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AN APPROXIMATE SOLUTION FOR THE
LINEARIZED SPINUP DYNAMICS OF A
DUAL-SPIN SPACECRAFT

THESIS

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AN APPROXIMATE SOLUTION FOR THE LINEARIZED SPINUP
DYNAMICS OF A DUAL-SPIN SPACECRAFT

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Astronautical Engineering

David L. Kinney, B.S.

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Approved for public release; distribution unlimited

Preface

This thesis contains approximate solutions to the linearized equations of motion for the spinup of an ideal dual-spin spacecraft. For your convenience, a list of the symbols used in the development of these approximations can be found on page viii.

I would like to thank all of my instructors for the clarity and understanding with which they taught their subjects. I would especially like to thank two faculty members. First, I am grateful to Col R. L. Bagley for making dynamics really understandable so that it became my favorite area of study. Second, I thank my thesis advisor, Capt C. D. Hall, for his timely assignment to the faculty and his interest in spacecraft dynamics. Without him I would have had to choose a thesis topic in which I may not have been interested.

David L. Kinney

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List of Symbols and Notation

Roman Symbols

A	Amplitude function
C_j	Constant, $j=1,2$
$\hat{e}_j$	Unit vectors defining the principal axes of P+R, $j=1,2,3$
g_a	Spinup torque
h	Magnitude of the total angular momentum of P+R
$\vec{h}$	Total angular momentum vector of P+R
h_a	Angular momentum of R about $\hat{e}_1$
h_j	Angular momentum of P+R about $\hat{e}_j$, $j=1,2,3$
i	Square root of negative one
i_j	Dimensionless moment of inertia of P+R about $\hat{e}_j$, $j=1,2,3$
I_j	Moment of inertia of P+R about $\hat{e}_j$, $j=1,2,3$
I_p	Moment of inertia of P about $\hat{e}_1$
I_s	Moment of inertia of R about $\hat{e}_1$
k_j	$\mu_0 - i_j$, $j=2,3$
P	Platform of the spacecraft
R	Rotor of the spacecraft
S	Phase function
S_q	The qth WKB term, $q=0,1,2 \dots$
$S_{q,0}$	Value of S_q at τ_0
t	Dimensionless time
$\tilde{t}$	Time
x_j	Small perturbation from equilibrium point of X_j

x_{j0}	Value of x_j at τ_0
X_j	Dimensionless angular momentum of P+R about $\hat{e}_j$, $j=1,2,3$
X_{10}	Value of X_1 at $t=0$

Greek Symbols

δ	Boundary-layer thickness
ϵ	Dimensionless spinup torque
μ	Dimensionless angular momentum of R about $\hat{e}_1$
μ_0	Value of μ at $t=0$
τ	ϵt
τ_0	Value of τ at some initial time
ω_j	Angular velocity of P about $\hat{e}_j$, $j=1,2,3$
ω_s	Angular velocity of R relative to P about $\hat{e}_1$

Other Symbols and Notation

$(\dot{})$	$d()/dt$
$()'$	$d()/d\tau$
$ $	Absolute value
$\sim$	Asymptotic approximation
$\%$	Percent

Abstract

An approximate solution is obtained for the linearized equations of motion for the spinup dynamics of a dual-spin spacecraft. The approximation is obtained from the WKB (Wentzel, Kramers, and Brillouin) method, which usually generates a divergent series. Comparison between the WKB solution and the numerically integrated solution are used to study the effects of spinup torque magnitude and the number of WKB terms used on the accuracy of the approximation. The WKB method generates accurate approximations for small spinup torques using as few as two terms in the series when no singularities are present in the WKB terms during the spinup time interval. For an axisymmetric spacecraft, the WKB series truncates after one term and provides the exact solution to the equations of motion.

AN APPROXIMATE SOLUTION FOR THE LINEARIZED SPINUP
DYNAMICS OF A DUAL-SPIN SPACECRAFT

I. Introduction

Background

Dual-spin spacecraft models consist of a platform and a rotor initially spinning together. To get the platform to despin, angular momentum is transferred from the platform to the rotor causing the rotor to spinup. The equations of motion for these dynamics are generally only solvable by developing an approximate solution or by numerical integration. The approximate solution has the advantage over numerical integration in that if the state of a system needs to be known at a particular time in the future, only that time needs to be substituted into the approximation to obtain the answer. The numerical method must be integrated from some initial time when the state is known to the time in question. In 1976, Gebman and Mingori published such an approximation using perturbation methods (reference 2). Their method examined flat spin recovery which will not be discussed in this thesis.

Problem and Scope

This thesis contains the examination of the dynamics of a dual-spin spacecraft using a perturbation technique called

the WKB (Wentzel, Kramers, and Brillouin) method. The spacecraft is modeled as a rigid body gyrost consisting of a rotor and a platform initially spinning together. A small constant torque is applied by the platform on the rotor until the platform is no longer spinning with respect to an inertial reference frame. The equations of motion for this system are linearized and the WKB method is applied to obtain an approximate solution. Both oblate and prolate spinup are examined. The linear equations of motion are solved using a numerical integrator. The WKB approximation and the linear solutions are compared to determine the accuracy and limitations of the WKB solution. Dimensionless torques of $\epsilon=0.1$, 0.01 , and 0.001 are used in this comparison as well as from one to six terms in the WKB solution.

Assumptions

The following assumptions are made in this problem:

1. There are no external torques on the spacecraft.
2. There are no losses due to friction.
3. Both the platform and the rotor are rigid bodies.
4. The rotor is axisymmetric about its spin axis.
5. The platform is asymmetric, but the axisymmetric case is also examined.
6. The spinup torque is constant during spinup and zero at all other times.

General Approach

First the equations of motion for the system were developed. The form of these equations came from a paper by Hall and Rand (reference 4) and its related PhD dissertation by Hall (reference 3). The equations were linearized and put into a form appropriate for application of the WKB method. The WKB approximation was then developed with the help of a mathematical manipulation program. Next, numerical integration was used to compare the linear and nonlinear solutions. Then, the WKB solution was compared to the linear solution. This comparison consisted of one example spacecraft for oblate spinup and one for prolate spinup. The spinup torques and the number of terms in the WKB approximation were varied during each comparison.

Sequence of Presentation

Chapter II contains the derivation of the equations of motion of the dual-spin spacecraft. These equations are put into a dimensionless form, linearized, then put into a form that can be used for the WKB method. Spacecraft spinup is also discussed. Chapter III contains a description of the WKB method and its application to the spinup problem. Approximate solutions for the spinup problem are developed. Chapter IV contains the comparisons between the WKB and linear solutions using specific examples. Chapter V contains the major conclusions drawn from the comparison and recommendations for further study.

II. Dynamics of a Dual-Spin Spacecraft

This chapter contains the description of the spacecraft model to be examined. From this the equations of motion are derived, put into terms of dimensionless variables, and then linearized. Finally, spacecraft spinup and the significance of the moments of inertia to spinup are discussed. For a full discussion of spacecraft dynamics, the reader is referred to the book Spacecraft Attitude Dynamics by Peter C. Hughes (Reference 5).

Spacecraft Model (3:24-26;4:641-642)

The dual-spin spacecraft to be examined will be modelled as the gyrostat shown in Fig 1. The platform, P , and the rotor, R , are rigid bodies connected by a rigid shaft with frictionless bearings. A constant spinup torque is applied to both bodies along the shaft. There are no external torques on the gyrostat and it is free to rotate in space. The two bodies however are constrained to rotate about the axis of the shaft which is a principal axis, $\hat{e}_1$, of the gyrostat. To keep the gyrostat as general as possible, the platform will be asymmetric while the rotor will be axisymmetric about the $\hat{e}_1$ axis.

Equations of Motion (3:25-29;4:642-643)

In Fig 1 the unit vectors, $\hat{e}_j$, $j=1,2,3$, are the principal axes of the gyrostat and are fixed in the platform. The

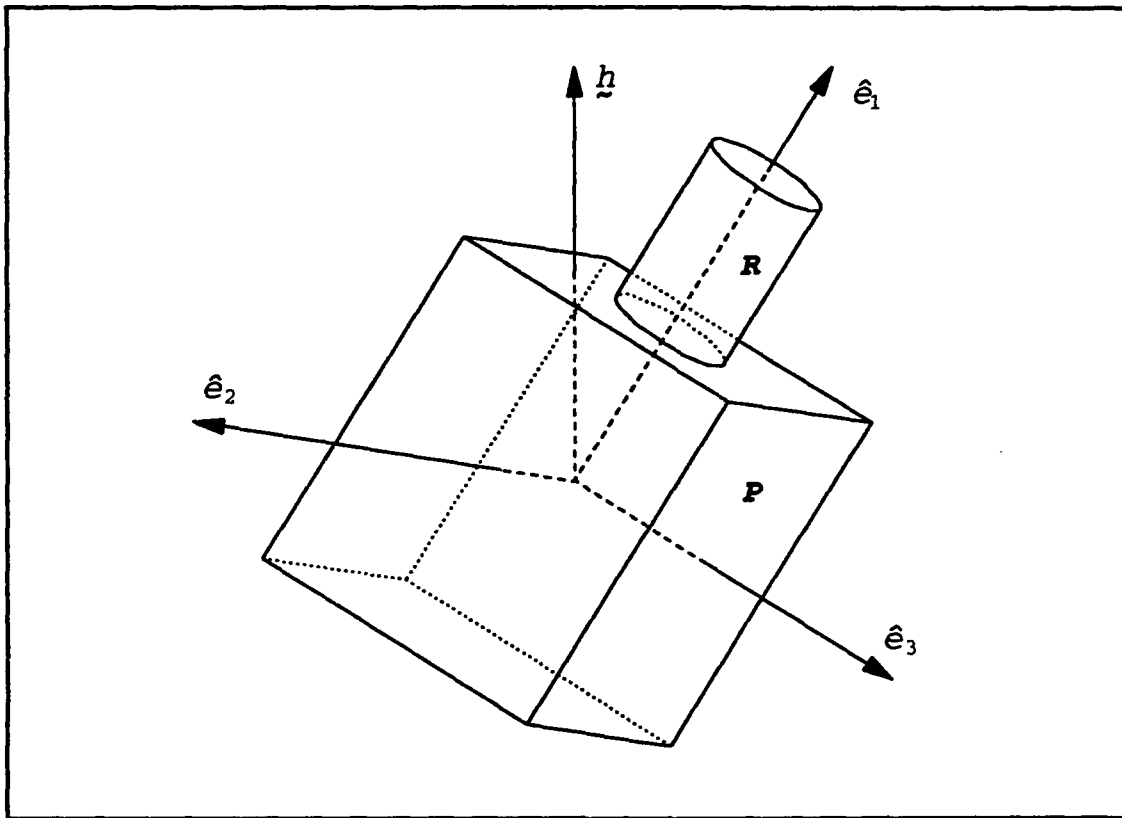


Figure 1. Dual-Spin Spacecraft $P + R$ (3:26;4:642)

vector, $\mathbf{h}$, is the total angular momentum of the gyrost. Since there are no external torques, the angular momentum vector is fixed in the inertial reference frame.

Before the equations of motion can be determined, several angular velocities must be defined. The angular velocities of the platform are defined as ω_1 , ω_2 , and ω_3 which correspond to the three principal axes. These angular velocities are relative to the inertial frame, but are expressed in terms of the body-fixed unit vectors. The variable ω_g is the angular velocity of the rotor relative to the platform. Therefore, $\omega_1 + \omega_g$ is the angular velocity of the

rotor about its axis of symmetry relative to the inertial frame.

Like the angular velocity of the platform, the angular momentum of the gyrostat, $\underline{h}$, can be written in terms of the body-fixed coordinates to obtain h_1 , h_2 , and h_3 along the principal axes. Now the angular momentum can be written as follows:

$$\underline{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} I_s \omega_s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_1 \omega_1 + I_s \omega_s \\ I_2 \omega_2 \\ I_3 \omega_3 \end{bmatrix} \quad (1)$$

where I_1 , I_2 , and I_3 are the moments of inertia about the gyrostat's principal axes and I_s is the moment of inertia of the rotor about the $\hat{e}_1$ axis. I_s is included in I_1 and the transverse moments of inertia of the rotor are included in I_2 and I_3 .

Next, the derivative of Eq (1) is taken with respect to time, $\tilde{t}$, to give

$$\frac{d\underline{h}}{d\tilde{t}} = \begin{bmatrix} \frac{dh_1}{d\tilde{t}} \\ \frac{dh_2}{d\tilde{t}} \\ \frac{dh_3}{d\tilde{t}} \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} \frac{dh_1}{d\tilde{t}} - \omega_3 h_2 + \omega_2 h_3 \\ \frac{dh_2}{d\tilde{t}} + \omega_3 h_1 - \omega_1 h_3 \\ \frac{dh_3}{d\tilde{t}} - \omega_2 h_1 + \omega_1 h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

Note that Eq (2) is set equal to zero since there are no external moments.

Two more terms are now defined: the moment of inertia of the platform about $\hat{e}_1$ is I_p ; and the angular momentum of the rotor in the $\hat{e}_1$ direction is h_a to give

$$I_1 = I_p + I_s \quad (3)$$

and

$$h_a = I_s(\omega_s + \omega_1) \quad (4)$$

Combining Eqs (3) and (4) and the relationship for h_1 from Eq (1) gives

$$\omega_1 = \frac{h_1 - h_a}{I_p} \quad (5)$$

Then using Eq (5) and the relationships for h_2 and h_3 from Eq (1), ω_1 , ω_2 , and ω_3 can be eliminated from Eq (2) to give the equations of motion:

$$\frac{dh_1}{d\tilde{t}} = \left(\frac{I_2 - I_3}{I_2 I_3} \right) h_2 h_3 \quad (6)$$

$$\frac{dh_2}{d\tilde{t}} = \left(\frac{I_3 - I_p}{I_3 I_p} h_1 - \frac{h_a}{I_p} \right) h_3 \quad (7)$$

$$\frac{dh_3}{d\tilde{t}} = \left(\frac{I_p - I_2}{I_2 I_p} h_1 + \frac{h_a}{I_p} \right) h_2 \quad (8)$$

and the spinup torque, g_a , is defined as

$$\frac{dh_a}{d\tilde{t}} = g_a \quad (9)$$

A first integral of motion may be obtained by noting that there are no external moments and thus the angular

momentum is conserved. This yields the following relationship:

$$h^2 = h_1^2 + h_2^2 + h_3^2 = \text{constant} \quad (10)$$

Dimensionless Equations (3:30-31;4:643-644)

The equations of motion can be further simplified by transforming the variables into dimensionless form. Using the transformations for the dimensionless angular momenta

$$x_j = \frac{h_j}{h} \quad j=1,2,3 \quad (11)$$

the dimensionless moments of inertia

$$i_j = 1 - \frac{I_p}{I_j} \quad j=1,2,3 \quad (12)$$

the dimensionless angular momentum of the rotor about $\hat{e}_1$

$$\mu = \frac{h_a}{h} \quad (13)$$

the dimensionless time

$$t = \frac{h\tilde{t}}{I_p} \quad (14)$$

and the dimensionless torque applied by the platform to the rotor about $\hat{e}_1$

$$\varepsilon = \frac{g_a I_p}{h^2} \quad (15)$$

Eqs (6)-(9) can be put into a dimensionless form:

$$\dot{X}_1 = (i_2 - i_3) X_2 X_3 \quad (16)$$

$$\dot{X}_2 = (i_3 X_1 - \mu) X_3 \quad (17)$$

$$\dot{X}_3 = (\mu - i_2 X_1) X_2 \quad (18)$$

$$\dot{\mu} = \epsilon \quad (19)$$

where

$$(\dot{}) = \frac{d()}{dt} \quad (20)$$

Now the first integral of motion in Eq (10) becomes

$$X_1^2 + X_2^2 + X_3^2 = 1 \quad (21)$$

Linearized Equations

Before Eqs (16)-(19) can be linearized, the equilibrium points must first be found. The equilibrium points occur where the first derivatives with respect to time, t , of the momenta are all equal to zero. This is true when X_2 and X_3 are equal to zero for the case $\epsilon \neq 0$, and therefore $X_1 = \pm 1$ from Eq (21) (3:32-33). Only $X_1 = +1$ will be examined since this is the equilibrium point about which spacecraft typically operate.

Eqs (16), (17), and (18) may now be linearized about the equilibrium point

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (22)$$

where

$$\dot{X}_1 = \dot{X}_2 = \dot{X}_3 = 0 \quad (23)$$

Substituting

$$\begin{aligned} \dot{X}_1 &= \dot{x}_1 \\ \dot{X}_2 &= \dot{x}_2 \\ \dot{X}_3 &= \dot{x}_3 \end{aligned} \quad (24)$$

and

$$\begin{aligned} X_1 &= 1 + x_1 \\ X_2 &= x_2 \\ X_3 &= x_3 \end{aligned} \quad (25)$$

into Eqs (16), (17), and (18) and noting that the lower case x 's are small perturbations from the nominal values so quadratic terms in x can be dropped gives

$$\dot{x}_1 = 0 \quad (26)$$

$$\dot{x}_2 + (\mu - i_3) x_3 = 0 \quad (27)$$

$$\dot{x}_3 + (i_2 - \mu) x_2 = 0 \quad (28)$$

For an axisymmetric spacecraft ($i_2=i_3$), the WKB method gives the exact solutions for Eqs (26)-(28) and the nonlinear equations. This will be shown in Chapter IV.

Next, taking the derivatives of Eqs (27) and (28) gives

$$\ddot{x}_2 + (\mu - i_3) \dot{x}_3 + \dot{\mu} x_3 = 0 \quad (29)$$

$$\ddot{x}_3 + (i_2 - \mu) \dot{x}_2 - \dot{\mu} x_2 = 0 \quad (30)$$

Then combining Eqs (27), (28), (29), and (30) to decouple the variables gives

$$\ddot{x}_2 + \frac{\dot{\mu}}{i_3 - \mu} \dot{x}_2 + (\mu - i_3) (\mu - i_2) x_2 = 0 \quad (31)$$

$$\ddot{x}_3 + \frac{\dot{\mu}}{i_2 - \mu} \dot{x}_3 + (i_2 - \mu) (i_3 - \mu) x_3 = 0 \quad (32)$$

In order to use the WKB method to solve the linear differential equations in x_2 and x_3 , Eq (19), its first integral

$$\mu = \varepsilon t + \mu_0 \quad (33)$$

and the change of variable

$$\begin{aligned} \tau &= \varepsilon t \\ \frac{d(\cdot)}{dt} &= \varepsilon \frac{d(\cdot)}{d\tau} \\ \frac{d^2(\cdot)}{dt^2} &= \varepsilon^2 \frac{d^2(\cdot)}{d\tau^2} \end{aligned} \quad (34)$$

must be substituted into Eq (31) to give

$$\epsilon^2 x_2'' + \frac{\epsilon^2}{i_3 - \mu_0 - \tau} x_2' + (\tau + \mu_0 - i_3)(\tau + \mu_0 - i_2) x_2 = 0 \quad (35)$$

where

$$() ' = \frac{d()}{d\tau} \quad (36)$$

and μ_0 is the value of μ at $t=0$. The zero subscript notation on any other variable in the thesis will be its value at some initial time. The only exception is S_0 which will be defined later. It should be noted that Eq (35) is the same as that for x_3 if the subscripts 2 and 3 on all variables are exchanged.

Finally, the substitutions

$$\begin{aligned} k_2 &= \mu_0 - i_2 \\ k_3 &= \mu_0 - i_3 \end{aligned} \quad (37)$$

are made into Eq (35) as a simplification to obtain

$$\epsilon^2 x_2'' - \frac{\epsilon^2}{\tau + k_3} x_2' + (\tau + k_2)(\tau + k_3) x_2 = 0 \quad (38)$$

Before discussing the WKB method, a few words need to be said about spinup and dimensionless inertia parameters.

Spinup (3:57-59)

Spinup of an ideal dual-spin spacecraft (i.e., no friction) usually starts when the platform and rotor are spinning together as one rigid body called the all-spun condition where $\omega_s=0$. The spinup motor is then started so there

is a small constant torque between the platform and rotor. Angular momentum is transferred from the platform to the rotor such that the rotor is spunup and the platform is despun. The motor is then switched off when $\omega_1=0$. Combining the all-spun condition, $\omega_s=0$, with Eqs (4) and (5) gives

$$h_a = \frac{I_s h_1}{I_1} \quad (39)$$

or its dimensionless form

$$\mu = i_1 X_1 \quad (40)$$

Combining the final despun condition, $\omega_1=0$, and Eq (5) gives

$$h_a = h_1 \quad (41)$$

or its dimensionless form

$$\mu = X_1 \quad (42)$$

The linearized counterpart of Eq (40) becomes

$$\mu = i_1 \quad (43)$$

and Eq (42) becomes

$$\mu = 1 \quad (44)$$

Eqs (43) and (44) are the initial and final spinup conditions, respectively, for linearized spinup.

Dimensionless Moments of Inertia (3:33-34;4:644)

According to Eq (12), the relative value relationships between the actual moments of inertia and their dimensionless counterparts hold true such that

$$I_j > I_k \Leftrightarrow i_j > i_k, \quad j, k = 1, 2, 3 \quad (45)$$

The following relationships also can be seen from Eq (12):

$$\begin{aligned} I_p > I_j &\Leftrightarrow i_j < 0, \quad j = 2, 3 \\ I_p < I_j &\Leftrightarrow i_j > 0, \quad j = 2, 3 \end{aligned} \quad (46)$$

Now the dynamical shape of the gyrostat can be defined.

Note that in Eqs (6)-(8) only i_2 and i_3 appear. So the possible dynamical shape of the gyrostat depends only on these two parameters and oblate and prolate are now defined as follows:

oblate: $i_2 < 0$ and $i_3 < 0$

prolate: $i_2 > 0$ and $i_3 > 0$

To obtain the range of values for the other moment of inertia, i_1 , Eqs (3) and (12) can be combined to give

$$i_1 = \frac{I_s}{I_1} = \frac{I_s}{I_p + I_s} \quad (47)$$

which is the ratio of the moment of inertia of the rotor to that of the entire gyrostat about the $\hat{e}_1$ axis. This means i_1 can take on the values of $0 < i_1 < 1$ depending on the relationship between I_p and I_s .

Although i_1 does not appear in the equations of motion, it does have an effect on the spinup dynamics. In the previous section, the spin up was seen to begin where the initial value of μ was equal to i_1 for the linearized case. It also limits the initial rotor spin axis to the major or minor axis where:

major axis: $i_1 > \{i_2, i_3\}$

minor axis: $i_1 < \{i_2, i_3\}$.

If the value of i_1 falls between i_2 and i_3 , the all-spun condition is about the intermediate axis and is therefore unstable.

Also, the physically possible values of i_2 and i_3 are limited by i_1 . The inequalities

$$\begin{aligned} I_1 &< I_2 + I_3 \\ I_2 &< I_1 + I_3 \\ I_3 &< I_1 + I_2 \end{aligned} \tag{48}$$

must hold true for a physically possible system. For more information on these physical limitations, the reader is referred to reference 3 (3:176-180).

III. WKB Approximation

This chapter contains the description of the WKB method and how it is applied to the spinup of a dual-spin spacecraft. Approximate solutions are obtained for oblate spinup and then for prolate spinup.

Background

WKB theory is a method for obtaining an approximate solution to a linear differential equation where the highest derivative is multiplied by a small parameter (1:484). It is sometimes referred to as WKBJ for Wentzel, Kramers, Brillouin, and Jeffreys who independently used this method in quantum mechanics during the 1920s. It was also independently discovered by many others so that it is also sometimes called Liouville-Green or phase integral (6:244).

WKB theory contains boundary-layer theory as a special case. Boundary-layer theory will not be discussed here. The reader is referred to reference 1 for a discussion of boundary-layer theory. The WKB approximation starts by assuming solutions of the form:

$$x(\tau) \sim A(\tau) \exp\left[\frac{S(\tau)}{\delta}\right], \quad \delta \rightarrow 0^+ \quad (49)$$

where $S(\tau)$ is the phase, δ is the boundary layer thickness, and $A(\tau)$ is an amplitude function. The symbol " $\sim$ " is used here to denote the asymptotic nature of the approximation. In this case, asymptotic means as δ gets closer to 0^+ the

difference between the approximation and $x(\tau)$ becomes much less than $x(\tau)$. Eq (49) is not in a suitable form to obtain an asymptotic approximation because there is an implicit dependence on δ in both the phase and amplitude functions. This implicit dependence can be made explicit by expanding $A(\tau)$ and $S(\tau)$ as a power series in δ . The two series can then be combined to form a single power series so that Eq (49) becomes

$$x(\tau) \sim \exp \left[\frac{1}{\delta} \sum_{q=0}^{\infty} \delta^q S_q(\tau) \right], \quad \delta \rightarrow 0 \quad (50)$$

from which an approximate solution can now be derived (1:484-486).

There are several limitations to the WKB method. The differential equation must be linear and its highest derivative must be multiplied by a small parameter (1:484). This is the reason the equations of motion for the dual-spin spacecraft were put into the form of Eq (38). Also, the WKB series within the exponential of Eq (50) is not convergent (1:493). This means that some finite number of terms will give the best approximation.

WKB Method

This section contains the derivation of the first six terms in the WKB series so that solutions may be approximated for x_2 for both oblate and prolate spinup. The steps for this method are taken from reference 1 (1:486-490). The

derivations of all the WKB terms were obtained using the symbolic manipulation program Mathematica 2.0 for SPARC by Wolfram Research, Inc.

The first step to obtain the first six terms of the WKB series is to take Eq (50) and its derivatives

$$x_2 = \exp \left(\sum_{q=0}^5 \delta^{q-1} S_q \right) \quad (51)$$

$$x_2' = \left(\sum_{q=0}^5 \delta^{q-1} S_q' \right) \exp \left(\sum_{q=0}^5 \delta^{q-1} S_q \right) \quad (52)$$

$$x_2'' = \left[\left(\sum_{q=0}^5 \delta^{q-1} S_q' \right)^2 + \sum_{q=0}^5 \delta^{q-1} S_q'' \right] \exp \left(\sum_{q=0}^5 \delta^{q-1} S_q \right) \quad (53)$$

and substitute them into Eq (38):

$$\begin{aligned} \epsilon^2 \left[\left(\sum_{q=0}^5 \delta^{q-1} S_q' \right)^2 + \sum_{q=0}^5 \delta^{q-1} S_q'' \right] \exp \left(\sum_{q=0}^5 \delta^{q-1} S_q \right) \\ - \frac{\epsilon^2}{\tau + k_3} \left(\sum_{q=0}^5 \delta^{q-1} S_q' \right) \exp \left(\sum_{q=0}^5 \delta^{q-1} S_q \right) \\ + (\tau + k_2)(\tau + k_3) \exp \left(\sum_{q=0}^5 \delta^{q-1} S_q \right) = 0 \quad (54) \end{aligned}$$

After eliminating the exponentials and expanding Eq (54), the dominant terms are found to be

$$\begin{aligned} (\tau + k_2)(\tau + k_3) \\ \frac{\epsilon^2 S_0'^2}{\delta^2} \quad (55) \end{aligned}$$

For these terms to be of the same order of magnitude, δ must be of the same order as ϵ . For simplicity $\delta=\epsilon$ is chosen.

The next step is to make the substitution $\delta=\epsilon$ into Eq (54) and group terms by powers of ϵ :

$$\begin{aligned}
& \epsilon^0 \left[(\tau+k_2)(\tau+k_3) + S_0'^2 \right] + \epsilon^1 \left(\frac{-1}{\tau+k_3} S_0' + 2 S_0' S_1' + S_0'' \right) \\
& + \epsilon^2 \left(\frac{-1}{\tau+k_3} S_1' + S_1'^2 + 2 S_0' S_2' + S_1'' \right) \\
& + \epsilon^3 \left(\frac{-1}{\tau+k_3} S_2' + 2 S_1' S_2' + 2 S_0' S_3' + S_2'' \right) \\
& + \epsilon^4 \left(S_2'^2 + \frac{-1}{\tau+k_3} S_3' + 2 S_1' S_3' + 2 S_0' S_4' + S_3'' \right) \\
& + \epsilon^5 \left(2 S_2' S_3' + \frac{-1}{\tau+k_3} S_4' + 2 S_1' S_4' + 2 S_0' S_5' + S_4'' \right) \\
& + O(\epsilon^6) = 0 \quad (56)
\end{aligned}$$

Now the terms for each power of ϵ are individually set equal to zero in order to obtain six equations, one for each of the derivatives of the WKB series terms:

$$S_0' = \pm [-(\tau+k_2)(\tau+k_3)]^{1/2} \quad (57)$$

$$S_1' = \frac{(\tau+k_3)^{-1} S_0' - S_0''}{2 S_0'} \quad (58)$$

$$S_2' = \frac{(\tau+k_3)^{-1} S_1' - S_1'^2 - S_1''}{4 S_0'} \quad (59)$$

$$S_3' = \frac{(\tau+k_3)^{-1} S_2' - 2 S_1' S_2' - S_2''}{2 S_0'} \quad (60)$$

$$S_4' = \frac{(\tau+k_3)^{-1} S_3' - S_2'^2 - 2 S_1' S_3' - S_3''}{2 S_0'} \quad (61)$$

$$S_5' = \frac{(\tau+k_3)^{-1} S_4' - 2 S_2' S_3' - 2 S_1' S_4' - S_4''}{2 S_0'} \quad (62)$$

Note that there are two solutions for S_0' and thus two solutions to the second order differential equation as would be expected. To determine these solutions the sign of the term $(\tau+k_2)(\tau+k_3)$ must be determined. This is done in the following sections for oblate and prolate spinup.

WKB Oblate Solution

For oblate spinup, both i_2 and i_3 are negative and μ_0 is assumed to be positive. This means the derivative of the zeroth term in Eq (57) is imaginary and the derivatives of the first six terms of the WKB series may be written as follows:

$$S_0' = \pm i [(\tau+k_2)(\tau+k_3)]^{1/2} \quad (63)$$

$$S_1' = \frac{k_2 - k_3}{4(\tau+k_2)(\tau+k_3)} \quad (64)$$

$$S_2' = \pm i \frac{(k_3 - k_2)(7k_2 + 5k_3 + 12\tau)}{32[(\tau+k_2)(\tau+k_3)]^{5/2}} \quad (65)$$

$$S_3' = \frac{3(k_3 - k_2)(7k_2^2 + 8k_2k_3 + 5k_3^2 + 22k_2\tau + 18k_3\tau + 20\tau^2)}{64[(\tau+k_2)(\tau+k_3)]^4} \quad (66)$$

$$S'_4 = \frac{\pm i}{2048} (k_2 - k_3) [(\tau + k_2)(\tau + k_3)]^{-11/2} (1463k_2^3 + 2163k_2^2k_3 \\ + 1989k_2k_3^2 + 1105k_3^3 + 6552k_2^2\tau + 8304k_2k_3\tau \\ + 5304k_3^2\tau + 10704k_2\tau^2 + 9456k_3\tau^2 + 6720\tau^3) \quad (67)$$

$$S'_5 = \frac{3}{1024} (k_2 - k_3) [(\tau + k_2)(\tau + k_3)]^{-7} (707k_2^4 + 1246k_2^3k_3 \\ + 1392k_2^2k_3^2 + 1130k_2k_3^3 + 565k_3^4 + 4074k_2^3\tau \\ + 6522k_2^2k_3\tau + 6174k_2k_3^2\tau + 3390k_3^3\tau + 9372k_2^2\tau^2 \\ + 12696k_2k_3\tau^2 + 8172k_3^2\tau^2 + 10480k_2\tau^3 + 9680k_3\tau^3 + 5040\tau^4) \quad (68)$$

Now Eqs (63)-(68) can be integrated to obtain the first six terms of the WKB series for oblate spinup:

$$S_0 = \pm i \left\{ \left(\frac{k_2 + k_3}{4} + \frac{\tau}{2} \right) [(\tau + k_2)(\tau + k_3)]^{1/2} \right. \\ \left. - \frac{(k_3 - k_2)^2}{8} \ln \left\{ k_2 + k_3 + 2\tau + 2[(\tau + k_2)(\tau + k_3)]^{1/2} \right\} \right\} \quad (69)$$

$$S_1 = \frac{\ln(\tau + k_3) - \ln(\tau + k_2)}{4} \quad (70)$$

$$S_2 = \frac{\pm i}{48} (7k_2^3 - 27k_2^2k_3 + 9k_2k_3^2 - 5k_3^3 - 6k_2^2\tau - 36k_2k_3\tau - 6k_3^2\tau \\ - 24k_2\tau^2 - 24k_3\tau^2 - 16\tau^3) (k_3 - k_2)^{-2} [(\tau + k_2)(\tau + k_3)]^{-3/2} \quad (71)$$

$$S_3 = \frac{(k_2 - k_3)(7k_2 + 5k_3 + 12\tau)}{64[(\tau + k_2)(\tau + k_3)]^3} \quad (72)$$

$$\begin{aligned}
S_4 = \pm i [(\tau + k_2)(\tau + k_3)]^{1/2} & \left[\frac{1105}{9216(k_2 - k_3)^2(\tau + k_2)^5} \right. \\
& + \frac{1547}{9216(k_2 - k_3)^3(\tau + k_2)^4} + \frac{1627}{15360(k_2 - k_3)^4(\tau + k_2)^3} \\
& + \frac{1633}{46080(k_2 - k_3)^5(\tau + k_2)^2} + \frac{127}{5760(k_2 - k_3)^6(\tau + k_2)} \\
& - \frac{1463}{9216(k_2 - k_3)^2(\tau + k_3)^5} + \frac{2653}{9216(k_2 - k_3)^3(\tau + k_3)^4} \\
& - \frac{4973}{15360(k_2 - k_3)^4(\tau + k_3)^3} + \frac{15767}{46080(k_2 - k_3)^5(\tau + k_3)^2} \\
& \left. - \frac{4223}{5760(k_2 - k_3)^6(\tau + k_3)} \right] \quad (73)
\end{aligned}$$

$$\begin{aligned}
S_5 = \frac{k_3 - k_2}{2048[(\tau + k_2)(\tau + k_3)]^6} & (707k_2^3 + 1071k_2^2k_3 + 1017k_2k_3^2 \\
& + 565k_3^3 + 3192k_2^2\tau + 4176k_2k_3\tau + 2712k_3^2\tau + 5280k_2\tau^2 \\
& + 4800k_3\tau^2 + 3360\tau^3) \quad (74)
\end{aligned}$$

Since only the even numbered terms are imaginary for the oblate spinup, the WKB series can be divided into imaginary and real parts:

$$\sum_{q=0}^Q \epsilon^{q-1} S_q = \sum_{m=0}^M \epsilon^{2m-1} S_{2m} + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1} \quad (75)$$

where

$$\begin{aligned}
Q &= M + N \\
M &\leq N \leq M+1
\end{aligned}$$

The m summation contains the imaginary terms, while the n summation contains the real terms. Using Eq (75) and noting that the two WKB series solutions are complex conjugates of

each other, the general WKB solution for oblate spinup may be written as

$$x_2 = C_1 \exp \left(\sum_{m=0}^M \epsilon^{2m-1} S_{2m} + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1} \right) + C_2 \exp \left(- \sum_{m=0}^M \epsilon^{2m-1} S_{2m} + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1} \right) \quad (76)$$

where

$$\begin{aligned} C_1 &= \text{constant} \\ C_2 &= \text{constant} \end{aligned}$$

The solution for x_3 could just as easily be written by noting the symmetry in Eq (35) that was mentioned in Chapter II. C_1 and C_2 can be obtained by imposing the following initial conditions on Eq (76):

$$\begin{aligned} x_2(\tau_0) &= x_{20} \\ x_2'(\tau_0) &= x_{20}' \end{aligned} \quad (77)$$

The result is

$$C_1 = \frac{x_{20}' + x_{20} \left(\sum_{m=0}^M \epsilon^{2m-1} S_{2m,0}' - \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1,0}' \right)}{2 \left(\sum_{m=0}^M \epsilon^{2m-1} S_{2m,0}' \right) \exp \left(\sum_{m=0}^M \epsilon^{2m-1} S_{2m,0} + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1,0} \right)} \quad (78)$$

$$C_2 = \frac{x_{20}' - x_{20} \left(\sum_{m=0}^M \epsilon^{2m-1} S_{2m,0}' + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1,0}' \right)}{-2 \left(\sum_{m=0}^M \epsilon^{2m-1} S_{2m,0}' \right) \exp \left(- \sum_{m=0}^M \epsilon^{2m-1} S_{2m,0} + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1,0} \right)} \quad (79)$$

where the notation $S_{q,0}$ means the initial value of S_q . Using the results of Eqs (76), (78), and (79), and eliminating the imaginary numbers by substituting the identities

$$\begin{aligned}\cos \theta &= \frac{\exp(i\theta) + \exp(-i\theta)}{2} \\ \sin \theta &= \frac{\exp(i\theta) - \exp(-i\theta)}{2i}\end{aligned}\tag{80}$$

gives

$$\begin{aligned}x_2 = & \exp \left[\sum_{n=1}^N \epsilon^{2(n-1)} (S_{2n-1} - S_{2n-1,0}) \right] \\ & \left\{ x_{20} \cos \left[-i \sum_{m=0}^M \epsilon^{2m-1} (S_{2m} - S_{2m,0}) \right] \right. \\ & \left. + \left(\frac{x'_{20} - x_{20} \sum_{n=1}^N \epsilon^{2(n-1)} S'_{2n-1,0}}{-i \sum_{m=0}^M \epsilon^{2m-1} S'_{2m,0}} \right) \sin \left[-i \sum_{m=0}^M \epsilon^{2m-1} (S_{2m} - S_{2m,0}) \right] \right\}\end{aligned}\tag{81}$$

which is the WKB solution for the oblate spinup problem.

WKB Prolate Solution

For prolate spinup, both i_2 and i_3 are positive and the magnitude of μ_0 is assumed to be less than the magnitude of either i_2 or i_3 . So from Eq (37) both k_2 and k_3 are negative. This means the derivative of the zeroth term in Eq (57) can be either real or imaginary. The prolate spinup can therefore be divided into three regions, each with a WKB approximation. To determine these regions, the assumption $k_2 < k_3$ is made so the three regions are as follows:

Region 1: $0 \leq \tau < -k_3$
 Region 2: $-k_3 < \tau < -k_2$
 Region 3: $-k_2 < \tau \leq 1 - \mu_0$

It should be noted that Eqs (64)-(68) have singular points at $\tau = -k_2$ and $\tau = -k_3$ so that solutions near these points may not be valid. No generality is lost when making the assumption $k_2 < k_3$ since if $k_2 > k_3$ then a coordinate transformation would bring the situation back to the original assumption.

Region 1 Solution. Since $\tau < -k_2$ and $\tau < -k_3$ in region 1, the term $(\tau + k_2)(\tau + k_3)$ is always positive. Therefore, the derivative of the zeroth WKB term is imaginary. This yields the same solution as for oblate spinup.

Region 2 Solution. Since $\tau > -k_3$ and $\tau < -k_2$ in region 2, the term $(\tau + k_2)(\tau + k_3)$ is always negative. Therefore, the derivative of the zeroth WKB term is real and the derivatives of the first six terms of the WKB series may be written as follows:

$$S'_0 = \pm [-(\tau + k_2)(\tau + k_3)]^{1/2} \quad (82)$$

$$S'_1 = \frac{k_2 - k_3}{4(\tau + k_2)(\tau + k_3)} \quad (83)$$

$$S'_2 = \pm \frac{(k_2 - k_3)(7k_2 + 5k_3 + 12\tau)}{32[-(\tau + k_2)(\tau + k_3)]^{5/2}} \quad (84)$$

$$S'_3 = \frac{3(k_3 - k_2)(7k_2^2 + 8k_2k_3 + 5k_3^2 + 22k_2\tau + 18k_3\tau + 20\tau^2)}{64[(\tau + k_2)(\tau + k_3)]^4} \quad (85)$$

$$S'_4 = \frac{\pm 1}{2048} (k_2 - k_3) [-(\tau + k_2)(\tau + k_3)]^{-11/2} (1463k_2^3 + 2163k_2^2k_3 \\ + 1989k_2k_3^2 + 1105k_3^3 + 6552k_2^2\tau + 8304k_2k_3\tau \\ + 5304k_3^2\tau + 10704k_2\tau^2 + 9456k_3\tau^2 + 6720\tau^3) \quad (86)$$

$$S'_5 = \frac{3}{1024} (k_2 - k_3) [(\tau + k_2)(\tau + k_3)]^{-7} (707k_2^4 + 1246k_2^3k_3 \\ + 1392k_2^2k_3^2 + 1130k_2k_3^3 + 565k_3^4 + 4074k_2^3\tau \\ + 6522k_2^2k_3\tau + 6174k_2k_3^2\tau + 3390k_3^3\tau + 9372k_2^2\tau^2 \\ + 12696k_2k_3\tau^2 + 8172k_3^2\tau^2 + 10480k_2\tau^3 + 9680k_3\tau^3 + 5040\tau^4) \quad (87)$$

Now Eq (82)-(87) can be integrated to obtain the first six terms of the WKB series for prolate spinup in region 2:

$$S_0 = \pm \left\{ \left(\frac{k_2 + k_3}{4} + \frac{\tau}{2} \right) [-(\tau + k_2)(\tau + k_3)]^{1/2} \right. \\ \left. + \frac{(k_3 - k_2)^2}{8} \arcsin \left(\frac{k_2 + k_3 + 2\tau}{|k_2 - k_3|} \right) \right\} \quad (88)$$

$$S_1 = \frac{\ln(\tau + k_3) - \ln(\tau + k_2)}{4} \quad (89)$$

$$S_2 = \pm \frac{1}{48} (7k_2^3 - 27k_2^2k_3 + 9k_2k_3^2 - 5k_3^3 - 6k_2^2\tau - 36k_2k_3\tau - 6k_3^2\tau \\ - 24k_2\tau^2 - 24k_3\tau^2 - 16\tau^3) (k_3 - k_2)^{-2} [-(\tau + k_2)(\tau + k_3)]^{-3/2} \quad (90)$$

$$S_3 = \frac{(k_2 - k_3)(7k_2 + 5k_3 + 12\tau)}{64[(\tau + k_2)(\tau + k_3)]^3} \quad (91)$$

$$\begin{aligned}
S_4 = \pm [- (\tau + k_2) (\tau + k_3)]^{1/2} & \left[\frac{1105}{9216 (k_2 - k_3)^2 (\tau + k_2)^5} \right. \\
& + \frac{1547}{9216 (k_2 - k_3)^3 (\tau + k_2)^4} + \frac{1627}{15360 (k_2 - k_3)^4 (\tau + k_2)^3} \\
& + \frac{1633}{46080 (k_2 - k_3)^5 (\tau + k_2)^2} + \frac{127}{5760 (k_2 - k_3)^6 (\tau + k_2)} \\
& - \frac{1463}{9216 (k_2 - k_3)^2 (\tau + k_3)^5} + \frac{2653}{9216 (k_2 - k_3)^3 (\tau + k_3)^4} \\
& - \frac{4973}{15360 (k_2 - k_3)^4 (\tau + k_3)^3} + \frac{15767}{46080 (k_2 - k_3)^5 (\tau + k_3)^2} \\
& \left. - \frac{4223}{5760 (k_2 - k_3)^6 (\tau + k_3)} \right] \quad (92)
\end{aligned}$$

$$\begin{aligned}
S_5 = \frac{k_3 - k_2}{2048 [(\tau + k_2) (\tau + k_3)]^6} & (707k_2^3 + 1071k_2^2k_3 + 1017k_2k_3^2 \\
& + 565k_3^3 + 3192k_2^2\tau + 4176k_2k_3\tau + 2712k_3^2\tau + 5280k_2\tau^2 \\
& + 4800k_3\tau^2 + 3360\tau^3) \quad (93)
\end{aligned}$$

As with the oblate spinup problem, the WKB series for region 2 of prolate spinup can be divided into even and odd terms except all terms are real this time:

$$\sum_{q=0}^{M+N} \epsilon^{q-1} S_q = \sum_{m=0}^M \epsilon^{2m-1} S_{2m} + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1} \quad (94)$$

Using Eq (94) and noting that the two WKB series solutions only differ by a factor of negative one in their even terms, the general WKB solution for prolate spin-up in region 2 may be written as

$$\begin{aligned}
x_2 = C_1 \exp \left(\sum_{m=0}^M \epsilon^{2m-1} S_{2m} + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1} \right) \\
+ C_2 \exp \left(- \sum_{m=0}^M \epsilon^{2m-1} S_{2m} + \sum_{n=1}^N \epsilon^{2(n-1)} S_{2n-1} \right) \quad (95)
\end{aligned}$$

C_1 and C_2 can be obtained by imposing the following initial conditions on Eq (95):

$$\begin{aligned}
x_2(\tau_0) &= x_{20} \\
x_2'(\tau_0) &= x_{20}' \quad (96)
\end{aligned}$$

The result is

$$C_1 = \left[\frac{x_{20}' - x_{20} \sum_{q=0}^{M+N} (-1)^{q+1} \epsilon^{q-1} S_{q,0}'}{2 \sum_{m=0}^M \epsilon^{2m-1} S_{2m,0}'} \right] \exp \left[- \sum_{q=0}^{M+N} \epsilon^{q-1} S_{q,0} \right] \quad (97)$$

$$C_2 = - \left[\frac{x_{20}' - x_{20} \sum_{q=0}^{M+N} \epsilon^{q-1} S_{q,0}'}{2 \sum_{m=0}^M \epsilon^{2m-1} S_{2m,0}'} \right] \exp \left[- \sum_{q=0}^{M+N} (-1)^{q+1} \epsilon^{q-1} S_{q,0} \right] \quad (98)$$

Using the results of Eqs (95), (97), and (98) and the identities

$$\begin{aligned}
\sinh(\theta) &= \frac{\exp(\theta) - \exp(-\theta)}{2} \\
\cosh(\theta) &= \frac{\exp(\theta) + \exp(-\theta)}{2} \quad (99)
\end{aligned}$$

gives

$$\begin{aligned}
x_2 = & \exp \left[\sum_{n=1}^N \epsilon^{2(n-1)} (S_{2n-1} - S_{2n-1,0}) \right] \\
& \left\{ x_{20} \cosh \left[\sum_{m=0}^M \epsilon^{2m-1} (S_{2m} - S_{2m,0}) \right] \right. \\
& \left. + \left(\frac{x'_{20} - x_{20} \sum_{n=1}^N \epsilon^{2(n-1)} S'_{2n-1,0}}{\sum_{m=0}^M \epsilon^{2m-1} S'_{2m,0}} \right) \sinh \left[\sum_{m=0}^M \epsilon^{2m-1} (S_{2m} - S_{2m,0}) \right] \right\} \quad (100)
\end{aligned}$$

which is the WKB solution for region 2 of the prolate spinup problem. Note how this solution parallels the oblate solution in Eq (81).

Region 3 Solution. Since $\tau > -k_2$ and $\tau > -k_3$ in region 3, the term $(\tau + k_2)(\tau + k_3)$ is always positive. Therefore, the derivative of the zeroth WKB term is imaginary. Like the solution for region 1, this yields the same solution as for oblate spinup.

Now that the WKB approximations have been derived, they can be compared to the linear solutions to determine their accuracy. This is done in the next chapter.

IV. Comparison of WKB and Linear Solutions

This chapter contains the results of the study of the WKB method. After a brief description of the procedure used; a comparison of the nonlinear and linear solutions for oblate spinup is presented. Then, the WKB solution is compared to the linear solution for oblate spinup. Next is a presentation of the comparison of the nonlinear and linear solutions for prolate spinup. Finally, a comparison of the WKB and linear solutions for prolate spinup is presented.

Procedure

All of the numerical integrations of the nonlinear equations of motion, Eqs (16)-(19), and linear equations of motion, Eqs (26)-(28), were accomplished using a fourth order predictor-corrector algorithm developed by the numerical analyst Hamming. This algorithm was used in a computer program coded in FORTRAN (7:107-113). A sample of the type of program used is in the Appendix. The comparison between different solutions was done in the same program and the data output to a file. This file was then imported to a spreadsheet program to develop the graphs in this chapter.

Linear vs Nonlinear Oblate Solutions

This section contains an examination of the accuracy of the linear solution as X_1 gets farther from the equilibrium point. The effects of asymmetry are also explained.

In Fig 2, the magnitude of the maximum relative error and the maximum phase error between the nonlinear and linear solution for axisymmetric ($i_2=i_3$) oblate spinup is plotted as a function of the value of X_1 for several values of ϵ .

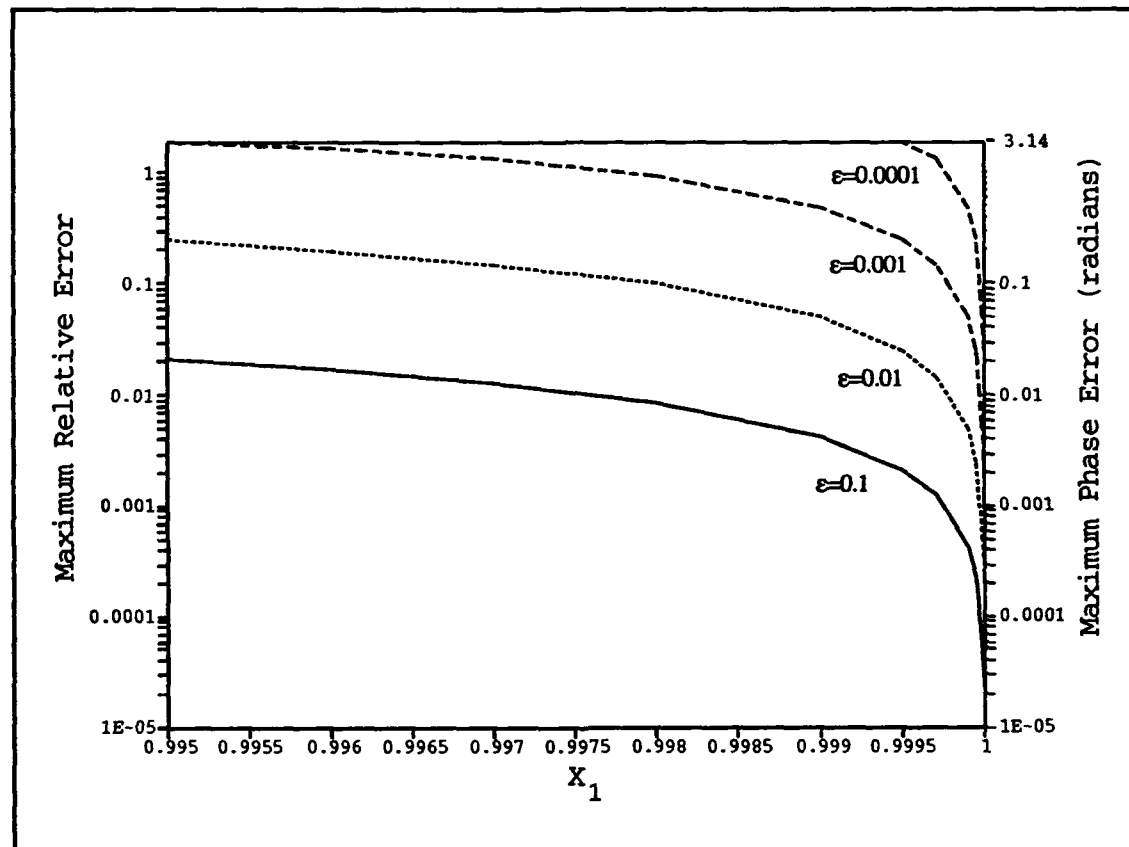


Figure 2. Linear Error in x_2 for $i_2=i_3=-0.5$

The maximum error is relative to the maximum amplitude of x_2 , a constant (i.e., the magnitude of the linear solution minus the nonlinear solution all divided by the maximum value of x_2). Spinup is taken to start at $\mu=0$ and end at $\mu=1$ to obtain the worst case error. Fig 2 shows that for each ϵ the error gets worse as X_1 departs from the $X_1=1$ value at the equilibrium point. Also, smaller ϵ 's require that

X_1 be closer to the equilibrium point because as ϵ decreases by a factor of ten, the spinup time increases by a factor of ten which gives more time for error to build up.

Fig 3 is the same type of graph as Fig 2 except only one value of ϵ and three values of $i_2=i_3$ are plotted. Fig 3

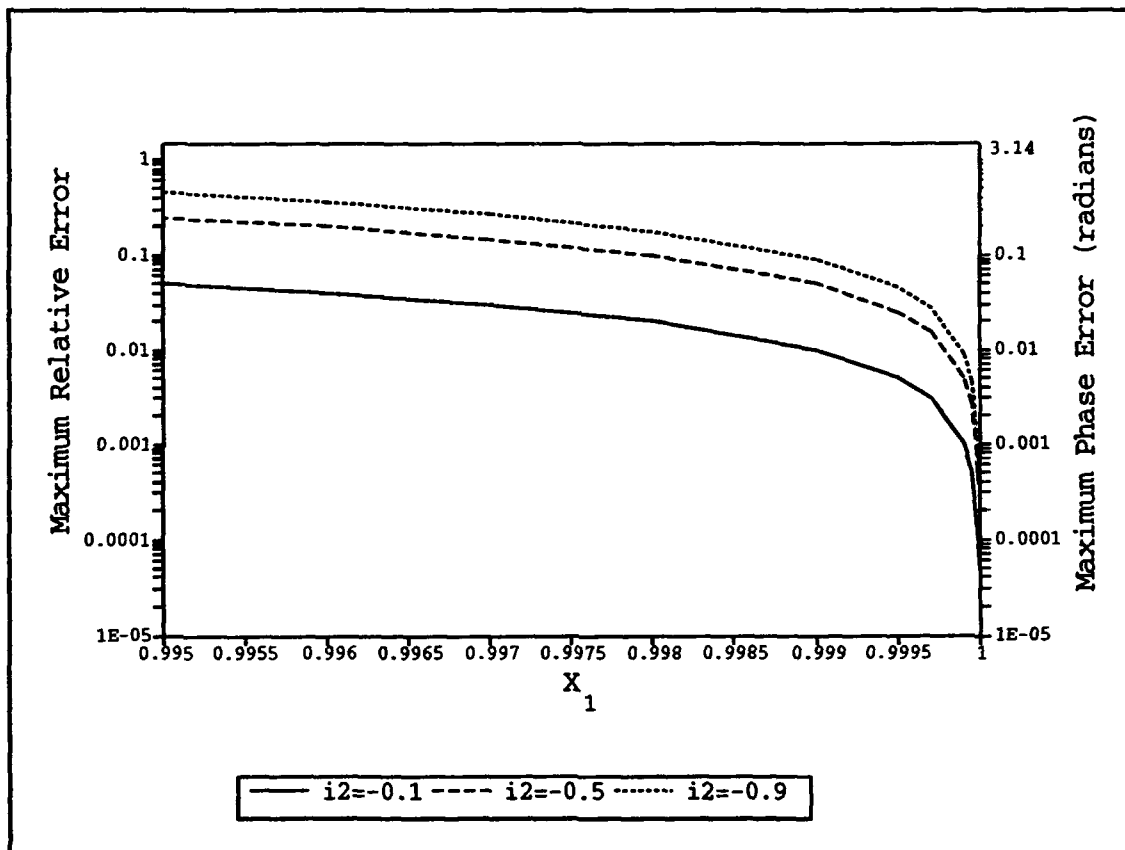


Figure 3. Linear Error in x_2 for $\epsilon=0.01$

shows that the smaller the inertia parameter the larger the error for a given X_1 .

Fig 4 shows the error between the nonlinear and linear solutions (i.e., linear minus nonlinear) for the following conditions:

$$X_1 = 0.9999$$

$$\begin{aligned}\epsilon &= 0.01 \\ i_2 &= -0.9 \\ i_3 &= -0.1 \\ x_{30} &= 0\end{aligned}$$

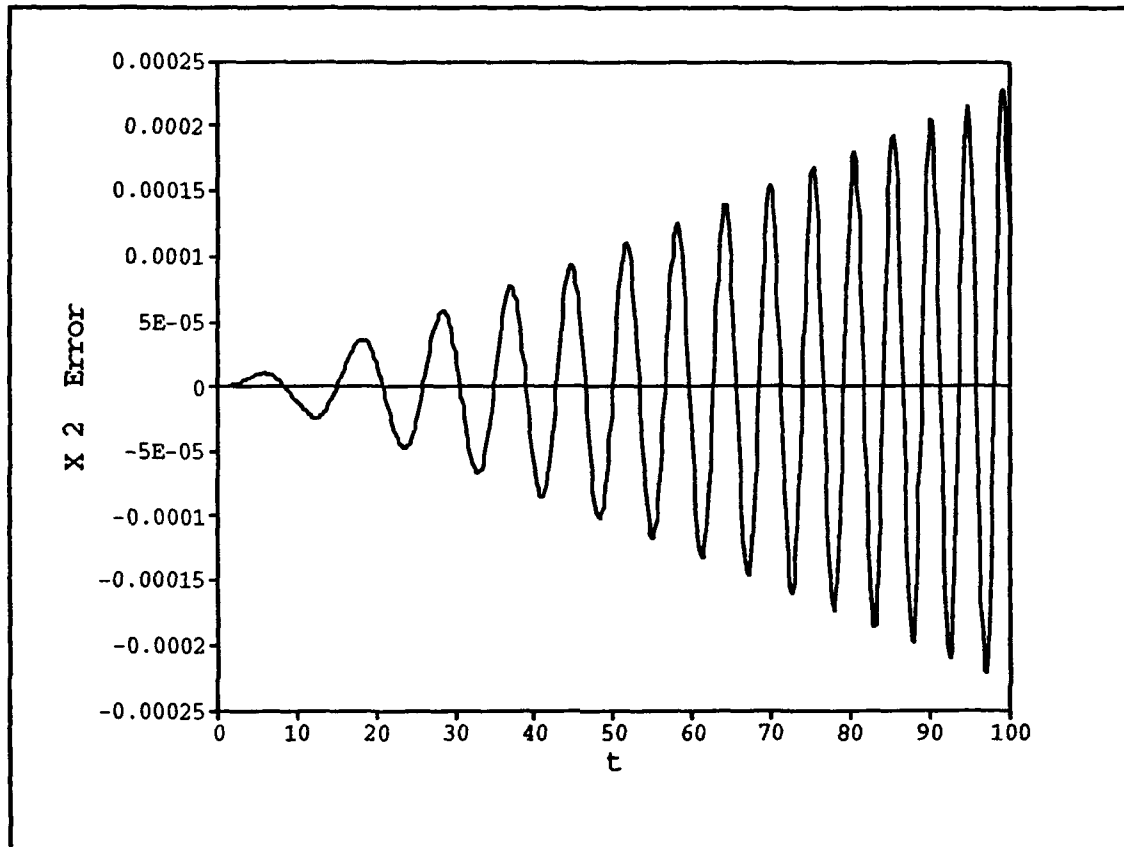


Figure 4. Linear Error in x_2 for $X_1=0.9999$, $\epsilon=0.01$, $i_2=-0.9$, and $i_3=-0.1$

This is a highly asymmetric body and departure from the nonlinear solution would be expected. From Fig 3 for $X_1=0.9999$ and $i_2=i_3=-0.9$, the relative error is approximately 0.009. Multiplying the relative error by the maximum value of x_2 calculated from Eq (21) gives an error of approximately 0.00013 for the axisymmetric gyrostat. The maximum error from Fig 4 is approximately 0.00023 which is the same order of magnitude as for the axisymmetric case, but larger. This means that for asymmetric gyrostats, the

value of X_1 must be closer to one than for an axisymmetric gyrostat (with moments of inertia the same as the lesser of i_2 or i_3) to produce the same linear error. Linear error may also be reduced by making the gyrostat less asymmetric. From this information, the range of X_1 can be determined for a particular gyrostat so that the linear solution approximates the nonlinear solution.

WKB vs Linear Oblate Solutions

This section contains the results of a comparison of the WKB and linear solutions for oblate spinup. The following conditions were used throughout this section for comparison:

$$\begin{aligned} X_1 &= 0.9999 \\ \mu_0 &= 0.1 \\ i_2 &= -0.6 \\ i_3 &= -0.4 \end{aligned}$$

The comparisons were made using three values for ϵ (0.1, 0.01, and 0.001) and from one to six WKB terms. The value of X_1 and the choice of i_2 and i_3 should give a good linear approximation for all ϵ . The program used to run this comparison is in the Appendix.

Fig 5 shows the linear solution and WKB approximation for $\epsilon=0.1$ and one WKB term. For all values of ϵ , the approximation using one WKB term is the only case to show any significant error. Using only one WKB term only gives an estimate of the phase while the amplitude is held constant. The time varying amplitude correction does not appear until a second WKB term is added to the approximation. The third

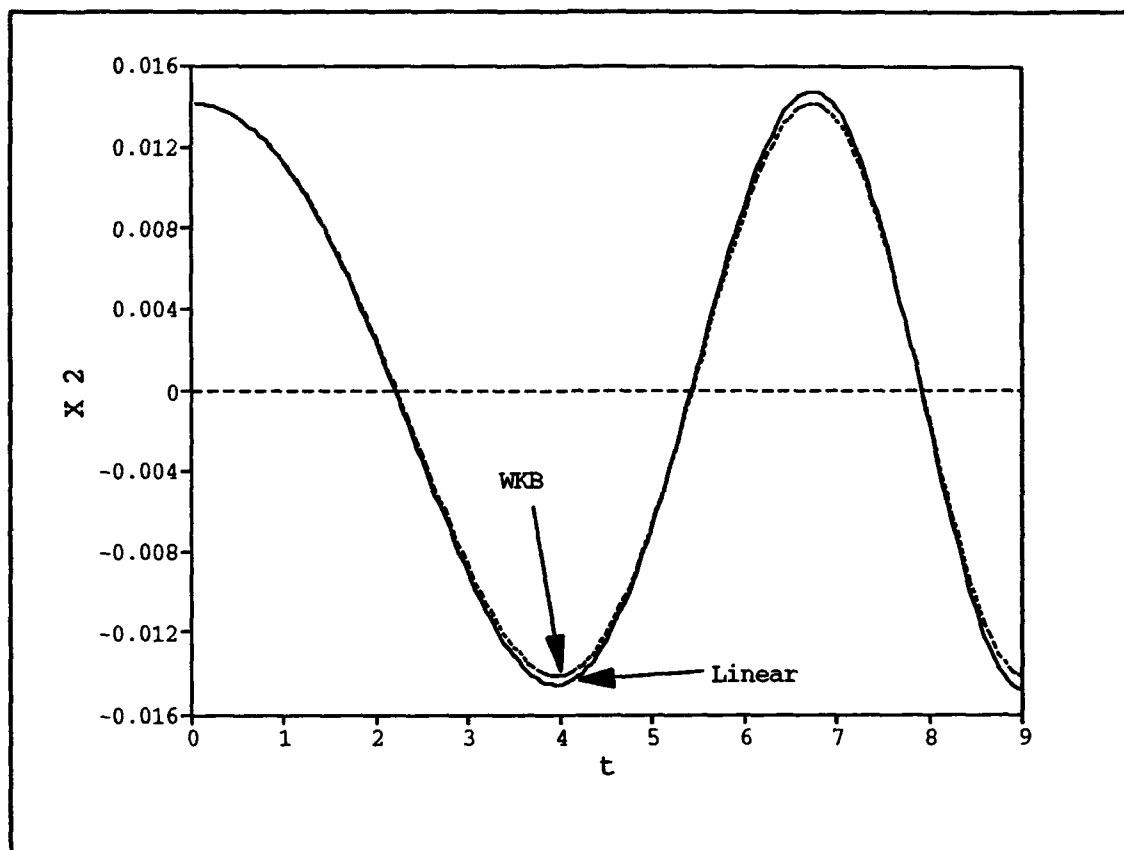


Figure 5. Linear and WKB Solutions for $\epsilon=0.1$ and One WKB Term

term mainly adjusts the varying phase again. Then the fourth term adjusts the varying amplitude, etc. This is evident from Eq (81).

Fig 6 is a summary of all of the comparisons. It shows the error in the last maximum (or minimum) of the WKB solution relative to the last maximum (or minimum) of the linear solution while varying the number of WKB terms. Fig 6 only contains only amplitude error. It shows that there is little increase in accuracy of the amplitude after two terms for $\epsilon=0.1$ and 0.001 . But there is still significant increase in accuracy for $\epsilon=0.01$ through the sixth WKB term.

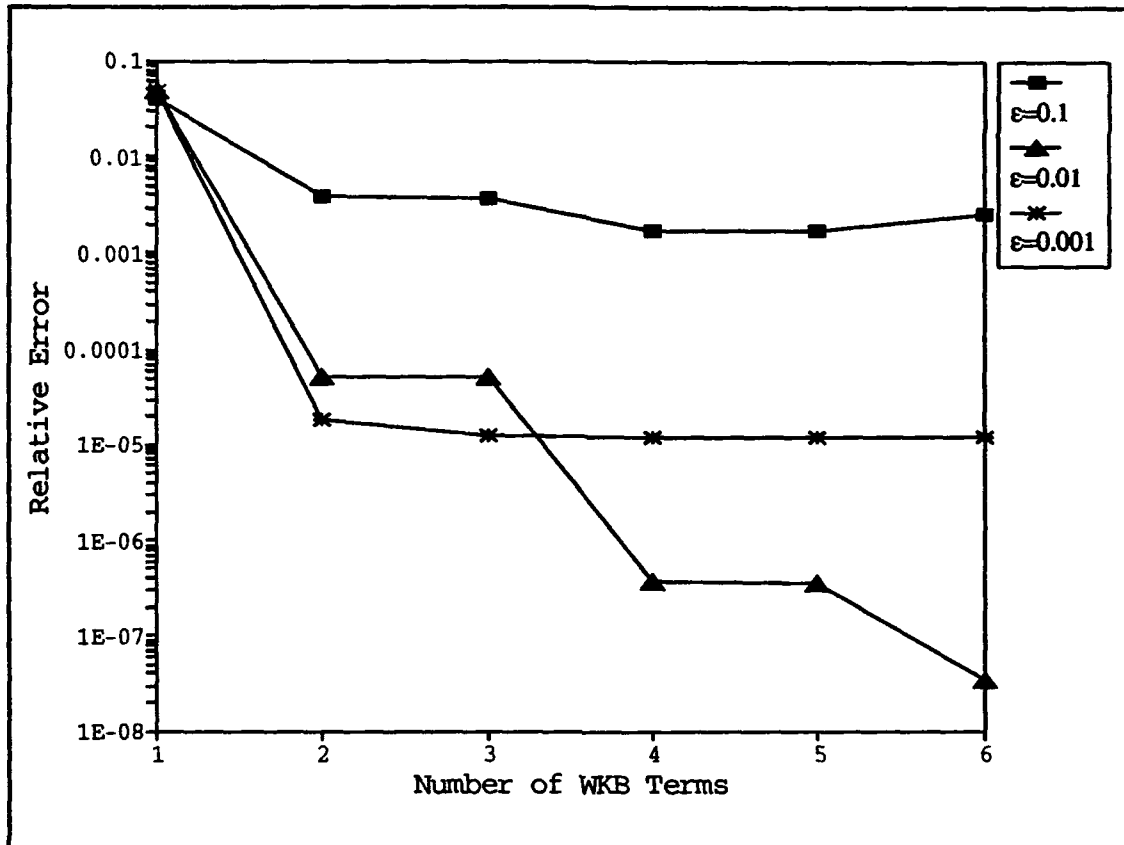


Figure 6. Relative Error in the Last Maximum of x_2 as ϵ and the Number of WKB Terms Change

Fig 7 shows the maximum error in the WKB solution over the entire spinup relative to the maximum magnitude of the linear solution during the spinup. This error includes the effects of both magnitude and phase. It indicates that for $\epsilon=0.1$, the WKB solution eventually diverges from the linear solution or at least oscillates around some final value as the number of terms increases. For $\epsilon=0.01$, there is a steady improvement in the WKB approximation until after the fifth term. To tell whether it will reach some final value or not would require the evaluation of more WKB terms. For $\epsilon=0.001$, the error of the WKB solution reaches a constant

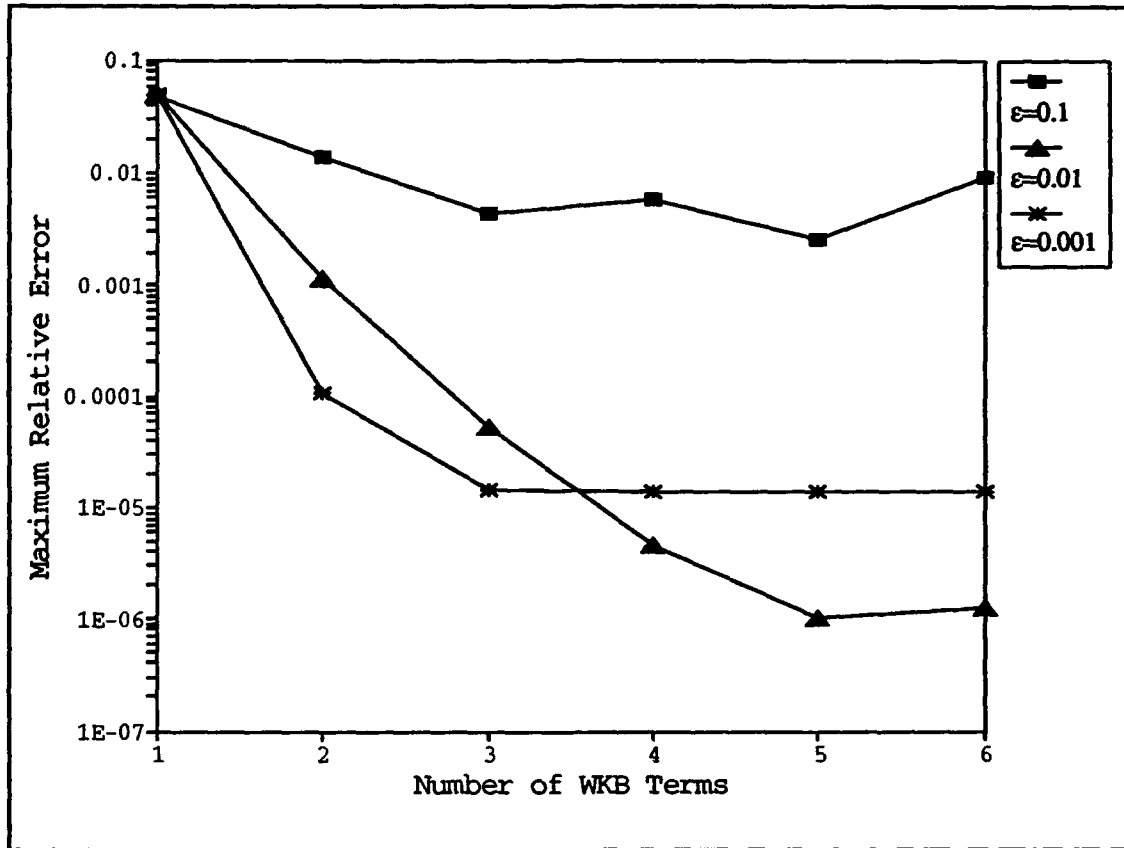


Figure 7. Maximum Relative Error in x_2 as ϵ and the Number of WKB Terms Change

value after the fourth term. The apparent oscillatory behavior of the $\epsilon=0.1$ error is probably due to ϵ being not much less than one as WKB theory requires. The higher order terms are multiplied by increasing powers of ϵ . To reach a final value quickly, ϵ must be much less than one. This is why the approximation with $\epsilon=0.001$ reaches its final value so quickly. After three terms, the maximum relative error for $\epsilon=0.01$ is less than for $\epsilon=0.001$. However, if the spinup for $\epsilon=0.001$ is examined over the same period of time as the total spinup for $\epsilon=0.01$, error in the $\epsilon=0.001$ approximation is less.

This example demonstrates that the WKB solution is very good approximation to linear oblate spinup dynamics. Except for the $\epsilon=0.1$ case, relative errors of less than 1% can be obtained with just two terms.

The effect of changing the inertia parameters was briefly examined. The cases investigated had the same conditions as the previous oblate example except the inertia parameters were varied. Only $\epsilon=0.01$ and two and four terms were used. Table 1 shows the results. It lists the inertia parameters used and the approximate factor by which the error increased over the $i_2=-0.6$, $i_3=-0.4$ case.

Table 1
Error Increase as Inertia Parameters are Varied

i_2	i_3	<u>2 TERMS</u>	<u>4 TERMS</u>
-0.9	-0.4	1	1
-0.5	-0.4	1	1
-0.6	-0.5	0.5	0.5
-0.6	-0.2	5	70
-0.9	-0.1	10	100
-0.6	-0.1	10	100
-0.2	-0.1	10	1000

The trend seems to indicate that inertia parameters that are closer to zero than those used in the baseline case will produce significantly larger errors than the baseline case. This is most likely due to spinup starting closer to the singularity at $t=i_3-\mu_0=-k_3$ in the WKB solution.

Nonlinear vs Linear Prolate Solution

This section contains a brief look at two of the factors that cause the linear solution to diverge from the nonlinear solution for prolate spinup.

Fig 8 shows a comparison of the nonlinear and linear solutions for the following conditions:

$$\begin{aligned}\epsilon &= 0.01 \\ X_{10} &= 0.9999 \\ i_2 &= 0.6 \\ i_3 &= 0.4 \\ \mu_0 &= 0.1\end{aligned}$$

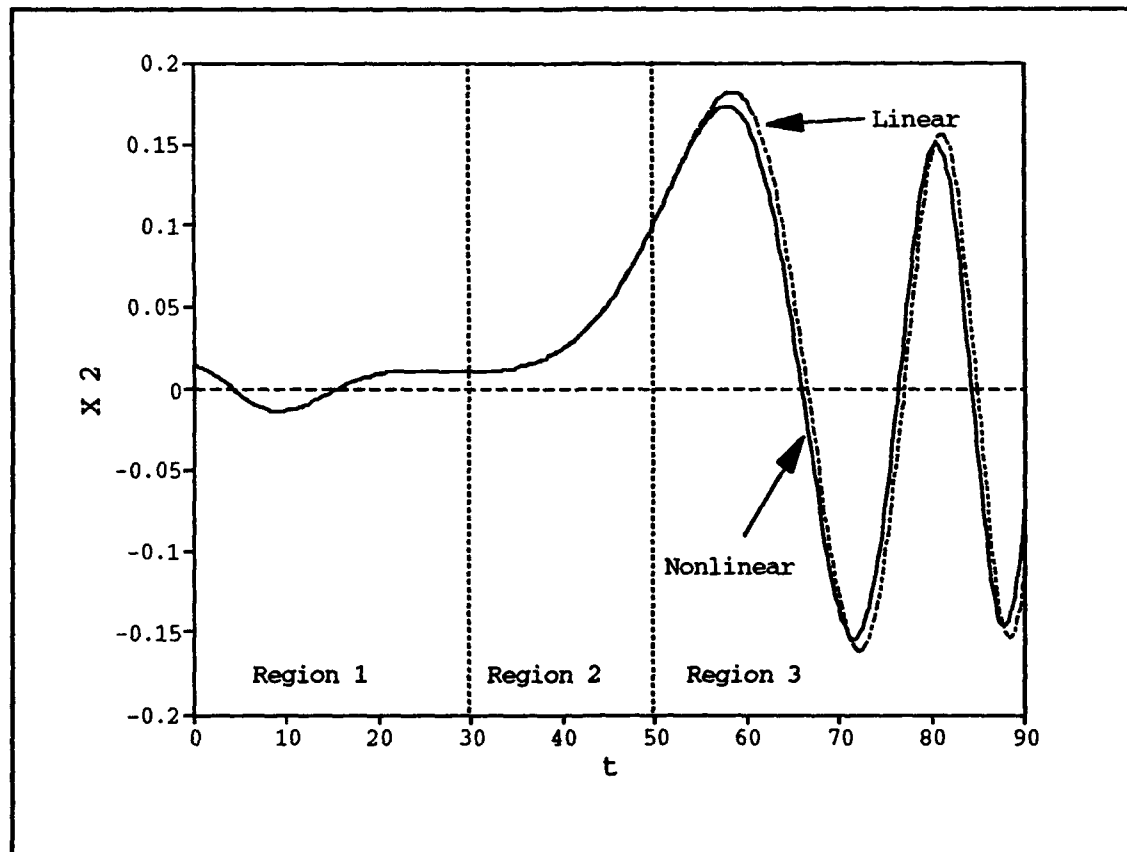


Figure 8. Nonlinear and Linear Solutions for $\epsilon=0.01$

The graph has been divided into the three regions defined in Chapter III. The above conditions cause the linear solution to deviate from the nonlinear solution. There is an

overshoot of the first maximum in region 3 which causes errors in both the extreme amplitudes and in the phase of the oscillation through the remainder of the spinup.

Fig 9 demonstrates the effect of changing only the initial value of X_1 from the conditions of Fig 8. The value

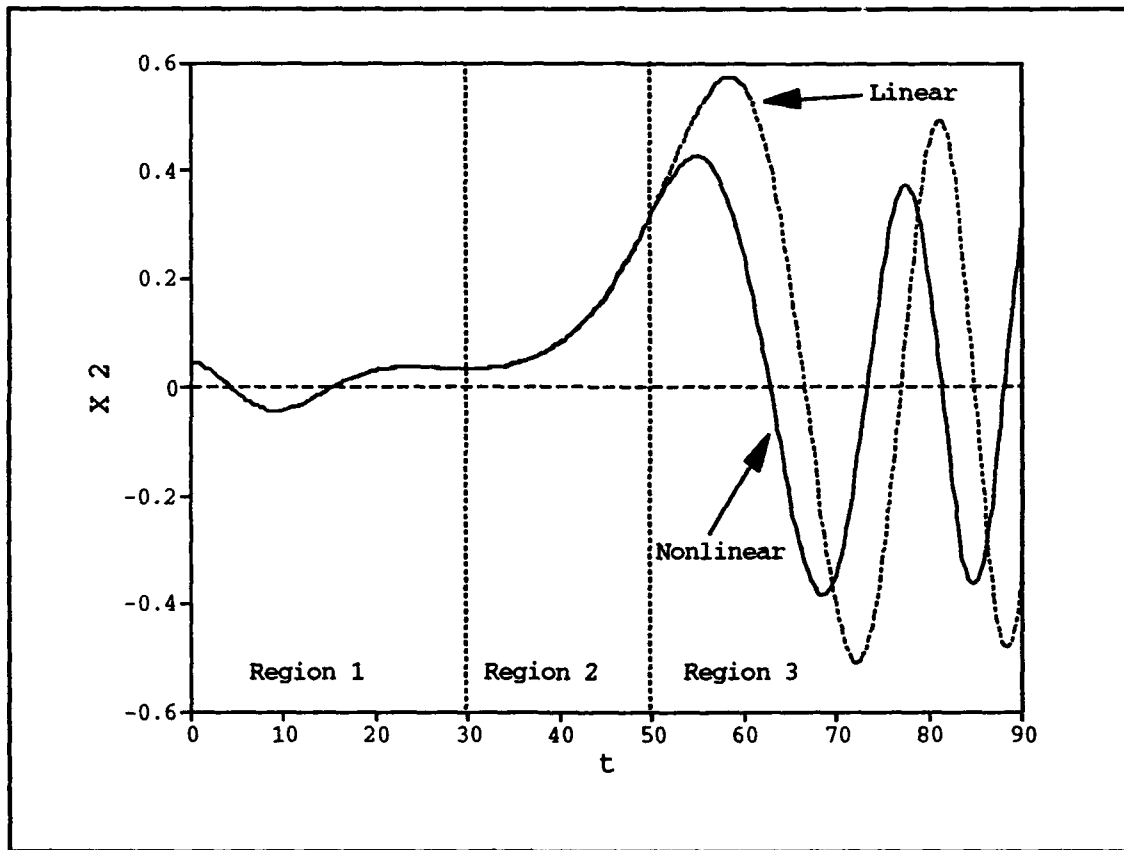


Figure 9. Nonlinear and Linear Solutions for $X_{10}=0.999$

of X_{10} is now 0.999 which is farther from the equilibrium point. The overshoot of the first maximum in region 3 has now increased. This causes more error in the extreme amplitude value and the phase over that seen in Fig 8.

Fig 10 demonstrates the effect of making the spacecraft more asymmetric than the case in Fig 8. The value of i_3 has

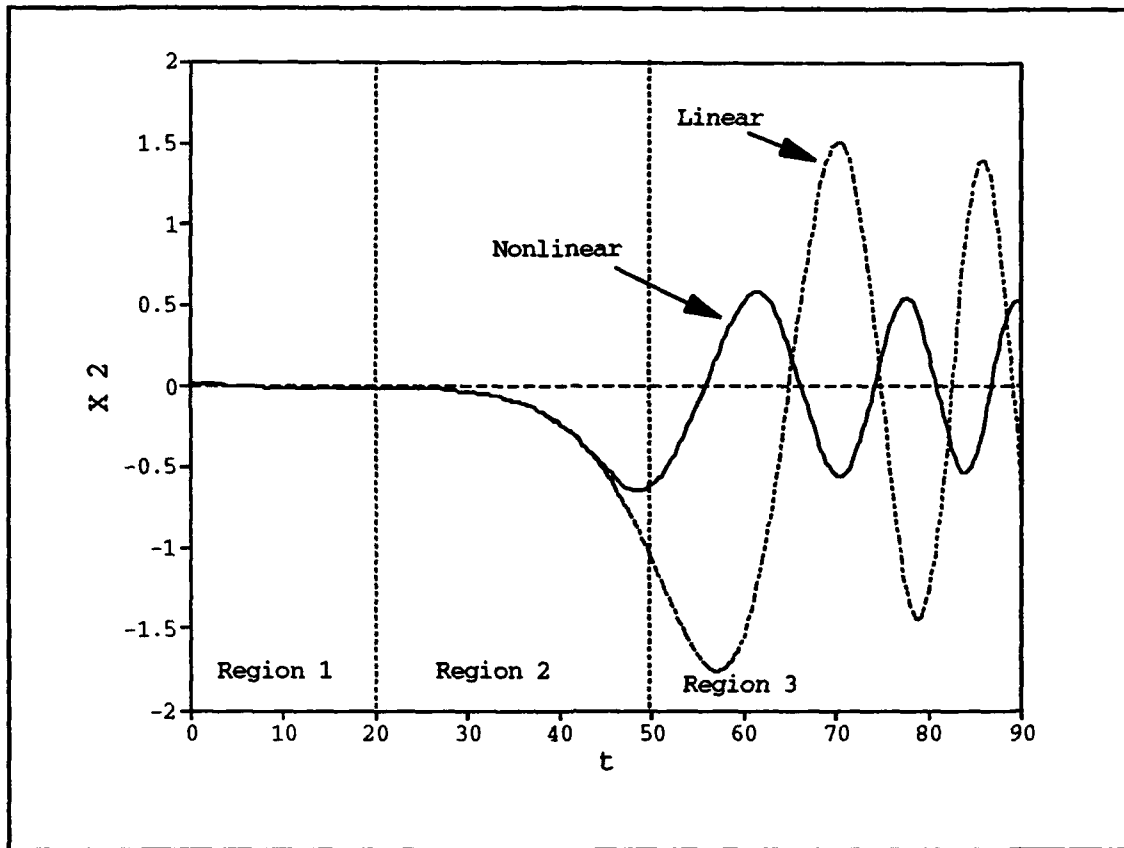


Figure 10. Nonlinear and Linear Solutions for $i_3=0.3$

now been changed to 0.3 which causes a deviation from the nonlinear solution which is even greater than the case in Fig 9.

Using figures similar to Figs 8-10, it can be shown that as ϵ increases the value of X_{10} must be closer to the equilibrium point and the spacecraft must be less asymmetric for the linear solution to better approximate the nonlinear solution.

It appears that region 2 is where exponential growth takes place in the linear solution which agrees with the approximate solution in Eq (100). Any error in the linear solution will rapidly grow in region 2 and will become pro-

nounced if region 2 is too wide. As ϵ increases, the time spent in region 2 increases, which will all amplify the error.

WKB vs Linear Prolate Solutions

This section contains the results of a comparison of the WKB and linear solutions for prolate spinup. The following conditions are used throughout this comparison:

$$\begin{aligned}x_1 &= 0.99999 \\ \mu_0 &= 0.1 \\ i_2 &= 0.6 \\ i_3 &= 0.5\end{aligned}$$

Comparisons are done using three values for ϵ (0.1, 0.01, and 0.001) and from one to six WKB terms. Although the conditions are just out of the linear region for $\epsilon=0.001$, the comparison still shows of how well the WKB method approximates the linear solution. The actual comparisons are divided into the three regions for prolate spinup defined in Chapter III.

Region 1 Comparison. Region 1 of prolate spinup starts at $\tau=0$ and ends near $\tau=0.4$ which is a singularity for all but the zeroth term of the WKB series (S_0). Therefore, one comparison between WKB and linear solutions could be as shown in Fig 11. Fig 11 is a plot that shows how close the WKB solutions get to the $\tau=0.4$ point before they deviate from the linear solution by 1% relative to x_{20} . Fig 11 shows that for $\epsilon=0.1$ the WKB solution with one term came closest to the singularity. For $\epsilon=0.01$, four terms came closest. For $\epsilon=0.001$, there was no substantial change after

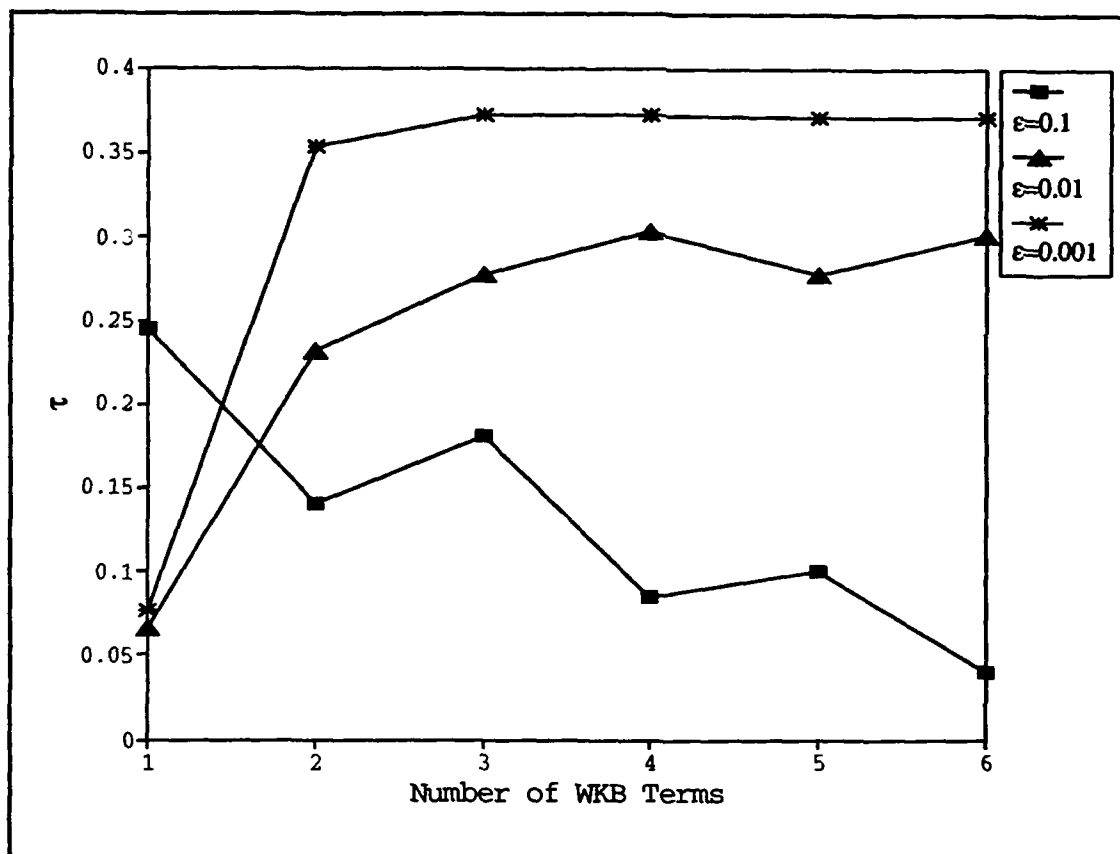


Figure 11. WKB vs Linear Comparison for 1% Relative Error Criterion in Region 1

the third terms, however the solution with four terms was slightly better than the others. Using this criterion, small ϵ 's were better than larger ones.

Another comparison is shown in Fig 12. It plots the maximum error between the WKB and linear solutions relative to x_{20} as a function of the number of WKB terms used for the three ϵ 's. This error was only measured between $\tau=0$ and $\tau=0.2$ to try to eliminate the error caused by the singularity at $\tau=0.4$. Fig 12 shows that on this interval, the best approximation for $\epsilon=0.1$ is still one WKB term. For $\epsilon=0.01$, five terms is now best. For $\epsilon=0.001$, six terms shows the

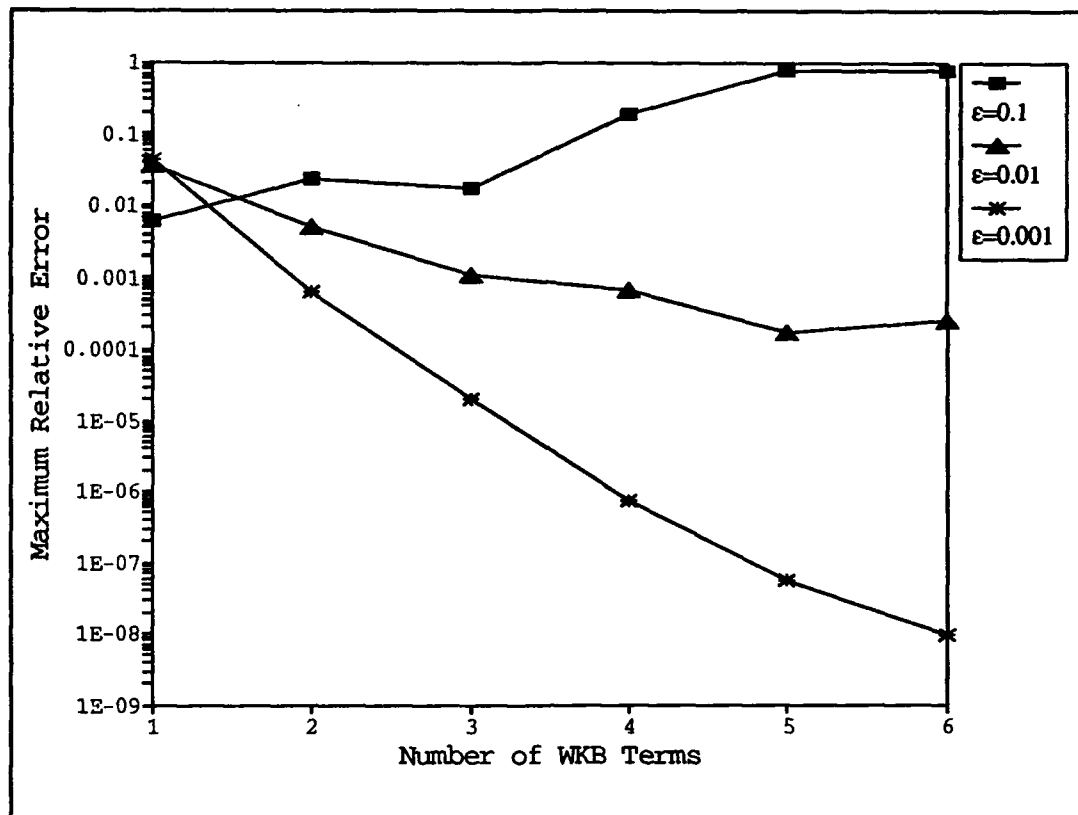


Figure 12. Relative WKB Error in First Half Region 1

least error.

Fig 13, 14, and 15 are three examples that show the comparison of the WKB and linear solutions for prolate spin-up in region 1. Fig 13 is the comparison for $\epsilon=0.1$ for one WKB term which was the best approximation for both the Fig 11 and 12 criteria. Fig 14 is the comparison for $\epsilon=0.01$ and four WKB terms which was the best for the Fig 11 criterion. And finally, Fig 15 is the comparison for $\epsilon=0.001$ and four terms which was also the best for the Fig 11 criterion.

The two comparison criteria in Figs 11 and 12 were completely arbitrary and other similar comparison criteria

could have been used, but overall the smaller the ϵ used the better the approximation of the linear solution.

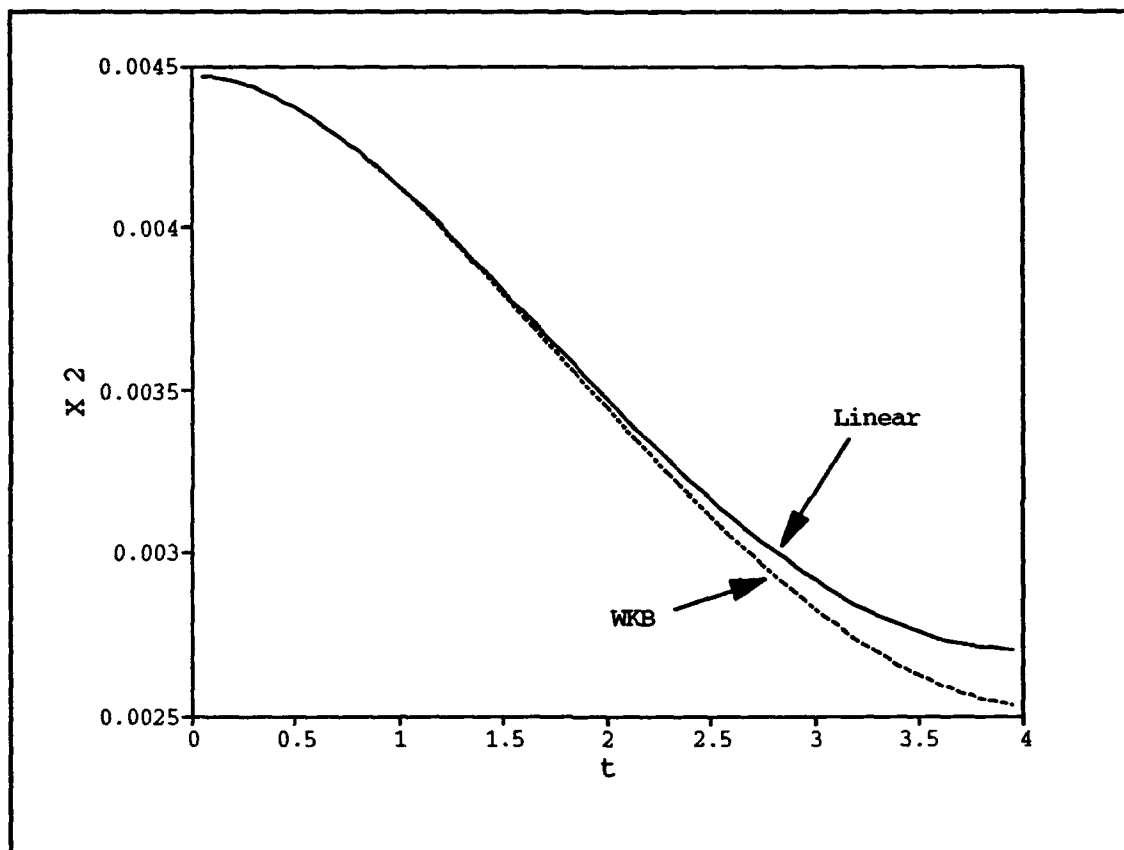


Figure 13. WKB and Linear Solutions in Region 1 for $\epsilon=0.1$ and One Term

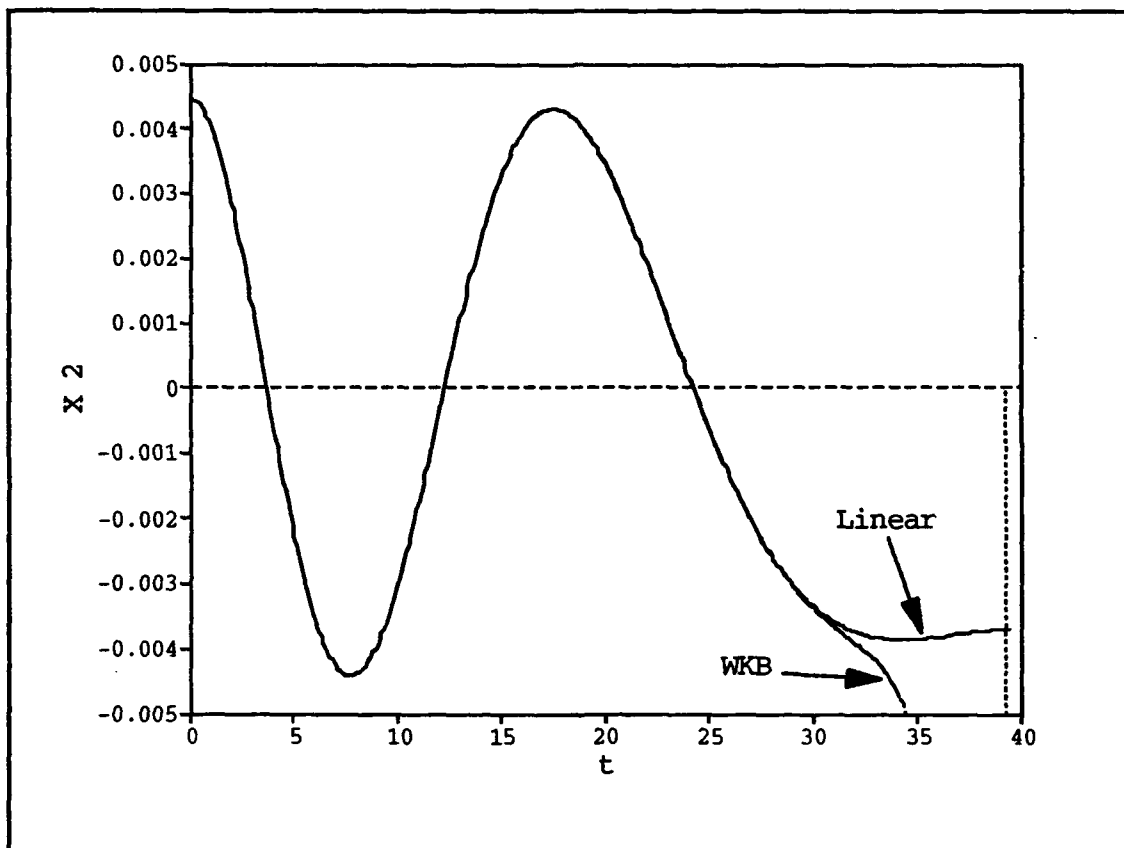


Figure 14. WKB and Linear Solutions in Region 1 for $\epsilon=0.01$ and Four Terms

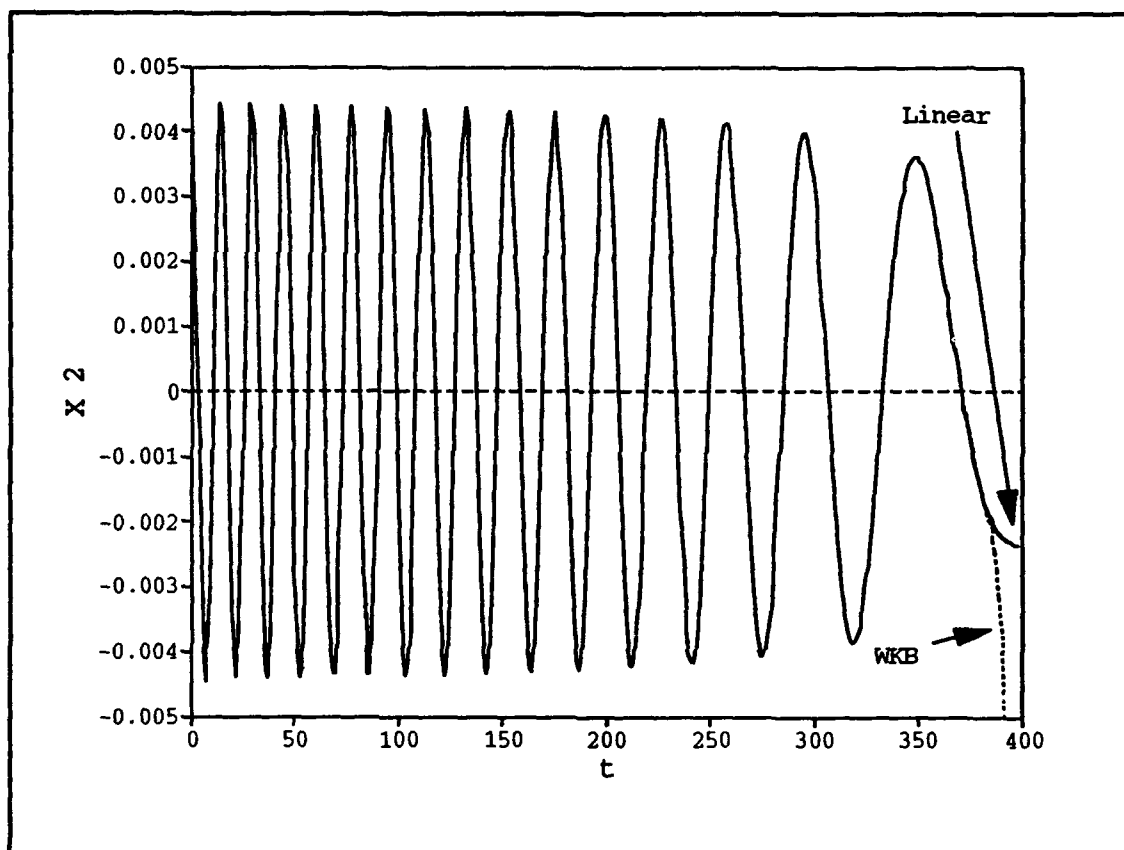


Figure 15. WKB and Linear Solutions in Region 1 for $\epsilon=0.001$ and Four Terms

Region 2 Comparison. Region 2 comparison is more complicated than for region 1. Region 2 is bounded by two singularities in the WKB solution at $\tau=0.4$ and $\tau=0.5$. Also, an initial condition for the WKB solution in region 2 can not be obtained from the WKB solution in region 1. The initial condition used for the WKB solution in region 2 came from numerically integrating the linear equations of motion out to the point $\tau=0.45$ which is the half way point between the singularities. This point was chosen to try to minimize the effects of the singularities on the WKB approximation in this region. The WKB solutions were then propagated backward and forward from the $\tau=0.45$ point and compared to the linear solution.

Fig 16 shows the same type of graph as Fig 11. Fig 16 is a plot that shows how close the WKB solutions get to the $\tau=0.4$ and 0.5 points before they deviate from the linear solution by 1% relative to the magnitude of the extreme value of the linear solution in region 2. Since the extreme values are different for each ϵ , care should be taken when trying to compare one ϵ to another. The closer the points are to the top and bottom of the graph, the better the approximation is. For $\epsilon=0.1$, the WKB solution with one term was the best. For $\epsilon=0.01$, two WKB terms were best. Two terms were also best for $\epsilon=0.001$ case based on this criterion.

Figs 17, 18, and 19 are examples of the comparisons between the WKB and linear solutions for prolate spinup in

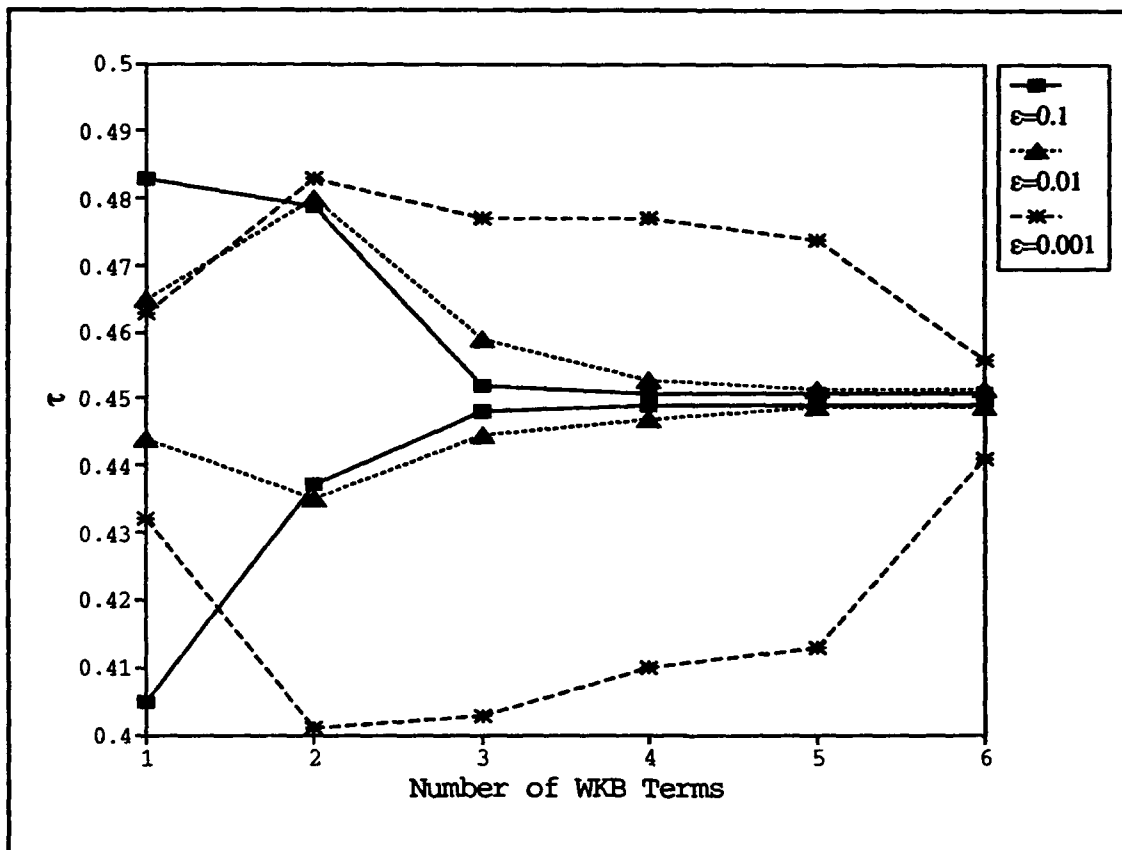


Figure 16. WKB vs Linear Comparison for 1% Relative Error Criterion in Region 2

region 2. Fig 17 is the best approximation for $\epsilon=0.1$ (one WKB term). Fig 18 is the best approximation for $\epsilon=0.01$ (two WKB terms). Fig 19 is the best approximation for $\epsilon=0.001$ (two WKB terms).

The WKB solutions in region 2 are not as accurate as those for region 1. If region 2 were larger, the WKB solution would probably have been better than it was, but this would have made the linear solution deviate farther from the nonlinear solution.

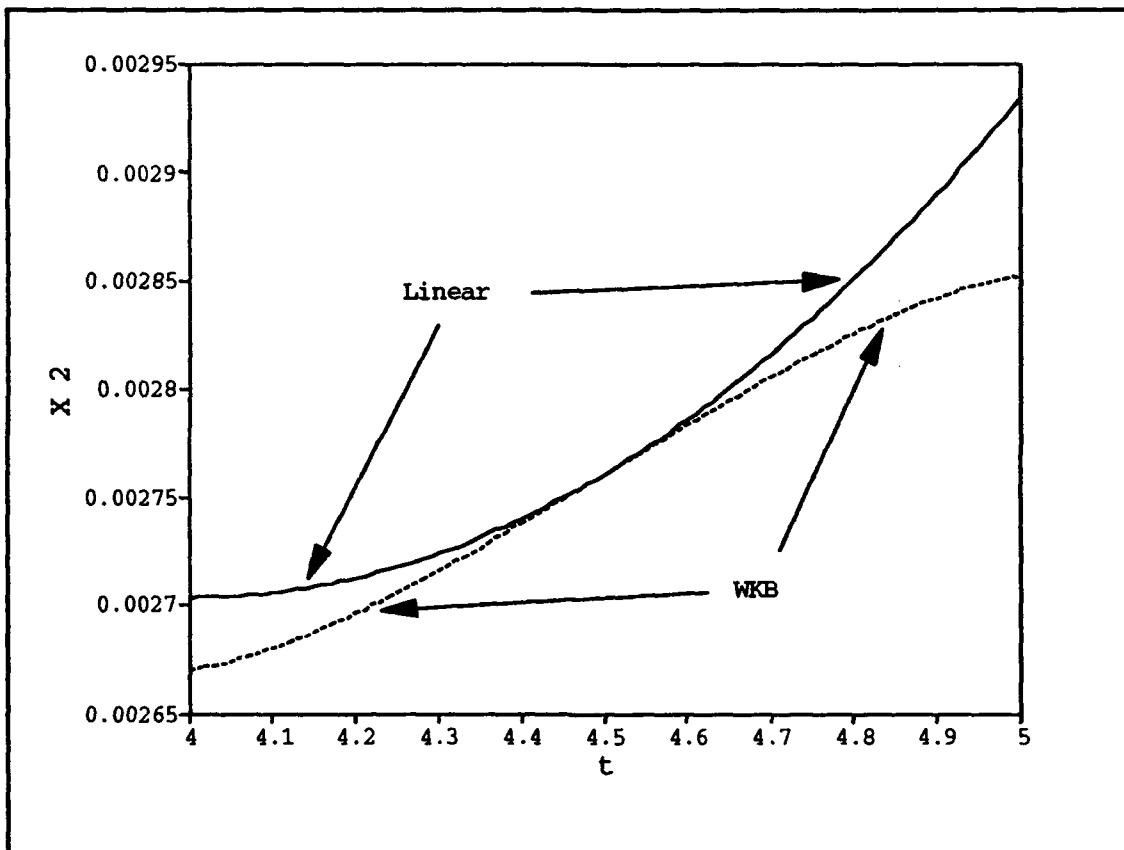


Figure 17. WKB and Linear Solutions in Region 2 for $\epsilon=0.1$ and One Term

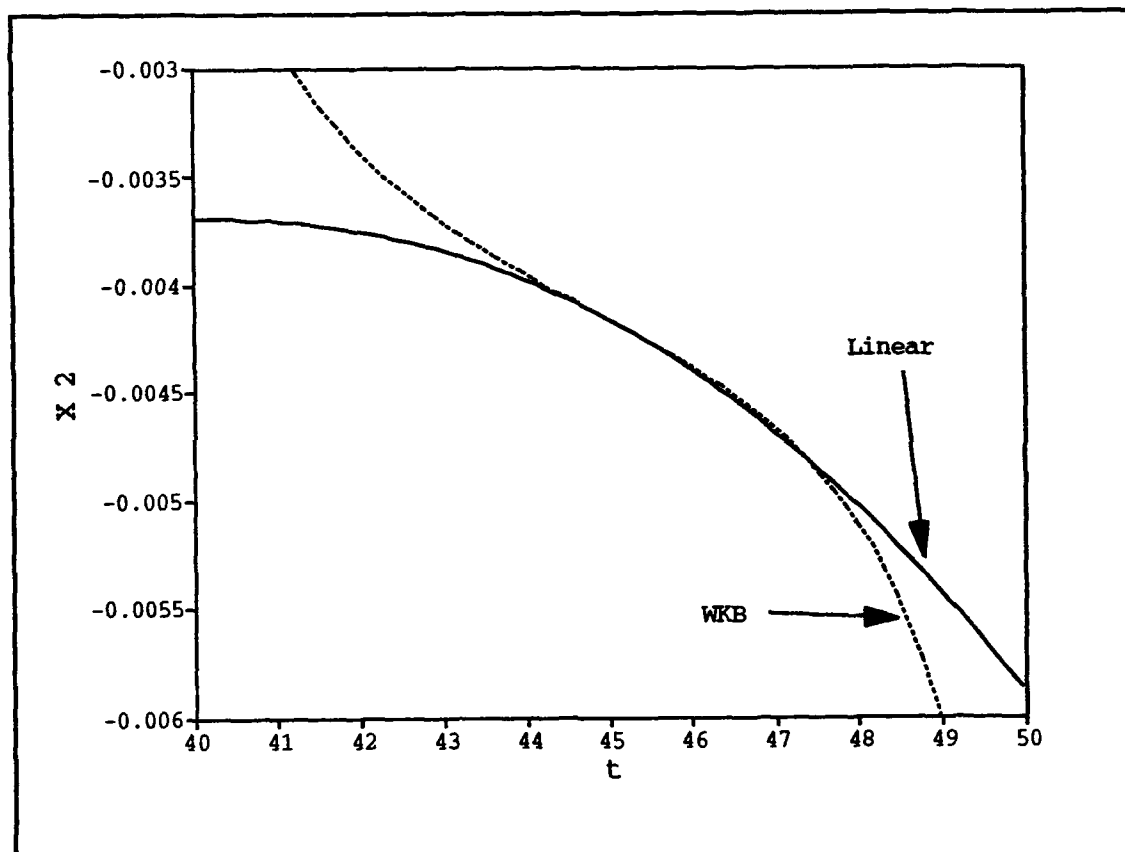


Figure 18. WKB and Linear Solutions in Region 2 for $\epsilon=0.01$ and Two Terms

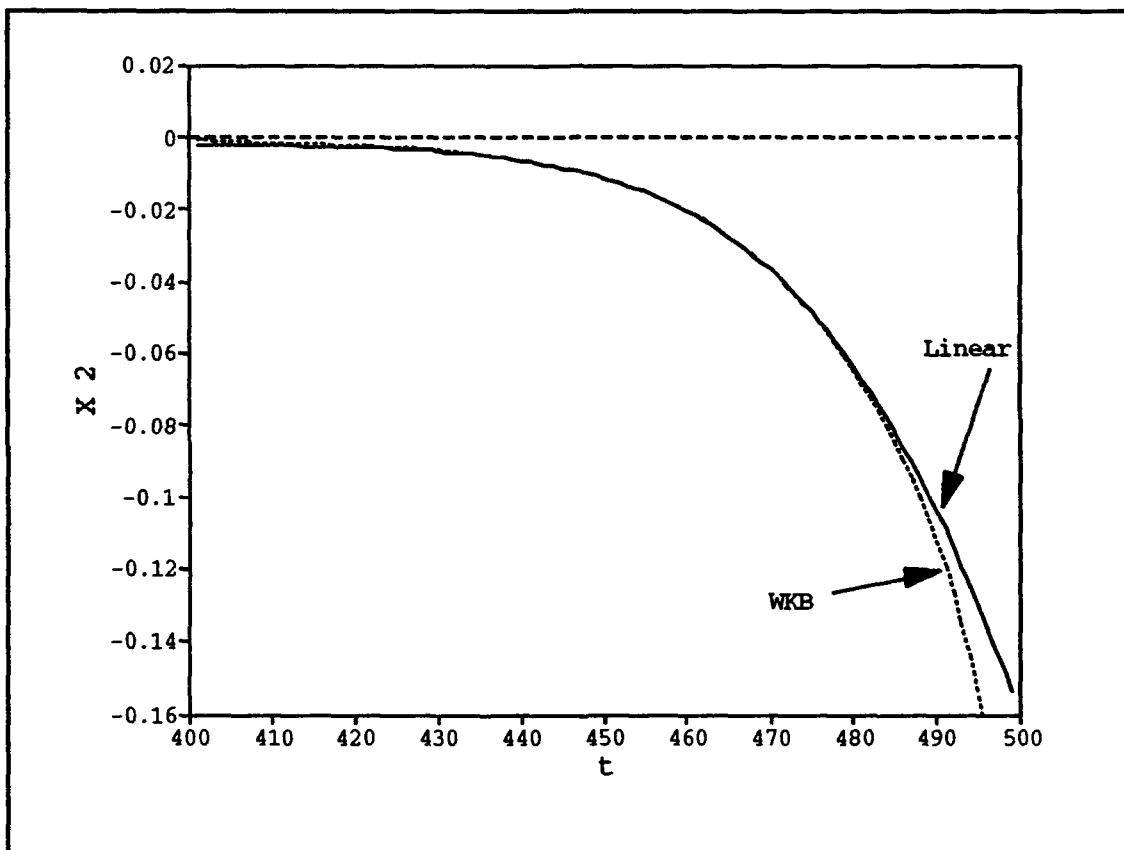


Figure 19. WKB and Linear Solutions in Region 2 for $\epsilon=0.001$ and Two Terms

Region 3 Comparison. In region 3 the initial conditions for the WKB approximation can not be obtained from region 2 because of the singularity at $\tau=0.5$. So as in region 2, the linearized equations of motion were numerically integrated to $\tau=0.9$ and then the WKB solution was propagated backward to obtain the comparison.

Like Fig 11 for region 1, Fig 20 is a plot that shows how close the WKB solutions get to the $\tau=0.5$ point before

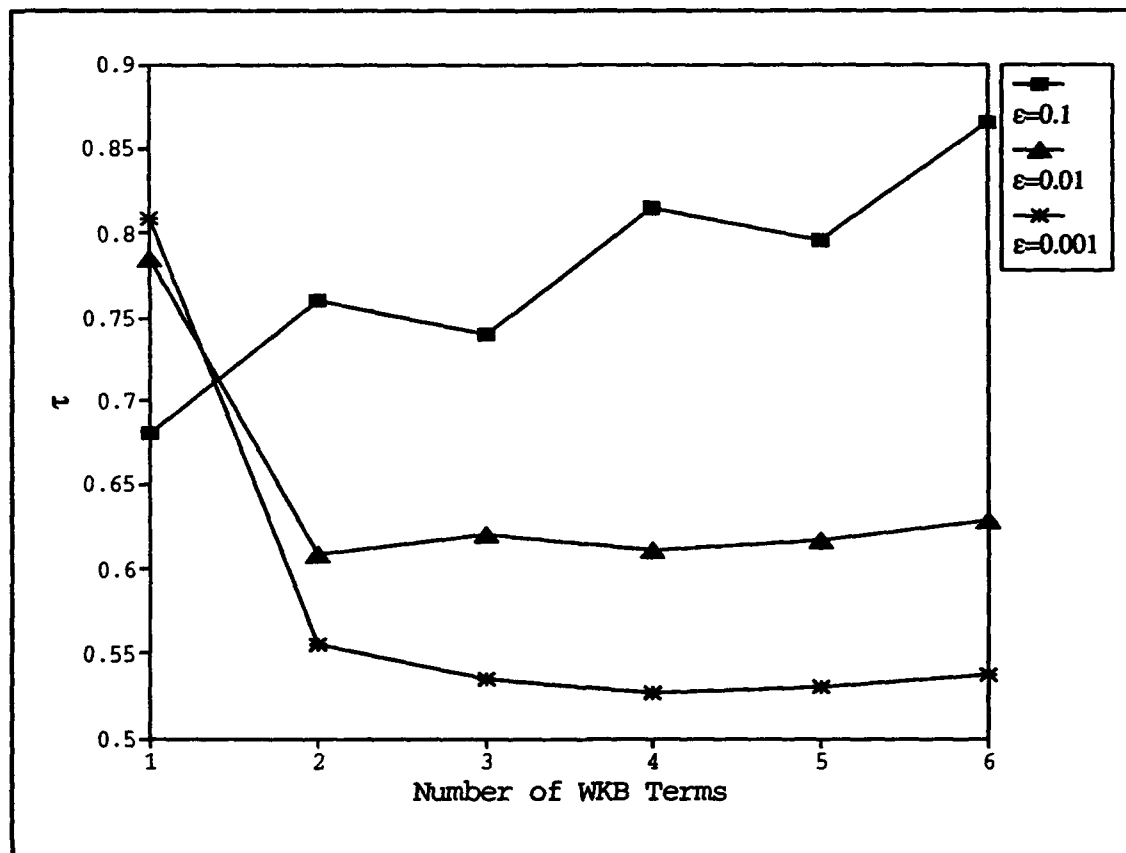


Figure 20. WKB vs Linear Comparison for 1% Relative Error Criterion in Region 3

they deviate from the linear solution by 1% relative to the magnitude of the extreme value of the linear solution in

region 3. Fig 20 shows that the best solution for this criterion for $\epsilon=0.1$ is one WKB term, For $\epsilon=0.01$, two terms came closest to the singularity. For $\epsilon=0.001$, four terms was best.

Fig 21 is similar to Fig 12 for region 1. Fig 21 plots

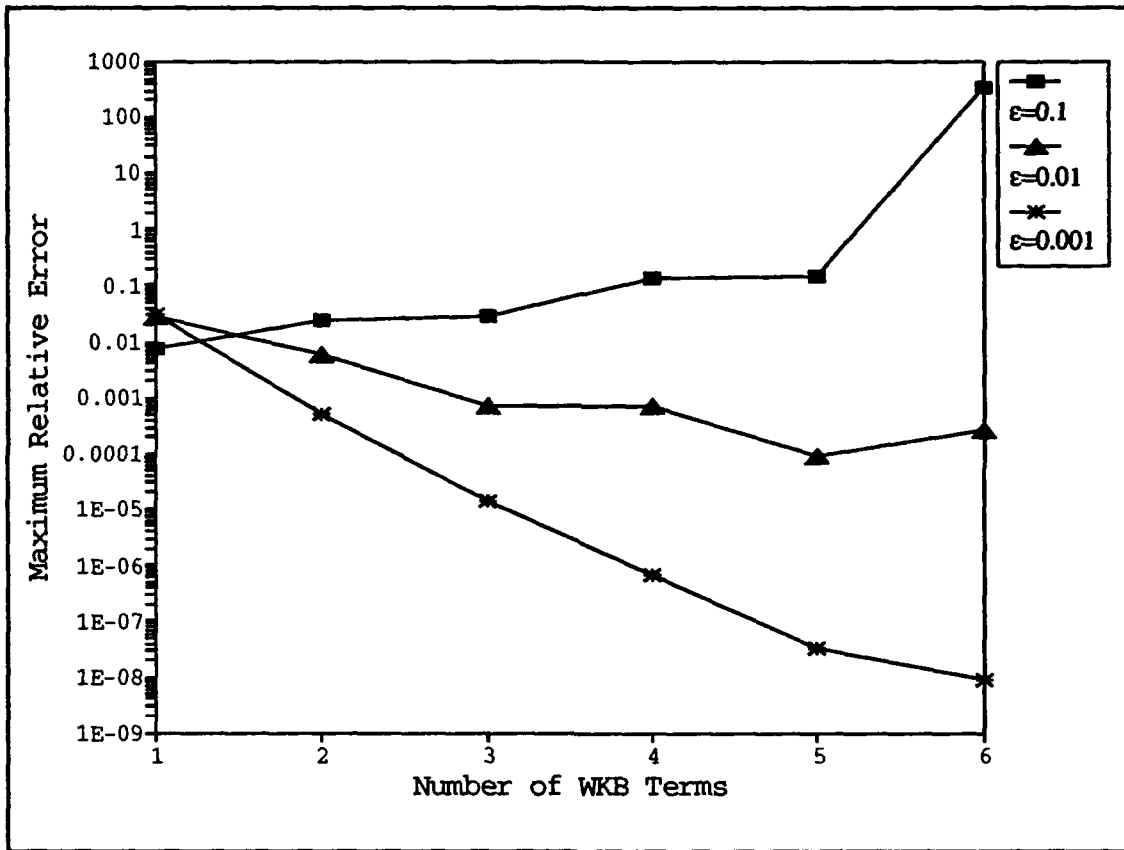


Figure 21. Relative WKB Error in Last Half of Region 3

the maximum error between the WKB and linear solutions relative to the magnitude of the extreme value of the linear solution in region 3. The error is shown as a function of the number of WKB terms used in the approximation. The interval used for the error data was from $\tau=0.9$ to $\tau=0.7$ to try to eliminate the effect of the singularity at $\tau=0.5$ in

the WKB approximation. For this criterion, the best approximation for $\epsilon=0.1$ is the same as for Fig 20 (one term). Now the best approximation for $\epsilon=0.01$ on this interval is five WKB terms. The best approximation for $\epsilon=0.001$ is six terms.

Fig 22, 23, and 24 are examples of the comparison between the WKB and linear solutions. Fig 22 shows the comparison for $\epsilon=0.1$ and one WKB term. Fig 23 shows the comparison for $\epsilon=0.01$ and two WKB terms. Fig 24 shows the comparison for $\epsilon=0.001$ and four WKB terms.

The criteria used in Figs 20 and 21 are arbitrary and different criteria could give different numbers of terms as being best. Like region 2, care should be taken when trying to compare one ϵ to another, since the errors in Fig 20 and 21 are relative to different values for each ϵ .

Spinup Starting in Region 3. One case of prolate spinup not examined is when $\mu_0 > i_2$ and i_3 . This spinup begins in region 3. It was not examined because it is the same as oblate spinup. Like oblate spinup, the WKB approximation will get worse as the start of spinup is moved closer to the WKB singularity.

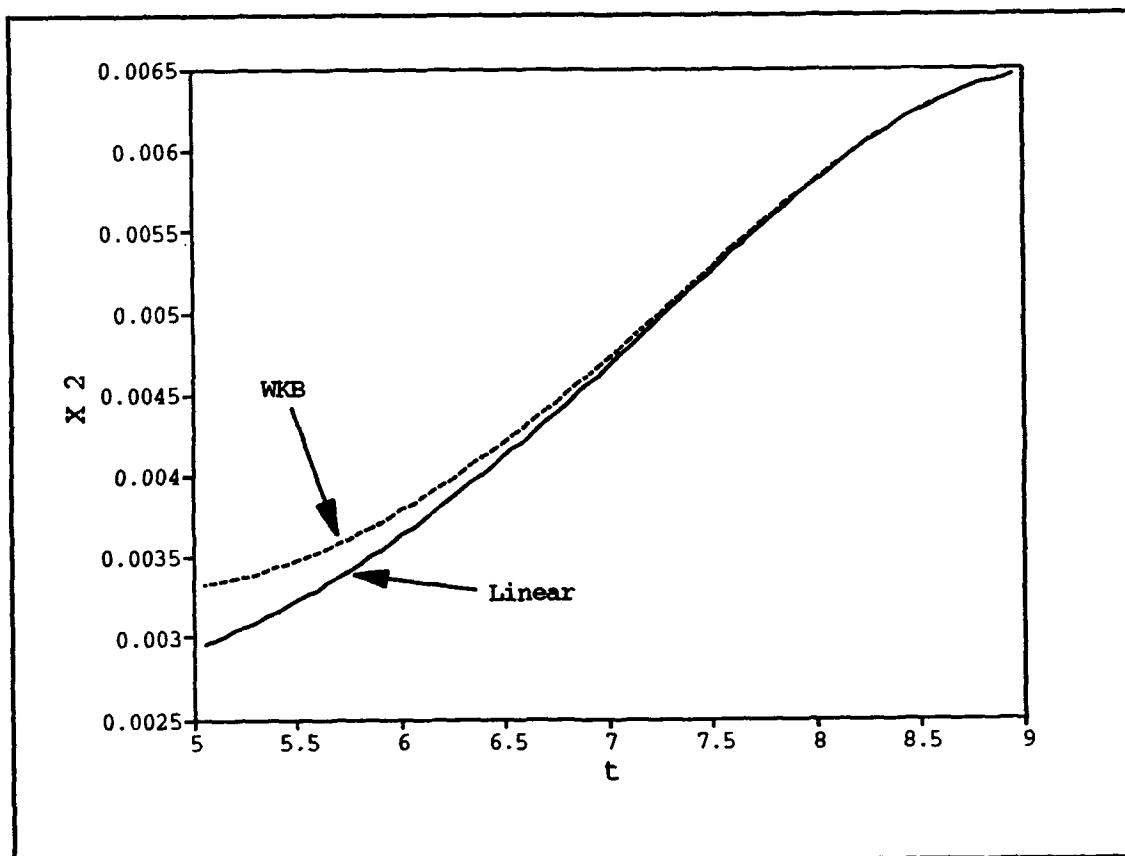


Figure 22. WKB and Linear Solutions in Region 3 for $\epsilon=0.1$ and One Term

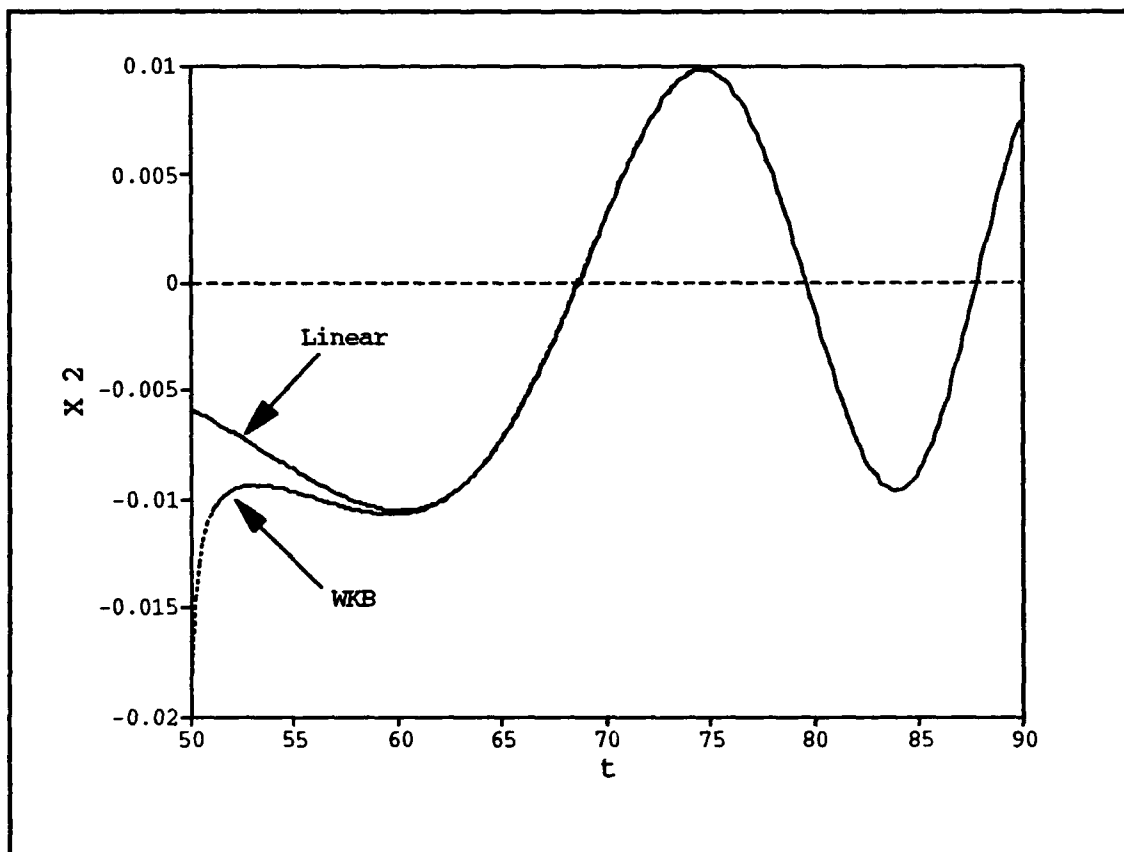


Figure 23. WKB and Linear Solutions in Region 3 for $\epsilon=0.01$ and Two Terms

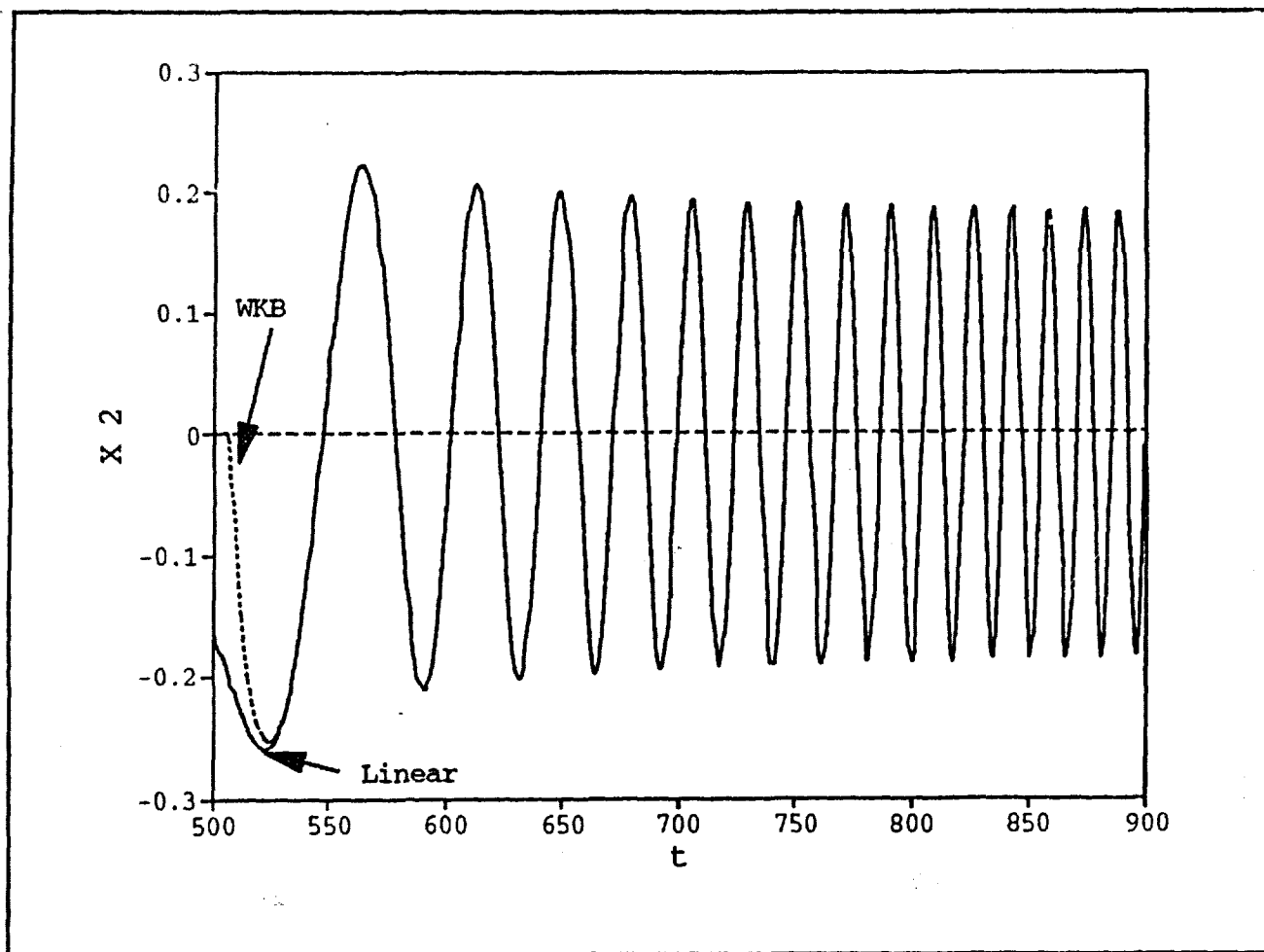


Figure 24. WKB and Linear Solutions in Region 3 for $\epsilon=0.001$ and Four Terms

WKB vs Linear Axisymmetric Solutions

The final case to be examined is the special case of axisymmetric spinup where $i_2=i_3$ and $k_2=k_3$. For this condition, the solution to the linear equations of motion, Eqs (26)-(28), takes the form

$$x_2 = C_1 \exp \left[i \int (\mu - i_2) dt \right] + C_2 \exp \left[-i \int (\mu - i_2) dt \right] \quad (101)$$

Substituting Eqs (33), (34), and (37) into Eq (101) gives

$$x_2 = C_1 \exp \left[\frac{i}{\epsilon} \left(\frac{\tau^2}{2} + k_2 \tau \right) \right] + C_2 \exp \left[\frac{-i}{\epsilon} \left(\frac{\tau^2}{2} + k_2 \tau \right) \right] \quad (102)$$

When $k_2=k_3$, all of the derivatives of the WKB terms in Eqs (63)-(68) are zero except

$$S'_0 = \pm i (\tau + k_2) \quad (103)$$

Taking the integral of Eq (103) with respect to τ gives

$$S_0 = \pm i \left(\frac{\tau^2}{2} + k_2 \tau \right) \quad (104)$$

Then substituting Eq (104) into Eq (76) gives the same equation as Eq (102). Therefore, the WKB solution is exactly the same as the linear solution for the axisymmetric case.

Also, note that for $i_2=i_3$ the nonlinear equations, Eqs (16)-(18), become linear. Now, if the definition of k_2 is changed to

$$k_2 = \mu_0 - i_2 X_1 \quad (105)$$

the WKB solution will also be the same as the solution to Eqs (16)-(18) since X_1 is now a constant.

V. Conclusions and Recommendations

Conclusions

In this thesis, the WKB method is used to approximate the linear solutions for oblate and prolate spinup of a dual-spin spacecraft. Based on the comparisons between the WKB and linear solutions, the following conclusions are drawn:

1. The WKB solution is a good approximation to the linear solution for oblate spinup. For $\epsilon=0.01$ and 0.001 , the WKB solution with only two terms has a relative error of less than 1% for the example problem. For $\epsilon=0.1$, three terms are required to reach this accuracy.
2. The WKB solution for prolate spinup is not practical. The WKB solution contains two singularities around which the approximation is not valid. These points divide the WKB solution into three parts. To obtain each succeeding WKB solution the linear equations of motion must be numerically integrated to obtain initial conditions for the new WKB solution. Also, the linear solution is only valid for spacecraft that are nearly axisymmetric and for very small perturbations from the

equilibrium point which greatly restricts the conditions under which the WKB method can be used.

3. The WKB method provides the exact solution for the axisymmetric spacecraft. Only one WKB term is required for the WKB solution to be the same as either the solution of the linear or nonlinear equations of motion.

Recommendations

The following are some recommendations for further study:

1. Find some method to patch the three solutions of the WKB approximation for prolate spinup together to give a solution that is not dependent upon a numerically integrated solution.
2. Use a different perturbation method to approximate the nonlinear equations of motion. This would be less restrictive than using the WKB approximations which are only valid where the linear solution is valid.
3. Investigate the optimum range of ϵ for the minimum error between the nonlinear and WKB solutions for spinup. The error between the WKB and linear solution varies with ϵ for a set of initial conditions. Generally as ϵ decreases, the error de-

creases depending on the number of WKB terms used in the approximation. However, as ϵ decreases, the spinup time grows longer and the error increases between the linear and nonlinear solution. This is shown qualitatively in Fig 25.

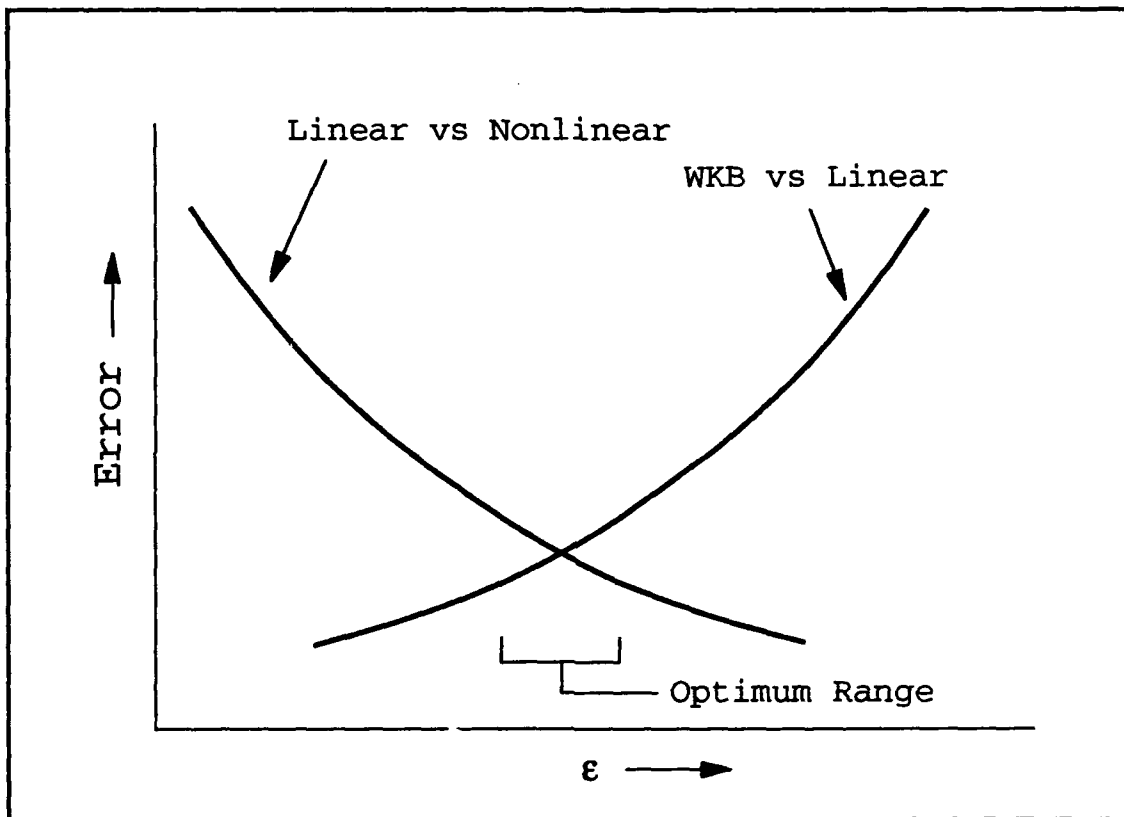


Figure 25. Error as a Function of ϵ

Appendix: Computer Program

Below is the FORTRAN program that was used to generate the comparison between the linear and WKB solutions of oblate spinup. All of subroutine HAMING and the design for the main program and subroutine RHS were taken from reference 7 (7:108-113):

```

      program dualspin
c
      common /ham/ t,x(42,4),f(42,4),err(42),n,h,mode
      common /dual/ i2,i3,e,mu0,z,z0,zp0,iterms
      double precision t, x,f,err,h,i2,i3,e
      double precision z,mu0,z0,zp0
      open (unit=20,file='C:\qpro\bin\wkb.txt')
c
      n=6
      t=0.d0
c
      write(*,*) 'Compare WKB with exact(type: 2); linear(type: 5).'
      read(*,*) m
      write(*,*) 'Type 1-6 for number of WKB terms.'
      read(*,*) iterms
      write(*,*) 'What is i2 ?'
      read(*,*) i2
      write(*,*) 'What is i3 ?'
      read(*,*) i3
      write(*,*) 'What is epsilon (mu dot) ?'
      read(*,*) e
      write(*,*) 'What is the initial value of x1 ?'
      read(*,*) x(1,1)
      x(2,1)=dsqrt(1.d0-x(1,1)**2.d0)
      zp0=0.d0
      x(3,1)=0.d0
      write(*,*) 'What is the initial value of x4 (mu) ?'
      read(*,*) x(4,1)
      h=.025d0
      write(*,*) 'This will produce ',nint((1.d0-x(4,1))/(e*h)),
+ ' data points.'
      write(*,*) 'To print every Ith data point, ',
+ 'enter I as an integer value.'
      read(*,*) ith
      z0=x(2,1)
      mu0=x(4,1)
      x(5,1)=x(2,1)
      x(6,1)=x(3,1)
c
      ii=nint((1.d0-x(4,1))/(e*h*dble(ith)))
      nxt=0
      call haming(nxt)
      if(nxt.eq.0) stop 99
```

```

      do 100 i=1,i1
        do 50 j=1,ith
          call haming(nxt)
50    continue
      call wkb
      write(20,1000) t,x(m,nxt),z,z-x(m,nxt)
1000  format(f8.3,2x,e16.9,2x,e16.9,2x,e16.9)
100  continue
c
      close(20)
      end
c
$INCLUDE:'rhsolin.for'
$INCLUDE:'haming.for'
$INCLUDE:'wkbo.for'

      subroutine rhs(nxt)
c
      common /ham/ t,x(42,4),f(42,4),err(42),n,h,mode
      common /dual/ i2,i3,e,mu0,z,z0,zp0,iterms
      double precision t,x,f,err,h,i2,i3,e,mu0,z,z0,zp0
c These are the actual EOMs:
      f(1,nxt)=(i2-i3)*x(2,nxt)*x(3,nxt)
      f(2,nxt)=(i3*x(1,nxt)-x(4,nxt))*x(3,nxt)
      f(3,nxt)=-(i2*x(1,nxt)-x(4,nxt))*x(2,nxt)
      f(4,nxt)=e
c These are the linearized EOMs:
c This is X2 dot:
      f(5,nxt)=(i3-x(4,nxt))*x(6,nxt)
c This is X3 dot:
      f(6,nxt)=(x(4,nxt)-i2)*x(5,nxt)
      return
      end

      subroutine haming(nxt)
      common /ham/ x,y(42,4),f(42,4),errest(42),n,h,mode
      double precision x,y,f,errest,h
      double precision hh,xo,tol
      tol=0.0000000000001d0
      if(nxt) 190,10,200
10  xo=x
      hh=h/2.0d0
      call rhs(1)
      do 40 l=2,4
        x=x+hh
        do 20 i=1,n
          y(i,l)=y(i,l-1)+hh*f(i,l-1)
          call rhs(1)
          x=x+hh
          do 30 i=1,n
            y(i,l)=y(i,l-1)+h*f(i,l)
40    call rhs(1)
        jsw=-10
50    isw=1
        do 120 i=1,n
          hh=y(i,1)+h*(9.0d0*f(i,1)+19.0d0*f(i,2)
1    -5.0d0*f(i,3)+f(i,4))/24.0d0
          if(dabs(hh-y(i,2)).lt.tol) go to 70
          isw=0
70    y(i,2)=hh
          hh=y(i,1)+h*(f(i,1)+4.0d0*f(i,2)
1    + f(i,3))/3.0d0

```

```

        if(dabs(hh-y(i,3)).lt.tol) go to 90
        isw=0
90    y(i,3)=hh
        hh=y(i,1)+h*(3.0d0*f(i,1)+9.0d0*f(i,2)
1    +9.0d0*f(i,3)+3.0d0*f(i,4))/8.0d0
        if(dabs(hh-y(i,4)).lt.tol) go to 110
        isw=0
110   y(i,4)=hh
120   continue
        x=xo
        do 130 l=2,4
            x=x+h
130   call rhs(l)
        if(isw) 140,140,150
140   jsw=jsw+1
        if(jsw) 50,280,280
150   x=xo
        isw=1
        jsw=1
        do 160 i=1,n
160   errest(i)=0.0d0
        nxt=1
        go to 280
190   jsw=2
        nxt=labs(nxt)
200   x=x+h
        np1=mod(nxt,4)+1
        go to (210,230),isw
210   go to (270,270,270,220),nxt
220   isw=2
230   nm2=mod(np1,4)+1
        nm1=mod(nm2,4)+1
        npo=mod(nm1,4)+1
        do 240 i=1,n
            f(i,nm2)=y(i,np1)+4.0d0*h*(2.0d0*f(i,npo)
1            -f(i,nm1)+2.0d0*f(i,nm2))/3.0d0
240   y(i,np1)=f(i,nm2)-0.925619835d0*errest(i)
        call rhs(np1)
        do 250 i=1,n
            y(i,np1)=(9.0d0*y(i,npo)-y(i,nm2)
1            +3.0d0*h*(f(i,np1)+2.0d0*f(i,npo)
2            -f(i,nm1)))/8.0d0
            errest(i)=f(i,nm2)-y(i,np1)
250   y(i,np1)=y(i,np1)+0.0743801653d0*errest(i)
        go to (260,270), jsw
260   call rhs(np1)
270   nxt=np1
280   return
        end

```

subroutine wkb

```

c      common /ham/ t,x(42,4),f(42,4),err(42),n,h,mode
c      common /dual/ i2,i3,e,mu0,z,z0,zp0,items

c      double precision t,x,f,err,h
c      double precision mu0,i2,i3,e,z,z0,zp0,k2,k3,et
c      double precision rel,rel0,relp0,img,img0,imgp0
c      double precision s0,s1,s2,s3,s4,s5
c      double precision s00,s10,s20,s30,s40,s50
c      double precision s0p0,s1p0,s2p0,s3p0,s4p0,s5p0

```

```

k2=mu0-12
k3=mu0-13
c
s0p0=dsqrt(k2*k3)
c
s1p0=(k2-k3)/(4.d0*k2*k3)
c
s2p0=((k3-k2)*(7.d0*k2+5.d0*k3))/
+ (32.d0*(k2*k3)**2.5d0)
c
s3p0=(3.d0*(k3-k2)*(7.d0*k2**2.d0
+ 8.d0*k2*k3+5.d0*k3**2.d0))/
+ (64.d0*(k2*k3)**4.d0)
c
s4p0=((k2-k3)*(1.463d3*k2**3.d0+2.163d3*k2**2.d0*k3
+ 1.989d3*k2*k3**2.d0+1.105d3*k3**3.d0))/
+ (2.048d3*(k2*k3)**5.5d0)
c
s5p0=(3.d0*(7.07d2*k2**5.d0+5.39d2*k2**4.d0*k3
+ 1.46d2*k2**3.d0*k3**2.d0
+ -2.62d2*k2**2.d0*k3**3.d0
+ -5.65d2*k2*k3**4.d0-5.65d2*k3**5.d0))/
+ (1.024d3*(k2*k3)**7.d0)
c
do 10 i=1,2
c
if(i.eq.1) then
et=0.d0
else
et=e*t
end if
c
s0=((k2+k3)/4.d0+et/2.d0)*dsqrt((k2+et)*(k3+et))
+ -(k3-k2)**2.d0/8.d0*dlog(k2+k3+2.d0*et
+ 2.d0*dsqrt((k2+et)*(k3+et)))
c
s1=(dlog(k3+et)-dlog(k2+et))/4.d0
c
s2=(-(-7.d0*k2**3.d0+27.d0*k2**2.d0*k3
+ -9.d0*k2*k3**2.d0+5.d0*k3**3.d0
+ +6.d0*k2**2.d0*et+36.d0*k2*k3*et
+ +6.d0*k3**2.d0*et+24.d0*k2*et**2.d0
+ +24.d0*k3*et**2.d0+16.d0*et**3.d0)/
+ (48.d0*(k3-k2)**2.d0*((k2+et)*(k3+et))**1.5d0)
c
s3=((k2-k3)*(7.d0*k2+5.d0*k3+12.d0*et))/
+ (64.d0*((k2+et)*(k3+et))**3.d0)
c
s4=(1.105d3/(9.216d3*(k2-k3)**2.d0*(k2+et)**5.d0)
+ 1.547d3/(9.216d3*(k2-k3)**3.d0*(k2+et)**4.d0)
+ 1.626d3/(1.536d4*(k2-k3)**4.d0*(k2+et)**3.d0)
+ 1.633d3/(4.608d4*(k2-k3)**5.d0*(k2+et)**2.d0)
+ 1.27d2/(5.76d3*(k2-k3)**6.d0*(k2+et))
+ -1.463d3/(9.216d3*(k2-k3)**2.d0*(k3+et)**5.d0)
+ 2.653d3/(9.216d3*(k2-k3)**3.d0*(k3+et)**4.d0)
+ -4.973d3/(1.536d4*(k2-k3)**4.d0*(k3+et)**3.d0)
+ 1.5767d4/(4.608d4*(k2-k3)**5.d0*(k3+et)**2.d0)
+ -4.223d3/(5.76d3*(k2-k3)**6.d0*(k3+et)))*
+ dsqrt((k2+et)*(k3+et))
c
s5=((k3-k2)*(7.07d2*k2**3.d0+1.017d3*k2**2.d0*k3
+ 1.017d3*k2*k3**2.d0+5.65d2*k3**3.d0
+ +3.192d3*k2**2.d0*et+4.176d3*k2*k3*et
+ +2.712d3*k3**2.d0*et+5.28d3*k2*et**2.d0

```

```

+   +4.8d3*k3*et**2.d0+3.36d3*et**3.d0))/
+   (2.048d3*((k2+et)*(k3+et))**6.d0)
c
  if(i.eq.1) then
    s00=s0
    s10=s1
    s20=s2
    s30=s3
    s40=s4
    s50=s5
  end if
c
10  continue
c
  if(terms.eq.1) then
    rel0=0.d0
    relp0=0.d0
    img0=s00/e
    imgp0=s0p0/e
    rel=0.d0
    img=s0/e
  end if
c
  if(terms.eq.2) then
    rel0=s10
    relp0=s1p0
    img0=s00/e
    imgp0=s0p0/e
    rel=s1
    img=s0/e
  end if
c
  if(terms.eq.3) then
    rel0=s10
    relp0=s1p0
    img0=s00/e+e*s20
    imgp0=s0p0/e+e*s2p0
    rel=s1
    img=s0/e+e*s2
  end if
c
  if(terms.eq.4) then
    rel0=s10+e**2.d0*s30
    relp0=s1p0+e**2.d0*s3p0
    img0=s00/e+e*s20
    imgp0=s0p0/e+e*s2p0
    rel=s1+e**2.d0*s3
    img=s0/e+e*s2
  end if
c
  if(terms.eq.5) then
    rel0=s10+e**2.d0*s30
    relp0=s1p0+e**2.d0*s3p0
    img0=s00/e+e*s20+e**3.d0*s40
    imgp0=s0p0/e+e*s2p0+e**3.d0*s4p0
    rel=s1+e**2.d0*s3
    img=s0/e+e*s2+e**3.d0*s4
  end if
c
  if(terms.eq.6) then
    rel0=s10+e**2.d0*s30+e**4.d0*s50
    relp0=s1p0+e**2.d0*s3p0+e**4.d0*s5p0
    img0=s00/e+e*s20+e**3.d0*s40
    imgp0=s0p0/e+e*s2p0+e**3.d0*s4p0

```

```

rel=s1+e**2.d0*s3+e**4.d0*s5
img=s0/e+e*s2+e**3.d0*s4
end if
c
if(items.gt.6 .OR. items.lt.1) then
write(*,*) 'Number of WKG terms invalid!! #terms=',items
end if
c
z=dexp(rel-rel0)*(z0*dcos(img-img0)
+ +(zp0-z0*relp0)*dsin(img-img0)/imgp0)
c
return
c
end

```

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Vita

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