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Singular Perturbation Methods

for Nonlinear Dynamical Systems and Waves

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Table of Contents

1. Abstract

Progress has been made on understanding the complex behavior of physical processes described by nonlinear ordinary and partial differential equations through the use of singular perturbation methods. Modulation equations for the amplitude and phase of dissipatively perturbed strongly nonlinear oscillators and traveling waves have been derived from the action equation using the usual method of multiple scales. Equivalent results have been obtained using the method of averaging developed for the first time for a nonlinear partial differential equation, the Klein-Gordon equation, describing dispersive waves. In another study, Whitham's averaged Lagrangian principle has been generalized to account for arbitrary perturbations of the initial conditions. In other work, Bourland and Haberman analyzed the slow crossing of an unperturbed homoclinic orbit (separatrix) for dynamical systems. Solutions in the neighborhood of the separatrix are matched to the nonlinear slowly varying oscillations, resulting in the determination of accurate analytic formulas for the boundaries of the basin of attraction and connection formulas across the separatrix for the amplitude and phase. Under current investigation are generalizations of the slow crossing of a separatrix to arbitrary Hamiltonian systems and to nonchaotic situations in which small periodic forcing causes the existence of an infinite sequence of resonance layers that coalesce on the separatrix.

2. Status of Research

During the period of the first NSF grant (1985-87), research on two problems was completed:
1. The leading tail of solitary waves for the Korteweg-de Vries equation [10]
2. Structure of two-dimensional diffusive shock waves [1]

In addition, preliminary work began on two problems:
1. Crossing of a separatrix (see subsequent discussion)
2. Nonlinear penumbral caustics [7]

We will present a brief summary of our results since 1987 on the following two related problems:
1. Phase shift for perturbed strongly nonlinear oscillators and dispersive waves
   (summarized in greater detail in [15])
2. Analytic formulas for capture:
   the boundaries of the basin of attraction and the slow transition across a separatrix
   (summarized in greater detail in [13]).

Other completed work concerns slowly varying traveling waves for reaction-diffusion equations [16].

Major efforts continue on the crossing of a separatrix. An investigation ([A] with E. Ho) has begun on general Hamiltonian systems under quasi-autonomous dissipative perturbations. So far it has been shown that all of the work of Bourland and Haberman on separatrix crossing can be extended to the more general case. Formulas for the phase shift have been determined. Accurate criteria for capture has been obtained, that is, similar analytic formulas exist for the perturbed boundaries of the basin of attraction. Numerical computations have motivated an asymptotic investigation of higher order corrections to the usual Melnikov energy bounds. Preliminary work has also been done on the Hamiltonian system that describes the usual weakly nonlinear resonance. An additional investigation ([B] with J. Brothers) has begun on the effect of small periodic perturbations on strongly nonlinear oscillatory dynamical systems with double-well potentials. The object is to extend the ideas of Bourland and Haberman for the crossing of a separatrix by including a small but fast periodic forcing. Chaos can result (with a corresponding fractal boundary of the basin of attraction), but our interest will be on the nonchaotic behavior preliminary to the larger amplitude bifurcation to chaos. The usual slow variation procedures fail at each resonance band where the forcing frequency is a fractional multiple of the nonlinear natural frequency. There are an infinity number of such subharmonic resonance bands that coalesce on the unperturbed homoclinic orbit. We have carefully calculated the strongly nonlinear oscillations away from each resonance using multi-phase averaging. In addition we have calculated the solution in each resonance layer and matched the solution across each resonance layer in order to calculate the phase shift. The forcing amplitude acts as a bifurcation parameter. For sufficiently small amplitudes (the ones we wish to study at first) subharmonic resonant periodic solutions do not exist and the solution merely makes a slow passage through resonance (called transient resonance). The more frequently studied case of sustained resonance where periodic solutions exists will not be particularly harder to analyze, but we still restrict our attention at first to the more elementary case. A separate analysis in the neighborhood of the unstable periodic solution will be necessary but will be quite similar to the analysis of the homoclinic orbit or separatrix of an unstable fixed point as performed by Cary, Escande, and Tennyson and looked at in more detail by Bourland and Haberman. We are planning to do this type of analysis so that the near homoclinic solution can be matched to the slowly varying solution away from the homoclinic orbit. Again, great care must be entertained since the subharmonic resonance bands coalesce on this homoclinic orbit. We hope to obtain formulas for the boundaries of the basin of attraction in the case in which the solution is not chaotic. We anticipate that a proposal to continue work on this project will be forthcoming in early Fall 1992.

3. Phase Shift for Perturbed Strongly Nonlinear Oscillators and Dispersive Waves

1. Method of multiple scales. Modulations of traveling waves for nonlinear partial differential equations can be obtained by the method of multiple scales in which the solution is assumed to depend on a fast phase $\psi = \frac{\theta(X,T)}{\epsilon} + \phi(X,T)$ and the slow variables $X = \epsilon\, x$ and $T = \epsilon\, t$, where $\phi(X,T)$ is the phase shift. To leading-order, a nonlinear ordinary differential equation in the traveling wave coordinate ψ is obtained, from which one derives the local amplitude dependent dispersion relation. At $O(\epsilon)$ in the perturbation expansion, $L(u_1) = R_1$, where L is the linearized operator and R_1 is the corresponding nonhomogeneous term. Eliminating the secular terms yields the leading-order wave action equation. It is more difficult to eliminate secular terms from the $O(\epsilon^2)$ equation, $L(u_2) = R_2$, but in [3] it was shown that the resulting solvability condition determines the phase shift. It was also shown there that an easier way to derive the phase shift equation is based on Whitham's exact equation for wave action:

$$\frac{\partial}{\partial T} I + \frac{\partial}{\partial X} Q = -D, \tag{1.1}$$

where I is the exact wave action, Q its flux, and D its dissipation. If a perturbation expansion is introduced ($I = I_0+\epsilon I_1+...$, $Q = Q_0+\epsilon Q_1+...$, and $D = D_0+\epsilon D_1+...$), then the amplitude parameter E is determined by $\frac{\partial}{\partial T} I_0 + \frac{\partial}{\partial X} Q_0 = -D_0$. Evaluating the exact wave action equation to $O(\epsilon)$ yields

$$\frac{\partial}{\partial T} I_1 + \frac{\partial}{\partial X} Q_1 = -D_1. \tag{1.2}$$

I_1, Q_1, and D_1 depend on u_1 (the higher order perturbation of the solution). However, if the $O(\epsilon)$ perturbation is purely dissipative, then in [3] it was shown that $O(\epsilon)$ perturbations of wave action, its flux, and its dissipation are only due to perturbations of the wave number k and frequency ω:

$$I_1 = \hat{D}(I_0), \quad Q_1 = \hat{D}(Q_0), \quad \text{and} \quad D_1 = \hat{D}(D_0) + \int_0^1 g\, u_{0_\psi} d\psi, \tag{1.3}$$

where $\hat{D} = \omega_1 \frac{\partial}{\partial \omega} + k_1 \frac{\partial}{\partial k} = -\frac{\partial \phi}{\partial T}\frac{\partial}{\partial \omega} + \frac{\partial \phi}{\partial X}\frac{\partial}{\partial k}$ is the linearized operator corresponding to a Taylor series in ω and k. The term $\int_0^1 g\, u_{0_\psi} d\psi$ represents the dissipation due to the $O(\epsilon^2)$ perturbation. Thus, the higher-order wave action equation yields a linear PDE for the phase shift:

$$\frac{\partial}{\partial T}\left(-\frac{\partial \phi}{\partial T} I_{0_\omega} + \frac{\partial \phi}{\partial X} I_{0_k}\right) + \frac{\partial}{\partial X}\left(-\frac{\partial \phi}{\partial T} Q_{0_\omega} + \frac{\partial \phi}{\partial X} Q_{0_k}\right) = -\left(-\frac{\partial \phi}{\partial T} D_{0_\omega} + \frac{\partial \phi}{\partial X} D_{0_k}\right) - \int_0^1 g u_{0_\psi} d\psi, \tag{1.4}$$

generalizing earlier work on ordinary differential equations ([2] and [6]). For non-purely dissipative perturbations, the governing equation must be slightly modified.

2. Lagrangian formulation. In [5] it was also shown that the phase shift can be determined this way for perturbed nonlinear partial differential equations formulated in terms of a Lagrangian:

$$\frac{\partial}{\partial t} L_1 + \frac{\partial}{\partial x} L_2 - L_3 = -\epsilon\, h - \epsilon^2 g, \tag{2.1}$$

where the Lagrangian satisfies $L = L\,(u_t, u_x, u; X, T)$ and, for example, L_1 stands for the partial

derivative of L with respect to the first argument u_t. Whitham showed that the modulation equations correspond to the exact conservation of wave action modified by the inclusion of dissipation:

$$-\frac{\partial}{\partial T}\mathcal{L}_\omega + \frac{\partial}{\partial x}\mathcal{L}_k = -D, \tag{2.2}$$

where $\mathcal{L}$ is the averaged Lagrangian $\mathcal{L} = \int_0^1 L\, d\psi$. Leading-order wave action determines the amplitude in the well-known way($-\frac{\partial}{\partial T}\mathcal{L}_{\omega_0} + \frac{\partial}{\partial x}\mathcal{L}_{k_0} = -D_0$). For purely dissipative $O(\epsilon)$ perturbations, (1.3) were again shown to be valid in [5] so that the next order of the wave action equation becomes:

$$-\frac{\partial}{\partial T}\hat{D}\,\mathcal{L}_{\omega_0} + \frac{\partial}{\partial x}\hat{D}\,\mathcal{L}_{k_0} = -\hat{D}(D_0) - \int_0^1 g\, u_{0_\psi}\, d\psi, \tag{2.3a}$$

where $\hat{D} = -\phi_T \frac{\partial}{\partial \omega_0} + \phi_X \frac{\partial}{\partial k_0}$. Thus, the phase shift ϕ satisfies the following linear partial differential equation written in terms of partial derivatives of the averaged Lagrangian:

$$-\frac{\partial}{\partial T}\left(-\frac{\partial \phi}{\partial T}\mathcal{L}_{\omega\omega} + \frac{\partial \phi}{\partial X}\mathcal{L}_{\omega k}\right) + \frac{\partial}{\partial X}\left(-\frac{\partial \phi}{\partial T}\mathcal{L}_{\omega k} + \frac{\partial \phi}{\partial X}\mathcal{L}_{kk}\right) = \frac{\partial \phi}{\partial T}D_{0_\omega} - \frac{\partial \phi}{\partial X}D_{0_k} - \int_0^1 g u_{0_\psi} d\psi. \tag{2.3b}$$

Similarly, the phase shift was obtained in [5] for strongly nonlinear multi-phase oscillatory waves.

3. Korteweg-de Vries equation. In [4] these ideas were applied to oscillatory single-phase solutions of the Korteweg-de Vries (KdV) equation. Since there are two amplitude parameters, there are two exact action equations, each of which can be utilized. The modulation equations for the phase shift can be obtained, but consists of a coupled system of two linear partial differential equations in two unknowns since only one amplitude parameter can be eliminated using the dispersion relation.

4. Generalization to two fast scales. The usual method of multiple scales assumes the perturbations are slowly varying traveling waves. This puts a severe restriction on the perturbations of the initial conditions. To overcome this restriction, in [12] a method of multiple scales was developed with two fast scales, the usual traveling wave coordinate and time ($\psi = \frac{\theta(X,T)}{\epsilon} + \phi(X,T)$ and $t = t$) and the usual two slow variables ($T = \epsilon t$ and $X = \epsilon x$). It has been shown that the partial differential equation (2.1) is equivalent to an exact equation of wave action:

$$-\frac{\partial}{\partial T}\bar{\mathcal{L}}_\omega + \frac{\partial}{\partial X}\bar{\mathcal{L}}_k = -D, \tag{4.1}$$

where the Lagrangian is now averaged over time as well as the phase:

$$\bar{\mathcal{L}} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{L}\, dt = \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_0^1 L\, d\psi\, dt; \tag{4.2}$$

it is the time average of the usual averaged Lagrangian. When the wave action equation is evaluated to leading order, the usual equation for the wave amplitude results. To evaluate the wave action to the

next order requires knowledge of the perturbation of the solution u_1. Here, $u_1 = \bar{u}_1(\psi) + \tilde{u}_1(\psi,t)$. In general the time dependence is difficult to obtain explicitly since it corresponds to the equations that arise in analyzing the stability of the traveling wave. However, for the nonlinear Klein-Gordon equation, the perturbed wave action I_1 satisfies

$$I_1 = \hat{D}(I_0) + \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_0^1 (u_{0_\psi} \tilde{u}_{1_t} - 2\omega_0 u_{0_\psi} \tilde{u}_{1_\psi})\, d\psi\, dt. \tag{4.3}$$

In [12] we showed the limit in (4.3) vanishes due to the boundedness of $\tilde{u}_1$ (corresponding to the modest assumption that the traveling wave is stable). Thus,

$$I_1 = \hat{D}(I_0)\,; \tag{4.4}$$

the time-dependent part of u_1 does not contribute to perturbations of the wave action, its flux, and its dissipation. Consequently, (1.4) or(2.3) is not altered when initial conditions are properly analyzed.

5. Method of averaging. For ordinary differential equations, methods of averaging (based on energy-angle variables and near identity transformations) have proved useful to derive the long time behavior of perturbed periodic systems. Recently, a similar formalism has been introduced in [14] for oscillatory dispersive wave solutions of nonlinear partial differential equations. Amplitude parameter E and angle ψ variables are introduced, motivated by the corresponding expressions derived by the method of multiple scales. Two "energy" equations have been derived in terms of the appropriately defined wave number k and frequency ω, which are referred to as standard form for the amplitude equations:

$$\underline{\underline{A}} \begin{bmatrix} k \\ \omega \end{bmatrix}_t + \underline{\underline{B}} \begin{bmatrix} k \\ \omega \end{bmatrix}_x = \epsilon[\,] + \epsilon^2[\,], \tag{5.1}$$

where $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are related two-by-two matrices. The expressions for these matrices and the perturbation terms can be found in [14]. Equations (5.1) form a hyperbolic system (which can be shown to have characteristic velocities $\pm c$ as does the pde). It has been shown that the traveling wave assumption corresponds to $u_t = -\omega u_\psi$ and $u_x = k u_\psi$, from which the standard form for the angle equations has been derived:

$$\psi_t = -\omega + f_0(k_t,\omega_t,k_x,\omega_x) + \epsilon f_1 + \epsilon^2 f_2 \tag{5.2a}$$

$$\psi_x = k + g_0(k_t,\omega_t,k_x,\omega_x) + \epsilon g_1 + \epsilon^2 g_2. \tag{5.2b}$$

Detail expressions for f_0 and g_0 also appear in [14]. The four equations, (5.1) and (5.2), which involve the three unknowns (k,ω,ψ) are linear in spatial and temporal derivatives of k and ω. They are well-posed since the two consistency equations have been shown to be equivalent. Although these equations appear complex, after the introduction of near identity transformations and averaging, modulation equations for the amplitude, wave number, frequency, and phase shift have been derived [14] with some effort which are equivalent to the ones obtained by the method of multiple scales.

4. Analytic Formulas for Capture: the Boundaries of the Basin of Attraction and the Slow Transition Across a Separatrix

1. Before capture: strongly nonlinear slowly varying oscillations. To determine the slow variation of a nonlinear oscillator due to small perturbations, various techniques have been developed. It is well-known that these asymptotic techniques fail if the trajectory approaches an unperturbed homoclinic orbit, the infinite period limit of a periodic solution. For Hamiltonian systems (without dissipation), Timofeev, Neishtadt, and Tennyson, Cary, and Escande represented the solution in the neighborhood of the unperturbed homoclinic orbit as a large sequence of nearly homoclinic orbits. They showed that the action changed by a small, but important amount after the solution crossed the separatrix. We have analyzed the slow crossing of a separatrix when small dissipation is present:

$$\frac{d^2y}{dt^2} + V_y(y) = -\epsilon h^*(y, \frac{dy}{dt}) \ , \tag{1.1}$$

where V is a double-well potential with a saddle point at y=0. We assume the perturbation is purely dissipative, $h^*(y, -\frac{dy}{dt}) = -h^*(y, \frac{dy}{dt})$. With the inclusion of small dissipation, it is well known that the two branches of the stable manifold are tightly wound (separating solutions captured in the left well from those captured in the right). In [8] we analyzed the slow passage through a separatrix due to dissipation using the method of multiple scales, determining the boundaries of the basins of attraction and connection formula. Generalizations to slowly varying potentials $V(y,\epsilon t)$ and dissipative perturbations $h^*(y,dy/dt,\epsilon t)$ are discussed in [9]. In [11] we used the equivalent method of averaging with the energy E and angle ψ variables (easily related to the usual action-angle variables):

$$E = \frac{1}{2}(\frac{dy}{dt})^2 + V(y) \quad \text{and} \quad \frac{|\psi|}{\Omega(E)} = \int_{y_{min}}^{y} \frac{dy}{\sqrt{2}[E-V(y)]^{1/2}} \ , \tag{1.2a,b}$$

where $\Omega(E)$ is the usual frequency. The following averaged equations were derived in [11] using a near identity transformation, if the perturbation is purely dissipative:

$$\frac{de}{dt} = -\epsilon\, \Omega(e)\, D(e) + O(\epsilon^3) \quad \text{and} \quad \frac{d\phi}{dt} = \Omega(e) + O(\epsilon^2), \tag{1.3a,b}$$

where e is the average of E. We introduce a perturbation expansion, $e=e_0+\epsilon e_1+ O(\epsilon^2)$. The phase has been derived in [2] using the method of multiple scales and in [11] using the method of averaging:

$$\phi = \frac{1}{\epsilon} \int_0^T \Omega_0 \, dT + A\, \Omega_0(T) + B + O(\epsilon) \ , \tag{1.4a}$$

where $\Omega_0 \equiv \Omega(e_0)$. From a careful analysis of the initial conditions ([2] and [11]),

$$B = \frac{1}{\epsilon\, D_0(e_0(0))} \left(\epsilon \int_0^{\phi(0)} h y_{0\psi} d\psi + \text{initial perturbation of the physical energy E}\right) . \tag{1.4b}$$

The slow variation of the leading-order energy (equivalent to the usual action equation),

$$\frac{de_0}{dT} = -\Omega_0 D_0(e_0), \tag{1.5}$$

follows for (1.3a). As the homoclinic orbit is approached ($e_0 \to 0$), $1/\Omega_0 \sim c_1 \ln|e_0| + c_2$, where c_2 is shown in [8] to depend on the entire potential. For captured oscillations in the right or left well, $c_1^R = -\sqrt{2}/2\lambda$, where λ is given by (3.1), and c_2 is different in each case. The time T_c (when $e_0 = 0$) may be determined from (1.5). At this time, the corresponding phase follows from (1.4a).

2. Capture and the boundary of the basin of attraction. From (1.1), $dE/dt = -\epsilon h^* dy/dt$. As the solution approaches the critical energy ($e_0 \to 0$), the solution is composed of nearly solitary pulses. The Melnikov result is that the energy decreases by $\epsilon D_R > 0$ ($\epsilon D_L > 0$) over a right (left) homoclinic while over a complete (figure-eight shaped) pulse, it decreases by $\epsilon D_C = \epsilon D_R + \epsilon D_L$. The last saddle approach (see § 3) has positive energy, $W_0 > 0$; the energy will become negative at the next saddle approach. Since the energy will diminish by ϵD_R on the next right homoclinic orbit (the well-known separation of the two stable manifolds near the separatrix), the criteria for right capture is that $0 < W_0 < \epsilon D_R$. Corresponding initial conditions can be determined from slow variation theory, since when the phase is an integer we ([8], and [11]) have shown that e_0 of the method of averaging and the actual energy E are sufficiently close near the separatrix. (At other phases, the difference is $O(\epsilon)$, which is too large.) Near the separatrix for the double-pulsed solution, the dissipation $D_0 \to D_R + D_L$, so that, from (1.5), $\Delta e_0/\Delta\phi = -(D_R + D_L)$. We use the time of the last saddle approach (where the phase is an integer) and the time T_c (where $e_0 = 0$ and the phase equals ϕ_c). When the phase is an integer, e_0 is sufficiently close to the actual energy, which is $W_0 + \frac{1}{2}\epsilon D_L$ using the Melnikov idea. Thus,

$$W_0 + \tfrac{1}{2}\epsilon D_L = \epsilon (D_R + D_L)\, \phi_c^{mod}, \tag{2.1}$$

where $\phi_c^{mod} \equiv \phi_c - [\phi_c]$ and where $[\phi_c]$ is the integral part of ϕ_c. By an appropriate integration of (1.5), we determine the initial energy e(0) whose energy at the last saddle approach is W_0:

$$\frac{1}{\epsilon}\int_0^{e(0)} \frac{dE}{D_0(E)} = [\phi_c] + \frac{\frac{1}{2}D_L + (W_0/\epsilon)}{D_R + D_L} - \frac{1}{D_0(e(0))}\int_0^{\phi(0)} h y_{0\psi}\, d\psi, \tag{2.2}$$

where (here only) the perturbation of the initial energy (see(1.4b) is assumed to vanish. The boundaries of the basin of attraction (for the right well) correspond to $W_0 = 0$ and $W_0 = \epsilon D_R$.

3. Separatrix Region: a Sequence of Solitary Pulses. A transition region near the unperturbed homoclinic orbit consists of a large sequence of solitary pulses (separated by a close approach to a saddle point), as discussed for Hamiltonian systems (without dissipation) by Cary, Escande, and Tennyson. The solution in the vicinity of a saddle point with energy W_n is

$$y = \pm \frac{|W_n|^{\frac{1}{2}}}{\lambda} \sinh \text{ or } \cosh \lambda\sqrt{2}(t - t_n), \qquad \text{where } \lambda^2 = -\tfrac{1}{2}V_{yy}(0) > 0, \tag{3.1}$$

and where $t = t_n$ is the time of closest approach to the saddle. The energy dissipation depends on whether the solution is nearly a right or left homoclinic orbit, $W_{n-1}-W_n = \epsilon D_R$ or ϵD_L. To determine the time of the saddle approaches, we consider the approximate homoclinic orbit connecting two saddle approaches occurring at $t=t_{n-1}$ and $t=t_n$. The exponential growth specified by (3.1) must correspond to the exponential decay (calculated in [8]) of the tails of the corresponding solitary pulses. In this manner, the "period" for a sequence of solitary pulses is the average of the periods for the surrounding saddles:

$$t_n - t_{n-1} = -\frac{\sqrt{2}}{2\lambda} \ln \left| W_n \, W_{n-1} \right|^{1/2} + c_2 \ , \tag{3.2}$$

where for left homoclinic orbits $c_2=c_2^L$ in (3.2) ,while for right ones $c_2= c_2^R$. It is not difficult to solve (3.2) in terms of the unknown t_0 using the basic property of the gamma function, $\Gamma(x + 1) = x\Gamma(x)$.

4. Matching and the Connection Formulas Across a Separatrix. After the solution is captured into the right well, we can insist that e_0 be the unique solution of (1.5) which starts at the critical energy, $e_0(T_R) = 0$, at some as yet unknown time $T=T_R= \epsilon t_R$. The phase (see (1.4a) contains the second constant $\phi(T_R)$. Using the same reasoning used to obtain (2.1), we derive $\phi(T_R)=W_0/\epsilon\ D_R$. Eliminating W_0 from (2.1), yields $D_R\phi(T_R) = \phi_c^{\mathrm{mod}}(D_R+ D_L) - \frac{1}{2} D_L$, determining the captured phase from initial conditions. The asymptotic expansion for nonlinear oscillators fails when $e_0 =O(\ )$, i.e. as the unperturbed separatrix is approached, suggesting the method of matched asymptotic expansions (even though the energy is predicted correctly at extrema). Similarly, the sequence of solitary pulses will fail when $n=O(1/\epsilon)$, since there the solution can no longer be approximated by homoclinic orbits. We [8] have shown that the energy already matches to leading order before capture due to (2.1), which determines W_0. The careful matching [8] of the phase angles before capture determines the time t_0 of the last saddle approach. In the same manner, the phase angle near homoclinic orbits was matched to the phase angle of the nonlinear oscillator after capture, determining the starting time $T_R = \epsilon\, t_R$:

$$t_R - t_C = \frac{c_1}{D_C}\left(\frac{D_R-D_L}{4} - \frac{W_0}{\epsilon}\right) \ln\left(\epsilon D_C\right) + \frac{c_1}{2D_R}\frac{W_0}{\epsilon} \ln\left(\epsilon D_R\right) + \frac{1}{2}\, c_2^L - (c_2^R + c_2^L)\left(\frac{D_L}{2D_C} + \frac{W_0}{\epsilon D_C}\right) - A + \frac{c_2^R}{D_R}\frac{W_0}{\epsilon}$$

$$- \frac{c_1}{2}\left[\ln \left| \frac{\Gamma\left(\frac{W_0}{\epsilon D_C}\right)\Gamma\left(\frac{W_0}{\epsilon D_C} + \frac{D_L}{D_C}\right)}{2\Gamma\left(\frac{W_0}{\epsilon D_R}\right)\sin\frac{\pi W_0}{\epsilon D_R}} \right| - \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln\frac{D_R}{D_C}\right] \ , \tag{4.1}$$

where $c_1 = c_1^R+c_1^L =2c_1^R=-\sqrt{2}/\lambda$ and $D_C=D_R+D_L$, completing the connection formulas for the slow passage through the separatrix.

5. List of Research Publications

Earlier Grants

[1] R. Haberman, The initial formation and structure of two-dimensional diffusive shock waves, Wave Motion 8 (1986) pp. 267-276.

[2] F. J. Bourland and R. Haberman, The modulated phase shift for strongly nonlinear, slowly varying, and weakly damped oscillators, SIAM J. Appl. Math. 48 (1988) pp.737-748.

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Future Publication Plans

[A] It is anticipated that R. Haberman will write papers with E. Ho on (a) phase shift for perturbed Hamiltonian systems, (b) analytic formulas for the boundary of the basin of attraction for Hamiltonian systems, (c) capture in 1-1 resonance, (d) logarithmic correction to Melnikov energy dissipation for autonomous systems, and (e) logarithmic correction to Melnikov energy dissipation using time-dependent methods.

[B] It is anticipated that R. Haberman will write papers with J. Brothers on (a) multiphase averaging, (b) phase shift jumps across sequences of resonant layers, and (c) crossing a separatrix with periodic forcing.

6. Participating Professionals

(i) F. Jay Bourland (AFOSR support ended August 1989)
M.S. Applied Mathematics, SMU, May 1987
Ph.D. Applied Mathematics, SMU, May 1989
present address (through August 1992)
Department of Mathematics
Stanford University

(ii) Jerry Brothers
M.S. Applied Mathematics, SMU, May 1990
passed Ph.D. qualifying examination January 1991
Ph.D. expected, SMU, May 1993

(iii) Eric Ho
M.S. Applied Mathematics, SMU, May 1990
passed Ph.D. qualifying examination January 1991
Ph.D. expected, SMU, May 1993

7. List of Presentations by R. Haberman (since January 1989)

(a) Phase Shift for Strongly Nonlinear Oscillators and Dispersive Waves

1. UNAM (Mexico City)
   Department of Mathematics and Mechanics (July 3, 1989)
2. SIAM National Meeting
   San Diego (July 20, 1989)
3. Nonlinear Dispersive Wave Conference
   University of Central Florida (March 12, 1991)
4. ICIAM International Meeting
   Washington, D.C. (July 9, 1991)

(b) Crossing a Separatrix

1. Dynamics Days
   Houston (January 4, 1989)
2. UNAM (Mexico City)
   Department of Mathematics and Mechanics (July 5, 1989)
3. SIAM National Meeting - presented by F. J. Bourland
   San Diego (July 20, 1989)
4. Workshop on Asymptotic Analysis
   Argonne National Laboratory (February 27, 1990)
5. Texas A&M University
   Department of Mathematics (March 27, 1990)
6. University of Washington
   Department of Applied Mathematics (March 30, 1990)
7. SIAM Conference on Dynamical Systems
   Orlando (May 10, 1990)
8. Nonlinear Science: The Next Decade
   Los Alamos (May 23, 1990)
9. New Jersey Institute of Technology
   Department of Mathematics (October 21, 1991)
10. Dynamics Days Conference - poster sessions by J. Brothers and E. Ho
    Austin (January 9, 1992)
11. University of Chicago
    Program in Applied Mathematics (April 24, 1992)
12. University of Texas at Dallas
    Workshop on Mathematical Modeling (April 30, 1992)

(c) Slowly Varying Traveling Waves for Reaction-Diffusion Equations

1. Waves Conference at the Canadian Applied Mathematics Society
   Edmonton, Alberta (June 15, 1992)
2. SIAM National Meeting - presented by M. Booty
   Los Angeles (scheduled July 22, 1992)