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Projection with a
Minimal System of Inequalities

by
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Carnegie Mellon University

July, 1992

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Abstract

Projection of a polyhedron involves the use of a cone whose extreme rays induce the inequalities defining the projection. These inequalities need not be facet defining.

We introduce a transformation that produces a cone whose extreme rays induce facets of the projection.

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Given a polyhedron

$$Q := \{(u, x) \in \mathcal{R}^p \times \mathcal{R}^q : Au + Bx \leq d\},$$

where A, B and d have m rows, the *projection* of Q onto the subspace of the x -variables is

$$P_x(Q) := \{x \in \mathcal{R}^q : \exists u \in \mathcal{R}^p \text{ with } (u, x) \in Q\}.$$

We are discussing the projection of a polyhedron, but the projection of a more general set can often be reduced to that of a polyhedron: for instance, if instead of Q we consider the nonpolyhedral set

$$S := \{(u, x) \in \mathcal{R}^p \times \mathcal{R}^q : Au + Bx \leq d, x \in X\}$$

where X is arbitrary, then

$$P_x(S) = P_x(Q) \cap X.$$

It is well known (see, for instance, [2], Section IV, or [5], Chapter I.4.4) that

$$P_x(Q) = \{x \in \mathcal{R}^q : (vB)x \leq vd \text{ for all } v \in \text{extr } W\}$$

where *extr* W denotes the set of extreme rays of the *projection cone*

$$W := \{v \in \mathcal{R}^m : vA = 0, v \geq 0\}.$$

It is also well known that although the inequalities defining $P_x(Q)$ are in 1-1 correspondence with the extreme rays of W , they do not necessarily define facets of $P_x(Q)$, i.e. the system defining $P_x(Q)$ is not necessarily minimal. As pointed out by M. Goemans in his recent talk at IPCO II [4], in practice often large numbers of redundant inequalities are generated, even though only extreme rays of W are used. It would be nice to be able to tell which extreme rays of W induce facets of $P_x(Q)$, but that question has no answer in terms of the properties of W only: whether the inequality $(vB)x \leq vd$ defines a facet of $P_x(Q)$ depends on B and d as well as on v itself. Instead of trying to answer that question, therefore, we address a related problem: is there an alternative representation of Q , whose projection cone (or its projection on a subspace) has the property that *all* of its extreme rays induce facets of the projection? In the sequel we give such a representation. Its key feature, which brings about the desired property, is that in this representation the coefficient matrix

of x is the identity matrix plus possibly some zero rows, while the right hand side is the unit vector with 1 in the last position.

Let $\text{rank}(B) = r$, and w.l.o.g. assume that the first r rows and columns of B are linearly independent. (Recall that B is $m \times q$). Perform the following sequence of operations on the system defining Q :

1. If $r = m$, let $B_1 := B$ and go to 2. Otherwise add to B $m - r$ new columns $b^{q+1}, \dots, b^{q+m-r} \in \mathcal{R}^m$, where for $j = 1, \dots, m - r$, $b^{q+j} := e_{r+j}$, and where e_ℓ is the unit vector in $\mathcal{R}^m$ with 1 in position ℓ . Call the resulting matrix B_1 and go to 2.
2. If $r = q$, i.e. if B_1 is $m \times m$ with $\text{rank}(B_1) = m$, let $B_0 := B_1$, $A_0 := A$, $d_0 = d$, and go to 3. Otherwise add to B_1 $q - r$ new rows $b_{m+1}, \dots, b_{m+q-r} \in \mathcal{R}^{m+q-r}$, where for $i = 1, \dots, q - r$, $b_{m+i} = e_{r+i}^T$, and where e_ℓ^T is the transpose of the unit vector in $\mathcal{R}^{m+q-r}$ with 1 in position $r + i$. Call the resulting matrix B_0 . Add $q - r$ zero rows to A and call the resulting matrix A_0 . Finally, add to d $q - r$ components equal to M (a number sufficiently large for the resulting inequality to be redundant), and call the resulting vector d_0 . Then go to 3.
3. Let B_0 be $n \times n$. By construction, B_0 is nonsingular. Consider now the polyhedron

$$\begin{aligned} Q' &:= \left\{ (u, s, x') \in \mathcal{R}^p \times \mathcal{R}^n \times \mathcal{R}^n \left| \begin{array}{l} A_0 u + I s + B_0 x' = d_0, \quad s \geq 0 \\ x'_{q+j} = 0 \text{ for } j = 1, \dots, n - q \end{array} \right. \right\} \\ &= \left\{ (u, s, x') \in \mathcal{R}^p \times \mathcal{R}^n \times \mathcal{R}^n \left| \begin{array}{l} B_0^{-1} A_0 u + B_0^{-1} s + x' = B_0^{-1} d_0, \quad s \geq 0 \\ x'_{q+j} = 0 \text{ for } j = 1, \dots, n - q \end{array} \right. \right\} \end{aligned}$$

and rewrite Q' as

$$Q'' = \left\{ (u, s, u_0, x') \in \mathcal{R}^p \times \mathcal{R}^n \times \mathcal{R} \times \mathcal{R}^n \left| \begin{array}{l} B_0^{-1} A_0 u + B_0^{-1} s - B_0^{-1} d_0 u_0 + x' = 0, \quad s \geq 0 \\ u_0 = 1 \\ x'_{q+j} = 0 \text{ for } j = 1, \dots, n - q \end{array} \right. \right\}$$

4. Let $B_0^{-1} = \begin{pmatrix} B_{0q}^{-1} \\ B_{0,n-q}^{-1} \end{pmatrix}$, where B_{0q}^{-1} and $B_{0,n-q}^{-1}$ are the submatrices containing the first

q and the last $n - q$ rows of B_0^{-1} , respectively, and let

$$Q^0 := \left\{ (u, s, u_0, x) \in \mathcal{R}^p \times \mathcal{R}^n \times \mathcal{R} \times \mathcal{R}^q \left| \begin{array}{llll} B_{0q}^{-1} A_0 u & + & B_{0q}^{-1} s & - & B_{0q}^{-1} d_0 u_0 & + & x & = & 0 \\ B_{0,n-q}^{-1} A_0 u & + & B_{0,n-q}^{-1} s & - & B_{0,n-q}^{-1} d_0 u_0 & & & = & 0 \\ & & & & u_0 & & & = & 1 \\ & & & & & s & & \geq & 0 \end{array} \right. \right\}.$$

5. To project Q^0 onto the x -space, define the projection cone

$$W^0 := \left\{ (v, w, v_0) \in \mathcal{R}^q \times \mathcal{R}^{n-q} \times \mathcal{R} \left| \begin{array}{lll} v B_{0q}^{-1} A_0 & + & w B_{0,n-q}^{-1} A_0 & = & 0 \\ v(-B_{0q}^{-1}) A_0 & + & w(-B_{0,n-q}^{-1}) A_0 & + & v_0 & = & 0 \\ v B_{0q}^{-1} & + & w B_{0,n-q}^{-1} & \geq & 0 \end{array} \right. \right\}.$$

Note that, since B_0^{-1} is nonsingular, W^0 is pointed. The projection of Q^0 is then

$$P_x(Q^0) = \left\{ x \in \mathcal{R}^q \left| \begin{array}{l} vx \leq v_0 \text{ for all } (v, v_0) \in \mathcal{R}^q \times \mathcal{R} \text{ such that} \\ (v, w, v_0) \in \text{extr } W^0 \text{ for some } w \in \mathcal{R}^{n-q} \end{array} \right. \right\},$$

where $\text{extr } W^0$ denotes the set of extreme rays of W^0 .

It is easy to see that $P_x(Q^0) \equiv P_x(Q)$, since there is a 1-1 correspondence between the points of Q and those of Q^0 .

Consider now the polyhedral cone polar to $P_x(Q^0)$, namely

$$P_x^*(Q^0) = \left\{ (v, v_0) \in \mathcal{R}^q \times \mathcal{R} \mid vx \leq v_0 \text{ for all } x \in P_x(Q^0) \right\}.$$

By construction,

$$\begin{aligned} P_x^*(Q^0) &= \{ (v, v_0) \in \mathcal{R}^q \times \mathcal{R} : (v, w, v_0) \in W^0 \text{ for some } w \in \mathcal{R}^{n-q} \}, \\ &\equiv P_{(v, v_0)}(W^0), \end{aligned}$$

i.e. the polar cone of the projection of Q^0 on the x -space is the projection of W^0 on the (v, v_0) -space.

But then from basic properties of polarity, we have the following.

Theorem 1 *Let $P_x(Q)$ be full dimensional. Then the inequality $vx \leq v_0$ defines a facet of $P_x(Q)$ if and only if (v, v_0) is an extreme ray of the cone $P_{(v, v_0)}(W^0)$.*

Proof. If $P_x(Q)$ is full dimensional, so is $P_x(Q^0)$; and from basic properties of polarity, there is a 1-1 correspondence between facets of the polyhedron $P_x(Q^0)$ and extreme rays of its polar cone $P_x^*(Q^0)$. But $P_x^*(Q^0) \equiv P_{(v,v_0)}(W^0)$. $\square$

Note that if $(\bar{v}, \bar{v}_0)$ is an extreme ray of $P_{(v,v_0)}(W^0)$, then there exists $\bar{w} \in \mathcal{R}^{n-q}$ such that $(\bar{v}, \bar{w}, \bar{v}_0)$ is an extreme ray of W^0 . The converse, however, is not always true.

On the other hand, in the important special case when the matrix B is of full row rank, i.e. when $r = m$ and $n = q$, we have

Corollary 2 *Let $P_x(Q)$ be full dimensional, and $\text{rank}(B) = m$. Then the inequality $vx \leq v_0$ defines a facet of $P_x(Q)$ if and only if (v, v_0) is an extreme ray of the cone W^0 .*

A few comments are in order.

First, note that the definition of Q does not contain explicitly constraints of the form $u \geq 0$; if they are present, they are part of the system $Au + Bx \leq d$, i.e. A and B are of the form $A = \begin{pmatrix} A' \\ -I \end{pmatrix}$ and $B = \begin{pmatrix} B' \\ 0 \end{pmatrix}$, respectively, where I is the $p \times p$ identity matrix and 0 is the $p \times q$ zero matrix.

Second, while the above described transformation produces a projection cone with a very desirable property, there is a price to pay for this: if the matrix A has a structure that makes it easy to generate the extreme rays of W , that structure gets lost in the transformation, and the extreme rays of W^0 , or $P_{(v,v_0)}(W^0)$, may be much harder to generate.

Third, in many important cases the matrix B is of the form $B = \begin{pmatrix} I \\ 0 \end{pmatrix}$, which voids the need for the above transformation and produces directly a projection cone $W := \{(v, w, v_0) : (v, w)A = 0, (v, w) \geq 0\}$ such that the extreme rays of $P_{(v,v_0)}(W)$, the projection of W onto the (v, v_0) -space, yield facets of $P_x(Q)$. This is the case encountered, for instance, in the characterization of the perfectly matchable subgraph polytope of a graph in [3]; as well as in the recent lift-and-project cutting plane procedure of [1]. In these cases, it is important to know how to use the cone W to generate extreme rays of $P_{(v,v_0)}(W)$ (see [1] and its references for a discussion of this issue, which was investigated in the 70's).

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