

Naval Research Laboratory

Washington, DC 20375-5320

AD-A254 735



2

NRL/MR/6386-92-7104

Fluid-Structure Interaction Using Retarded Potential and ABAQUS

C. T. DYKA

*Geo-Centers, Inc.
Fort Washington, MD 20744*

and

M. A. TAMM

*Computer Operations and Communications Branch
Research Computation Division*

August 19, 1992

1708
251950

DTIC
ELECTE
SEP 02 1992
S B D

02 8 31 052

Approved for public release; distribution unlimited.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503				
1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE August 19, 1992	3. REPORT TYPE AND DATES COVERED Interim		
4. TITLE AND SUBTITLE Fluid-Structure Interaction Using Retarded Potential and ABAQUS		5. FUNDING NUMBERS 63569N 6604		
6. AUTHOR(S) C. T. Dyka* and M. A. Tamm				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Geo-Centers, Inc. 10903 Indian Head Highway Fort Washington, MD 20744		8. PERFORMING ORGANIZATION REPORT NUMBER NRL/MR/6386-92-7104		
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) DARPA Arlington, VA 22209		10. SPONSORING / MONITORING AGENCY REPORT NUMBER		
11. SUPPLEMENTARY NOTES *Geo-Centers, Inc., Fort Washington, MD 20744				
12a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution unlimited.		12b. DISTRIBUTION CODE		
13. ABSTRACT (Maximum 200 words) The transient analysis of coupled fluid-structure systems is a very challenging and important area of particular concern to the naval community. However, the general state of affairs in this class of problems is far from fully satisfactory. In this paper, a brief discussion concerning first transient structural analysis and then fluid-structure interaction is undertaken. Following this, the nonlinear finite element method (FEM) program ABAQUS is used to model the structure, while the fluid realm is discretized with boundary elements (BEM) employing the retarded potential (RP) formulation. The RP method has not been extensively applied in the past to such classes of problems due primarily to storage requirements. However, the RP approach does offer important advantages over both the use of FEM for the fluid or the widely used Doubly Asymptotic Approximation (DAA), which is also a boundary element method. Discussion concerning time stepping (including automatic considerations), modeling and applications, accuracy, storage, stability, and the overall efficiency of this coupled technique in a MICRO-VAX environment is included. We conclude that the RP method is quite viable for the fluid and that the coupling to ABAQUS to form ABAQUS-RP is a very promising tool for transient, nonlinear, fluid-structure interaction.				
14. SUBJECT TERMS Fluid-structure interaction Retarded potential Coupled problems Finite element Boundary element Staggered solutions Transient			15. NUMBER OF PAGES 18	
			16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL	

CONTENTS

INTRODUCTION	1
FORMULATION	3
IMPLEMENTATION	4
APPLICATION	7
SUMMARY AND FUTURE DIRECTIONS	8
REFERENCES	13

DTIC QUALITY INSPECTED 3

Accession For	
NTIS GRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	
By _____	
Distribution/	
Availability Codes	
DTIC	Avail and/or Special
A-1	

FLUID-STRUCTURE INTERACTION USING RETARDED POTENTIAL AND ABAQUS

INTRODUCTION

Transient analysis of coupled systems is in general a difficult and demanding problem. Even with current levels of computing, a good deal of improvement is still required to efficiently analyze many real world problems. In this paper, we will be concerned with coupled, fluid-structure interaction problems, in particular, with submerged structures subjected to weak shock waves. We begin first with a brief review of the transient analysis of structures using finite element methods (FEM). This is followed by a discussion of the merits of various approaches for discretizing the fluid media, including boundary element methods (BEM) and the retarded potential (RP) approach.

Transient analysis of linear and nonlinear structures using the finite element method has developed to a very sophisticated level, see for instance [1-3,22]. Finite difference operators are typically employed for the time domain and can be grouped into either explicit or implicit methods. Explicit techniques do not require the forming and factoring of a global stiffness matrix but do have stability and accuracy concerns which severely restrict the size of the time steps. They are primarily applicable to wave propagation analyses, and have been very efficiently implemented on some vector machines. In addition, explicit approaches hold much promise for coarse grain parallel processing environments. Unconditionally stable implicit operators, such as the trapezoidal and Hilber-Hughes rules [1-3], have been widely applied for structural dynamics types of linear and nonlinear analyses in which the time interval of concern is typically much longer than in wave propagation type problems. Implicit methods can employ much larger time steps, but require the formation and factoring of a global stiffness matrix. This is very costly and even prohibitive especially for

large nonlinear structural problems where the stiffness matrix must be repeatedly formed and factored. Explicit /implicit methods in which part of the structure is treated as explicit and part as implicit have also been developed but they too have similar restrictions.

Overall the transient FEM analysis of especially large, nonlinear transient structures remains a difficult and expensive process. Explicit and implicit methods both have advantages and disadvantages which can greatly restrict their efficiency – even on the largest present day supercomputers.

For transient fluid-structure interaction problems, FEM is the obvious choice for discretization of the structure, which may respond in a linear or nonlinear manner. In the fluid media, which is typically approximated as an acoustic media, several choices exist for discretization [21]. The handling of the interface between the fluid and the structure can also be quite involved. Typically, due to efficiency considerations, the coupled system is solved in a staggered fashion [17].

The treatment of the fluid media is motivated by the fact that the analyst is concerned primarily with the response of the structure and what effect the fluid has upon it. Thus additional approximations in the fluid realm can be introduced. Two approaches are evident. The first is domain discretization with techniques such as finite element, finite difference or finite volume. Of these, FEM seems to have been the most popular. The discretization of the fluid domain with finite elements, however, represents a costly approach that can also introduce further approximations. To achieve a suitable level of accuracy, a large number of elements and degrees of freedom needs to be included particularly for 3D problems, and typically implicit methods must be applied. Thus even with staggered solutions, this can be a very costly if not prohibitive approach. With domain methods such as FEM, the modeling of infinite boundary conditions raises significant difficulties. Silent boundary conditions [3] have been applied with some success. However, general purpose infinite type finite elements [22] for transient problems are not available.

The second approach for the fluid media is the use of boundary element methods (BEM) - see [4]. The introduction of BEM, while eliminating the need to discretize the domain of the fluid, carries with it an assumption of homogeneity. Thus thermal variations, cavitation, etc. can not be easily dealt with if at all. The retarded potential method represents an exact formulation of the transient BEM problem [4,7,15,16,19]. However, difficulties especially with storage have greatly limited its

more widespread application. The doubly asymptotic approximation (DAA), represents an approximation to the RP method and has been widely employed for modeling the transient fluid media – see for instance [5,6,9,11,12]. In [10], DAA was coupled with a finite element capability that models cavitation in the fluid. In general, however, there is significant concern over DAA accuracy especially for highly transient, nonlinear applications of fairly long durations. DAA correctly calculates the fluid loading at early and late time. In the intermediate time region or for the radiation by the structure vibrations in the intermediate frequency range, DAA's accuracy is suspect. DAA2 [12] represents a generalization of DAA with some improved accuracy in the intermediate frequency range.

Overall, the solution of transient fluid-structure interaction problems is certainly not at a fully satisfactory level. A great deal of work remains to develop efficient, accurate techniques for this very important class of problems.

In what follows, we first discuss the retard potential formulation and then its coupling to the nonlinear finite element code ABAQUS for the solution of submerged structures subjected to weak shock waves.

FORMULATION

Consider a submerged structure subjected to an incident pressure wave. We assume that the fluid media can be approximated as an ideal compressible medium in linear wave motion (acoustic fluid). The incident wave p^{inc} impinges on the structure and is scattered to create p^{sca} . In addition, the structure's response initiates a radiation pressure p^{rad} in the surrounding fluid. The total pressure p is then represented as:

$$p = p^{inc} + p^{sca} + p^{rad}. \quad (1)$$

The boundary condition on the "wetted" surface of the structure can be written as [15]

$$\frac{\partial p}{\partial n} = -\rho a, \quad (2)$$

where n is the unit normal into the structure; ρ is the fluid density; and a is the normal acceleration of the structure. The retarded potential integral is the solution of the linear wave equation,

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}, \quad (3)$$

subject to the boundary conditions of eq. (2). In eq. (3), c is the sonic velocity in the fluid.

A discussion of the retarded potential equation and its discretized approximation as applied to submerged structures can be found in [15,16,19]. The form of the equation for the calculation of the total pressure p on the "wetted" surface of the structure subjected to a continuous incident pressure field p^{inc} is as follows:

$$p(x, t) = 2p^{inc}(x, t) - \left(\frac{\rho}{2\pi}\right) \int_S \left(a \frac{(x, t')}{R} dS'\right) \quad (4)$$

$$+ \left(\frac{1}{2\pi}\right) \int_S \left(p(x', t) + \frac{R}{c} \frac{\partial}{\partial t} p(x', t)\right) \left(\frac{1}{R^2} \frac{\partial R}{\partial n'}\right) (dS')$$

in which

$$t' = t - R/c \quad (5)$$

$$R = \|x - x'\| \quad (6)$$

and where t' is the retarded time; x is the position of the observation point on the surface; x' is the location of an integration point on the surface; R is the distance between the observation point and the integration point; and n' represents the surface normal direction at the integration point orientated away from the fluid.

IMPLEMENTATION

For computational purposes, eq. (4) is discretized in a boundary element sense and linked to the structural analysis code ABAQUS to form the program ABAQUS-RP. The structure will be allowed to respond in a linear or nonlinear fashion. The surface pressure field is approximated by subdividing the surface into K zones or elements of constant pressure. These constant boundary elements coincide with the "wetted" surface of the structure which is discretized with ABAQUS 8 node quadratic shell elements (SR8). Thus the boundary elements although constant in the pressure are quadratic in terms of the geometry [20]. Figure 1 indicates a 8 node ABAQUS shell element (such as the S8R) and the corresponding boundary element divided into subzones. The pressures on the fluid "wetted" surface are discretized in time to coincide with the time steps of ABAQUS. Prior to conducting the time history computation, a matrix of influence coefficients must be calculated for the BEM grid which expresses the relations of current pressures on past pressure, and past and present accelerations.

Each zone or boundary element k is subdivided into a mesh of subzones l . This subdivision permits accurate numerical integration of the geometric influence coefficients as well as determination of the time delayed pressures and accelerations from each proceeding time step. For the purpose of measuring the distance between the integration point l and the observation point j (thereby the time retardation), point j is located at the center of each boundary element. The integration point l is located at the center of subzone l within boundary element k . When the observation point j is within the zone k being integrated, R can go to zero, and singular integration techniques must be employed - see [16,19].

The time step dt is assumed to be constant, and the current time can then be expressed as:

$$t = m \cdot dt . \quad (7)$$

The retardation time between points j and l (within zone k) is expressed as t'_{jkl} and follows from eq. (5). In general t'_{jkl} will not be a multiple of dt and may be expressed as:

$$t'_{jkl} = n_{jkl} + \gamma_{jkl} , \quad (8)$$

where n is an integer and $0 \leq \gamma \leq 1$. In eq. (4) p , $\frac{\partial p}{\partial t}$ and a can now be interpolated as follows:

$$p_k(mdt - t') = (1 - \gamma) p_k^{m-n} + \gamma p_k^{m-n-1} . \quad (9)$$

The term $\frac{\partial p_k}{\partial t}$ is typically expressed by a three point backward difference formula [15,16,19] and we have employed this form. However in [13], Groenenboom indicates that this represents an explicit assumption and requires for stability that

$$dt > \max \frac{R}{c} . \quad (10)$$

Groenenboom [13] presents an implicit formulation of the retarded potential which requires a matrix inversion.

The retarded potential integral eq. (4) can be discretized and arranged into the form [15,16,19]:

$$p_j^m = 2(p_j^{inc})^m - \frac{\rho}{2\pi} \sum_{k=1}^K \sum_{i=0}^{\Gamma} A_{ij}^k \cdot a_k^{m-i} - \frac{1}{2\pi} \sum_{k=1}^K \sum_{i=0}^I B_{ij}^k p_k^{m-i} \quad (11)$$

where

$$\Gamma = \frac{\max R_{jkl}}{cdt} + 1, \quad I = \frac{\max R_{jkl}}{cdt} + 3 . \quad (12)$$

The terms A_{ij}^k and B_{ij}^k depend only on geometry and are computed and stored prior to calculating the pressure. Since the pressure p_k^m on the right side of eq. (11) are not yet known, a convergence criteria that implies that a disturbance from one zone cannot affect any other zone in less than one time step is enforced, $B_{0j}^k = 0$ for all $j \neq k$.

Equation (11) represents the response of the fluid media on the "wetted" surface of the structure. Both the current pressures p_j^m and current structural accelerations a_k^m are unknown, and must be determined. The fluid pressure represents a loading term on the governing FEM equations of the structure. The coupled equations for the fluid-structure interaction can be solved by elimination of one field variable, in a fully coupled sense (solve both sets of equations simultaneously), or as is done in this paper in a staggered fashion [17] – which is much more efficient. With a staggered approach the current pressures p_j^m in eq. (11) are solved using matrix multiplications of past pressures and accelerations, and current acceleration estimates obtained from the structural equations. The solution of the pressures p_j^m thus requires storage of a large number of pressures and accelerations at previous time steps. Therein lies the difficulty that has handicapped the application of the retarded potential approach.

The following staggered scheme is employed in this paper:

1. Apply $(p_j^{inc})^0$ at $m = 0$
2. Solve the FEM equations and calculate a_j^1
3. $m = m + 1$
4. From eq. (11) calculate p_j^m
5. Apply p_j^m to the FEM equations and calculate a_j^{m+1}
6. If the maximum time step is reached, stop.
7. Go to 3.

We note that in using a staggered approach, a phenomenon known in control theory as dead time feedback can arise [17]. In [18], Tamm stabilized the solution for a combined retarded potential - FEM technique by simply eliminating the added step of time delay in the structural acceleration values relative to the fluid pressures. This approach was effective because of the rather minor influence of the current accelerations on the current pressures at the observation point j - see [18] for further details. In this paper, therefore, we have included this feature.

APPLICATION

A retarded potential (RP) capability has been coupled to the ABAQUS program, through the DLOAD user written subroutine, to form ABAQUS - RP. The initial implementation, which is the subject of this paper, has been in a MICRO-VAX 3600 environment. The fluid-structure interaction problems investigated, therefore, have been relatively small in size and coarse in the time stepping, and serve only to validate the overall capability. An important feature in the ABAQUS - RP program is the Hiber-Hughes implicit time operator with controllable numerical damping [1]. Previous experience [15,19] has shown that some numerical damping has helped stabilize the fluid response. In addition, a variable time stepping capability has been included in the RP program. However, it has proven to be unstable with the present adaptive time stepping algorithm employed in ABAQUS [1]. Also to help minimize storage, the RP subprogram contains a capability to limit how far back in time previous responses are stored. This is an important approximation especially for longer time histories or larger structures, but our experience with it is limited.

To demonstrate the ABAQUS-RP program, consider Figure 2 in which an elastic sphere is subjected to an incident step, plane wave along the x axis. An exact solution is presented in [14]. Figure 3 indicates the doubly symmetric ABAQUS model which consists of 54 S8R quadratic shell elements [1] overlaid with constant pressure boundary elements. Due to symmetry only 1/4 of the sphere is modeled. The radius of the sphere r is 1 m, while the thickness $2h$ is 0.02 m. The properties of the steel are: Young's modulus $E = 2.0684 \text{ E}11 \text{ Pa}$, Poisson's ratio $\nu = 0.3$, and the mass density $\rho_s = 7784.5 \text{ kg/m}^3$. The properties of the surrounding water are: the speed of sound $c = 1461.2 \text{ m/sec}$ and the mass density $\rho_w = 999.6 \text{ kg/m}^3$. The magnitude of the incident wave is assumed to be $14.0\text{E}6 \text{ Pa}$. The time step employed is $dt = 0.10 \text{ } r/c$.

Figure 4 compares the ABAQUS - RP predictions for the dimensionless displacement history at $\Theta = 45$ and 90 degrees with a closed form solution [14]. With the above dt , this model took approximately 8.5 hours to go 200 time steps on the MICRO-VAX. In general, the agreement is only fair due probably to the coarseness of the time step and the grid, and also to the relatively high numerical damping employed with the Hilber-Hughes operator ($\alpha = 0.3$ [1] was used). As discussed earlier, the use of numerical damping can be an important consideration for the stability of the finite element/retarded potential calculations. In addition, to simulate a step loading, as was done in [15], a ramp loading with a rise time of $t = .5 c/r$ was employed. We note also that in [15] the agreement between NASTRAN - RP results and the closed form solution [14] tended to deteriorate at later times.

In an effort to apply ABAQUS-RP to a nonlinear problem, the sphere is allowed to yield ($\sigma_{yield} = 345E6$ Pa and perfect plasticity is assumed) and large displacements are considered. The magnitude of the plane wave is reduced to $8.5E6$ Pa (otherwise the sphere would collapse). The displacement histories are plotted in Figure 5. Comparisons to DAA predictions etc. are not available so this problem represents only a demonstration of the nonlinear capability of the program.

SUMMARY AND FUTURE DIRECTIONS

In this paper, we have presented a brief discussion concerning transient analysis of coupled fluid-structure interaction systems. Motivated by the present state-of-the-art, an advanced retarded potential capability has been coupled to the ABAQUS nonlinear finite element program to produce ABAQUS-RP. This code, which is executed in a staggered fashion, is currently implemented in a MICRO-VAX environment, and has been successfully applied to smaller degree of freedom (dof) fluid-structure interaction problems.

Overall, the RP method offers significant advantages over both total FEM and FEM/DAA approaches. It has not been extensively investigated in the past due primarily to storage requirements. We have demonstrated that it can be fairly efficiently implemented on a MICRO-VAX computer and applied to nonlinear structures with a relatively small number of dof. Clearly, additional computer runs are necessary to further study the selection of time steps, the use of numerical damping,

and the overall question of stability. A more thorough investigation concerning stability, along the lines of [17], is probably in order.

The remaining challenge is the implementation of ABAQUS-RP in a supercomputer (CRAY) environment. This will greatly increase both the size of the problems that could be efficiently analyzed and the length of the time history able to be considered. In addition, finer time steps can be used. Because the retarded potential capability is coupled to ABAQUS through a user written subroutine, however, this will handicap our ability to streamline the storage and retrieval of past variables [7]. We would also like to implement a capability with adaptive time stepping similar to [8] and this may be difficult. ABAQUS's adaptive time stepping algorithm often "jumps" around unnecessarily, and especially with expansions can destabilize the fluid. In addition, ABAQUS's implicit time stepping approach is based on a costly full Newton formulation.

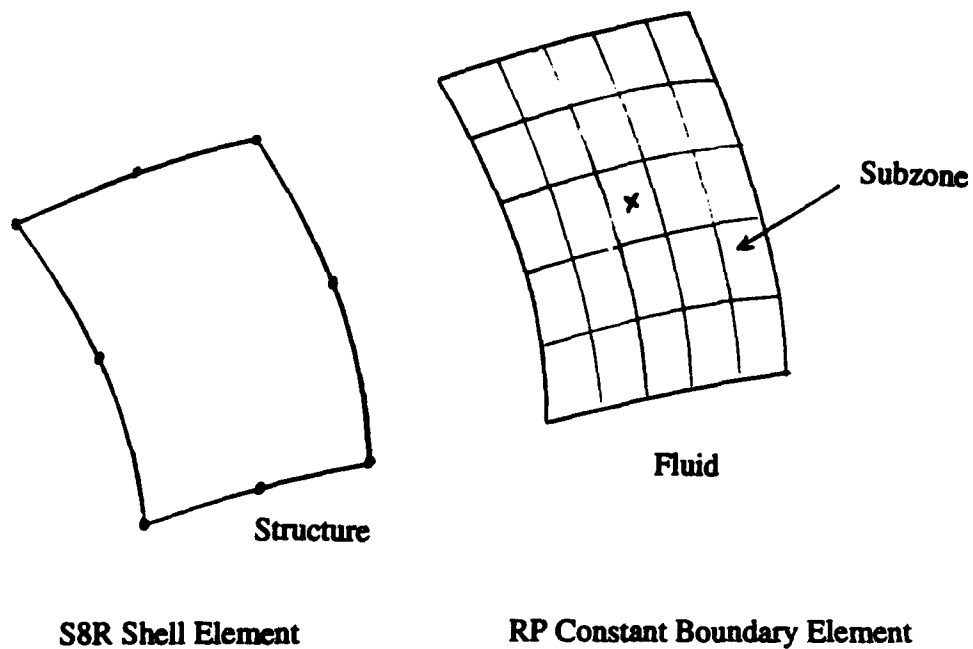


Figure 1. Quadratic ABAQUS shell element overlaid by a constant RP boundary element.

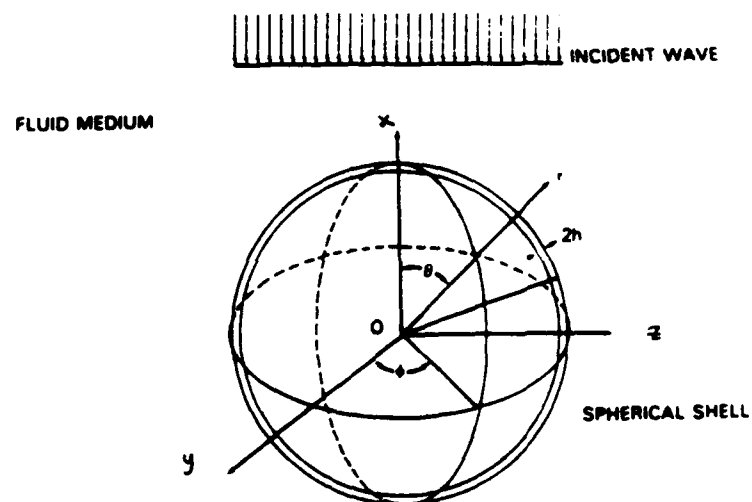


Figure 2. Elastic sphere subject to an incident, step, plane wave.

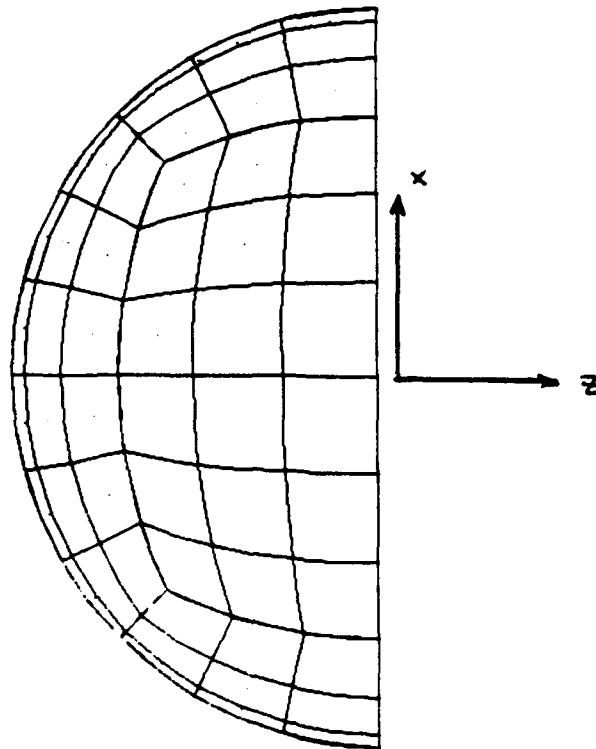


Figure 3. Sphere modeled with S8R ABAQUS shell elements.

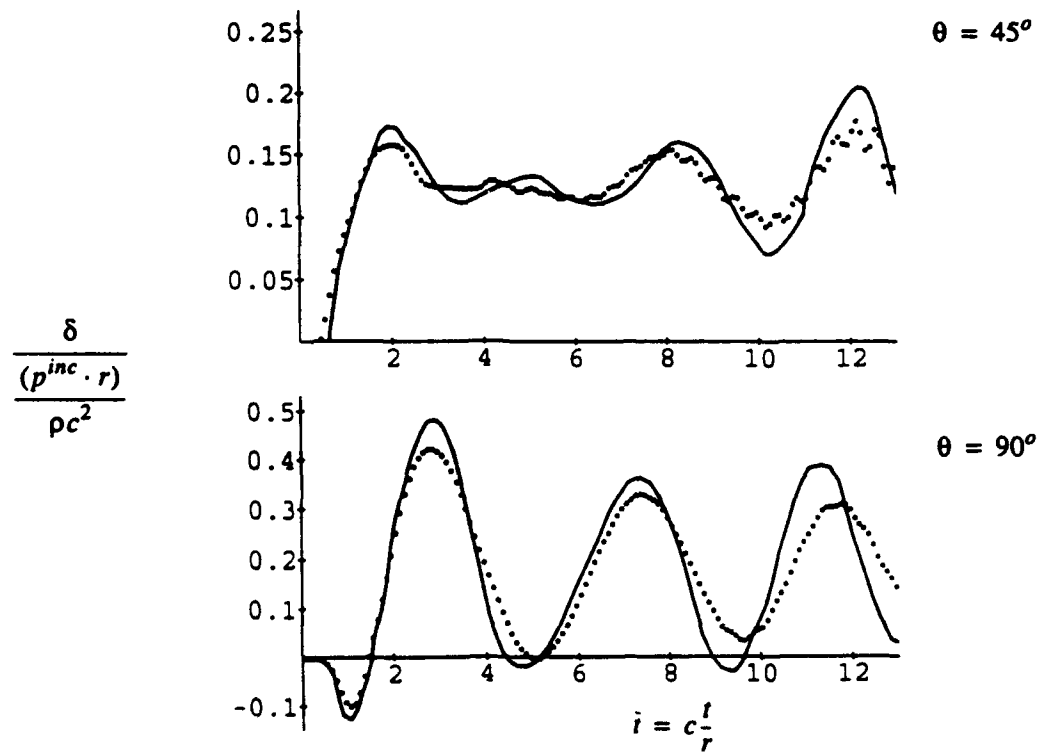


Figure 4. Shell deflection δ in the x direction, _____ classical solution,
 ABAQUS - RP (elastic sphere).

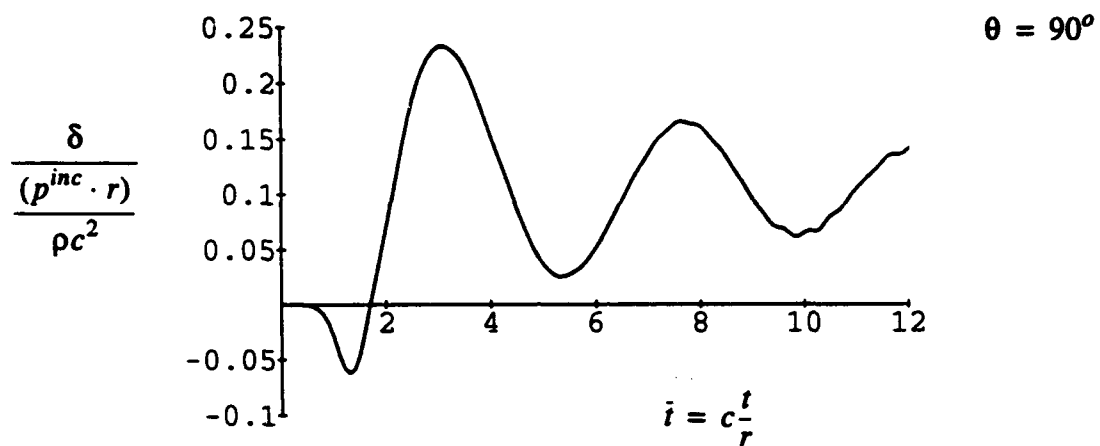


Figure 5. Shell deflection δ in the x direction, ABAQUS - RP results (elastic-plastic sphere).

REFERENCES

1. ABAQUS User's and Theoretical Manuals, Hibbitt, Karlsson and Sorensen, Providence, RI, 1989
2. Bathe, K. J., Finite Element Methods in Engineering, Prentice-Hall, 1982
3. Belytschko, T. and Hughes, T. J. R. ,editors, Computational Methods for Transient Analysis, North-Holland, 1983.
4. Brebbia, C. A., Telles, J. C. F., and Wrobel, L. C., Boundary Element Techniques Theory and Applications in Engineering, Springer-Verlag, 1983.
5. Deruntz, J. A., The Underwater Shock Analysis Code and Its Applications, Lockheed Palo Alto Research Lab Report, 1989.
6. Deruntz, J. A. and Rankin, C. C., "Applications of the USA-STAGS-CFA Code to Nonlinear Fluid-Structure Interaction Problems in Underwater Shock of Submerged Structures, " Lockheed Palo Alto Research Lab, Report, 1989.
7. Dohner, J. L., Shoureshi, R., and Bernard, R. J. , "Transient Analysis of Three -Dimensional-Wave Proagation Using the Boundary Element Method," Int. J. Num. Meth. Engin., Vol. 24,pp 621-634,1987.
8. Dyka, C. T. and Remondi, A. "An Efficient Adaptive Dynamic Algorithm," Computers and Structures, Vol. 24, No. 5, pp. 781-790, 1986.
9. Felippa, C. A., "Top-Down Derivation of Doubly Asymptotic Approximations for Structure-Fluid Interaction Analysis," in Innovative Numerical Analysis for the Applied Engineering Sciences, R. P. Shaw editor, University Press of Virginia, 1980.
10. Felippa, C. A. and DeRuntz, J. A., "Finite Element Analysis of Shock Induced Hull Cavitation," Comp. Meth. Appl. Mech. Eng., 44, pp 297-337,1984.
11. Geers, T. L., "Residual Potential and Approximation Methods for Three-Dimensional Fluid-Structure Interaction Problems," J. Acout. Soc. Amer., Vol. 49, No. 2,pp. 1505-1510,1971.
12. Geers, T. L., "Doubly Asymptotic Approximations for Transient Motions of Submerged Structures," J. Acoust. Soc. Amer., Vol. 64, No. 5, pp. 1500-1508, 1978.

13. Groenenboom, P. H. L., "Solution of the Retarded Potential Problem and Numerical Stability," in *Boundary Element 6*, C. A. Brebbia editor, Springer-Verlag, 1984.
14. Huang, H., "Transient Interaction of Plane Acoustic Waves with a Spherical Elastic Shell," *J. Acoust. Soc. Amer.*, Vol. 45, No. 3, pp. 661-670, 1969.
15. Huang, H., Everstine, G. C. and Wang, Y. F., "Retarded Potential Techniques for the Analysis of Submerged Structures Impinged by Weak Shock Waves," in *Computational Methods for Fluid-Structure Interaction Problems*, T. Belytschko and T. L. Geers editors, ASME, AMD, Vol. 26, 1977.
16. Mitzner, K. M., "Numerical Solution for Transient Scattering from a Hard Surface of Arbitrary Shape - Retarded Potential Technique," *J. Acoust. Soc. Amer.*, Vol. 42, No.2, 1967.
17. Park, K. C., Felippa, C. A. and DeRuntz, J. A., "Stabilization of Staggered Solution Procedures for Fluid-Structure Interaction Analysis," in *Computational Methods for Fluid-Structure Interaction Problems*, T. Belytschko and T. L. Geers editors, ASME,AMD, Vol. 26, 1977.
18. Tamm, M. A., *Stabilization of the Coupled Retarded Potential - Finite Element Procedure for Submerged Structural Analysis*, NRL Memorandum Report 5902, Dec. 1986.
19. Tamm, M. A. and Webbon, W. W., *Submerged Structural Response to Weak Shock by Coupled Three-Dimensional Retarded Potential Fluid - Finite Element Structural Analysis*, NRL Memorandum, Report 5903, May 1987.
20. Tamm, M. A., *A Parametric Patch Surface Geometry Definition for the Three Dimensional Retarded Potential Technique*, NRL Memorandum Report 5904, June 1987.
21. Zienkiewicz, O. C. and Bettess, P., "Fluid -Structure Dynamic Interaction and Some Unified Approximation Processes," in *Fifth Symposium on the Unification of Finite Element, Finite Differences and Calculus of Variations*, H. Kardestuncer editor, University of Connecticut, May 1980.
22. Zienkiewicz, O. C. and Taylor, R. L., *The Finite Element Method*, Volumes 1 and 2, fourth edition, McGraw-Hill, 1989 and 1990.