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STRATIFIED PLATE MODEL
FOR
COMPOSITE PLATE ANALYSIS

THESIS

Richard A. Coveno, Capt., USAF

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THESIS

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In Partial Fulfillment of the

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Preface

The stratified concept being presented here was originally theorized by Col. Ronald Bagley at the Air Force Institute of Technology. Captain Al Lesmerises developed the details of the theory in his Masters Thesis completed in March 1992 [2]. This work is a continuation of that development.

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Notation

Symbol	Value
A,B,C	Strata Model Displacement Field Coefficients
a	Plate Length (x)
b	Plate Width (y)
c	Stiffness Matrix Constants
E	Youngs Modulus
F	Number of Fiber Strata
G	Shear Modulus
h_f	Idealized fiber height
i,j	Summation Indices
k_i	Matrix Strata Stiffness Matrix Constant
M	Number of Matrix Strata
[M]	Coefficient Matrix for the linear System of Equations
N	Order of the Displacement Polynomial
n	Number of Plies
p(x,y)	Load function
p,q	Mode Shape Parameters
r	fiber radius
[S]	Stiffness Matrix
s_i	Fiber Strata Stiffness Matrix Constant
T	Kinetic Energy
[T]	Transformation Matrix
t	Laminate Thickness
t_p	Ply thickness
t_m	Strata Model matrix thickness

u,v,w	Displacements in x,y,z
W	Work
x,y,z	Coordinate Directions
z_{r+}	Top z coordinate of the r^{th} fiber stratum
z_{r-}	Bottom z coordinate of the r^{th} fiber stratum
z_f	Fiber Strata midplane coordinate
α,β	Matrix Strata Mapping Parameters
θ	Fiber Orientation Angle
Ψ,ζ	The x and y dependence functions of the assumed displacement
ν	Poisson's ratio
ε	Normal Strain
σ	Normal Stress
τ	Shear Stress
γ	Shear Strain
π	Pi (3.14159256..)
II	Potential Energy
GV_f	Global Volume Fraction of fiber
GV_m	Global Volume Fraction of matrix
V_f	Local fiber strata volume fraction of fiber
V_m	Local fiber strata volume fraction of matrix

Subscripts

f,m	Fiber and Matrix Strata Property
i,j,r,t	Summation Indices
,x	Derivative with respect to x

Abstract

This study continues the investigation of the stratified plate model for the analysis of composites plates. The strata theory models a composite plate as a stack of isotropic matrix and orthotropic, fiber dominated layers. Energy methods are used to derive a set of partial differential equations and boundary conditions as a function of the displacement. An assumed displacement field, of a polynomial in z and a Sine/Cosine function in x and y, generates a solution to a simply supported composite plate with cross plies. A cylindrical bending example is also solved with a polynomial in z and Sine/Cosine function in x and y, displacement. A Ritz polynomial x and y displacement function is used to generate a solution to a composite plate with angle plies. The strata model is expected to be useful for predicting stresses in composites where the matrix material has become viscoelastic.

A STRATIFIED PLATE MODEL
FOR
COMPOSITE ANALYSIS

I. *Introduction*

Composites Background

Composites currently in use today generally consist of very strong and stiff continuous fibers surrounded and held together by a softer, more ductile, matrix material. The fiber is the main load carrying material and the purpose of the matrix is hold the fibers together, transfer load from fiber to fiber, and protect the fibers from the environment. A composite ply or lamina consists of a single layer of aligned fibers held together in a flat sheet by the matrix. Most composite structures are manufactured with composite laminates composed of many plys or laminae, layered together with different fiber orientation. Because of the unique and almost infinitely variable arrangement of stiff fibers and soft matrix the mechanics of composite materials is difficult to model mathematically and, except for a few simple cases, exact closed form solutions to the differential equations of motion have not been found.

Current Composite Analysis Methods

The axis system used for the plates in this work is as follows:

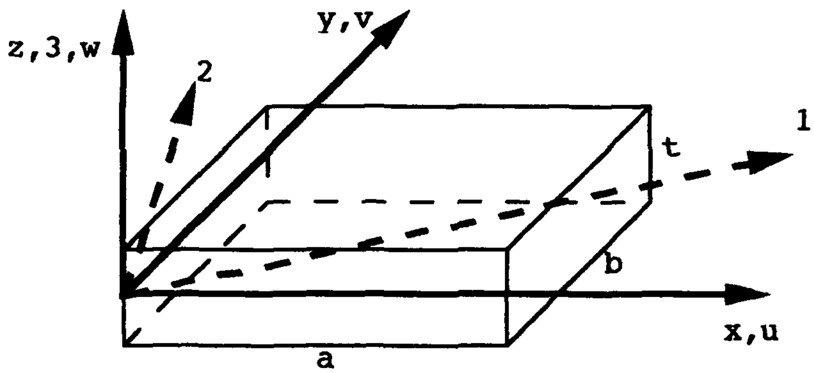

Figure 1. Coordinate System of Plate

The x, y and z direction are the primary coordinates. The 1 direction is the off primary axis, fiber orientation and the 2 direction is transverse to the fiber direction.

The state of stress in a plate is related to the strain by the generalized, Hooke's law stiffness matrix [S] as follows:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = [S] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \qquad (1)$$

The 6x6 stiffness matrix [S], coefficients are a function of the twelve 3-D engineering constants tensile moduli (E_{11}, E_{22}, and E_{33}), shear moduli (G_{12}, G_{13}, and G_{23}) and Poisson's ratios (ν_{12}, ν_{13}, ν_{23}, ν_{21}, ν_{31}, and ν_{32}). The subscripting on the stress, strain and tensile and shear moduli has the first subscript being the load face and the second being the load direction. The subscripting on the Poisson ratios has the first subscript being the load direction and the second subscript being the deformation direction (ie. $v21=\varepsilon_1/\varepsilon_2$ with the load in the 2 direction). Due to symmetry of the stiffness matrix, the Poisson ratios are related by the relationships, $\nu_{21}=(E_{22}/E_{11})*\nu_{12}$ which reduce the number of required constants from twelve to nine [9:10].

The classic laminated plate method of composite analysis involves idealizing the composite ply consisting of fibers and matrix as a homogeneous transversely isotropic plate. That is, a plate with three planes of elastic symmetry (orthotropic) in which one of the planes is isotropic.

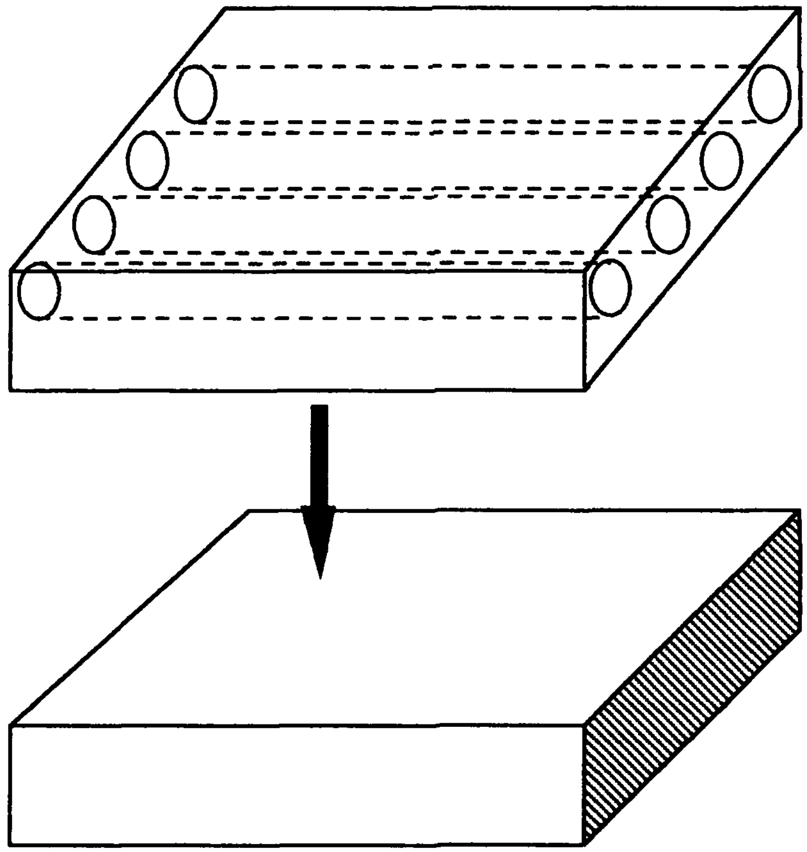

Figure 2. Classic Laminated Plate Idealization

Five engineering constants (E_{11}, E_{22}, v_{12}, v_{23}, G_{12}) are required to derive the generalized Hooke's law, 6x6, stiffness matrix coefficients of a transversely isotropic plate [9:11]. The engineering constants are derived from volume fraction averaging of the homogeneous properties of the fiber and the matrix. The fiber and matrix moduli and Poison ratios combine, in proportion to their volume fraction, in series or in parallel to generate the averaged composite engineering constant [1: 61-86]. The stiffness matrix [S] coefficients are derived from the 5 engineering constants [9: 11-12]. Most classic laminated plate models reduce the Hooke's, 6x6 stiffness matrix to a 3x3 matrix by assuming each ply is in a state of plane stress. This reduces the number of required composite engineering constants to four (E_{11}, E_{22}, v_{12}, and G_{12}). The entire laminate stiffness is the summation of the ply stiffnesses. With the stiffness known, the strain on each ply can be calculated as a function of the strain on the mid plane, the curvatures, and the distance from the mid plane. The strain on the mid plane and the curvatures are calculated from the inverse of the composite stiffness and the applied load and moment vector [1: 190-194]. This method has proven useful and accurate in predicting the performance of composite laminates at low to moderate operating temperatures [1].

One drawback of this approach is that the area of each ply where the softer matrix is dominant is assigned the averaged higher composite stiffness. Epoxy and thermoplastic matrix moduli are usually on the order of 2-4 GPA while graphite, fiber moduli are on the order of 200-550 GPA [1: 31-33; 23]. The difference in stiffness is even more apparent when the laminate is subjected to high temperature. The matrix materials generally have glass transition temperatures in the range of 50-200 degrees Celsius [1: 28]. Even high heat resistant, thermoplastic, matrix materials have a maximum continuous use temperature in the range of 150-300 degrees Celsius [3: 156]. Glass fiber materials, on the other hand, have a softening point on the order of 771-970 degrees Celsius [7: 2.4-2.5] and properly treated carbon fibers have operating temperatures of over 1500 degrees Celsius [8: 62-68]. The large difference in stiffness, undoubtedly plays a role in the actual behavior of the material. Under high temperature loading, due to the softening of the matrix material, matrix dominant areas must experience higher displacements while fiber dominant areas will experience lower displacements but higher stresses. The averaging of properties within a ply can not address the differences in displacement, strain and stress. The fact that most models assume each ply is in a state of plane stress also eliminates the ability to predict through-the-thickness displacements, strains and stresses.

Debonding or delamination of plys is a common type of failure in composites and can be attributed to through-the-thickness stresses and strain. The classic Laminated Plate Theory assumes that each plate is in a state of plane stress and does not generate any through-the-thickness stress. This is another drawback of classic laminated plate method.

Strata Concept

As an alternative to modeling each ply as a homogeneous transversely isotropic plate, the strata theory breaks each ply into matrix and fiber dominated layers, each of which will be referred to as a stratum. This idealization requires the assumption that the fibers are evenly disperesed and alligned in each ply.

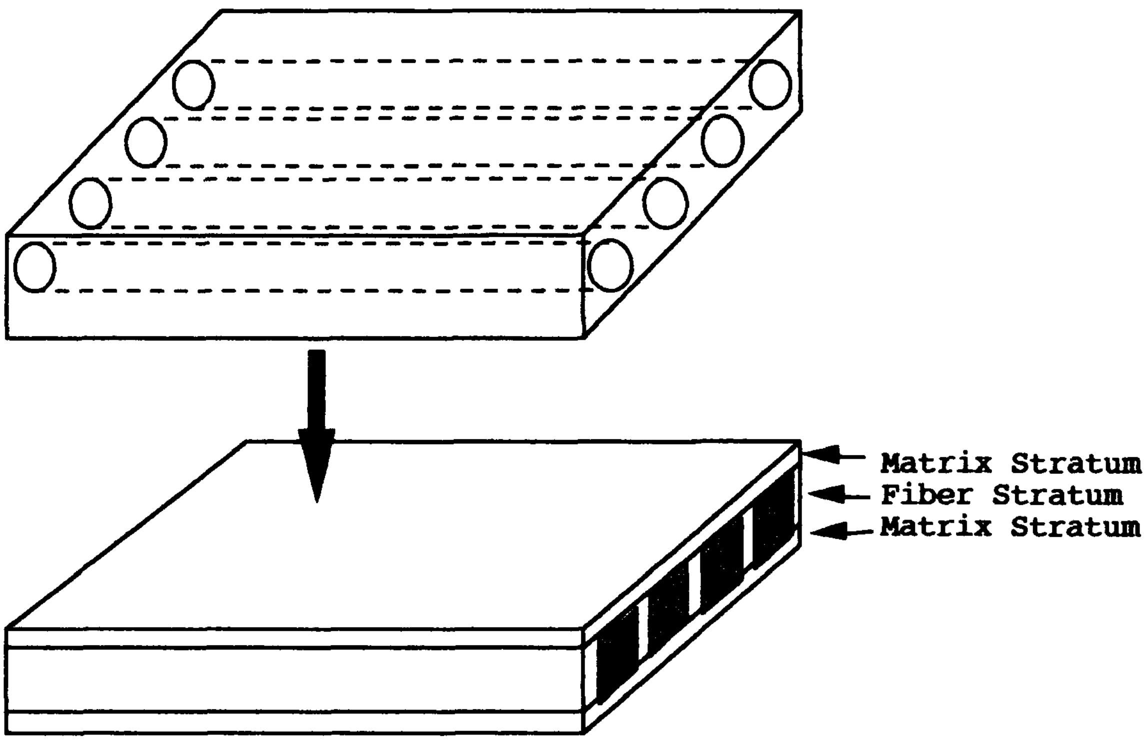

Figure 3. Strata Theory Idealization

In order to clearly distinguish between the matrix and the fiber strata, each fiber is idealized as a square cross section which has equal area to the actual fiber. Each matrix stratum is entirely matrix material and has the properties of a homogeneous isotropic plate. Each fiber stratum is a combination of matrix and fiber, dominated by the stiffer fiber. As will be shown in the following sections, this idealization allows the matrix strata to experience different displacements than the fiber strata. This model is expected to more accurately predict the displacements, strains and stresses of composite plates with dissimilar matrix and fiber constituents.

II. *The Strata Model*

Strata Laminate Idealization

Each lamina in a laminate of composite plies is idealized as a stack of matrix and fiber strata as shown below:

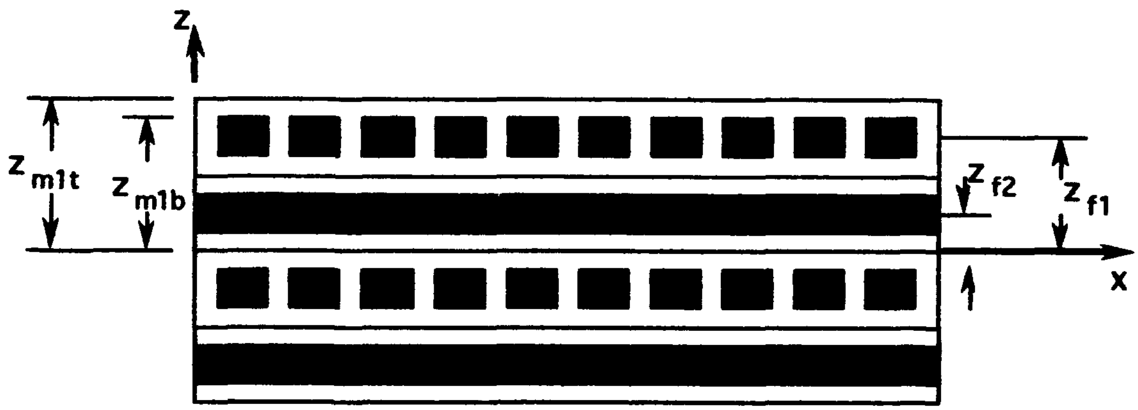

Figure 4. Strata Laminate Idealization

The lower matrix strata of one ply combine with the upper matrix strata of the next lower ply to make a single matrix stratum of the combined thicknesses. A laminate of N plies is broken into N+1 matrix strata and N fiber strata.

Strata Stiffness Properties

Each matrix stratum is a homogeneous layer of isotropic material. Only three engineering constants are required to determine the stiffness matrix coefficients of an isotropic homogeneous plate [Whitney: 12-13]. The stiffness matrix, of matrix strata t, $[K_m]_t$, will have the form:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}_t = \begin{bmatrix} k_1 & k_2 & k_2 & 0 & 0 & 0 \\ k_2 & k_1 & k_2 & 0 & 0 & 0 \\ k_2 & k_2 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_3 \end{bmatrix}_t \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_t \qquad t = 1..N+1 \tag{2}$$

The tensile modulous E, Poisson's ratio ν, and the shear modulous G are required to define the stiffness matrix coefficients of the matrix strata. G is customarily determined from the tensile modulous and Poisson's ratio by the relationship $G=E/2(1+\nu)$. This reduces the required number of constants to two, E and ν.

Each fiber stratum is a combination of matrix and fiber, dominated by the fiber. The original ply thickness (t_p) and the fiber radius (r) are required to define the global volume fraction (V_f) of the ply.

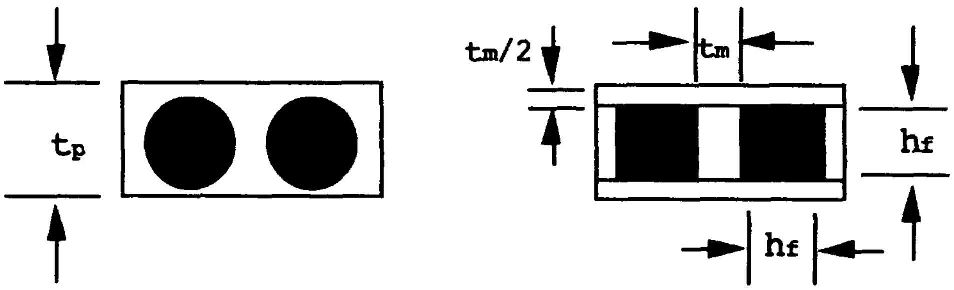

Figure 5. Fiber Strata Parameters

The idealized fiber height (h_f) of each stratum is calculated by setting the fiber volume (πr^2) equal to the idealized fiber height squared. The global volume fraction (GV_f)is unchanged and the local, fiber stratum, volume fraction (V_f) can be defined from the fiber radius (r) and the matrix thickness (t_m).

$$GV_f = \frac{\pi r^2}{t_p^2} \qquad h_f^2 = \pi r^2 \qquad \text{Local } V_f = \frac{\pi r^2}{(h_f + t_m)h_f} \tag{3}$$

Each fiber stratum is modeled as a homogeneous, specially orthotropic plate. An orthotropic plate requires 9 independent engineering properties E_{11}, E_{22}, E_{33}, ν_{12}, ν_{13}, ν_{23}, G_{12}, G_{13}, and G_{23} to define the stiffness matrix coefficients. A representative volume element shows that $E_{11}=E_{33}$, $\nu_{13}=\nu_{12}$, $\nu_{23}=\nu_{21}=(E_{22}/E_{11})*\nu_{12}$, and $G_{12}=G_{23}$ which reduces the number of required constants to 5.

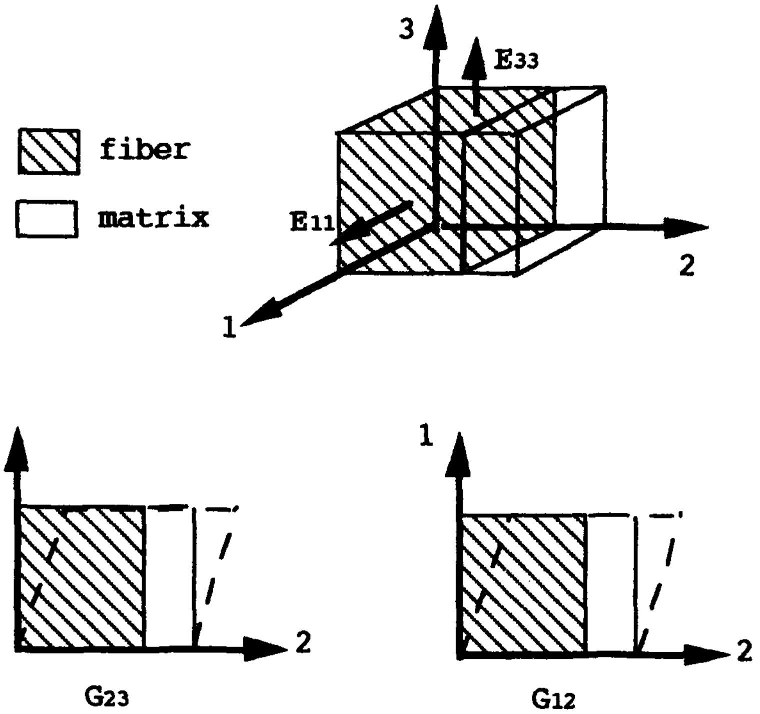

Figure 6. Fiber Strata Representative Volume Element

The modulus and Poisson's ratio in each direction can be determined by applying the fiber and matrix moduli and Poisson's ratio in series or parallel with the volume fraction, as the geometry dictates. The engineering constants are determined as follows:

$$E_{11} = V_f E_f + V_m E_m \qquad G_{13} = V_f G_f + V_m G_m \qquad \upsilon_{12} = V_f \upsilon_f + V_m \upsilon_m$$

$$E_{22} = \frac{E_f E_m}{V_f E_m + V_m E_f} \qquad G_{12} = \frac{G_f G_m}{V_f G_m + V_m G_f} \qquad \upsilon_{23} = \upsilon_{12} \frac{E_{22}}{E_{11}}$$

$$E_{33} = E_{11} \qquad G_{23} = G_{12} \qquad \upsilon_{13} = V_f \upsilon_f + V_m \upsilon_m \qquad (4)$$

Using the relationship $G = E/2(1+\nu)$, the 3-D stiffness matrix of the fiber strata in each ply can be generated if the constants E and ν are known for the matrix and the fibers involved.

The on fiber axis (x,y,z), stiffness matrix, of fiber stratum r, $\left[S_f\right]_r'$, will have the specially orthotropic form:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}_r = \begin{Bmatrix} s_1 & s_3 & s_3 & 0 & 0 & 0 \\ s_3 & s_2 & s_4 & 0 & 0 & 0 \\ s_3 & s_4 & s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_5 \end{Bmatrix}_r' \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}_r \quad r = 1..N \tag{5}$$

In cases when the fibers are not aligned with the primary axis system the off-axis stiffnesses will be required to define the ply stiffness. The off fiber axis stiffness [S], may be calculated by the tensor transformation of [S]' as follows:

$$[S] = \{T\}^{-1}[S]'\{T\} \tag{6}$$

Using the transformation matrix:

$$\{T\} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & +n & m & 0 \\ -mn & mn & 0 & 0 & 0 & m^2 - n^2 \end{bmatrix}$$

$$m = \cos(\theta), \qquad n = \sin(\theta) \tag{7}$$

The resultant stiffness matrix, of the fiber stratum r, will be of the form:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}_r = \begin{bmatrix} s_1 & s_3 & s_{10} & 0 & 0 & s_5 \\ s_3 & s_2 & s_{11} & 0 & 0 & s_6 \\ s_{10} & s_{11} & s_{12} & 0 & 0 & s_{13} \\ 0 & 0 & 0 & s_7 & s_9 & 0 \\ 0 & 0 & 0 & s_9 & s_8 & 0 \\ s_5 & s_6 & s_{13} & 0 & 0 & s_4 \end{bmatrix}_r \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_r \qquad r = 1..N \tag{8}$$

The stiffness matrices of both the fiber and matrix strata can be fully defined if the fiber radius (r), the thickness for each ply (t_p), the global volume fraction (V_f), and the tensile modulus (E) and Poisson's ratio (ν) of the fiber and matrix, are known.

Generation of the Governing Equations

Using Hamilton's energy principle, the variation of the volume integral of the internal strain energy (II) minus the kinetic energy (T) is equal to the work of the applied forces (W).

$$\int_0^a \int_0^b \int_{-t/2}^{t/2} (\delta\Pi - \delta T)\,dz\,dy\,dx = \int_0^a \int_0^b \delta W\,dy\,dx \tag{9}$$

For simplicity, this document only examines static cases where the kinetic energy is equal to zero. The internal strain energy is broken into fiber strata and matrix strata contributions:

$$\Pi = \Pi_m + \Pi_f \tag{10}$$

Internal strain energy of the fiber stratum r and matrix stratum t are functions of the strain vectors and stiffness matrices:

$$\Pi_{fr} = \frac{1}{2}[\varepsilon_f]_r^T[S_f]_r[\varepsilon_f]_r \qquad \Pi_{mt} = \frac{1}{2}[\varepsilon_m]_t^T[K_m]_t[\varepsilon_m]_t \tag{11}$$

The strain vector for each stratum is a function of the displacements u, v, and w. Only the linear strain terms are used and the thin plate assumption, where the displacement in the z direction does not vary in z, is used:

$$\varepsilon_{fr} = \begin{Bmatrix} u_{r,x} \\ v_{r,y} \\ 0 \\ u_{r,y} + v_{r,x} \\ u_{r,z} + w_{r,x} \\ v_{r,z} + w_{r,y} \end{Bmatrix} \qquad \varepsilon_{mt} = \begin{Bmatrix} u_{t,x} \\ v_{t,y} \\ 0 \\ u_{t,y} + v_{t,x} \\ u_{t,z} + w_{t,x} \\ v_{t,z} + w_{t,y} \end{Bmatrix} \tag{12}$$

The total strain energy is the summation of the strain energy of all the strata. M is the number of matrix strata (N+1) and F is the number of fiber strata (N).

$$\Pi_f = \sum_{r=1}^{F}\Pi_r \qquad \Pi_m = \sum_{t=1}^{M}\Pi_t \tag{13}$$

Work is a function of the load and displacement. The Strata Theory assumes small displacements and only considers the displacements in the z direction (w) for the calculation of work.

$$W = p(x,y)*w(x,y) \tag{14}$$

The entire expression becomes a function of the displacement functions u, v, and w and the load function p(x,y). The first variation can be taken with respect to the displacements (u, v, and w) and set equal to zero in order to obtain the minimum energy level or the equilibrium energy state. This leaves an expression with the variation of the derivatives in x, y, and z of the displacement functions. Integration by parts is used to eliminate the derivatives of variational terms. The following expression results plus a set of boundary conditions, generated from the integration by parts.

$$\int_0^a\int_0^b \sum_{t=1}^{M} {}^{*} \int_{z_{t-}}^{z_{t+}} \begin{bmatrix} \{k_{1t}u_{t,xx} + k_{2t}v_{t,xy} + \\ k_{3t}(u_{t,yy} + u_{t,zz} + v_{t,xy} + w_{t,xz})\}\delta u_t + \\ \\ \{k_{1t}v_{t,yy} + k_{2t}u_{t,xy} + \\ k_{3t}(v_{t,xx} + u_{t,xy} + v_{t,zz} + w_{t,yz})\}\delta v_t + \\ \\ \{k_{3t}(u_{t,xz} + v_{t,yz} + w_{t,yy} + w_{t,xx})\}\delta w_t \end{bmatrix} dzdydx +$$

$$\int_0^a\int_0^b \sum_{r=1}^{F} {}^{*} \int_{z_{r-}}^{z_{r+}} \begin{bmatrix} \{s_{1r}u_{r,xx} + s_{4r}u_{r,yy} + s_{6r}v_{r,yy} + \\ (s_{3r} + s_{4r})v_{r,xy} + s_{5r}(2u_{r,xy} + v_{r,xx}) + \\ s_{7r}(u_{r,zz} + w_{r,xz}) + s_{9r}(v_{r,zz} + w_{r,yz})\}\delta u_r + \\ \\ \{s_{4r}u_{r,xx} + s_{2r}u_{r,yy} + s_{5r}u_{r,xx} + \\ (s_{3r} + s_{4r})u_{r,xy} + s_{6r}(u_{r,yy} + 2v_{r,xy}) + \\ s_{8r}(v_{r,zz} + w_{r,yz}) + s_{9r}(u_{r,zz} + w_{r,xz})\}\delta v_r + \\ \\ \{s_{7r}(w_{r,xx} + u_{r,xz}) + \\ s_{8r}(w_{r,yy} + v_{r,yz}) + s_{9r}(2w_{r,xy} + v_{r,xz} + u_{r,yz})\}\delta w_r \end{bmatrix} dzdydx$$

$$= \int_0^a\int_0^b [\, p(x,y) * \delta w \,]dydx \qquad (15)$$

The above equation can be broken into 3 equations each containing the factor of δu, δv or δw. The relationships must hold for an arbitrary variation of unity, so the variational terms may be factored out leaving the following set of differential equations and boundary condition equations from the integration by parts.

$$\int_0^a\int_0^b \sum_{r=1}^{f}\sum_{t=1}^{M}\left[\begin{array}{l}\displaystyle\int_{z_{t-}}^{z_{t+}}\left\{\begin{array}{l}k_{3t}(u_{t,yy}+v_{t,xy}+u_{t,zz}+w_{t,xz})+\\ k_{2t}v_{t,xy}+k_{1t}u_{t,xx}\end{array}\right\}dz\\ \displaystyle\int_{z_{r-}}^{z_{r+}}\left\{\begin{array}{l}s_{1r}u_{r,xx}+s_{4r}u_{r,yy}+(s_{3r}+s_{4r})v_{r,xy}+\\ s_{5r}(2u_{r,xy}+v_{r,xx})+s_{6r}v_{r,yy}+\\ s_{7r}(u_{r,zz}+w_{r,xz})+s_{9r}(v_{r,zz}+w_{r,yz})\end{array}\right\}dz\end{array}\right]dydx=0 \tag{16}$$

$$\int_0^a\int_0^b \sum_{r=1}^{F}\sum_{t=1}^{M}\left[\begin{array}{l}\displaystyle\int_{z_{t-}}^{z_{t+}}\left\{\begin{array}{l}k_{3t}(u_{t,xy}+v_{t,xx}+v_{t,zz}+w_{t,yz})+\\ k_{2t}u_{t,xy}+k_{1t}v_{t,yy}\end{array}\right\}dz\\ \displaystyle\int_{z_{r-}}^{z_{r+}}\left\{\begin{array}{l}s_{2r}v_{r,yy}+(s_{3r}+s_{4r})u_{r,xy}+s_{4r}v_{r,xx}+\\ s_{6r}(2v_{r,xy}+u_{r,yy})+s_{5r}u_{r,xx}+\\ s_{8r}(v_{r,zz}+w_{r,yz})+s_{9r}(u_{r,zz}+w_{r,xz})\end{array}\right\}dz\end{array}\right]dydx=0 \tag{17}$$

$$\int_0^a\int_0^b \sum_{r=1}^{F}\sum_{t=1}^{M}\left[\begin{array}{l}\displaystyle\int_{z_{t-}}^{z_{t+}}\left\{\begin{array}{l}k_{3t}(u_{t,xz}+v_{t,yz}+\\ w_{t,xx}+w_{t,yy})\end{array}\right\}dz+\\ \displaystyle\int_{z_{r-}}^{z_{r+}}\left\{\begin{array}{l}s_{7r}(w_{r,xx}+u_{r,xz})+\\ s_{8r}(v_{r,yz}+w_{r,yy})+\\ s_{9r}(u_{r,yz}+2w_{r,xy})\end{array}\right\}dz\end{array}\right]=\int_0^a\int_0^b p(x,y)dydx \tag{18}$$

Boundary Conditions:

$$
\delta u \sum_{t=1}^{M} \sum_{r=1}^{F} \left[\begin{array}{l}
\displaystyle\int_0^b \left[\begin{array}{l} \displaystyle\int_{z_{t-}}^{z_{t+}} \left\{ k_{1t}u_{t,x} + k_{2t}v_{t,y} \right\} dz \; + \\ \displaystyle\int_{z_{r-}}^{z_{r+}} \left\{ S_{1r}u_{r,x} + S_{3r}v_{r,y} + S_{5r}(u_{r,y} + v_{r,x}) \right\} dz \end{array} \right]_0^a dy \; + \\
\displaystyle\int_0^a \left[\begin{array}{l} \displaystyle\int_{z_{t-}}^{z_{t+}} \left\{ k_{3t}u_{t,y} + k_{3t}v_{t,x} \right\} dz \; + \\ \displaystyle\int_{z_{r-}}^{z_{r+}} \left\{ S_{5r}u_{r,x} + S_{6r}v_{r,y} + S_{4r}(u_{r,y} + v_{r,x}) \right\} dz \end{array} \right]_0^b dx \; + \\
\displaystyle\int_0^b \int_0^a \left[\begin{array}{l} \Big[k_{3t}u_{t,z} + k_{3t}w_{t,x} \Big]_{z_{r-}}^{z_{t+}} \; + \\ \Big[S_{7r}(u_{r,z} + w_{r,x}) + S_{9r}(w_{r,y} + v_{r,z}) \Big]_{z_{r-}}^{z_{r+}} \end{array} \right] dxdy \; +
\end{array} \right] = 0
\tag{19}
$$

$$
\delta v \sum_{t=1}^{M} \sum_{r=1}^{F} \left[\begin{array}{l}
\displaystyle\int_0^b \left[\begin{array}{l} \displaystyle\int_{z_{t-}}^{z_{t+}} \left\{ k_{3t}u_{t,y} + k_{3t}v_{t,x} \right\} dz \; + \\ \displaystyle\int_{z_{r-}}^{z_{r+}} \left\{ S_{5r}u_{r,x} + S_{6r}v_{r,y} + S_{4r}(u_{r,y} + v_{r,x}) \right\} dz \end{array} \right]_0^a dy \; + \\
\displaystyle\int_0^a \left[\begin{array}{l} \displaystyle\int_{z_{t-}}^{z_{t+}} \left\{ k_{2t}u_{t,x} + k_{1t}v_{t,x} \right\} dz \; + \\ \displaystyle\int_{z_{r-}}^{z_{r+}} \left\{ S_{3r}u_{r,x} + S_{2r}v_{r,y} + S_{6r}(u_{r,y} + v_{r,x}) \right\} dz \end{array} \right]_0^b dx \; + \\
\displaystyle\int_0^b \int_0^a \left[\begin{array}{l} \Big[k_{3t}v_{t,z} + k_{3t}w_{t,y} \Big]_{z_{t-}}^{z_{t+}} \; + \\ \Big[S_{8r}(v_{r,z} + w_{r,y}) + S_{9r}(w_{r,x} + u_{r,z}) \Big]_{z_{r-}}^{z_{r+}} \end{array} \right] dxdy \; +
\end{array} \right]
\tag{20}
$$

$$\delta w \sum_{t=1}^{M} \sum_{r=1}^{F} \left[\begin{array}{l} \displaystyle\int_0^b \left[\begin{array}{l} \displaystyle\int_{z_{t-}}^{z_{t+}} \left\{ k_{3t} u_{t,z} + k_{3t} w_{t,x} \right\} dz + \\ \displaystyle\int_{z_{r-}}^{z_{r+}} \left\{ s_{9r}(v_{r,y} + w_{r,x}) + s_{7r}(u_{r,z} + w_{r,x}) \right\} dz \end{array} \right]_0^a dy + \\ \displaystyle\int_0^a \left[\begin{array}{l} \displaystyle\int_{z_{t-}}^{z_{t+}} \left\{ k_{3t} v_{t,z} + k_{3t} w_{t,y} \right\} dz + \\ \displaystyle\int_{z_{r-}}^{z_{r+}} \left\{ s_{8r}(v_{r,z} + w_{r,y}) + s_{9r}(u_{r,z} + w_{r,x}) \right\} dz \end{array} \right]_0^b dx + \\ \displaystyle\int_0^b \int_0^a \left[k_{3t} u_{t,z} + k_{3t} w_{t,x} \right]_{z_{t-}}^{z_t} dx dy \end{array} \right] = 0 \qquad (21)$$

If displacement functions can be found that satisfy the differential equations and the boundary conditions, then compatibility and equilibrium will be satisfied and the strain and stress in each stratum can be calculated.

Strata Model Displacement Assumptions

The assumptions of the strata model are as follows:

1) Only linear strain terms are considered significant.
2) The displacement in w does not vary in z thus $\varepsilon_z=0$.
3) There is constant strain through the thickness of the fiber strata.

The main premise of the strata model is that the fibers and thus the fiber strata are much stiffer than the matrix. The strata model assumes constant strain through the thickness of the fiber strata. The displacement functions in x and y, of the fiber strata, will not vary in z. Strain in the matrix strata will vary in z. The displacement, in x and y, of each position in the matrix strata will be a function of its z coordinate.

The above assumptions will generate different displacement functions (u and v), for each stratum. To maintain continuity, a continuous displacement function must be maintained through the entire thickness of the plate.

The fiber and matrix strata interface must coincide after deformation. The z coordinate of the bottom of the matrix strata above each fiber strata ($z_{mi\,bottom}$), and the top of the matrix strata below the fiber strata ($z_{m(i+1)top}$) must be mapped to the fiber stratum z coordinate z_{fi}.

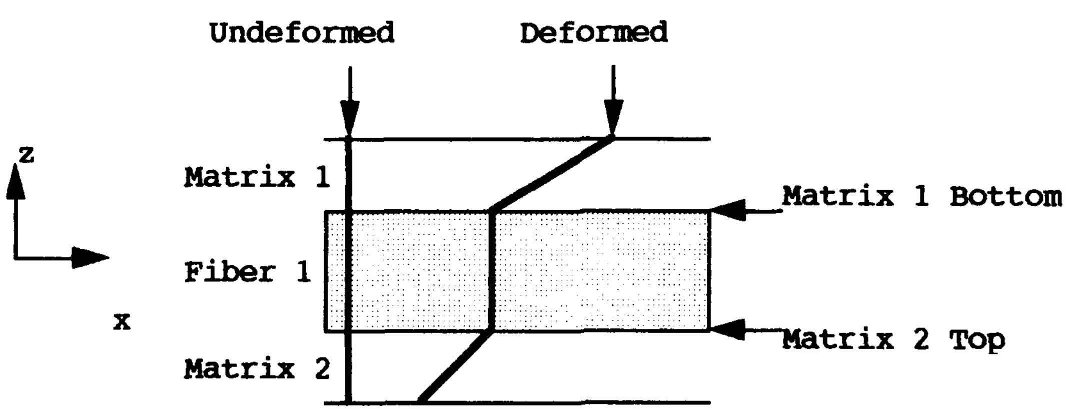

Figure 7. Mapping Parameters

The mapping is accomplished with the mapping parameters α and β. Each matrix strata will have it's own α and β mapping parameters, that are a function of the surrounding fiber strata coordinates

z_{fi-1} and z_{fi}. The linear mapping is accomplished by the following equations.

$$\alpha_{m_i} + \beta_{m_i} z_{m_i top} = z_{f_{i-1}}$$
$$\alpha_{m_i} + \beta_{m_i} z_{m_i bottom} = z_{f_i} \qquad (22)$$

By solving both equations simultaneously the α and β of each matrix stratum can be calculated as follows.

$$\beta_i = \frac{z_{f_{i-1}} - z_{f_i}}{z_{m_i top} - z_{m_i bottom}}$$
$$\alpha_i = \left[\frac{z_{f_{i-1}}}{z_{m_i top}} - \frac{z_{f_i}}{z_{m_i bottom}}\right]\left[\frac{z_{m_i top} * z_{m_i bottom}}{z_{m_i bottom} - z_{m_i top}}\right] \qquad (23)$$

The upper most and lower most matrix surfaces will be mapped to the z coordinate of the free surface +/- t/2. In the above algorithm this gives a z_{f0} of t/2 and a $z_{f(P+1)}$ of -t/2. In lay-ups with all plies having the same thickness (t_p), and idealized fiber strata height (h_f), the β parameter will be constant. Matrix strata with positive z coordinates will have negative α parameters and matrix strata with negative z coordinates will have positive α parameters. The linear mapping results in the z coordinate for matrix strata t, being represented with ($\alpha_t + \beta_t * z$) and the z coordinate of fiber strata r, being represented with the constant z_{fr}.

To generate solutions, the Strata Theory assumes that the through-the-thickness displacement (in the z direction), is a polynomial function of z, and solves for the displacement in x and y. The constant displacement through the fiber strata can be modeled with a polynomial function of the stratum mid plane distance (z_f) and an unknown function of x and y. The displacement through the matrix strata varies in z and is assumed to be a polynomial function of ($\alpha+\beta z$) and the same unknown function of x and y. The displacement in the z direction (w) is assumed to be a function of x and y only. The assumed displacement relationships are represented mathematically, as follows:

$$u_f(x,y,z) = \sum_{i=0}^{N} z_f^i \cdot \psi_i(x,y)$$

$$v_f(x,y,z) = \sum_{i=0}^{N} z_f^i \cdot \varsigma_i(x,y)$$

$$w_f(x,y,z) = w(x,y)$$

$$u_m(x,y,z) = \sum_{i=0}^{N} (\alpha_m + \beta_m z)^i \cdot \psi_i(x,y)$$

$$v_m(x,y,z) = \sum_{i=0}^{N} (\alpha_m + \beta_m z)^i \cdot \varsigma_i(x,y)$$

$$w_m(x,y,z) = w(x,y) \tag{24}$$

The order of the polynomial (N) can vary to model the complexity of the loading and geometry. More complex loading and geometry require more complex displacement functions. The displacement function must have sufficient order to survive two derivatives and generate complex strain functions. A polynomial order of 5 has been found to have sufficient complexity to represent the displacements and strains of the simple cases examined in this document and shall be used in all the example cases unless otherwise stated.

Solution Procedures

Using the above displacement functions to derive the strain energy vectors, the z differentials in the governing equations can be evaluated before the first variation. The governing equations become functions of z, the functions $\psi(x,y)$, $\zeta(x,y)$, $w(x,y)$ and the load function $p(x,y)$. The first variation is taken on the Ψ, ζ, and w functions, generating the Strata Theory governing equations as follows:

$$\int_0^a\int_0^b \sum_{i=0}^{N}\sum_{r=1}^{F}\sum_{t=1}^{M} \left[\begin{array}{l} k_{3t}(\psi_{i,yy}Z_m^2 + \psi_i Z_{m,z}^2 + w_{,x}Z_{m,z}) + \\ (k_{2t} + k_{3t})\zeta_{i,xy}Z_f^2 + k_{1t}\psi_{i,xx}Z_m^2 + \\ (s_{3r} + s_{4r})\zeta_{i,xy}Z_f^2 + s_{4r}\psi_{i,yy}Z_f^2 + \\ s_{1r}\psi_{i,xx}Z_f^2 + s_{5r}\zeta_{i,xx}Z_f^2 + \\ 2s_{5r}\psi_{i,xx}Z_f^2 + s_{6r}\zeta_{i,xx}Z_f^2 \end{array}\right] dydx = 0 \qquad (25)$$

$$\int_0^a\int_0^b \sum_{i=0}^{N}\sum_{r=1}^{F}\sum_{t=1}^{M} \left[\begin{array}{l} k_{3t}(\zeta_{i,xx}Z_m^2 + \zeta_i Z_{m,z}^2 + w_{,y}Z_{m,z}) + \\ (k_{2t} + k_{3t})\psi_{i,xy}Z_f^2 + k_{1t}\zeta_{i,yy}Z_m^2 + \\ (s_{3r} + s_{4r})\psi_{i,xy}Z_f^2 + s_{4r}\zeta_{i,xx}Z_f^2 + \\ s_{2r}\zeta_{i,yy}Z_f^2 + s_{5r}\psi_{i,xx}Z_f^2 + \\ s_{6r}(\psi_{i,yy}Z_f^2 + 2\zeta_{i,xy}Z_f^2) \end{array}\right] dydx = 0 \qquad (26)$$

$$\int_0^a\int_0^b \sum_{i=0}^{N}\sum_{r=1}^{F}\sum_{t=1}^{M} \begin{bmatrix} k_{3t}(w_{,xx} + w_{,yy}) + \\ k_{3t}(\psi_{i,x} + \zeta_{i,y})Z_{m,z} + \\ s_{7r}w_{,xx} + s_{8r}w_{,yy} + 2s_{9r}w_{,yy} \end{bmatrix} dydx = \int_0^a\int_0^b p(x,y)dydx \tag{27}$$

where

$$Z_m = \int_{z_{t-}}^{z_{t+}} (\alpha_t + \beta_t z)^i dz \qquad Z_f = \int_{z_{r-}}^{z_{r+}} z_{fr}^i dz \tag{28}$$

and

$$Z_m^2 = \int_{z_{t-}}^{z_{t+}} (\alpha_t + \beta_t z)^{(i+j)} dz \qquad Z_f^2 = \int_{z_{r-}}^{z_{r+}} z_{fr}^{(i+j)} dz$$

$$Z_{m,z} = \int_{z_{t-}}^{z_{t+}} i * (\alpha_t + \beta_t z)^{(i-1)} dz \qquad Z_{f,z} = \int_{z_{r-}}^{z_{r+}} i * z_{fr}^{(i-1)} dz$$

$$Z_{m,z}^2 = \int_{z_{t-}}^{z_{t+}} -ij * \beta_t(\alpha_t + \beta_t z)^{(i+j-2)} dz \qquad Z_{f,z}^2 = \int_{z_{r-}}^{z_{r+}} -ij * z_{fr}^{(i+j-2)} dz$$

$$j = 0..N \tag{29}$$

The derivation also generates the following set of boundary conditions from the integration by parts:

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=0}^{N} \begin{bmatrix} \int_0^b \delta\psi_i \begin{bmatrix} k_{1t}\psi_{i,x}Z_m^2 + k_{2t}\zeta_{i,y}Z_m^2 + s_{1r}\psi_{i,x}Z_f^2 + \\ s_{3r}\zeta_{i,y}Z_f^2 + s_{5r}(\psi_{i,y} + \zeta_{i,x})Z_f^2 \end{bmatrix}_0^a dy + \\ \int_0^a \delta\psi_i \begin{bmatrix} k_{3t}(\psi_{i,y} + k_{3t}\zeta_{i,x})Z_m^2 + s_{5r}\psi_{i,x}Z_f^2 + \\ s_{6r}\zeta_{i,y}Z_f^2 + s_{4r}(\psi_{i,y} + \zeta_{i,x})Z_f^2 \end{bmatrix}_0^b dx \end{bmatrix} = 0 \tag{30}$$

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=0}^{N}\left[\begin{array}{l}\int_0^b \delta\zeta_i \left[\begin{array}{l} k_{3t}\psi_{i,y}Z_m^2 + k_{3t}\zeta_{i,x}Z_m^2 + s_{5r}\psi_{i,x}Z_f^2 + \\ s_{4r}(\psi_{i,y} + \zeta_{i,x})Z_f^2 + s_{6r}\zeta_{i,y}Z_f^2 \end{array}\right]_0^a dy + \\ \int_0^a \delta\zeta_i \left[\begin{array}{l} k_{2t}\psi_{i,x}Z_m^2 + k_{1t}\zeta_{i,x}Z_m^2 + s_{3r}\psi_{i,x}Z_f^2 + \\ s_{6r}(\psi_{i,y} + \zeta_{i,x})Z_f^2 + s_{7r}\zeta_{i,y}Z_f^2 \end{array}\right]_0^b dx \end{array}\right] = 0 \tag{31}$$

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=0}^{N}\left[\begin{array}{l}\int_0^b \delta w\left[k_{3t}\psi_i Z_{m,z} + k_{3t}w_{,x}Z_m + s_{7r}w_{,x}Z_f + s_{9r}w_{,x}Z_f \right]_0^a dy + \\ \int_0^a \delta w\left[k_{3t}\zeta_i Z_{m,z} + k_{3t}w_{,y}Z_m + s_{9r}w_{,x}Z_f + s_{8r}w_{,y}Z_f \right]_0^b dx \end{array}\right] = 0 \tag{32}$$

With the correct choice of the $\psi(x,y)$, $\zeta(x,y)$ and $w(x,y)$ functions and the load functions $p(x,y)$, the x and y dependence can be evaluated directly and factored or integrated out of the governing equations. The governing equations can then be solved and the displacement functions determined.

III. *Strata Model For a Simply Supported Homogeneous Plate*

Simply Supported Homogeneous Plate Equations

For verification of the theory, the strata model can be solved for the simple case of a simply supported homogeneous plate. For a homogeneous plate the fiber strata stiffness constants (s_f) reduce to the form of the matrix strata coefficients (k_m) where $s_1=s_2=k_1$, $s_3=k_2$, $s_4=s_7=s_8=k_3$ and $s_5=s_6=s_9=0$. The governing equations 25-27 reduce to:

$$\int_0^a\int_0^b \sum_{t=1}^{M}\sum_{i=0}^{N} \begin{bmatrix} k_{3t}(2\psi_{i,yy}Z_m^2 + \psi_i Z_{m,z}^2 + w_{,x}Z_{m,z}) + \\ (k_{2t} + k_{3t})2\zeta_{i,xy}Z_m^2 + k_{1t}2\psi_{i,xx}Z_m^2 \end{bmatrix} dydx = 0 \tag{33}$$

$$\int_0^a\int_0^b \sum_{t=1}^{M}\sum_{i=0}^{N} \begin{bmatrix} k_{3t}(2\zeta_{i,xx}Z_m^2 + \zeta_i Z_{m,z}^2 + w_{,y}Z_{m,z}) + \\ (k_{2t} + k_{3t})2\psi_{i,xy}Z_m^2 + k_{1t}2\zeta_{i,yy}Z_m^2 \end{bmatrix} dydx = 0 \tag{34}$$

$$\int_0^a\int_0^b \sum_{t=1}^{M}\sum_{i=0}^{N} \begin{bmatrix} k_{3t}(2w_{,xx} + 2w_{,yy}) + \\ k_{3t}(\psi_{i,x} + \zeta_{i,y})Z_{m,z} \end{bmatrix} dydx = \int_0^a\int_0^b p(x,y)dydx \tag{35}$$

Equation 28 must be modified to allow displacement to vary in z through the entire thickness. Equation 28 becomes:

$$Z_m = \int_{z_{t-}}^{z_{t+}} z^i dz \tag{36}$$

The boundary conditions reduce to:

$$\sum_{t=1}^{M}\sum_{i=0}^{N}\left[\begin{array}{l} \int_0^b \delta\psi_i \left[k_{1t}\, 2\psi_{i,x} Z_m^2 + k_{2t}\, 2\zeta_{i,y} Z_m^2 \right]_0^a dy \; + \\ \int_0^a \delta\psi_i \left[k_{3t}\, 2(\psi_{i,y} + k_{3t}\zeta_{i,x}) Z_m^2 \right]_0^b dx \end{array}\right] = 0 \tag{37}$$

$$\sum_{t=1}^{M}\sum_{i=0}^{N}\left[\begin{array}{l} \int_0^b \delta\zeta_i \left[k_{3t}\, 2(\psi_{i,y} Z_m^2 + k_{3t}\zeta_{i,x} Z_m^2) \right]_0^a dy \; + \\ \int_0^a \delta\zeta_i \left[k_{2t}\, 2\psi_{i,x} Z_m^2 + k_{1t}\, 2\zeta_{i,x} Z_m^2 \right]_0^b dx \end{array}\right] = 0 \tag{38}$$

$$\sum_{t=1}^{M}\sum_{i=0}^{N}\left[\begin{array}{l} \int_0^b \delta w \left[k_{3t} (\psi_i Z_{m,z} + 2 w_{,x} Z_m) \right]_0^a dy \; + \\ \int_0^a \delta w \left[k_{3t} (\zeta_i Z_{m,z} + 2 w_{,y} Z_m) \right]_0^b dx \end{array}\right] = 0 \tag{39}$$

Functions to represent the x and y dependence of the displacement are required to solve the homogeneous plate governing equations (33-39). Using proper Sine and Cosine series functions for the Ψ, ζ and load functions, can force the x and y dependence to become constant throughout each of the governing equations. The displacement and load relationships [2: 30]:

$$\psi_i(x,y) = A_i \cdot Cos(px)Sin(qy)$$
$$\zeta_i(x,y) = B_i \cdot Sin(px)Cos(qy)$$
$$w(x,y) = C \cdot Sin(px)Sin(qy)$$
$$p(x,y) = P \cdot Sin(px)Sin(qy)$$

$$i = 0,1,..N \qquad (40)$$

generate x and y dependent derivatives in equations 33-35 that go to zero over the integral of the plate. The x and y dependence is constant in all the terms of each governing equation and can be factored out, leaving a set of governing equations as a function of z and the coefficients A_i, B_i and C. The z integrals can be evaluated directly, leaving the governing equations as function of the coefficients A_i, B_i and C.

The load p(x,y) in equation 40, represents a distributed pressure that varies in x and y with a maximum pressure at the center of the plate. For a plate simply supported on all four edges, this should create a domed displacement function with the maximum deflection at the center. More complex loading can be modeled with a full Fourier Sine series [6: 549].

The governing equations can be transformed into a linear system of equations as follows:

$$[M]\begin{Bmatrix} A_i \\ B_i \\ C \end{Bmatrix} = \begin{Bmatrix} 0_i \\ 0_i \\ p \end{Bmatrix} \qquad i = 0..N \qquad (41)$$

M will be [2*(N+1)+1] x [2*(N+1)+1] matrix while the other two matrices will be 2*(N+1) x 1 vectors. Guassian elimination can be used to solve for the coefficients A_i, B_i, and C. The coefficients are then used to generate the displacements, strains and stress throughout the laminate.

The boundary conditions for a simply supported plate have v, w and σ_x equal zero a' x=0 and x=a, and u, w, and σ_y equal to zero at y=0 and y=b. This satisfies boundary conditionof equation 39. By definition $\delta u=0$ at y=0, and y=b and $\delta v=0$ at x=0 and x=a, which satisfies boundary conditions of equations 37 and 38.

Simply Supported Homogeneous Plate Example

A homogeneous plate test case was solved using the above procedure. The following parameter are required to define the plate.

Table 1. Homogeneous Plate Parameters

Parameter	Value
Number of plies (n)	2
Ply thickness (h)	1/2 in
Laminate thickness (t)	1 in
Plate length (a)	10 in
Plate width (b)	10 in
Tensile Modulus(E)	$3.0*10^5$ psi
Poisson's ratio (ν)	0.4
Shear Modulus $G=E/2(1+\nu)$	$1.071*10^5$ psi
Polynomial Order (N)	5

With the above parameters the stiffness and displacements of a homogeneous plate can be fully defined and the system of equations solved for the x and y dependence coefficients. The laminate thickness (t) and the load (P) were chosen to normalize the plate dimensions and displacements to the thickness and to normalize the plate moduli to the load. The stratified plate model displacements, strains and stresses, of a simply supported homogeneous plate are shown in Figures 8 through 22. The x and y dependence has been factored out to give the through the thickness distributions. All the length dimensions have been normalized to the plate thickness (h) and the modulous dimensions have been normalized to the load (P).

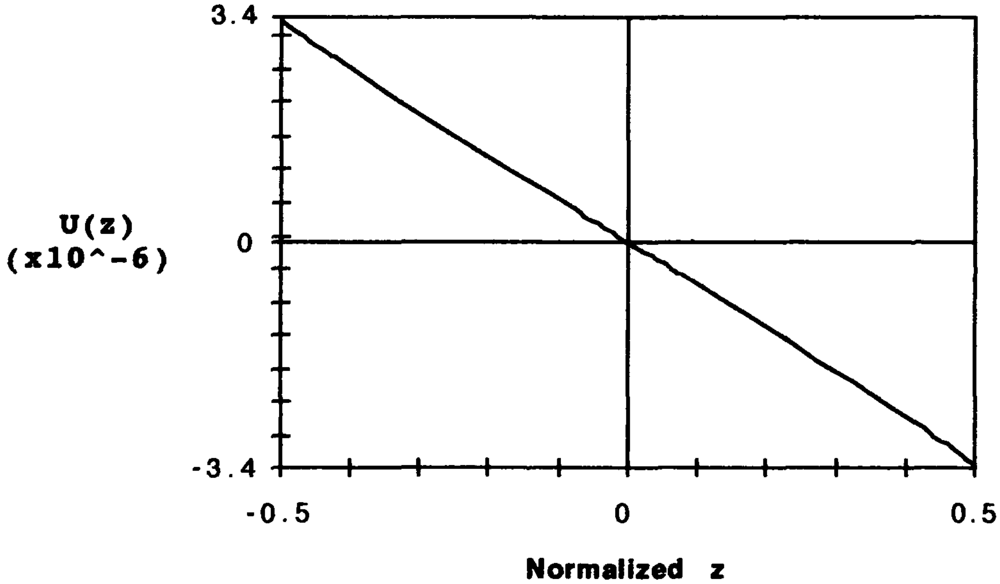

Figure 8. U(z) Displacement for a Homogeneous Plate

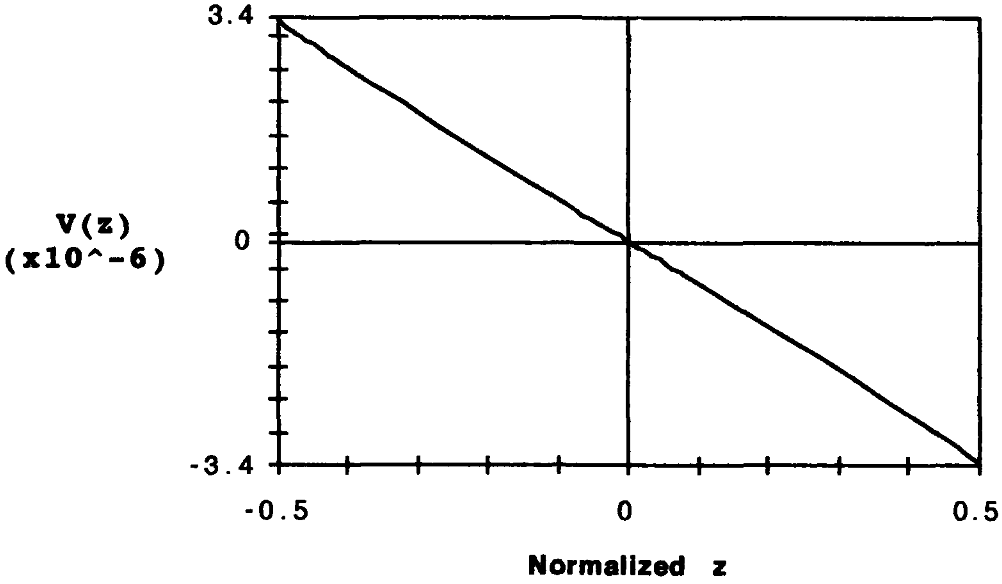

Figure 9. V(z) Displacement for a Homogeneous Plate

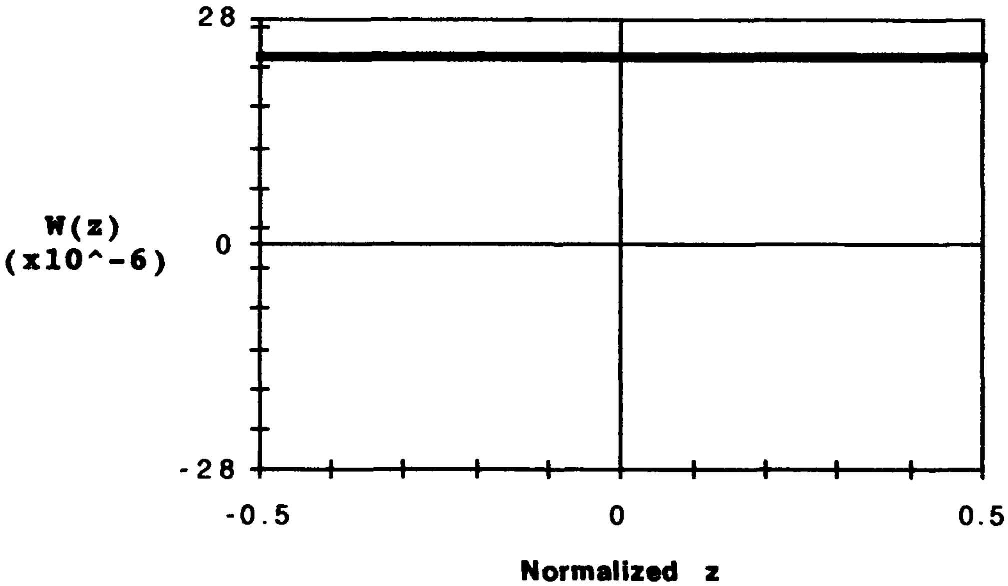

Figure 10. W(z) Displacement for a Homogeneous Plate

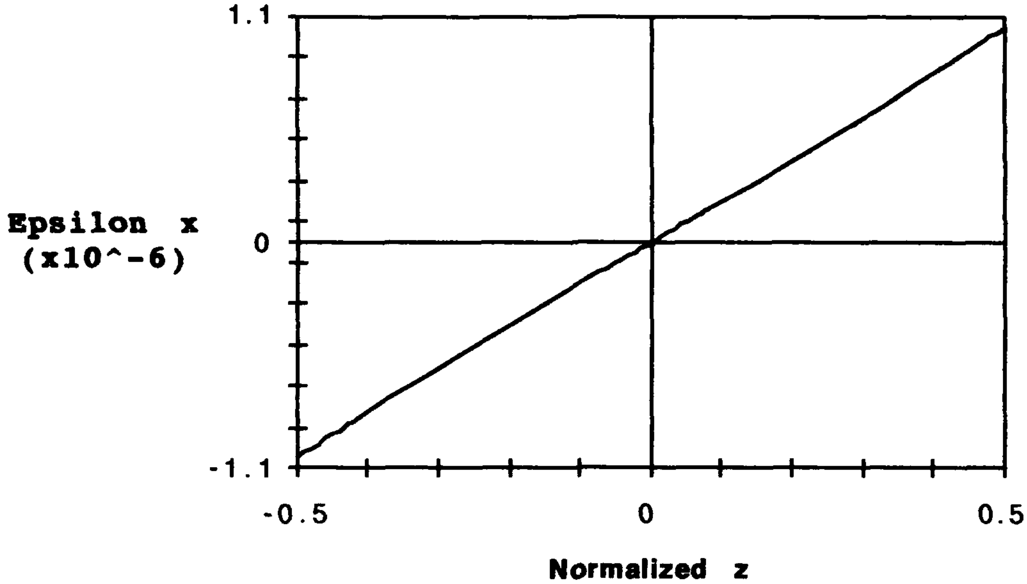

Figure 11. Epsilon x (ε_x) Strain for a Homogeneous Plate

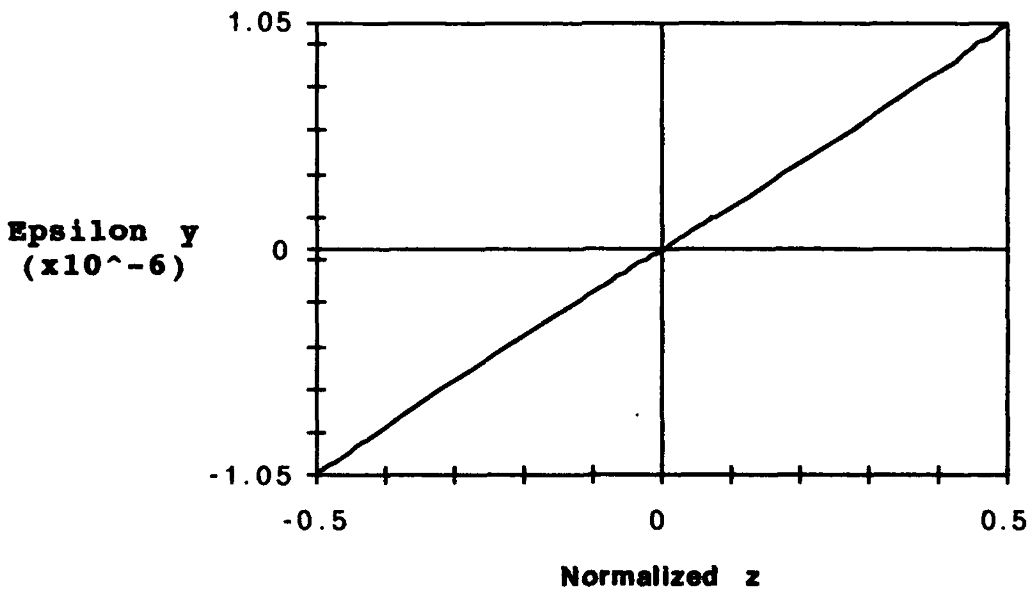

Figure 12. Epsilon y (ε_y) Strain for a Homogeneous Plate

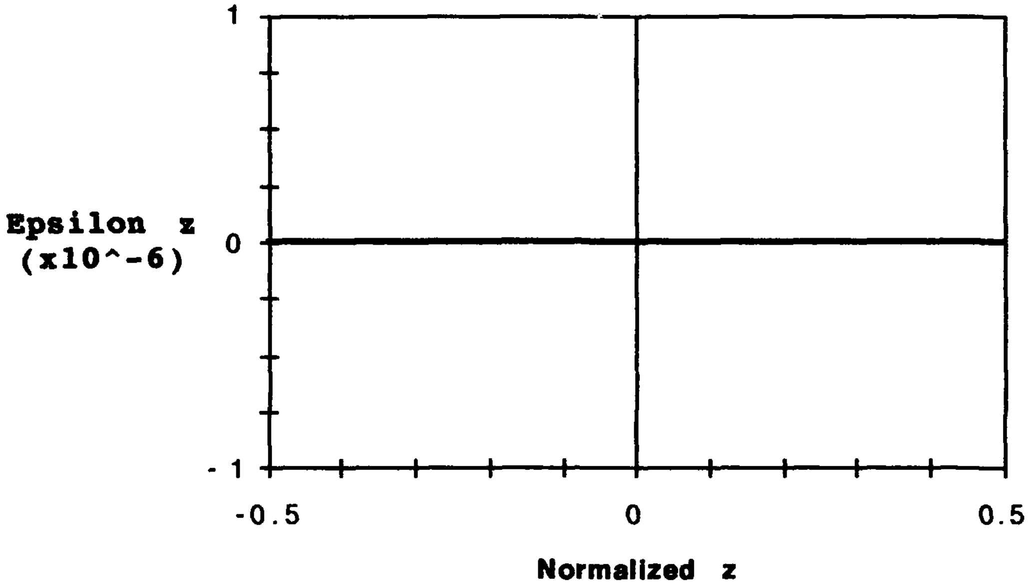

Figure 13. Epsilon z (ε_z) Strain for a Homogeneous Plate

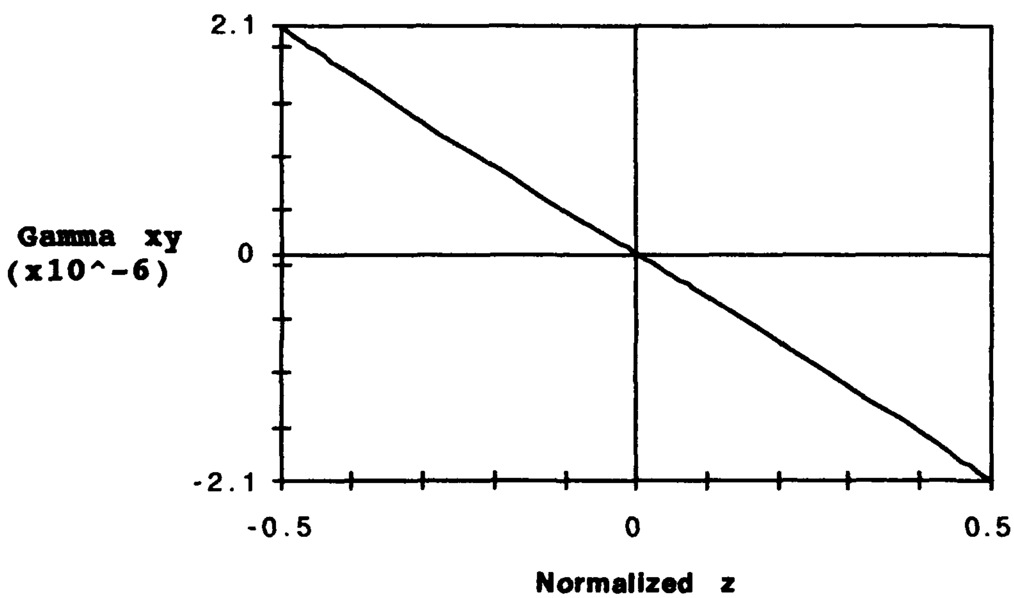

Figure 14. Gamma xy (γ_{xy}) for a Homogeneous Plate

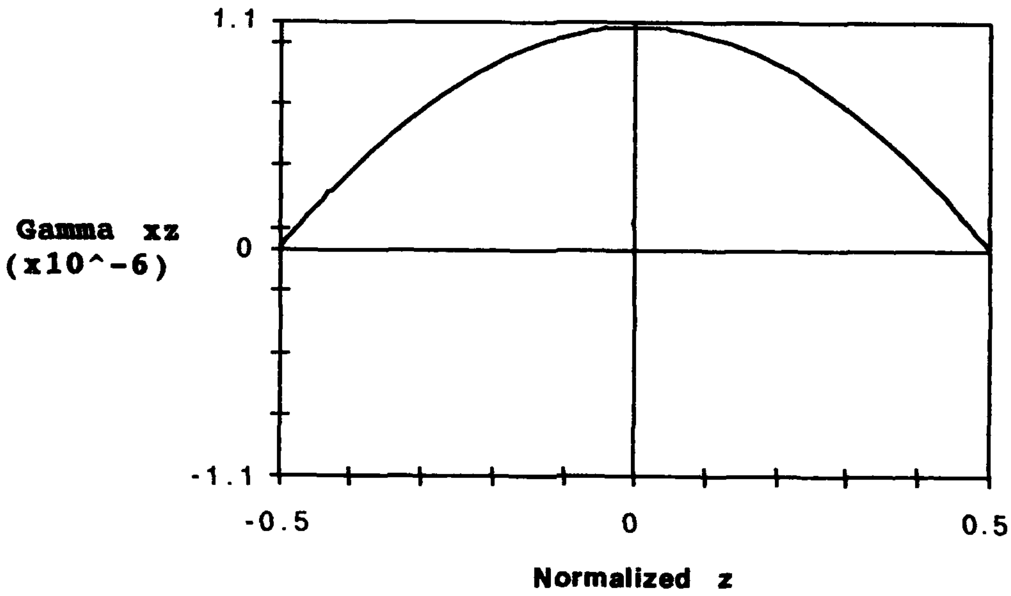

Figure 15. Gamma xz (γ_{xz}) for a Homogeneous Plate

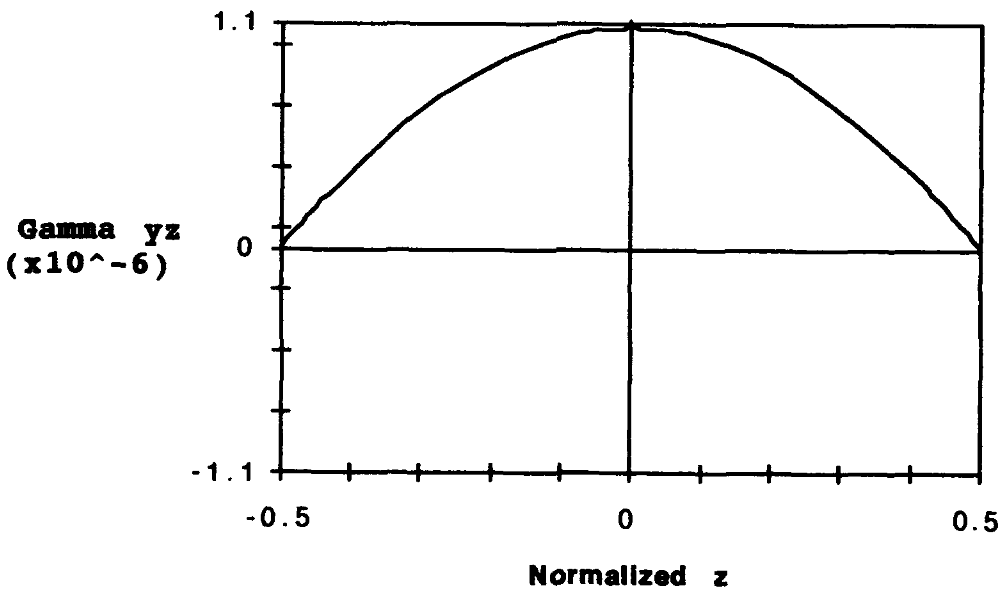

Figure 16. Gamma yz (γ_{yz}) for a Homogeneous Plate

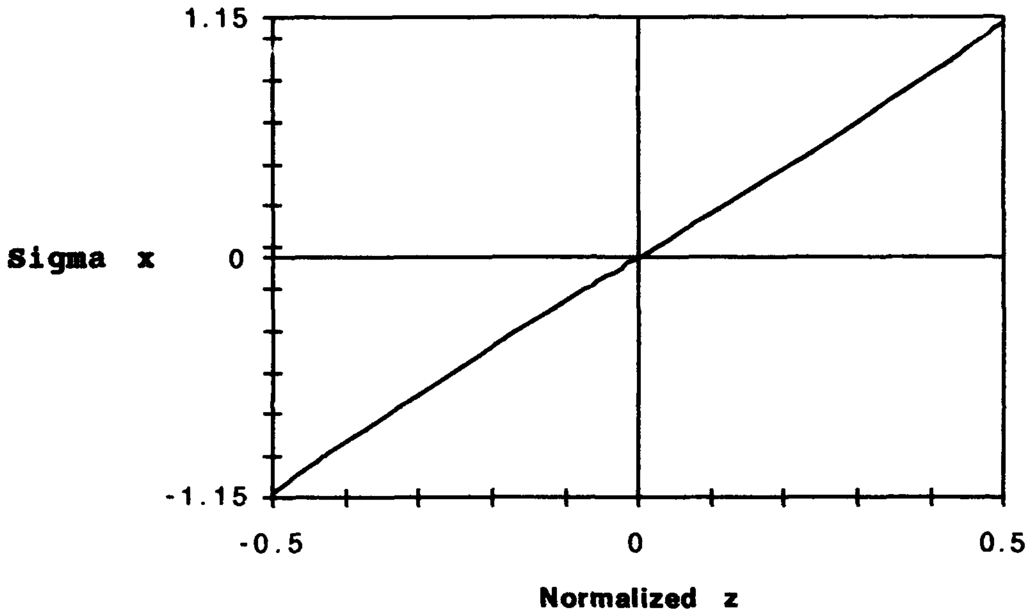

Figure 17. Sigma x (σ_x) for a Homogeneous Plate

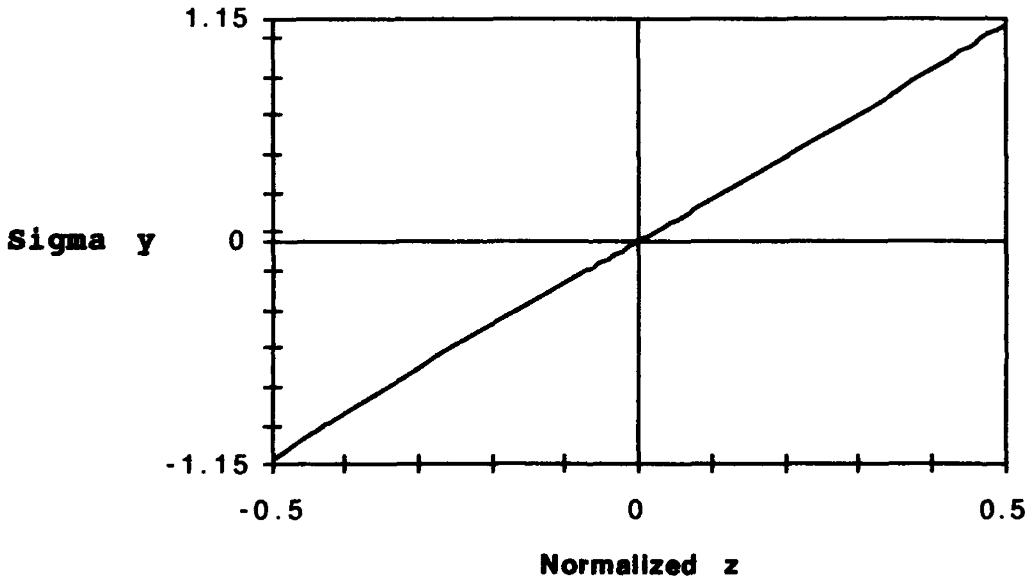

Figure 18. Sigma y (σ_y) for a Homogeneous Plate

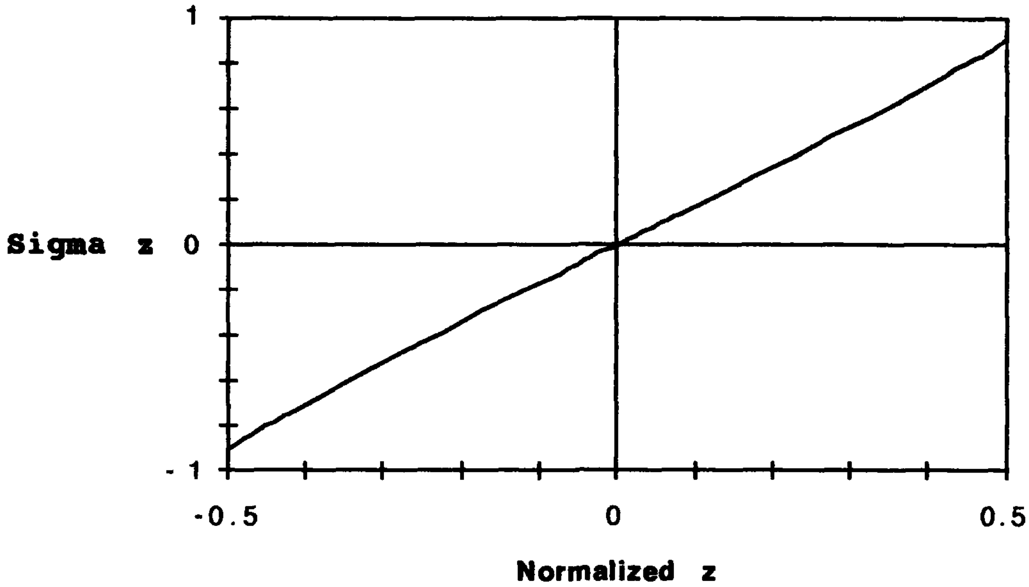

Figure 19. Sigma z (σ_z) for a Homogeneous Plate

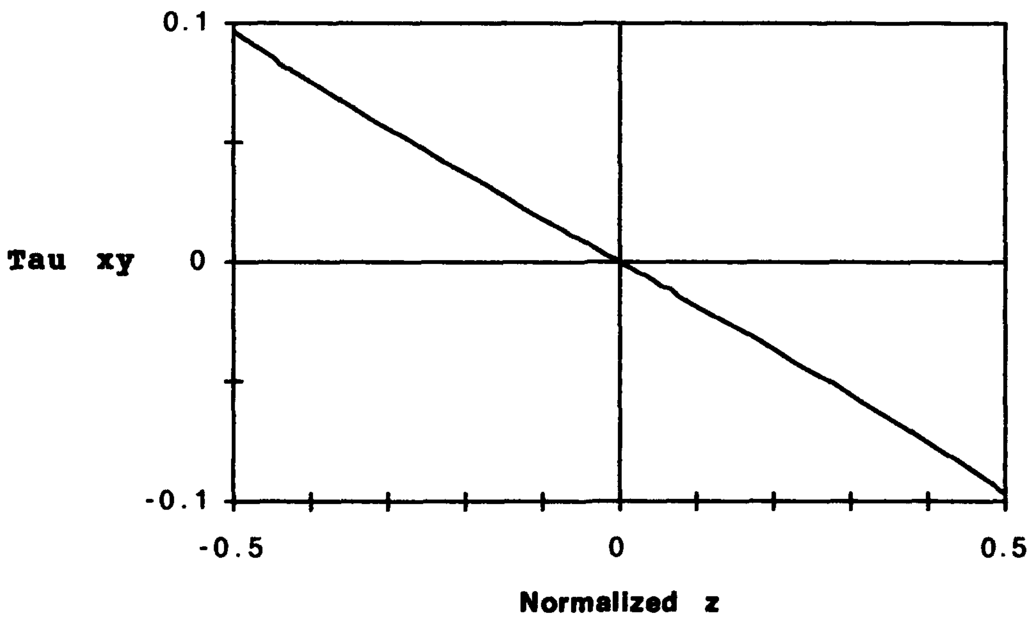

Figure 20. Tau xy (τ_{xy}) for a Homogeneous Plate

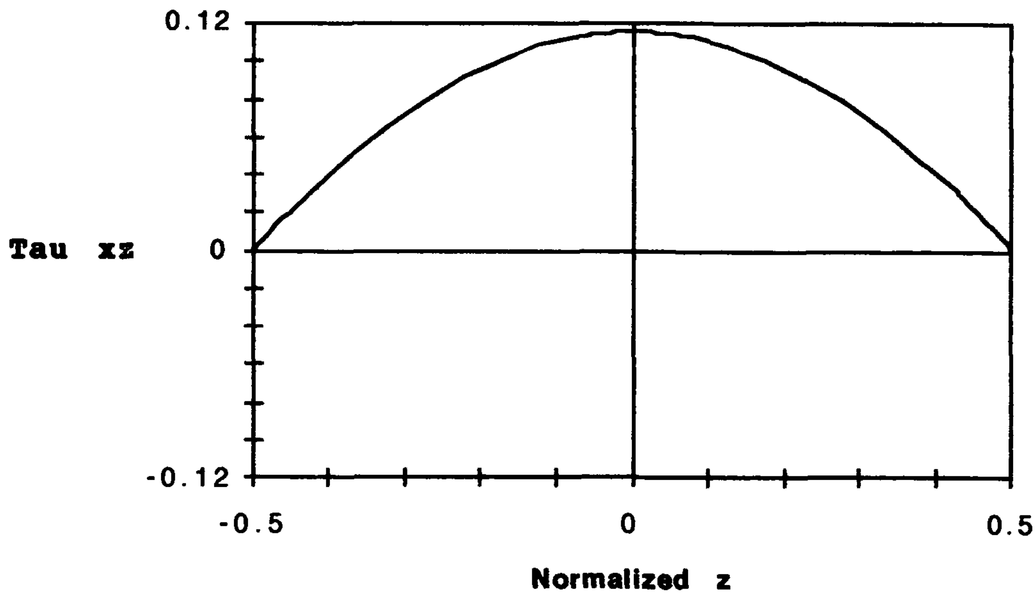

Figure 21. Tau xz (τ_{xz}) for a Homogeneous Plate

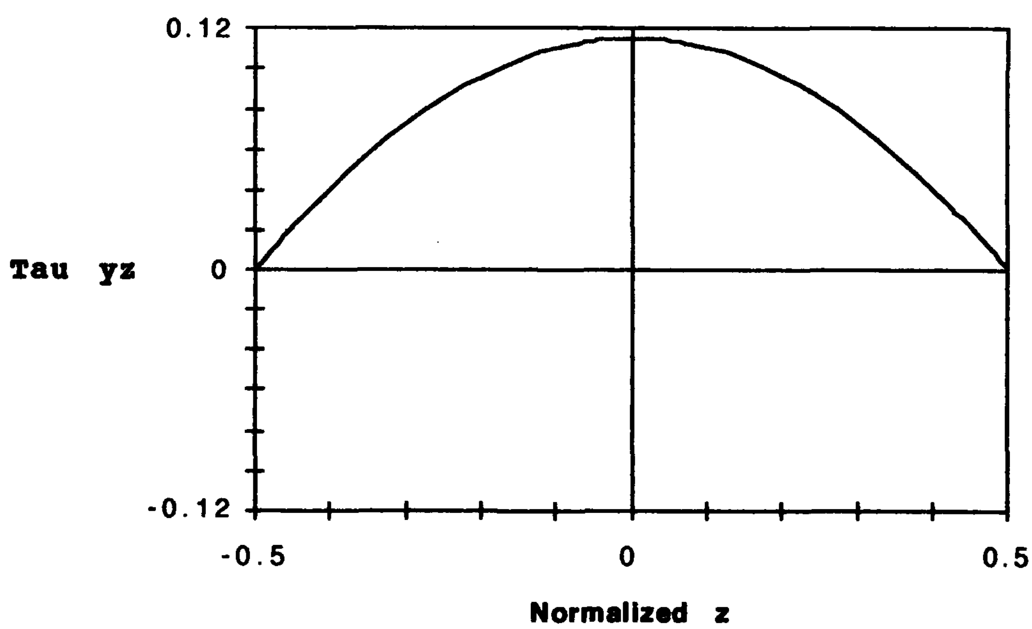

Figure 22. Tau yz (τ_{yz}) for a Homogeneous Plate

Simply Supported Homogeneous Plate Discussion

The results shown in figures 8-13 are the expected linear displacement and normal strain functions for a homogeneous plate. The upper half of the plate is in tension from the bending, but experiences a negative displacement due to the rotation allowed by the B.C.. The constant displacement in z is a result of the strata model displacement assumptions and corresponds with most linear, small displacement plate models. The parabolic stress τ_{xz} and τ_{yz} strain and stress distribution is a result of mid plane carrying the maximum transverse shear, and the transverse shear going to zero on the top and bottom surfaces. The existance of σ_z is a result of the e_x, ε_y and γ_{xy} terms of the strain vector multiplying out with terms in the stiffess matrix. The stress distribution can only be explained by assuming that half the load is applied to the lower surface and half the load is applied to the upper surface.

IV. *Solution to a Simply Supported Plate with Cross Plies*

Composite Plate with Cross Plies Equations

For a simply supported square composite plate with only 0 or 90 degree plies, the fiber strata stiffness constants s_5, s_6, and s_9 are equal to zero and equations 25-27 reduce to:

$$\int_0^a\int_0^b \sum_{r=1}^{F}\sum_{t=1}^{M}\sum_{i=0}^{N} \begin{bmatrix} k_{3t}(\psi_{i,yy}Z_m^2 + \psi_i Z_{m,z}^2 + w_{,x}Z_{m,z}) + \\ (k_{2t} + k_{3t})\zeta_{i,xy}Z_f^2 + k_{1t}\psi_{i,xx}Z_m^2 + \\ (s_{3r} + s_{4r})\zeta_{i,xy}Z_f^2 + s_{4r}\psi_{i,yy}Z_f^2 + s_{1r}\psi_{i,xx}Z_f^2 \end{bmatrix} dydx = 0 \tag{42}$$

$$\int_0^a\int_0^b \sum_{r=1}^{F}\sum_{t=1}^{M}\sum_{i=0}^{N} \begin{bmatrix} k_{3t}(\zeta_{i,xx}Z_m^2 + \zeta_i Z_{m,z}^2 + w_{,y}Z_{m,z}) + \\ (k_{2t} + k_{3t})\psi_{i,xy}Z_f^2 + k_{1t}\zeta_{i,yy}Z_m^2 + \\ (s_{3r} + s_{4r})\psi_{i,xy}Z_f^2 + s_{4r}\zeta_{i,xx}Z_f^2 + s_{2r}\zeta_{i,yy}Z_f^2 \end{bmatrix} dydx = 0 \tag{43}$$

$$\int_0^a\int_0^b \sum_{r=1}^{F}\sum_{t=1}^{M}\sum_{i=0}^{N} \begin{bmatrix} k_{3t}(w_{,xx} + w_{,yy}) + \\ k_{3t}(\psi_{i,x} + \zeta_{i,y})Z_{m,z} + \\ s_{7r}w_{,xx} + s_{8r}w_{,yy} \end{bmatrix} dydx = \int_0^a\int_0^b p(x,y)dydx \tag{44}$$

$$Z_m = \int_{z_{t-}}^{z_{t+}} (\alpha_t + \beta_t z)^i dz \qquad Z_f = \int_{z_{r-}}^{z_{r+}} z_{fr}^i dz \tag{45}$$

The boundary condition equations 20-32 reduce to:

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=0}^{N}\left[\begin{array}{l}\int_0^b \delta\psi_i \left[\begin{array}{l}k_{1t}\psi_{i,x}Z_m^2 + k_{2t}\zeta_{i,y}Z_m^2 + \\ s_{1r}\psi_{i,x}Z_f^2 + s_{3r}\zeta_{i,y}Z_f^2\end{array}\right]_0^a dy + \\ \int_0^a \delta\psi_i \left[\begin{array}{l}k_{3t}(\psi_{i,y} + k_{3t}\zeta_{i,x})Z_m^2 + \\ s_{4r}(\psi_{i,y} + \zeta_{i,x})Z_f^2\end{array}\right]_0^b dx\end{array}\right] = 0 \qquad (46)$$

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=0}^{N}\left[\begin{array}{l}\int_0^b \delta\zeta_i \left[\begin{array}{l}k_{3t}\psi_{i,y}Z_m^2 + k_{3t}\zeta_{i,x}Z_m^2 + \\ s_{4r}(\psi_{i,y} + \zeta_{i,x})Z_f^2\end{array}\right]_0^a dy + \\ \int_0^a \delta\zeta_i \left[\begin{array}{l}k_{2t}\psi_{i,x}Z_m^2 + k_{1t}\zeta_{i,x}Z_m^2 + \\ s_{3r}\psi_{i,x}Z_f^2 + s_{2r}\zeta_{i,y}Z_f^2\end{array}\right]_0^b dx\end{array}\right] = 0 \qquad (47)$$

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=0}^{N}\left[\begin{array}{l}\int_0^b \delta w\left[k_{3t}\psi_i Z_{m,z} + k_{3t}w_{,x}Z_m + s_{7r}w_{,x}Z_f\right]_0^a dy + \\ \int_0^a \delta w\left[k_{3t}\zeta_i Z_{m,z} + k_{3t}w_{,y}Z_m + s_{8r}w_{,y}Z_f\right]_0^b dx\end{array}\right] = 0 \qquad (48)$$

The boundary conditions for a simply supported plate have v, w and σ_x equal zero at x=0 and x=a, and u, w, and σ_y equal to zero at y=0 and y=b. This satisfies boundary condition of equation 48. By the simply supported definition $\delta u=0$ at y=0, and y=b and $\delta v=0$ at x=0 and x=a, which satisfies boundary conditions of equations 46 and 47.

The ψ, ζ, w and p functions of equation 40 can be used in equations 42-45. The x and y dependence derivatives once again go to zero but are constant in each equation and can be factored out. The z integral can be evaluated directly, leaving the equations as a function of A_i, B_i and C. The system of equations, can then be solved for the A_i, B_i and C coefficients.

Simply Supported Square Plate with 0/0 Plies Example

The example plate in table 1 is used for the composite plate test case. Additional composite parameters must be supplied to fully define the composite plate.

Table 2. Composite Plate Parameters

Parameter		Value
Number of plies (n)		2
Ply thickness (h)		1/2 in
Laminate thickness		1 in
Plate length (a)		10 in
Plate width (b)		10 in
Ply orientation angles	θ_1	0 radians
	θ_2	0 radians
Matrix Tensile Modulous(E_m)		$3.0*10^5$ psi
Matrix Poisson's ratio (ν_m)		0.4
Matrix Shear Modulous G_m=E/2(1+ν_m)		$1.071*10^5$ psi
Fiber Tensile Modulous(E_f)		$2.0*10^7$ psi
Fiber Poisson's ratio (ν_f)		0.35
Fiber Shear Modulous G_f=E_f/2(1+ν_f)		$1.407*10^7$ psi
Global Fiber Volume Fraction		0.5
Global Matrix Volume Fraction		0.5
Fiber radius		$\sqrt{2} / 4\sqrt{\pi}$ in

The ply thickness was chosen to generate a total thickness of 1, to automatically normalize the results to the thickness. The fiber radius (not realistic) was derived from a global volume fraction of 1/2. From the supplied information in Table 2 the required strata model parameters may be calculated as follows:

Table 3. Strata Model Parameters

Parameter		Value
Polynomial Order (N)		5
Number of Matrix Strata (n+1)		3
Number of Fiber Strata (n)		2
Idealized fiber height (b)		$\sqrt{2} / 4$
Matrix thickness (t_m)		$0.5 - \sqrt{2} / 4$
Local Fiber Strata Volume Fraction		$\sqrt{2} / 2$
Local Fiber Strata Volume Matrix		$1 - \sqrt{2} / 2$
Mapping Parameter β_i		$2 + \sqrt{2}$
Mapping Parameter	α_1	$-(1 + \sqrt{2}) / 2$
	α_2	0
	α_3	$(1 + \sqrt{2}) / 2$
p (π/a)		$\pi/10$
q (π/b)		$\pi/10$

The idealized composites plate is is pictured as shown below:

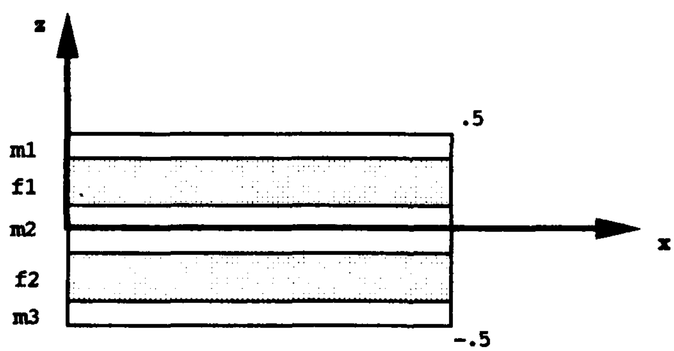

Figure 23. Example Composite Plate.

The displacements, strains and stresses of a simply supported composite plate with 0/0 degree plies are shown in figures 23-37. The x and y dependence has been factored out.

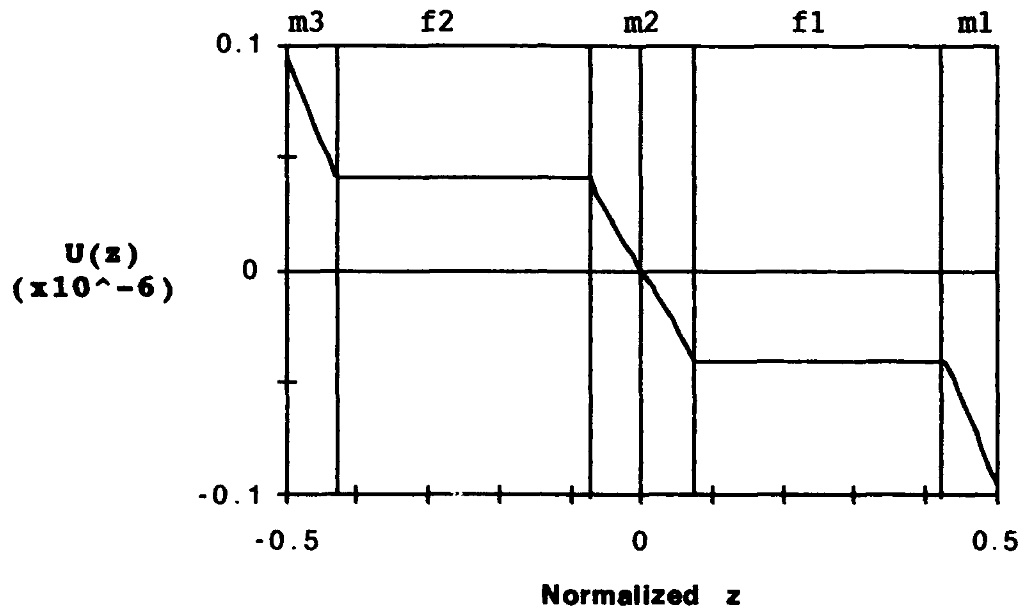

Figure 24. U(z) Displacement for a 0/0 Ply Plate

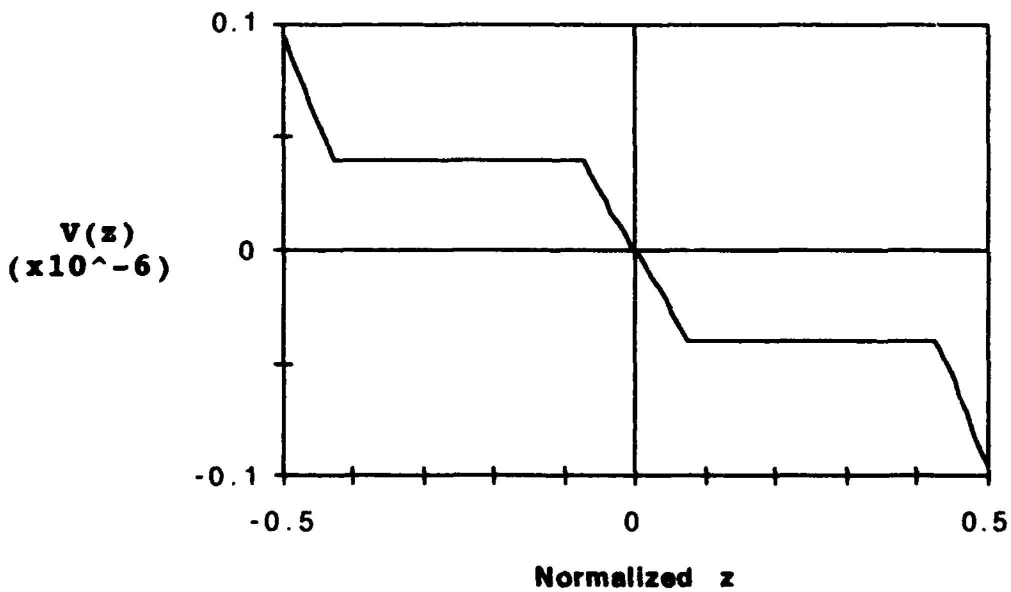

Figure 25. V(z) Displacement for a 0/0 Ply Plate

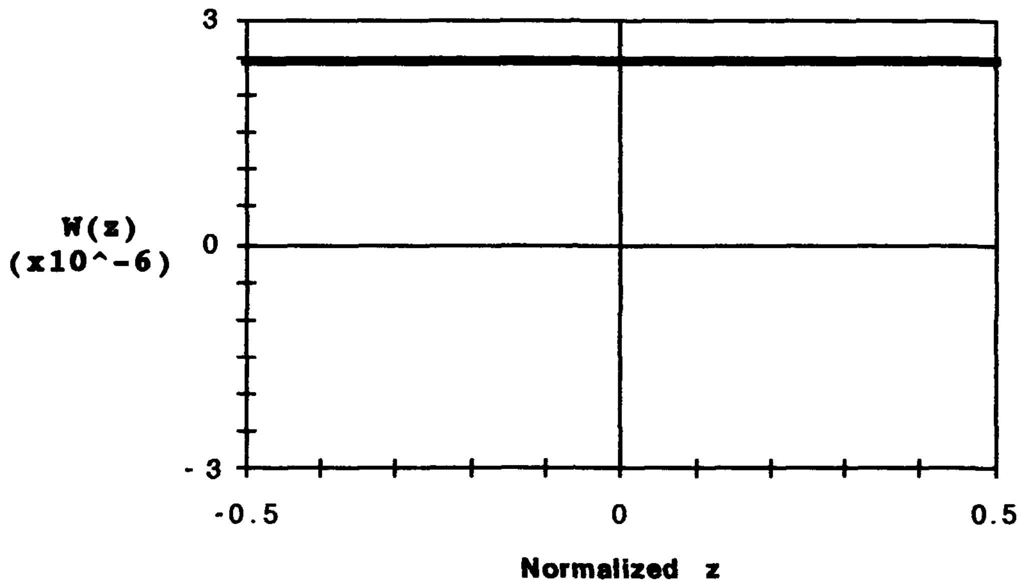

Figure 26. W(z) Displacement for a 0/0 Ply Plate

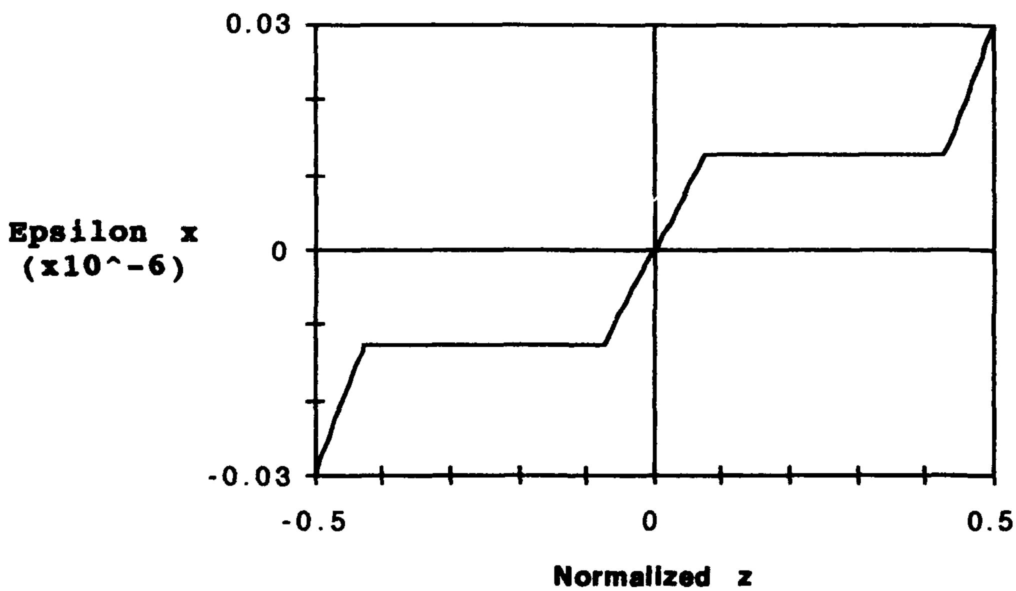

Figure 27. Epsilon x (ε_x) Strain for a 0/0 Ply Plate

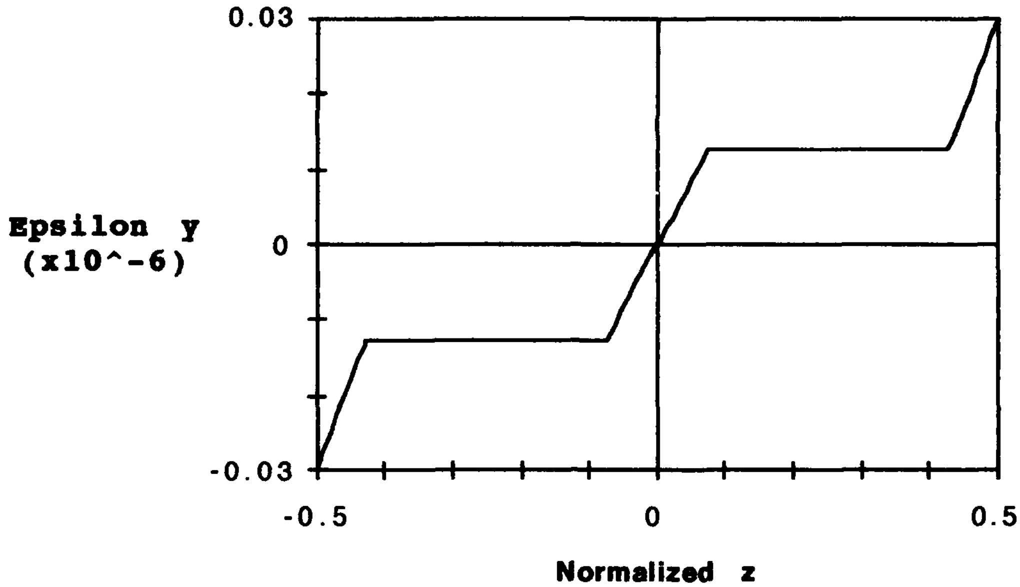

Figure 28. Epsilon y (ε_y) Strain for a 0/0 Ply Plate

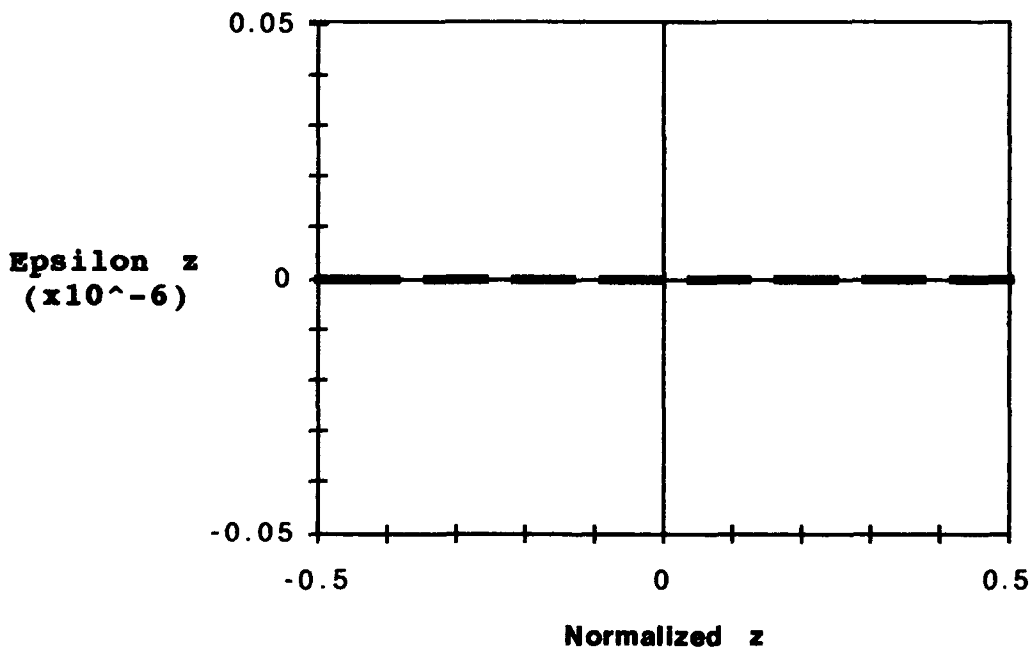

Figure 29. Epsilon z (ε_z) Strain for a 0/0 Ply Plate

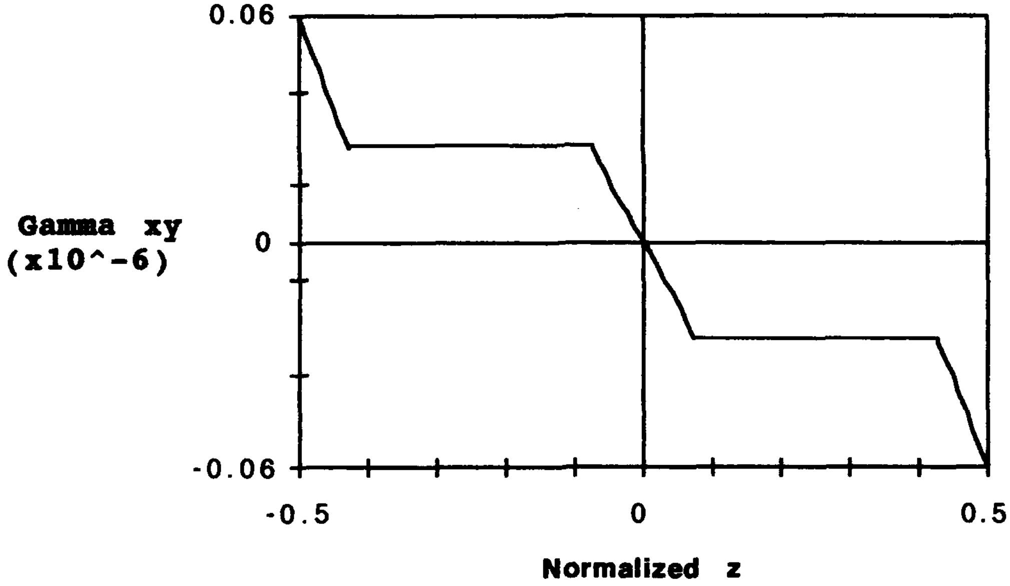

Figure 30. Gamma xy (γ_{xy}) for a 0/0 Ply Plate

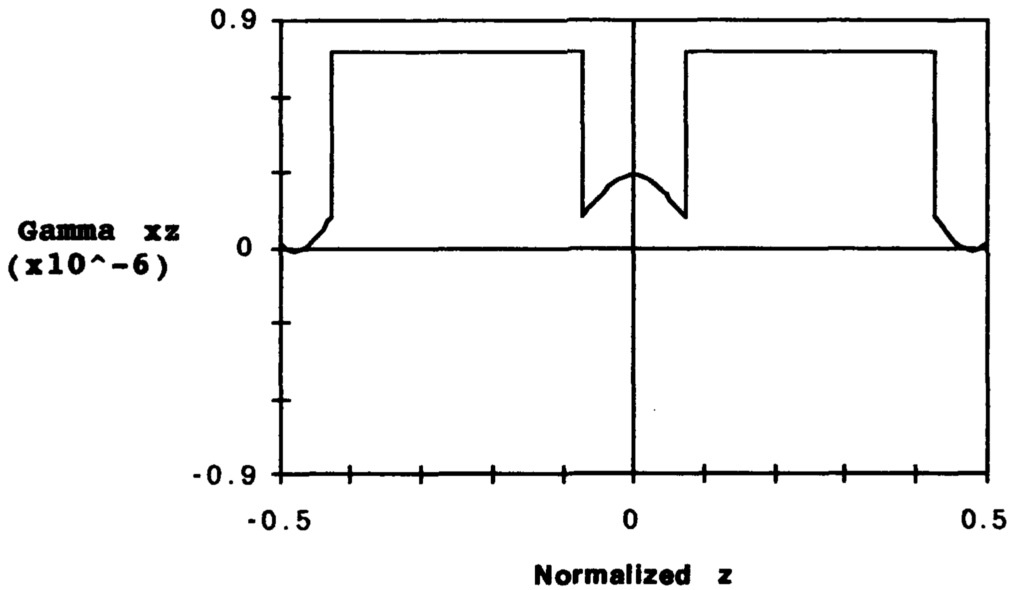

Figure 31. Gamma xz (γ_{xz}) for a 0/0 Ply Plate

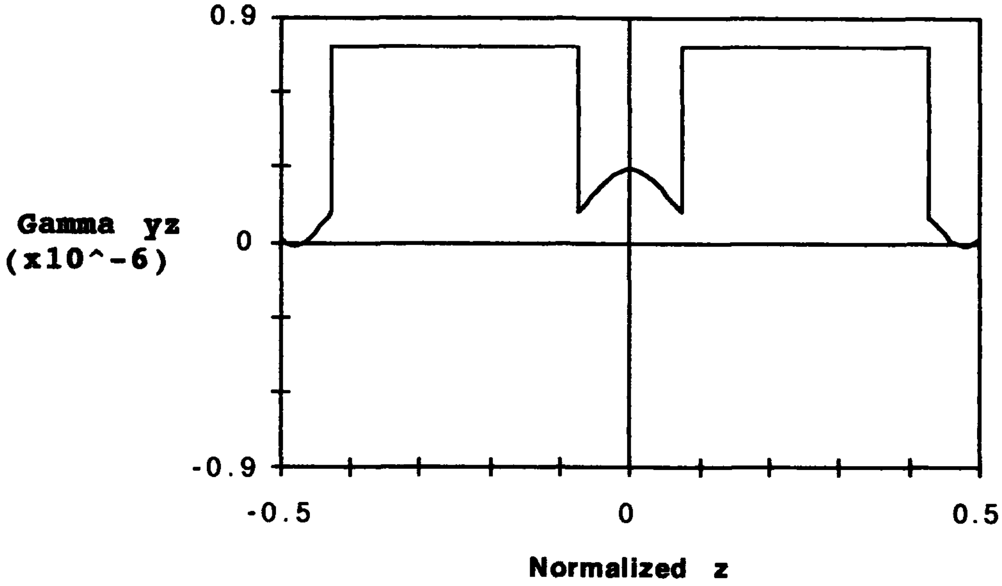

Figure 32. Gamma yz (γ_{yz}) for a 0/0 Ply Plate

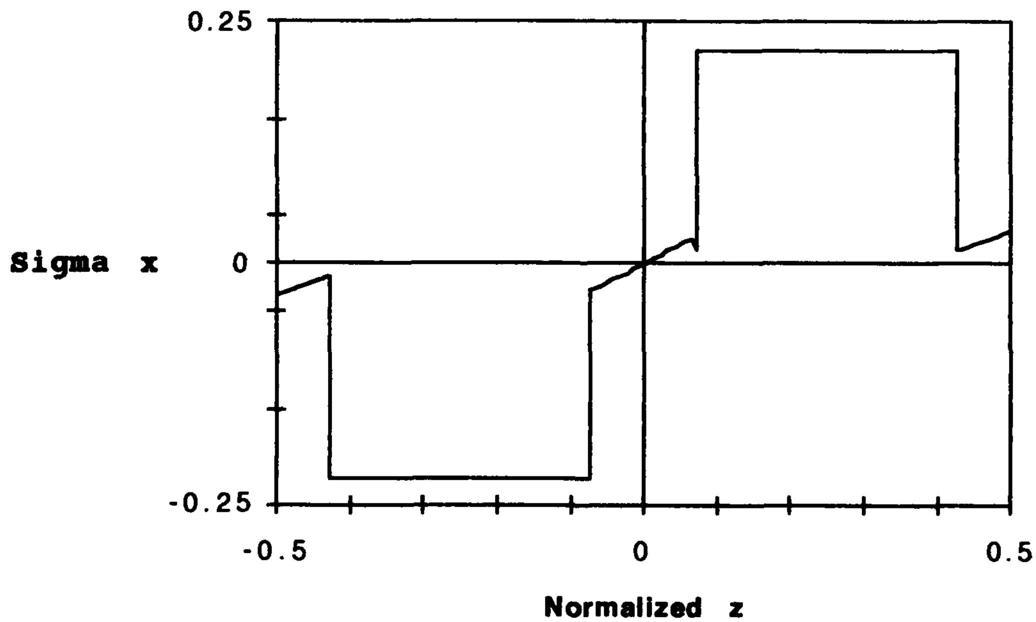

Figure 33. Sigma x (σ_x) for a 0/0 Ply Plate

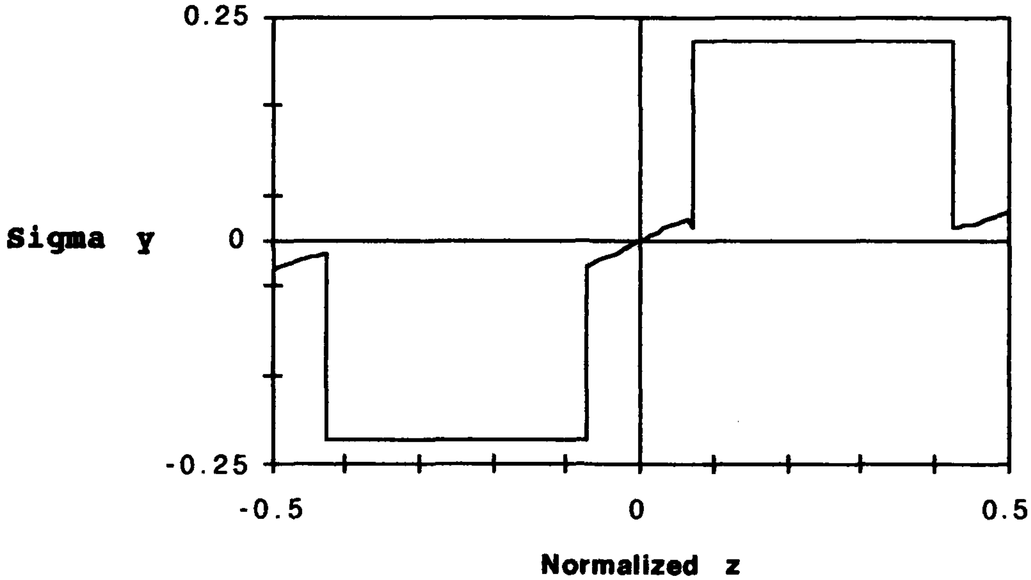

Figure 34. Sigma y (σ_y) for a 0/0 Ply Plate

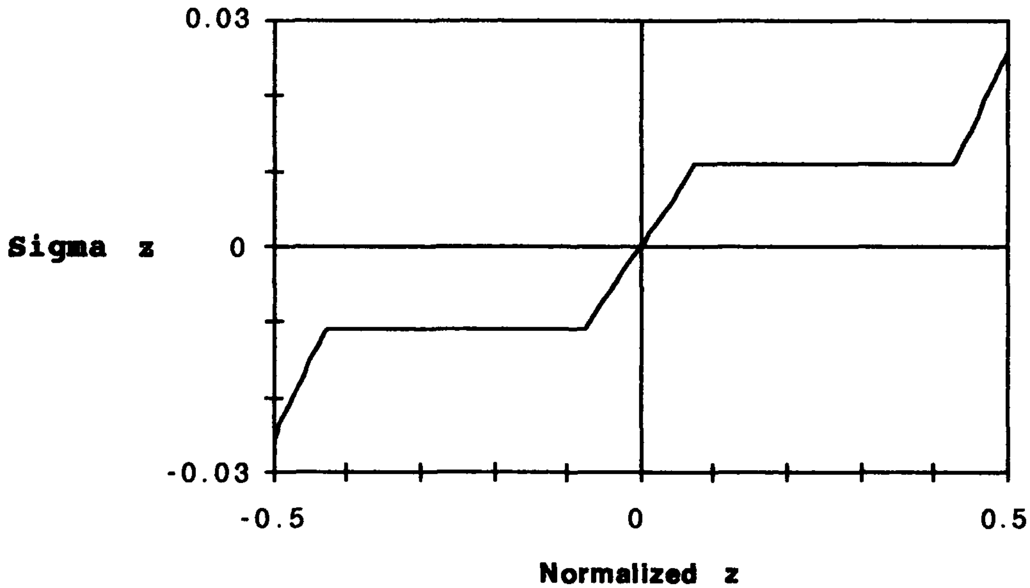

Figure 35. Sigma z (σ_z) for a 0/0 Ply Plate

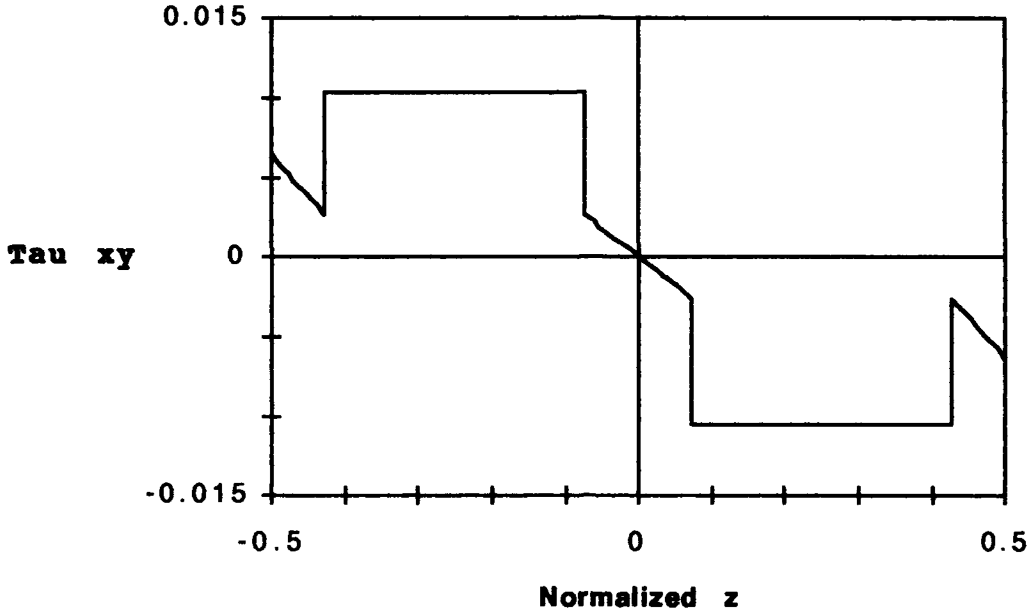

Figure 36. Tau xy (τ_{xy}) for a 0/0 Plate

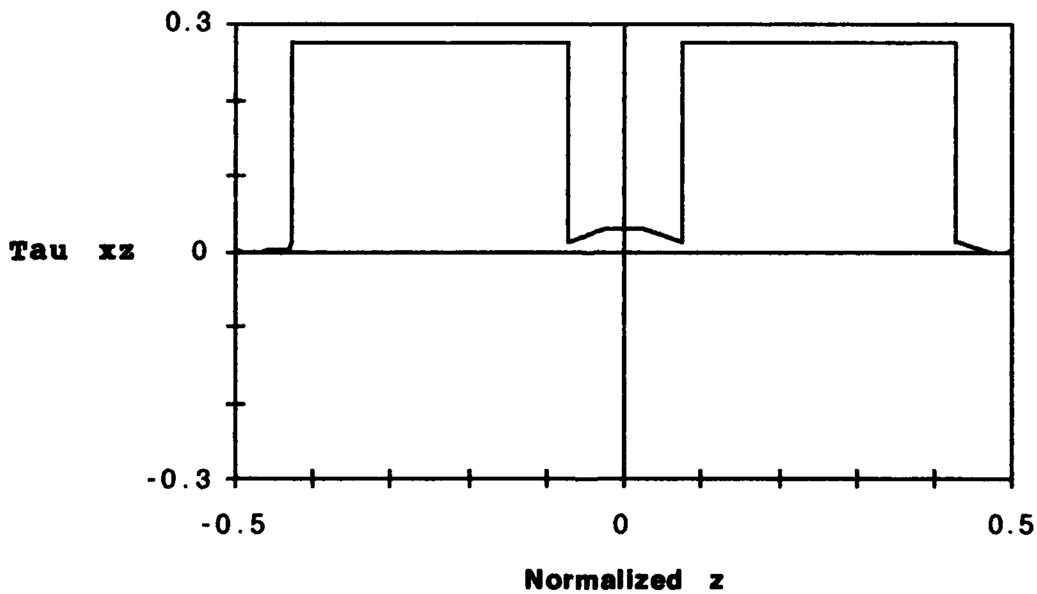

Figure 37. Tau xz (τ_{xz}) for a 0/0 Plate

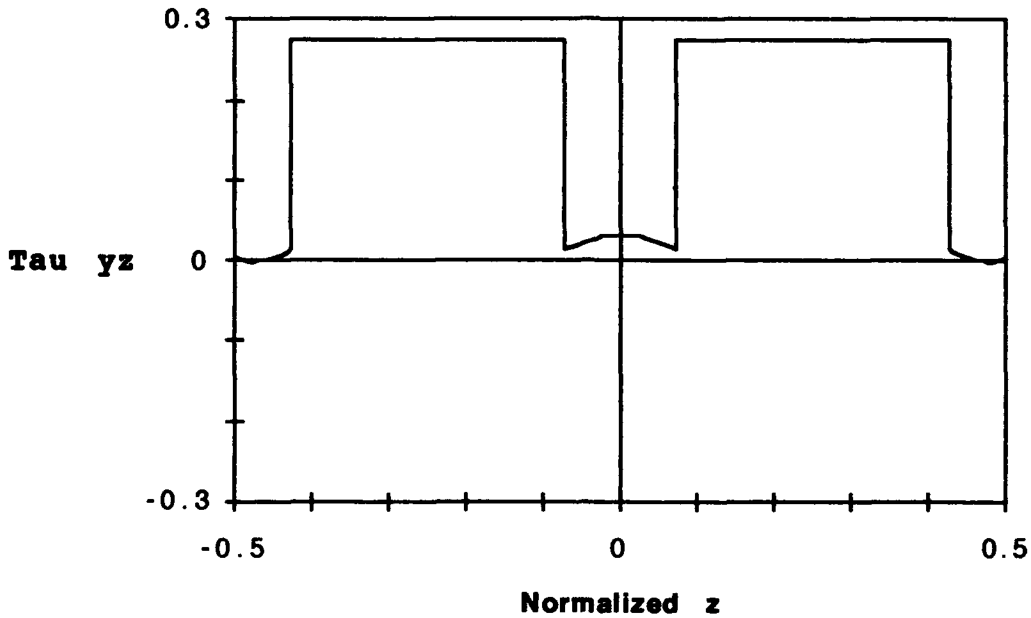

Figure 38. Tau yz (τ_{yz}) for a 0/0 Plate

Simply Supported 0/0 Composite Plate Discussion

The displacements shown in figures 21-23 are expected for a composite plate. The outer matrix strata exhibit greater displacements, in x and y, than the stiffer fiber strata. The ductile matrix's displacement increases with the distance from the mid plane. The maximum displacement is in the z direction as can be expected from the loading. The maximum strain is found to be the γ_{xy} shear strain from, the coupling of the normal strains. The shear strains, γ_{xz} and γ_{yz} are discontinuous functions because the strata model assumption that the fiber strata strain does not vary in z. This creates a plain strain assumption in z of the fiber strata. Because of this anomaly the γ_{xz} and γ_{yz} strains from the strata model should not be relied on. The existance of σ_z is a result of the e_x, ε_y and γ_{xy} terms of the strain vector multiplying out with the s_{10}, s_{11} and s_{13} terms of the fiber strata stiffess matrix and the k_2 and k_3 terms of the matrix strata stiffness matrix. The stress distribution can only be explained by assuming that half the load is applied to the lower surface and half the load is applied to the upper surface. The model produces fairly significant normal stresses in z. The transverse normal stress, σ_z, is on the same order of magnitude as the shear stress τ_{xy}. This a peculiarity of the loading assumptions in the strata model. The presence of the stiffer fibers creates a discontinuous stress distribution in all directions. The maximum stress is experienced by the fiber strata while the maximum normal strain is experienced by the outer matrix strata.

The stress and strain distributions generated by the strata model parallel those generated by a Classic Laminated Plate model analysis of a square plate under a uniform moment about the x and in y axes. The ε_x, ε_y and γ_{xy} strains and the σx, σ_y and τ_{xy} stresses generated by a Classic Laminated Plate model analysis with a plane stress assumption, exhibit the same patterns of stress and strain as the strata model.

Square Plate with 0/90 Degree Plies Example

The displacements, strains and stresses for a 0/90 plate are shown in figures 38-52. The x and y dependence has been factored out to give the through the thickness distributions.

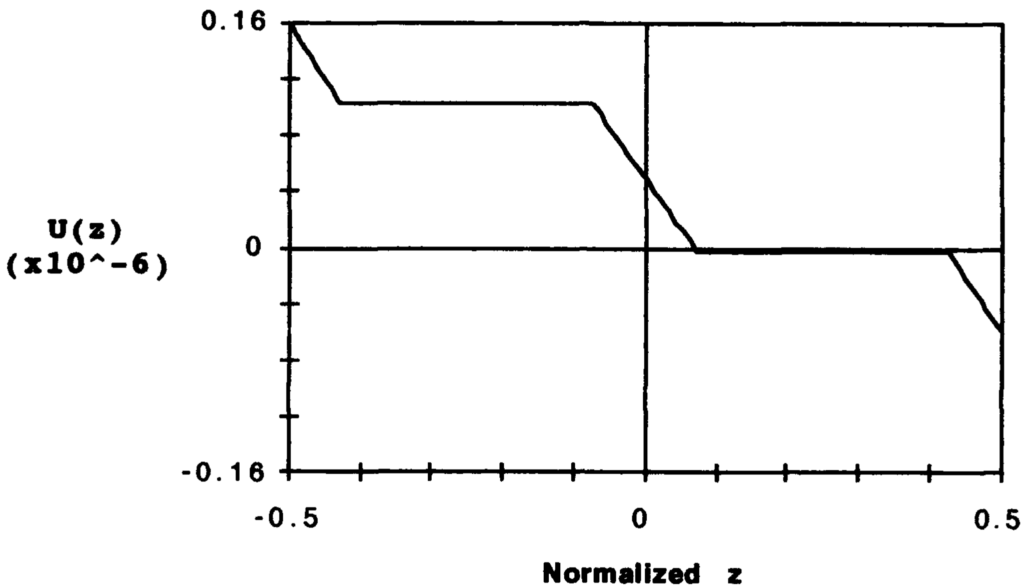

Figure 39. U(z) Displacement for a 0/90 Laminate Plate

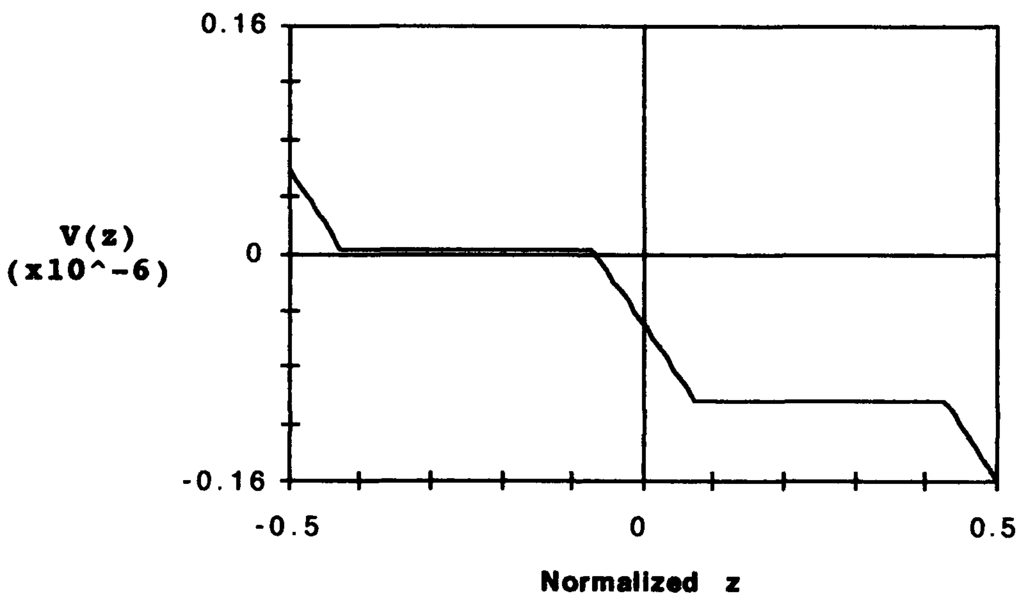

Figure 40. V(z) Displacement for a 0/90 Laminate Plate

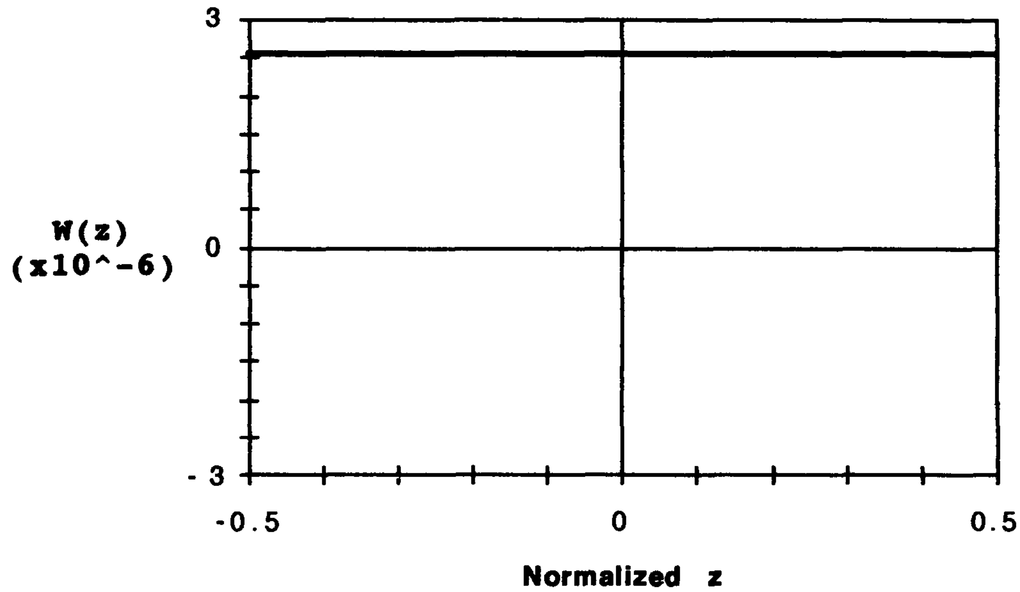

Figure 41. W(z) Displacement for a 0/90 Laminate Plate

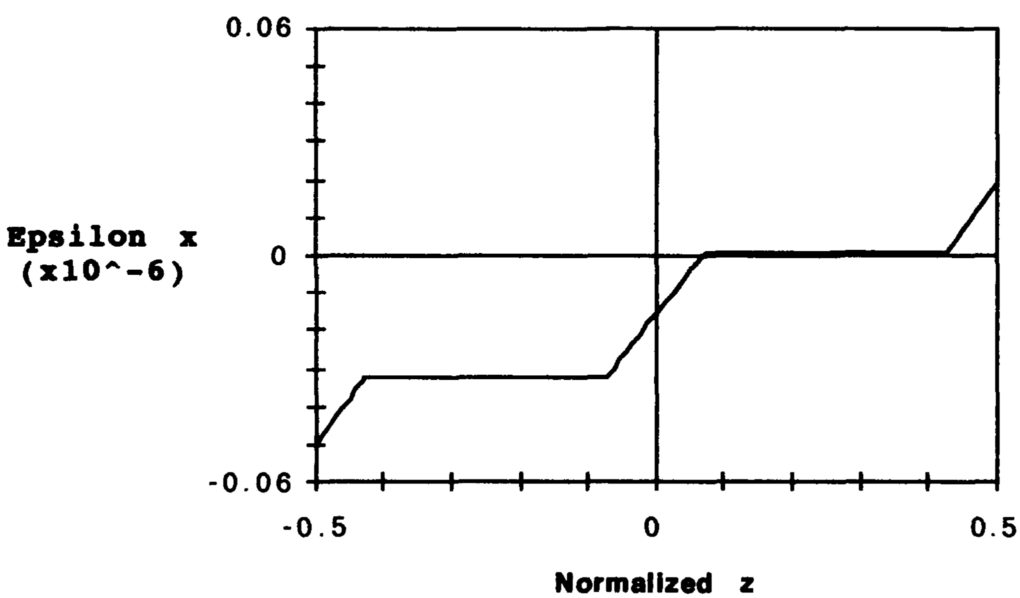

Figure 42. Epsilon x (ε_x) Strain for a 0/90 Laminate Plate

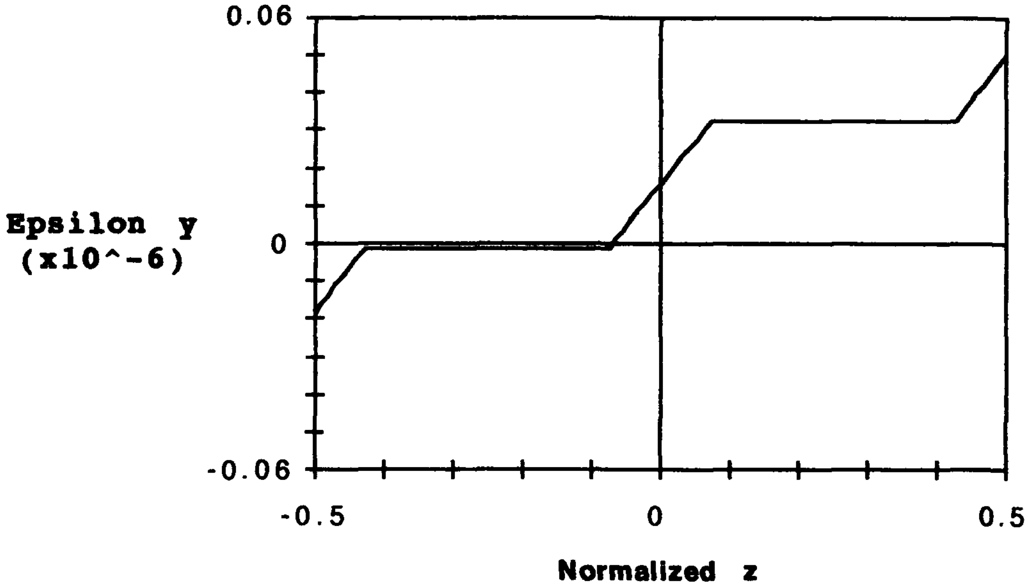

Figure 43. Epsilon y (e_y) Strain for a 0/90 Laminate Plate

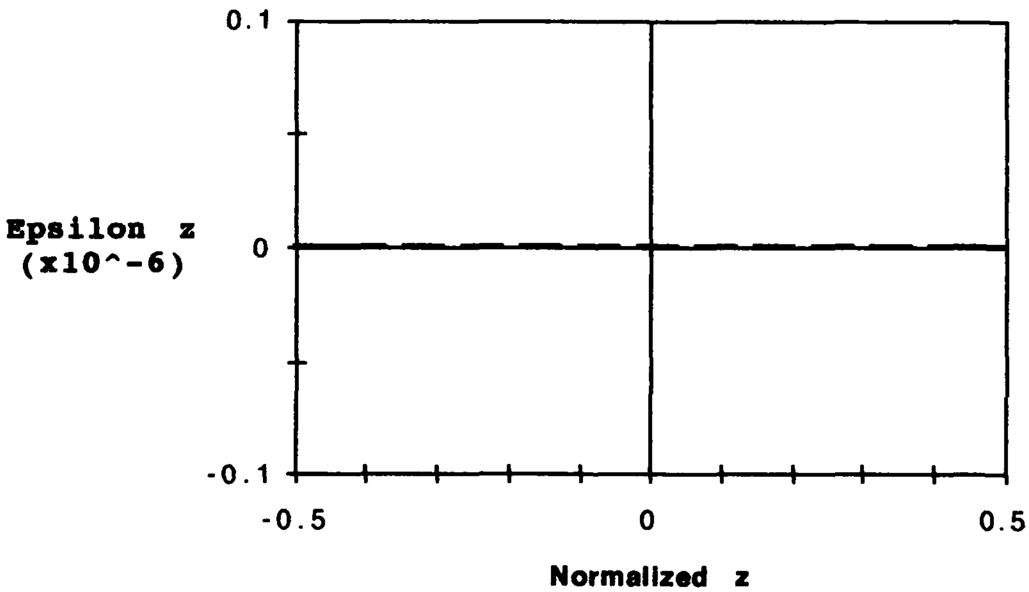

Figure 44. Epsilon z (e_z) Strain for a 0/90 Laminate Plate

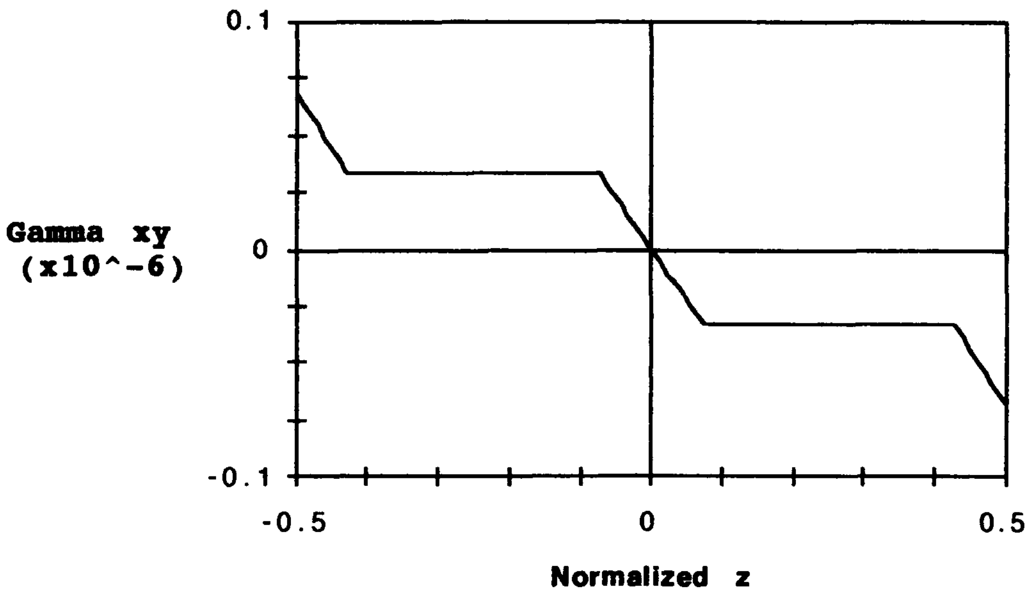

Figure 45. Gamma xy (γ_{xy}) for a 0/90 Laminate Plate

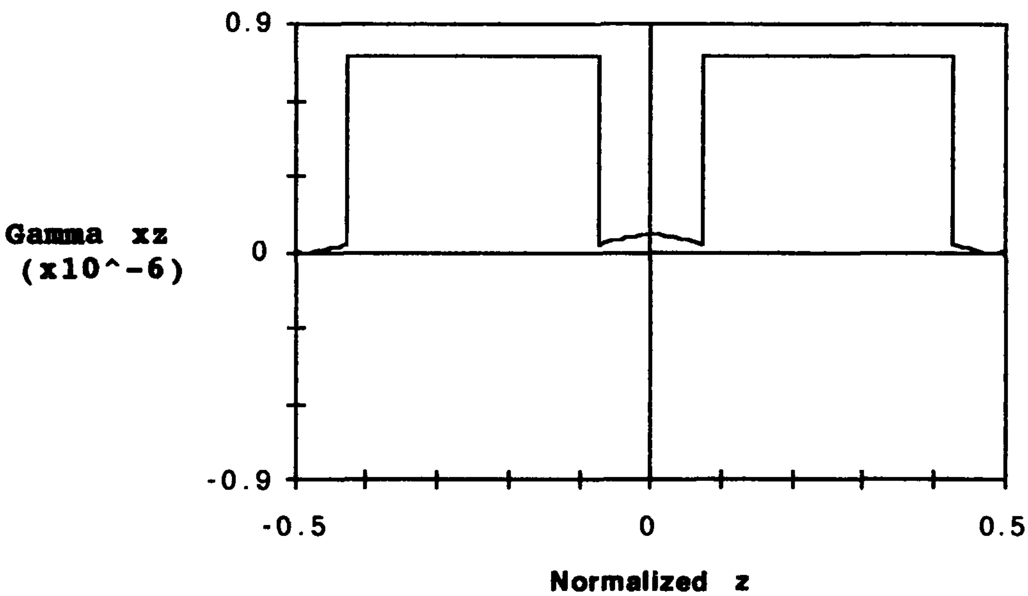

Figure 46. Gamma xz (γ_{xz}) for a 0/90 Laminate Plate

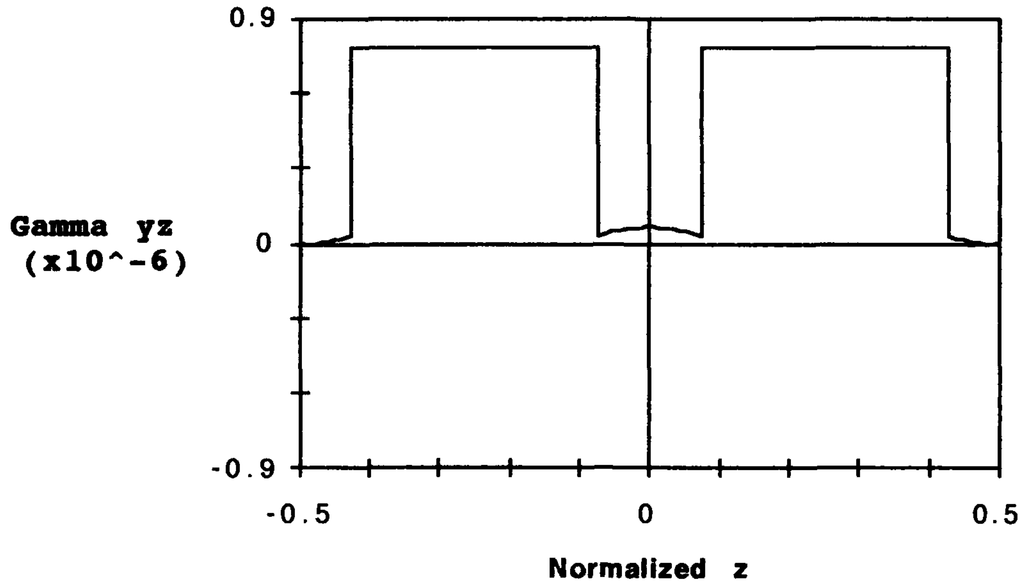

Figure 47. Gamma yz (γ_{yz}) for a 0/90 Laminate Plate

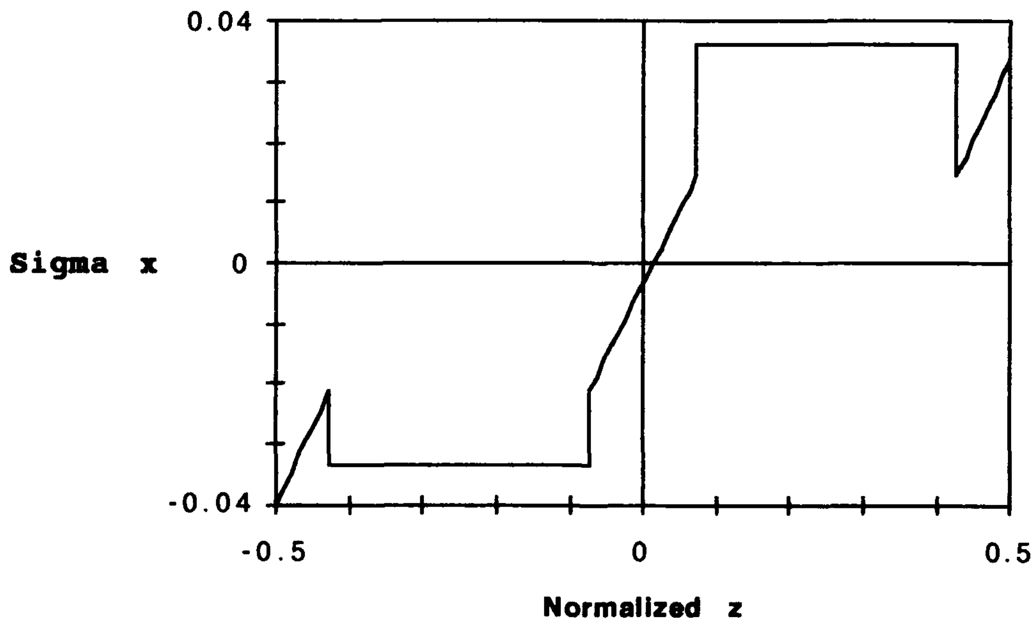

Figure 48. Sigma x (σ_x) for a 0/90 Laminate Plate

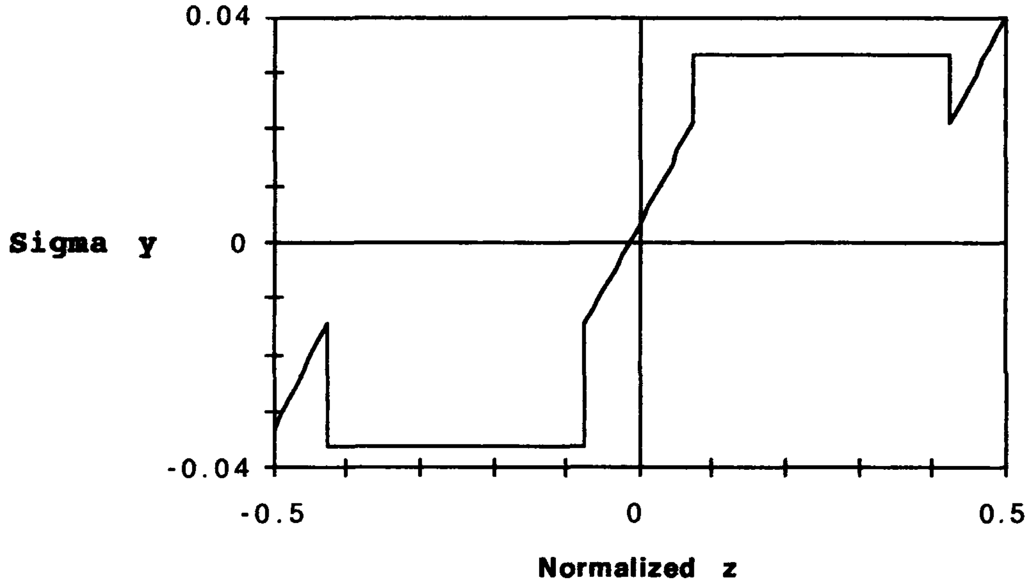

Figure 49. Sigma y (σ_y) for a 0/90 Laminate Plate

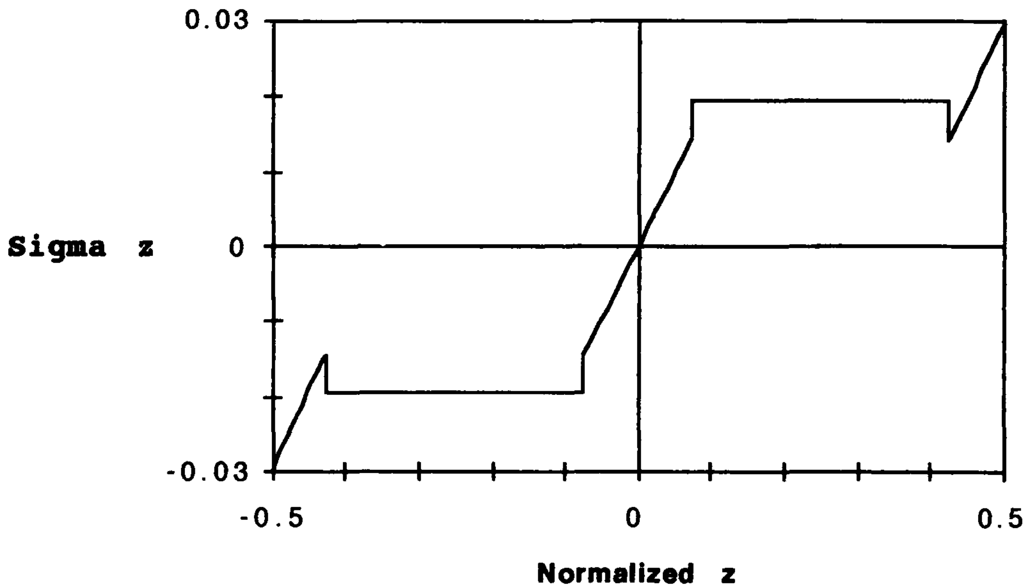

Figure 50. Sigma z (σ_z) for a 0/90 Laminate Plate

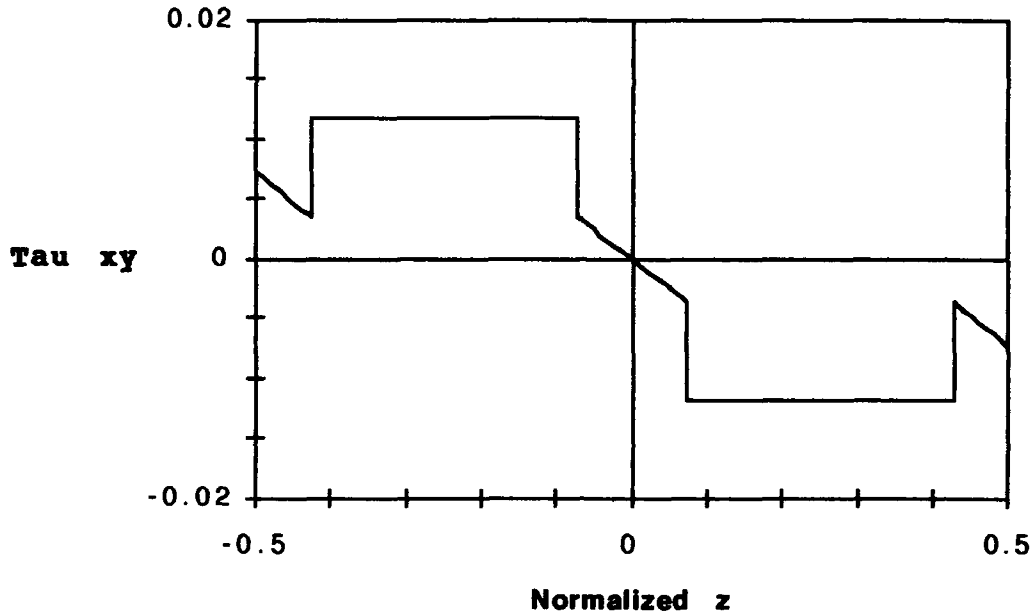

Figure 51. Tau xy (τ_{xy}) for a 0/90 Laminate Plate

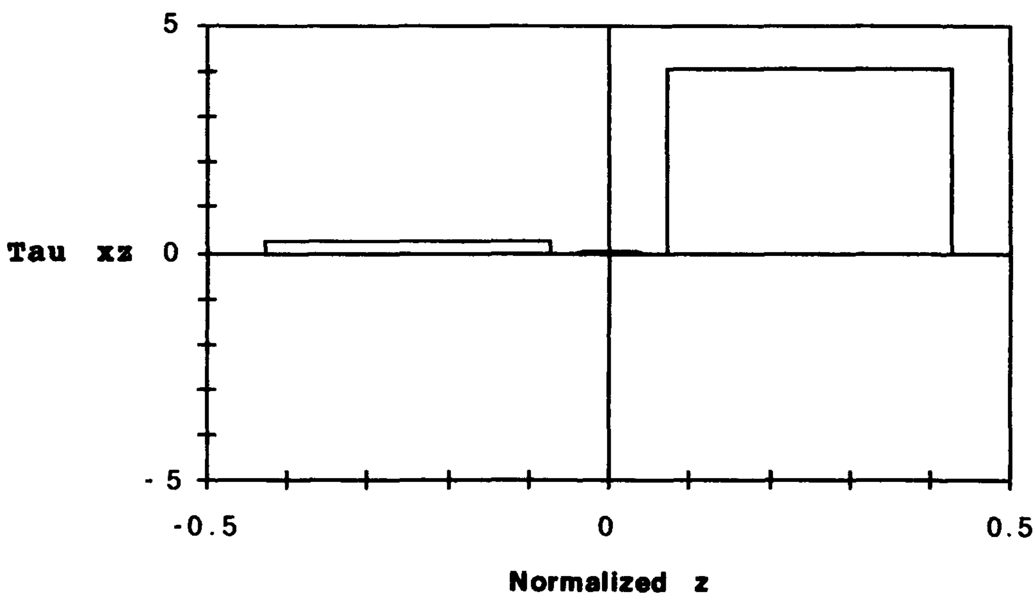

Figure 52. Tau xz (τ_{xz}) for a 0/90 Laminate Plate

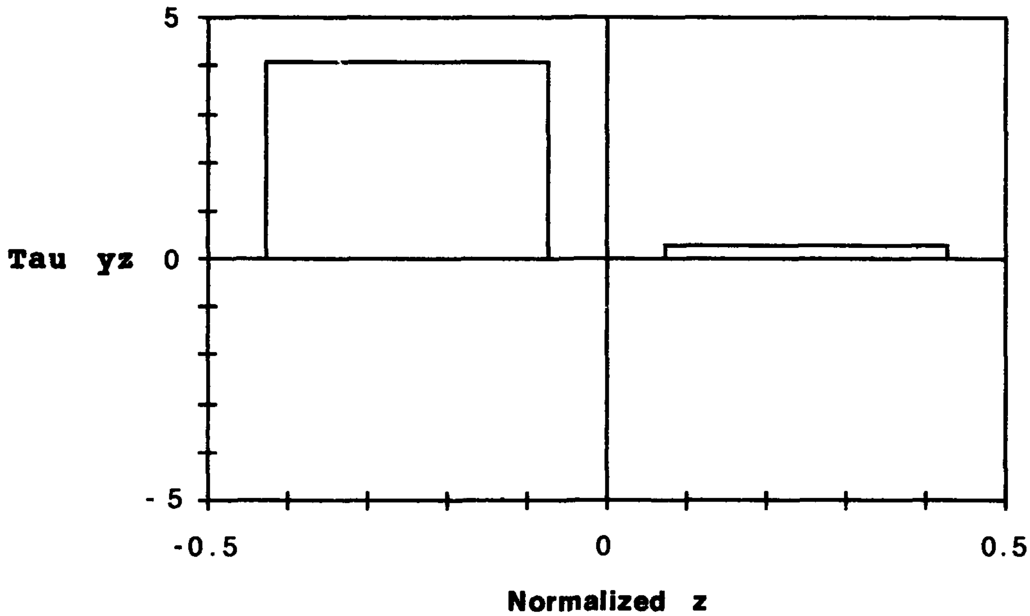

Figure 53. Tau yz (τ_{yz}) for a 0/90 Laminate Plate

Simply Supported 0/90 Composite Plate Discussion

As expected, a larger displacement and normal strain in x is experienced by the 90 degree ply, because of the orientation of the fibers. The shear strain γ_{xy} is equal in both plies because of the counter acting effects of the fiber strata. The shear strains, γ_{xz} and γ_{yz} are discontinuous functions because of the strata model assumption that the fiber strata strain does not vary in z. The maximum strain is experienced by the fiber strata in the transverse shear direction (γ_{xy}). Discontinuous stress distributions are experienced in both plies because of the presence of the stiffer fiber strata. The maximum stress is experienced by the fibers in the transverse shear direction, because of the plain strain condition.

V. *Cylindrical Bending*

The simply supported boundary conditions of sections III and IV are only useful for plates supported on all four edges. A plate supported on two opposing edges exhibits cylindrical bending where displacements and strains in one directions go to zero. This type of boundary condition can be modeled with the strata theory by letting the y dimension (b) go to infinity and examining the plate at the center [2: 57-58]. Letting b go to infinity makes the displacement v and derivatives with respect to y go to zero. This simplifies the governing equations and produces zero values for the displacement v, and the strains ε_y, τ_{xy} and τ_{yz}.

Cylindrical Bending Governing Equations

The governing equations (25-27) for a plate with cross plies in cylindrical bending reduce to:

$$\int_0^a \sum_{i=0}^{N} \sum_{r=1}^{F} \sum_{t=1}^{M} \begin{bmatrix} k_{3t}(\psi_i Z_{m,z}^2 + w_{,x} Z_{m,z}) + k_{1t}\psi_{i,xx} Z_m^2 + \\ s_{1r}\psi_{i,xx} Z_f^2 + +2 s_{5r}\psi_{i,xx} Z_f^2 \end{bmatrix} dx = 0 \tag{49}$$

$$\int_0^a \sum_{i=0}^{N} \sum_{r=1}^{F} \sum_{t=1}^{M} \left[k_{3t} w_{,xx} + k_{3t}\psi_{i,x} Z_{m,z} + s_{7r} w_{,xx} \right] dx = \int_0^a p(x) dx \tag{50}$$

$$Z_m = \int_{z_{t-}}^{z_{t+}} (\alpha_t + \beta_t z)^i dz \qquad Z_f = \int_{z_{r-}}^{z_{r+}} z_{fr}^i dz \tag{51}$$

The boundary Conditions 30-32 reduce to

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=0}^{N}\left[\int_0^b \delta\psi_i \left[k_{1t}\psi_{i,x}Z_m^2 + s_{1r}\psi_{i,x}Z_f^2 \right]_0^a dy\right] = 0 \tag{52}$$

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=0}^{N}\left[\delta w\left[k_{3t}\psi_i Z_{m,z} + k_{3t}w_{,x}Z_m + s_{7r}w_{,x}Z_f + s_{9r}w_{,x}Z_f \right]_0^a\right] = 0 \tag{53}$$

The ψ, ζ, w and p functions of equation 40 can be used if the y dependence is deleted. The x and y dependence takes on the following form

$$\psi_i(x,y) = A_i \bullet \mathrm{Cos}(px)$$
$$w(x,y) = C \bullet \mathrm{Sin}(px)$$
$$p(x,y) = P \bullet \mathrm{Sin}(px)$$

$$i = 0,1,..N \tag{54}$$

Using the above relationship in equations 49-51, allows the x and y dependence derivatives to be factored out, and the system of equations solved for the A_i and C coefficients.

The boundary conditions for a plate in cylindrical bending have v, w and σ_x equal to zero at x=0 and x=a. This satisfies boundary condition 53. A simply supported plate has zero curvature at x=0 and x=a, so the $\psi_{i,x}$ and $\zeta_{i,x}$ terms in equation 52 go to zero satisfying the boundary condition.

0/0 Composite Plate in Cylindrical Bending Example

The same sample plate described in Table 2 and 3 is used for cylindrical bending accept that the y dimension (b) is set at infinity. The displacements, strains and stresses of a composite plate with 0/0 plies in cylindrical bending are presented in figures 53-60. The x and y dependence has been factored out to give the through the thickness distributions.

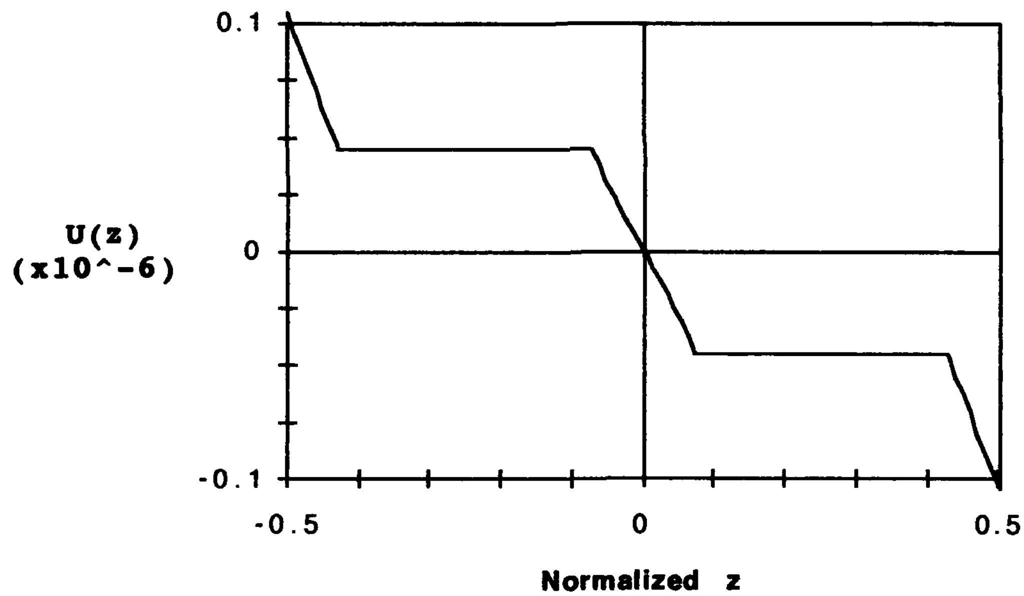

Figure 54. U(z) for a 0/0 Plate in Cylindrical Bending

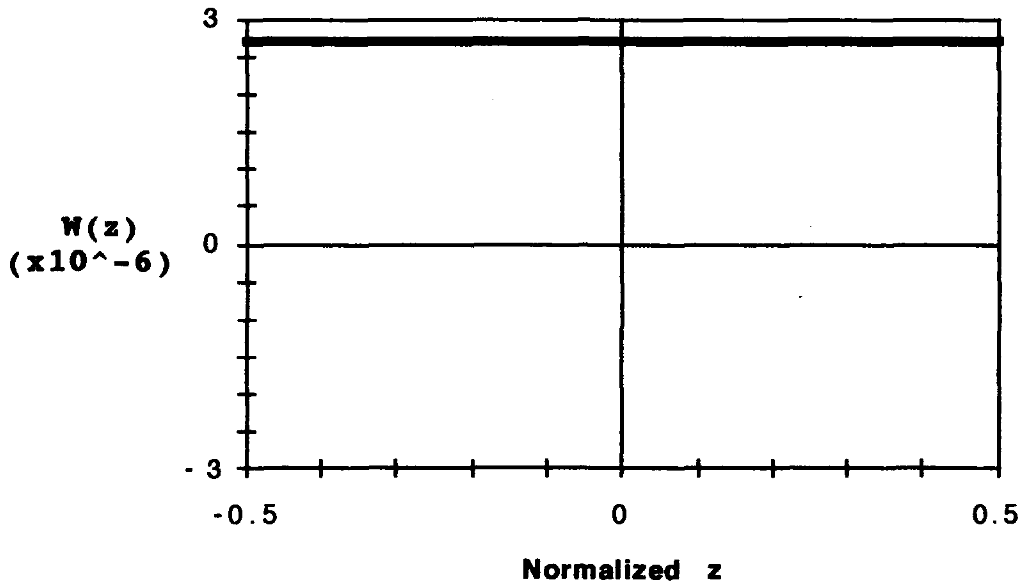

Figure 55. W(z) for a 0/0 Plate in Cylindrical Bending

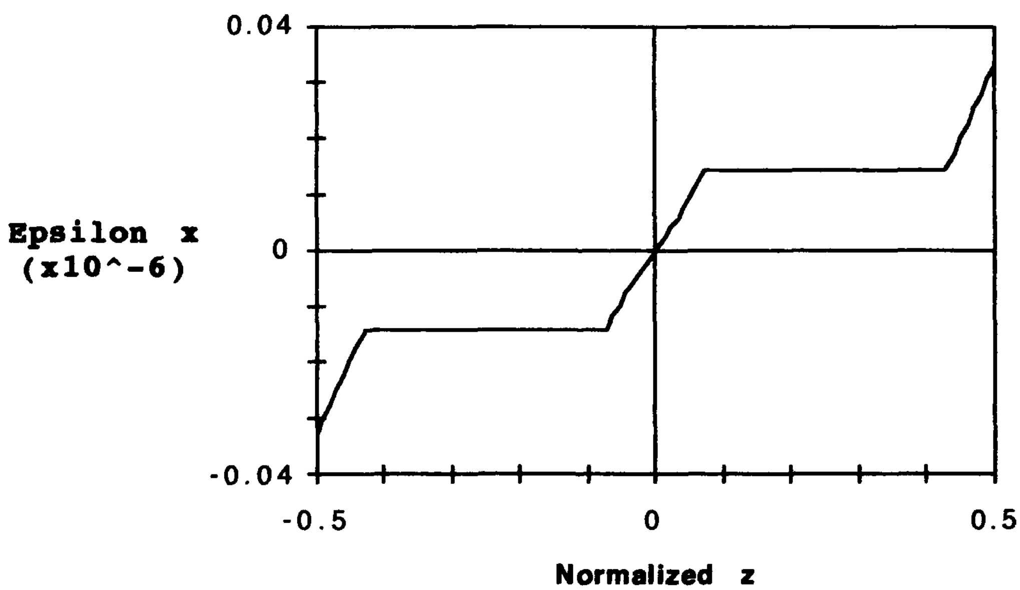

Figure 56. $\varepsilon_x(z)$ for a 0/0 Plate in Cylindrical Bending

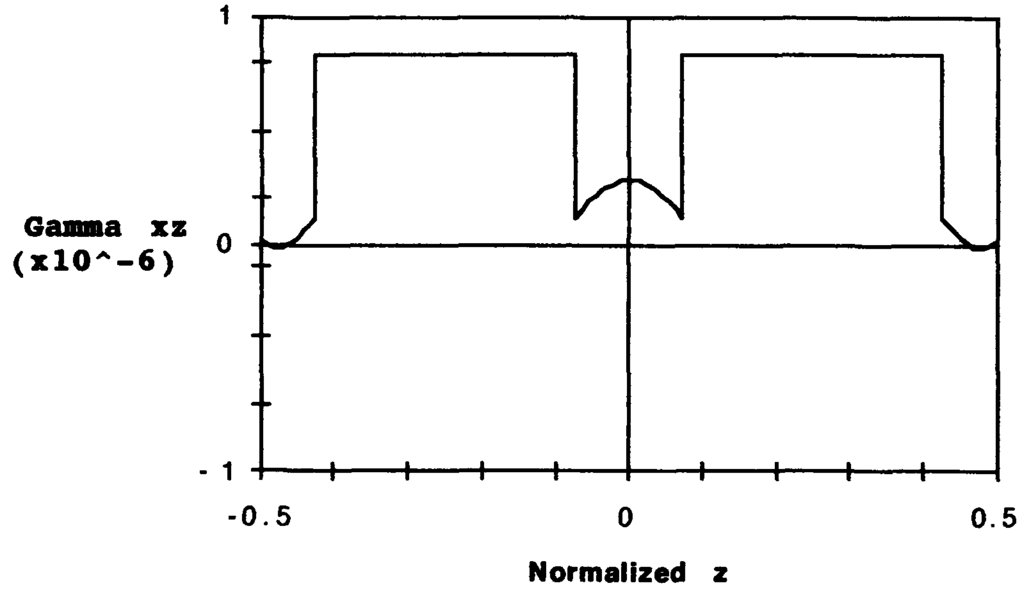

Figure 57. $\gamma_{xz}(z)$ for a 0/0 Plate in Cylindrical Bending

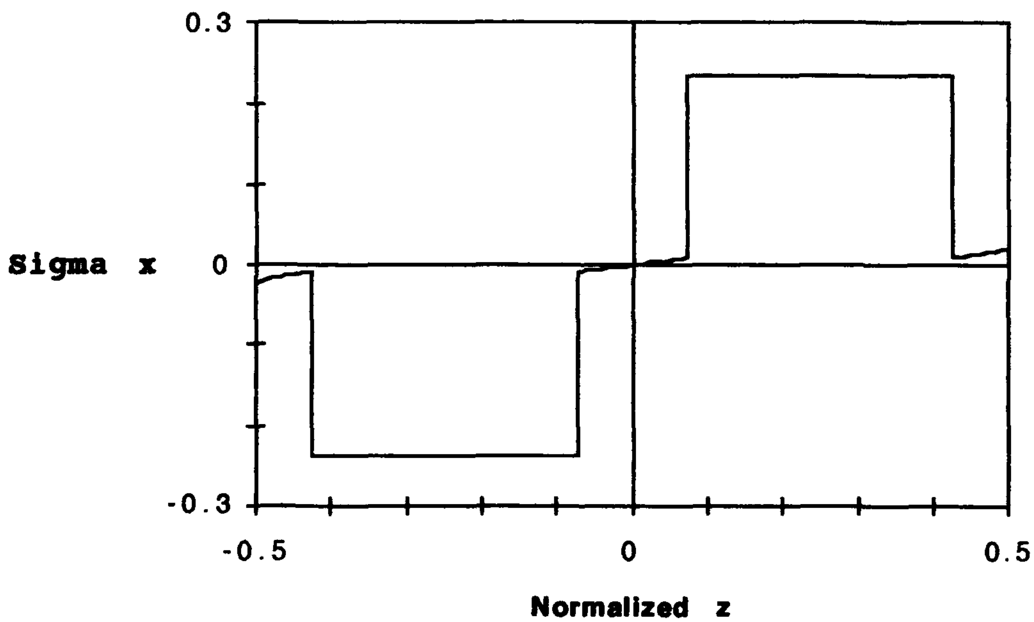

Figure 58. $\sigma_x(z)$ for a 0/0 Plate in Cylindrical Bending

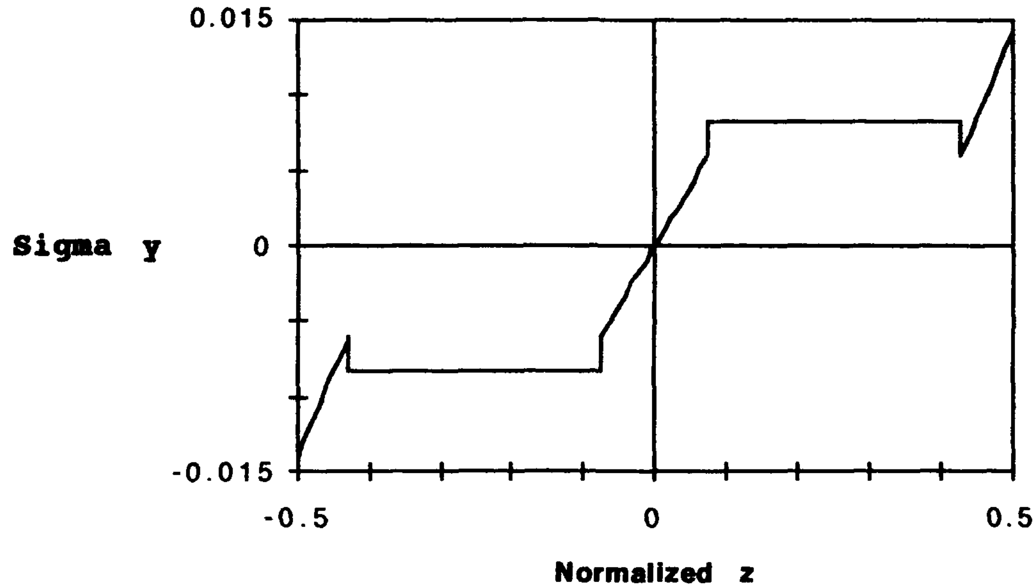

Figure 59. $\sigma_y(z)$ for a 0/0 Plate in Cylindrical Bending

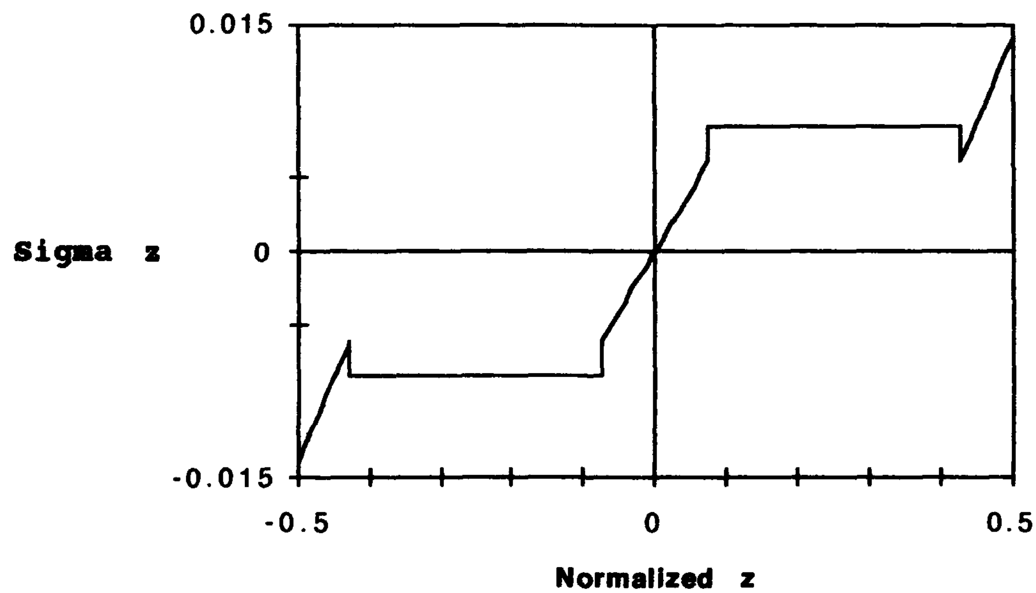

Figure 60. $\sigma_z(z)$ for a 0/0 Plate in Cylindrical Bending

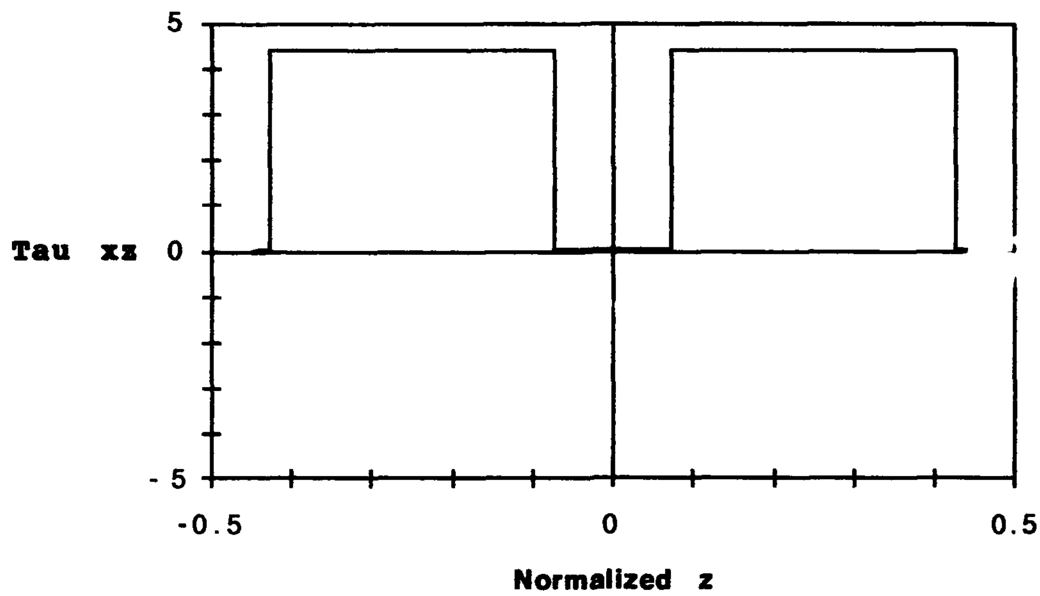

Figure 61. τ_{xz} for a 0/0 Plate in Cylindrical Bending

Cylindrical Bending 0/0 Plate Discussion

The displacements shown in figures 52-53 are expected for a simply supported composite plate in cylindrical bending. The outer matrix strata exhibit greater displacements than the stiffer fiber strata. The ductile matrix's, displacement increases with the distance from the mid plane. The presents of the stiffer fibers creates a discontinuous stress distribution in all directions. The plate still experiences stress in the y direction from a Poisson's contraction effect. The maximum normal stress is experienced by the fiber strata while the maximum normal strain is experienced by the outer matrix strata.

0/90 Composite Plate in Cylindrical Bending Example

The same sample plate described in Table 2 and 3 is used for cylindrical bending accept that the y dimension (b) is set at infinity and θ_2 is set a 90 degrees. The displacements, strains and stresses, independent of x and y, of a composite plate with 0/90 plies in cylindrical bending are presented in figures 61-68.

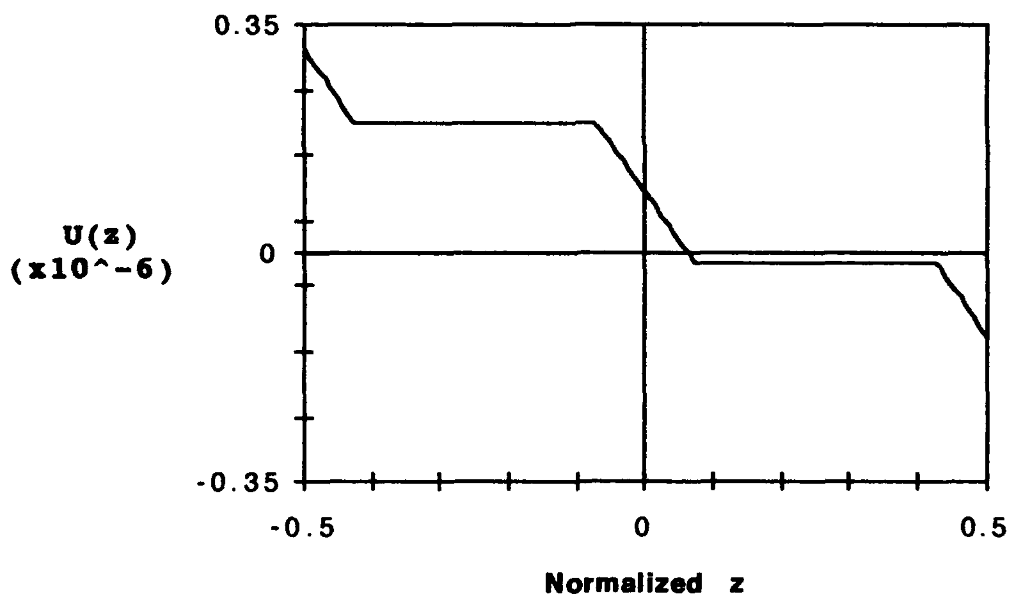

Figure 62. U(z) for a 0/90 Plate in Cylindrical Bending

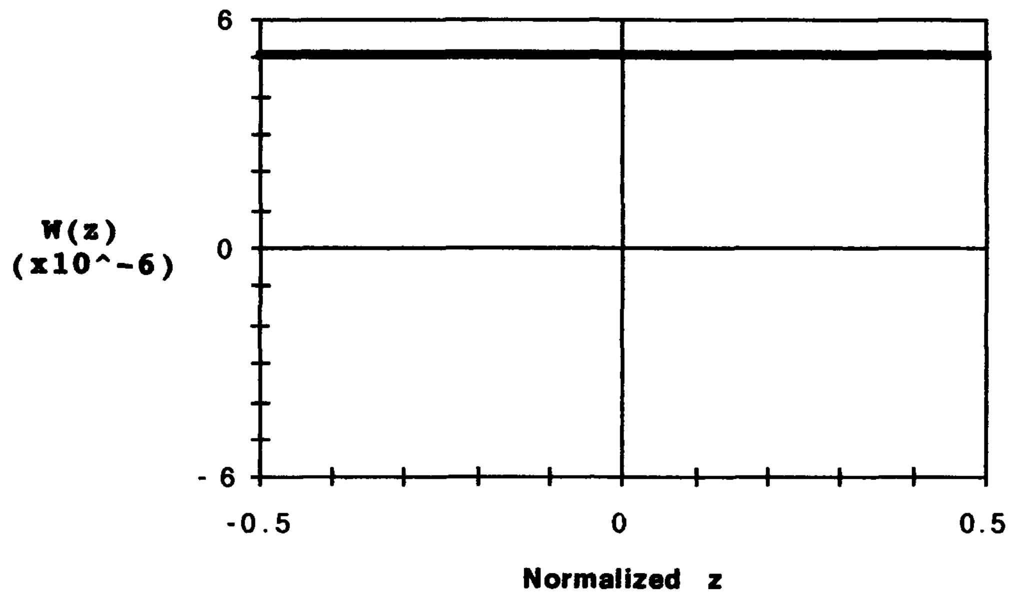

Figure 63. W(z) for a 0/90 Plate in Cylindrical Bending

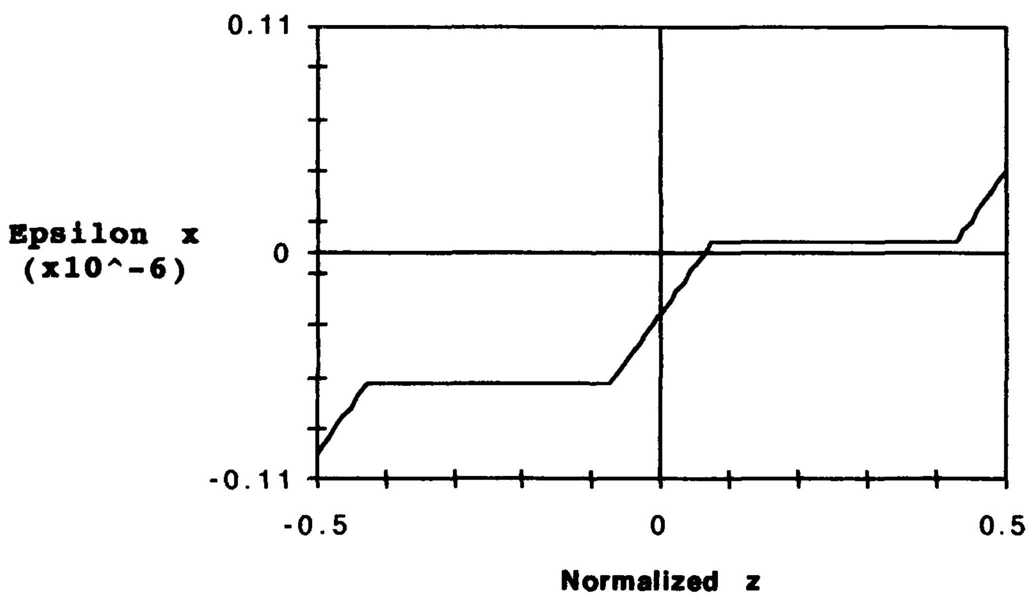

Figure 64. $\varepsilon_x(z)$ for a 0/90 Plate in Cylindrical Bending

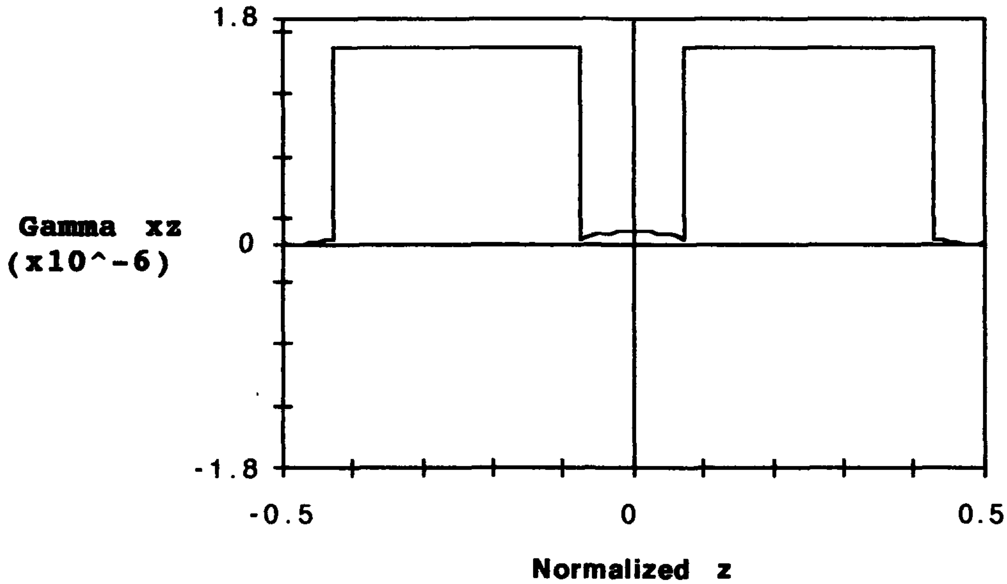

Figure 65. γ_{xz} for a 0/90 Plate in Cylindrical Bending

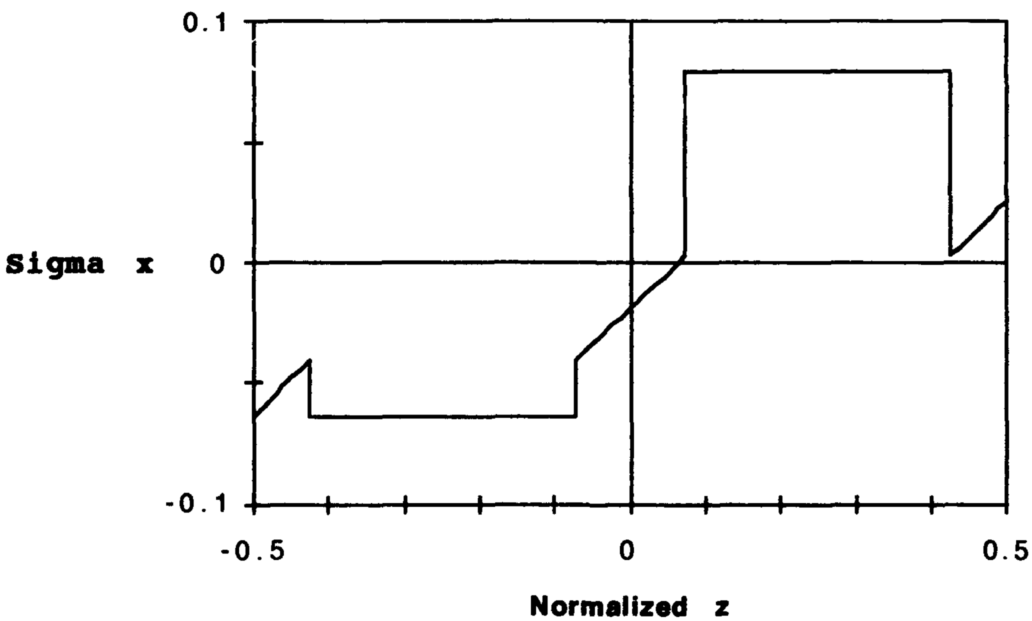

Figure 66. $\sigma_x(z)$ for a 0/90 Plate in Cylindrical Bending

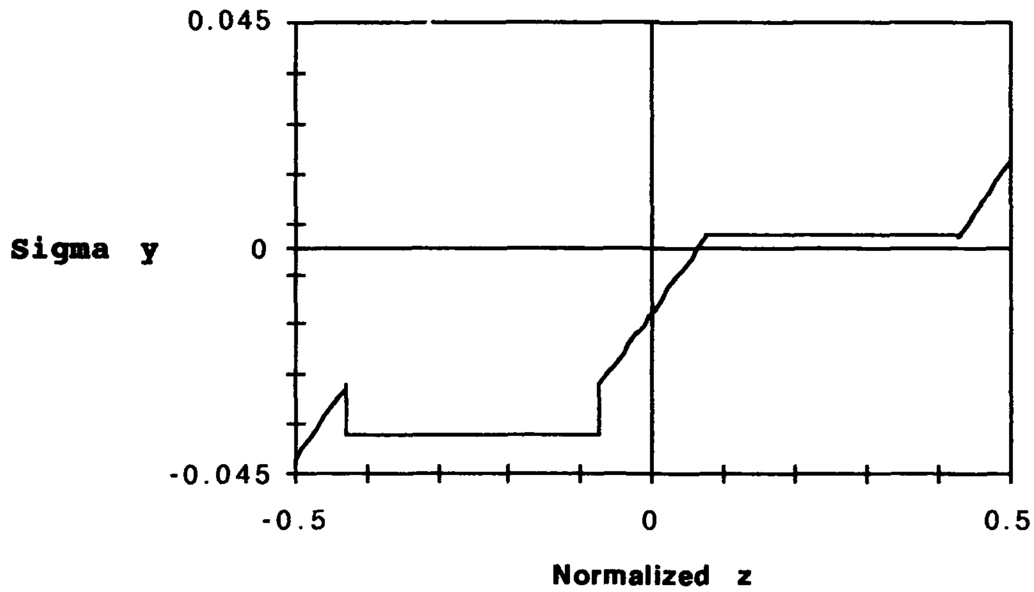

Figure 67. $\sigma_y(z)$ for a 0/90 Plate in Cylindrical Bending

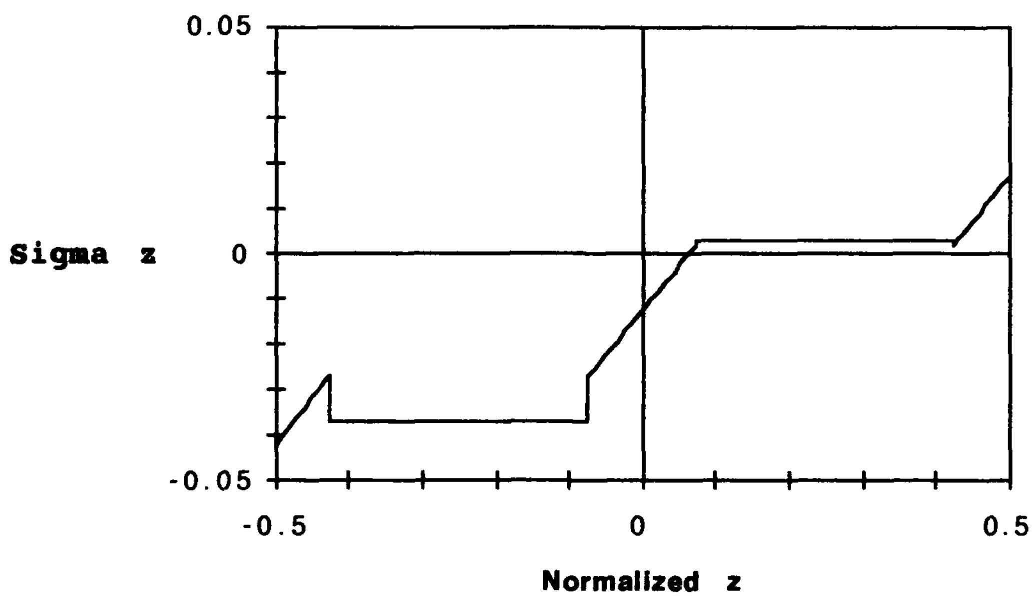

Figure 68. $\sigma_z(z)$ for a 0/90 Plate in Cylindrical Bending

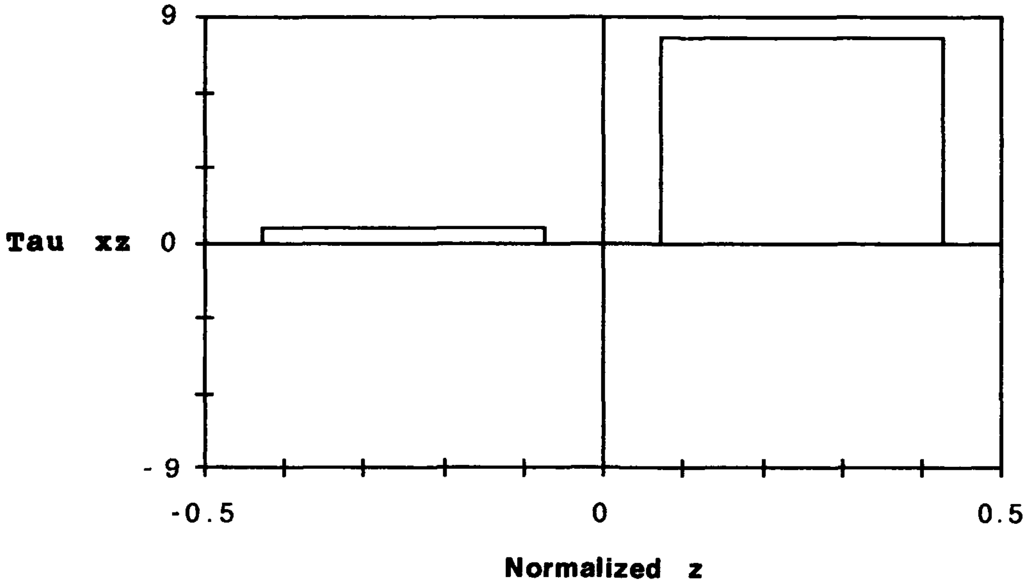

Figure 69. τ_{xz}(.' for a 0/90 Plate in Cylindrical Bending

Cylindrical Bending 0/90 Plate Discussion

As expected, a larger normal displacement and strain in x is experienced by the 90 degree ply, because of the orientation of the fibers. The reverse is true for the 0 degree ply in the y direction. Once again we get discontinuous γ_{xz} because of the zero strain gradient assumption in z. Discontinuous stress distributions are experienced in both plies because of the presence of the stiffer fiber strata. The maximum stress is experienced by the fibers in the transverse shear direction.

VI. *Ritz Polynomial Solution to Composites with Angle Plies*

With angle plies included in the governing equations the fiber strata stiffness matrix coefficients s_5, s_6 and s_9 remain in the governing equations. With the Sine/Cosine Ψ and ζ functions, the x and y dependent derivatives are not constant in each of the equations and can not be factored out. Complete functions to represent the x and y dependence of the displacement are required to solve the governing equations. A Ritz polynomial series can be used for the x and y dependence functions Ψ and ζ. The x and y dependent derivatives can then be evaluated and integrated out of the governing equations and the equations solved. The Ritz displacement relationships are represented as follows

$$\psi_{i,j}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{N} A_{i,j} \cdot x^i y^j$$

$$\zeta_{i,j}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{N} B_{i,j} \cdot x^i y^j$$

$$w_{i,j}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{N} C_{i,j} \cdot x^i y^j \qquad (55)$$

The double series can be transformed into a single series of the form:

$$\psi_i(x,y) = \sum_{i=1}^{N^2} A_i \cdot f_i(x,y)$$

$$\zeta_i(x,y) = \sum_{i=1}^{N^2} B_i \cdot f_i(x,y)$$

$$w_i(x,y) = \sum_{i=1}^{N^2} C_i \cdot f_i(x,y)$$

$$p(x,y) = P \tag{56}$$

Different applied loads p(x,y) can also be represented by a Ritz polynomial series. A single constant load term independent of x and y represents a constant distributed pressure.

For the Ritz polynomial displacement assumptions the governing equations (25-28) become:

$$\int_0^a\int_0^b \sum_{i=1}^{N^2}\sum_{r=1}^{F}\sum_{t=1}^{M} \begin{bmatrix} k_{3t}(\psi_{i,yy}Z_m^2 + \psi_i Z_{m,z}^2 + w_{i,x}Z_{m,z}) + \\ (k_{2t} + k_{3t})\zeta_{i,xy}Z_f^2 + k_{1t}\psi_{i,xx}Z_m^2 + \\ (s_{3r} + s_{4r})\zeta_{i,xy}Z_f^2 + s_{4r}\psi_{i,yy}Z_f^2 + \\ s_{1r}\psi_{i,xx}Z_f^2 + s_{5r}\zeta_{i,xx}Z_f^2 + \\ 2s_{5r}\psi_{i,xx}Z_f^2 + s_{6r}\zeta_{i,xx}Z_f^2 \end{bmatrix} dydx = 0 \tag{57}$$

$$\int_0^a\int_0^b \sum_{i=1}^{N^2}\sum_{r=1}^{F}\sum_{t=1}^{M} \begin{bmatrix} k_{3t}(\zeta_{i,xx}Z_m^2 + \zeta_i Z_{m,z}^2 + w_{i,y}Z_{m,z}) + \\ (k_{2t} + k_{3t})\psi_{i,xy}Z_f^2 + k_{1t}\zeta_{i,yy}Z_m^2 + \\ (s_{3r} + s_{4r})\psi_{i,xy}Z_f^2 + s_{4r}\zeta_{i,xx}Z_f^2 + \\ s_{2r}\zeta_{i,yy}Z_f^2 + s_{5r}\psi_{i,xx}Z_f^2 + \\ s_{6r}(\psi_{i,yy}Z_f^2 + 2\zeta_{i,xy}Z_f^2) \end{bmatrix} dydx = 0 \tag{58}$$

$$\int_0^a\int_0^b \sum_{i=1}^{N^2}\sum_{r=1}^{F}\sum_{t=1}^{M}\begin{bmatrix} k_{3t}(w_{i,xx} + w_{i,yy}) + \\ k_{3t}(\psi_{i,x} + \zeta_{i,y})Z_{m,z} + \\ s_{7r}w_{i,xx} + s_{8r}w_{i,yy} + 2s_{9r}w_{i,yy}\end{bmatrix} dydx = \int_0^a\int_0^b p(x,y)dydx \tag{59}$$

where

$$Z_m = \int_{z_{t-}}^{z_{t+}} (\alpha_t + \beta_t z)^i dz \qquad Z_f = \int_{z_{r-}}^{z_{r+}} z_{fr}^i dz \tag{60}$$

The boundary conditions of equations 30-32, for a Ritz Polynomial, become:

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=1}^{N^2}\begin{bmatrix} \int_0^b \delta\psi_i \begin{bmatrix} k_{1t}\psi_{i,x}Z_m^2 + k_{2t}\zeta_{i,y}Z_m^2 + s_{1r}\psi_{i,x}Z_f^2 + \\ s_{3r}\zeta_{i,y}Z_f^2 + s_{5r}(\psi_{i,y} + \zeta_{i,x})Z_f^2 \end{bmatrix}_0^a dy + \\ \int_0^a \delta\psi_i \begin{bmatrix} k_{3t}(\psi_{i,y} + k_{3t}\zeta_{i,x})Z_m^2 + s_{5r}\psi_{i,x}Z_f^2 + \\ s_{6r}\zeta_{i,y}Z_f^2 + s_{4r}(\psi_{i,y} + \zeta_{i,x})Z_f^2 \end{bmatrix}_0^b dx \end{bmatrix} = 0 \tag{61}$$

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=1}^{N^2}\begin{bmatrix} \int_0^b \delta\zeta_i \begin{bmatrix} k_{3t}\psi_{i,y}Z_m^2 + k_{3t}\zeta_{i,x}Z_m^2 + s_{5r}\psi_{i,x}Z_f^2 + \\ s_{4r}(\psi_{i,y} + \zeta_{i,x})Z_f^2 + s_{6r}\zeta_{i,y}Z_f^2 \end{bmatrix}_0^a dy + \\ \int_0^a \delta\zeta_i \begin{bmatrix} k_{2t}\psi_{i,x}Z_m^2 + k_{1t}\zeta_{i,x}Z_m^2 + s_{3r}\psi_{i,x}Z_f^2 + \\ s_{6r}(\psi_{i,y} + \zeta_{i,x})Z_f^2 + s_{2r}\zeta_{i,y}Z_f^2 \end{bmatrix}_0^b dx \end{bmatrix} = 0 \tag{62}$$

$$\sum_{t=1}^{M}\sum_{r=1}^{F}\sum_{i=1}^{N^2}\left[\begin{matrix}\int_0^b \delta w_i \begin{bmatrix} k_{3t}\Psi_i Z_{m,z} + k_{3t} w_{i,x} Z_m + \\ s_{7r} w_{i,x} Z_f + s_{9r} w_{i,x} Z_f \end{bmatrix}_0^a dy + \\ \int_0^a \delta w_i \begin{bmatrix} k_{3t}\zeta_i Z_{m,z} + k_{3t} w_{i,y} Z_m + \\ s_{9r} w_{i,x} Z_f + s_{8r} w_{i,y} Z_f \end{bmatrix}_0^b dx \end{matrix}\right] = 0 \tag{63}$$

The Ritz polynomial displacement functions do not satisfy the boundary conditions of a plate, simply supported on four sides. The displacement of the mid surface at x=a and y=b are not zero. The rotations Ψ and ζ, at x=0 and y=0 are equal to zero. The Ritz polynomial represents a plate, fixed or clamped on two sides at x=0 and y=0. This satisfies the conditions that Ψ=u=0 at x=0 and ζ=v=0 at y=0. Also, at the free edge x=a, u and Ψ do not equal zero but $\Psi_{,x}=0$ and $\Psi_{,y}=0$ and at y=b, $\zeta_{,x}=0$ and $\zeta_{,y}=0$. This satisfies boundary conditions 61 and 62. Boundary condition 63 is satisfied by the condition that w=0 at x=0 and y=0. Only the geometric boundary conditions are satisfied by the Ritz Polynomial displacement assumptions.

The governing equations can be transformed into a linear system of equations as follows:

$$[M]\begin{Bmatrix} A_i \\ B_i \\ C_i \end{Bmatrix} = \begin{Bmatrix} 0_i \\ 0_i \\ p \end{Bmatrix} \qquad i = 0..N^2 \tag{64}$$

M will be $[2*N^2+1]$ x $[3*N^2]$ matrix while the A_i, B_i, and C_i matrix will be $3*N^2$ vector and the right hand matrix will be a $2*N^2+1$

vector. The x, y and z integrals of the governing equations can be evaluated directly leaving a system of equations as function of the A_i, B_i and C_i coefficients only. Guassian elimination can be used to solve for the coefficients A_i, B_i, and C_i. The coefficients are then used to generate the displacements, strains and stress throughout the laminate.

The polynomial order N, for a Ritz polynomial displacement, must include enough terms to represent the complex strain derivatives. A higher order of N is required for a Ritz polynomial than a Sine/Cosine function to properly represent the displacement. An N of six creates 36th order polynomial and 73 X 108 [M] matrix. The Ritz polynomial solution does not converge monotonically on the correct solution as N is increased. The solutions generated oscillate between very good and very poor.

Simply Supported 0/60 Composite Plate Example

The same sample plate described in Table 2 and 3 is used for the Ritz polynomial angle ply example accept that the θ_2 is 60 degrees. An N of 6 was used creating 36th order polynomials in each direction. The displacements, strains and stresses of a simply supported composite plate with 0/60 plies are presented in figures 68-82. The x and y dependence has been factored out to give the through the thickness distributions.

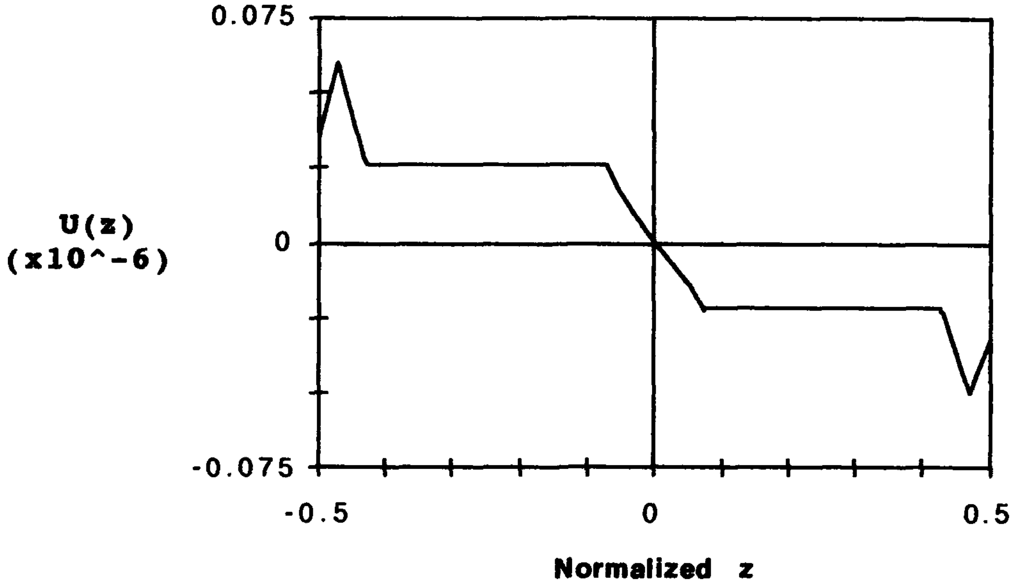

Figure 70. U(z) Displacement for a 0/60 Laminated Plate

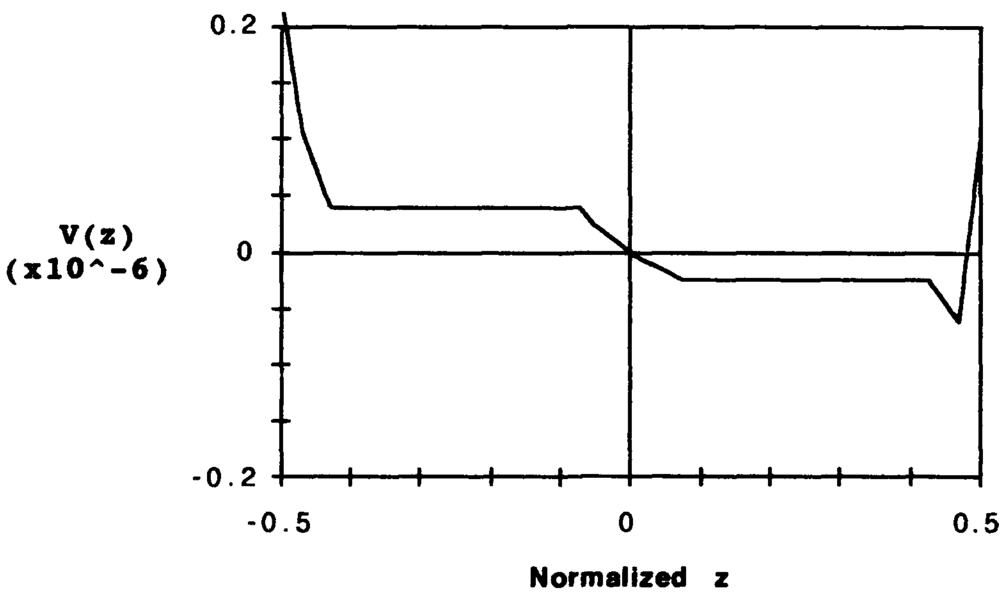

Figure 71. V(z) Displacement for a 0/60 Laminated Plate

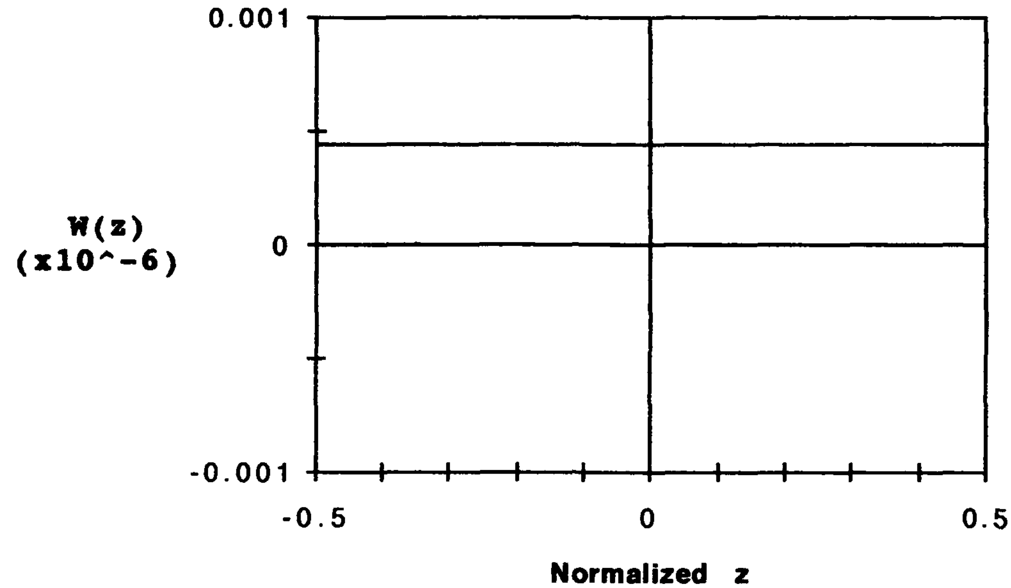

Figure 72. W(z) Displacement for a 0/60 Laminated Plate

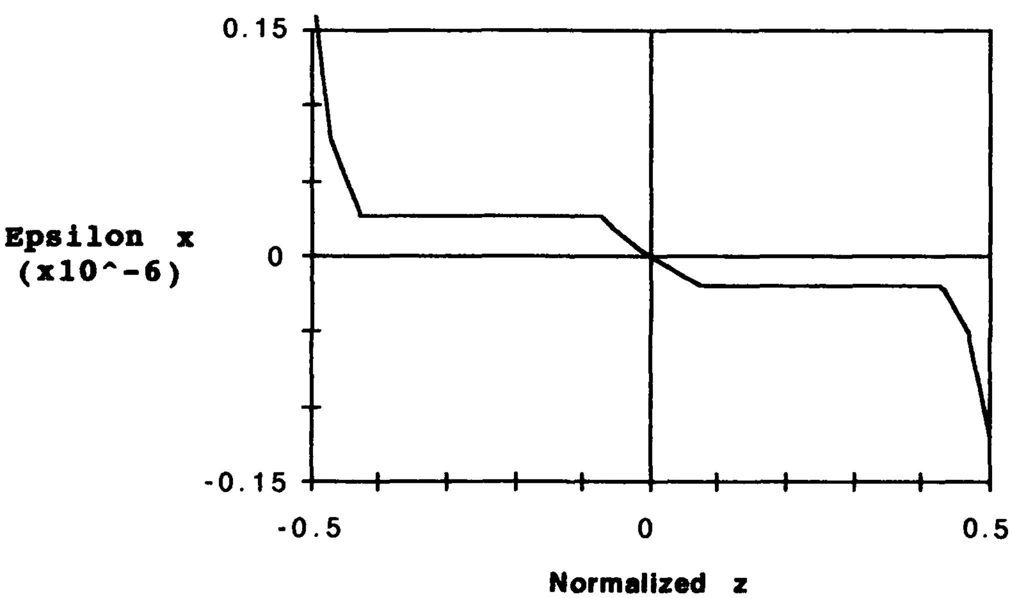

Figure 73. Epsilon (ε_x) for a 0/60 Laminated Plate

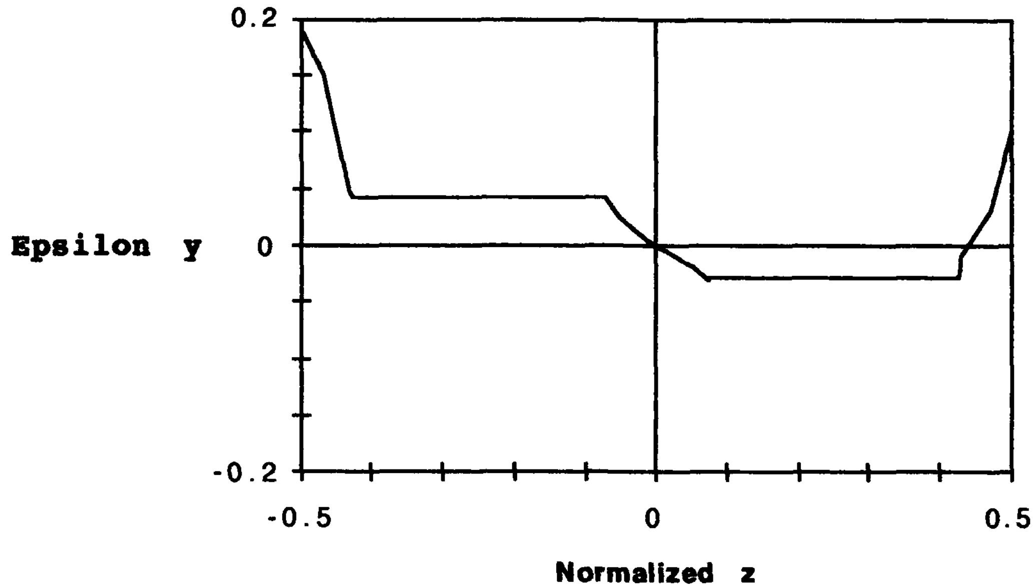

Figure 74. Epsilon (ε_y) for a 0/60 Laminated Plate

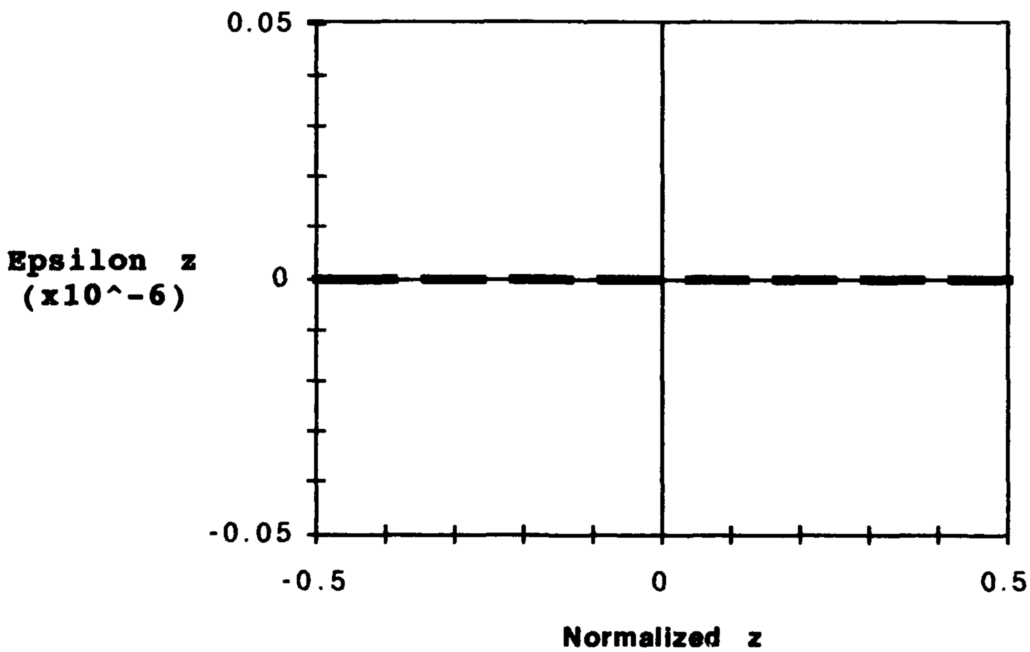

Figure 75. Epsilon (ε_z) for a 0/60 Laminated Plate

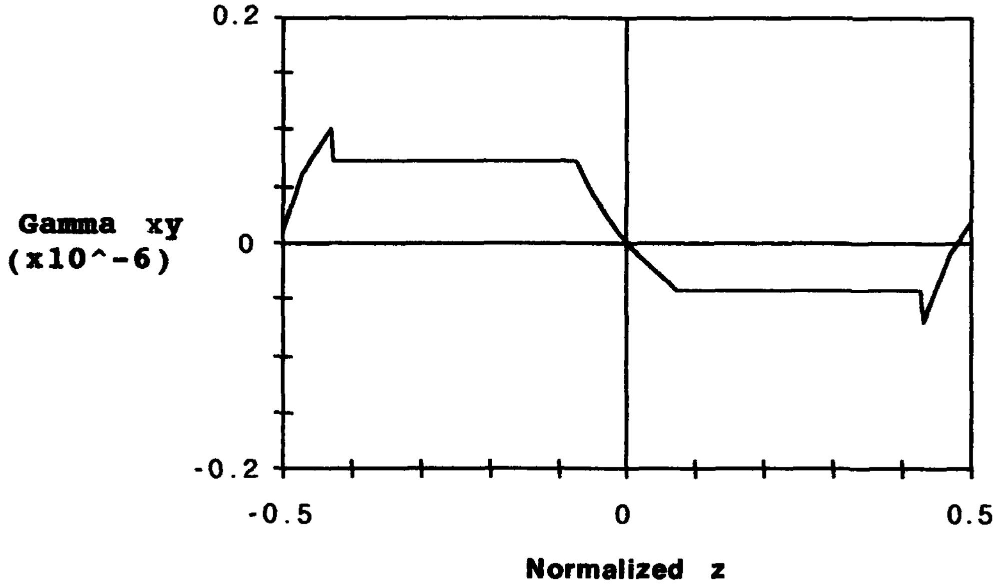

Figure 76. Gamma (γ_{xy}) for a 0/60 Laminated Plate

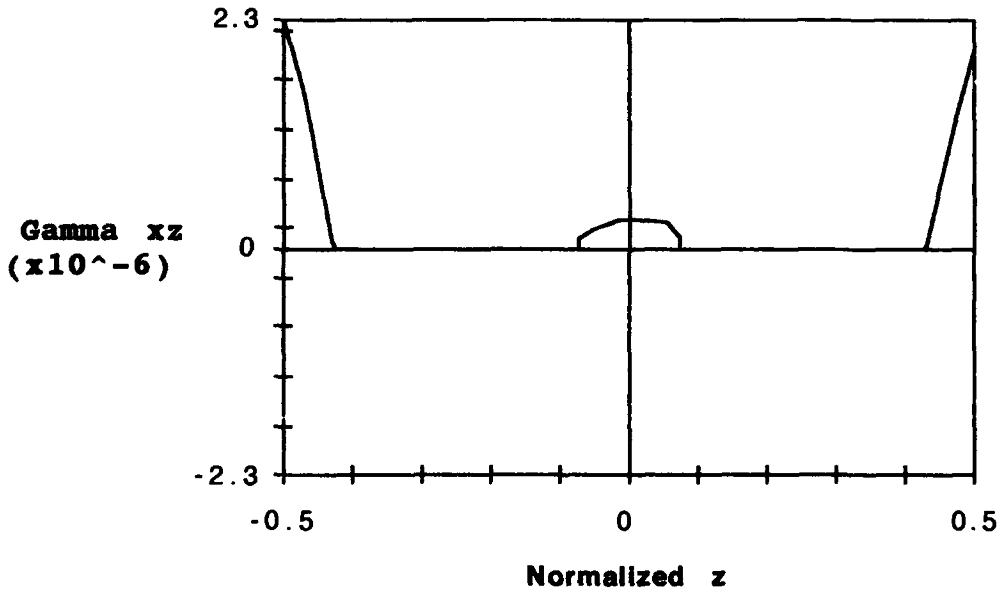

Figure 77. Gamma (γ_{xz}) for a 0/60 Laminated Plate

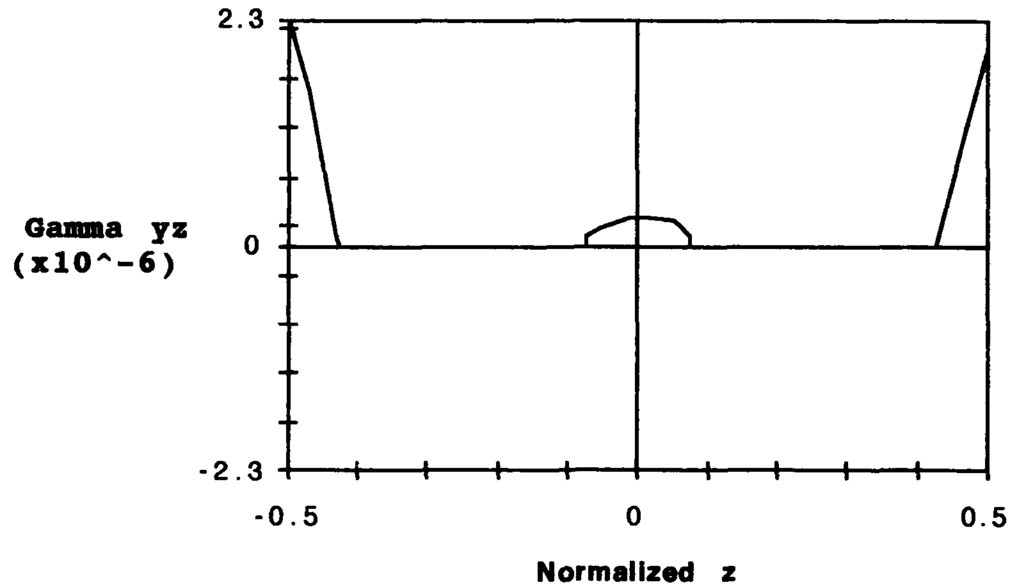

Figure 78. Gamma (γ_{yz}) for a 0/60 Laminated Plate

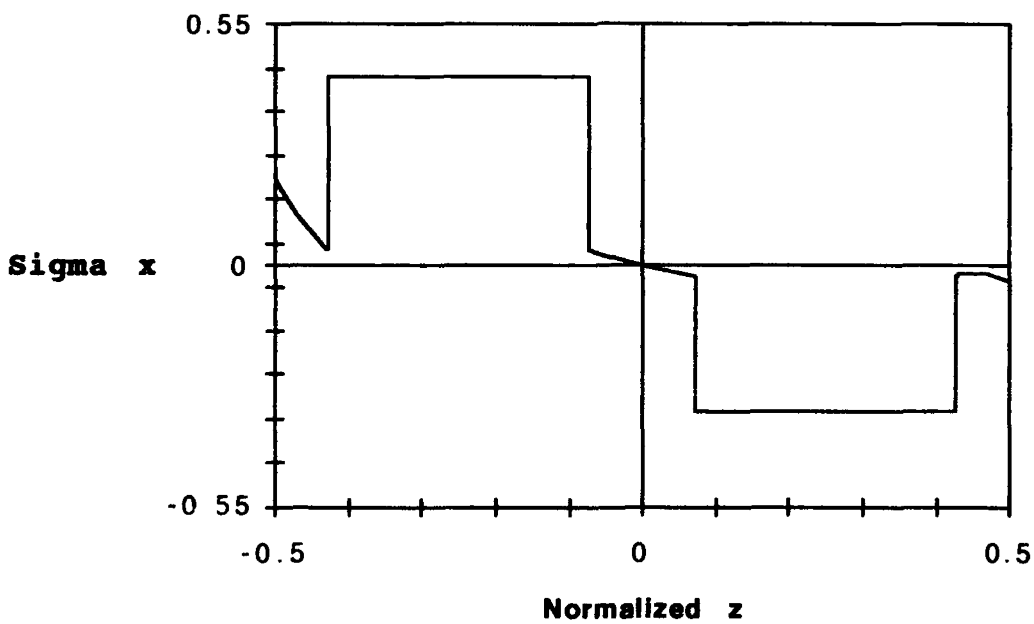

Figure 79. Sigma (σ_x) for a 0/60 Laminated Plate

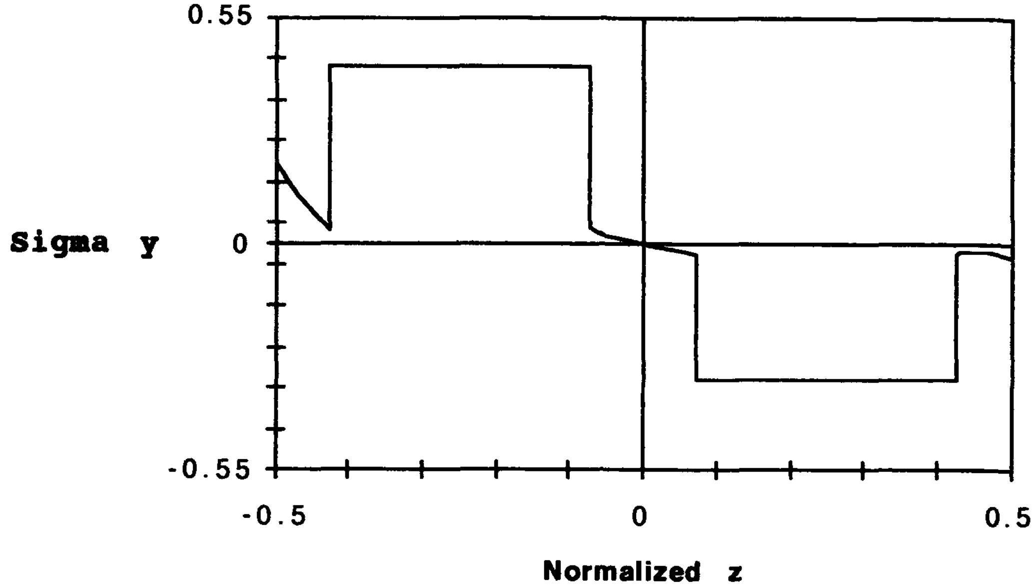

Figure 80. Sigma (σ_y) for a 0/60 Laminated Plate

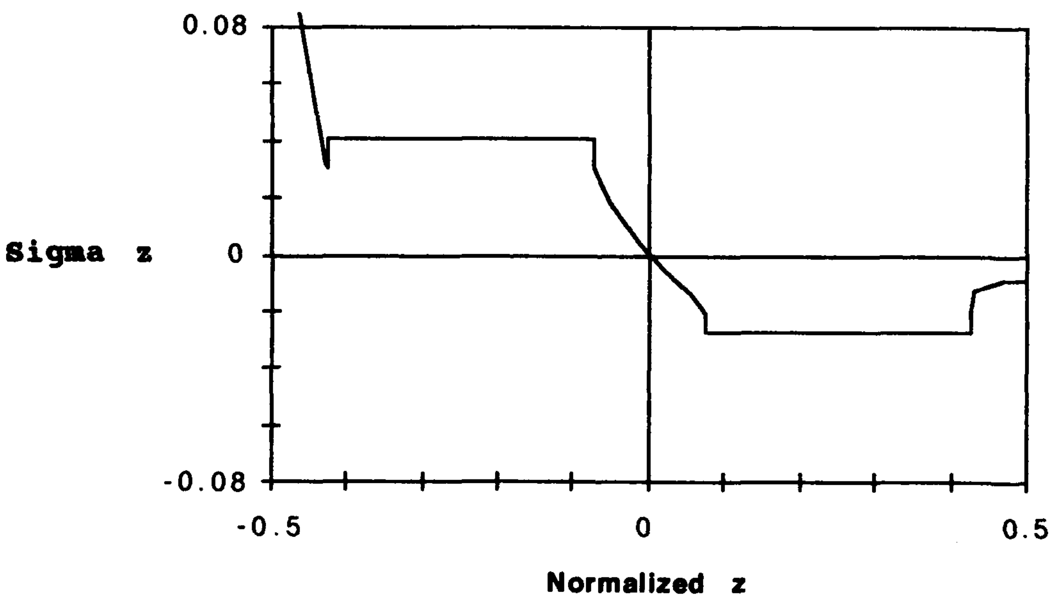

Figure 81. Sigma (σ_z) for a 0/60 Laminated Plate

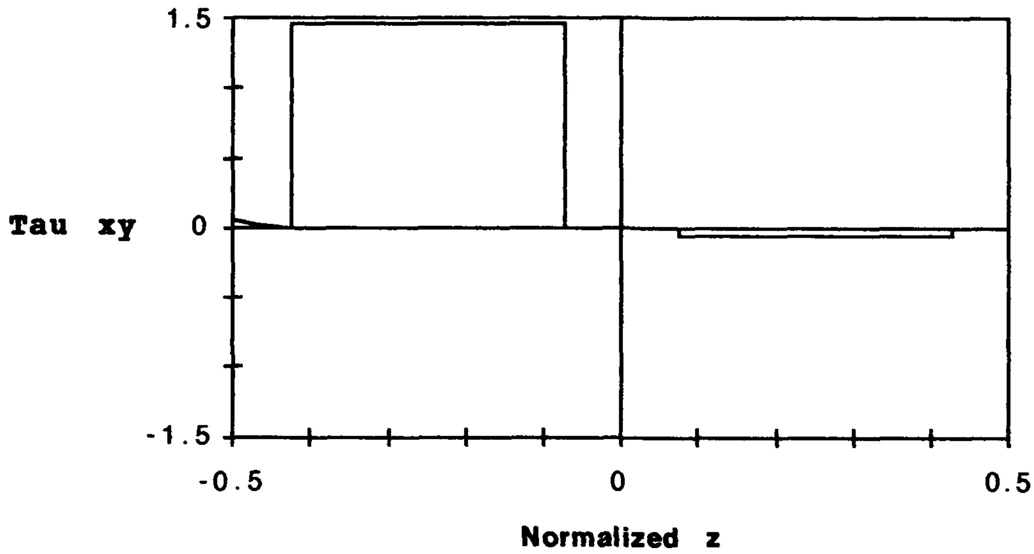

Figure 82. Tau (τ_{xy}) for a 0/60 Laminated Plate

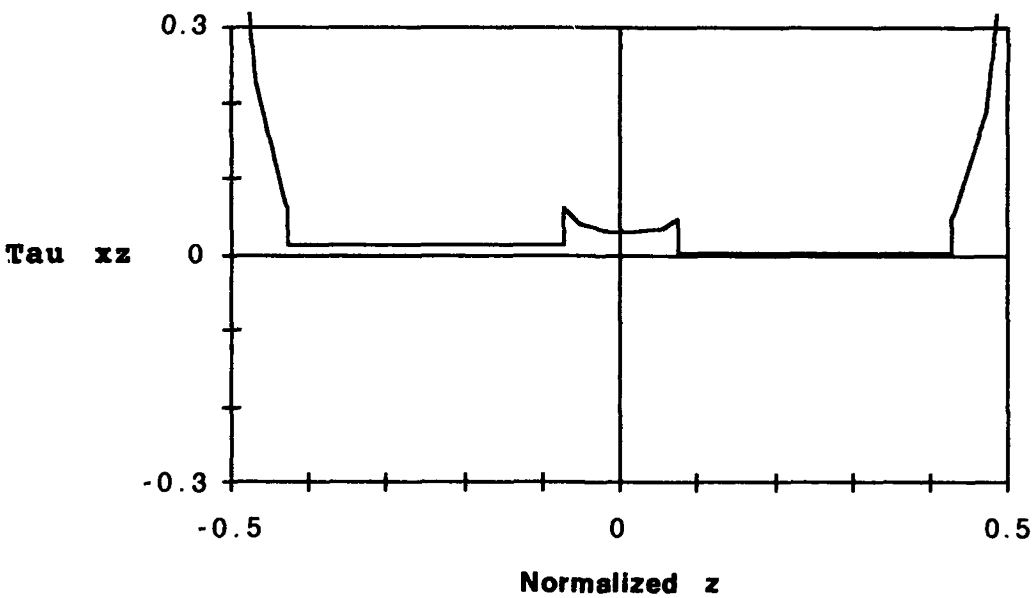

Figure 83. Tau (τ_{xz}) for a 0/60 Laminated Plate

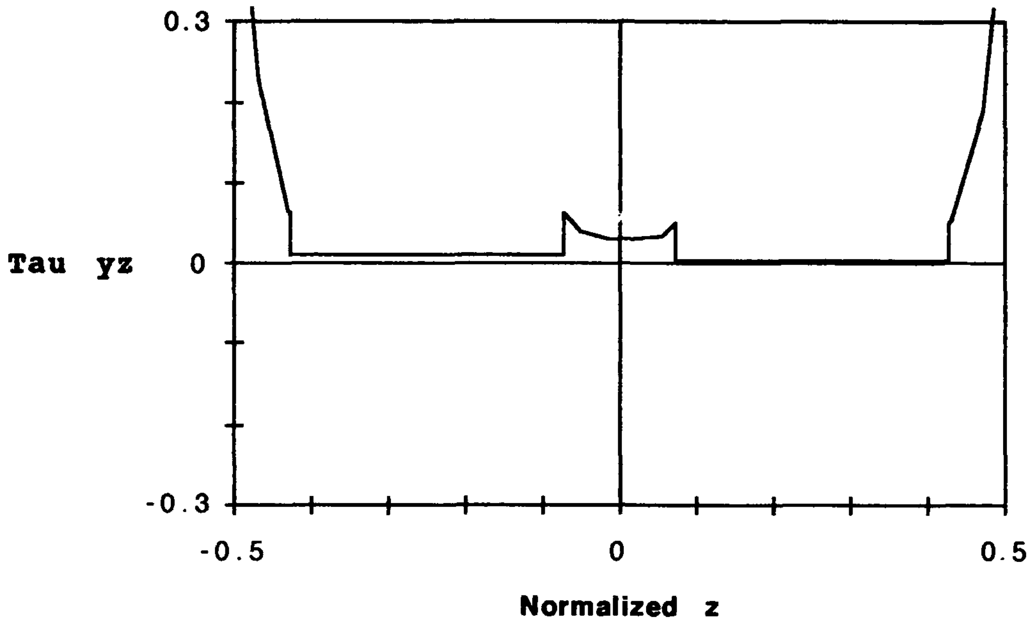

Figure 84. Tau (τ_{yz}) for a 0/60 Laminated Plate

Ritz Polynomial 0/60 Composite Plate Discussion

As can be seen in the displacement functions in figures 69 and 70, an N value of 6 creating a 36th order polynomial representation of the displacement function, falls short of the Sine/Cosine function 5th order polynomial modeling of the displacement. The outer matrix strata exhibit displacements that are not consistent with the loading and the boundary conditions. The strain functions exhibit somewhat better results but still are not consistent with classical solutions. An additional problem arises with the neutral axis of the plate in the Ritz polynomial representation. The integral of the stress over the area of the plate should produce the stress in each direction. Figures 79 and 80 show the neutral axis remaining at the mid plane. The integral of the stress generates an in plane force which should not be present. The Ritz polynomial also contain the same discontinuous transverse shear strain and illogical transverse stress result found in the Sine/Cosine solution. The above problems in the Ritz polynomial solution make it a questionabe method for solving the Strata model.

VII. *Conclusions and Recommendations*

Conclusions

The purpose of the strata model is to more accurately predict the strain and stress distributions with in an individual composite ply and throughout the thickness of a composite laminate. The model is expected to be most useful when the fiber and matrix materials used have dissimilar stiffness properties or high temperature brings that about. For the simple cases run in this document, the strata model predicts fiber strata normal stresses 5 times as high as those experienced by the matrix strata. The model generates transverse normal stresses σ_z on the same order of magnitude as the in plane shear stresses. This suggests that transverse normal stresses should not be assumed negligible, as in some models. The strata model assumption of a zero z gradient in the fiber strata places the fiber strata in a state of plain strain. This produces discontinuous γ_{xz} and γ_{yz} shear strains. This anomaly of the strata model can not be removed without making the solution procedure vastly more complicated. From the plain strain condition in the fiber strata, the model predicts significant transverse shear stresses in the fiber strata. Most models deem transverse shear as negligible. Overall the model predicts a significant variation of the stresses through the thickness of an individual ply. This is a sharp constrast to the constant strain and stress predicted by current methods.

The strata model is based on an assumed displacement of a polynomial in z and solves for the x and y dependent displacements. The strata model can be solved with assumed x and y displacement functions of either, a Sine/Cosine series, or a Ritz polynomial series.

The Sin/Cosine series solution used in this work limits the application to composites with only cross plies, in order to factor out the x and y dependence instead of integrating them out of the differential equations. The Sine/Cosine series solution requires a moderate polynomial order to model the x and y dependence, but requires a full series expansion to represent simple loading. Factoring out the x and y dependence allows the system of equations to be solved without losing any displacment mode shsapes. A double series expansion of Sin/Cosine functions could be used to model composites with angle plies but would require a large number of Sine/Cosine integrals to integrate out the x and y dependence.

A Ritz polynomial series requires a much higher polynomial order, but allows the analysis of composites with angle plies. This is possible because the x and y dependence can be integrated out of the governing equations without losing any displacement modes. An extremely high order Ritz polynomial is required to model a simple, two ply, composite plate and obtain results consistent with the loading and other models. The Ritz polynomial solution does not conerge monotonically on the correct solution as the order N is increased. The computational time becomes prohibitive as the polynomial order grows. Ritz polynomial displacement function assumptions are not recommended to solve strata model problems.

Areas for Further Research

This thesis only investigated simply supported plates with distributed pressures. How other types of boundary conditions can be modelled with the strata model should be investigated further.

Sin/Cosine terms in a double series expansion should be investigated as a displacement assumptions for the strata model. This type of displacement assumption has the potential to enable the analysis of composites with cross plies, without requiring an inordinate polynomial order.

The effects of the order of the displacement polynomial were investigated in a very limited sense in this work. More complex loading and an increased number of plies would require higher order polynomials to represent the deformation and strain. Further research in this area is necessary.

One of the reasons the strata model was developed is to better model dynamic loading. Dynamic terms can be included in the derivation of the governing equations and a larger more complex system of equations established and solved.

The through the thickness stress distributions generated by the strata model should be investigated as a causes of known failure mechanisms. The unique stress and strain distributions generated by the strata model could be used to explain known failure mechanisms in current composites.

One of the primary advantages of the strata model is it allows matrix dominated areas to deform with some independence from the stiffer fiber strata. In high temperature applications the matrix material can become viscoelastic and its properties degrade. Further investigation is required on the effects of degrading the matrix properties.

Appendix A: Mathmatica Routine to Generate the Strain Energy Terms

```
(*This Routine generates the strain Energy matrices *)
Clear[all]

(*Enter the stiffness Matrices *)

s={{ s1, s3,s10, s5,  0,  0},
   { s3, s2,s11, s6,  0,  0},
   {s10,s11,s12,s13,  0,  0},
   { s5, s6,s13, s4,  0,  0},
   {  0,  0,  0,  0, s7, s9},
   {  0,  0,  0,  0, s9, s8}}

k={{k1,k2,k2, 0, 0, 0},
   {k1,k1,k2, 0, 0, 0},
   {k1,k2,k1, 0, 0, 0},
   { 0, 0, 0,k3, 0, 0},
   { 0, 0, 0, 0,k3, 0},
   { 0, 0, 0, 0, 0,k3},

(* Enter the strain vectors *)

em = {{umx}, {vmy}, {wmz}, {(umy+vmx)}, {(umz+wmx)}, {(vmz+wmy)}}

ef = {{ufx}, {vfy}, {wfz}, {(ufy+vfx)}, {(ufz+wmx)}, {(vfz+wfy)}}

emt =Transpose[em]
eft =Transpose[ef]

(* Create the Strain energy Matricies *)

MatrixStrataStrainEnergy=(1/2)*emt.k.em

FiberStrataStrainEnergy =(1/2)*eft.s.ef
```

Appendix B: Mathmatica Routine to Solve a Homogeneous Plate

```
Clear[all]

Np=5
j=Range[0,Np]

nplies=2
plythick=.5
LaminateHeight=plythick*nplies

hf={.3535,.3535,.3535}

Do [mt[[f]]=plythick-hf[[f]],{f,nplis+1}]

Do[zf[[f]]=((nplies/2)-(f-1/2)*plythick,{f,nplies+1}]

zm={.5,.42675,.07325,-.07325,-.42675,-.5}
zmt={.5,.07325,-.4267}
zmb={42675,-.07325,-.5}

volf=0
volm=1-volf
a=10
b=10
p=Pi/a
q=Pi/b
theta={0,0}
P1=1

Ef= {20000000,20000000}
vf= {.35,.35}
Gf= {Ef[[1]]/(2*(1+vf[[1]])),Ef[[2]]/(2*(1+vf[[2]]))}
Ef= {300000,300000}
vf= {.4,.4}
Gf= {Em[[1]]/(2*(1+vm[[1]])),Em[[2]]/(2*(1+vm[[2]]))}

zfl[[1]]=Laminateheight/2
zfl[[nplies+2]]=-Laminateheight/2

Be=plythick/(plythick-hf)
Do[al[m]=((zfl[[m]]/zmt[[m]])-(zfl[[m+1]]/zmb[[m]]))*
         ((zmt[[m]]*zmb[[m]])/(zmb[[m]]-zmt[[m]])),{m,nplies}]

s= Array[sf,{7,nplies}]
k= Array[km,{3,nplies+1}]

Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(1-vm[[m]])),
             {m,nplies+1}]
Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(  vm[[m]])),
             {m,nplies+1}]
Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(1-2*vm[[m]])/2),
             {m,nplies+1}]
```

```
(*Calculate the Fiber strata Engineering Properties *)

Do[E11[[f]]=volf*Ef[[f]]+volm*Em[[f]],{f,nplies}]
Do[E33[[f]]=volf*Ef[[f]]+volm*Em[[f]],{f,nplies}]
Do[E22[[f]]=Em[[f]]*Ef[[f]]/(volf*Ef[[f]]+volm*Em[[f]]),
     {f,nplies}]

Do[v12[[f]]=volf*vf[[f]]+volm*vm[[f]],{f,nplies}]
Do[v13[[f]]=volf*vf[[f]]+volm*vm[[f]],{f,nplies}]
Do[v23[[f]]=v12[[f]]*E22[[f]]/E11[[f]],{f,nplies}]

Do[G13[[f]]=volf*Gf[[f]]+volm*Gm[[f]],{f,nplies}]
Do[G12[[f]]=Em[[f]]*Ef[[f]]/(volf*Ef[[f]]+volm*Em[[f]]),
          {f,nplies}]
Do[E23[[f]]=G12[[f]],{f,nplies}]

(*Calculate the fiber strata stiffness matrix form the Eng Con*)

Do [ V[[f]]=1-
v12[[f]]*(v12[[f]]*E22[[f]]/E11[[f]]+v23[[f]]*v13[[f]])-
v13[[f]]^2*1-v23[[f]]^2*E33[[f]]/E22[[f]],{f,nplies}]

Do [sf[1,f]=(1-v23[[f]]^2*E33[[f]]/E22[[f]])*E11[[f]]/V[[f]],
            {f,nplies}]
Do [sf[2,f]=(1-v13[[f]]^2*E33[[f]]/E11[[f]])*E22[[f]]/V[[f]],
            {f,nplies}]
Do [sf[3,f]=(v12[[f]]+v13[[f]]*E33[[f]]/E22[[f]])*
            (E22[[f]]/V[[f]]), {f,nplies}]
Do [sf[4,f]=(v23[[f]]+v12[[f]]*v13[[f]]*E22[[f]]/E11[[f]])*
            (E33[[f]]/V[[f]]), {f,nplies}]
Do [sf[5,f]=G12[[f]], {f,nplies}]
Do [sf[6,f]=G13[[f]], {f,nplies}]
Do [sf[7,f]=G12[[f]], {f,nplies}]

(*Calculate the rotated fiber strata stiffness matrix (sr)  *)

sr=Array[sfr,{13,nplies}]

Do[sfr[1,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
                Sin[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*Sin[theta[[f]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[2,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
                Sin[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*Sin[theta[[f]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[3,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
```

```
            Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
            Sin[theta[[f]]^2*sf[3,f]+
            Cos[theta[[f]]^2*Sin[theta[[f]]^2*sf[5,f],
            {f,nplies}]
Do[sfr[4,f] =   2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[2,f]-
            Cos[theta[[f]]]^2+Sin[theta[[f]]]^2*sf[5,f],
            {f,nplies}]
Do[sfr[5,f] =   2*Cos[theta[[f]]]*Sin[theta[[f]]]*
        (Sin[theta[[f]]]^2*sf[2,f]+Cos[theta[[f]]]^2*sf[3,f])-
            2*Cos[theta[[f]]]*Sin[theta[[f]]]*
        (Cos[theta[[f]]]^2*sf[1,f]+Sin[theta[[f]]]^2*sf[3,f])+
            Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Cos[theta[[f]]]^2+Sin[theta[[f]]])*sf[3,f]],
        {f,nplies}]
Do[sfr[6,f] =   -2*Cos[theta[[f]]]*Sin[theta[[f]]]*
        (Sin[theta[[f]]]^2*sf[1,f]+Cos[theta[[f]]]^2*sf[3,f])-
            2*Cos[theta[[f]]]*Sin[theta[[f]]]*
        (Cos[theta[[f]]]^2*sf[2,f]+Sin[theta[[f]]]^2*sf[3,f])+
            Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Cos[theta[[f]]]^2+Sin[theta[[f]]])*sf[5,f]],
        {f,nplies}]
Do[sfr[7,f] =Cos[theta[[f]]]^2*sf[6,f]+Sin[theta[[f]]]^2*sf[7,f],
            {f,nplies}]
Do[sfr[8,f] =Sin[theta[[f]]]^2*sf[6,f]+Cos[theta[[f]]]^2*sf[7,f],
            {f,nplies}]
Do[sfr[9,f] =-(Cos[theta[[f]]]*Sin[theta[[f]]]*sf[6,f])+
            Cos[theta[[f]]]*Sin[theta[[f]]]*sf[7,f],{f,nplies}]
Do[sfr[10,f] =  Cos[theta[[f]]]^2*sf[3,f]+Sin[theta[[f]]]*sf[4,f]
            {f,nplies}]
Do[sfr[11,f] =Sin[theta[[f]]]^2*sf[3,f]+Cos[theta[[f]]]^2*sf[4,f],
            {f,nplies}]
Do[sfr[12,f] =  sf[1,f],{f,nplies}]
Do[sfr[13,f] =  2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[3,f])+
            2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[4,f],
            {f,nplies}]

(* Put the stiffness into matrix form *)

S=Table[SS[f],{f,nplis}]
K=Table[KK[m],{m,nplies+1}]

Do[SS[f]={{ sr[[1]][[f]], sr[[3]][[f]],sr[[10]][[f]],sr[[5]],0,0},
        { sr[[3]][[f]], sr[[2]][[f]],sr[[11]][[f]],sr[[6]],0,0},
        {sr[[10]][[f]],sr[[11]][[f]],sr[[12]][[f]],sr[[13]],0,0}
        { sr[[5]][[f]], sr[[3]][[f]],sr[[10]][[f]],sr[[5]],0,0},
        {           0,        0,0,0, sr[[7]][[f]],sr[[9]][[f]]},
        {           0,        0,0,0, sr[[9]][[f]],sr[[8]][[f]]}}
        ,{f,nplies}]

Do[KK[m]={{ k[[1]][[m]], k[[2]][[m]],k[[2]][[m]],0,0,0},
        { k[[2]][[m]], k[[1]][[m]],k[[2]][[m]],0,0,0},
        { k[[2]][[m]], k[[2]][[m]],k[[1]][[m]],0,0,0},
        {          0,            0,           0,k[[3]][[m]],0,0},
```

```
          {           0,             0,            0,0,k[[3]][[m]],0},
          {           0,             0,            0,0,0,k[[3]][[m]]}}
          ,{m,nplies+1}]

(*Enter expressions for the load and phi, chi and w functions*)

P = P1*Sin[p*x]*Sin[q*y]

B1 = Table[BB[i],{i,0,Np}]
C1 = Table[CC[i],{i,0,Np}]

phi = B1*Cos[p*x]*Sin[q*y]
chi = C1*Sin[p*x]*Cos[q*y]
w   = A *Sin[p*x]*Sin[q*y]

(*Generate the derivatives of phi, chi and w *)

chix  = D[chi,x]
chiy  = D[chi,y]
chixx = D[chix,x]
chiyy = D[chiy,y]
chixy = D[chix,y]

phix  = D[phi,x]
phiy  = D[phi,y]
phixx = D[phix,x]
phiyy = D[phiy,y]
phixy = D[phix,y]

wx  = D[w,x]
wy  = D[wy]
wxx = D[wx,x]
wyy = D[w,y]
wxy = D[wx,y]

(* Create the Differential Equations for a Homogeneous Plate*)

edu = j'(
     Sum[Integrate[ k[[3,m]]*z^(j-1),{z,zmb[[m]],zmt[[m]]}],
                                             {m,nplies+1}]*  wx +
     Sum[Integrate[sr[[7,f]]*z^(j-1),{z,zmt[[f+1]],zmb[[f]]}],
                                             {f,nplies}]*    wx +
     Sum[Integrate[sr[[9,f]]*z^(j-1),{z,zmt[[f+1]],zmb[[f]]}],
                                        {f,nplies}]*   wy)
+ (Sum[-i*j*
    (Sum[Integrate[ k[[3,m]]*z^(j+i-2),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}]* phi[[i+1]]+
     Sum[Integrate[sr[[7,f]]*z^(j+i-2),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*   phi[[i+1]]+
     Sum[Integrate[sr[[9,f]]*z^(j+i-2),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*   chi[[i+1]] )+

    (Sum[Integrate[sr[[1,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]+
```

```
    Sum[Integrate[ k[[1,m]]*z^(j+i),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}]*  phixx[[i+1]]+
  2*Sum[Integrate[sr[[5,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*    phixy[[i+1]]+
   (Sum[Integrate[sr[[4,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}] +
    Sum[Integrate[ k[[3,m]]*z^(j+i),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}])*  phiyy[[i+1]]+
    Sum[Integrate[sr[[5,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*    chixx[[i+1]] +
    Sum[Integrate[sr[[6,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*    chiyy[[i+1]] +
   (Sum[Integrate[sr[[3,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]+
    Sum[Integrate[sr[[4,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}] +
    Sum[Integrate[ k[[2,m]]*z^(j+i),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}]+
    Sum[Integrate[ k[[3,m]]*z^(j+i),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}] )*  chixy[[i+1]]
    , {i,0,Np}] )


edv = j*(
    Sum[Integrate[k[[3,m]]*z^(j-1),{z,zmb[[m]],zmt[[m]]}],
                                       {m,nplies+1}]*        wy +
    Sum[Integrate[sr[[9,f]]*z^(j-1),{z,zmt[[f+1]],zmb[[f]]}],
                                       {f,nplies}]*          wx +
    Sum[Integrate[sr[[8,f]]*z^(j-1),{z,zmt[[f+1]],zmb[[f]]}],
                                       {f,nplies}]*          wy)
+ (Sum[-i*j*
   (Sum[Integrate[ k[[3,m]]*z^(j+i-2),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}]*  chi[[i+1]]+
    Sum[Integrate[sr[[8,f]]*z^(j+i-2),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*    chi[[i+1]]+
    Sum[Integrate[sr[[9,f]]*z^(j+i-2),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*    chi[[i+1]] )+

   (Sum[Integrate[sr[[2,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]+
    Sum[Integrate[ k[[1,m]]*z^(j+i),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}]*  chiyy[[i+1]]+
  2*Sum[Integrate[sr[[6,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*    chixy[[i+1]]+
   (Sum[Integrate[sr[[4,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}] +
    Sum[Integrate[ k[[3,m]]*z^(j+i),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}])*  chixx[[i+1]]+
    Sum[Integrate[sr[[6,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*    phiyy[[i+1]] +
    Sum[Integrate[sr[[5,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]*    phixx[[i+1]] +
   (Sum[Integrate[sr[[3,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}]+
```

```
    Sum[Integrate[sr[[4,f]]*z^(j+i),{z,zmt[[f+1]],zmb[[f]]}],
                                   {f,nplies}] +
    Sum[Integrate[ k[[2,m]]*z^(j+i),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}]+
    Sum[Integrate[ k[[3,m]]*z^(j+i),{z,zmb[[m]],zmt[[m]]}],
                                   {m,nplies+1}] )*  phixy[[i+1]]
    , {i,0,Np}] )

edw =
  -(Sum[Integrate[ k[[3,m]],{z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[Integrate[sr[[7,f]],{z,zmt[[f+1]],zmb[[f]]}],{f,nplies}])
                                                        * wxx  +
  2*Sum[Integrate[sr[[9,f]],{z,zmt[[f+1]],zmb[[f]]}],{f,nplies}]
                                                        * wxy
   (Sum[Integrate[ k[[3,m]],{z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[Integrate[sr[[8,f]],{z,zmt[[f+1]],zmb[[f]]}],{f,nplies}])
                                                        * wyy   +
  Sum[-i(
    Sum[Integrate[ k[[3,m]]*z^(i-1),{z,zmb[[m]],zmt[[m]]}],
                          {m,nplies+1}]* wy +
    Sum[Integrate[sr[[9,f]]*z^(j-1),{z,zmt[[f+1]],zmb[[f]]}],
                          {f,nplies}]  * (phix[[i+1]]+chiy[[i+1]])+
    Sum[Integrate[sr[[7,f]]*z^(j-1),{z,zmt[[f+1]],zmb[[f]]}],
                          {f,nplies}]  * phix[[i+1]] +
    Sum[Integrate[sr[[8,f]]*z^(j-1),{z,zmt[[f+1]],zmb[[f]]}],
                          {f,nplies}]  * chiy[[i+1]]
    ,{i,0,Np}]


(* Construct a system of equations for the Diff Equations *)

m = Array[mm,{2*(Np+1)+1,2*(Np+1)+1}]

fedu=edu/Cos[p*x]*Sin[q*y]
Do[M[[j+1,i+1]] =
    Coefficient[fedu[[j+1]],BB[i]],{j,0,Np},{i,0,Np}]
Do[M[[j+1,i+1+Np+1]] =
    Coefficient[fedu[[j+1]],CC[i]],{j,0,Np},{i,0,Np}]
Do[M[[j+1,2*(Np+1)+1]] =
    Coefficient[fedu[[j+1]],A],{j,0,Np},{i,0,Np}]

fedv=edu/Cos[q*x]*Sin[p*y]
Do[M[[j+Np+2,i+1]] =
    Coefficient[fedu[[j+1]],BB[i]],{j,0,Np},{i,0,Np}]
Do[M[[j+Np+2,i+1+Np+1]] =
    Coefficient[fedu[[j+1]],CC[i]],{j,0,Np},{i,0,Np}]
Do[M[[j+Np+2,2*(Np+1)+1]] =
    Coefficient[fedu[[j+1]],A],{j,0,Np},{i,0,Np}]

fedw=edu/Sin[p*x]*Sin[q*y]
Do[M[[2*(Np+1)+1,i+1]] =
    Coefficient[fedu[[j+1]],BB[i]],{j,0,Np},{i,0,Np}]
Do[M[[2*(Np+1)+1,i+1+Np+1]] =
    Coefficient[fedu[[j+1]],CC[i]],{j,0,Np},{i,0,Np}]
```

```
Do[M[[2*(Np+1)+1,2*(Np+1)+1]] =
     Coefficient[fedu[[j+1]],A],{j,0,Np},{i,0,Np}]

(* Create the answer vector (bm) for the m*sol=bm system *)

bm = Array [bbm,{2*(Np+1),1}]
Do [bm[[i]]=0, {i,2*(Np+1)}]
Do [bm[[2*(Np+1)+1]]=P1, {i,2*(Np+1)}]

(* Solve the system of Equations for the sol vector *)

sol=LinearSolve[m,bm]

(* Determine the B1, C1, A coefficients *)

Do[BB[i]=sol[[i+1],{i,0,Np}]
Do[CC[i]=sol[[i+Np+2},{i,0,Np}]
A=sol[[2*(Np+1)+1]]

(* Creat the Displacement functions for the fiber strata *)

Uf[z_]=Table[uff[z][f],{f,nplies}]
Vf[z_]=Table[vff[z][f],{f,nplies}]
Wf[z_]=Table[wff[z][f],{f,nplies}]
Do[uff[z_][f] = Sum[zf[[f]]^i*BB[i],{i,0,Np},{f,nplies}]
Do[vff[z_][f] = Sum[zf[[f]]^i*CC[i],{i,0,Np},{f,nplies}]
Do[wff[z_][f] = A,{f,nplies}]

(* Create the Displacement Functions for the matrix strata *)

Um[z_]=Table[umm[z][m],{m,nplies+1}]
Vm[z_]=Table[vmm[z][m],{m,nplies+1}]
Wm[z_]=Table[wmm[z][m],{m,nplies+1}]
Do[umm[z_][f] = Sum[(al[[m]]+Bem[[m]]*z)^i*BB[i],
                {i,0,Np},{m,nplies+1}]
Do[vmm[z_][f] = Sum[(al[[m]]+Bem[[m]]*z)^i*BB[i],
                {i,0,Np},{m,nplies+1}]
Do[wmm[z_][f] = A,{m,nplies+1}]

(* Create the Strain Functions for the matrix strata *)

efx[z_]=Table[effx[z][f],{f,nplies}]
efy[z_]=Table[effy[z][f],{f,nplies}]
efz[z_]=Table[effz[z][f],{f,nplies}]
efxy[z_]=Table[effxy[z][f],{f,nplies}]
efxz[z_]=Table[effxz[z][f],{f,nplies}]
efyz[z_]=Table[effyz[z][f],{f,nplies}]

Do [effx[z_][f]=p*Uf[z][[f]],{f,nplies}]
Do [effy[z_][f]=p*Vf[z][[f]],{f,nplies}]
Do [effz[z_][f]=0,{f,nplies}]
Do [effx[z_][f]=q*Uf[z][[f]]+p*Vf[z][[f]],{f,nplies}]
Do [effy[z_][f]=D[Uf[z][[f]],z]+p*Wf[z][[f]],{f,nplies}]
Do [effy[z_][f]=D[Vf[z][[f]],z]+q*Wf[z][[f]],{f,nplies}]
```

```
(* Create the Strain Functions for the matrix strata *)

emx[z_]=Table[emmx[z][m],{m,nplies+1}]
emy[z_]=Table[emmy[z][m],{m,nplies+1}]
emz[z_]=Table[emmz[z][m],{m,nplies+1}]
emxy[z_]=Table[emmxy[z][m],{m,nplies+1}]
emxz[z_]=Table[emmxz[z][m],{m,nplies+1}]
emyz[z_]=Table[emmyz[z][m],{m,nplies+1}]

Do [emmx[z_][f]=-p*Um[z][[f]],{f,nplies}]
Do [emmy[z_][f]=-q*Vm[z][[f]],{f,nplies}]
Do [emmz[z_][f]=0,{f,nplies}]
Do [emmx[z_][f]=q*Um[z][[m]]+p*Vf[z][[m]],{m,nplies+1}]
Do [emmy[z_][f]=D[Um[z][[m]],z]+p*Wf[z][[m]],{m,nplies+1}]
Do [emmy[z_][f]=D[Vm[z][[m]],z]+q*Wf[z][[m]],{m,nplies+1}]

(* Create the Strain vector for the fiber strata *)

Fstrain[z_]=Table[fstrain[z][f],{f,nplies}]
Do[fstrain[z_][f]={efx[z],efy[z][f],efz[z][f],efxy[z][f],
                    efxz[z][f],efyz[z][f]},{f,nplies}]

(* Create the Strain vector for the matrix strata *)

Mstrain[z_]=Table[mstrain[z][m],{m,nplies+1}]
Do[mstrain[z_][f]={emx[z][m],emy[z][m],emz[z][m],emxy[z][m],
                    emxz[z][m],emyz[z][m]},{m,nplies+1}]

(* Calculate the Stress in each stratum by Stiffness*strain *)

Fstress[z_]= Table[fstress[z][f],{f,nplies}]
Do[fstress[z_][f]= S[[f]].Fstrain[z][[f]],{f,nplies}]

Mstress[z_]= Table[mstress[z][m],{m,nplies+1}]
Do[mstress[z_][m]= S[[m]].Fstrain[z][[m]],{m,nplies+1}]


(*  Place the stress componentes into plotable form *)

smx[z_]=Table[smxx[z][m],{m,nplies+1}]
smy[z_]=Table[smyy[z][m],{m,nplies+1}]
smz[z_]=Table[smzz[z][m],{m,nplies+1}]
smxy[z_]=Table[smxyy[z][m],{m,nplies+1}]
smxz[z_]=Table[smxzz[z][m],{m,nplies+1}]
smyz[z_]=Table[smyzz[z][m],{m,nplies+1}]

sfx[z_]=Table[sfxx[z][f],{f,nplies}]
sfy[z_]=Table[sfyy[z][f],{f,nplies}]
sfz[z_]=Table[sfzz[z][f],{f,nplies}]
sfxy[z_]=Table[sfxyy[z][f],{f,nplies}]
sfxz[z_]=Table[sfxzz[z][f],{f,nplies}]
sfyz[z_]=Table[sfyzz[z][f],{f,nplies}]
```

```
Do[smxx[z_][m] =Mstress[z][[m]][[1]],{m,nplies+1}]
Do[smyy[z_][m] =Mstress[z][[m]][[2]],{m,nplies+1}]
Do[smzz[z_][m] =Mstress[z][[m]][[3]],{m,nplies+1}]
Do[smxyy[z_][m]=Mstress[z][[m]][[4]],{m,nplies+1}]
Do[smxzz[z_][m]=Mstress[z][[m]][[5]],{m,nplies+1}]
Do[smyzz[z_][m]=Mstress[z][[m]][[6]],{m,nplies+1}]

Do[sfxx[z_][f] =Fstress[z][[f]][[1]],{f,nplies}]
Do[sfyy[z_][f] =Fstress[z][[f]][[2]],{f,nplies}]
Do[sfzz[z_][f] =Fstress[z][[f]][[3]],{f,nplies}]
Do[sfxyy[z_][f]=Fstress[z][[f]][[4]],{f,nplies}]
Do[sfxzz[z_][f]=Fstress[z][[f]][[5]],{f,nplies}]
Do[sfyzz[z_][f]=Fstress[z][[f]][[6]],{f,nplies}]
```

Appendix C: Mathmatica Routine for Strata Model Problems

```
Clear[all]

Np=5
j=Range[0,Np]

nplies=2
plythick=.5
LaminateHeight=plythick*nplies

hf={.3535,.3535}

Do [mt[[f]]=plythick-hf[[f]],{f,nplis+1}]

Do[zf[[f]]=((nplies/2)-(f-1/2)*plythick,{f,nplies+1}]

zm={.5,.42675,.07325,-.07325,-.42675,-.5}
zmt={.5,.07325,-.4267}
zmb={42675,-.07325,-.5}

volf=1.4142/2
volm=1-volf
a=10
b=10
p=Pi/a
q=Pi/b
theta={0,0}
P1=1

Ef= {20000000,20000000}
vf= {.35,.35}
Gf= {Ef[[1]]/(2*(1+vf[[1]])),Ef[[2]]/(2*(1+vf[[2]]))}
Ef= {300000,300000}
vf= {.4,.4}
Gf= {Em[[1]]/(2*(1+vm[[1]])),Em[[2]]/(2*(1+vm[[2]]))}

zf1[[1]]=Laminateheight/2
zf1[[nplies+2]]=-Laminateheight/2

Be=plythick/(plythick-hf)
Do[al[m]=((zf1[[m]]/zmt[[m]])-(zf1[[m+1]]/zmb[[m]]))*
         ((zmt[[m]]*zmb[[m]])/(zmb[[m]]-zmt[[m]])),{m,nplies}]

s= Array[sf,{7,nplies}]
k= Array[km,{3,nplies+1}]

Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(1-vm[[m]])),
              {m,nplies+1}]
Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(  vm[[m]])),
              {m,nplies+1}]
Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(1-2*vm[[m]])/2),
              {m,nplies+1}]
```

```
(*Calculate the Fiber strata Engineering Properties *)

Do[E11[[f]]=volf*Ef[[f]]+volm*Em[[f]],{f,nplies}]
Do[E33[[f]]=volf*Ef[[f]]+volm*Em[[f]],{f,nplies}]
Do[E22[[f]]=Em[[f]]*Ef[[f]]/(volf*Ef[[f]]+volm*Em[[f]]),
      {f,nplies}]

Do[v12[[f]]=volf*vf[[f]]+volm*vm[[f]],{f,nplies}]
Do[v13[[f]]=volf*vf[[f]]+volm*vm[[f]],{f,nplies}]
Do[v23[[f]]=v12[[f]]*E22[[f]]/E11[[f]],{f,nplies}]

Do[G13[[f]]=volf*Gf[[f]]+volm*Gm[[f]],{f,nplies}]
Do[G12[[f]]=Em[[f]]*Ef[[f]]/(volf*Ef[[f]]+volm*Em[[f]]),
            {f,nplies}]
Do[E23[[f]]=G12[[f]],{f,nplies}]

(*Calculate the fiber strata stiffness matrix form the Eng Con*)

Do [ V[[f]]=1-
v12[[f]]*(v12[[f]]*E22[[f]]/E11[[f]]+v23[[f]]*v13[[f]])-
v13[[f]]^2*1-v23[[f]]^2*E33[[f]]/E22[[f]],{f,nplies}]

Do [sf[1,f]=(1-v23[[f]]^2*E33[[f]]/E22[[f]])*E11[[f]]/V[[f]],
            {f,nplies}]
Do [sf[2,f]=(1-v13[[f]]^2*E33[[f]]/E11[[f]])*E22[[f]]/V[[f]],
            {f,nplies}]
Do [sf[3,f]=(v12[[f]]+v13[[f]]*E33[[f]]/E22[[f]])*
            (E22[[f]]/V[[f]]), {f,nplies}]
Do [sf[4,f]=(v23[[f]]+v12[[f]]*v13[[f]]*E22[[f]]/E11[[f]])*
            (E33[[f]]/V[[f]]), {f,nplies}]
Do [sf[5,f]=G12[[f]], {f,nplies}]
Do [sf[6,f]=G13[[f]], {f,nplies}]
Do [sf[7,f]=G12[[f]], {f,nplies}]

(*Calculate the rotated fiber strata stiffness matrix (sr)  *)

sr=Array[sfr,{13,nplies}]

Do[sfr[1,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
                Sin[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*Sin[theta[[f]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[2,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
                Sin[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*Sin[theta[[f]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[3,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
```

```
                Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
                Sin[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]]^2*Sin[theta[[f]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[4,f] =   2*Cos[theta[[f]]]*Sin[theta[[f]]]*
                (2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[2,f]-
                Cos[theta[[f]]]^2+Sin[theta[[f]]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[5,f] =   2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Sin[theta[[f]]]^2*sf[2,f]+Cos[theta[[f]]]^2*sf[3,f])-
                2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Cos[theta[[f]]]^2*sf[1,f]+Sin[theta[[f]]]^2*sf[3,f])+
                Cos[theta[[f]]]*Sin[theta[[f]]]*
                (Cos[theta[[f]]]^2+Sin[theta[[f]]])*sf[3,f]],
            {f,nplies}]
Do[sfr[6,f] =   -2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Sin[theta[[f]]]^2*sf[1,f]+Cos[theta[[f]]]^2*sf[3,f])-
                2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Cos[theta[[f]]]^2*sf[2,f]+Sin[theta[[f]]]^2*sf[3,f])+
                Cos[theta[[f]]]*Sin[theta[[f]]]*
                (Cos[theta[[f]]]^2+Sin[theta[[f]]])*sf[5,f]],
            {f,nplies}]
Do[sfr[7,f] =Cos[theta[[f]]]^2*sf[6,f]+Sin[theta[[f]]]^2*sf[7,f],
                {f,nplies}]
Do[sfr[8,f] =Sin[theta[[f]]]^2*sf[6,f]+Cos[theta[[f]]]^2*sf[7,f],
                {f,nplies}]
Do[sfr[9,f] =-(Cos[theta[[f]]]*Sin[theta[[f]]]*sf[6,f])+
                Cos[theta[[f]]]*Sin[theta[[f]]]*sf[7,f],{f,nplies}]
Do[sfr[10,f] =  Cos[theta[[f]]]^2*sf[3,f]+Sin[theta[[f]]]*sf[4,f]
                {f,nplies}]
Do[sfr[11,f] =Sin[theta[[f]]]^2*sf[3,f]+Cos[theta[[f]]]^2*sf[4,f],
                {f,nplies}]
Do[sfr[12,f] =  sf[1,f],{f,nplies}]
Do[sfr[13,f] =  2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[3,f])+
                2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[4,f],
                {f,nplies}]

(* Put the stiffness into matrix form *)

S=Table[SS[f],{f,nplis}]
K=Table[KK[m],{m,nplies+1}]

Do[SS[f]={{ sr[[1]][[f]], sr[[3]][[f]],sr[[10]][[f]],sr[[5]],0,0},
          { sr[[3]][[f]], sr[[2]][[f]],sr[[11]][[f]],sr[[6]],0,0},
          {sr[[10]][[f]],sr[[11]][[f]],sr[[12]][[f]],sr[[13]],0,0}
          { sr[[5]][[f]], sr[[3]][[f]],sr[[10]][[f]],sr[[5]],0,0},
          {            0,       0,0,0, sr[[7]][[f]],sr[[9]][[f]]},
          {            0,       0,0,0, sr[[9]][[f]],sr[[8]][[f]]}}
          ,{f,nplies}]

Do[KK[m]={{ k[[1]][[m]], k[[2]][[m]],k[[2]][[m]],0,0,0},
          { k[[2]][[m]], k[[1]][[m]],k[[2]][[m]],0,0,0},
          { k[[2]][[m]], k[[2]][[m]],k[[1]][[m]],0,0,0},
          {           0,           0,          0,k[[3]][[m]],0,0},
```

```
          {          0,             0,           0,0,k[[3]][[m]],0},
          {          0,             0,           0,0,0,k[[3]][[m]]}}
          ,{m,nplies+1}]

(*Enter expressions for the load and phi, chi and w functions*)

P = P1*Sin[p*x]*Sin[q*y]

B1 = Table[BB[i],{i,0,Np}]
C1 = Table[CC[i],{i,0,Np}]

phi = B1*Cos[p*x]*Sin[q*y]
chi = C1*Sin[p*x]*Cos[q*y]
w   = A *Sin[p*x]*Sin[q*y]

(*Generate the derivatives of phi, chi and w *)

chix  = D[chi,x]
chiy  = D[chi,y]
chixx = D[chix,x]
chiyy = D[chiy,y]
chixy = D[chix,y]

phix  = D[phi,x]
phiy  = D[phi,y]
phixx = D[phix,x]
phiyy = D[phiy,y]
phixy = D[phix,y]

wx  = D[w,x]
wy  = D[wy]
wxx = D[wx,x]
wyy = D[w,y]
wxy = D[wx,y]

(* Create the Differential Equations for a Composite Plate *)

edu = j*Be(
     Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j-1),
                {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]  )*wx +
+ (Sum[-i*j*
    (Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i-2),
                {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]*  phi[[i+1]]+

    (Sum[sr[[1,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]+
     Sum[Integrate[ k[[1,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
                {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}])*phixx[[i+1]]+
   2*Sum[sr[[5,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*phixy[[i+1]]+
    (Sum[sr[[4,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}] +
     Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
                {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}])*phiyy[[i+1]]+
     Sum[sr[[5,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*chixx[[i+1]]+
     Sum[sr[[6,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*chiyy[[i+1]]+
    (Sum[sr[[3,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]+
```

```
    Sum[sr[[4,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}] +
    Sum[Integrate[ k[[2,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}] )*chixy[[i+1]]
    , {i,0,Np}] )


edv = j*(
    Sum[Integrate[k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j-1),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]*    wy +
+ (Sum[-i*j*
   (Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i-2),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]*chi[[i+1]]+
    Sum[sr[[8,f]]*hf[[f]]*zf[[f]]^(j+i-2),{f,nplies}]*chi[[i+1]]+
    Sum[sr[[9,f]]*hf[[f]]*zf[[f]]^(j+i-2),{f,nplies}]
                                               *chi[[i+1]])+
   (Sum[sr[[2,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]+
    Sum[Integrate[ k[[1,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]* chiyy[[i+1]]+
  2*Sum[sr[[6,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*chixy[[i+1]]+
   (Sum[sr[[4,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}] +
    Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}])*chixx[[i+1]]+
    Sum[sr[[6,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*phiyy[[i+1]]+
    Sum[sr[[5,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*phixx[[i+1]]+
   (Sum[sr[[3,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]+
    Sum[sr[[4,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}] +
    Sum[Integrate[ k[[2,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}] )*phixy[[i+1]]
    , {i,0,Np}] )

edw =
  -(Sum[Integrate[k[[3,m]],{z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[sr[[7,f]]*hf[[f]],{f,nplies}])       * wxx  +
  2*Sum[sr[[9,f]]*hf[[f]],{f,nplies}]                * wxy
   (Sum[Integrate[k[[3,m]],{z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[sr[[8,f]]*hf[[f]],{f,nplies}])       * wyy    +
  Sum[-i(
    Sum[Integrate[k[[3,m]]*(al[[m]]+bem[[m]]*z)^(i-1),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]* wy +
    ,{i,0,Np}]

(* Construct a system of equations for the Diff Equations *)

m = Array[mm,{2*(Np+1)+1,2*(Np+1)+1}]

fedu=edu/Cos[p*x]*Sin[q*y]
Do[M[[j+1,i+1]] =
    Coefficient[fedu[[j+1]],BB[i]],{j,0,Np},{i,0,Np}]
Do[M[[j+1,i+Np+2]] =
    Coefficient[fedu[,[j+1]],CC[i]],{j,0,Np},{i,0,Np}]
```

```
Do[M[[j+1,2*(Np+1)+1]] =
     Coefficient[fedu[[j+1]],A],{j,0,Np},{i,0,Np}]

fedv=edu/Cos[q*x]*Sin[p*y]
Do[M[[j+Np+2,i+1]] =
     Coefficient[fedu[[j+1]],BB[i]],{j,0,Np},{i,0,Np}]
Do[M[[j+Np+2,i+Np+2]] =
     Coefficient[fedu[[j+1]],CC[i]],{j,0,Np},{i,0,Np}]
Do[M[[j+Np+2,2*(Np+1)+1]] =
     Coefficient[fedu[[j+1]],A],{j,0,Np},{i,0,Np}]

fedw=edu/Sin[p*x]*Sin[q*y]
Do[M[[2*(Np+1)+1,i+1]] =
     Coefficient[fedu[[j+1]],BB[i]],{j,0,Np},{i,0,Np}]
Do[M[[2*(Np+1)+1,i+Np+2]] =
     Coefficient[fedu[[j+1]],CC[i]],{j,0,Np},{i,0,Np}]
Do[M[[2*(Np+1)+1,2*(Np+1)+1]] =
     Coefficient[fedu[[j+1]],A],{j,0,Np},{i,0,Np}]

(* Create the answer vector (bm) for the m*sol=bm system *)

bm = Array [bbm,{2*(Np+1),1}]
Do [bm[[i]]=0, {i,2*(Np+1)}]
Do [bm[[2*(Np+1)+1]]=P1, {i,2*(Np+1)}]

(* Solve the system of Equations for the sol vector *)

sol=LinearSolve[m,bm]

(* Determine the B1, C1, A coefficients *)

Do[BB[i]=sol[[i+1],{i,0,Np}]
Do[CC[i]=sol[[i+Np+2},{i,0,Np}]
A=sol[[2*(Np+1)+1]]


(* Creat the Displacement functions for the fiber strata *)

Uf[z_]=Table[uff[z][f],{f,nplies}]
Vf[z_]=Table[vff[z][f],{f,nplies}]
Wf[z_]=Table[wff[z][f],{f,nplies}]
Do[uff[z_][f] = Sum[zf[[f]]^i*BB[i],{i,0,Np},{f,nplies}]
Do[vff[z_][f] = Sum[zf[[f]]^i*CC[i],{i,0,Np},{f,nplies}]
Do[wff[z_][f] = A,{f,nplies}]

(* Create the Displacement Functions for the matrix strata *)

Um[z_]=Table[umm[z][m],{m,nplies+1}]
Vm[z_]=Table[vmm[z][m],{m,nplies+1}]
Wm[z_]=Table[wmm[z][m],{m,nplies+1}]
Do[umm[z_][f] = Sum[(al[[m]]+Bem[[m]]*z)^i*BB[i],
                {i,0,Np},{m,nplies+1}]
Do[vmm[z_][f] = Sum[(al[[m]]+Bem[[m]]*z)^i*BB[i],
                {i,0,Np},{m,nplies+1}]
```

```
Do[wmm[z_][f] = A,{m,nplies+1}]

(* Create the Strain Functions for the matrix strata *)

efx[z_]=Table[effx[z][f],{f,nplies}]
efy[z_]=Table[effy[z][f],{f,nplies}]
efz[z_]=Table[effz[z][f],{f,nplies}]
efxy[z_]=Table[effxy[z][f],{f,nplies}]
efxz[z_]=Table[effxz[z][f],{f,nplies}]
efyz[z_]=Table[effyz[z][f],{f,nplies}]

Do [effx[z_][f]=-p*Uf[z][[f]],{f,nplies}]
Do [effy[z_][f]=-q*Vf[z][[f]],{f,nplies}]
Do [effz[z_][f]=0,{f,nplies}]
Do [effx[z_][f]=q*Uf[z][[f]]+p*Vf[z][[f]],{f,nplies}]
Do [effy[z_][f]=D[Uf[z][[f]],z]+p*Wf[z][[f]],{f,nplies}]
Do [effy[z_][f]=D[Vf[z][[f]],z]+q*Wf[z][[f]],{f,nplies}]

(* Create the Strain Functions for the matrix strata *)

emx[z_]=Table[emmx[z][m],{m,nplies+1}]
emy[z_]=Table[emmy[z][m],{m,nplies+1}]
emz[z_]=Table[emmz[z][m],{m,nplies+1}]
emxy[z_]=Table[emmxy[z][m],{m,nplies+1}]
emxz[z_]=Table[emmxz[z][m],{m,nplies+1}]
emyz[z_]=Table[emmyz[z][m],{m,nplies+1}]

Do [emmx[z_][f]=-p*Um[z][[f]],{f,nplies}]
Do [emmy[z_][f]=-q*Vm[z][[f]],{f,nplies}]
Do [emmz[z_][f]=0,{f,nplies}]
Do [emmx[z_][f]=q*Um[z][[m]]+p*Vf[z][[m]],{m,nplies+1}]
Do [emmy[z_][f]=D[Um[z][[m]],z]+p*Wf[z][[m]],{m,nplies+1}]
Do [emmy[z_][f]=D[Vm[z][[m]],z]+q*Wf[z][[m]],{m,nplies+1}]

(* Create the Strain vector for the fiber strata *)

Fstrain[z_]=Table[fstrain[z][f],{f,nplies}]
Do[fstrain[z_][f]={efx[z],efy[z][f],efz[z][f],efxy[z][f],
                    efxz[z][f],efyz[z][f]},{f,nplies}]

(* Create the Strain vector for the matrix strata *)

Mstrain[z_]=Table[mstrain[z][m],{m,nplies+1}]
Do[mstrain[z_][f]={emx[z][m],emy[z][m],emz[z][m],emxy[z][m],
                    emxz[z][m],emyz[z][m]},{m,nplies+1}]

(* Calculate the Stress in each stratum by Stiffness*strain *)

Fstress[z_]= Table[fstress[z][f],{f,nplies}]
Do[fstress[z_][f]= S[[f]].Fstrain[z][[f]],{f,nplies}]

Mstress[z_]= Table[mstress[z][m],{m,nplies+1}]
Do[mstress[z_][m]= S[[m]].Fstrain[z][[m]],{m,nplies+1}]
```

```
(*  Place the stress componentes into plotable form *)

smx[z_]=Table[smxx[z][m],{m,nplies+1}]
smy[z_]=Table[smyy[z][m],{m,nplies+1}]
smz[z_]=Table[smzz[z][m],{m,nplies+1}]
smxy[z_]=Table[smxyy[z][m],{m,nplies+1}]
smxz[z_]=Table[smxzz[z][m],{m,nplies+1}]
smyz[z_]=Table[smyzz[z][m],{m,nplies+1}]

sfx[z_]=Table[sfxx[z][f],{f,nplies}]
sfy[z_]=Table[sfyy[z][f],{f,nplies}]
sfz[z_]=Table[sfzz[z][f],{f,nplies}]
sfxy[z_]=Table[sfxyy[z][f],{f,nplies}]
sfxz[z_]=Table[sfxzz[z][f],{f,nplies}]
sfyz[z_]=Table[sfyzz[z][f],{f,nplies}]

Do[smxx[z_][m] =Mstress[z][[m]][[1]],{m,nplies+1}]
Do[smyy[z_][m] =Mstress[z][[m]][[2]],{m,nplies+1}]
Do[smzz[z_][m] =Mstress[z][[m]][[3]],{m,nplies+1}]
Do[smxyy[z_][m]=Mstress[z][[m]][[4]],{m,nplies+1}]
Do[smxzz[z_][m]=Mstress[z][[m]][[5]],{m,nplies+1}]
Do[smyzz[z_][m]=Mstress[z][[m]][[6]],{m,nplies+1}]

Do[sfxx[z_][f] =Fstress[z][[f]][[1]],{f,nplies}]
Do[sfyy[z_][f] =Fstress[z][[f]][[2]],{f,nplies}]
Do[sfzz[z_][f] =Fstress[z][[f]][[3]],{f,nplies}]
Do[sfxyy[z_][f]=Fstress[z][[f]][[4]],{f,nplies}]
Do[sfxzz[z_][f]=Fstress[z][[f]][[5]],{f,nplies}]
Do[sfyzz[z_][f]=Fstress[z][[f]][[6]],{f,nplies}]
```

Appendix D: Mathmatica Routine with Ritz Polynomial

```
Clear[all]

Np=5
j=Range[0,Np^2]

nplies=2
plythick=.5
LaminateHeight=plythick*nplies

hf={.3535,.3535}

Do [mt[[f]]=plythick-hf[[f]],{f,nplis+1}]

Do[zf[[f]]=((nplies/2)-(f-1/2)*plythick,{f,nplies+1}]

zm={.5,.42675,.07325,-.07325,-.42675,-.5}
zmt={.5,.07325,-.4267}
zmb={42675,-.07325,-.5}

volf=1.4142/2
volm=1-volf
a=10
b=10
p=Pi/a
q=Pi/b
theta={0,0}
P1=1

zf1[[1]]=Laminateheight/2
zf1[[nplies+2]]=-Laminateheight/2

Be=plythick/(plythick-hf)
Do[al[m]=((zf1[[m]]/zmt[[m]])-(zf1[[m+1]]/zmb[[m]]))*
        ((zmt[[m]]*zmb[[m]])/(zmb[[m]]-zmt[[m]])),{m,nplies}]

Ef= {20000000,20000000}
vf= {.35,.35}
Gf= {Ef[[1]]/(2*(1+vf[[1]])),Ef[[2]]/(2*(1+vf[[2]]))}
Ef= {300000,300000}
vf= {.4,.4}
Gf= {Em[[1]]/(2*(1+vm[[1]])),Em[[2]]/(2*(1+vm[[2]]))}

s= Array[sf,{7,nplies}]
k= Array[km,{3,nplies+1}]

Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(1-vm[[m]])),
             {m,nplies+1}]
Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(  vm[[m]])),
             {m,nplies+1}]
Do [km[1,m]=(Em[[m]]/((1-2*vm[[m]])*(1+vm[[m]]))*(1-2*vm[[m]])/2),
             {m,nplies+1}]
```

```
(*Calculate the Fiber strata Engineering Properties *)

Do[E11[[f]]=volf*Ef[[f]]+volm*Em[[f]],{f,nplies}]
Do[E33[[f]]=volf*Ef[[f]]+volm*Em[[f]],{f,nplies}]
Do[E22[[f]]=Em[[f]]*Ef[[f]]/(volf*Ef[[f]]+volm*Em[[f]]),
     {f,nplies}]

Do[v12[[f]]=volf*vf[[f]]+volm*vm[[f]],{f,nplies}]
Do[v13[[f]]=volf*vf[[f]]+volm*vm[[f]],{f,nplies}]
Do[v23[[f]]=v12[[f]]*E22[[f]]/E11[[f]],{f,nplies}]

Do[G13[[f]]=volf*Gf[[f]]+volm*Gm[[f]],{f,nplies}]
Do[G12[[f]]=Em[[f]]*Ef[[f]]/(volf*Ef[[f]]+volm*Em[[f]]),
          {f,nplies}]
Do[E23[[f]]=G12[[f]],{f,nplies}]

(*Calculate the fiber strata stiffness matrix form the Eng Con*)

Do [ V[[f]]=1-
v12[[f]]*(v12[[f]]*E22[[f]]/E11[[f]]+v23[[f]]*v13[[f]])-
v13[[f]]^2*1-v23[[f]]^2*E33[[f]]/E22[[f]],{f,nplies}]

Do [sf[1,f]=(1-v23[[f]]^2*E33[[f]]/E22[[f]])*E11[[f]]/V[[f]],
            {f,nplies}]
Do [sf[2,f]=(1-v13[[f]]^2*E33[[f]]/E11[[f]])*E22[[f]]/V[[f]],
            {f,nplies}]
Do [sf[3,f]=(v12[[f]]+v13[[f]]*E33[[f]]/E22[[f]])*
            (E22[[f]]/V[[f]]), {f,nplies}]
Do [sf[4,f]=(v23[[f]]+v12[[f]]*v13[[f]]*E22[[f]]/E11[[f]])*
            (E33[[f]]/V[[f]]), {f,nplies}]
Do [sf[5,f]=G12[[f]], {f,nplies}]
Do [sf[6,f]=G13[[f]], {f,nplies}]
Do [sf[7,f]=G12[[f]], {f,nplies}]

(*Calculate the rotated fiber strata stiffness matrix (sr)  *)

sr=Array[sfr,{13,nplies}]

Do[sfr[1,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
                Sin[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*Sin[theta[[f]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[2,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
                Sin[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]^2*Sin[theta[[f]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[3,f] =   Sin[theta[[f]]^2*(Sin[theta[[f]]^2*sf[2,f]+
                Cos[theta[[f]]^2*sf[3,f]+
```

```
                Cos[theta[[f]]^2*(Cos[theta[[f]]^2*sf[1,f]+
                Sin[theta[[f]]^2*sf[3,f]+
                Cos[theta[[f]]]^2*Sin[theta[[f]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[4,f] =   2*Cos[theta[[f]]]*Sin[theta[[f]]]*
                (2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[2,f]-
                Cos[theta[[f]]]^2+Sin[theta[[f]]]^2*sf[5,f],
                {f,nplies}]
Do[sfr[5,f] =   2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Sin[theta[[f]]]^2*sf[2,f]+Cos[theta[[f]]]^2*sf[3,f])-
                2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Cos[theta[[f]]]^2*sf[1,f]+Sin[theta[[f]]]^2*sf[3,f])+
                Cos[theta[[f]]]*Sin[theta[[f]]]*
                (Cos[theta[[f]]]^2+Sin[theta[[f]]])*sf[3,f]],
            {f,nplies}]
Do[sfr[6,f] =   -2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Sin[theta[[f]]]^2*sf[1,f]+Cos[theta[[f]]]^2*sf[3,f])-
                2*Cos[theta[[f]]]*Sin[theta[[f]]]*
            (Cos[theta[[f]]]^2*sf[2,f]+Sin[theta[[f]]]^2*sf[3,f])+
                Cos[theta[[f]]]*Sin[theta[[f]]]*
                (Cos[theta[[f]]]^2+Sin[theta[[f]]])*sf[5,f]],
            {f,nplies}]
Do[sfr[7,f] =Cos[theta[[f]]]^2*sf[6,f]+Sin[theta[[f]]]^2*sf[7,f],
                {f,nplies}]
Do[sfr[8,f] =Sin[theta[[f]]]^2*sf[6,f]+Cos[theta[[f]]]^2*sf[7,f],
                {f,nplies}]
Do[sfr[9,f] =-(Cos[theta[[f]]]*Sin[theta[[f]]]*sf[6,f])+
                Cos[theta[[f]]]*Sin[theta[[f]]]*sf[7,f],{f,nplies}]
Do[sfr[10,f] =  Cos[theta[[f]]]^2*sf[3,f]+Sin[theta[[f]]]*sf[4,f]
                {f,nplies}]
Do[sfr[11,f] =Sin[theta[[f]]]^2*sf[3,f]+Cos[theta[[f]]]^2*sf[4,f],
                {f,nplies}]
Do[sfr[12,f] =  sf[1,f],{f,nplies}]
Do[sfr[13,f] =  2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[3,f])+
                2*Cos[theta[[f]]]*Sin[theta[[f]]]*sf[4,f],
                {f,nplies}]

(* Put the stiffness into matrix form *)

S=Table[SS[f],{f,nplis}]
K=Table[KK[m],{m,nplies+1}]

Do[SS[f]={{ sr[[1]][[f]], sr[[3]][[f]],sr[[10]][[f]],sr[[5]],0,0},
          { sr[[3]][[f]], sr[[2]][[f]],sr[[11]][[f]],sr[[6]],0,0},
          {sr[[10]][[f]],sr[[11]][[f]],sr[[12]][[f]],sr[[13]],0,0}
          { sr[[5]][[f]], sr[[3]][[f]],sr[[10]][[f]],sr[[5]],0,0},
          {           0,        0,0,0, sr[[7]][[f]],sr[[9]][[f]]},
          {           0,        0,0,0, sr[[9]][[f]],sr[[8]][[f]]}}
          ,{f,nplies}]

Do[KK[m]={{ k[[1]][[m]], k[[2]][[m]],k[[2]][[m]],0,0,0},
          { k[[2]][[m]], k[[1]][[m]],k[[2]][[m]],0,0,0},
          { k[[2]][[m]], k[[2]][[m]],k[[1]][[m]],0,0,0},
          {          0,            0,          0,k[[3]][[m]],0,0},
```

```
        {           0,            0,            0,0,k[[3]][[m]],0},
        {           0,            0,            0,0,0,k[[3]][[m]]}}
        ,{m,nplies+1}]

(*Enter expressions for the load and phi, chi and w functions*)

P = P1

A1 = Array[AA1,{Np,Np}]
B1 = Array[BB1,{Np,Np}]
C1 = Array[CC1,{Np,Np}]

A2 = Table[AA[i],{i,Np^2}]
B2 = Table[BB[i],{i,Np^2}]
C2 = Table[CC[i],{i,Np^2}]

Do [A1[[n,l]]=A2[[l+(n-1)*Np]],{n,Np},{l,Np}]
Do [B1[[n,l]]=B2[[l+(n-1)*Np]],{n,Np},{l,Np}]
Do [C1[[n,l]]=C2[[l+(n-1)*Np]],{n,Np},{l,Np}]

phi = Sum [Sum[B1[[l,n]]*x^n*y^l,{n,NP}],{l,Np}]
chi = Sum [Sum[C1[[l,n]]*x^n*y^l,{n,NP}],{l,Np}]
w   = Sum [Sum[A1[[l,n]]*x^n*y^l,{n,NP}],{l,Np}]

(*Generate the derivatives of phi, chi and w *)

Do [chix1[i]  = D[chi[[i]],x], {i,Np^2}]
Do [chiy1[i]  = D[chi[[i]],y], {i,Np^2}]
Do [chixx1[i] = D[D[chi[[i]],x],x], {i,Np^2}]
Do [chiyy1[i] = D[D[chi[[i]],y],y], {i,Np^2}]
Do [chixy1[i] = D[D[chi[[i]],x],y], {i,Np^2}]

Do [phix1[i]  = D[phi[[i]],x], {i,Np^2}]
Do [phiy1[i]  = D[phi[[i]],y], {i,Np^2}]
Do [phixx1[i] = D[D[phi[[i]],x],x], {i,Np^2}]
Do [phiyy1[i] = D[D[phi[[i]],y],y], {i,Np^2}]
Do [phixy1[i] = D[D[phi[[i]],x],y], {i,Np^2}]

Do [wx1[i]  = D[w[[i]],x], {i,Np^2}]
Do [wy1[i]  = D[w[[i]],y], {i,Np^2}]
Do [wxx1[i] = D[D[w[[i]],x],x], {i,Np^2}]
Do [wyy1[i] = D[D[w[[i]],y],y], {i,Np^2}]
Do [wxy1[i] = D[D[w[[i]],x],y], {i,Np^2}]

(* Create the Differential Equations for a Composite Plate *)

edu = Integrate[
 (Sum[
     i*Be(
     Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(i-1),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]  )*wx[[i]] +
     -i*j*
    (Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i-2),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]*  phi[[i]]+
```

```
   (Sum[sr[[1,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]+
    Sum[Integrate[ k[[1,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}])*phixx[[i]]+
  2*Sum[sr[[5,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*phixy[[i]]+
   (Sum[sr[[4,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}] +
    Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}])*phiyy[[i]]+
    Sum[sr[[5,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*chixx[[i]]+
    Sum[sr[[6,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*chiyy[[i]]+
   (Sum[sr[[3,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]+
    Sum[sr[[4,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}] +
    Sum[Integrate[ k[[2,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}] )*chixy[[i]]
    , {i,Np^2}] ), {x,0,a},{y,0,b}]


edv = Integrate[
    (Sum[
     i*(
     Sum[Integrate[k[[3,m]]*(al[[m]]+bem[[m]]*z)^(i-1),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]*    wy[[i]] +
     -i*j*
   (Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i-2),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]*chi[[i+1]]+
    Sum[sr[[8,f]]*hf[[f]]*zf[[f]]^(j+i-2),{f,nplies}]*chi[[i]]+
    Sum[sr[[9,f]]*hf[[f]]*zf[[f]]^(j+i-2),{f,nplies}]
                                                  *chi[[i]])+
   (Sum[sr[[2,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]+
    Sum[Integrate[ k[[1,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]* chiyy[[i]]+
  2*Sum[sr[[6,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*chixy[[i]]+
   (Sum[sr[[4,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}] +
    Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}])*chixx[[i]]+
    Sum[sr[[6,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*phiyy[[i]]+
    Sum[sr[[5,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]*phixx[[i]]+
   (Sum[sr[[3,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}]+
    Sum[sr[[4,f]]*hf[[f]]*zf[[f]]^(j+i),{f,nplies}] +
    Sum[Integrate[ k[[2,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[Integrate[ k[[3,m]]*(al[[m]]+bem[[m]]*z)^(j+i),
              {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}] )*phixy[[i]]
    , {i,Np^2}] ), {x,0,a},{y,0,b}]

edw = -Integrate[
  (Sum[Integrate[k[[3,m]],{z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[sr[[7,f]]*hf[[f]],{f,nplies}])      * wxx[[i]]  +
  2*Sum[sr[[9,f]]*hf[[f]],{f,nplies}]       * wxy[[i]]
   (Sum[Integrate[k[[3,m]],{z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]+
    Sum[sr[[8,f]]*hf[[f]],{f,nplies}])      * wyy[[i]]   +
  Sum[i(
```

```
     Sum[Integrate[k[[3,m]]*(al[[m]]+bem[[m]]*z)^(i-1),
               {z,zmb[[m]],zmt[[m]]}],{m,nplies+1}]* wy +
     ,{i,Np^2}], {x,0,a},{y,0,b}]

(* Construct a system of equations for the Diff Equations *)

m = Array[mm,{2*Np^2+1,3*Np^2}]

fedu=Expand[edu]
Do[M[[j,i]] =  Coefficient[fedu[[j]],BB[i]],{j,Np^2},{i,Np^2}]
Do[M[[j,i+Np^2]] =
               Coefficient[fedu[[j]],CC[i]],{j,Np^2},{i,Np^2}]
Do[M[[j,i+2*Np^2]] =
               Coefficient[fedu[[j]],AA[i]],{j,Np^2},{i,Np^2}]

fedv=Expand[edv]
Do[M[[j+Np^2,i]] =
               Coefficient[fedu[[j]],BB[i]],{j,Np^2},{i,Np^2}]
Do[M[[j+Np^2,i+Np^2]] =
               Coefficient[fedu[[j]],CC[i]],{j,Np^2},{i,Np^2}]
Do[M[[j+Np^2,2*Np^2]] =
               Coefficient[fedu[[j]],AA[i]],{j,Np^2},{i,Np^2}]

fedw=Expand[edw]
Do[M[[2*Np^2+1,i]] =
               Coefficient[fedu[[j]],BB[i]],{j,Np^2},{i,Np^2}]
Do[M[[2*Np^2+1,i+Np^2]] =
               Coefficient[fedu[[j]],CC[i]],{j,Np^2},{i,Np^2}]
Do[M[[2*Np^2+1,i+2*Np^2]] =
               Coefficient[fedu[[j]],AA[i]],{j,Np^2},{i,Np^2}]

(* Create the answer vector (bm) for the m*sol=bm system *)

bm = Array [bbm,{2*Np^2+1,1}]
Do [bm[[i]]=0, {i,2*Np^2}]
Do [bm[[2*Np^2+1]]=Integrate [P ,{x,0,a}, {y,0,b}]

(* Solve the system of Equations for the sol vector *)

sol=LinearSolve[m,bm]

(* Determine the B1, C1, A1 coefficients *)

Do[BB[i]=sol[[i],{i,Np^2}]
Do[CC[i]=sol[[i+Np^2},{i,Np^2}]
Do[AA[i]=sol[[i+2*Np^2]],{i,Np^2}]


(* Creat the Displacement functions for the fiber strata *)

Uf[z_]=Table[uff[z][f],{f,nplies}]
Vf[z_]=Table[vff[z][f],{f,nplies}]
Wf[z_]=Table[wff[z][f],{f,nplies}]
Do[uff[z_][f] = Sum[zf[[f]]^i*phi[[i]],{i,Np^2},{f,nplies}]
```

```
Do[vff[z_][f] = Sum[zf[[f]]^i*chi[[i]],{i,Np^2},{f,nplies}]
Do[wff[z_][f] = Sum[            w[[1]],{i,Np^2},{f,nplies}]

(* Create the Displacement Functions for the matrix strata *)

Um[z_]=Table[umm[z][m],{m,nplies+1}]
Vm[z_]=Table[vmm[z][m],{m,nplies+1}]
Wm[z_]=Table[wmm[z][m],{m,nplies+1}]
Do[umm[z_][f] = Sum[(al[[m]]+Bem[[m]]*z)^i*phi[[i]],
                                        {i,Np^2},{m,nplies+1}]
Do[vmm[z_][f] = Sum[(al[[m]]+Bem[[m]]*z)^i*chi[[i]],
                                        {i,Np^2},{m,nplies+1}]
Do[wmm[z_][f] = Sum[                       w[[1]],
                                        {i,Np^2},{m,nplies+1}]

(* Create the Strain Functions for the matrix strata *)

efx[z_]=Table[effx[z][f],{f,nplies}]
efy[z_]=Table[effy[z][f],{f,nplies}]
efz[z_]=Table[effz[z][f],{f,nplies}]
efxy[z_]=Table[effxy[z][f],{f,nplies}]
efxz[z_]=Table[effxz[z][f],{f,nplies}]
efyz[z_]=Table[effyz[z][f],{f,nplies}]

Do [effx[z_][f]=D[Uf[z][[f]],x],{f,nplies}]
Do [effy[z_][f]=D[Vf[z][[f]],y],{f,nplies}]
Do [effz[z_][f]=0,{f,nplies}]
Do [effx[z_][f]=D[Uf[z][[f]],y]+D[Vf[z][[f]],x],{f,nplies}]
Do [effy[z_][f]=D[Uf[z][[f]],z]+D[Wf[z][[f]],x],{f,nplies}]
Do [effy[z_][f]=D[Vf[z][[f]],z]+D[Wf[z][[f]],y],{f,nplies}]

(* Create the Strain Functions for the matrix strata *)

emx[z_]=Table[emmx[z][m],{m,nplies+1}]
emy[z_]=Table[emmy[z][m],{m,nplies+1}]
emz[z_]=Table[emmz[z][m],{m,nplies+1}]
emxy[z_]=Table[emmxy[z][m],{m,nplies+1}]
emxz[z_]=Table[emmxz[z][m],{m,nplies+1}]
emyz[z_]=Table[emmyz[z][m],{m,nplies+1}]

Do [emmx[z_][f]=D[Um[z][[f]],x],{f,nplies}]
Do [emmy[z_][f]=D[Vm[z][[f]],y],{f,nplies}]
Do [emmz[z_][f]=0,{f,nplies}]
Do [emmx[z_][f]=D[Um[z][[m]],y]+D[Vf[z][[m]],x],{m,nplies+1}]
Do [emmy[z_][f]=D[Um[z][[m]],z]+D[Wf[z][[m]],x],{m,nplies+1}]
Do [emmy[z_][f]=D[Vm[z][[m]],z]+D[Wf[z][[m]],y],{m,nplies+1}]

(* Create the Strain vector for the fiber strata *)

Fstrain[z_]=Table[fstrain[z][f],{f,nplies}]
Do[fstrain[z_][f]={efx[z],efy[z][f],efz[z][f],efxy[z][f],
                   efxz[z][f],efyz[z][f]},{f,nplies}]

(* Create the Strain vector for the matrix strata *)
```

```
Mstrain[z_]=Table[mstrain[z][m],{m,nplies+1}]
Do[mstrain[z_][f]={emx[z][m],emy[z][m],emz[z][m],emxy[z][m],
                   emxz[z][m],emyz[z][m]},{m,nplies+1}]

(* Calculate the Stress in each stratum by Stiffness*strain *)

Fstress[z_]= Table[fstress[z][f],{f,nplies}]
Do[fstress[z_][f]= S[[f]].Fstrain[z][[f]],{f,nplies}]

Mstress[z_]= Table[mstress[z][m],{m,nplies+1}]
Do[mstress[z_][m]= S[[m]].Fstrain[z][[m]],{m,nplies+1}]

(*  Place the stress componentes into plotable form *)

smx[z_]=Table[smxx[z][m],{m,nplies+1}]
smy[z_]=Table[smyy[z][m],{m,nplies+1}]
smz[z_]=Table[smzz[z][m],{m,nplies+1}]
smxy[z_]=Table[smxyy[z][m],{m,nplies+1}]
smxz[z_]=Table[smxzz[z][m],{m,nplies+1}]
smyz[z_]=Table[smyzz[z][m],{m,nplies+1}]

sfx[z_]=Table[sfxx[z][f],{f,nplies}]
sfy[z_]=Table[sfyy[z][f],{f,nplies}]
sfz[z_]=Table[sfzz[z][f],{f,nplies}]
sfxy[z_]=Table[sfxyy[z][f],{f,nplies}]
sfxz[z_]=Table[sfxzz[z][f],{f,nplies}]
sfyz[z_]=Table[sfyzz[z][f],{f,nplies}]

Do[smxx[z_][m] =Mstress[z][[m]][[1]],{m,nplies+1}]
Do[smyy[z_][m] =Mstress[z][[m]][[2]],{m,nplies+1}]
Do[smzz[z_][m] =Mstress[z][[m]][[3]],{m,nplies+1}]
Do[smxyy[z_][m]=Mstress[z][[m]][[4]],{m,nplies+1}]
Do[smxzz[z_][m]=Mstress[z][[m]][[5]],{m,nplies+1}]
Do[smyzz[z_][m]=Mstress[z][[m]][[6]],{m,nplies+1}]

Do[sfxx[z_][f] =Fstress[z][[f]][[1]],{f,nplies}]
Do[sfyy[z_][f] =Fstress[z][[f]][[2]],{f,nplies}]
Do[sfzz[z_][f] =Fstress[z][[f]][[3]],{f,nplies}]
Do[sfxyy[z_][f]=Fstress[z][[f]][[4]],{f,nplies}]
Do[sfxzz[z_][f]=Fstress[z][[f]][[5]],{f,nplies}]
Do[sfyzz[z_][f]=Fstress[z][[f]][[6]],{f,nplies}]
```

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Vita

Captain Richard A. Coveno was born on June 13, 1965 in Burlington Massachusetts. He graduated from Burlington High School in 1983 and attended the Air Force Academy in Colorado Springs Colorado. He majored in Engineering Mechanics and graduated in the top 10 percent of his class in 1987. He was assigned to Hill AFB Utah in the Ogden Air Logistics Center as an F-4 Systems Engineer. As an engineering program manager he wrote system specifications and managed engineering contracts. He also managed the test and evaluation program for a major RF-4C modification program.

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