[\frac{\rho-s}{N\theta}\right]^{jk-i}\left[\frac{r-\rho}{N\theta}\right]^{i-1}\right\} * \\
& \quad * \left\{\left[1-\frac{r-\rho}{N\theta}\right]^{N-jk} - \left[1-\frac{r-s}{N\theta}\right]^{N-jk}\right\}, \qquad (23)
\end{aligned}
$$

where $r - s \leq N\theta$.

In a similar manner we can get with theorem 5 of the Appendix the probability of failure during the jth inspection interval, given θ, while a preventive repair was not needed at the $(j-1)$th inspection:

$$
\Pr\left\{R_{(j-1)k} \geq \rho \cap R_{jk} < s \,\Big|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =
$$

$$= \sum_{n=(j-1)k+1}^{jk} \binom{N-1}{n-1} \left[1 - \frac{r-s}{N\theta}\right]^{N-n} *$$

$$* \left\{ \left[\frac{r-s}{N\theta}\right]^{n-1} - \sum_{i=1}^{(j-1)k} \binom{n-1}{i-1} \left[\frac{\rho - s}{N\theta}\right]^{n-i} \left[\frac{r-\rho}{N\theta}\right]^{i-1} \right\}, \tag{24}$$

$r - s \leq N\theta$. The probability of failure in the time-interval from the last inspection until the end of the time-horizon, given in (22), can be calculated by summing the failure probabilities over the time-units $(Kk+1)\Delta, \ldots, N\Delta$.

As in (14) we can take the inverted gamma distribution as a prior $\pi(\theta)$ to calculate the posterior probabilities in (21). As a result we have closed-form expressions.

For the infinite case the probability of preventive repair in the jth inspection interval, conditional on θ, is with theorem 12 of the Appendix

$$\lim_{N\to\infty} \Pr\left\{ R_{(j-1)k} \geq \rho \cap s \leq R_{jk} < \rho \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta \right\} =$$

$$= \frac{1}{(jk-1)!} \left\{ \left[\frac{r-s}{\theta}\right]^{jk-1} - \sum_{i=1}^{(j-1)k} \binom{jk-1}{i-1} \left[\frac{\rho-s}{\theta}\right]^{jk-i} \left[\frac{r-\rho}{\theta}\right]^{i-1} \right\} *$$

$$* \left\{ \exp\left\{-\frac{r-\rho}{\theta}\right\} - \exp\left\{-\frac{r-s}{\theta}\right\} \right\}, \tag{25}$$

and the probability of failure in the same interval, given θ, is

$$\lim_{N\to\infty} \Pr\left\{ R_{(j-1)k} \geq \rho \cap R_{jk} < s \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta \right\} =$$

$$= \sum_{n=(j-1)k+1}^{jk} \frac{1}{(n-1)!} \exp\left\{-\frac{r-s}{\theta}\right\} *$$

$$* \left\{ \left[\frac{r-s}{\theta}\right]^{n-1} - \sum_{i=1}^{(j-1)k} \binom{n-1}{i-1} \left[\frac{\rho-s}{\theta}\right]^{n-i} \left[\frac{r-\rho}{\theta}\right]^{i-1} \right\}, \tag{26}$$

according to theorem 11 of the Appendix.

The optimal value of $C(k)$ can be found by minimising over $k = 1, \ldots, N$. The restriction that $k \geq 1$ does not lead to big problems. If the optimum inspection interval length will be lower than Δ, then our model is wrong and we need to subdivide the time-axis into smaller time-units.

As a result of symmetry in the detrioration process we are able to express the probabilities of preventive repair and failure in closed-form results. Hence $C(k)$ can be evaluated easily for every k.

4 Conclusions

In this report we used symmetric properties of the deterioration process to build maintenance optimisation models. A consequence of the symmetry is that the expected accumulated damage to a hydraulic structure is linear in time. The Bayesian approach uses a prior for the average amount of damage with regard to a finite or infinite time-horizon. The amounts of damage are assumed to be l_1 symmetric.

Advantages of the approach are that we base our model on an observable quantity. Instead of a lifetime or a transition probability matrix we use the structure's resistance suffering stochastic deterioration in every time-unit. Furthermore it is possible to use data to update our prior of the average amount of damage, both in case of perfect and imperfect inspection. We used symmetry in the physics to simplify our model considerably, while not assuming mutually independent damages.

We applied the Bayesian approach to two decision problems from the field of hydraulic engineering. The first one is concerned with determining a preventive repair interval given a certain safety norm. The second is an inspection problem. How often do we have to inspect a hydraulic structure to ensure that the sum of expected inspection, repair and failure costs is minimal?

The main advantage of the foregoing approach is that the model is not built on ad hoc assumptions, while the probabilities of preventive repair and failure in a certain (inspection) interval still can be expressed in closed-form results.

A Technical proofs

Theorem 1

$$I(n,x) = \int_{\delta_1=0}^{x}\int_{\delta_2=0}^{x-\delta_1}\cdots\int_{\delta_n=0}^{x-\sum_{l=1}^{n-1}\delta_l} 1 d\delta_n\cdots d\delta_2 d\delta_1 = \frac{x^n}{n!},$$

$$n = 1,2,\ldots,\ x \in \mathbb{R}_+. \tag{27}$$

Proof:
$I(n,x)$ is the classical multi-dimensional Dirichlet integral. □

Theorem 2

$$\Pr\left\{\sum_{l=1}^{n-1} D_l \leq x \cap \sum_{l=1}^{n} D_l > x \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$

$$= \binom{N-1}{n-1}\left[1-\frac{x}{N\theta}\right]^{N-n}\left[\frac{x}{N\theta}\right]^{n-1}, \tag{28}$$

$n = 1,\ldots,N$, $x \leq N\theta$, $x \in \mathbb{R}_+$, *which is the binomial density with parameters* $x/N\theta$ *and* N, *with expectation* $\frac{N-1}{N}(x/\theta)+1$.

Proof:
With $l_N(\delta_1,\delta_2,\cdots,\delta_n|\theta)$ given in (6) it follows that

$$\Pr\left\{\sum_{l=1}^{n-1} D_l \leq x \cap \sum_{l=1}^{n} D_l > x \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$

$$= \int_{\delta_1=0}^{x}\int_{\delta_2=0}^{x-\delta_1}\cdots\int_{\delta_n=x-\sum_{l=1}^{n-1}\delta_l}^{N\theta-\sum_{l=1}^{n-1}\delta_l} l_N(\delta_1,\delta_2,\cdots,\delta_n|\theta)\, d\delta_n\cdots d\delta_2 d\delta_1$$

$$= \int_{\delta_1=0}^{x}\cdots\int_{\delta_n=x-\sum_{l=1}^{n-1}\delta_l}^{N\theta-\sum_{l=1}^{n-1}\delta_l} \frac{N-1}{N\theta}\cdots\frac{N-n}{N\theta}\left[1-\frac{\sum_{l=1}^{n}\delta_l}{N\theta}\right]^{N-n-1} d\delta_n\cdots d\delta_1$$

$$= \int_{\delta_1=0}^{x}\cdots\int_{\delta_{n-1}=0}^{x-\sum_{l=1}^{n-2}\delta_l} \frac{N-1}{N\theta}\cdots\frac{N-n+1}{N\theta}\left[1-\frac{x}{N\theta}\right]^{N-n} d\delta_{n-1}\cdots d\delta_1$$

$$= \frac{N-1}{N\theta}\cdots\frac{N-n+1}{N\theta}\left[1-\frac{x}{N\theta}\right]^{N-n}\frac{x^{n-1}}{(n-1)!}$$

$$= \binom{N-1}{n-1}\left[1-\frac{x}{N\theta}\right]^{N-n}\left[\frac{x}{N\theta}\right]^{n-1},$$

where we used the result (27) in the last step but one. □

Theorem 3

$$\begin{aligned}\Pr\left\{\sum_{l=1}^{n} D_l \le x \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} &= \\ &= 1 - \sum_{l=1}^{n}\binom{N-1}{l-1}\left[1-\frac{x}{N\theta}\right]^{N-l}\left[\frac{x}{N\theta}\right]^{l-1},\end{aligned} \tag{29}$$

$n = 1, \ldots, N$, $x \le N\theta$, $x \in \mathbb{R}_+$, which is the beta distribution in $x/N\theta$ with parameters n and $N-n$.

Proof:
The proof follows directly from theorem 2:

$$\begin{aligned}\Pr\left\{\sum_{l=1}^{n} D_l \le x \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} &= \\ &= 1 - \sum_{l=1}^{n}\Pr\left\{\sum_{m=1}^{l-1} D_m \le x \cap \sum_{m=1}^{l} D_m > x \,\middle|\, \frac{1}{N}\sum_{m=1}^{N} D_m = \theta\right\} \\ &= 1 - \sum_{l=1}^{n}\binom{N-1}{l-1}\left[1-\frac{x}{N\theta}\right]^{N-l}\left[\frac{x}{N\theta}\right]^{l-1},\end{aligned}$$

$n = 1, \ldots, N$, $x \le N\theta$, $x \in \mathbb{R}_+$.

We get the probability density function by differentiating (29) with respect to x:

$$\begin{aligned}\Pr\left\{\sum_{l=1}^{n} D_l = x \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} &= \\ &= \frac{1}{\mathrm{B}(n, N-n)}\left[1-\frac{x}{N\theta}\right]^{N-n-1}\left[\frac{x}{N\theta}\right]^{n-1}\frac{1}{N\theta}\end{aligned}$$

which is the beta density with parameters n and $N-n$. □

Theorem 4

$$\begin{aligned}J(j,n,x,y) &= \int_{\delta_1=0}^{x}\cdots\int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l}\int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l}\cdots\int_{\delta_n=0}^{y-\sum_{l=1}^{n-1}\delta_l} 1\,d\delta_n\cdots d\delta_1 \\ &= \frac{1}{n!}\left[y^n - \sum_{i=1}^{j}\binom{n}{i-1}(y-x)^{n-i+1}x^{i-1}\right], \\ &\qquad j \le n-1,\ j,n = 1,2,\ldots,\ x \le y \le N\theta,\ x,y \in \mathbb{R}_+.\end{aligned} \tag{30}$$

Proof:
The integral can be determined by successively integrating out the variables δ_n, δ_{n-1}, $\ldots$, δ_1.

$$
\begin{aligned}
J(j,n,x,y) &= \\
&= \int_{\delta_1=0}^{x}\cdots\int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l}\int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l}\cdots\int_{\delta_n=0}^{y-\sum_{l=1}^{n-1}\delta_l} 1 d\delta_n\cdots d\delta_1 \\
&= \int_{\delta_1=0}^{x}\cdots\int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l}\int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l}\cdots\int_{\delta_{n-2}=0}^{y-\sum_{l=1}^{n-3}\delta_l} \frac{1}{2!}\left[y-\sum_{l=1}^{n-2}\delta_l\right]^2 d\delta_{n-2}\cdots d\delta_1 \\
&= \cdots \\
&= \int_{\delta_1=0}^{x}\cdots\int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l} \frac{1}{(n-j)!}\left[y-\sum_{l=1}^{j}\delta_l\right]^{n-j} d\delta_j\cdots d\delta_1 \\
&= -\int_{\delta_1=0}^{x}\cdots\int_{\delta_{j-1}=0}^{x-\sum_{l=1}^{j-2}\delta_l} \frac{1}{(n-j+1)!}\left[y-x\right]^{n-j+1} d\delta_{j-1}\cdots d\delta_1 + \\
&\quad + \int_{\delta_1=0}^{x}\cdots\int_{\delta_{j-1}=0}^{x-\sum_{l=1}^{j-2}\delta_l} \frac{1}{(n-j+1)!}\left[y-\sum_{l=1}^{j-1}\delta_l\right]^{n-j+1} d\delta_{j-1}\cdots d\delta_1.
\end{aligned}
$$

The first integral can be solved by using (27) and becomes

$$
-\frac{(y-x)^{n-j+1}x^{j-1}}{(n-j+1)!(j-1)!}. \tag{31}
$$

By successively solving the second integral we get:

$$
-\frac{(y-x)^{n-j+2}x^{j-2}}{(n-j+2)!(j-2)!}\cdots-\frac{(y-x)^{n-1}x^1}{(n-1)!1!}+\int_{\delta_1=0}^{x}\frac{1}{(n-1)!}\left[y-\delta_1\right]^{n-1}d\delta_1, \tag{32}
$$

where we used (27) $(j-2)$ times. The last integral leads to

$$
\int_{\delta_1=0}^{x}\frac{1}{(n-1)!}\left[y-\delta_1\right]^{n-1}d\delta_1 = -\frac{(y-x)^n}{n!}+\frac{y^n}{n!}. \tag{33}
$$

By summing (31), (32) and (33) and multiplying them with $n!/n!$ we see immediately that $J(j,n,x,y)$ equals

$$
\frac{1}{n!}\left[y^n-\sum_{i=0}^{j-1}\binom{n}{i}(y-x)^{n-i}x^i\right] = \frac{1}{n!}\left[y^n-\sum_{i=1}^{j}\binom{n}{i-1}(y-x)^{n-i+1}x^{i-1}\right],
$$

$j\le n-1$, $j,n=1,2,\ldots$, $x\le y\le N\theta$, $x,y\in\mathbb{R}_+$.

□

Theorem 5

$$\Pr\left\{\sum_{l=1}^{j} D_l \le x \cap \sum_{l=1}^{n-1} D_l \le y \cap \sum_{l=1}^{n} D_l > y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$

$$= \binom{N-1}{n-1}\left[1-\frac{y}{N\theta}\right]^{N-n}\left\{\left[\frac{y}{N\theta}\right]^{n-1} - \sum_{i=1}^{j}\binom{n-1}{i-1}\left[\frac{y-x}{N\theta}\right]^{n-i}\left[\frac{x}{N\theta}\right]^{i-1}\right\},$$

$$j \le n-1,\ j,n = 1,\ldots,N,\ x \le y \le N\theta,\ x,y \in \mathbb{R}_+. \qquad (34)$$

Proof:
In evaluating (34) we use the likelihood $l_N(\delta_1, \delta_2, \cdots, \delta_n|\theta)$ derived in (6). The integral can be determined by first integrating out the variable δ_n.

Instead of the notation $(N-1)\cdots(N-1-n+1)$ we will use the usual shorthand $(N-1)_n$. In the last step but one we use $J(j, n-1, x, y)$, following theorem 4.

$$\Pr\left\{\sum_{l=1}^{j} D_l \le x \cap \sum_{l=1}^{n-1} D_l \le y \cap \sum_{l=1}^{n} D_l > y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$

$$= \int_{\delta_1=0}^{x} \cdots \int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l} \int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l} \cdots \int_{\delta_n=y-\sum_{l=1}^{n-1}\delta_l}^{N\theta-\sum_{l=1}^{n-1}\delta_l} l_N(\delta_1,\delta_2,\cdots,\delta_n|\theta)\,d\delta_n \cdots d\delta_1$$

$$= \int_{\delta_1=0}^{x} \cdot\cdot \int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l} \int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l} \cdot\cdot \int_{\delta_n=y-\sum_{l=1}^{n-1}\delta_l}^{N\theta-\sum_{l=1}^{n-1}\delta_l} \frac{(N-1)_n}{(N\theta)^n}\left[1-\frac{\sum_{l=1}^{n}\delta_l}{N\theta}\right]^{N-n-1} d\delta_n \cdot\cdot d\delta_1$$

$$= \int_{\delta_1=0}^{x} \cdot\cdot \int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l} \int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l} \cdot\cdot \int_{\delta_{n-1}=0}^{y-\sum_{l=1}^{n-2}\delta_l} \frac{(N-1)_{n-1}}{(N\theta)^{n-1}}\left[1-\frac{y}{N\theta}\right]^{N-n} d\delta_{n-1} \cdot\cdot d\delta_1$$

$$= \frac{(N-1)_{n-1}}{(N\theta)^{n-1}}\left[1-\frac{y}{N\theta}\right]^{N-n}\frac{1}{(n-1)!}\left\{y^{n-1} - \sum_{i=1}^{j}\binom{n-1}{i-1}(y-x)^{n-i}x^{i-1}\right\}$$

$$= \binom{N-1}{n-1}\left[1-\frac{y}{N\theta}\right]^{N-n}\left\{\left[\frac{y}{N\theta}\right]^{n-1} - \sum_{i=1}^{j}\binom{n-1}{i-1}\left[\frac{y-x}{N\theta}\right]^{n-i}\left[\frac{x}{N\theta}\right]^{i-1}\right\},$$

$j \le n-1,\ j,n = 1,\ldots,N,\ x \le y \le N\theta,\ x,y \in \mathbb{R}_+.$ □

Theorem 6

$$\Pr\left\{\sum_{l=1}^{j} D_l \leq x \cap \sum_{l=1}^{n} D_l \leq y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$

$$= 1 - \sum_{l=1}^{j} \binom{N-1}{l-1} \left[1 - \frac{x}{N\theta}\right]^{N-l} \left[\frac{x}{N\theta}\right]^{l-1} +$$

$$+ \sum_{l=j+1}^{n} \binom{N-1}{l-1} \left[1 - \frac{y}{N\theta}\right]^{N-l} \left\{\sum_{i=1}^{j} \binom{l-1}{i-1} \left[\frac{y-x}{N\theta}\right]^{l-i} \left[\frac{x}{N\theta}\right]^{i-1} - \left[\frac{y}{N\theta}\right]^{l-1}\right\},$$

$$j \leq n-1,\ j,n = 1,\ldots,N,\ x \leq y \leq N\theta,\ x,y \in \mathbb{R}_{+}. \tag{35}$$

Proof:
The proof follows from theorems 2 and 5:

$$\Pr\left\{\sum_{l=1}^{j} D_l \leq x \cap \sum_{l=1}^{n} D_l \leq y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$

$$= 1 - \sum_{l=1}^{j} \Pr\left\{\sum_{m=1}^{l-1} D_m \leq x \cap \sum_{m=1}^{l} D_m > x \,\middle|\, \frac{1}{N}\sum_{m=1}^{N} D_m = \theta\right\} +$$

$$+ \sum_{l=j+1}^{n} \Pr\left\{\sum_{m=1}^{j} D_m \leq x \cap \sum_{m=1}^{l-1} D_m \leq y \cap \sum_{m=1}^{l} D_m > y \,\middle|\, \frac{1}{N}\sum_{m=1}^{N} D_m = \theta\right\}$$

$$= 1 - \sum_{l=1}^{j} \binom{N-1}{l-1} \left[1 - \frac{x}{N\theta}\right]^{N-l} \left[\frac{x}{N\theta}\right]^{l-1} +$$

$$+ \sum_{l=j+1}^{n} \binom{N-1}{l-1} \left[1 - \frac{y}{N\theta}\right]^{N-l} \left\{\sum_{i=1}^{j} \binom{l-1}{i-1} \left[\frac{y-x}{N\theta}\right]^{l-i} \left[\frac{x}{N\theta}\right]^{i-1} - \left[\frac{y}{N\theta}\right]^{l-1}\right\},$$

$j \leq n-1,\ j,n = 1,\ldots,N,\ x \leq y \leq N\theta,\ x,y \in \mathbb{R}_{+}$. □

Theorem 7

$$\Pr\left\{\sum_{l=1}^{j} D_l \le x \cap \sum_{l=1}^{n-1} D_l \le y \cap x < \sum_{l=1}^{n} D_l \le y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$
$$= \binom{N-1}{n-1}\left\{\left[\frac{y}{N\theta}\right]^{n-1} - \sum_{i=1}^{j}\binom{n-1}{i-1}\left[\frac{y-x}{N\theta}\right]^{n-i}\left[\frac{x}{N\theta}\right]^{i-1}\right\} *$$
$$* \left\{\left[1-\frac{x}{N\theta}\right]^{N-n} - \left[1-\frac{y}{N\theta}\right]^{N-n}\right\},$$
$$j \le n-1,\ j,n = 1,\ldots,N,\ x \le y \le N\theta,\ x,y \in \mathbb{R}_+. \qquad (36)$$

Proof:
In evaluating the probability in (36) we use the likelihood $l_N(\delta_1,\delta_2,\cdots,\delta_n|\theta)$ derived in (6). The integral can be determined by first integrating out the variable δ_n. Thereafter application of theorem 4 in the form of $J(j,n-1,x,y)$ leads to the required result. Again we will use the shorthand $(N-1)_n$ to denote $(N-1)\cdots(N-1-n+1)$.

$$\Pr\left\{\sum_{l=1}^{j} D_l \le x \cap \sum_{l=1}^{n-1} D_l \le y \cap x < \sum_{l=1}^{n} D_l \le y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta\right\} =$$
$$= \int_{\delta_1=0}^{x}\cdots\int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l}\int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l}\cdots\int_{\delta_n=x-\sum_{l=1}^{n-1}\delta_l}^{y-\sum_{l=1}^{n-1}\delta_l} l_N(\delta_1,\delta_2,\cdots,\delta_n|\theta)\,d\delta_n\cdots d\delta_1$$
$$= \int_{\delta_1=0}^{x}\cdot\cdot\int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l}\int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l}\cdot\cdot\int_{\delta_n=x-\sum_{l=1}^{n-1}\delta_l}^{y-\sum_{l=1}^{n-1}\delta_l} \frac{(N-1)_n}{(N\theta)^n}\left[1-\frac{\sum_{l=1}^{n}\delta_l}{N\theta}\right]^{N-n-1} d\delta_n\cdot\cdot d\delta_1$$
$$= -\int_{\delta_1=0}^{x}\cdot\cdot\int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l}\int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l}\cdot\cdot\int_{\delta_{n-1}=0}^{y-\sum_{l=1}^{n-2}\delta_l} \frac{(N-1)_{n-1}}{(N\theta)^{n-1}}\left[1-\frac{y}{N\theta}\right]^{N-n} d\delta_{n-1}\cdot\cdot d\delta_1$$
$$+\int_{\delta_1=0}^{x}\cdot\cdot\int_{\delta_j=0}^{x-\sum_{l=1}^{j-1}\delta_l}\int_{\delta_{j+1}=0}^{y-\sum_{l=1}^{j}\delta_l}\cdot\cdot\int_{\delta_{n-1}=0}^{y-\sum_{l=1}^{n-2}\delta_l} \frac{(N-1)_{n-1}}{(N\theta)^{n-1}}\left[1-\frac{x}{N\theta}\right]^{N-n} d\delta_{n-1}\cdot\cdot d\delta_1$$
$$= \binom{N-1}{n-1}\left\{\left[\frac{y}{N\theta}\right]^{n-1} - \sum_{i=1}^{j}\binom{n-1}{i-1}\left[\frac{y-x}{N\theta}\right]^{n-i}\left[\frac{x}{N\theta}\right]^{i-1}\right\} *$$
$$* \left\{\left[1-\frac{x}{N\theta}\right]^{N-n} - \left[1-\frac{y}{N\theta}\right]^{N-n}\right\},$$

$j \le n-1,\ j,n = 1,\ldots,N,\ x \le y \le N\theta,\ x,y \in \mathbb{R}_+.$ □

Theorem 8

$$\lim_{N\to\infty} \Pr\left\{ \sum_{l=1}^{n} D_l \le x \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta \right\} =$$
$$= 1 - \sum_{l=1}^{n} \frac{1}{(l-1)!}\left[\frac{x}{\theta}\right]^{l-1} \exp\left\{-\frac{x}{\theta}\right\}, \tag{37}$$

$n = 1, 2, \ldots,$ *which is the gamma distribution in x with parameters n and $1/\theta$.*

Proof:
Follows from theorem 3 by evaluating the logarithm in the expression

$$\left[1 - \frac{x}{N\theta}\right]^{N-n} \;=\; \exp\left\{(N-n)\log\left[1 - \frac{x}{N\theta}\right]\right\}$$

with a Taylor series and taking the limit of N to infinity. □

Theorem 9

$$\lim_{N\to\infty} \Pr\left\{ \sum_{l=1}^{n-1} D_l \le x \cap \sum_{l=1}^{n} D_l > x \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta \right\} =$$
$$= \frac{1}{(n-1)!}\left[\frac{x}{\theta}\right]^{n-1} \exp\left\{-\frac{x}{\theta}\right\}, \tag{38}$$

$n = 1, 2, \ldots,$ *which is the poisson density with parameter x/θ and expectation $(x/\theta)+1$.*

Proof:
The proof follows from theorem 2 in a similar way as the proof of theorem 8. □

Theorem 10

$$\lim_{N\to\infty} \Pr\left\{ \sum_{l=1}^{j} D_l \le x \cap \sum_{l=1}^{n} D_l \le y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta \right\} =$$
$$= 1 - \sum_{l=1}^{j} \frac{1}{(l-1)!}\left[\frac{x}{\theta}\right]^{l-1} \exp\left\{-\frac{x}{\theta}\right\} +$$
$$+ \sum_{l=j+1}^{n} \frac{1}{(l-1)!}\left\{ \sum_{i=1}^{j} \binom{l-1}{i-1} \left[\frac{y-x}{\theta}\right]^{l-i} \left[\frac{x}{\theta}\right]^{i-1} - \left[\frac{y}{\theta}\right]^{l-1} \right\} \exp\left\{-\frac{y}{\theta}\right\},$$
$$j \le n-1,\ j, n = 1, 2, \ldots,\ x \le y,\ x, y \in \mathbb{R}_+. \tag{39}$$

Proof:
The proof follows from theorem 6 in a similar way as the proof of theorem 8. □

Theorem 11

$$\lim_{N\to\infty} \Pr\left\{ \sum_{l=1}^{j} D_l \le x \cap \sum_{l=1}^{n-1} D_l \le y \cap \sum_{l=1}^{n} D_l > y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta \right\}$$
$$= \frac{1}{(n-1)!}\left\{ \left[\frac{y}{\theta}\right]^{n-1} - \sum_{i=1}^{j} \binom{n-1}{i-1} \left[\frac{y-x}{\theta}\right]^{n-i} \left[\frac{x}{\theta}\right]^{i-1} \right\} \exp\left\{-\frac{y}{\theta}\right\}, \quad (40)$$

$j \le n-1$, $j,n = 1,2,\ldots$, $x \le y$, $x,y \in \mathbf{R}_+$.

Proof:
The proof follows from theorem 5 in a similar way as the proof of theorem 8. □

Theorem 12

$$\lim_{N\to\infty} \Pr\left\{ \sum_{l=1}^{j} D_l \le x \cap \sum_{l=1}^{n-1} D_l \le y \cap x < \sum_{l=1}^{n} D_l \le y \,\middle|\, \frac{1}{N}\sum_{l=1}^{N} D_l = \theta \right\} =$$
$$= \frac{1}{(n-1)!}\left\{ \left[\frac{y}{\theta}\right]^{n-1} - \sum_{i=1}^{j} \binom{n-1}{i-1} \left[\frac{y-x}{\theta}\right]^{n-i} \left[\frac{x}{\theta}\right]^{i-1} \right\} *$$
$$* \left\{ \exp\left\{-\frac{x}{\theta}\right\} - \exp\left\{-\frac{y}{\theta}\right\} \right\},$$
$$j \le n-1,\ j,n = 1,2,\ldots,\ x \le y,\ x,y \in \mathbf{R}_+. \quad (41)$$

Proof:
The proof follows from theorem 7 in a similar way as the proof of theorem 8. □

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