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Spectral Multiplicity for Second-Order Stochastic Processes

C.R. Baker
Department of Statistics, CB# 3260
University of North Carolina
Chapel Hill, N.C. 27599-3260

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The theory of spectral multiplicity for second-order stochastic processes is developed from first principles. Each of the representations originally obtained by Cramer and by Hida is developed. The Hellinger-Hahn theorem on multiplicity in Hilbert space is obtained as a corollary, instead of being used to provide the representations.

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1. Introduction.

In this paper we give a compact development of the basic theory of spectral multiplicity for second-order stochastic processes. This theory gives a representation of a large class of such processes in terms of a sum of filtered mutually-orthogonal orthogonal increment (o.i.) processes. The representation has been found very useful in applications to nonGaussian signal detection [1] and to communication through channels perturbed by additive noise [2], [3], especially when the channel has feedback.

The theory to be developed here originated in the work of Cramer [4] and Hida [5]. Our development is directed toward obtaining the spectral representation in the elegant form given by Hida. Several paths are available: one may simply quote Hida's results; one may quote the Hellinger-Hahn theorem (on which Hida's results are based) and then adapt that result to the representation of a second-order stochastic process; one may prove the Hellinger-Hahn theorem, then do the adaptation; or one may give a direct proof of the spectral representation. It is this last path that will be followed here. It has some advantages over simply proving the Hellinger-Hahn theorem, since some readers may be more comfortable with second-order stochastic processes than with abstract Hilbert space; our development also gives some nice applications of RKHS (reproducing kernel Hilbert spaces) theory. Moreover, as a consequence of the development, we actually prove the Hellinger-Hahn theorem, and it is stated at the end of the paper. Although the spectral representation is well-known, there does not seem to be a readily-accessible development that yields the form given by Hida. Since the development here is explicit and based on RKHS theory, it may be useful in extensions to second-order random fields.

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2. Basic Definitions and Structure

Let T be an interval in $\mathbb{R}$. For simplicity, we denote the end points as $-\infty$ and $+\infty$, but actually T can be any interval, finite or infinite, open, half-open, or closed. (X_t), t in T, is a second-order stochastic process (SOSP) on the probability space (Ω,β,P). $\langle\cdot,\cdot\rangle$ will denote the inner product in $L_2[\Omega,\beta,P]$, with $\|\cdot\|$ the corresponding norm. $L_t(X)$ will denote the closure in $L_2(\Omega,\beta,P)$ of $\text{span}\{X_s, s\leq t\}$, with L(X) the closure of $\text{span}\{X_s, s\in T\}$. All equalities given here are in the sense of $L_2(\Omega,\beta,P)$; thus x = y means that x and y are equal a.e. dP; $x = \Sigma_1^\infty y_n$ means that $(\Sigma_1^N y_n)$ converges to x in $L_2(\Omega,\beta,P)$ as $N \to \infty$. The following two assumptions will be made:

(A1) (X_t) is mean-square left-continuous on T, the mean-square right limit X_{t^+} exists whenever $t^+ \in T$, and $\sup_{t\in T} \|X_t\|^2 < \infty$;

(A2) $\bigcap_{s\in T} L_s(X) = \{0\}$.

Assumption A1 implies that the reproducing kernel Hilbert space (RKHS) of the correlation function of (X_t), which we denote here as H(X), is separable, as is L(X). A SOSP which satisfies A2 is said to be purely nondeterministic. These two assumptions give rise to a left-continuous resolution of the identity, as follows. Let P_t be the projection operator mapping the Hilbert space L(X) onto $L_t(X)$. The following result is an immediate consequence of assumptions A1 and A2.

<u>Lemma 1</u>: The family (P_t) has the following properties:

(a) If s > t, then $P_sP_t = P_tP_s = P_t$;

(b) $P_t = P_{t-}$ for all t such that $t- \in T$, where $P_{t-}x = \lim_{s\uparrow t} P_s x$ for all x in L(X);

(c) $P_{-\infty} \equiv \lim_{s\downarrow-\infty} P_s = 0$ (operator);

(d) $P_{+\infty} \equiv \lim_{s\uparrow\infty} P_s = I$ (identity operator in L(X)).

The limits in (c) and (d) are in the strong operator topology; e.g.,

$P_{-\infty}x = \lim_{x\downarrow-\infty} P_s x = \underline{0}$ (element) for all x in L(X).

It should perhaps be noted that m.s. left-continuity of (X_t) is not necessary for left-continuity of the family $\{P_t, t \in T\}$. For example, let $T = [0,1]$, let (W_t) be the standard Wiener process, and define

$$
\begin{aligned}
X_t &= -W_t \text{ for } t < \tfrac{1}{2} \\
&= +W_t \text{ for } t \geq \tfrac{1}{2}.
\end{aligned}
$$

Then $\|X_t - X_{½}\|^2 = 3t + \frac{1}{2}$ for $t < \frac{1}{2}$, so (X_t) is not m.s. left-continuous at $t = \frac{1}{2}$. However, $X_{½} \in \overline{\text{sp}\{X_s, s < ½\}}$, so $L_{½}(X) = L_{½-}(X)$, and $L_t(X) = L_{t-}(X)$ for t in (0,T].

The left-continuity of (P_t) is thus a weaker property than m.s. left-continuity of (X_t). This property gives rise to other m.s. left-continuous processes, as follows.

<u>Lemma 2</u>: Let y be in L(X), and define $Z_t = P_t y$. Then (Z_t) is m.s. left-continuous, Z_{t+} exists for all t+ in T, and $\sup_{t\in T} |Z_t| < \infty$. Moreover, (Z_t) is purely non-deterministic and has orthogonal increments.

<u>Proof</u>: The fact that (Z_t) is m.s. left-continuous follows by left continuity of (P_t). Z_{t+} exists since the limit $P_{t+} = \lim_{s\downarrow t} P_s$ exists. $\sup_{t\in T} \|Z_t\| \leq \|y\|$. (Z_t) is purely non-deterministic since

$$\bigcap_{t\in T} L_t(Z) \subset \bigcap_{t\in T} L_t(X) = \underline{0}.$$

If $t_3 > t_2 \geq t_1$, then

$$E(Z_{t_3} - Z_{t_2})Z_{t_1} = \langle (P_{t_3} - P_{t_2})y, P_{t_1}y \rangle = 0$$

since $(P_{t_3} - P_{t_2})x \perp L_{t_1}(X) = \mathrm{range}(P_{t_1})$. □

<u>Lemma 3</u>: (P_t) has a finite or countably-infinite number of points of discontinuity. If t and s are two such points, distinct, with $P_t x = 0$, $P_{t+}x = x$ and $P_{s+}y = y$, $P_s y = 0$, then $x \perp y$.

<u>Proof</u>: If $P_{t+}x = x$, $P_t x = 0$, then $x \perp L_v(X)$, $v \leq t$, proving the second statement. The first statement then follows, since separable L(X) contains at most a countably-infinite number of mutually orthogonal elements. □

3. <u>Canonical Representations</u>

Suppose now that one could find a family $\{(Z_t^n),\ n \geq 1\}$ of mutually orthogonal stochastic processes, each with orthogonal increments, such that the closed linear span of $\{(Z_s^n),\ n \geq 1;\ s \in (-\infty, t]\} \equiv L_t(Z)$ contains $L_t(X)$ for all t in T. Then, since the subspaces $L_t(Z^n)$ are orthogonal for fixed t and different values of n, $L_t(X) \subset \bigoplus_{n \geq 1} L_t(Z^n)$ (i.e., x in $L_t(X) \Rightarrow x = \Sigma_n x_n$, $x_n \in L_t(Z^n)$). Processes (Z_t^n) having such properties are called *innovations processes* for (X_t). The number of terms in the index set is the multiplicity of the innovations process. We will first show that there always exists an innovations process for (X_t) and then prove the existence of an innovations process of minimal multiplicity. For example, if (X_t) has orthogonal increments, then the minimal multiplicity is one.

It can be that $L_t(X) \subset \bigoplus_{n=1}^{M} L_t(Z_n)$ where (Z_n) are o.i.processes, but with the reverse inclusion not holding. A representation of the form $L_t(X) = \bigoplus_{n=1}^{M} L_t(Z_n)$ for all t in T, with (Z_n) being o.i.processes, is called a *canonical representation* for (X_t).

The natural question: when does a SOSP (X_t) have a canonical representation? We note first that if $T = \mathbb{R}$ and (X_t) is m.s. continuous and wide-sense stationary, then $X_t = \int_{\mathbb{R}} e^{i\lambda t} dY(\lambda)$ where $Y(\lambda)$ is a PND $0 \pm P$.

<u>Theorem 1</u>: Suppose that (X_t) is a SOSP satisfying assumptions (A-1) and (A-2). Then (X_t) has a canonical representation.

<u>Proof</u>: Since the RKHS $H(X)$ of X is separable, then by the isometry between $L(X)$ and $H(X)$ (Theorem IX.2), $L(X)$ is also separable. Thus, there exists a countable CONS (Z^n) for $L(X)$. Define elements (Z_n) as follows. Let $Z_1 = Z^1$. Given $Z_1, Z_2, \ldots, Z_n$, let $\tilde{P}_n$ be the projection operator in $L(X)$ with range space equal to the closed linear span of $\bigcup_{i=1}^{n} L(Z_i)$, where $L(Z_i) = \overline{\text{span}\{P_t Z_i,\ t \in T\}}$. If $\text{range}(\tilde{P}_n) = L(X)$, the process is terminated. Otherwise, Z_{n+1} is defined by $Z_{n+1} = Z^{n+1} - \tilde{P}_n Z^{n+1}$.

If the above procedure terminates for some finite n, then necessarily $L(X) = \overline{\text{span}\{\bigcup_{i=1}^{n} L(Z_i)\}}$. If the process does not terminate for some finite n, suppose that $y \in L(X)$ and $y \perp \overline{\text{span}\{\bigcup_{i\geq 1} L(Z_i)\}}$. Then $y \perp Z^i$ for all $i \geq 1$ by the construction of $\{Z_i, i\geq 1\}$ and so $\|y\| = 0$. Thus, $L(X) = \overline{\text{span}\{\bigcup_{n\geq 1} L(Z_n)\}}$.

Define $L_t(Z_i) = \overline{\text{span}\{P_s Z_i,\ s \leq t\}}$, $i \geq 1$, $t \in T$. To see that $L_t(Z_i) \perp L_s(Z_j)$ for $i \neq j$ and all s,t in T, one proceeds as follows. By construction, $Z_j \perp L(Z_k)$ for all $k \neq j$, and $P_t Z_j = Z_j - y_0$, where $y_0 \perp \text{range}(P_t)$. Thus, $y_0 \perp L_t(Z_k)$, and since $Z_j \perp L(Z_k) \supset L_t(Z_k)$, $P_t Z_j \perp L_t(Z_k)$. Hence $P_t Z_j \perp P_s Z_k$ for all $x \leq t$ and for all t in T. By symmetry, $P_t Z_k \perp P_s Z_j$, for all $s \leq t$, all t in T. This gives $L_t(Z_j) \perp L_s(Z_k)$ for $j \neq k$ and all s,t in T, so $\overline{\text{span}\{\bigcup_{n\geq 1} L_t(Z_n)\}} = \bigoplus_{n\geq 1} L_t(Z_n)$ for all t in T and $L(X) = \bigoplus_{n\geq 1} L(Z_n)$.

Since $L_t(Z_n) \subset L_t(X)$ for all $n \geq 1$ and all t in T, it remains only to show that $L_t(X) \subset \bigoplus_n L_t(Z_n)$ for all t in T. We already have that $L(X) = \bigoplus_n L(Z_n)$ and $L_t(Z_n) \perp L_s(Z_m)$ for $n \neq m$, all s,t in T. Thus, by the continuity of the operator P_t,

$$\begin{aligned} P_t L(X) = L_t(X) &= P_t[\bigoplus_n L(Z_n)] \\ &= P_t\, \overline{\text{span} \bigcup_n L(Z_n)} \\ &= \overline{\text{span} \bigcup_n L_t(Z_n)} = \bigoplus_n L_t(Z_n). \end{aligned} \qquad \square$$

<u>Definition 1</u>: Let (y_t), $t \in T$ be an orthogonal increment process on (Ω, β, P), such that the right m.s. limit y_{a+} exists for all a in T (except at the right end-point). Then, μ_y will denote the Lebesgue-Stieltjes measure on the Borel sets of T defined by $\mu_y(a,b] = \|y_{b+}\|^2 - \|y_{a+}\|^2$. If (y_t) is also m.s. left continuous, then $\mu_y[a,b) = \|y_b\|^2 - \|y_a\|^2$. $L_2[\mu_y]$ will denote the set of all Borel-measurable functions $f\colon T \to \mathbb{R}^{\#}$ (extended real line) such that $\int_T |f(t)|^2 d\mu_y(t) < \infty$. The Lebesgue-Stieltjes measure μ_y defined by y will be called the *spectral measure* for y.

<u>Corollary</u>: From Chapter VIII, the preceding theorem shows that (X_t) has the canonical representation

$$X_t = \sum_{i=1}^{M} \int_0^t F_i(t,s)dB_i(s), \qquad \text{all } t \in T,$$

where $L_t(X) = L_t(\underline{B})$ for all $t \in T$, $L_t(\underline{B}) = \bigoplus_{i=1}^{M} L_t(B_i)$, the (B_i) are mutually orthogonal o.i. processes, $B_i(t) = P_t Z_i$, $F_i(t,s) = 0$ for $s > t$, and $F_i(t,\cdot)$ belongs to $L_2[d\mu_{B_i}]$ for all $t \in T$, all $i \leq M$.

Remark. Suppose that (y_t) is such that $y_{t^+} \in L_t(Z)$ for all t, where $Z \in L(X)$. Then $y_{t+} = \int_{-\infty}^{t} f(t,s)dP_s Z$ for all $t \in T$. This gives $\|y_{t+}\|^2 = \int_{-\infty}^{t} |f(t,s)|^2 d\mu_Z(s)$. Thus, $\mu_y << \mu_Z$. This result will be used frequently.

Proposition 1: Suppose (X_t) has the representation given in the preceding Corollary, except that it is not known whether or not $L_t(\underline{B}) = L_t(X)$ for all t in T. This equality then holds, for all t in T, if and only if the following condition is satisfied: For every t in T,

$$\sum_{i=1}^{M} \int_{-\infty}^{s} F_i(s,u)g_i(u)d\mu_{B_i}(u) = 0 \qquad \text{for all } s \leq t$$

implies $\sum_{i=1}^{M} \int_{-\infty}^{t} |g_i(u)|^2 d\mu_{B_i}(u) = 0.$

This condition can be restated as follows: For every t in T, and every $i \leq M$, $\{F_i(s,\cdot),\ s \leq t\}$ is complete in $L_2[(-\infty,t],\mu_{B_i}]$.

Proof: Every element in $L_t(B_i)$ has the form $\int_{-\infty}^{t} g_i(s)dB_i(s)$ for some g_i in $L_2[\mu_{B_i}]$, from Chapter VIII. If $L_t(\underline{B}) \neq L_t(X)$ for some $t \in T$, then there exists an element $Z_t = \sum_{i=1}^{M} \int_{-\infty}^{t} g_i(s)dB_i(s)$ in $L_t(\underline{B})$ such that $Z_t \perp L_t(X)$. But then, for $s \leq t$, $\langle Z_t, X_s \rangle = \sum_{i=1}^{M} \int_{-\infty}^{s} F_i(s,u)g_i(u)d\mu_{B_i}(u)$. Thus, in order that $L_t(\underline{B}) \neq L_t(X)$, it is necessary (and, obviously, sufficient) that there exist elements $\{g_i,\ i \leq M\}$, $g_i \in L_2(\mu_{B_i})$, such that $\sum_{i=1}^{M} \int_{-\infty}^{t} |g_i(u)|^2 d\mu_{B_i}(u) \neq 0$, while $\sum_{i=1}^{M} \int_{-\infty}^{s} F_i(s,u)g_i(u)d\mu_{B_i}(u) = 0$ for all $s \leq t$.

This proves the statement for the first condition. The alternate (equivalent) condition then clearly holds, using the fact that every element g_i in $L_2[\mu_{B_i}]$ defines an element in $L_t(B_i)$ having the representation $\int_{-\infty}^{t} g_i(s)dB_i(s)$.

□

4. Proper Canonical Representations

Theorem 1 proves the existence of a canonical representation for (X_t). The Corollary gives a very useful representation of X_t for each $t \in T$, when a canonical representation exists. However, the number of terms, M, appearing in the series is not specified, nor do the measures (μ_{B_i}) have any particular relationship to each other, nor is anything said about uniqueness. These points will all be addressed as we now proceed to obtain a *proper canonical representation*.

Definition 2: Let $\mathscr{D}(X)$ be the closed linear span of all elements x in L(X) such that for some discontinuity point t_i of $\{P_t, t \in T\}$, $P_{t_i} x = 0$, $P_s x = s$ for $s > t_i$. Define the subspace $\mathscr{C}(X) \subset L(X)$ by $\mathscr{C}(X) = \mathscr{D}(X)^{\perp}$, the set of all elements y in L(X) such that $y \perp \mathscr{D}(X)$.

We have already seen that right-m.s. continuity of (X_t) at t_0 is not a necessary condition in general, for $P_{t_0} = P_{t_0+}$. However, if (X_t) is an OIP, then right-m.s. continuity of (X_t) at t_0 is a necessary and sufficient condition for $P_{t_0+} = P_{t_0}$. One notes that for $t \leq t_0$, for (X_t) an OIP,

$$E|X_{t_0+} - X_{t_0}|^2 = E|X_{t_0+} - X_{t_0} + X_{t_0} - X_t|^2$$

$$= E|X_{t_0+} - X_{t_0}|^2 + E|X_{t_0} - X_t|^2 \geq E|X_{t_0+} - X_{t_0}|^2.$$

From a Hilbert space viewpoint, X_{t_0} is the unique element in $L_{t_0}(X)$ which is nearest to X_{t+} when (X_t) is an OIP. In the general case, of course, this need not be true, so that X_{t_0+} can belong to $L_{t_0}(X)$ even though $E|X_{t_0+} - X_{t_0}|^2 = \|X_{t_0+} - X_{t_0}\|^2 > 0$.

Lemma 4: Let P_{t_0+} be the operator in $L(X)$ defined by $P_{t_0+}u = \lim_{s \downarrow t_0} P_s u$. Then P_{t_0+} is well-defined for each $t_0 \in T$ (except the right endpoint) and P_{t_0+} is a projection operator.

Proof: See Theorem IV.10.

Proposition 2: $x \in \mathscr{C}(X) \iff t \to \|P_t x\|$ is continuous on T.

Proof: Note first that $t \to \|P_t x\|$ continuous at t_0 is equivalent to $P_t x \to P_{t_0} x$ as $t \downarrow t_0$, since (P_t) is already half-continuous on T and for $t > t_0$, $\|P_t x - P_{t_0} x\|^2 = \|P_t x\|^2 - \|P_{t_0} x\|^2$. Thus, $t \to \|P_t x\|$ continuous on T is equivalent to $P_{t_0} x = P_{t_0+} x$ for all t_0 such that t_0+ is defined. Now, suppose that $t \to \|P_t x\|$ is continuous on T and that $P_{t_0+} y = y$, $P_{t_0} y = 0$. Then for $t > t_0$, $\langle x,y \rangle = \langle x, P_t y \rangle = \langle P_t x, y \rangle$. Since $P_{t_0} x = P_{t_0+} x$, this gives $\langle x,y \rangle = \langle x, P_{t_0+} y \rangle = \langle P_{t_0+} x, y \rangle = \langle P_{t_0} x, y \rangle = \langle x, P_{t_0} y \rangle = 0$. Thus, $x \perp \mathscr{D}(X)$.

Conversely, suppose that $x \in \mathscr{C}(X)$, and that $t \to \|P_t x\|$ is discontinuous at $t = t_0$. Then $\|P_{t_0+} x\| = \|P_{t_0} x\| + \alpha$, where $\alpha > 0$. Let $x_0^{\perp} = P_{t_0+} x - P_{t_0} x$; then $\|x_0^{\perp}\| = \alpha$ and $x_0^{\perp} \perp \text{range}(P_{t_0})$, so $P_{t_0+} x_0^{\perp} = x_0^{\perp}$, $P_{t_0} x_0^{\perp} = 0$; $x_0^{\perp}$ belongs to $\mathscr{D}(X)$. Then $\langle x, x_0^{\perp} \rangle = \langle x, P_{t_0+} x_0^{\perp} \rangle = \langle P_{t_0+} x, x_0^{\perp} \rangle = \|x_0^{\perp}\|^2 = \alpha$. tThis contradicts $x \perp \mathscr{D}(X)$. □

Proposition 3 : Suppose that $Z \in \mathscr{C}(X)$. Then $L_t(Z) = \bigoplus_{n=1}^{M} L_t(Z_n)$, $M \leq \infty$, where $L_t(Z) = \overline{\text{span}\{P_s Z,\ s \leq t\}}$; $\{(P_t Z_n),\ n \geq 1\}$ is a family of mutually orthogonal m.s. continuous processes, each with orthogonal increments.

Moreover, if $y \in L_t(Z)$, then there exists $\{F_n(t,\cdot),\ n \le M\}$ depending on y and such that

$$y = \sum_{i=1}^{M} \int_{-\infty}^{t} F_n(t,s)d(P_s Z_n)$$

where $F_n(t,s) = 0$ for $s > t$ and $F_n(t,\cdot)$ belongs to $L_2[d\mu_{Z^n}]$, for each t in T.

Proof: If $Z \in \mathscr{C}(X)$, then by Prop. 2, $t \to \|P_t Z\|$ is continuous, so $(P_t Z)$ is a purely-nondeterministic m.s. continuous process. Moreover, $\|P_t Z\|^2 \le \|Z\|^2$, all t in T. The existence of a canonical representation then follows from Theorem 1. The representation of y in $L_t(Z)$ follows from the previous results on SOSP. □

Suppose that t_i is a discontinuity point of the family $\{P_t,\ t \in T\}$ of projection operators. We use $M(t_i)$ to denote the dimensionality of the subspace of L(X) spanned by the elements y which satisfy $P_{t_i+}y = y$, $P_{t_i}y = 0$. $M(t_i)$ is thus the multiplicity of the eigenvalue 1 for the projection operator $P_{t_i+} - P_{t_i}$.

Proposition 4: For each t in T,

$$X_t = \sum_{n=1}^{M} \int_{-\infty}^{\infty} F_n(t,s)dB_n(s) + \sum_{t_i<t} \sum_{j=1}^{M(t_i)} \langle X_t, \psi_{ij}\rangle \psi_{ij}$$

where $\{B_n,\ n \ge 1\}$ is a mutually orthogonal family of o.i. processes, each m.s. continuous; $F_n(t,s) = 0$ for $s > t$ and $F_n(t,\cdot)$ is in $L_2[d\mu_{B_n}]$ for $n \ge 1$; (t_i) are the discontinuities of (P_t), and the $\{\psi_{ij},\ j \le M(t_i),\ i \ge 1\}$ are o.n. random variables such that for all discontinuity points t_i, $P_{t_i+} \ne P_{t_i}$,

$$P_s\psi_{ij} = \psi_{ij} \quad \text{for } s > t_i$$
$$P_{t_i}\psi_{ij} = 0$$

for $j = 1,2,\ldots,M(t_i)$.

Proof: Immediate, using Prop. 3 and the definitions of $\mathscr{C}(X)$ and $\mathscr{D}(X)$.

Proposition 5: For the representation of X_t given in Prop. 4, and each discontinuity point t_i

$$|\langle X_{t_i+}, \psi_{ij}\rangle| > 0 \quad \text{all } j \leq M(t_i)$$

where $M(t_i)$ and $\{\psi_{ij},\ j \leq M(t_i),\ i \geq 1\}$ are defined as in Prop. 4.

Proof: If not, then since $\psi_{ij} \perp \text{range}(P_s)$ for $s \leq t_i$, $\langle X_s, \psi_{ij}\rangle = 0$, all $s \leq t_i$; if also $\langle X_{t_i+}, \psi_{ij}\rangle = 0$ then $P_{t_i+}\psi_{ij} = 0$, since $\text{range}(P_{t_i+}) = \bigcap_{s>t_i} L_s(X)$. This contradicts Prop. 4. □

Corollary: Let (Z_t) be the projection of (X_t) onto $\mathscr{D}(X)$. Then $(\|Z_t\|)$ has jumps at all t_i; the jump at t_i has magnitude equal to

$$\sum_{j=1}^{M(t_i)} |\langle X_{t_i+}, \psi_{ij}\rangle|^2.$$

Proof: The sum is the squared norm of the projection of (X_t) onto the subspace spanned by the eigenvectors of $P_{t_i+} - P_{t_i}$. □

Proposition 6: Suppose (X_t) has orthogonal increments. If Z_t is the projection of X_t onto $\mathscr{D}(X)$, then $t \to \|Z_t\|$ is a step function.

Proof: Consider $\sum_{j=1}^{M(t_i)} |\langle X_t - X_s, \psi_{ij}\rangle|^2$ for $t > s \geq t_i$. As (X_t) is PND and has orthogonal increments, $E(X_t - X_s)\bar{u} = 0$ if $u \in L_s(X)$, so that $\langle X_t - X_s, \psi_{ij}\rangle^2 = 0$ for $t > s > t_i$. Thus, for $t > t_i$,

$$\sum_{j=1}^{M(t_i)} |\langle X_t, \psi_{ij}\rangle|^2 = \sum_{j=1}^{M(t_i)} |\langle X_t - X_{t_i+} + X_{t_i+}, \psi_{ij}\rangle|^2 = \sum_{j=1}^{M(t_i)} |\langle X_{t_i+}, \psi_{ij}\rangle|^2.$$

Corollary: If (X_t) has orthogonal increments, then for each discontinuity point t_i the value of $\sum_{j=1}^{M(t_i)} \langle X_t, \psi_{ij}\rangle \psi_{ij}$ is independent of t for $t > t_i$ and is zero for $t \leq t_i$.

Lemma 4: Suppose $Y_t \in L_t(X)$ and that Y_t is orthogonal to $\mathfrak{D}(X)$. Then there exists a family (B_n) of mutually-orthogonal m.s. continuous processes, each with orthogonal increments, and functions $\{F_n(t,\cdot);\ n \leq M\}$ on T such that

(a) the measures (μ_{B_n}) are ordered by absolute continuity:

$\mu_{B_n} \gg \mu_{B_{n+1}}$ for $n \leq M$;

(b) $F_n(t,s) = 0$ for $s > t$ and $F_n(t,\cdot)$ belongs to $L_2[\mu_{B_n}]$ for all $n \leq M$;

(c) $Y_t = \sum_{n=1}^{M} \int_{-\infty}^{t} F_n(t,s)dB_n(s)$ (if $M = \infty$, then the equality holds as a limit in the mean as $M \to \infty$).

Proof: We need only show the existence of a representation (c) having property (a), since the existence of a representation as in (c) and (b) has already been shown.

Let (Z_n) be orthogonal elements in $\mathscr{C}(X)$ such that $P_{\mathscr{C}}L_t(X) = \bigoplus_{n\geq 1} L_t(Z_n)$, all t in T, $P_{\mathscr{C}}$ the projection of $\underset{+}{L}(X)$ onto $\mathscr{C}(X)$, as in Theorem 1. Note that $P_{\mathscr{C}}L_t(X) = \overline{\text{span}\{P_{\mathscr{C}}X_s,\ s\leq t\}}$. Let $B_1 = \sum_{i\geq 1} \alpha_i Z_i$, where $|\alpha_i| \neq 0$ for all $i \geq 1$ and $\sum_{n\geq 1} |\alpha_i|^2 \|Z_i\|^2 < \infty$. Then, denoting F_Z as the map $t \to \|P_t Z\|^2$,

$$F_{B_1}(t) = \|P_t B_1\|^2 = \sum_{i\geq 1} |\alpha_i|^2 \|P_t Z_i\|^2 = \sum_{i\geq 1} |\alpha_i|^2 F_{Z_i}(t).$$

Thus, $\mu_{B_1} \gg \mu_{Z_i}$ for $i \geq 1$. In fact, $\mu_{B_1} \gg \mu_v$ for all v in $\mathscr{C}(X)$, since any such v has the representation $v = \sum_n P^n v$, P^n the projection of L(X) onto $L(Z_n)$, and $\mu_{P^n v} \ll \mu_{Z_n}$ by previous results (see the Remark following Theorem 1).

We thus have B_1 in $\mathscr{C}(X)$ such that $\mu_{B_1} \gg \mu_Y$ for all Y in $\mathscr{C}(X)$. Let $L(B_1)^{\perp}$ be the orthogonal complement in $\mathscr{C}(X)$ of $L(B_1)$. Then $L(B_1)^{\perp} = \oplus_n L(y^n)$, where (y^n) is constructed from an o.n. set in $L(B_1)^{\perp}$. Define $B_2 = \Sigma_i \alpha_{2i} y_i$, where $|\alpha_{2i}| > 0$ for all i, and $\sum_{i\geq 1} |\alpha_{2i}|^2 \|y_1^i\|^2 < \infty$. Then $\mu_{B_2} \gg \mu_Y$ for all Y in $L(B_1)^{\perp}$, and $\mu_{B_1} \gg \mu_{B_2}$.

Continuing in this way, we obtain a sequence of elements $\{B_n, n\leq M\}$ such that $\mu_{B_n} \gg \mu_{B_{n+1}}$ for $n \geq 1$, and $L_s(B_n) \perp L_t(B_m)$ for $n \neq m$, all s and t in T; the last statement is proved as in the proof of Theorem 1. Moreover, since $\{L(B_n), n\geq 1\}$ are orthogonal subspaces of $\mathbb{C}(X)$ and $\mathbb{C}(X)$ is separable, there exists an at-most countable number of such (B_n), and by Zorn's Lemma $\mathbb{C}(X) = \bigoplus_{n=1}^{M} L(B_n)$ where B can be infinite. This gives $P_{\mathscr{C}} L_t(X) = \bigoplus_{n=1}^{M} L_t(B_n)$ for all t in T. Thus, if $Y_t \in P_{\mathscr{C}} L_t(X)$, we have that $Y_t = \Sigma_1^M Y_t^n$, where Y_t^n is the projection of Y_t onto $L_t(B_n)$. Since $Y_t^n = \int_{-\infty}^{t} F_n(t,s) dP_s B_n$, with $E|Y_t^n|^2 = \int_T |F_n(t,s)|^2 d\mu_{B_n}$, the representation of part (c) follows. □

A distinguishing characteristic of the measures (μ_{B_n}) defined in Lemma 4 is that $\mu_{B_1} \gg \mu_Y$ for all Y in $\mathscr{C}(X)$, $\mu_{B_2} \gg \mu_Y$ for all $Y \perp L(B_1)$, and $\mu_{B_{n+1}} \gg \mu_Y$ for all $Y \perp \oplus_1^n L(B_i)$. Such a set will be called a *maximal chain* of measures. The preceding result shows the existence of a canonical representation such that the measures defined by the projection of (X_t) onto $\mathscr{C}(X)$ form a maximal chain. Nothing has been said about uniqueness. This will now be addressed.

<u>Lemma 5</u>: Suppose $L_t(X) = \bigoplus_{n=1}^{M} L_t(Z_n)$ for all t in T with $\mu_{Z_n} \gg \mu_{Z_{n+1}}$ for $n\geq 1$, where (μ_{Z_n}) is a maximal chain of measures, $M \leq \infty$. Consider $\bigoplus_{i=1}^{N} L(Y^i) \subset L(X)$,

where $\mu_{Y^i} >> \mu_{Y^{i+1}}$ for $i \geq 1$. Then $N \leq M$ and $\mu_{Y^i} << \mu_{Z^i}$ for $i \leq N$.

Proof: Clearly $\mu_{Y^i} << \mu_{Z_1}$ for all $i \leq N$, since Z_1 is a maximal element. We can assume $M < \infty$. Let $k \leq \min(N,M)$ be the smallest integer n such that $\mu_{Y^n} << \mu_{Z_n}$ is false (we will show that no such n can exist). Then μ_{Y^k} has the Lebesgue decomposition $\mu_{Y^k} = \lambda_k + \sigma_k$ where $\lambda_k << \mu_{Z_k}$, $\sigma_k \perp \mu_{Z_k}$. Thus, there exists a Borel $A \subset T$ such that $\sigma_k(A) = \sigma_k[T]$, $\mu_{Z_k}(A) = 0$. We will show that $\sigma_k[T] = 0$.

For $i < k$, $\mu_{Z_i} >> \mu_{Y^i} >> \mu_{Y^k} >> \sigma_k$. Define $u_i = \int_T \left[\frac{d\sigma_k}{d\mu_{Z_i}}\right]^{1/2}(s)dP_sZ_i$, $i < k$, and $v_i = \int_T \left[\frac{d\sigma_k}{d\mu_{Y^i}}\right]^{1/2}(s)dP_sY_i$, $i \leq k$. Since $v_i \in L(Y^i)$, we have $v_i \perp v_j$ and $P_tv_i \perp P_tv_j$ for $i \neq j$, $i,j \leq k$. Similarly, $u_i \perp u_j$ and $P_tu_i \perp P_su_j$ for all s,t in T and $i,j \leq k-1$ such that $i \neq j$. Note that $\mu_{u_i} = \sigma_k$ and $\mu_{v_i} = \sigma_k$ for $i \leq k$.

Claim: $\displaystyle\bigoplus_1^{k-1} L(u_i) = \{x \text{ in } L(X): \mu_x << \sigma_k\}$.

Proof of Claim: Let $W \in L(X)$, $\mu_W << \sigma_k$. Then $W = \Sigma_1^M w_i$, $w_i = \int_T g_i(s)dP_sZ_i$. $F_W(t) = \Sigma_1^M F_{w_i}(t)$, $F_{w_i}(t) = \int_{-\infty}^t |g_i|^2 d\mu_{Z_i}$, where $F_v(t) = \|P_tv\|^2$. From these relations and $\mu_W << \sigma_k$, $F_{w_i}(t) = \int_{-\infty}^t |g_i|^2 d\mu_{Z_i} = \int_{-\infty}^t |h_i|^2 d\sigma_k$ for all $i \leq M$. However, for $i \geq k$, $\mu_{Z_k} >> \mu_{Z_i}$ while by hypothesis $\sigma_k \perp \mu_{Z_k}$, so $\sigma_k \perp \mu_{Z_i}$ for $i \geq k$. Now, $\int_{-\infty}^t |g_i|^2 d\mu_{Z_i} = \int_{-\infty}^t |h_i|^2 d\sigma_k$ for all t in T, all $i \leq M$; for $i \geq k$, $\int_{-\infty}^t I_A |h_i|^2 d\sigma_k = \int_{-\infty}^t |h_i|^2 d\sigma_k$, all t, while $\int_{-\infty}^t |g_i|^2 I_A d\mu_{Z_i} = 0$, all t, since $\mu_{Z_k}(A) = 0$. Since $\mu_{w_i}[A \cap (-\infty,t)] = \int_{-\infty}^t |g_i|^2 I_A d\mu_{Z_i} = \int_{-\infty}^t |h_i|^2 I_A d\sigma_k$, $h_i = 0$ a.e. $d\sigma_k$, $i \geq k$. But since $\int_{-\infty}^t |h_i|^2 I_A d\sigma_k = \int_{-\infty}^t |h_i|^2 d\sigma_k = \int_{-\infty}^t |g_i|^2 d\mu_{Z_i}$, we have $g_i = 0$ a.e. $d\mu_{Z_i}$, $i \geq k$. This gives $\|w_i\|^2 = 0 \quad i \geq k$, since $\|w_i\|^2 = \int_T |g_i|^2 d\mu_{Z_i}$, and so $W = \sum_1^{k-1} w_i$, with $F_{w_i}(t) = \int_{-\infty}^t |g_i|^2 d\mu_{Z_i} = \int_{-\infty}^t |h_i|^2 d\sigma_k = \int_{-\infty}^t |h_i|^2 \frac{d\sigma_k}{d\mu_{Z_i}} d\mu_{Z_i}$; since the Radon-Nikodyn derivative $d\mu_{W_i}/d\mu_{Z_i}$ is unique up to a.e. $d\mu_{Z_i}$ equivalence, $g_i = |h_i| \left[\frac{d\sigma_k}{d\mu_{Z_i}}\right]^{1/2} [\text{sign } (g_i)]$, a.e. $d\mu_{Z_i}$.

We now show that $w_i \in L(u_i)$, where $u_i = \int_T \left[\frac{d\sigma_k}{d\mu_{Z_i}}\right]^{1/2} (s) dP_s Z_i$. We have $P_t u_i = \int_{-\infty}^t \left[\frac{d\sigma_k}{d\mu_{Z_i}}\right]^{1/2} (s) dP_s Z_i$, and so

$$\int_T |h_i(s)| (\text{sign } g_i)(s) dP_s u_i$$

$$= \int_T |h_i(s)| (\text{sign } g_i)(s) \left[\frac{d\sigma_k}{d\mu_{Z_i}}\right]^{\frac{1}{2}} (s) dP_s Z_i = \int_T g_i(s) dP_s Z_i = w_i.$$

Since w_i as in $L(u_i)$, $W = \sum_{i=1}^{M} w_i$ belong to $\bigoplus_1^{k-1} L(u_i)$, so that $\{x \in L(X): \mu_x << \sigma_k\} \subset \bigoplus_1^{k-1} L(u_i)$.

To prove the reverse inclusion, if $W = \sum_{i=1}^{k-1} \int_T h_i dP_t u_i =$ (by definition of u_i) $\sum_{i=1}^{k-1} \int_T h_i \left[\frac{d\sigma_k}{d\mu_{Z_i}}\right]^{1/2} (s) dP_s Z_i$, then $F_W(t) = \sum_{i=1}^{k-1} \int_{-\infty}^t h_i^2 d\sigma_k$, so $\mu_W << \sigma_k$. The Claim is proved.

By the definition of v_i, it is clear that $\mu_{v_i} = \sigma_k$ for $i \leq k$, since $P_t v_i = \int_{-\infty}^t \left[\frac{d\sigma_k}{d\mu_{Y^i}}\right]^{\frac{1}{2}} (s) dP_s Y_i$.

Thus, we have $\mu_{v_i} << \sigma_k$ for $i \leq k$ and $v_i \in L(X)$, so by the Claim, $v_i \in \bigoplus_1^{k-1} L(u_i) \quad i = 1, \ldots, k$.

This requires that $v_i = \sum_{j=1}^{k-1} \int_T C_{ij}(s) dP_s u_j$ all $i \le k$ where $\sum_{j=1}^{k-1} \int_T |C_{ij}(s)|^2 d\mu_{u_j}(s) < \infty$, $i \le k$. Since the definition of u_j gives $\mu_{u_j} = \sigma_k$ for $j \le k-1$, we have $F_{v_i}(t_{i^+}) = \mu_{v_i}(-\infty, t] = \sigma_k(-\infty, t] = \sum_{j=1}^{k-1} \int_{-\infty}^{t} |C_{ij}(s)|^2 d\sigma_k(s)$ for all t in T, all $i \le k$. Thus,

$$\sum_{j=1}^{k-1} |C_{ij}(s)|^2 = 1 \quad \text{a.e. } d\sigma_k(s) \text{ for } i = 1, \ldots, k.$$

Similarly, since $v_i \in L(Y_i)$, $v_j \in L(Y^j)$ and $L(Y^i) \perp L(Y^j)$, giving $P_s v_i \perp P_t v_j$ for $i, j \le k$, $i \ne j$ and all s, t in T, one has for $i \ne j$

$$\begin{aligned} 0 &= EP_t v_i \overline{P_t v_j} \\ &= (\text{since } EP_s u_m \overline{P_t u_{m'}} = 0 \text{ for } m \ne m',\ m, m' \le k-1, \text{ all } s, t) \\ &\quad \sum_{m=1}^{k-1} \int_{-\infty}^{t} C_{im}(s) \overline{C_{jm}(s)} d\mu_{u_m}(s) \\ &= (\text{since } \mu_{u_m} = \sigma_k,\ m \le k-1) \sum_{m=1}^{k-1} \int_{-\infty}^{t} C_{im}(s) \overline{C_{jm}(s)} d\sigma_k(s) \\ &= \int_{-\infty}^{t} \sum_{m=1}^{k-1} C_{im}(s) \overline{C_{jm}(s)} d\sigma_k(s), \end{aligned}$$

all t in T. Hence

$$\sum_{m=1}^{k-1} C_{im}(s) \overline{C_{jm}(s)} = 0 \quad \text{a.e. } d\sigma_k(s) \quad \text{for } i \ne j.$$

We now have a family of k vector-valued functions, $\underset{\sim}{C}_i$, each $\underset{\sim}{C}_i$ having k-1 components, $\underset{\sim}{C}_i(s) = (C_{im}(s), \ldots, C_{i,k-1}(s))$ such that a.e. $d\sigma_k(s)$

$$\sum_{j=1}^{k-1} |C_{ij}(s)|^2 = 1 \qquad i = 1, \ldots, k$$

and

$$\sum_{j=1}^{k-1} C_{ij}(s) C_{mj}(s) = 0 \qquad i \ne m.$$

Suppose that the measure σ_k gives positive measure to some Borel set in T. Then, for all s in a set of positive σ_k measure, we have a set of k elements in E^{k-1}, the ith being $(C_{i1}(s),\ldots,C_{i,k-1}(s))$ which are orthonormal. Since any orthonormal set of elements belonging to E^{k-1} can contain at most k-1 elements, this contradiction establishes that σ_k is identically zero. Hence, we have (from the original Lebesgue decomposition of μ_{Y^k}) $\mu_{Y^k} << \mu_{Z^k}$. Thus,

$$\mu_{Y^i} << \mu_{Z_i} \quad \text{for } i \leq \min(N,M).$$

Suppose now that $M < N$. Define

$$u_i = \int_T \left[\frac{d\mu_{Y^{M+1}}}{d\mu_{Z_i}}\right]^{1/2}(s)dP_sZ_i \qquad i \leq M$$

$$v_i = \int_T \left[\frac{d\mu_{Y^{M+1}}}{d\mu_{Y_i}}\right]^{1/2}(s)dP_sY_i \qquad i \leq M+1.$$

We then have $\mu_{u_i} = \mu_{Y^{M+1}} = \mu_{v_i}$ and

$$\oplus_1^M L(u_i) = \{w \in \oplus_1^M L(Z_i): \mu_w << \mu_{Y^{M+1}}\};$$

the equality being proved as in the case of the Claim, replacing σ_k by $\mu_{Y^{M+1}}$. Since $\mu_{v_i} << \mu_{Y^{M+1}}$ for $i \leq M+1$, this gives

$$\bigoplus_1^{M+1} L(v_i) \subset \bigoplus_1^{M} L(u_i).$$

Proceeding in the same manner as for σ_k previously, one obtains that $\mu_{Y^{M+1}}[T]$ must equal zero. This shows that $N \leq M$, proving the theorem. □

Corollary: Suppose that

$$L(X) = \oplus_1^M L(Z_i) \oplus \mathfrak{D}$$

where $(P_t Z_i)$ are m.s. continuous processes and have ordered spectral measures, $\mu_{Z_n} \gg \mu_{Z_{n+1}}$, $n < M$. Then if

$$L(X) = \oplus_1^N L(Y_i) \oplus \mathfrak{D}$$

where $\mu_{Y_n} \gg \mu_{Y_{n+1}}$ $n \geq 1$, it is necessary that $N = M$ and $\mu_{Y_n} \ll \mu_{Z_n} \ll \mu_{Y_n}$ $n = 1,\ldots,M$.

Proof: It remains only to show that if (X_t) is m.s. continuous and $L(X) = \bigoplus_{i=1}^{M} L(Z_i)$, with $\mu_{Z_n} \gg \mu_{Z_{n+1}}$, $n < M$, then $\mu_{Z_1} \gg \mu_y$ for any y in L(X). But y in L(X) requires that $y = \Sigma_{i=1}^M \int_0^T h_i dP_s Z_i$, so $F_y(t) = \Sigma_{i=1}^M \int_0^t |h_i|^2 dF_{Z_i}$ and since $\mu_{Z_i} \ll \mu_{Z_1}$ for all $i \geq 1$, one has that $\mu_y \ll \mu_{Z_1}$. □

Lemma 5 makes no assumptions on the discontinuities of (P_t); in fact, those discontinuities (if any) do not enter into the representation. Thus, one obtains another corollary to Lemma 5, stated in the following theorem. It is due to Cramér [ ] and the representation it contains will be termed "Cramér's representation."

Theorem 2. Suppose that (X_t) is a SOSP satisfying assumptions (A1) and (A2). Then there exists $\{Z_n, n \geq 1\}$ in L(X) such that

$$L_t(X) = \bigoplus_{i=1}^{M} L_t(Z_n)$$

and $\mu_{Z_n} \gg \mu_{Z_{n+1}}$, all $n < M$. If also there exists $\{Y_n, n \geq 1\}$ in L(X) with

$$L_t(X) = \bigoplus_{i=1}^{N} L_t(Y_n) \quad \text{for all } t \text{ in } T,$$

and $\mu_{Y_n} >> \mu_{Y_{n+1}}$ for $n < N$, then $N = M$ and $\mu_{Z_{n+1}} << \mu_{Y_{n+1}} \mu_{Z_{n+1}}$ for $0 \leq n < M$. The unique number M $\{0 < M \leq \infty\}$ is the *multiplicity* of (X_t). Any x in $L_t(X)$ has the representation

$$x = \sum_{i=1}^{M} \int_0^t G_n(t,s)\, dP_s Z_n,$$

where $G_n(t,\cdot)$ is in $\mathcal{L}_2[\mu_{Z_n}]$ and $\Sigma_{n=1}^{M} \int_0^1 G_n^2(t,s) d\mu_{Z_n} < \infty$.

The next theorem is essentially due to Hida [ ]; the representation given will be termed "Hida's representation".

Theorem 3: Let (X_t), $t \in T$ (an interval) be a second-order stochastic process satisfying assumptions (A1) and (A2). Then

$$X_t = \Sigma_{n=1}^{M} \int_{-\infty}^{t} F_n(t,s) dB_n(s) + \sum_{t_i < t} \sum_{j=1}^{d(t_i)} f_{ij}(t) Q_{ij}$$

for all t in T, where the equality holds in $L_2(\Omega,\beta,P)$, and with the RHS having the following properties:

(1) (B_n) is a mutually-orthogonal family of m.s. continuous orthogonal-increment stochastic processes, with spectral measures (μ_{B_n}) satisfying $\mu_{B_n} >> \mu_{B_{n+1}}$ for $n \geq 1$.

(2) $F_n(t,s) = 0$ for $s > t$ and $F_n(t,\cdot)$ belongs to $\mathcal{L}_2[\mu_{B_n}]$, all t in T, all $n \geq 1$, and $\sum_{n=1}^{M} \int_{-\infty}^{t} |F_n(t,s)|^2 d\mu_{B_n}(s) < \infty$ for each t in T.

(3) (t_i) are the discontinuities of $\{L_t(X),\ t \in T\}$, $L_t(X) = \overline{\text{span}\{X_s,\ s \leq t\}}^{L_2(\Omega,\beta,P)}$; i.e., those points t_i such that $L_{t_i}(X) \neq \bigcap_{s > t_i} L_s(X)$.

(4) $1 \le d(t_i) \le \infty$, for each t_i.

(5) $\{Q_{ij}, j \le d(t_i), i \ge 1\}$ are o.n. elements in $L_2(\Omega,\beta,P)$.

(6) Q_{ij} and $B_n(t)$ are orthogonal in $L_2(\Omega,\beta,P)$, for $n \le M$, all $t \in T$, $j \le d(t_i)$, $i \ge 1$.

(7) $f_{ij}(t) = \langle X_t, Q_{ij}\rangle$ all t, (i,j), and

(8) $L_t(X) = \oplus_1^M L_t(B_n) \oplus \overline{\text{span}\{Q_{ij}, j \le d(t_i), t_i < t\}}$.

Moreover, if (X_t) has a representation

$$X_t = \Sigma_1^N \int_{-\infty}^t F'_n(t,s)dB'_n(s) + \sum_{t_i \le t} \sum_{j=1}^{d(t_i)} f'_{ij}(t)Q'_{ij}$$

where (F'_n), (B'_n), (Q'_{ij}), (f'_{ij}) have properties (1)-(8), then $N = M$, the measures μ_{B_n} and $\mu_{B'_n}$ are mutually absolutely continuous for $n \le M$, $F'_n(t,s) = F_n(t,s)\dfrac{d\mu_{B_n}}{d\mu_{B'_n}}$ a.e. $d\mu_{B'_n}$, $n \le M$, and $Q'_{ij} = A_iQ_{ij}$ for $j \le d(t_i)$, each t_i, where A_i is a unitary matrix in $\mathbb{R}^{d(t_i)}$.

The *multiplicity* of (X_t) is then defined to be $\sup(M, \dim \mathfrak{D}(X))$.

To see that the multiplicity as defined in Theorem 3 is the same as that defined in Theorem 2, let $\{Q_{ij}, j \le M(t_i)$, discontinuities (t_i) of $(P_t)\}$ be a c.o.n. set in $\mathfrak{D}(X)$ such that $P_{t_i}Q_{ij} = 0$, $P_{t_i+}Q_{ij} = Q_{ij}$, $j \le M(t_i)$. For each ij, (P_tQ_{ij}) is a PND OI, with $\mu_{Q_{ij}}$ giving measure one to t_i, $\mu_{A_{ij}}\{T_i\}^c = 0$. Let (B_n) be such that $\mathfrak{C}(X) = \oplus_{i=1}^M L(B_n)$. Define (Z_n) by

$$\begin{aligned} Z_n &= B_n + Y_n && n \le \min(M, \dim \mathfrak{D}(X)) \\ &= B_n && \text{if } \dim \mathfrak{D}(X) < n \le M \\ &= Y_n && \text{if } M < n \le \dim \mathfrak{D}(X). \end{aligned}$$

where $n < \infty$ in all cases. The sequence (Y_n) is given by $Y_n = \Sigma_i \Sigma_{j=1}^{M(t_i)} \alpha_{ij}^n Q_{ij}$, where the $\{(\alpha_{ij}^n): n \geq 1,\ j \leq M(t_i),$ all discontinuities $t_i\}$ are defined as follows

$$\sum_i \sum_{j=1}^{M(t_i)} \alpha_{ij}^2 = 1; \qquad \text{and}$$

$$\alpha_{ij}^1 \neq 0, \quad \text{all } ij;$$

$$Y_{n+1} \perp sp\{Y_1, \ldots, Y_n\}, \quad \text{all } n > 1;$$

$$\alpha_{ij}^{n+1} \neq 0 \ \text{ if and only if } Q_{ij} \notin sp\{Y_1, Y_2, \ldots, Y_n\}, \quad n > 1.$$

To prove that $L(X) = \oplus_{i=1}^{M \vee \dim \mathfrak{D}(X)} L(Z_i)$, it is sufficient to show that $\mathfrak{D}(X) = \overline{sp\{Y_n, n \geq 1\}}$. Since (Q_{ij}) is a c.o.n. set in $\mathfrak{D}(X)$, $\mathfrak{D}(X) = \overline{sp\{Y_n, n \geq 1\}}$ if and only if $Q_{ij} \in \overline{sp\{Y_n, n \geq 1\}}$ for all ij. Let (Q'_{ij}) be the set of all (Q_{ij}) such that (Q'_{ij}) is not in $\overline{sp\{Y_n, n \geq 1\}}$. Then, by definition of (Y_n), there exists an element $Y = \Sigma_{ij} \alpha'_{ij} Q_{ij}$ in $\{Y_n, n \geq 1\}$ where $\alpha'_{ij} \neq 0$ if and only if $Q_{ij} \in \{Q'_{ij}: Q'_{ij} \notin \overline{sp\{Y_n, n \geq 1\}}\}$. This requires that $Y \perp Q'_{ij}$ for all Q'_{ij}, a contradiction.

The representation given in Theorem 3 is called the *proper canonical representation* of (X_t). Although the elements (B_n), (F_n), (Q_{ij}), and (f_{ij}) appearing in the representation are not unique, their number is unique and those appearing in any given proper canonical representation can be obtained from the corresponding elements of any other such representation.

The space $\mathfrak{D}(X)$ consists only of $\{0\}$ if (X_t) is m.s. continuous. If (P_t) has a discontinuity at t_0, then necessarily (X_t) is not m.s. right-continuous at t_0; we have seen that the converse does not hold. In order for (P_t) to have a discontinuity at t_0, it is necessary and sufficient that $\bigcap_{s>t_0} L_s(X)$ contain a non-zero element orthogonal to $L_{t_0}(X)$.

Corollary: Suppose that (X_t) is m.s. continuous on T. Then for all t in T, $X_t = \sum_{n=1}^{M} \int_{-\infty}^{t} F_n(t,s)dB_n(s)$ where $\{F_n(t,\cdot),\ n \leq M\}$ and $\{B_n,\ n \geq 1\}$ are as in Theorem 3. The covariance function of (X_t) has the representation $EX_t\overline{X}_s = \sum_{n=1}^{M} \int_{-\infty}^{t\wedge s} F_n(t,u)\overline{F_n(s,u)}d\mu_{B_n}(u)$ for $\{\mu_{B_n},\ n \leq M\}$ defined as in Theorem 3. If T is a finite interval, T = [a,b], then the covariance operator R_X of (X_t), $R_X: L_2[a,b] \to L_2[a,b]$, has trace equal to Trace $R_X = E\int_a^b |X_t|^2 dt = \sum_{n=1}^{M} \int_a^b \int_a^t |F_n(t,u)|^2 d\mu_{B_n}(u)$.

Proof: Follows directly from Theorem 3.

The development as given by Hida [5] replaces assumption (A1) with the assumption that L(X) is separable and that the m.s. limits X_{t+} and X_{t-} exist for all t. $L_t^*(X)$ is then defined as $\bigcap_{s>t} L_s(X)$. The projection operator P_t^* is the operator with range equal to $L_t^*(X)$. The family $\{P_t^*,\ t \in T\}$ is then a right-continuous resolution of the identity. This defines a self-adjoint linear operator T in L(X), $T = \int_{-\infty}^{\infty} \lambda dP_\lambda^*$. The Hellinger-Hahn Theorem [6, p. 247 ff.] is then applied to obtain the proper canonical representation.

5. General Formulation of Spectral Multiplicity

The preceding results were derived for a second-order stochastic process. However, they can be formulated purely in terms of a iven self-adjoint operator in Hilbert space. The following result contains the Hellinger-Hahn theorem and related results ([6], Sec. VII.2), and consequences.

Theorem 4: Let T be a self-adjoint bounded linear operator in the real separable Hilbert space H. Let $\{P_\lambda, \lambda \in \mathbb{R}\}$ be a left-continuous resolution of the identity determined by T. Then:

(1) $H = \mathscr{C} \oplus \mathscr{D}$, where $\mathscr{D}$ is the closed linear span of all eigenvectors of T, and $\mathscr{C} = \mathscr{D}^\perp$;

(2) For any x in $\mathscr{C}$, the map $\lambda \to \|P_\lambda x\|$ is continuous;

(3) There exists an orthonormal basis $\{e_n, n \geq 1\}$ for $\mathscr{C}$ such that $\mu_n >> \mu_{n+1}$, all $n \geq 1$, where μ_n is the Lebesgue-Stieltjes measure on $\mathbb{R}$ determined by the non-decreasing function $\lambda \to \|P_\lambda e_n\|^2$;

(4) If $\mathscr{D} \neq H$, $\{v_n, n \geq 1\}$ is any other basis for $\mathscr{C}$ such that $\nu_n >> \nu_{n+1}$ for all $n \geq 1$, where ν_n is determined by $\lambda \to \|P_\lambda v_n\|^2$, then $\nu_n \sim \mu_n$ for all $n > 1$;

(5) Suppose that $\mathscr{C}$ has dimension $M \geq 1$, and that $\{e_n, n \geq 1\}$ and $\{\mu_n, n \geq 1\}$ are as in (3). Let $\{\lambda_n, n \geq 1\}$ be the set of *distinct* eigenvalues of T, let $m(n)$ be the dimensionality of the subspace spanned by the eigenvectors for the eigenvalue λ_n, and let $\{u_n^i, i \leq m(n)\}$ be orthonormal eigenvectors corresponding to λ_n. Then, for any x in H and any λ in $\mathbb{R}$, there exists a family of Borel-measurable functions $\{F_n(\lambda,\cdot), n \leq M\}$, depending on x, such that $\sum_{n=1}^{M} \int_{-\infty}^{\lambda} |F_n(\lambda,s)|^2 d\mu_n(s) < \infty$, and

$$P_\lambda x = \sum_{n=1}^{M} \int_{-\infty}^{\lambda} F_n(\lambda,s) dP_s e_n + \sum_{\lambda_n < \lambda} \sum_{i=1}^{m(n)} \langle u_n^i, x\rangle u_n^i.$$

Each summand in the first term on the RHS of this expression is the limit of partial sums of the form $\sum_{k=0}^{K-1} \int_{-\infty}^{\lambda} F_n(\lambda,s_k')[P_{s_{k+1}} - P_{s_k}]e_n$, where $s_i < s_{i+1}$ for $0 \leq i \leq K-1$, $s_k' \in (s_k, s_{k+1}]$, and the limit is taken as $\sup\{s_{k+1} - s_k: 0 \leq k \leq K-1\} \to 0$ and $s_K \uparrow \infty$, $s_0 \downarrow -\infty$.

Proof: This theorem is simply a reformulation of the preceding results on the spectral representation of a purely deterministic m.s. left-continuous SOSP, replacing L(X) by H. The only use made of the assumptions (A1) and (A2) was to show that L(X) is separable and to establish a left-continuous resolution of the identity, the family $\{P_t, t \in \mathbb{R}\}$ of Lemma 1. Mean-square left continuity of the stochastic process $(P_t Z)$ for Z in L(X) corresponds to left-continuity of $t \to \|P_t Z\|$.

In the present theorem, the given operator T has a left-continuous resolution of the identity. The space H is assumed to be separable. The theorem then follows from Theorem 3 above. □

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