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HYPERBOLIC CONSERVATION LAWS WITH UMBILIC POINTS I

Gui-Qiang Chen and Pui Tak Kan

UNIVERSITY OF WISCONSIN

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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Hyperbolic Conservation Laws with Umbilic Points I

Gui-Qiang Chen[1]
Department of Mathematics, University of Chicago
Pui Tak Kan[2]
Department of Mathematics, University of Wisconsin at Madison

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Abstract

In this paper a compactness framework for approximate solutions to nonlinear hyperbolic systems with umbilic points is established by combining ideas in modern nonlinear analysis with classical methods, and by a detailed analysis of a highly singular Euler-Poisson-Darboux-type equation. Then this framework is successfully applied to prove the convergence of the Lax-Friedrichs scheme, the Godunov scheme and the viscosity method, and the existence of global entropy solutions for the Cauchy problem with large initial data for a canonical class of the quadratic flux systems and other related systems. In forthcoming papers [CK1, CK2], we apply (a variant of) this framework to solve the other three canonical classes of the quadratic flux systems, the system of three-phase flow in porous media and other related systems with umbilic points.

Key Words. Conservation laws, nonstrict hyperbolicity, umbilic points, compactness framework, Lax-Friedrichs scheme, Godunov scheme, viscosity method

AMS(MOS) Subject Classifications. 35L65, 35L80, 35D05, 35A35, 65M10, 76T05, 35Q05

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[1] Email address: cheng@zaphod.uchicago.edu
[2] Email address: kan@math.wisc.edu

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1. Introduction

We are concerned with hyperbolic systems of conservation laws with umbilic points. A point in state space $U_0 \in \mathbf{R}^n$ is called an umbilic point for a system of hyperbolic conservation laws

$$\partial_t U + \partial_x F(U) = 0, \qquad U \in \mathbf{R}^n, \tag{1.1}$$

if some wave speeds coincide at this point U_0, that is, at least two eigenvalues $\lambda_i(U)$ and $\lambda_j(U), i \neq j$, among the n real eigenvalues of the matrix $\nabla F(U)$ are such that $\lambda_i(U_0) = \lambda_j(U_0)$. Such umbilic points allow a degree of interaction, or nonlinear resonance, between distinct modes, and lead to high singularities, which is missing in the strictly hyperbolic case.

The study of singular hyperbolic equations has an extensive history dating back at least to the work of Euler [Eu] two hundred years ago, when he proposed a singular equation — the Euler-Poisson-Darboux equation. Its close relations with wave theory, fluid dynamics as well as geometry have attracted great attentions from mathematicians for two centuries including Poisson (1823), Darboux (1914), Riemann (1860), Volterra (1892), Beltrami (1880) and also Weinstein, Erdélyi, Lions and others in the 60's (see [Y]). Recently, a study of the behavior of its solutions led to a solution of the nonlinear system of isentropic gas dynamics [Ch1, DCL] (also see [Di2, Ch2]) because of its close relation with entropy information. On the other hand, the theory of linear equations with multiple characteristics is also well developed. An important feature of such equations is the loss of differentiability [DeG], which leads to ill-posedness in Sobolev spaces and to the use of Gevrey Classes [Ge]. Another feature is that sign conditions on the subprincipal symbol play an important role (cf. [Fr, Oh, H]).

Recently, nonlinear hyperbolic systems with umbilic points have arisen from such areas as multiphase flows in porous media, elasticity, water wave problems, and maganetohydrodynamics. Such umbilic points appear naturally in multidimensional systems of conservation laws. In particular, Lax showed that in three space variables there must be umbilic points if the number of equations is $2(mod\ 4)$ [La1]. The theory of local solutions for such systems is well developed because many tools for linear equations still can be used. However, since such systems are nonlinear, even if the initial data are smooth, the solutions of the Cauchy problem generally develop singularities and become discontinuous in finite time. This is an exact reflection of the physical phenomena of the breaking of waves and the development of shock waves. An effort has been made to understand the Riemann solutions for such systems (cf. [G1, IMPT, IT, SS2, SSMP]). Two kinds of degeneracy are classified, which govern different behavior of solutions near umbilic points. A typical example of parabolic degeneracy is the system of isentropic gas dynamics (cf. [Ch2]). The most simple example of hyperbolic degeneracy is the system with a rotational symmetry, one linear degenerate characteristic field, and one contact characteristic field (cf. [KK, LW, F, Ch3]).

This is the first in a series of our papers. In this series of papers we focus on isolated umbilic points with hyperbolic degeneracy. Near such an isolated umbilic point one can scale and blow up singularity to yield generically a homogeneous polynomial flux, determined by the lowest-order nontrivial terms, by a Galilean transformation. For the 2×2 case, this process generically leads to a homogeneous quadratic polynomial flux. Such a polynomial flux contains some inessential scaling parameters and the selection of a unique flux from each equivalence class is the problem of normal forms. It was solved by Issacson, Plohr, and Temple and in a more satisfactory form by Schaeffer and Shearer [SS1]. The classification of the geometry of rarefaction curves and some preliminary tools for the analysis of Riemann problems for the quadratic flux systems are also presented in [SS1]. The Riemann solutions for such systems were constructed by Issacson, Marchesin, Paes-Leme, Plohr, Schaeffer, Shearer, Temple and others (cf. [IMPT, IT, SS2, SSMP]).

The global existence of weak solutions to the Cauchy problem for a special case of such quadratic flux systems was solved in [K] using the viscosity method. The method of compensated compactness was used and the main analysis involves detailed estimates and characterization of the singularities of solutions to the associated entropy equation which is of Euler-Poisson-Darboux type. Classes of Goursat data were then carefully chosen to cancel these singularities in the construction of general classes of regular entropies. In [Rb], the analysis in [K] was applied to a slightly different system. And in [Lu], a different proof was given independently to the same problem studied in [K].

In this first paper of this series, we establish an L^∞ compactness framework for sequences of approximate solutions to general nonstrictly hyperbolic systems with umbilic points. Some techniques in compensated compactness [Ta, Mu, Se] are used and the analysis of singularities in [K] is generalized to achieve this framework. Under this framework approximate solution sequences, which are apriori bounded in L^∞ and which produce correct entropy dissipations, lead to the compactness of the corresponding Riemann invariant sequences. In another series of papers, we are going to develop the corresponding L^p theory. One of the principal difficulties associated with such systems is the general lack of enough classes of entropy functions that can be verified to satisfy certain weak compactness conditions in the div-curl lemma of Tartar [Ta] and Murat [Mu]. This is due to possible singularities of entropy functions near the regions of nonstrictly hyperbolicity. The analysis leading to the compactness involves two major steps:

In the first step, we construct regular entropy functions governed by a highly singular entropy equation. There are two main difficulties. The first is that, in general, the coefficients of the entropy equation are multiple-valued functions near the umbilic points in the Riemann invariant coordinates, which is missing in the special cases [K, Rb]. This difficulty is overcome by a detailed analysis of the singularities of the Riemann function of the entropy equation in Section 3 and Section 4. This analysis involves a study of a corresponding Euler-Poisson-Darboux equation using a majorant idea and requires very complicated estimates and calculations. Finally, an appropriate choice of Goursat data leads to a cancellation of singularities and we obtain regular entropies in the Riemann invariant coordinates. The second difficulty is that the nonlinear correspondence between the U-coordinates and the Riemann invariant coordinates is, in general, irregular. A regular entropy function in the Riemann invariant coordinates is usually no longer regular in the physical coordinates U. We overcome this by a detailed analysis of the correspondence between these two coordinates.

In the second step, we study the structure of the Young measures associated with the approximate sequences, and prove that the support of the Young measures lies in finite isolated points or separate lines in the Riemann invariant space. This is achieved by a delicate use of Serre's technique [Se] and regular entropy functions, constructed in the first step, in the Tartar-Murat commutation equation [Ta] for Young measures associated with the approximate solution sequences.

This compactness framework is successfully applied to prove the convergence of the Lax-Friedrichs scheme [La3], the Godunov scheme [Go] and the viscosity method for a canonical class of the quadratic flux systems and other related systems in Section 7. Corresponding existence theorems of global entropy solutions for such systems are established. The compactness is achieved by reducing the support of the corresponding Young measures to a Dirac mass in the physical space. This involves a careful construction of many other special solutions to the nonstrictly hyperbolic entropy equations and a detailed analysis of finite Tartar-Murat functional equations.

In the forthcoming second paper [CK1] we will apply this framework to solve the other three remaining canonical classes of quadratic flux systems and related systems. Several convergence theorems of L^∞ approximate solutions and existence theorems of global solutions will be presented.

In the third paper of this series [CK2], we will solve the system of three-phase incompressible flows in porous media, studied by many authors, with the aid of a variant of this framework. By a detailed analysis of the geometry of wave curves and the dissipation of entropy waves we obtain the convergence of approximate solutions (such as the Lax-Friedrichs scheme, the Godunov scheme, and the viscosity method) and the existence of global entropy solutions for the Cauchy problem with large initial data for this physical system.

In connection with earlier work on compactness frameworks on approximate solutions to strictly hyperbolic conservation laws we refer the reader to the work of Tartar [Ta] for scalar conservation laws, and to the work of DiPerna [Di1] and Serre [Se] for 2×2 systems. DiPerna's analysis [Di1] is based on a study of the Lax progressing entropy waves in state space [La1], and in particular, on relationships between their structure and the structure of the Young measures by using the strict hyperbolicity and the convexity of the systems. Serre [Se] provided another approach to establish such a compactness framework for general systems by using certain kinds of Goursat entropies whose regularities are ensured by the strict hyperbolicity of the systems.

Regarding work on other related topics on hyperbolic conservation laws with umbilic points we refer the reader to Isaacson, Marchesin, and Plohr [IMP] for a general description on transitional shock waves, to Liu [Liu] for a discussion on asymptotic stability of such waves, and to Glimm [G1, G3] for a review of applications of bifurcation theory and geometry to the analysis of Riemann solutions.

2. The Classification of Quadratic Fluxes and Wave Curves

Consider a hyperbolic system of conservation laws

$$\partial_t U + \partial_x F(U) = 0, \qquad U \in \mathbf{R}^2, \tag{2.1}$$

with an isolated umbilic point. An point $U_0 \in \mathbf{R}^2$ is called an isolated umbilic point if $\nabla F(U_0)$ is diagonalizable, and there is a neighborhood $\mathbf{N}$ of U_0 such that $\nabla F_T(U)$ has distinct eigenvalues for all $U \in \mathbf{N} - U_0$, where

$$\nabla F_T(U) = F(U_0) + \nabla F(U_0)(U - U_0) + \frac{1}{2}(U - U_0)^\top \nabla^2 F(U_0)(U - U_0).$$

Take the Taylor expansion for $F(U)$ about $U = U_0$:

$$F(U) = F(U_0) + \nabla F(U_0)(U - U_0) + \frac{1}{2}(U - U_0)^\top \nabla^2 F(U_0)(U - U_0) + h.o.t. \tag{2.2}$$

where $h.o.t.$ represents the remainder. The flux function

$$Q(U - U_0) = F(U_0) + \nabla F(U_0)(U - U_0) + \frac{1}{2}(U - U_0)^\top \nabla^2 F(U_0)(U - U_0) \tag{2.3}$$

determines the local behavior of hyperbolic singularity near the umbilic point U_0. Since $\nabla F(U_0)$ is diagonalizable, we can make a coordinate transformation to eliminate the linear term from (2.3) and relabel $U - U_0$ as U to obtain

$$\partial_t U + \partial_x Q(U) = 0, \tag{2.4}$$

from (2.1) and (2.3), where $Q(U) = \frac{1}{2}U^T \nabla^2 F(U_0)U$. From the normal form theorem in [SS1], there is a nonsingular linear coordinate transformation to transform the system (2.4) into

$$\partial_t U + \partial_x (dC(U)) = 0, \tag{2.5}$$

where

$$C(U) = \frac{1}{2}(\frac{1}{3}au^3 + bu^2 v + uv^2), \qquad a \neq 1 + b^2.$$

We now analyze the geometry of rarefaction wave curves and the genuine nonlinearity of the quadratic flux system (2.5) in the sense of Lax [La4]. We will make use of this information to understand the classification of the canonical form into four regions in the (a, b)-plane and thereby rederiving some of the results in [SS1] from a different viewpoint.

This analysis is also essential in obtaining an L^∞ apriori estimate for the sequences of viscous and finite difference approximate solutions via invariant region techniques. We establish such estimates in Section 7. For the symmetric case of (2.5) ($b = 0$), the existence of invariant regions and the geometry of wave curves were analyzed in [K]. We remark here that our analysis will be consistent with Darboux's local analysis near an isolated umbilic point (see [Da]).

Recall from (2.5) that the flux vector and matrix take the form:

$$F(U) = dC(U) = \frac{1}{2}(au^2 + 2buv + v^2, bu^2 + 2uv)^T, \tag{2.6}$$

and

$$\nabla F(U) = \begin{pmatrix} au + bv & bu + v \\ bu + v & u \end{pmatrix}. \tag{2.7}$$

The eigenvalues and eigenvectors are from (2.7),

$$\lambda_i = \frac{1}{2}[(a+1)u + bv + (-1)^i \sqrt{((a-1)u + bv)^2 + 4(bu+v)^2}], \quad i = 1, 2. \tag{2.8}$$

and

$$\mathbf{r}_i = ((a-1)u + bv + (-1)^i \sqrt{((a-1)u + bv)^2 + 4(bu+v)^2}\ ,\ 2(bu+v))^T, \tag{2.9}$$

respectively. It is then immediate that as long as $a \neq 1 + b^2$, $\lambda_1 = \lambda_2 \iff (u,v) = (0,0)$, so that $(0,0)$ is the unique umbilic point for (2.5).

The j^{th} family of rarefaction curves $\mathbf{R}_j$ is defined as the family of integral curves of the vector field given by $\mathbf{r}_j$. Therefore, $\mathbf{R}_j$ is defined by the following ordinary differential equation:

$$
\begin{aligned}
\frac{du}{dv} &= \frac{(a-1)u + bv + (-1)^j \sqrt{((a-1)u+bv)^2 + 4(bu+v)^2}}{2(bu+v)} \\
&\triangleq F_j(u,v) \\
&= \frac{(a-1)\alpha + b + (-1)^j sign(v) \sqrt{[(a-1)+b\sigma]^2 + 4(b+\sigma)^2}}{2(b\alpha+1)}, \quad v \neq 0,
\end{aligned}
\tag{2.10}
$$

where $\alpha = \frac{u}{v}$. Define

$$
g_j(\alpha) \triangleq \frac{[(a-1)\alpha + b] + (-1)^j \sqrt{[(a-1)+b\sigma]^2 + 4(b+\sigma)^2}}{2(b\alpha+1)}.
\tag{2.11}
$$

Then we have

$$
F_j(u,v) = \begin{cases} g_j(\alpha), & v > 0, \\ g_i(\alpha), & v < 0. \end{cases}
\tag{2.12}
$$

To analyze the geometry of the $\mathbf{R}_j$ curves, we start by noting that

$$
\frac{du}{uv} = F_j(u,v),
\tag{2.13}
$$

$$
\begin{aligned}
\frac{d^2u}{d^2v} &= \frac{1}{v}\partial_\alpha F_j(u,v)\left(\frac{du}{dv} - \alpha\right) \\
&= \frac{1}{v}\partial_\alpha F_j(u,v)(F_j - \alpha), \quad \text{when} \quad v \neq 0.
\end{aligned}
\tag{2.14}
$$

We collect, in the following lemma, some basic properties of g_j.

Lemma 2.1. *The function $g_j(\alpha), j = 1, 2$, satisfies*

$$(1) \quad g_1(\alpha)g_2(\alpha) = -1,$$

$$(2) \quad g_j(0) = \frac{b + (-1)^j\sqrt{b^2+4}}{2},$$

$$(3) \quad g_j(\pm\infty) = \frac{a - 1 \pm (-1)^j\sqrt{(a-1)^2 + 4b^2}}{2b},$$

$$(4) \quad \lim_{\alpha \to -\frac{1}{b} \pm 0} g_1(\sigma) = \begin{cases} \mp\infty, & a > 1 + b^2,\ b > 0, \\ 0, & a > 1 + b^2,\ b < 0, \\ 0, & a < 1 + b^2,\ b > 0, \\ \pm\infty, & a > 1 + b^2,\ b < 0, \end{cases}$$

$$(5) \quad \lim_{\alpha \to -\frac{1}{b} \pm 0} g_2(\alpha) = \begin{cases} 0, & a > 1 + b^2,\ b > 0, \\ \mp\infty, & a > 1 + b^2,\ b < 0, \\ \pm\infty, & a < 1 + b^2,\ b > 0, \\ 0, & a > 1 + b^2,\ b < 0, \end{cases}$$

$$(6) \quad g_j{}'(\alpha) = \begin{cases} > 0, & a > 1 + b^2, \\ < 0, & a < 1 + b^2. \end{cases}$$

Next, we make use of (2.11) – (2.14) and Lemma 2.1 to obtain a qualitative picture of the $\mathbf{R}_j$ curves. We distinguish several cases. We remark that from (4)-(6) in lemma 2.1, $a = 1 + b^2$ defines a boundary curve in the (a, b)-plane separating qualitatively different wave curve geometries. We shall elaborate on this below.

First, we assume that $a > 1 + b^2$. Using Lemma 2.1, we obtain the following graphs of g_j in the two subcases $b > 0$, and $b < 0$.

(note to printer: place Fig. 2.1 here in text)

For simplicity, we first consider the case $b > 0$. From (2.14), the change in convexity (as a function of the slope) of the wave curves depends on the location and number of roots of $g_j(\alpha) - \alpha$. A computation using the formulae for the g_j's shows that, in the present case, these locations are given by the roots of the cubic polynomial

$$h(\alpha) = -b\alpha^3 + (a-2)\alpha^2 + 2b\alpha + 1.$$

The discriminant of h is given by $\Delta = -32b^4 + b^2(27 + 36(a-2) - 4(a-2)^2) + 4(a-2)^3$. Thus, $\Delta = 0$ gives a new boundary in the (a,b)-plane which distinguishes different wave curve geometries. This corresponds to the division between regions III and IV in [SS1]. When $\Delta < 0$, h has three real roots α_0, α_1, and α_2. We therefore obtain the following figures for the $\mathbf{R}_j$ curves in the case $b > 0, a > 1 + b^2, \Delta < 0$.

(note to printer: place Fig. 2.2 here in text)

The case $b < 0, a > 1 + b^2, \Delta < 0$ is completely similar. We show the corresponding graphs below in Fig. 2.3.

(note to printer: place Fig. 2.3 here in text)

When $\Delta > 0$, h has only one real root and the wedge-shaped regions in Fig. 2.2 and Fig. 2.3 collapse into a line. This corresponds to region IV in [SS1].

(note to printer: place Fig. 2.4 here in text)

Next, we consider the case $a < 1 + b^2$. Using Lemma (2.1), the g_j diagrams are as follows:

(note to printer: place Fig. 2.5 here in text)

By completely similar reasoning as in previous cases, we obtain the following $\mathbf{R}_j$ curves diagrams:

(note to printer: place Fig. 2.6 here in text)

We now turn to investigate the genuine nonlinearity in the sense of Lax [La4] for the quadratic system (2.5). It will turn out that genuine nonlinearity allows a breakdown of the last case above ($a < 1 + b^2$) into two subcases. By definition, (2.5) is genuinely nonlinear in the j^{th} characteristic field at a point (u,v) if $\mathbf{r}_j \cdot \nabla\lambda_j \neq 0, i \neq j$, at (u,v). A calculation using $(2.8)-(2.9)$ shows that

$$\mathbf{r}_j \cdot \nabla\lambda_j = a\zeta + 3by + (-1)^j \frac{a\zeta^2 + 3by\zeta + 2(a+3)y^2}{\sqrt{\zeta^2 + 4y^2}},$$

$$\zeta = (a-1)u + bv,$$

$$y = bu + v. \tag{2.16}$$

Using (2.16), we find that

$$\text{(2.17)} \qquad \mathbf{r}_j \cdot \nabla\lambda_j \iff \begin{cases} y = 0, (-1)^j\zeta < 0, & a > \frac{3}{4}b^2, \\ y = 0, y = \frac{b(a-3)}{2a}\zeta, & a = \frac{3}{4}b^2, \\ y = 0, y = \frac{3b(a-3)\pm\sqrt{D}}{6a}\zeta, & a < \frac{3}{4}b^2. \end{cases}$$

where $D = -12(a - \frac{3}{4}b^2)(a+3)^2$. (2.17) shows that the curve $a = \frac{3}{4}b^2$ divides the region $\{(a,b)|a < 1 + b^2\}$ into two subregions according to a global change in loci of loss of genuine nonlinearity. This corresponds to the division between region I and II in [SS1].

We summarize the four boundaries in the (a,b)-plane separating different behavior of the wave curves and the qualitative change in the system (2.5) as one crosses these boundaries in the following table:

(note to printer: insert Table 2.1 here in text)

(note to printer: insert Fig 2.7: Four Boundary Curves here in text)

3. Riemann Invariants and Genuinely Nonlinearity for the Quadratic Flux System

In this section, we study the Riemann invariants of the quadratic flux system (2.5). We also study the monotonicity of $\lambda_i, i = 1,2$, as a function of Riemann invariants $w_j, j = 1,2$. We remark that this does not follow from our knowledge of genuine nonlinearity (Section 2) as the map $\mathcal{J} : (u,v) \to (w_1, w_2)$ is neither C^1 nor globally invertible in general.

Riemann invariants $w_j = w_j(u,v), j = 1,2$, are defined as functions that are constants along any rarefaction wave curves of the i^{th} family $\mathbf{R}_i$ where $i \neq j$. On regions where w_j is differentiable, it is easy to check that since $\mathbf{R}_i$ curves are integral curves of the vector field $\mathbf{r}_i$, $\mathbf{r}_i \cdot \nabla w_j = 0, i \neq j$.

We shall use the ordinary differential equation (2.10) to define w_j. Care must be taken in its definition to make sure that it comes out as a single-valued function globally. For simplicity, we restrict ourselves to a half plane domain defined below. This domain is also an invariant region for the viscous system associated with (2.5) (see Subsection 7.1). We remark that the $w_j's$ are not uniquely defined. We shall make the choice that guarantees maximal regularity of certain quantities. We elaborate on this below. The advantage of this choice will become clear in Section 4 when we study the entropy functions. From now on, we assume that $b \geq 0$ for simplicity. Another case is completely similar.

Proposition 3.1. *Consider (2.5) in region IV, i.e., $\Delta > 0$. In this case, the cubic polynomial $h(\alpha)$ has only one real root α_0. Denote*

$$\beta = \frac{|3 + 4b\alpha_0 + (a-2)\alpha_0^2|}{|\alpha_0|\sqrt{((a-1)\alpha_0 + b)^2 + 4(\alpha_0+1)^2}}.$$

Consider the half plane domain $\mathcal{I}_k \triangleq \{ (u,v) \mid (-1)^k(u - \alpha_0 v) \geq 0 \}$, $k = 1, 2$. Then the following formulae define a pair of Riemann invariants for (2.5) on $\mathcal{I}_k$:

$$w_j(\tilde{u},\tilde{v}) = (-1)^j |\tilde{v}|^{\beta} \left|\frac{(\alpha_0 + \tilde{\alpha})\alpha_0}{1+\alpha_0^2}\right|^{\beta} exp\{-\beta \int_0^{\tilde{\alpha}} H_i(\tilde{\alpha})\, d\tilde{\alpha}\}, \quad i \neq j, \tag{3.1.1}$$

with

$$H_i(\tilde{\alpha}) = -\frac{(1+\alpha_0^2)}{2(\alpha_0+\tilde{\alpha})}\frac{E(\tilde{\alpha})}{D(\tilde{\alpha})}, \tag{3.1.2}$$

where

$$\begin{aligned}
E(\tilde{\alpha}) =& -2b(\alpha_0\tilde{\alpha} - 1)^2 + (a-3)(\alpha_0+\tilde{\alpha})(\alpha_0\tilde{\alpha}-1) + b(\alpha_0+\tilde{\alpha})^2 \\
& + (-1)^{i+1}(\alpha_0+\tilde{\alpha})\sqrt{Q(\tilde{\alpha})}, \\
Q(\tilde{\alpha}) =& [(a-1)(\alpha_0\tilde{\alpha}-1) + b(\alpha_0+\tilde{\alpha})]^2 + 4[b(\alpha_0\tilde{\alpha}-1) + (\alpha_0+\tilde{\alpha})]^2, \\
D(\tilde{\alpha}) =& -b(\alpha_0\tilde{\alpha}-1)^3 + (a-2)(\alpha_0+\tilde{\alpha})(\alpha_0\tilde{\alpha}-1)^2 \\
& + 2b(\alpha_0+\tilde{\alpha})^2(\alpha_0\tilde{\alpha}-1) + (\alpha_0+\tilde{\alpha})^3, \\
\tilde{u} =& u + \frac{1}{\alpha_0}v, \\
\tilde{v} =& v - \frac{1}{\alpha_0}u, \\
\tilde{\alpha} =& \frac{\tilde{u}}{\tilde{v}}.
\end{aligned}$$

Next, we study the ratio $\frac{w_2}{w_1}$ as a function of $\tilde{\alpha} = \frac{\tilde{u}}{\tilde{v}}$. We show that our choice of the definition of $w_j's$ are natural in the sense that such Riemann invariants are globally well-defined on $\mathcal{I}_k$ and this ratio is an real analytic function of $\tilde{\alpha}$ in a maximal domain. This analyticity is crucial in analyzing the entropy equations of (2.5) and is indispensable in the proof of some of our main results (Theorem 4.4.3 and 4.5.2).

Proposition 3.2. *Let w_1 and w_2 be defined as in Proposition 3.1. Then*

$$\frac{w_j}{w_i} = \Gamma_j(\tilde{\alpha}, sgn(\tilde{v}))$$

(3.1.3)

$$\equiv -exp\{(-1)^{i+1}\beta \int_0^{\tilde{\alpha}} \frac{sign(\tilde{v})(1+\alpha_0^2)\sqrt{Q(\tilde{\alpha})}}{D(\tilde{\alpha})} d\tilde{\alpha}\}, \quad i \neq j, \ i,j = 1,2,$$

where the functions Γ_1 and Γ_2 are real analytic in $\tilde{\alpha} \in \mathbf{R} \cup \{(-1)^i\infty\}$.

The next proposition concerns the monotonicity of the wave speed λ_i in the variable w_i. This is important in the reduction of the Young measures for approximate solution sequences in Section 7.

Proposition 3.3. *Suppose that $\Delta > 0$. Given w_1 and w_2 as defined in Proposition 3.1, the eigenvalues $\lambda_i, i = 1,2$, are well-defined functions in $(w_1, w_2) \in \mathcal{J}(\mathcal{I}_k)$. Moreover, we have*

$$\frac{\partial \lambda_i}{\partial w_i} \neq 0, \ \text{for all } (w_1, w_2) \in \mathcal{J}(\mathcal{I}_k) - \{w_i = 0\}, \ i = 1,2.$$

4. Entropy Functions

4.1. The Entropy Equation

Following the standard definition in [La2], we call a pair of scalar functions $(\eta(u,v),\ q(u,v))$ an entropy-entropy flux pair for (2.1) if for smooth solutions $(u(x,t), v(x,t))$ of (2.1), we have the additional conservation law

$$\partial_t \eta(u(x,t), v(x,t)) + \partial_x q(u(x,t), v(x,t)) = 0.$$

It is easy to check that this happens iff η and q satisfy the compatibility condition

$$\nabla\eta\nabla F = \nabla q. \tag{4.1.1}$$

Eliminating q, we get a second order equation, the entropy equation,

$$g_u\eta_{uu} + (g_v - f_u)\eta_{uv} - f_v\eta_{vv} + (g_{uu} - f_{uv})\eta_u + (g_{uv} - f_{vv})\eta_v = 0, \tag{4.1.2}$$

where we denote $F(u,v) = (f(u,v), g(u,v))^\top$.

(4.1.2) is a linear hyperbolic equation whose characteristic variables turn out to be the Riemann invariants. A simple calculation gives the characteristic form as

$$\eta_{w_1w_2} + \frac{\lambda_{2w_1}}{\lambda_2 - \lambda_1}\eta_{w_2} - \frac{\lambda_{1w_2}}{\lambda_2 - \lambda_1}\eta_{w_1} = 0. \tag{4.1.3}$$

4.2. The Entropy Equation for the Quadratic Flux System

We now investigate the properties of the coefficients of (4.1.3) in the case of the quadratic flux system (2.5). It will turn out that these coefficients cannot be written down explicitly as functions of w_1 and w_2 in any reasonable closed form. Instead, we will compute them as functions of $\tilde{\alpha} = \frac{\tilde{u}}{\tilde{v}}$ and study their properties using information from Section 3. We summarize the main results in the following proposition:

Proposition 4.2.1. *Consider the quadratic flux system* (2.5). *Suppose that we are in region IV in the* (a,b)*-plane so that* $\Delta > 0$. *Then the coefficients of* (4.1.3) *satisfy, for* $i \neq j$,

(1)

$$\frac{\lambda_{jw_i}}{\lambda_2 - \lambda_1} = \frac{T_j(\tilde{\alpha}, sign(\tilde{v}))}{w_i},$$
$$T_1(\tilde{\alpha}, sign(\tilde{v})) = -T_1(\tilde{\alpha}, -sign(\tilde{v})),$$

where $T_j(\tilde{\alpha}, \pm 1)$ *and* $T_j(\tilde{\alpha}, \pm 1)\Gamma_j(\tilde{\alpha}, \pm 1)$ *are real analytic in* $\tilde{\alpha} \in \mathbf{R} \cup \{\pm\infty\}$;

(2) *If $\tilde{v} \geq 0$, then*

$$\frac{T_j(\tilde{\alpha}, sgn(\tilde{\alpha}))}{w_i} = \frac{\mathcal{A}_j(\frac{w_2}{w_1})}{w_2 - w_1} = \frac{\bar{\mathcal{A}}_j(\frac{w_1}{w_2})}{w_2 - w_1}.$$

Here, $\mathcal{A}_j(\sigma) = \overline{\mathcal{A}_j}(\frac{1}{\sigma})$, $j = 1, 2$, are real analytic in $\sigma \in \mathbf{R} \cup \{\pm\infty\}$ and $\frac{1}{\sigma} \in \mathbf{R} \cup \{\pm\infty\}$, respectively;

(3) *There exists $M > 0$ such that $||\mathcal{A}_i(\frac{w_2}{w_1})||_\infty$, $||w_j \partial_{w_j} \mathcal{A}_i(\frac{w_2}{w_1})||_\infty$, and $||(w_1, w_2)\nabla^2 \mathcal{A}_i(\frac{w_2}{w_1})(w_1, w_2)^\top||_\infty$, $i, j = 1, 2$, are all bounded by M for all w_1 and w_2.*

The bounds on $\mathcal{A}_i$ in (3) of Proposition 4.2.1 will prove to be crucial in analyzing the singular behavior of η. We shall elaborate on this in Section 4. As a corollary of (2) above, we obtain

Lemma 4.2.2. *The entropy equation for the quadratic flux system (2.5) takes the form*

$$\eta_{w_1 w_2} + \frac{\mathcal{A}_2(\frac{w_2}{w_1})}{w_2 - w_1}\eta_{w_2} - \frac{\mathcal{A}_1(\frac{w_2}{w_1})}{w_2 - w_1}\eta_{w_1} = 0. \tag{4.1.4}$$

It is clear from (4.1.4) that the coefficients of the entropy equation become singular along the line $w_1 = w_2$. For the quadratic flux system, we have the relation $w_1 \leq 0 \leq w_2$ (see Section 3). Therefore, upon restriction to the physical domain, the coefficients are singular only at the umbilic point. The singular nature of the coefficients also suggest that general solutions η cannot be expected to be C^2. This presents major difficulties in verifying H^{-1} compactness conditions in the div-curl lemma. This is reminiscent of similar difficulties for proving convergence of approximate solutions to the gas dynamics equations (see [Ch2, Di2]). However, the singularities of the entropies are of somewhat different characters in the two cases and the methods for resolving these difficulties are not the same. The method we shall present generalizes that in [K] and consists of constructions of very general classes of nonsingular (C^2) entropies. This analysis is not restricted to the quadratic flux systems and can be generalized to obtain a compactness framework theorem (Theorem 6.3) for convergence of approximate solutions to general nonstrictly hyperbolic systems with an isolated umbilic point.

We remark that (2) in Proposition 4.2.1 implies that $\mathcal{A}_i(\frac{w_2}{w_1})$ is real analytic in w_1 and w_2 except at the umbilic point $(w_1, w_2) = (0, 0)$ and that $\mathcal{A}_i(\frac{w_2}{w_1})$ is multi-valued at the umbilic point. We now interpret this by comparing (4.1.4) to the classical Euler-Poisson-Darboux equation. The classical EPD equation in characteristic form is

$$\eta_{w_1 w_2} + \frac{\beta_1}{w_2 - w_1}\eta_{w_2} - \frac{\beta_2}{w_2 - w_1}\phi_{w_1} = 0. \tag{4.1.5}$$

where β_1 and β_2 are constants. The significance of these constants lie in the fact that they completely determine the singular behavior of solutions to (4.1.5). The EPD equation arises as the entropy equation for the isentropic gas dynamics equations (see [Di2, Ch2]). A comparison between (4.1.4) and (4.1.5) indicates that, heuristically, we should expect the singularity of the entropy near the umbilic point depends on the angle of approach and the "size" of $\mathcal{A}_i$. This turns out to be the correct picture and motivates much of the work on the analysis and cancellation of singularities in Subsection 4.4.3.

4.3 Polynomial Entropies for the Quadratic Flux System

The quadratic flux system (2.5) admits entropy functions that are homogeneous polynomials in the physical variables u and v of arbitrary high degrees. This was observed in [K] for the symmetric case of (2.5), i.e., when $b = 0$. We now generalize this result for general a and b. We remark that the simple function $u^2 + v^2$ is a strictly convex entropy function for (2.5) for all a and b. This function plays a special role in obtaining H^{-1} compactness estimates for the dissipation of entropy for approximate solution sequences.

Using (4.1.2) and the form of the flux matrix in (2.5), we obtain the entropy equation for (2.5) as follows:

$$((a-1)u + bv)\eta_{uv} + (bu + v)(\eta_{vv} - \eta_{uu}) = 0. \tag{4.3.1}$$

Propositon 4.3.1. *Given any a and b, there exists an infinite sequence of solutions to* (4.3.1), η_k *($k = 1, 2, 3, \cdots$) that is a homogeneous polynomial in u and v of degree k.*

We look for special entropy functions of the form

$$\eta_k = v^k \phi(\alpha), \quad \alpha = \frac{u}{v}.$$

Then (4.2.1) reduces to an ordinary differential equation for $\phi(\alpha)$. It is then easy to check that for each positive integer k, this equation admits a polynomial solution in α of degree k. Thus, η_k is a homogeneous polynomial in u and v of degree k.

4.4. The Riemann Function for the Entropy Equation

Next, we study the general properties of entropy in the (w_1, w_2)-coordinates. In particular, we are interested in understanding the possible singularities of η and the construction of C^2 Goursat entropies. Goursat entropies are solutions of the entropy equation subjected to characteristic boundary conditions. To this end, we first study the Riemann function $\mathcal{R}$ for the hyperbolic equation (4.1.3). Recall that the Riemann function contains all information about the general solution and is defined as the solution to the following Goursat problem (characteristic boundary value problem) with special boundary data:

$$\begin{aligned} &\mathcal{R} = \mathcal{R}(w_1, w_2; \sigma, \tau), \\ &\mathcal{R}_{w_1 w_2} + \frac{\mathcal{A}_2(\frac{w_2}{w_1})}{w_2 - w_1}\mathcal{R}_{w_2} - \frac{\mathcal{A}_1(\frac{w_2}{w_1})}{w_2 - w_1}\mathcal{R}_{w_1} = 0, \\ &\mathcal{R}(w_1, \tau; \sigma, \tau) = exp\left\{-\int_\sigma^{w_1} \frac{\mathcal{A}_2(\frac{\tau}{s})}{s - \tau}\, ds\right\}, \\ &\mathcal{R}(\sigma, w_2; \sigma, \tau) = exp\left\{\int_\tau^{w_2} \frac{\mathcal{A}_1(\frac{y}{\sigma})}{\sigma - y}\, dy\right\}. \end{aligned} \tag{4.4.1}$$

Consider the general Goursat problem for the entropy equation (4.1.4)

$$\begin{aligned} &\eta_{w_1 w_2} + \frac{\mathcal{A}_2(\frac{w_2}{w_1})}{w_2 - w_1}\eta_{w_2} - \frac{\mathcal{A}_1(\frac{w_2}{w_1})}{w_2 - w_1}\eta_{w_1} = 0, \\ &\eta(\xi, w_2) = \phi(w_2), \\ &\eta(w_1, \kappa) = \theta(w_1), \end{aligned} \tag{4.4.2}$$

where ϕ and θ are prescribed functions, and ξ and κ fixed constants. It is well known that the solution η to (4.4.2) can be expressed in integral form in terms of $\mathcal{R}$, ϕ, and θ (at least in regions where η is C^2) as follows:

$$\eta(w_1,w_2) = \theta(\xi)\mathcal{R}(w_1,w_2;\xi,\kappa) + \int_\kappa^{w_2} \mathcal{R}(w_1,w_2;\xi,s)\left(\phi'(s) - \frac{\mathcal{A}_1(\frac{\xi}{s})}{s-\xi}\phi(s)\right) ds$$

$$+ \int_\xi^{w_1} \mathcal{R}(w_1,w_2;y,\kappa)\left(\theta'(y) + \frac{\mathcal{A}_2(\frac{y}{\kappa})}{\kappa-y}\theta(y)\right) dy. \tag{4.4.3}$$

Our strategy is first to analyze the existence and singular behavior of $\mathcal{R}$ and then make use of (4.4.3) to show how to choose general classes of data ϕ and θ to cancel these singularities. The existence of $\mathcal{R}$ is nontrivial as the coefficients of (4.4.1) are singular. In the case of strictly hyperbolic systems, the corresponding $\mathcal{R}$ and η equations will have regular coefficients only and existence of general solutions follows from standard iteration methods. This is not the case here and is one of the key difficulties in treating nonstrictly hyperbolic systems. Our analysis on $\mathcal{R}$ and η will be applicable to more general situations than (4.4.1) – (4.4.3). To state our results in their strongest form, we consider the following Goursat problem of which (4.4.1) – (4.4.3) are special cases.

$$\bar{\eta}_{w_1 w_2} + \frac{\mathcal{B}(w_1,w_2)}{w_2-w_1}\eta_{w_2} + \frac{\mathcal{C}(w_1,w_2)}{w_2-w_1}\eta_{w_1} = 0,$$

$$\bar{\eta}(\xi,w_2) = \phi(w_2),$$

$$\bar{\eta}(w_1,\kappa) = \theta(w_1). \tag{4.4.4}$$

Here, $\mathcal{B}$ and $\mathcal{C}$ are real analytic in w_1 and w_2 except at the origin $(0,0)$. The corresponding Riemann function satisfies

$$\bar{\mathcal{R}} = \bar{\mathcal{R}}(w_1,w_2;\sigma,\tau),$$

$$\bar{\mathcal{R}}_{w_1 w_2} + \frac{\mathcal{B}(w_1,w_2)}{w_2-w_1}\bar{\mathcal{R}}_{w_2} + \frac{\mathcal{C}(w_1,w_2)}{w_2-w_1}\bar{\mathcal{R}}_{w_1} = 0,$$

$$\bar{\mathcal{R}}(w_1,w_2;w_1,\tau) = exp\left\{-\int_\tau^{w_2} \frac{\mathcal{B}(w_1,s)}{s-w_1}\, ds\right\},$$

$$\bar{\mathcal{R}}(w_1,w_2;\sigma,w_2) = exp\left\{-\int_\sigma^{w_1} \frac{\mathcal{C}(y,w_2)}{w_2-y}\, dy\right\}. \tag{4.4.5}$$

Theorem 4.4.1. *Consider* (4.4.5) *under the assumptions*

(1) $\mathcal{B}$ *and* $\mathcal{C}$ *are real analytic everywhere except at* $(w_1, w_2) = (0,0)$*;*

(2) There exists $M > 0$ *such that* $|\mathcal{B}|_\infty$, $|\mathcal{C}|_\infty$, $|(w_1, w_2)\nabla^2\mathcal{B}(w_1, w_2)^\top|_\infty$, *and* $|(w_1, w_2)\nabla^2\mathcal{C}(w_1, w_2)^\top|_\infty$ *are bounded by* M *on any compact set* $(j = 1, 2)$*;*

(3) $\kappa = 0$.

Then there exists a unique solution $\bar{\mathcal{R}}(w_1, w_2; \xi, 0)$ *to* (4.4.5)*, with* $\sigma = \xi$ *and* $\tau = \kappa = 0$*, which is real analytic in the transformed variables* $\overline{w}_1$*, and* $\overline{w}_2$ *except on the line* $\overline{w}_1 + \overline{w}_2 = 1$*, were* $\overline{w}_2 = \frac{w_2}{\xi}$*, and* $\overline{w}_1 = \frac{\xi - w_1}{\xi}$*.*

The proof of Theorem 4.4.1 relies on a majorization process. A majorant problem corresponding to (4.4.5) will be constructed and the existence and analyticity of $\mathcal{R}$ follows from that of the majorant.

Lemma 4.4.2. *Under the assumptions (1) and (2) in Theorem 4.4.1, there exists* $M' > 0$ *such that*

$$\frac{\mathcal{B}(\overline{w}_1, \overline{w}_2)}{1 - \overline{w}_1 - \overline{w}_2} \ll \frac{M'}{1 - \overline{w}_1 - \overline{w}_2},$$
$$\frac{\mathcal{C}(\overline{w}_1, \overline{w}_2)}{1 - \overline{w}_1 - \overline{w}_2} \ll \frac{M'}{1 - \overline{w}_1 - \overline{w}_2}.$$

Here, $\ll$ *denotes majorization in the sense of real analytic power series.*

Proof of Lemma 4.4.2. We will prove that

$$\frac{\mathcal{B}(\overline{w}_1, \overline{w}_2)}{1 - \overline{w}_1 - \overline{w}_2} \ll \frac{M'}{1 - \overline{w}_1 - \overline{w}_2}. \tag{4.4.6}$$

The other majorizing inequality involving $\mathcal{C}$ can be proved in a similar fashion.

By assumption (1) in Theorem 4.4.1, $\mathcal{B}(\overline{w}_1, \overline{w}_2)$ is real analytic everywhere except at the point $(\overline{w}_1, \overline{w}_2) = (1, 0)$. Thus, away from this point, $\mathcal{B}$ admits a power series representation of the form

$$\mathcal{B}(\overline{w}_1, \overline{w}_2) = \sum_{m,n \geq 0} c_{mn} {\overline{w}_1}^m {\overline{w}_2}^n.$$

Moreover, in the region $|\overline{w}_1 + \overline{w}_2| < 1$, we have

$$\frac{1}{1 - \overline{w}_1 - \overline{w}_2} = \sum_{m,n\geq 0} \binom{m+n}{n} \overline{w}_1{}^m \overline{w}_2{}^n.$$

Combining the two power series, we obtain

$$\begin{aligned} \frac{\mathcal{B}(\overline{w}_1, \overline{w}_2)}{1 - \overline{w}_1 - \overline{w}_2} &= \sum_{m,n\geq 0} c_{mn} \overline{w}_1{}^m \overline{w}_2{}^n \sum_{k,l\geq 0} \binom{m+n}{n} \overline{w}_1{}^m \overline{w}_2{}^n \\ &= \sum_{p,q\geq 0} \overline{w}_1{}^p \overline{w}_2{}^q \sum_{\substack{0\leq m\leq p \\ 0\leq n\leq q}} \binom{m+n}{n} \overline{w}_1{}^m \overline{w}_2{}^n. \end{aligned} \tag{4.4.7}$$

The formula (4.4.7) is valid in the region $|\overline{w}_1 + \overline{w}_2| < 1$. Using (4.4.7), we translate the majorization inequality (4.4.6) into an equivalent inequality involving the coefficients $c_{m,n}$ in the power series expansion of $\mathcal{B}$,

$$\begin{aligned} &\frac{\mathcal{B}(\overline{w}_1, \overline{w}_2)}{1 - \overline{w}_1 - \overline{w}_2} \ll \frac{M'}{1 - \overline{w}_1 - \overline{w}_2} \\ \Leftrightarrow\ & |\sum_{\substack{p\geq 0 \\ 0\leq n\leq q}} c_{p-m,q-n}| \leq M' \binom{p+q}{q} \\ \Leftrightarrow\ & |\sum_{\substack{p\geq 0 \\ 0\leq n\leq q}} \frac{p!q!}{(p+q)!} \binom{m+n}{n} c_{p-m,q-n}| \leq M'. \end{aligned} \tag{4.4.8}$$

Our task is to verify the last inequality in (4.4.8). To this end, we consider the auxiliary function $\frac{g(z)}{(1-z)^{m+1}}$, where m is an integer and $g(z) = \sum_{k\geq 1} \hat{g}_k z^k$ is an analytic function of z in the disk $|z| < 1$. We have the following formulae:

$$\begin{aligned} \frac{g(z)}{(1-z)^{m+1}} &= \sum_{k\geq 1} \hat{g}_k z^k \sum_{n\geq 0} \binom{m+n}{n} z^n \\ &= \sum_{q\geq 1} z^q \sum_{0\leq n\leq q} \binom{m+n}{n} \hat{g}_{q-n}. \end{aligned} \tag{4.4.9}$$

We now make a useful choice of $g(z)$ that is related to $\mathcal{B}$. First, we rewrite $\mathcal{B}(\overline{w}_1, \overline{w}_2)$ in the following way:

$$\begin{aligned} \mathcal{B}(\overline{w}_1, \overline{w}_2) &= \sum_{l\geq 0} g_l(\overline{w}_2) \overline{w}_1{}^l, \\ g_l(\overline{w}_2) &= \sum_{k\geq 0} c_{lk} \overline{w}_2{}^k. \end{aligned} \tag{4.4.10}$$

We will take g to be g_l in (4.4.9). We claim that, on $|\overline{w}_2| < 1$,

$$\frac{g_{p-m}(\overline{w}_2)}{(1-\overline{w}_2)^{m+1}} \ll \frac{M}{(1-\overline{w}_2)^{m+1}}, \tag{4.4.11}$$

where M is a constant independent of p and m. The proof of the claim depends on certain decay estimates of the coefficients c_{mn}. Consider $\mathcal{B}(\rho_0 e^{i\theta}, z) = \sum_{l\geq 0} g_l(z)\rho_0^l e^{il\theta}$ where $0 < \rho_0 < 1$ and z belongs to the unit disk on the complex plane. $\mathcal{B}(\rho_0 e^{i\theta}, z)$ is analytic for all $|z| < 1$. Moreover, we have

$$g_l(z) = c\rho_0^{-l} \int_0^{2\pi} \mathcal{B}(\rho_0 e^{i\theta}, z)e^{-il\theta}\, d\theta,$$
$$|g_l(z)| \leq 2\pi c\rho_0^{-l}||\mathcal{B}||_\infty.$$

Here c is a universal constant. Now let $\rho_0 \to 1^-$, we obtain

$$|g_l(z)| \leq 2\pi c||\mathcal{B}||_\infty. \tag{4.4.12}$$

Likewise, we have

$$g_l''(z) = c\rho_0^{-l} \int_0^{2\pi} \partial_z^2\mathcal{B}(\rho_0 e^{i\theta}, z)e^{-il\theta}\, d\theta.$$

Using the bound $||z^2\partial_z^2\mathcal{B}(\rho_0 e^{i\theta}, z)||_\infty \leq c'$, where c' is independent of ρ_0 and θ, we obtain in a similar fashion,

$$||z^2 g_l''(z)||_\infty \leq 2\pi c'', \tag{4.4.13}$$

where c'' is a constant independent of l.

Combining (4.4.12) and (4.4.13), it is not difficult to obtain a decay estimate for the coefficients of $\mathcal{B}$

$$|c_{lk}| \leq M\ min(\frac{1}{(l+1)^2}, \frac{1}{(k+1)^2}). \tag{4.4.14}$$

Now (4.4.11) is equivalent to

$$|\sum_{0\leq n\leq q} \binom{m+n}{n} c_{p-m,q-n}| \leq M\binom{m+q}{q}.$$

The right side satisfies

$$
\begin{aligned}
M\binom{m+q}{q} &\geq M'\binom{m+q}{q}\sum_{n\geq 0}\frac{1}{(q-n+1)^2}\\
&\geq \binom{m+q}{q}\sum_{n\geq q}|c_{p-m,q-n}|\\
&\geq |\sum_{0\leq n\leq q}\binom{m+n}{n}c_{p-m,q-n}|.
\end{aligned}
$$

This proves (4.4.11).

We now show that (4.4.11) and (4.4.14) implies this lemma. Using (4.4.10) and (4.4.9), we see that (4.4.11) is equivalent to

$$
\sum_{q\geq 0}\overline{w}_2{}^q \sum_{\substack{0\leq m\leq p\\ 0\leq n\leq q}}\binom{m+n}{n}c_{p-m,q-n} \ll \sum_{0\leq m\leq p}\frac{M}{(1-\overline{w}_2)^{m+1}}. \tag{4.4.15}
$$

The right side of (4.4.15) can be summed to give

$$
\sum_{0\leq m\leq p}\frac{M}{(1-\overline{w}_2)^{m+1}} = \sum_{q\geq 0}\binom{p+q+1}{q+1}\overline{w}_2{}^q. \tag{4.4.16}
$$

(4.4.16) is now easily checked to imply (4.4.8) by virtue of (4.4.14).

This completes the proof of Lemma 4.4.2.

Using Lemma 4.4.2, we obtain a majorant problem of (4.4.5).

Theorem 4.4.3. *Suppose $\mathcal{G} = \mathcal{G}(\overline{w}_1, \overline{w}_2)$ is real analytic in a domain Ω and that $\mathcal{G}$ is the solution to the following majorant problem:*

$$
\begin{aligned}
\mathcal{G} &= \mathcal{G}(\overline{w}_1, \overline{w}_2),\\
\mathcal{G}_{\overline{w}_1\overline{w}_2} - \frac{M'}{1-\overline{w}_1-\overline{w}_2}&(\mathcal{G}_{\overline{w}_1}+\mathcal{G}_{\overline{w}_2}) = 0,\\
\mathcal{G}(\overline{w}_1, 0) &= \frac{M'}{1-\overline{w}_1},\\
\mathcal{G}(0, \overline{w}_2) &= \frac{M'}{1-\overline{w}_2}.
\end{aligned} \tag{4.4.17}
$$

Then, if $\mathcal{R}$ is the solution to (4.4.5) *with* $\tau = \kappa = 0$, *we have, on* Ω,

$$\mathcal{R} \ll \mathcal{G} \quad \textit{in the sense of power series in } \overline{w}_1 \textit{ and } \overline{w}_2.$$

We remark that (4.4.17) is the celebrated Euler-Poisson-Darboux equation. We summarize some of its most fundamental properties in the next lemma. We also note that although the analyticity of $\mathcal{G}$ (see Lemma 4.4.4 below) implies that of $\mathcal{R}$, $\mathcal{R}$ is not everywhere analytic on the domain of interest to our problem. It turns out that the line $w_1 = w_2$ (and hence the umbilic point) lies on the boundary of the region of convergence of the power series for $\mathcal{R}$ and $\mathcal{G}$. On this part of the boundary, $\mathcal{R}$ and its derivatives are singular. We characterize these singularities in Lemma 4.4.4 and Lemma 4.5.1.

Lemma 4.4.4.

(1) *In the variables* $\hat{w}_1 = 1 - \overline{w}_1$ *and* $\overline{w}_2$, (4.4.17) *is an Euler-Poisson-Darboux equation with special Goursat data:*

$$\begin{aligned} &\mathcal{G}_{\hat{w}_1\overline{w}_2} + \frac{M'}{\overline{w}_2 - \hat{w}_1}(\mathcal{G}_{\hat{w}_1} - \mathcal{G}_{\overline{w}_2}) = 0, \\ &\mathcal{G}(\hat{w}_1, 0) = \frac{M'}{\hat{w}_1}, \\ &\mathcal{G}(0, \overline{w}_2) = \frac{M'}{1 - \overline{w}_2}. \end{aligned} \tag{4.4.18}$$

(2) *The Riemann function of the first equation in* (4.4.18) *is*

$$\begin{aligned} \mathcal{H}(\hat{w}_1, \overline{w}_2; \xi, \kappa) &= \frac{(\hat{w}_1 - \overline{w}_2)^{2M'}}{(\hat{w}_1 - \kappa)^{M'}(\xi - \overline{w}_2)^{M'}} F(M', M', 1; \rho), \\ \rho &= \frac{(\xi - \hat{w}_1)(\overline{w}_2 - \kappa)}{(\hat{w}_1 - \kappa)(\xi - \overline{w}_2)}. \end{aligned} \tag{4.4.19}$$

Here, F is the Hypergeometric function defined by the power series

$$\begin{aligned} &F(\alpha, \beta\gamma; \rho) = 1 + \sum_{m=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)\beta(\beta+1)\cdots(\beta+m-1)}{1\cdot 2\cdots m\cdot\gamma(\gamma+1)\cdots(\gamma+m-1)}\rho^m, \\ &|\rho| < 1. \end{aligned} \tag{4.4.20}$$

(3) *The solution to* (4.4.18) *is given by*

$$\mathcal{G}(\hat{w}_1, \overline{w}_2) = M' + \int_1^{\hat{w}_1} \mathcal{H}(\hat{w}_1, \overline{w}_2; \xi, 0) M'(M'-1)\xi^{-2}\, d\xi$$

$$+ \int_0^{\overline{w}_2} \mathcal{H}(\hat{w}_1, \overline{w}_2; 1, y) M'(1-M')(1-y)^{-2}\, dy. \tag{4.4.21}$$

The proof of Theorem 4.4.1 follows from Theorem 4.4.3 and Lemma 4.4.4.

4.5. C^2 Goursat Entropies

In this subsection, we characterize the singularities of entropies and we show how to construct Goursat entropies that are regular. We continue to work on the general level of equations (4.4.4) and (4.4.5).

Similar to (4.4.3), we have an integral representation of $\bar{\eta}$ in terms of $\bar{\mathcal{R}}$.

$$\bar{\eta}(w_1, w_2) = \theta(\xi)\mathcal{R}(w_1, w_2; \xi, \kappa) + \int_\kappa^{w_2} \bar{\mathcal{R}}(w_1, w_2; \xi, s)(\phi'(s) + \frac{\mathcal{C}(\xi, s)}{s - \xi}\phi(s))ds$$

$$+ \int_\xi^{w_1} \bar{\mathcal{R}}(w_1, w_2; y, \kappa)(\theta'(y) + \frac{\mathcal{B}(y, \kappa)}{\kappa - y}\theta(y))dy. \tag{4.5.1}$$

Using (4.5.1) with $\kappa = 0$, it is clear that whatever singularities $\bar{\eta}$ might have are inherited from those of $\bar{\mathcal{R}}$. Since $\bar{\mathcal{R}} \ll \mathcal{G}$, the singularities of $\bar{\mathcal{R}}$ are dominated by those of $\mathcal{G}$. We make this precise with the aid of the next lemma.

Lemma 4.5.1. *Suppose that* F_1 *and* F_2 *are real analytic on an open domain* $\Omega \subset \mathbf{R}^2$. *Assume further that*

(1) $F_1 \ll F_2$ *on* Ω,

(2) $F_2 = \frac{1}{S}\hat{F}_2$ *where* $\hat{F}_2$ *is analytic on* $\overline{\Omega}$ *and* S *is analytic on* Ω.

Then we have

$$||SF_1||_{L^\infty(\Omega)} \leq ||\hat{F}_2||_{L^\infty(\Omega)},$$

$$F_1 = \frac{1}{S}\hat{F}_1, \quad \text{where } \hat{F}_1 = SF_1 \text{ is analytic on } \overline{\Omega}.$$

We have characterized the singularities in $\mathcal{G}$ and its derivatives. By repeated applications of Lemma 4.5.1, we can show that the singularities of $\mathcal{R}$ and its derivatives are dominated by those in $\mathcal{G}$ and corresponding derivatives. Thus, we have established a kind of comparison principle for the singularities of solutions to second order hyperbolic equations of Euler-Poisson-Darboux type One of our main objectives is to construct regular solutions to (4.4.4). It is now clear that this can be achieved if we can prescribe Goursat data that could cancel the type of singularities in $\mathcal{G}$ since they dominate the singularities in $\mathcal{R}$ which is the Riemann function for (4.4.4).

We now make use of what have been developed so far to construct general classes of entropies that are C^2 in w_1 and w_2. The domain of interest for our problem is $\{w_1 \leq 0 \leq w_2\}$. It is enough to focus our attentions on a compact set containing the umbilic point $\{w_1^- \leq w_1 \leq 0 \leq w_2 \leq w_2^+\}$ where $w_2^+ > 0$ and $w_1^- < 0$ are fixed constants. Also fix a constant $\delta > 0$ satisfying $|\delta| \ll w_2^+, -w_1^-$.

Without loss of generality, we consider (4.4.4) with the special choice of Goursat data $\phi(w_2) \equiv 0$:

$$
\begin{aligned}
&\bar{\eta}_{w_1 w_2} + \frac{\mathcal{B}(w_1, w_2)}{w_2 - w_1}\eta_{w_2} + \frac{\mathcal{C}(w_1, w_2)}{w_2 - w_1}\eta_{w_1} = 0,\\
&\bar{\eta}(\xi, w_2) \equiv 0,\\
&\bar{\eta}(w_1, \kappa) = \theta(w_1).
\end{aligned} \tag{4.5.2}
$$

There is no loss of generalities since an analysis similar to what is shown below will allow us to construct C^2 entropies with Goursat data $\theta \equiv 0$ and ϕ arbitrary (up to a finite number of conditions, see Theorem 4.5.2). The general case (as in (4.4.4)) then follows from a simple linear superposition of these two special cases.

Using the fact that $\bar{\mathcal{R}} \ll \mathcal{G}$ and combining Lemma 4.4.4 and Lemma 4.5.1, we obtain a decomposition of $\bar{\mathcal{R}}$ into a product of a singular and a regular part. The regular part is analytic everywhere. A similar decomposition exists for $\mathcal{G}$ using Lemma 4.4.4. Moreover, from Lemma 4.5.1 again, it is possible to set the singular parts to be identical for both $\bar{\mathcal{R}}$ and $\mathcal{G}$. Then using this decomposition with (4.5.1), it is possible to obtain conditions on θ that guarantee the cancellation of singularities. We now state our main result in this section.

Theorem 4.5.2. *Given a positive integer m and given a compact set K containing $(0,0)$ in $\mathbf{R}^2$. Consider the Goursat problem (4.5.2) with $\kappa = 0$. There exists a subspace $\mathcal{M} \subset L^2[w_1^-, -\delta]$ satisfying codim$\mathcal{M} < \infty$ such that if*

(1) $\theta \in \mathcal{M} \cap C^m$,

(2) $\theta(w_1) = 0, \quad$ *for all* $w_1 \geq -\delta$,

then there exists solution $\bar{\eta} \in C^m([w_1^-, 0] \times [0, w_2^+])$ to (4.5.2) satisfying

$$\max_{0 \leq k+l \leq m} \sup_{(w_1,w_2) \in K} ||w_1|^{-k}|w_2|^{-l}\partial^k_{w_1}\partial^l_{w_2}\bar{\eta}| \leq C.$$

Recall that (4.4.1) – (4.4.3) are special cases of (4.4.4), (4.4.5) and (4.5.1). We obtain immediately existence of many C^m Goursat entropies for the quadratic flux system (2.5).

Corollary 4.5.3. *Under the same assumptions on the Goursat data as in Theorem 4.5.2, there exists a solution $\eta \in C^m([w_1^-, 0] \times [0, w_2^+])$ satisfying*

$$\max_{0 \leq k+l \leq m} \sup_{(w_1,w_2) \in K} ||w_1|^{-k}|w_2|^{-l}\partial^k_{w_1}\partial^l_{w_2}\eta| \leq C.$$

We now turn to investigate the regularity of entropies constructed in Theorem 4.5.2 and Corollary 4.5.3 as function of the state variables u and v. This regularity is needed in our use of compensated compactness to prove the convergence of approximate solutions to (2.1).

In many strictly hyperbolic systems, such as the system of elasticity, the map $\mathcal{J} : (u, v) \to (w_1, w_2)$ is C^2. Therefore, C^2 regularity of entropies in the (u, v)-coordinates is a direct consequence of Theorem 4.5.2. For nonstrictly hyperbolic systems, the coincidence of eigenvalues λ_1 and λ_2 usually means that the geometry of the wave curves are very singular at the umbilic point. Thus,

$\mathcal{J}$ is usually not C^2 and additional work is necessary to get C^2 regularity for $\eta = \eta(u, v)$.

We now restrict ourselves to region IV of the quadratic flux system (2.5), i.e., where a and b satisfy $\Delta = -32b^4 + b^2[27+36(a-2)-4(a-2)^2]+4(a-2)^3 > 0$. Using Proposition 3.1, we have

Proposition 4.5.4. *Consider the quadratic flux system* (2.5) *in region IV, i.e.,* $\Delta > 0$. *Near the umbilic point* $(u, v) = (0, 0)$, *the derivatives of the Riemann invariants satisfy the following estimates:*

$$w_i = \mathcal{O}(1), \quad i = 1, 2,$$

$$\partial_u^m \partial_v^n w_i = \mathcal{O}\left(\frac{w_i}{|\tilde{v}|^{m+n}}\right), \quad 1 \le m + n \le 2. \tag{4.5.3}$$

Next, we recover from the proof of Theorem 4.5.3, certain fine estimates of the derivatives η in w_1 and w_2. As we shall see, these derivatives vanishe to certain orders as they approach the umbilic point.

Proposition 4.5.5. *Consider* (2.5) *in region IV. Given any nonnegative integers* m, n, k, l *with* $m \ge k, n \ge l$, *there exists entropy functions constructed using the method in Theorem 4.5.3 and which satisfy the following estimates near the umbilic point:*

$$\partial_{w_1}^k \partial_{w_2}^l \bar{\eta} = \mathcal{O}(|w_1|^{m-k} |w_2|^{n-l}).$$

Now using the chain rule and combining the estimates in Propositions 4.5.4 and 4.5.5, we obtain

Theorem 4.5.6. *Consider* (2.5) *with* a *and* b *satisfying* $\Delta = -32b^4 + b^2(27 + 36(a-2) - 4(a-2)^2) + 4(a-2)^3 > 0$. *Let* $\mathcal{J} : (u, v) \to (w_1, w_2)$ *denotes the map from the state space to the Riemann invariants plane. Suppose that* η *is the solution to* (4.4.2) *as constructed in Corollary 4.5.3. Then we have*

$$\eta \circ \mathcal{J} \in C^2(\mathbf{R}^2).$$

5. Approximate Solutions and Young Measures

5.1. Parabolic Approximations

We consider parabolic approximations to the general system (1.1) by adding artificial viscosities. Recall that (1.1) together with initial data $U_0(x)$ takes the form

$$\begin{aligned} &\partial_t U + \partial_x F(U) = 0, \qquad t > 0, x \in \mathbf{R}, \\ &U(x,0) = U_0(x). \end{aligned} \tag{5.1.1}$$

Consider a sequence of parabolic approximate solutions U^ϵ governed by the associated parabolic system

$$\begin{aligned} &\partial_t U^\epsilon + \partial_x F(U^\epsilon) = \epsilon \partial_x (D \partial_x U^\epsilon), \qquad t > 0, x \in \mathbf{R}, \\ &U^\epsilon(x,0) = U_0(x), \end{aligned} \tag{5.1.2}$$

where $D \geq 0$ is a non-negative matrix (viscosity matrix) and $\epsilon > 0$ measures the amount of artificial viscosities in (5.1.2).

We are concerned with the convergence of U^ϵ to a weak solution U of (5.1.1) as $\epsilon \to 0^+$. A compactness framework theorem (Theorem 6.3) will be established in Section 6 to attain this goal. It will then be applied to the quadratic flux system (2.5). This is made possible by a reduction of Young measure analysis (Theorem 6.2) using large classes of regular entropy functions (Theorem 4.5.2 and Theorem 4.5.3) constructed in Section 4. In this section, we dispense with certain preliminaries concerning existence and L^∞ apriori estimates of solutions to (5.1.2). For simplicity, we take $D = I$, the identity matrix.

Proposition 5.1.1. *Let c_1 and c_2 be constants. Let $\Omega_{c_1 c_2} \triangleq \{(u,v) \mid c_1 \leq w_1 \leq 0 \leq w_2 \leq c_2\}$. Suppose that the Riemann invariants w_1 and w_2 are quasi-convex on $\partial\Omega_{c_1 c_2}$. That is, on $\partial\Omega_{c_1 c_2}$,*

$$\begin{aligned} &\mathbf{r}_1^\top \cdot \nabla^2 w_2 \cdot \mathbf{r}_1 \geq 0, \\ -&\mathbf{r}_2^\top \cdot \nabla^2 w_1 \cdot \mathbf{r}_2 \geq 0. \end{aligned}$$

Then that $\Omega_{c_1c_2}$ is an invariant region for (5.1.2), i.e., $U_0(x) \in \Omega_{c_1c_2}$, for all $x \in \mathbf{R}$, implies $U^\epsilon(x,t) \in \Omega_{c_1c_2}$, for all $x \in \mathbf{R}$, $t > 0$. Moreover, if (5.1.2) admits a family of such invariant regions which spans $\mathbf{R}^2$, then we obtain an apriori L^∞ bound for solutions U^ϵ to (5.1.2).

Proposition 5.1.1 is a straightforward consequence of the invariant region theorem (see [CCS]). Granting such an L^∞ apriori estimate, a local in t solution to (5.1.2) obtained by standard iteration arguments can be extended globally in $t > 0$.

5.2. Finite Difference Approximations

As is known, it is very complicated to directly construct the Riemann solutions for nonstrictly hyperbolic systems of conservation laws (cf. [IMPT, IT, SS2, SSMP]). Fortunately, we can obtain the global entropy solutions of Riemann problems in any bounded domain in (x,t)-plane by proving the convergence of the viscous approximations and noting the finiteness of propagation speeds of the entropy solutions.

Now we construct Lax-Friedrichs approximations [La3] and Godunov approximations [Go] $U^\ell(x,t) = (u^\ell(x,t), v^\ell(x,t))$.

On the upper-left plane $t \geq 0$, we have the grid

$$t = nh\ , \quad x = j\ell\ ; \quad j = 0, \pm 1, \pm 2, \cdots\ , \quad n = 0, 1, 2, \cdots\ ,$$

where $n \geq 0$ and j are integers and the positive constants h and ℓ are the time step length and the space step length, respectively, that satisfy the inequality

$$\max_{i=1,2}\Big(\sup_{\substack{-\infty < x < \infty \\ 0 \leq t \leq T}} |\lambda_i(U^\ell)| \Big) < \frac{\ell}{h} \leq M\ , \tag{5.2.1}$$

for any given $T > 0$. We shall prove that $U^\ell(x,t)$ are uniformly bounded that (5.2.1) always holds.

5.2.1. Lax-Friedrichs Approximations

For integers $n \geq 1$, we set

$$J_n = \{j : j \text{ integers}, \ n + j = \text{even}\} \ .$$

On the rectangle $\{(x,t) : (j-1)\ell < x < (j+1)\ell, \ 0 \leq t < h, \ j \text{ odd}\}$, we define $U^\ell(x,t)$ as the solutions of Riemann problems

$$\begin{cases} (1.1) \ , \\ U|_{t=0} = \begin{cases} U_0^\ell((j-1)\ell) \ , & x < j\ell \ , \\ U_0^\ell((j+1)\ell) \ , & x > j\ell \ , \end{cases} \end{cases}$$

where $U_0^\ell(x) = U_0(x)X_\ell(x)$, and

$$X_\ell(x) = \begin{cases} 1 \ , & x \in [-\frac{1}{\ell}, \frac{1}{\ell}] \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$

and define

$$U_j^1 = \frac{1}{2\ell} \int_{(j-1)\ell}^{(j+1)\ell} U^\ell(x, h-0)dx \ .$$

Suppose that (u^ℓ, v^ℓ) have been defined for $t < nh$. Then, on the rectangle $\{(x,t) : j\ell < x < (j+2)\ell, \ nh < t < (n+1)h, \ j \in J_n\}$, we define $U^\ell(x,t)$ as the solutions of Riemann problems

$$\begin{cases} (1.1) \ , \\ U|_{t=nh} = \begin{cases} U_j^n \ , & x < (j+1)\ell \ , \\ U_{j+2}^n \ , & x > (j+1)\ell \ , \end{cases} \end{cases}$$

and define

$$U_j^{n+1} = \frac{1}{2\ell} \int_{(j-1)\ell}^{(j+1)\ell} U^\ell(x, (n+1)h - 0)dx \ . \tag{5.2.2}$$

In this manner we obtain the Lax-Friedrichs approximations on the upper-half plane, and the difference scheme (5.2.2) is just the Lax-Friedrichs scheme [La3].

5.2.2. Godunov Approximations

Similarly, on the rectangle $\{(x,t) : j\ell < x < (j+1)\ell,\ 0 \le t < h\}$, we define $U^\ell(x,t)$ as the solutions of Riemann problems

$$\begin{cases} (1.1)\ , \\ U|_{t=0} = \begin{cases} U_0^\ell(j\ell)\ , & x < (j+\frac{1}{2})\ell\ , \\ U_0^\ell((j+1)\ell)\ , & x > (j+\frac{1}{2})\ell\ , \end{cases} \end{cases}$$

where $U_0^\ell(x) = U_0(x)X_\ell(x)$, and

$$X_\ell(x) = \begin{cases} 1\ , & x \in [-\frac{1}{2}, \frac{1}{2}]\ , \\ 0\ , & \text{otherwise,} \end{cases}$$

and define

$$U_j^1 = \frac{1}{\ell}\int_{(j-1/2)\ell}^{(j+1/2)\ell} U^\ell(x, h-0)dx\ .$$

Suppose that U^ℓ have been defined for $t < nh$. Then, on the rectangle $\{(x,t) : j\ell < x < (j+1)\ell,\ nh < t < (n+1)h\}$, we define $U^\ell(x,t)$ as the solutions of Riemann problems

$$\begin{cases} (1.1)\ , \\ U|_{t=nh} = \begin{cases} U_j^n\ , & x < (j+\frac{1}{2})\ell\ , \\ U_{j+1}^n\ , & x > (j+\frac{1}{2})\ell\ , \end{cases} \end{cases}$$

and define

$$U_j^{n+1} = \frac{1}{\ell}\int_{(j-1/2)\ell}^{(j+1/2)\ell} U^\ell(x, (n+1)h-0)dx\ . \tag{5.2.3}$$

This completes the construction of the Godunov approximations.

Remark 1. The approximate solutions $U^\ell(x,t)$ constructed above have the same local structure as the random choice approximations of Glimm [G2].

Remark 2. If the Riemann solutions have convex invariant regions in U-space, it is easy to check that the Lax-Friedrichs and Godunov approximations have the same convex invariant regions in U-space from their construction and Jensen's inequality. This enables us to obtain L^∞ uniform bounds for these approximations.

5.3. Young Measures and Compensated Compactness

The Young measure representation (see [Ta]) for the sequence of bounded functions in an appropriate space is an efficient tool for studying the limit behavior of the approximate solutions of nonlinear problems, especially for conservation laws because of the lack of regularity of the limit problems. By combining the Young measure representation with compensated compactness [Ta, Mu], one can transfer the singular limit problems to the problems of solving some functional equations for the corresponding Young measures, that is, to studying the structure of the Young measures satisfying the functional equations. If one can solve these functional equations to clarify the structure of the Young measures, the limit behavior of the sequence can be well understood. Therefore, the essential difficulty is how to solve these functional equations for the Young measures. This difficulty is overcome for some important systems in conservation laws (cf. [Di, Ch1, Ch2, DCL, K, Mo, Se]). In this section we review some results on the Young measures and the compensated compactness for subsequent development to solve general nonstrictly hyperbolic systems.

Theorem 5.3.1([Ta]). Suppose that $U^\epsilon : \mathbf{R}^2_+ \to \mathbf{R}^n$ is a sequence of bounded measurable functions

$$U^\epsilon(x,t) \in K, \qquad \text{a.e.} \tag{5.3.1}$$

for a bounded set K in $\mathbf{R}^n$. Then there exists a subsequence (still labeled U^ϵ) and a family of Young measures

$$\nu_{x,t}(\lambda) \in \text{Prob.}(\mathbf{R}^n),$$

such that

(1) (i) for any continuous function g,

$$w^* - lim\ g(U^\epsilon) = < \nu_{x,t}(\lambda), g(\lambda) > = \int_{\mathbf{R}^n} g(\lambda) d\nu_{x,t}(\lambda);$$

(2) (ii) $U^\epsilon(x,t) \to U(x,t)$ strongly if and only if $\nu_{x,t}$ is a Dirac mass

$$\nu_{x,t} = \delta_{U(x,t)},$$

for almost all (x,t);

(3) (iii) $\nu_{x,t}$ satisfies

$$(5.3.2) \qquad < \nu_{x,t}, \begin{vmatrix} \eta_1 & q_1 \\ \eta_2 & q_2 \end{vmatrix} >= \begin{vmatrix} < \nu_{x,t}, \eta_1 > & < \nu_{x,t}, q_1 > \\ < \nu_{x,t}, \eta_1 > & < \nu_{x,t}, q_2 > \end{vmatrix}, \qquad \text{a.e.}$$

provided that

$$(5.3.3) \qquad \eta_i(U^\epsilon)_t + q_i(U^\epsilon)_x \qquad \text{compact in } H^{-1}_{\text{loc}},$$

for continuous function pairs $(\eta_i, q_i), i = 1, 2$.

This theorem ensures the existence of the Young measures uniquely determined by the sequence of bounded functions. The second result indicates that the strong convergence of the sequence is equivalent to the one point structure of the support of the Young measures. Furthermore, the boundedness of U^ϵ (5.3.1) automatically ensures that

$$\eta_i(U^\epsilon)_t + q_i(U^\epsilon)_x \in H^{-1}_{\text{loc}},$$

for continuous function pairs $(\eta_i, q_i), i = 1, 2$. This theorem indicates that an extra condition of weak composite compactness for $\eta_i(U^\epsilon)_t + q_i(U^\epsilon)_x$ can give us very useful information for the Young measures. These theorems provide a framework by which one can prove strong convergence of the sequence $U^\epsilon(x,t)$ satisfying (5.3.1) and (5.3.2) by deducing

$$\nu_{x,t}(\lambda) = \delta_{w^*-lim U^\epsilon(x,t)}(\lambda)$$

from the functional equations (5.3.3) for some continuous function pairs.

The following compactness embedding theorems are useful for obtaining the condition (5.3.2) for conservation laws.

Theorem 5.3.2. *Let $1 < q \le p < r < \infty$. Then*

$$(\textit{compact set of } W^{-1,q}_{loc}) \cap (\textit{bounded set of } W^{-1,r}_{loc}) \subset (\textit{compact set of } W^{-1,p}_{loc}).$$

The proof of Theorem 5.3.2 can be found in [DCL, Ch2].

Theorem 5.3.3([Mu]). The embedding of the positive cone of $W^{-1,p}$ in $W^{-1,q}$ is completely continuous for $q < p$.

Theorem 5.3.2 indicates that compactness in $W^{-1,q}_{\text{loc}}$ coupled with boundedness in $W^{-1,r}_{\text{loc}}$ yields compactness in $W^{-1,p}_{\text{loc}}$. Theorem 5.3.3 says that the uniformly lower (or upper) bound in the dual sense in $W^{-1,p}$ of the sequence in $W^{-1,q}$ leads to compactness in $W^{-1,q}, q < p$.

5.4 The Dissipation Measures $\eta(U^\ell)_t + q(U^\ell)_x$

5.4.1 Parabolic Approximate Solution Sequences

Consider a sequence of viscosity approximate solutions $\{U^\epsilon\}_{\epsilon>0}$ to (5.1.2). Now suppose that η_* is a C^2 strictly convex entropy for (5.1.1), and that $U_0(x)$ tends to a constant state $\bar{U}$ as $|x| \to \infty$ and $U_0 - \bar{U} \in L^2 \cap L^\infty$. In the case of (2.5) we may choose η_* to be $u^2 + v^2$. Multiplying (5.1.2) by $\nabla\eta_*(U^\epsilon) - \nabla\eta_*(\bar{U})$, a standard integration by parts argument gives the estimate

$$\epsilon \int_0^\infty \int_{-\infty}^\infty |U^\epsilon{}_x|^2 \, dx \, dt \leq C,$$

where C depends on U_0.

Consider any C^2 entropy-entropy flux pair (η, q) for the system (5.1.1). In the case of (2.5), such pairs were constructed in Section 4. Multiplying (5.1.2) by $\nabla\eta$, and integrating by parts, we get, after using the L^∞ bound on U^ϵ, the boundedness of $\nabla^2\eta$ on compact sets and a standard application of Theorem 5.3.3 [Mu], a weak compactness estimate for the dissipation measure

$$\eta(U^\epsilon)_t + q(U^\epsilon)_x \in \ \text{compact set of } H^{-1}_{loc}.$$

By the L^∞ bound of U^ϵ, as $\epsilon \to 0^+$, U^ϵ converges in weak star topology in L^∞. Then, by Theorem 5.3.1 [Ta], there exists a family of probability measures $\nu_{(x,t)}$, the Young measures, that describes this weak convergence in the following way. For all continuous function g, we have

$$w * -lim_{\epsilon\to 0^+} g(U^\epsilon(x,t)) = \int g(\lambda) \, d\nu_{(x,t)}(\lambda) \triangleq \ < \nu_{(x,t)}, g > .$$

Let (η, q) and $(\bar{\eta}, \bar{q})$ be two pairs of C^2 entropy-entropy flux. They satisfy compactness conditions in H^{-1} as stated above. Then, by Theorem 5.3.1 again, we get the commutation relation:

$$\text{(C)} \qquad <\nu, \eta\bar{q} - \bar{\eta}q> = <\nu, \eta><\nu, \bar{q}> - <\nu, \bar{\eta}><\nu, q> .$$

Here we have dropped the subscript (x,t) on $\nu_{(x,t)}$.

5.4.2 Finite Difference Approximate Solution Sequences

Consider approximate sequences $\{U^\ell\}_{\ell>0}$ generated by either the Godunov or the Lax-Friedrichs schemes. Using Green's formula, we obtain that, for any $\phi \in C_0^1(\Pi_T)$,

$$\iint_{0\le t\le T=mh} (\eta(U^\ell)\phi_t + q(U^\ell)\phi_x)dx\ dt = M(\phi) + L(\phi) + S(\phi)\ , \tag{5.4.1}$$

where

$$M(\phi) = \int_{-\infty}^{\infty} \phi(x,T)\eta(U^\ell(x,T))dx - \int_{-\infty}^{\infty} \phi(x,0)\eta(U^\ell(x,0))dx\ , \tag{5.4.2}$$

$$S(\phi) = \int_0^T \sum\{\sigma[\eta] - [q]\}\ \phi(x(t),t)\ dt\ , \tag{5.4.3}$$

$$L(\phi) = L_1(\phi) + L_2(\phi)\ . \tag{5.4.4}$$

For the Lax-Friedrichs approximations, $n + j =$ even,

$$\begin{cases} L_1(\phi) = \sum_{j,n} \phi_j^n \int_{(j-1)\ell}^{(j+1)\ell} (\eta(U_-^{\ell_n}) - \eta(U_j^{\ell n}))dx\ , \\ L_2(\phi) = \sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} (\eta(U_-^{\ell_n}) - \eta(U_j^{\ell_n}))(\phi - \phi_j^n)dx\ . \end{cases} \tag{5.4.5}$$

For the Godunov approximations,

$$\begin{cases} L_1(\phi) = \sum_{j,n} \phi_j^n \int_{(j-1/2)\ell}^{(j+1/2)\ell} (\eta(U_-^{\ell_n}) - \eta(U_j^{\ell_n}))dx\ , \\ L_2(\phi) = \sum_{j,n} \int_{(j-1/2)\ell}^{(j+1/2)\ell} (\eta(U_-^{\ell_n}) - \eta(U_j^{\ell_n}))\ (\phi - \phi_j^n)\ dx\ , \end{cases} \tag{5.4.6}$$

where $U_{-}^{\ell_n} = U^{\ell}(x, nh-0)$, $\phi_j^n = \phi(j\ell, nh)$, the summation in $\sum$ is taken over all shock waves s in U^{ℓ} at fixed time t, and σ is the propagating speed of the shock wave. If $\mathbf{s} = (x(t), t)$, then $[\eta]$ and $[q]$ denote the jump of $\eta(U^{\ell}(x,t))$ and $q(U^{\ell}(x,t))$ across s from left to right respectively, namely,

$$\begin{cases} [\eta] = \eta\{U^{\ell}(x(t)+0,t)\} - \eta\{U^{\ell}(x(t)-0,t)\} \ , \\ [q] = q\{U^{\ell}(x(t)+0,t) - q\{U^{\ell}(x(t)-0,t)\} \ . \end{cases}$$

To arrive at (4.4), we first notice that U have compact support in the region Π_T and, therefore, we may substitute $\eta = \eta_*, q = q_*$ and $\phi \equiv 1$ in the equality (5.4.1). Then

$$\begin{aligned} &\sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} (\eta_*(U_{-}^n) - \eta_*(U_j^n))dx + \int_0^T \sum \{\sigma[\eta_*] - [q_*]\} \ dt \\ &\quad = \int_{-\infty}^{\infty} \eta_*(U^{\ell}(x,0))dx - \int \eta_*(U^{\ell}(x,T)) \ dx \\ &\quad \le \int_{-\infty}^{\infty} \eta_*(U_0(x)) \ dx \le C \ , \end{aligned} \tag{5.4.7}$$

while

$$\begin{aligned} &\sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} (\eta_*(U_{-}^n) - \eta_*(U_j^n))dx \\ &= \sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} dx \int_0^1 (1-\theta)(U_{-}^n - U_j^n)^{\mathsf{T}} \ \nabla^2 \eta_*(U_j^n + \theta(U_{-}^n - U_j^n))(U_{-}^n - U_j^n) d\theta \ . \end{aligned} \tag{5.4.8}$$

Notice that the entropy inequality $\sigma[\eta_*] - [q_*] \ge 0$ is satisfied across the shock waves and η_* is a convex entropy. We obtain from (5.4.7)-(5.4.8) that

$$\int_0^T \sum \{\sigma[\eta_*] - [q_*]\} \ dt \le C \ , \tag{5.4.9}$$

and

$$\sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} dx \int_0^1 (1-\theta)(U_{-}^n - U_j^n)^{\mathsf{T}} \nabla^2 \eta_*(U_j^n + \theta(U_{-}^n - U_j^n))(U_{-}^n - U_j^n) d\theta \le C \ . \tag{5.4.10}$$

In particular, since $\mathbf{r}^\top \cdot \nabla^2\eta_* \cdot \mathbf{r} \geq C_0(\mathbf{r},\mathbf{r})$, $C_0 > 0$ constant, we obtain

$$\sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} |U_-^n - U_j^n|^2 dx \leq C \,. \tag{5.4.11}$$

For any bounded set $\Omega \subset \Pi_T$ and weak entropy pair (η, q), we derive from (5.4.2)-(5.4.5), and (5.4.9)-(5.4.11) that

$$\begin{aligned}
&|M(\phi)| \leq C||\phi||_{C_0(\Omega)} \,, \\
&|S(\phi)| \leq ||\phi||_{C_0} \int_0^T \sum |\sigma[\eta] - [q]| dt \\
&\quad \leq C||\phi||_{C_0} \int_0^T \sum \{\sigma[\eta_*] - [q_*]\} \, dt \\
&\quad \leq C||\phi||_{C_0(\Omega)}, \\
&|L_1(\phi)| = |\sum_{j,n} \phi_j^n \int_{(j-1)\ell}^{(j+1)\ell} (\eta(U_-^n) - \eta(U_j^n)) dx| \\
&\quad \leq ||\phi||_{C_0} \sum_{j,n} | \int_{(j-1)\ell}^{(j+1)\ell} (\eta(U_-^n) - \eta(U_j^n)) dx| \\
&\quad \leq C||\phi||_{C_0} \sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} dx \int_0^1 (1-\theta)(U_-^n - U_j^n)^\top \nabla^2\eta_*(U_j^n + \theta(U_-^n - U_j^n))(U_-^n - U_j^n) \, d\theta \\
&\quad \leq C||\phi||_{C_0(\Omega)} \,.
\end{aligned}$$

Hence

$$|(M + L_1 + S)(\phi)| \leq C||\phi||_{C_0} \,;$$

that is,

$$||M + L_1 + S||_{C_0^*} \leq C \,.$$

Using the embedding theorem $C_0^*(\Omega) \overset{\text{compact}}{\hookrightarrow} W^{-1,q_0}(\Omega)$, $1 < q_0 < \frac{n}{n-1}$, we have

$$M + L_1 + S \quad \text{compact in } W^{-1,q_0}(\Omega) \,. \tag{5.4.12}$$

Furthermore, for any $\phi \in C_0^\alpha(\Omega)$, $\frac{1}{2} < \alpha < 1$, we have from (4.31) that

$$
\begin{aligned}
|L_2(\phi)| &\le \sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} |\phi(x,nh) - \phi_j^n| \, |\eta(U_-^n) - \eta(U_j^n)| \, dx \\
&\le \ell^\alpha ||\phi||_{C_0^\alpha} \sum_n (\sum_j \int_{(j-1)\ell}^{(j+1)\ell} |\eta(U_-^n) - \eta(U_j^n)|^2 dx \;)^{1/2} \\
&\le ||\nabla \eta||_{L^\infty} \ell^{\alpha - 1/2} ||\phi||_{C_0^\alpha} (\sum_{j,n} \int_{(j-1)\ell}^{(j+1)\ell} |U_-^n - U_j^n|^2 dx \;)^{1/2} \\
&\le C\ell^{\alpha-1/2} ||\phi||_{C_0^\alpha(\Omega)} \\
&\le C\ell^{\alpha-1/2} ||\phi||_{W_0^{1,p}(\Omega)}, \quad p > \frac{n}{1-\alpha};
\end{aligned}
$$

that is,

(5.4.13)

$$
||L_2||_{W^{-1,q_0}(\Omega)} \le C\ell^{\alpha-1/2} \longrightarrow 0, \quad (\ell \to 0), \quad 1 < q_0 < \frac{n}{n-1+\alpha} < \frac{n}{n-1}.
$$

We obtain from (5.4.12) and (5.4.13) that

$$
M + L + S \quad \text{compact in} \quad W^{-1,q_0}(\Omega),
$$

with the aid of Theorem 5.3.2 and Theorem 5.3.3. Therefore,

$$
\eta(U^l)_t + q(U^l)_x \in \text{ compact set of } H_{loc}^{-1}. \tag{5.4.14}
$$

Similar to Section 5.4.1, we can obtain the commutation relation (C) from (5.4.14).

6. Compactness Framework for Approximate Solutions

Now we establish a structure framework for the Young measures, which are uniquely determined by the approximate solutions to the system (1.1) with an isolated umbilic point $P = (\bar{w}_1, \bar{w}_2)$:

$$
\lambda_1(P) = \lambda_2(P),
$$

in the Riemann coordinates. Then we conclude a corresponding compactness framework for approximate solution sequences to the system (1.1).

First, we list some basic assumptions on the structure of (1.1) for the framework theorems that we will state.

Hypothesis on the Hyperbolic System (1.1) for Theorem 6.1.

$(H1)\quad \dfrac{\lambda_{i w_j}}{\lambda_2-\lambda_1} = \dfrac{\mathcal{A}_i(w_1,w_2)}{w_2-w_1},$

where $i \neq j,\ i,j = 1,2,$ and $\mathcal{A}_i$ is real analytic in (w_1, w_2) except at $(0,0)$;

$(H2)\quad \displaystyle\sum_{i,j=1}^{2} (\|\mathcal{A}_i\|_{L^\infty(\Omega)} + \|w_j{}^2\partial^2_{w_j}\mathcal{A}_i\|_{L^\infty(\Omega)}) < +\infty,$

where Ω is any compact set containing the umbilic point P.

Theorem 6.1. *Assume that (1.1) satisfies (H1) and (H2). Suppose that a Young measure $\nu \in Prob.(\mathbf{R}^2)$ satisfies*

$$\text{supp}\ \nu(w_1,w_2) \subset [w_1^-, w_1^+] \times [w_2^-, w_2^+], \tag{6.1}$$

$$<\nu, \eta_1 q_2 - \eta_2 q_1> = <\nu,\eta_1><\nu,q_2> - <\nu,\eta_2><\nu,q_1>, \tag{6.2}$$

for all C^2 Goursat entropy-entropy pairs $(\eta_i, q_i), i = 1,2,$ as constructed in Section 4 for the systems (1.1) with an isolated umbilic point P. Then

$$\text{supp}\ \nu(w_1,w_2) \subset \{\ (w_1,w_2)\ |\ \lambda_{i w_i} = 0,\ i = 1,2\ \}.$$

Remark. For systems with more than one isolated umbilic points, a variant of Theorem 6.1 will be provided in [CK2].

Combining Theorem 6.1 with Theorem 5.3.1, we have

Theorem 6.2. *Suppose that $U^\epsilon(x,t)$ are measurable functions satisfying*

$$\|U^\epsilon\|_{L^\infty} \leq M < +\infty, \tag{6.3}$$

$$\eta(U^\epsilon)_t + q(U^\epsilon)_x \qquad \text{compact in } H^{-1}_{loc}, \tag{6.4}$$

for any C^2 Goursat entropy-entropy pair (η, q) to the systems (1.1) with an isolated umbilic point. Suppose that (1.1) satisfies (H1) and (H2) and that $\lambda_{i w_i} \neq 0,\ i = 1,2.$ Then

$$(w_1(U^\epsilon), w_2(U^\epsilon)) \longrightarrow (w_1(x,t), w_2(x,t)), \quad \text{a.e.} \tag{6.5}$$

where $(w_1(U), w_2(U))$ are the Riemann invariants.

Remark. Theorem 6.1 and Theorem 6.2 provide a framework for the compactness of the sequence of the Riemann invariants, determined by the sequence of approximate solutions satisfying the conditions (6.3) and (6.4).

We sketch the ideas in the proof of Theorem 6.1. Without loss of generality, we assume that the umbilic point $P = (0,0)$. Suppose that R is the minimal rectangle in (w_1, w_2) space containing the support of ν. There are two cases:

1. R does not contain the umbilic point $(0,0)$;
2. R contains the umbilic point $(0,0)$.

We will treat Case 2. Case 1 is similar and less complicated. We also assume that R is not a line segment parallel to any axis (this case can be dealt with using the method shown below), thus, $R = [w_1^-, 0] \times [0, w_2^+]$ for some $w_1^- < 0 < w_2^+$.

Let $\delta > 0$ denotes the constant used in the construction of entropies in Section 4. Assume that it is chosen a priori to satisfy $w_1^- < -\delta < 0$.

The proof will rely on the following Propositions and Lemmas.

Proposition 6.3. *Consider the entropy function constructed in Proposition 4.5.2. Assume that $w_1^* \in (w_1^-, -\delta)$ is a fixed constant and that the Goursat data θ is supported either on the interval (w_1^-, w_1^*) (west type entropy with limit w_1^*) or $(w_1^*, -\delta)$ (east type entropy with limit w_1^*). Then the entropy η and its flux q admit integral representations of the form*

$$\eta(w_1, w_2) = I(w_1, w_2)\theta(w_1) + \int_{w_1^*}^{w_1} J(w_1, w_2; y)\, dy,$$

$$q(w_1, w_2) = K(w_1, w_2)\theta(w_1) + \int_{w_1^*}^{w_1} L(w_1, w_2; y)\, dy,$$

where I, J, K, L are smooth as long as $y < 0$ and $w_1 < 0$. Moreover, $I \geq 1$.

Proposition 6.4. *Let (η, q) be of type east with limit $\alpha - \epsilon'$ and let $(\bar{\eta}, \bar{q})$ be of type west with limit $\alpha + \epsilon'$. Suppose that for all $\epsilon' > 0$ sufficiently small, $< \nu, \eta\bar{q} - \bar{\eta}q >= 0$. Then*

$$supp\ \nu \subset \{\ (w_1, w_2) \mid \frac{\partial \lambda_1}{\partial w_1}(\alpha, w_2) = 0\ \}.$$

If we assume also that $\frac{\partial \lambda_1}{\partial w_1}(\alpha, w_2)$ does not vanish. Then supp $\nu \cap \{(w_1, w_2) | w_2 \geq 0, w_1 = \alpha\}$ is empty.

The proof of Proposition 6.4 consists of a construction of the trace of ν on the line $w_1 = \alpha$. The argument is quite standard and we refer the reader to [Se].

For the sake of simplicity, we assume from now on that $\frac{\partial \lambda_i}{\partial w_i} \neq 0$, $i = 1, 2$, is satisfied. The general case is only slightly more tedious. With this assumption, ν can be reduced to a point mass in the Riemann plane below.

Next, using Proposition 6.3 and some properties of the kernel functions I, J, K, L, one can show that

Proposition 6.5. *Fix $\delta > 0$. Let w_1^* satisfy $w_1^- < w_1^* < -\delta < 0$. If for all east type entropy η with limit w_1^* we have $< \nu, \eta >= 0$ then supp $\nu \cap \{(w_1, w_2) | w_2 \geq 0, w_1^* \leq w_1 \leq -\delta\}$ is empty.*

Now, we will make use of Propositions 6.3 – 6.5 to reduce ν to a point mass. The reduction process will take several steps. First, we show that the support of ν must concentrate only at the four corners of R, i.e., ν is the sum of four delta functions. Then we further reduce the number of delta functions from four to one.

It turns out that after some detailed analysis, Propositions 6.4 and 6.5 allow us to concentrate the support of ν on the extreme left vertical edge of R and an arbitrarily narrow vertical strip containing the umbilic point.

Proposition 6.6. *supp $\nu \subseteq \{(w_1, w_2) | 0 \leq w_2 \leq w_2^+,$ and $w_1 = w_1^-$ or $-\delta \leq w_1 \leq 0\}$.*

By using entropies of type north and south and by performing a similar reduction process on horizontal line segments interior to the rectangle R, we conclude that $supp\nu$ is concentrated on a horizontal strip of width δ and on the line $w_2 = w_2^+$. Due to the complete similarity of the proof, we omit this argument but summarize the result as follows:

Proposition 6.7. *supp* $\nu \subseteq \{(w_1, w_2) | 0 \leq w_1 \leq w_1^-,$ *and* $w_2 = w_2^+$ *or* $\delta \geq w_2 \geq 0\}$.

Proposition 6.6 and 6.7 imply that the support of ν is in fact concentrated on the square $\{(w_1, w_2)|0 \leq w_2 \leq \delta, -\delta \leq w_1 \leq 0\}$, and the points $(w_1^-, 0), (0, w_2^+)$, and (w_1^-, w_2^+). Now, $\delta > 0$ is small but arbitrarily fixed. Let $\delta \to 0$. We obtain

Corollary 6.8. *The support of* ν *is concentrated at the four corners of* R, $(0,0)$, $(0, w_2^+)$, $(w_1^-, 0)$, *and* (w_1^-, w_2^+).

Knowing that the Young measure ν is a sum of four delta functions at the corners of R, we reduce ν further. Let A_i, $i = 1,2,3,4$, denote the corners of R, $A_1 = (0, w_2^+), A_2 = (0,0), A_3 = (w_1^-, 0), A_4 = (w_1^-, w_2^+)$.

By Corollary 6.8,

$$\nu = \sum_{1 \leq i \leq 4} \beta_i \delta_{A_i},$$

where

$$\beta_i \geq 0, \quad i = 1,2,3,4, \quad \text{and} \sum_{1 \leq i \leq 4} \beta_i = 1.$$

To reduce ν further, we apply an variant of Theorem 6.1 in [Se].

Proposition 6.9. *Suppose that* $\beta_i > 0$ *for all* $i = 1,2,3,4$. *Then*

$$\partial_{w_1} \lambda_1 = 0, \text{ on the line segments } A_1A_4,\ A_2A_3;$$

$$\partial_{w_2} \lambda_2 = 0, \text{ on the line segments } A_1A_2,\ A_3A_4.$$

Now, by assumption, $\partial_{w_1} \lambda_1$ and $\partial_{w_2} \lambda_2$ are nonzero everywhere. This contradicts the conclusion of the above Proposition. Thus we have

Lemma 6.10. *There exists an i, $1 \leq i \leq 4$, such that $\beta_i = 0$.*

Therefore, ν is only a sum of at most three Dirac masses. Further reduction of ν can be achieved by combining polynomial entropies (Proposition 3.1) and Goursat entropies. Finally, we conclude that the Young measure ν is a point mass on the (w_1, w_2)-plane.

This completes the sketch of the proof of Theorem 6.1.

7. Applications of the Compactness Framework

In this section, we apply the compactness framework theorem to the quadratic flux system in region IV ($\Delta > 0$). We prove the strong convergence of approximate solution sequences constructed by the viscosity method, the Godunov scheme, and the Lax Friedrichs scheme to the weak entropy solutions of (2.5) with large initial data for all positive time. As a corollary, we obtain the global existence of weak solutions to the Cauchy problem of (2.5) with large data in L^∞.

We first verify L^∞ apriori estimates for these approximate solution sequences. We then apply the compactness framework (Theorem 6.2) together with Theorems 4.5.2, 4.5.3, and 4.5.6 on the existence of regular entropies. To this end, the method of invariant regions will be used for the viscosity approximate sequence. In the case of the finite difference approximations, it is necessary to establish such apriori estimates for general Riemann problem solutions to (2.5) which are the building blocks of these difference schemes. We remark that the Riemann problem for (2.5) has not been solved by purely analytical methods before. Here we use the viscosity method to obtain existence of solutions to the Riemann problem on arbitrary compact sets in the (x,t)-plane. At the same time, we obtain the desired L^∞ apriori estimates.

7.1 Convergence of the Viscosity Method

Consider the Cauchy problem

$$
\begin{aligned}
&\partial_t U + \partial_x(dC(U)) = 0,\\
&U(x,0) = U_0(x),\\
&C(U) = \frac{1}{2}(\frac{1}{3}au^3 + bu^2v + uv^2),\\
&\Delta > 0.
\end{aligned} \tag{7.1}
$$

Using the convexity properties of the $\mathbf{R}_j$ curves or the explicit form of w_j, we obtain

Proposition 7.1.1. *Consider the quadratic flux system* (2.5). *Assume that we are in region IV, i.e.,* $\Delta > 0$. *Then for any pair of constants,* $c_1 < 0 < c_2$, $\Omega_{c_1c_2}$ *and* $\Omega_{c_1c_2} \cap \mathcal{I}_k$, *with* $\mathcal{I}_k$ *as defined in Proposition 5.1.1, are convex invariant regions for the parabolic systems with viscosity associated with* (2.5).

Using Proposition 7.1.1, we obtain an L^∞ apriori bound for viscosity approximate solutions U^ϵ.

Proposition 7.1.2. *Consider the Cauchy problem* (7.1) *with Cauchy data* $U_0 \in \mathcal{I}_k$, *and its associated viscosity approximation* $\{U^\epsilon\}_{\epsilon>0}$. *Suppose that there is a constant* $\bar{U}$ *such that* $U_0 - \bar{U} \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$. *Then, for any* $\epsilon > 0$, $U^\epsilon(x,t)$ *is well-defined for all* (x,t) *and moreover,* $|U^\epsilon(x,t)| \leq ||U_0||_{L^\infty}$.

Combining Proposition 7.1.2, Theorem 6.2, and Theorem 4.5.6, we obtain

Theorem 7.1.3. *Consider the Cauchy problem* (7.1) *with Cauchy data* $U_0 \in \mathcal{I}_k$, *and its associated viscosity approximation* $\{U^\epsilon\}_{\epsilon>0}$. *Suppose that* $\Delta > 0$. *Suppose that there is a constant* $\bar{U}$ *such that* $U_0 - \bar{U} \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$. *Then, as* $\epsilon \to 0^+$, *there exists a subsequence of* $U^\epsilon(x,t)$ *that converges a.e.* (x,t) *to a global weak solution of the hyperbolic system* (7.1).

7.2 Convergence of Finite Difference Schemes

The Godunov and the Lax-Friedrichs schemes use the solutions of the Riemann problem as their building blocks. In the case of the quadratic flux system (2.5), explicit constructions of the Riemann problem solution prove to be extremely complicated. And the Riemann problem has never been solved before by pure analytical methods. In order to prove the convergence of these finite difference schemes for the quadratic flux system with arbitrary data in L^∞, we first establish the existence of solutions for Riemann solutions and their L^∞ apriori bounds. To this end, we apply the viscosity method studied in Subsection 7.1.

Proposition 7.2.1. *Consider the Riemann problem for (2.5):*

$$\begin{aligned} &\partial_t U + \partial_x(dC(U)) = 0, \\ (7.2) \qquad &C(U) = \frac{1}{2}(\frac{1}{3}au^3 + bu^2v + uv^2), \qquad a \neq 1 + b^2, \end{aligned}$$

with

$$(7.3) \qquad U(0,x) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0. \end{cases}$$

Here, $U_L, U_R \in \mathcal{I}_k$ are constants. Suppose that $\Delta > 0$. Let $\Pi \subset \{ (x,t) \mid x \in \mathbf{R},\ t \geq 0 \}$ be a compact set. Then there exists a weak solution to the Riemann problem (7.2) *on the domain* Π.

The proof of Proposition 7.2.1 makes use of the finiteness of propagation speed for solutions of (7.2) and (7.3) and of Theorem 7.1.3. The invariant regions for the viscosity approximate solutions are convex. Thus, the L^∞ bound that the Riemann solutions inherit from the viscosity method are preserved during the averaging processes in the intermediate steps in the finite difference approximations.

Proposition 7.2.2. *Consider the Cauchy problem* (7.1) *with Cauchy data $U_0 \in \mathcal{I}_k$ for the quadratic flux system. Assume that $U_0 \in L^\infty$. Let $\{U^\ell\}_{\ell>0}$ be a sequence of approximate solutions to* (7.1) *generated* *by the Lax-Friedrichs scheme or the Godunov scheme. Then* $\{U^\ell\}_{\ell>0}$ *is well-defined and* $|U^\ell(x,t)| \leq ||U_0||_{L^\infty}$.

We obtain, after combining the above propositions with Theorem 4.5.6 and the analysis in Section 6,

Theorem 7.2.3. *Consider the Cauchy problem* (7.1) *for the quadratic flux system with Cauchy data* $U_0 \in \mathcal{I}_k$. *Assume that* $U_0 \in L^\infty$. *Let* $\{U^\ell\}_{\ell>0}$ *be a sequence of approximate solutions to* (7.1) *generated by the Lax-Friedrichs scheme or the Godunov scheme. Then, as* $\ell \to 0^+$, *there exists a subsequence of* $\{U^\ell\}_{\ell>0}$ *that converges to a global weak solution of* (7.1) *a.e. in* (x,t).

Remark. Our method yields the same convergence and existence theorems for another nonstrictly hyperbolic system (a special case of such system was studied in [Rb]): (1.1) with $F = (au^2+\varphi(v), uv)^\top, a \geq 1$, for all even regular, superlinear function $\varphi(v)$ satisfying $\varphi'(v)v > 0$ and $\varphi''(v)v > 0$ when $v \neq 0$.

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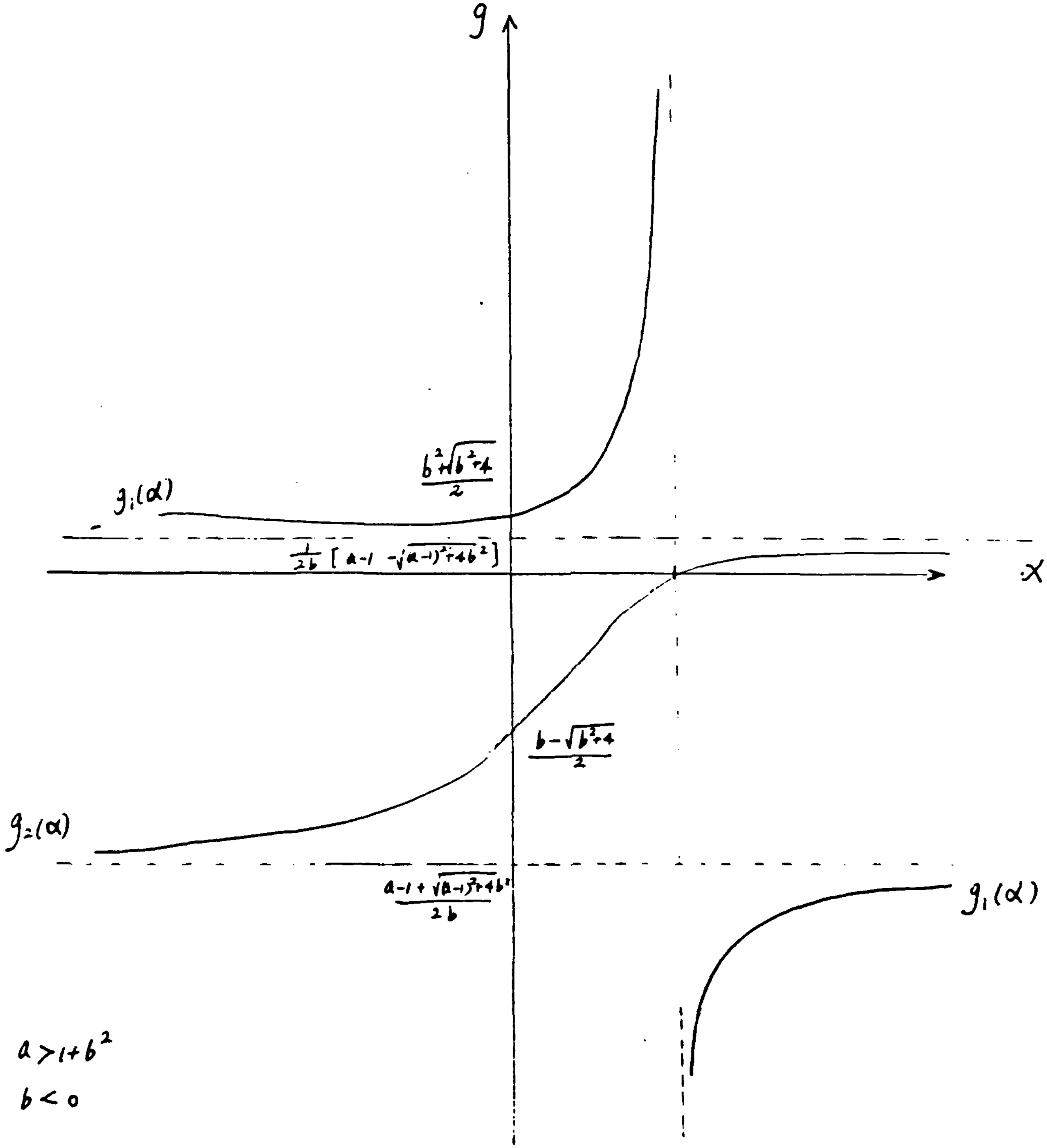

g
$g_1(\alpha)$
$\frac{b^2+\sqrt{b^2+4}}{2}$
$\frac{1}{2b}[a-1-\sqrt{(a-1)^2+4b^2}]$
x
$\frac{b-\sqrt{b^2+4}}{2}$
$g_2(\alpha)$
$\frac{a-1+\sqrt{(a-1)^2+4b^2}}{2b}$
$g_1(\alpha)$
$a > 1+b^2$
$b < 0$

FIGURE 2.1

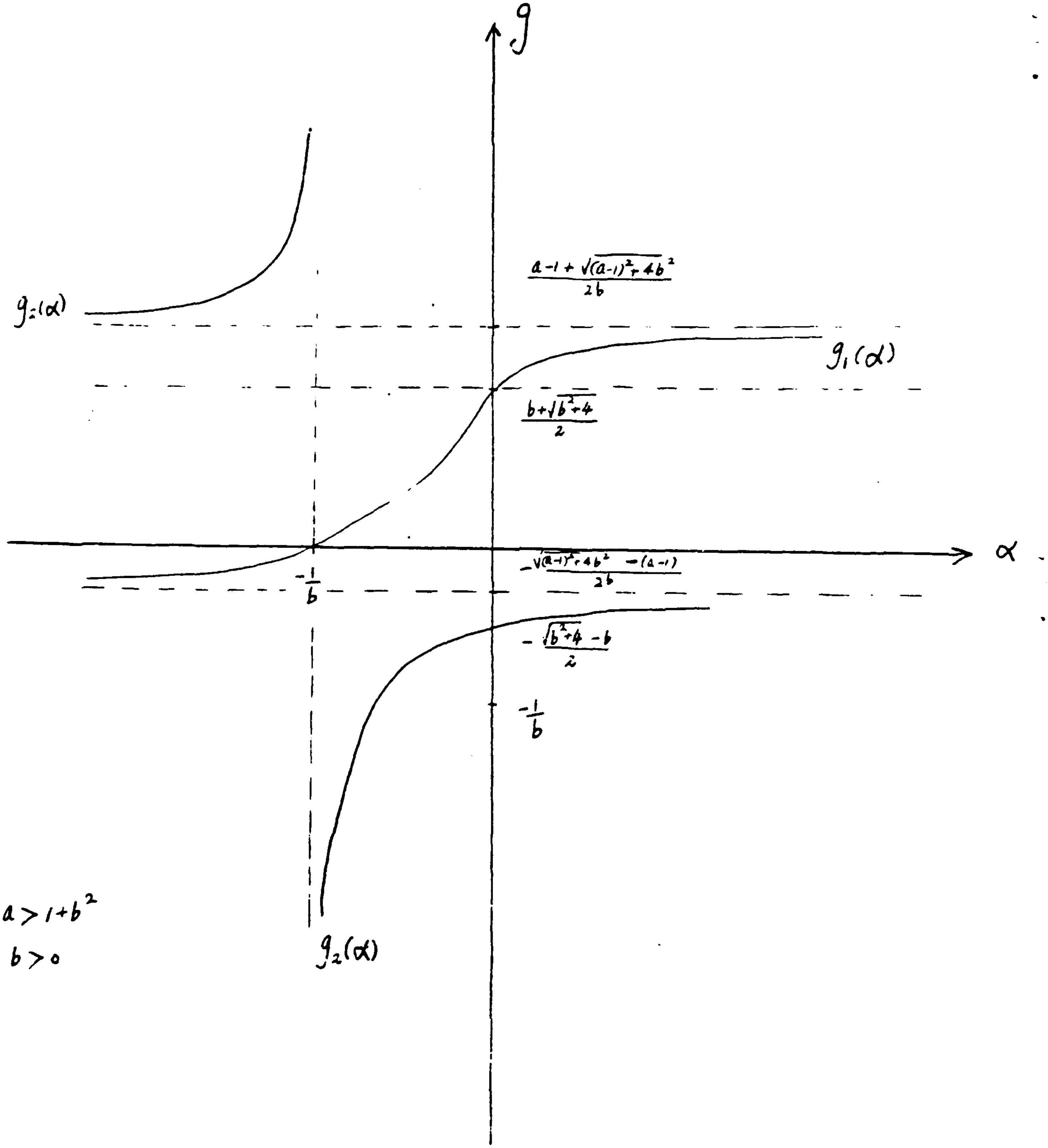
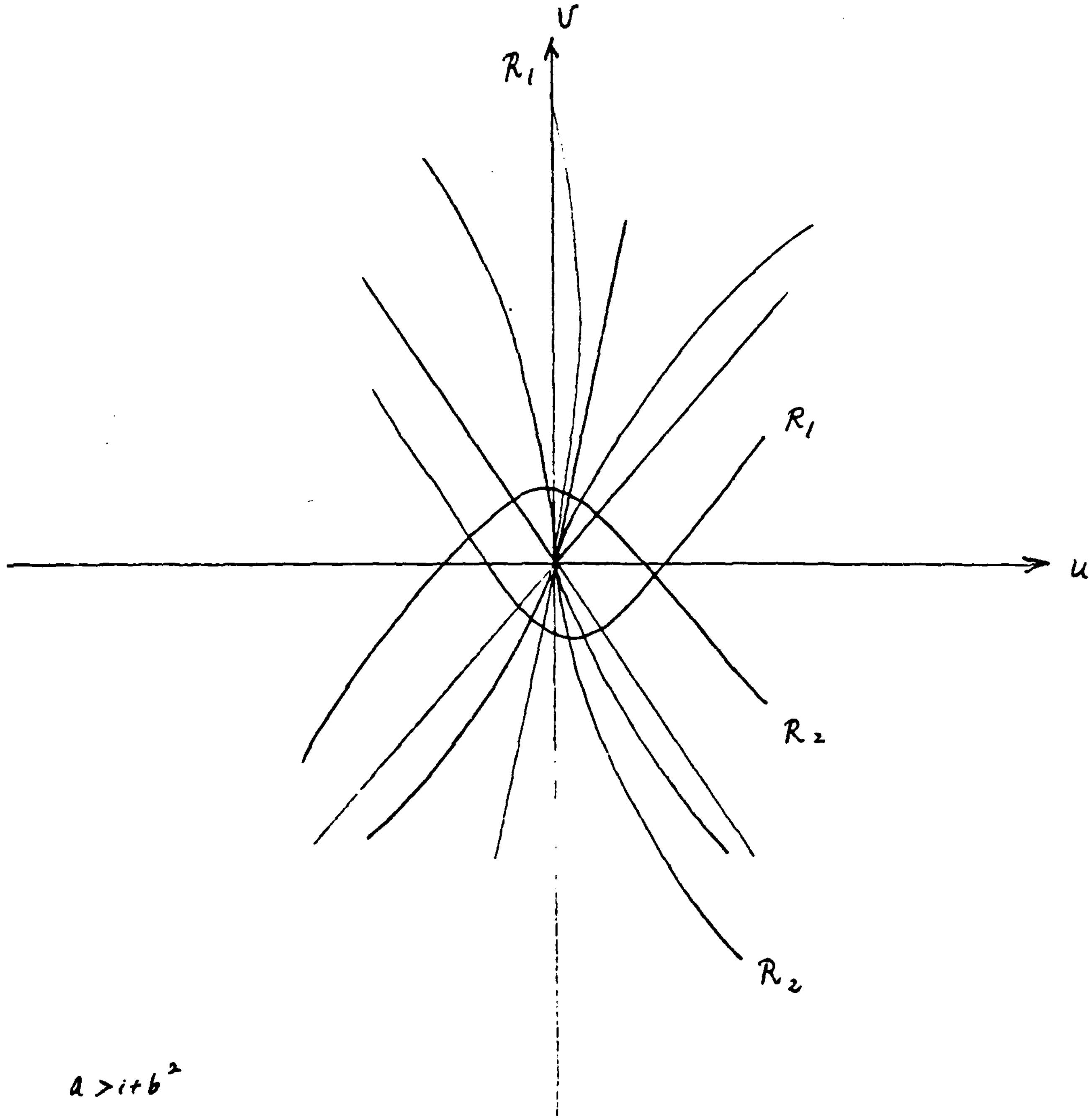


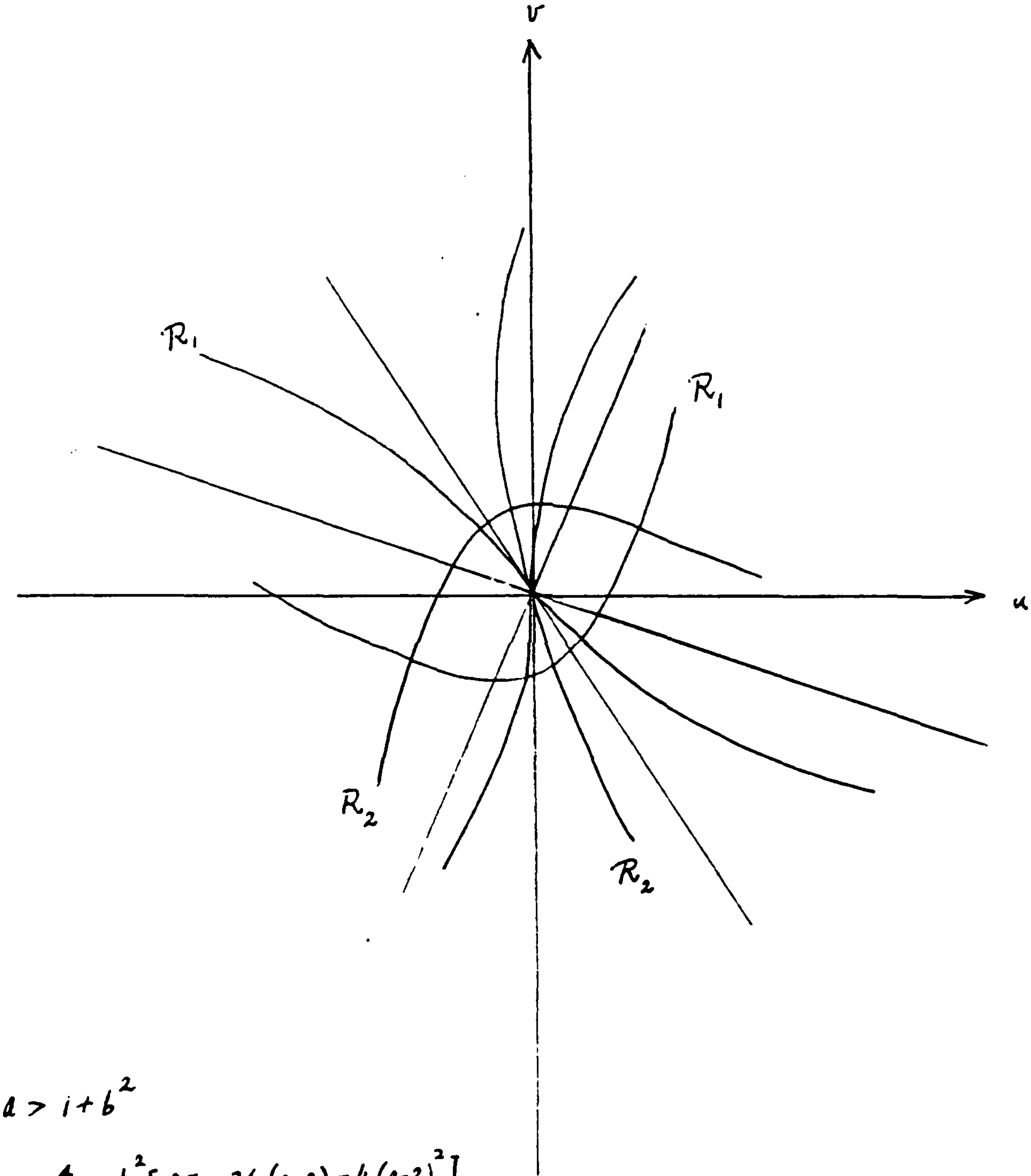
FIGURE 2.1 cont'd



$a > 1 + b^2$

$$-32b^4 + b^2[27 + 36(a-2) - 4(a-2)^3] + 4(a-2)^3 < 0$$

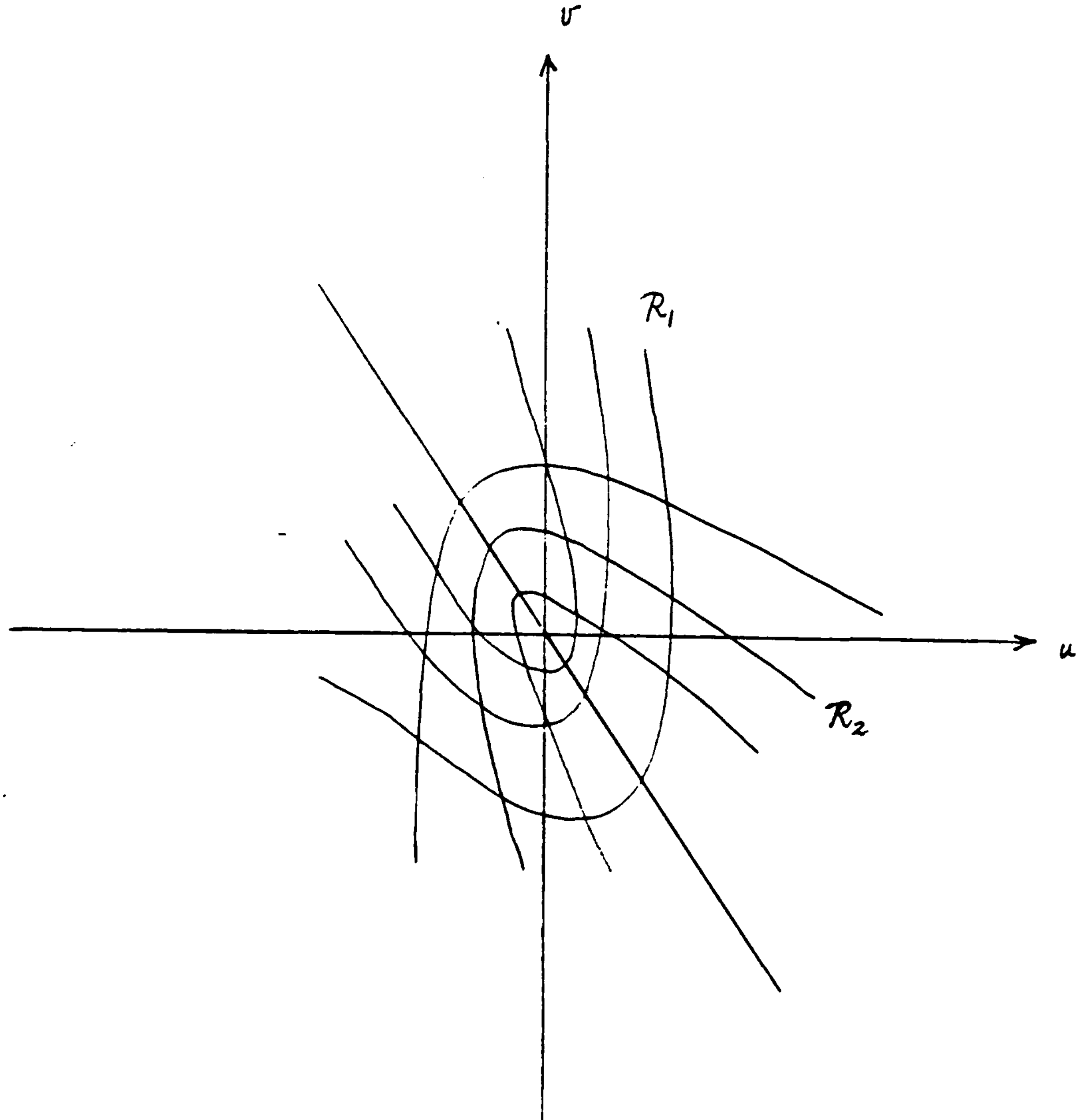
FIGURE 2.2



$a > 1 + b^2$

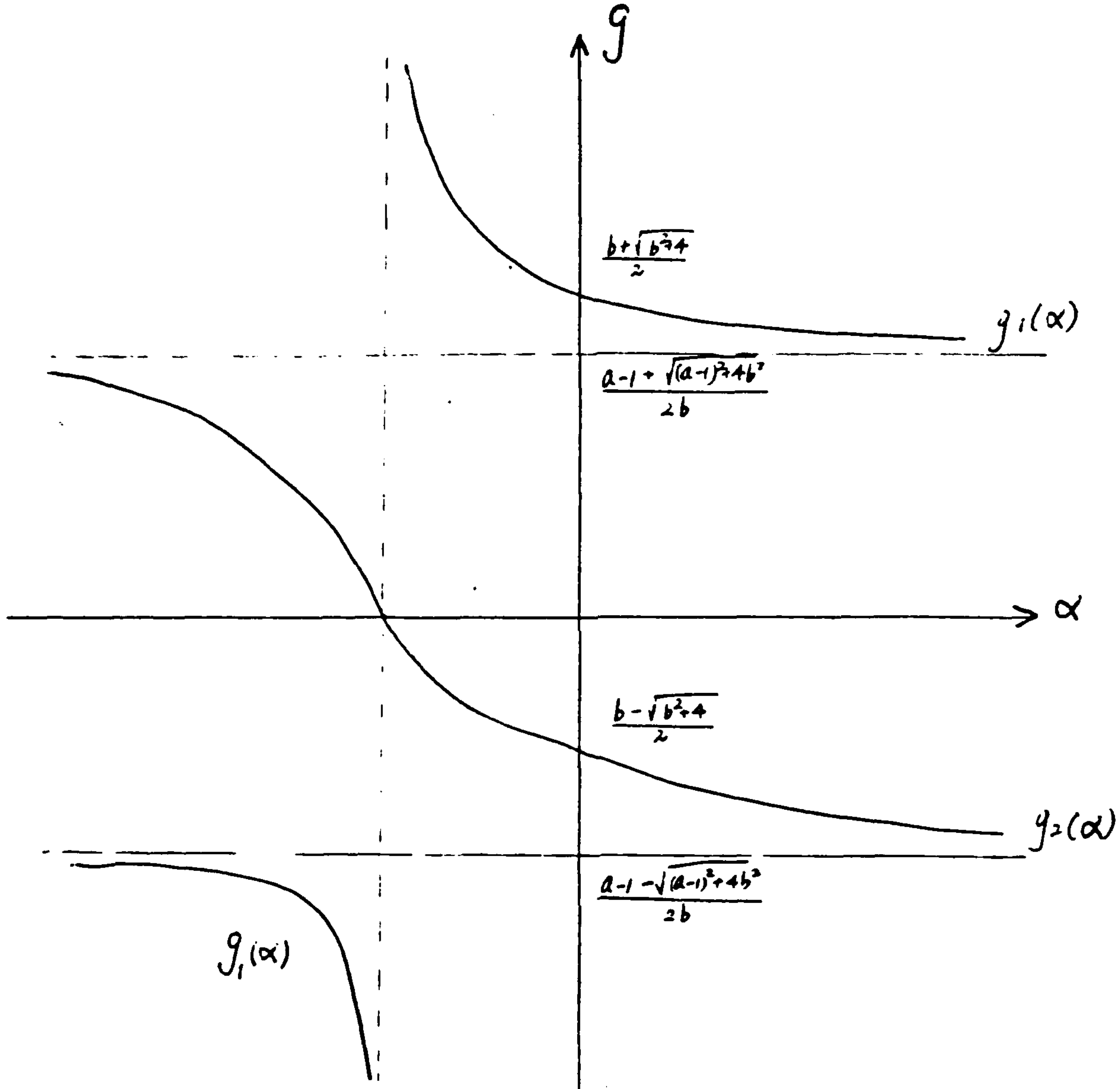
$-32\,b^4 + b^2[27 + 36(a-2) - 4(a-2)^2]$
$+ 4(a-2)^3 < 0$

FIGURE 2.3



$$-32b^4 + b^2[27 + 36(a-2) - 4(a-2)^2] + 4(a-2)^3 > 0$$

FIGURE 2.4



$a < 1+b^2, \quad b > 0$

FIGURE 2.5

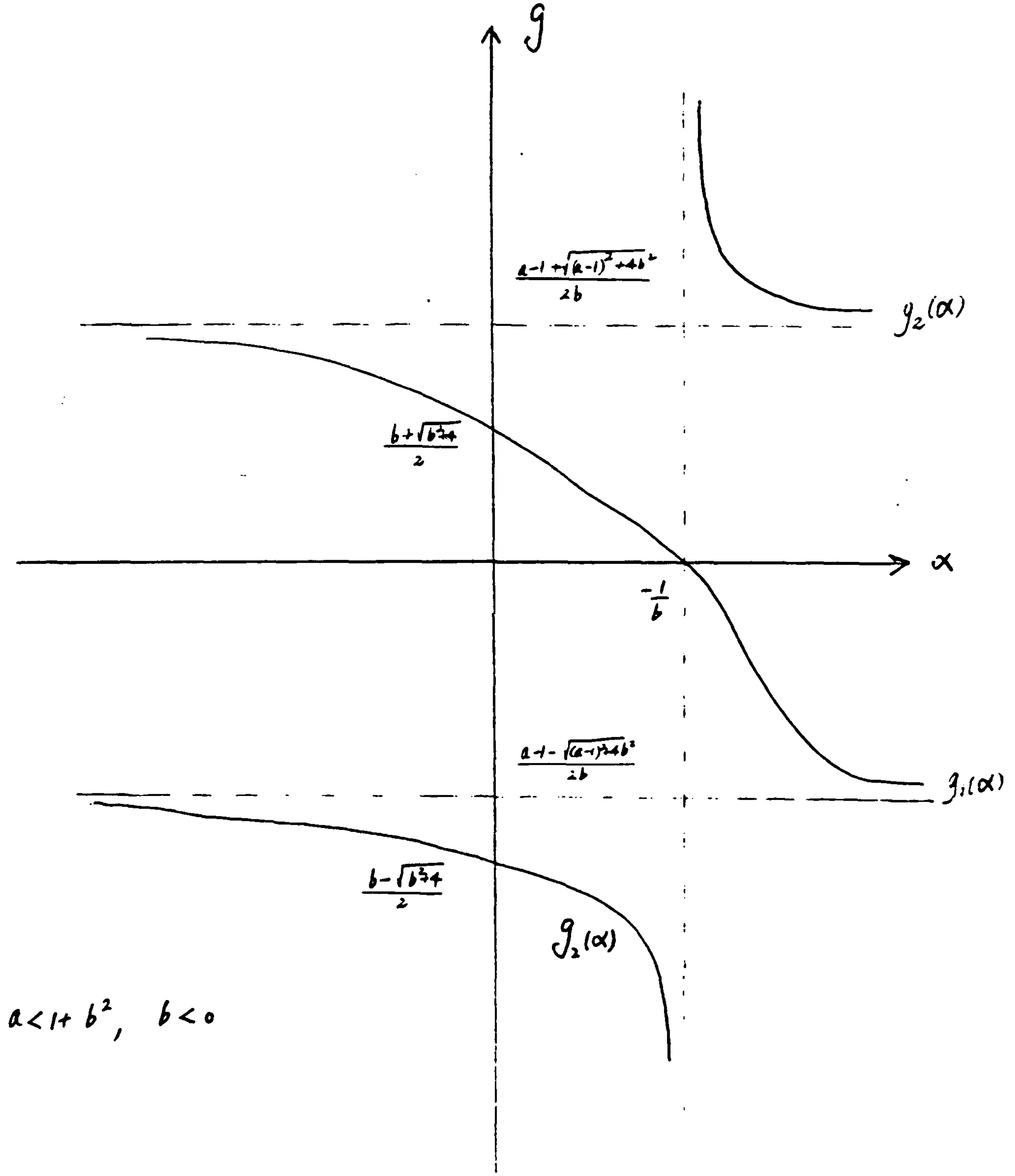
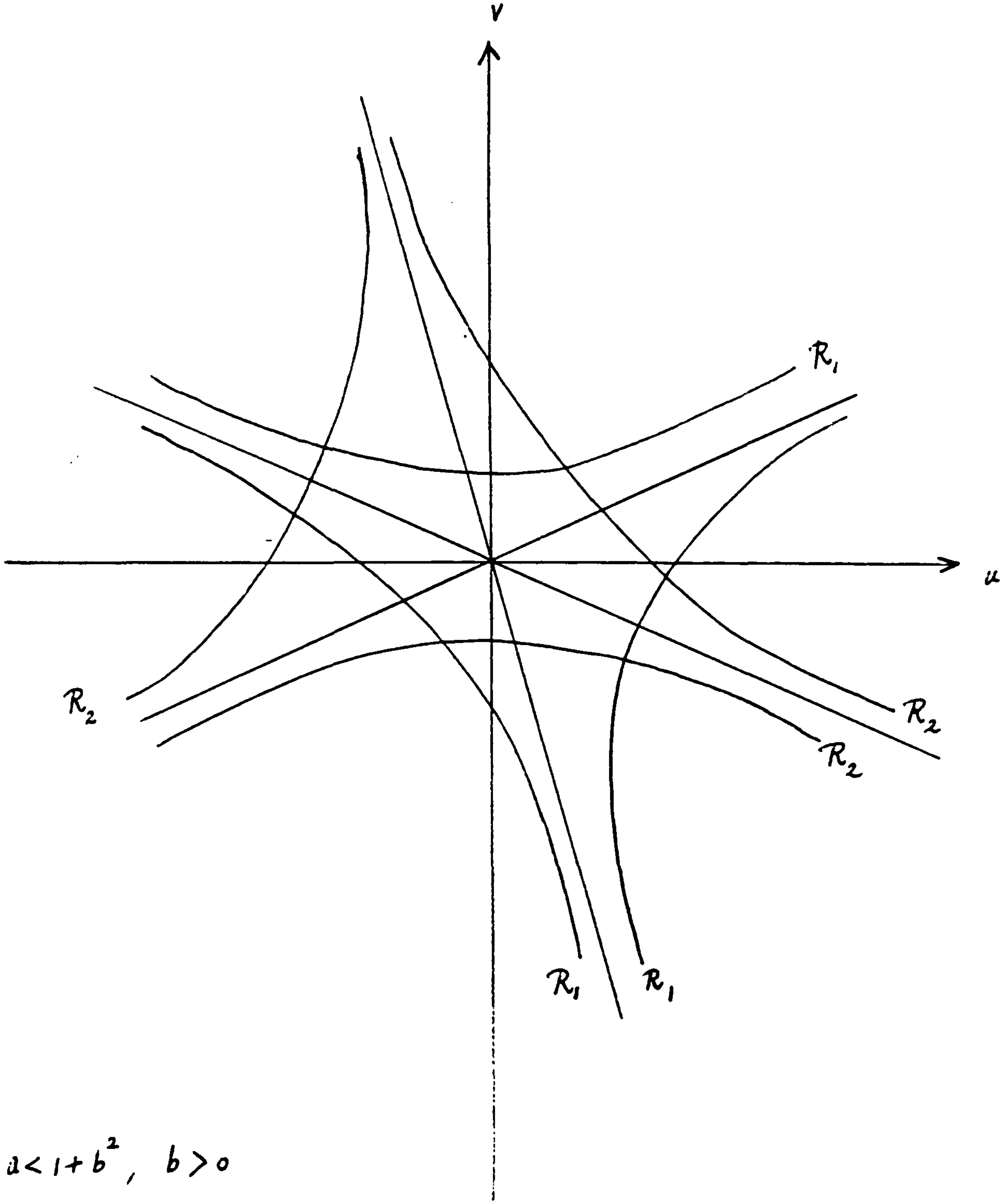


FIGURE 2.5 cont'd



v
u
$\mathcal{R}_1$
$\mathcal{R}_2$
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$\mathcal{R}_2$
$\mathcal{R}_1$
$\mathcal{R}_1$
$a < 1 + b^2, \ b > 0$

FIGURE 2.6

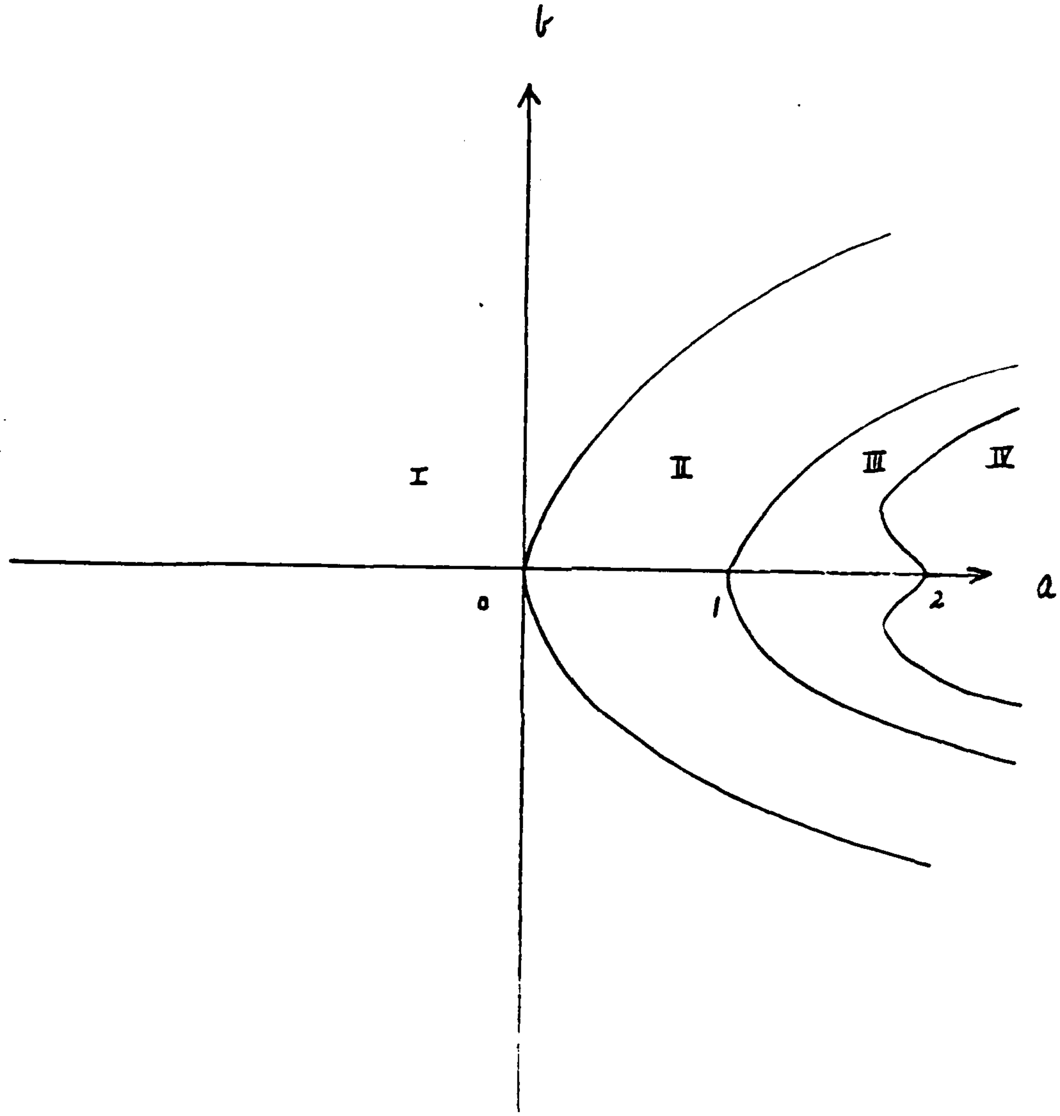

FIGURE 2.7

Boundary Curves	Qualitative Changes in (2.5) As One Crosses the Boundary
$a = \frac{3}{4} b^2$	Global Change in the Loci of Loss of Genuine Nonlinearity
$a = 1 + b^2$	Global Change in Wave Curves Geometry
Δ	Global Change in Wave Curves Geometry, Change in Multi-Valued-Ness of Riemann Invariants

TABLE 2.1