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13. ABSTRACT (Maximum 200 words):

A successful nonlinear partial differential equation based approach to restoration was carried out, ENO least squares, shock filters, feature detectors and total variation based deconvolution techniques were combined. Also rigorous morphological methods and wavelet analysis were developed and used to restore noisy, blurry images.

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I. Introduction.

A nonlinear partial differential based approach to some of the basic problems of image processing was initiated. Problems of noise removal, enhancement, and approximation of restored noisy, blurry images were attacked using this new approach. We have overcome many of the difficulties experienced by standard techniques such as spurious oscillations (ringing) and/or excessive blurring of edges. Here, we used essentially nonoscillatory (ENO) least squares approximation, (see e.g. [3]), together with feature detectors, shock filters [4] and total variation [5, 6, 21] based deconvolution to produce a state-of-the-art enhancement technique. Moreover the ENO preprocessing was used together with a flame filter based on Hamilton-Jacobi type ideas. This was generalized to the unique class of morphological requirements [19] satisfying operators using numerical techniques similar to those originating in [2]. Additionally, an anisotropic diffusion process originated by Malik and Perona [20] was turned into an anisotropic shock/rarefaction filter. The resulting evolution equation both enhances and denoises.

A major breakthrough came in our total variation based restoration of noisy, blurred images. The total variation of the image is minimized subject to constraints involving the point spread function of the blurring process and the statistics of the noise. The solution is obtained using Euler-Lagrange equation with artificial time evolution, and Lagrange multipliers enforce the constraints. This amounts to solving an interesting time dependent partial differential equation on a manifold determined by the constraints. As t increases the image is restored. The numerical algorithm is simple to implement and appears to be nonoscillatory (minimal ringing) and noninvasive (recovers sharp edges).

II. Restoration Algorithms.

Our first restoration algorithm involved additive noise:

We solve the following problem. Let

$$(2.1) \quad u_0(x, y) = (Au)(x, y) + \bar{n}(x, y)$$

where A is a linear integral operator and $\bar{n}$ is additive noise. Also $u_0(x, y)$ is the observed intensity function while $u(x, y)$ is the image to be restored. Both are defined on a region Ω in R^2 .

A may be a convolution type integral operator in which case we write:

$$(2.2) \quad (Au)(x, y) = (k * u)(x, y).$$

Examples we shall experiment with include

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Motion blur:

$$(2.3a) \quad \begin{aligned} k(x, y) &= \frac{1}{a} \quad \text{if} \quad -\frac{a}{2} \leq x \leq \frac{a}{2} \\ k(x, y) &= 0 \quad \text{elsewhere.} \end{aligned}$$

Diffraction limited blur:

$$(2.3b) \quad k(x, y) = a \operatorname{sinc} ax$$

where $\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$ and a is related to the aperture size and bandwidth limitation of the transmission system.

Defocus blur:

$$(2.3c) \quad \hat{k}(\omega_x, \omega_y) = \frac{J_1(a\sqrt{\omega_x^2 + \omega_y^2})}{a\sqrt{\omega_x^2 + \omega_y^2}}$$

where $\hat{\cdot}$ denotes the Fourier transform $J_1(x)$ is the Bessel function; and $a > 0$ measures the degree of blurring ($a = 0$ is just the identity map).

Gaussian blur:

$$(2.3d) \quad k(x, y) = (4\pi a)^{-1} \exp \left[-\frac{(x^2 + y^2)}{4a} \right]$$

which arises e.g., in atmospheric turbulence.

All these kernels result in severely blurred images whose restoration is an ill-posed problem. Inverting the equation

$$(2.4) \quad k * u = u_0$$

leads to the formula

$$u = \mathcal{F}^{-1}(\hat{k})^{-1} \mathcal{F} u_0$$

where $\mathcal{F}$ is the Fourier transform, $\mathcal{F}^{-1}$ its inverse and $\hat{k}$ is the transform of the kernel. The ill-posedness comes from the zeros of $\hat{k}(\xi, \eta)$ for ξ, η in bounded regions of R^2 and for $\xi^2 + \eta^2 \rightarrow \infty$. Even in the absence of noise, quantization error leads to severe ringing in this process.

The ill-posedness of this procedure has long been well known. Conventional variational approaches to this problem involve a least squares L^2 fit because this leads to linear equations. The first attempt along these lines was made by Phillips [11] and later refined by Twomey [12,13] in the one dimensional case. In our two dimensional continuous framework their constrained minimization problem becomes

$$(2.5a) \quad \text{Minimize} \quad \int_{\Omega} (u_{xx} + u_{yy})^2 dx dy$$

subject to constraints involving the mean

$$(2.5b) \quad \int_{\Omega} u dx dy = \int_{\Omega} u_0 dx dy$$

and standard deviation

$$(2.5c) \quad \int_{\Omega} (Au - u_0)^2 dx dy = \sigma^2.$$

The resulting linear system is easy to solve using modern numerical linear algebra. However, the results are rather disappointing, see section VI below.

We instead minimize the variation of the image, which is a direct measure of how oscillatory it is. The space of functions of bounded variation (BV) is appropriate for discontinuous functions. This is well known in the field of shock calculations, e.g., [3], and references therein.

Thus, our constrained minimization problem is

$$(2.6a) \quad \text{Minimize} \quad \int_{\Omega} \sqrt{u_x^2 + u_y^2} dx dy$$

subject to constraints involving u_0 .

In our work so far we have taken the same two constraints (2.5b) and (2.5c).

Again (2.5b) indicates that the white noise $\bar{n}(x, y)$ in (2.1) is of zero mean, and (2.5c) uses *a priori* information that the standard deviation of the noise $\bar{n}(x, y)$ is σ .

In this case we arrive at the Euler-Lagrange equations

$$(2.7a) \quad 0 = \frac{\partial}{\partial x} \left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{u_y}{\sqrt{u_y^2 + u_x^2}} \right) \\ - \lambda(A^*Au - A^*u) \text{ in } \Omega$$

with

$$(2.7b) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Omega.$$

Here A^* is just the adjoint integral operator.

The constant λ is a Lagrange multiplier chosen so that the constraint (2.5b) is satisfied. The first constraint (2.5a) will be shown in Remark 4 below to be satisfied if:

$$(2.8) \quad \int Au \equiv \int u \quad \text{and} \quad \int A^*u \equiv \int u$$

for each u . This is true up to normalization for A a convolution. Thus we assume (2.8) for simplicity only.

We shall use the gradient-projection method of Rosen [15] which, in this case, becomes the interesting "constrained" partial differential equation

$$(2.9a) \quad u_t = \frac{\partial}{\partial x} \left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}} \right) - \lambda A^*(Au - u_0)$$

for $t > 0$, $(x, y) \in \Omega$

$$(2.9b) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

and $u(x, y, 0)$ given so that (2.5b,c) are satisfied. If (2.5b) is satisfied initially, e.g. $u(x, y, 0) = u_0(x, y)$, then, by the conservation form of (2.9a,b) and by (2.8), it is always satisfied. Satisfying (2.5c) can be done through a process described in the next section.

The projection step in the gradient-projection method just amounts to updating $\lambda(t)$ so that (2.5c) remains true in time. One merely differentiates (2.5c) with respect to time, replaces u_t by the right side of (2.9) and then chooses $\lambda(t)$ so that this is set to be zero. This amounts to multiplying the right side of (2.9) by $A^*(Au - u_0)$, integrating over Ω , setting the result equal to zero, and solving for $\lambda(t)$.

We have

$$(2.10) \quad \lambda(t) = \frac{\int_{\Omega} \left(\frac{\partial}{\partial x} \left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}} \right) \right) \cdot A^*(Au - u_0) dx dy}{\int_{\Omega} (A^*(Au - u_0))^2 dx dy}$$

We thus have a dynamical procedure for restoring the image. As $t \rightarrow \infty$ the steady state solution is the desired restoration.

Remark (1.1). Equation (2.9a) can be written as

$$(2.11) \quad u_t = \mathcal{K}(u) - \lambda A^*(Au - u_0)$$

where $\mathcal{K}(u)$ is the curvature of the level set $u = \text{constant}$ at each point. This part of the operator can be viewed as moving each level set normal to itself with velocity equal to its curvature, divided by the magnitude of the gradient. The constraint term just acts to project the motion back so that (2.5c) is satisfied.

Remark (1.2). The method is quite general as regards nature and number of constraints. Generally one has many Lagrange multipliers and one must invert a Gram type of matrix to update their values. In particular, one may localize the constraints over space.

Remark (1.3). The method is also general as regards the nature of the noise. We have recently experimented successfully with both multiplicative and speckle noise. Again, we demonstrate the former in section VI and discuss the algorithm in section IV.

III. Numerical Approximations.

The easiest way to construct an efficient numerical approximation to (2.7) is to approximate the variational problem (2.6) with constraints (2.5b,c) directly.

We do this as follows: Let $x_i = i\Delta x$, $y_j = j\Delta x$, $i, j = 0, 1, \dots, N$, with $N\Delta x = 1$, and also $u_{ij} \simeq u(i\Delta x, j\Delta x)$. Our discrete variational problem is

$$(3.1a) \quad \text{Minimize} \quad \sum_{i,j=0}^{N-1} \sqrt{(\Delta_+^x u_{ij})^2 + (\Delta_+^y u_{ij})^2} + \epsilon$$

subject to the constraints:

$$(3.1b) \quad \sum_{i,j=0}^N u_{ij} \Delta x = \sum_{i,j=0}^N (u_0)_{ij} \Delta x$$

$$(3.1c) \quad \|Au - u_0\|^2 = \sum_{i,j=0}^N |(Au)_{ij} - (u_0)_{ij}|^2 (\Delta x) = \sigma^2.$$

Here $\Delta_{\pm}^x u_{ij} = \pm(u_{i\pm 1,j} - u_{ij})$, similarly for $\Delta_{\pm}^y u_{ij}$; and $\epsilon > 0$, of the order of round-off error, is used to avoid division by zero.

The discrete Euler-Lagrange equations come just by differentiating (3.1) with respect to u_{ij} and using a Lagrange multiplier. We arrive at:

$$(3.2a) \quad \begin{aligned} 0 = & \Delta_-^x \left(\frac{\Delta_+^x u_{ij}}{\sqrt{(\Delta_+^x u_{ij})^2 + (\Delta_+^y u_{ij})^2 + \epsilon}} \right) \\ & + \Delta_-^y \left(\frac{\Delta_+^y u_{ij}}{\sqrt{(\Delta_+^x u_{ij})^2 + (\Delta_+^y u_{ij})^2 + \epsilon}} \right) \\ & - \lambda [(A^* Au)_{ij} - (A^* u)_{ij}] \end{aligned}$$

with boundary conditions

$$(3.2b) \quad \Delta_+^x u_{0j} \equiv 0 \equiv \Delta_+^x u_{N-1,j}, \quad j = 0, 1, \dots, N$$

$$(3.2c) \quad \Delta_+^y u_{i0} \equiv 0 \equiv \Delta_+^y u_{i,N-1}, \quad i = 0, 1, \dots, N$$

We approximate this by the gradient-projection method:

$$(3.3) \quad \begin{aligned} u_{ij}^{n+1} = & u_{ij}^n + c \left[\Delta_-^x \left(\frac{\Delta_+^x u_{ij}^n}{\sqrt{(\Delta_+^x u_{ij}^n)^2 + (\Delta_+^y u_{ij}^n)^2 + \epsilon}} \right) \right. \\ & + \Delta_-^y \left(\frac{\Delta_+^y u_{ij}^n}{\sqrt{(\Delta_+^x u_{ij}^n)^2 + (\Delta_+^y u_{ij}^n)^2 + \epsilon}} \right) \Big] \\ & - \lambda [(A^* Au)_{ij}^n - (A^* u_0)_{ij}] . \end{aligned}$$

Typical values of $c = \frac{\Delta t}{\Delta x}$ range from .3 to 3.

Remark (1.4). We also used a more isotropic approximation of the discrete vari-

ation; i.e. minimize

$$(3.1c) \quad \frac{1}{2} \sum_{i,j=1}^{n-1} \sqrt{(\Delta_+^x u_{ij})^2 + (\Delta_+^y u_{ij})^2 + \epsilon} \\ + \frac{1}{2} \sum_{i,j}^{n-1} \sqrt{(\Delta_+^\xi u_{ij})^2 + (\Delta_+^\eta u_{ij})^2 + \epsilon}$$

where

$$(3.1e) \quad \Delta_+^\xi u_{ij} = \frac{u_{i+1,j+1} - u_{ij}}{\sqrt{2}}$$

$$(3.1f) \quad \Delta_+^\eta u_{ij} = \frac{u_{i+1,j-1} - u_{ij}}{\sqrt{2}}$$

yielding simple expressions corresponding to (3.2a) and (3.3). The results are slightly more pleasing.

Rather than use a discrete version of (2.10) we impose (3.1c) directly on u^{n+1} in (3.3). This amounts to solving a quadratic equation for λ^n as follows

$$(3.4a) \quad \|Au^{n+1} - u_0\|^2 = \sigma^2.$$

This means, among other things, that the constraint (3.1c) need not be satisfied by the initial guess $\{u_{ij}^0\}$. In our experiments we usually took $u_{ij}^0 = (u_0)_{ij}$. Also the resulting quadratic equation for λ derived below has (in general) two complex roots:

$$(3.5) \quad \lambda_{\pm} = c_1 \pm (\text{sign } c_1) \sqrt{c_2}$$

If $c_2 \leq 0$, we just set $\lambda = c_1$. If $c_2 > 0$ we take $\lambda = \lambda_-$. We have found in our experiments that this procedure eventually drives us to the manifold (3.1c) for $\sigma > 0$. Constraint (3.1b) is automatically satisfied by the procedure (3.3) when the initial data satisfies the constraint.

To simplify the solution procedure we rewrite (3.3) as

$$(3.6) \quad u_{ij}^{n+1} = p_{ij}^n - \lambda^n q_{ij}^n$$

and define $\{p_{ij}^n\} = p^n$, $\{q_{ij}^n\} = q^n$.

Then (3.4a) becomes

$$(3.7) \quad \begin{aligned} \|Au^{n+1} - u_0\|^2 &= \sigma^2 = \|Ap^n - \lambda^n Aq^n - u_0\|^2 \\ &= \lambda^2 \|Aq^n\|^2 - 2\lambda(Aq^n, Ap^n - u_0) + \|Ap^n - u_0\|^2 \end{aligned}$$

(dropping the superscript n).

$$(3.8) \quad \begin{aligned} \lambda_{\pm} &= (\|Aq\|)^{-2} \operatorname{sign}(Aq, Ap - u_0) \left[|(Aq, Ap - u_0)| \right. \\ &\quad \left. \pm \operatorname{sign}(Aq, Ap - u_0) \sqrt{(Aq, Ap - u_0)^2 - \|Aq\|^2(\|Ap - u_0\|^2 - \sigma^2)} \right] \end{aligned}$$

The definition of λ_- above approaches that obtained by the gradient-projection method in (2.10). Precisely we may write

$$(3.9) \quad \begin{aligned} \lambda_- &= \frac{1}{2} \frac{(\|Ap - u_0\|^2 - \sigma^2)}{(Aq, Ap - u_0)} \\ &\quad + O\left(\frac{\|Aq\|^2(\|Ap - u_0\|^2 - \sigma^2)^2}{|(Aq, Ap - u_0)^3|}\right). \end{aligned}$$

Using (3.6), (3.3), and Taylor's theorem we have (in smooth regions)

$$(3.10) \quad \begin{aligned} (Ap - u_0)_{ij} &= (Au)_{ij} - (u_0)_{ij} + \Delta t \left[A \left[\frac{\partial}{\partial x} \left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}} \right) \right] \right]_{ij} \\ &\quad + O(\Delta x) \Big]. \end{aligned}$$

So if (3.10) is satisfied at time level n we have

$$(3.11) \quad \begin{aligned} \|Ap - u_0\|^2 - \sigma^2 &= 2\Delta t(Au^n - u_0, A[\frac{\partial}{\partial x} \frac{u_x}{\sqrt{u_x^2 + u_y^2}} + \frac{\partial}{\partial y} \frac{u_y}{\sqrt{u_x^2 + u_y^2}}]) \\ &\quad + O((\Delta t)^2 + (\Delta x)^2) \end{aligned}$$

Also

$$(3.12) \quad (Aq)_{ij} = (AA^* \cdot (Au - u_0))_{ij}.$$

So

$$(3.13) \quad (Aq, Ap - u_0) = (A^*(Au^n - u_0), A^*(Au^n - u_0)) + O(\Delta t).$$

Finally, from (2.10), (3.11), (3.13) and (3.9) we have

$$(3.14) \quad \frac{\lambda^n}{\Delta t} = \lambda(n\Delta t) + O(\Delta t + \Delta x)$$

which is the desired result.

IV. Multiplicative Noise.

This time we shall solve the following problem. Let

$$(4.1) \quad u_0(x, y) = \bar{n}Au.$$

where A is as in (2.1) here $\bar{n}$ has mean one and standard deviation σ^2 .

We shall use the same class of convolution kernels as in the additive noise case. We shall again minimize the variation of the image, as in (2.6c), this time subject to constraints of the following type:

$$(4.2a) \quad \int_{\Omega} \left(\frac{Au}{u_0} \right) dx dy = \int_{\Omega} \left(\frac{1}{\bar{n}} \right) dx dy \quad (\text{involving the mean})$$

and

$$(4.2b) \quad \frac{1}{2} \int_{\Omega} \frac{(Au - u_0)^2}{(u_0)^2} dx dy = \frac{1}{2} \int_{\Omega} \left(\frac{(\bar{n} - 1)}{\bar{n}^2} \right)^2 dx dy \quad (\text{involving the standard deviation})$$

The resulting gradient-projection algorithm involves two Lagrange multipliers.

$$(4.3) \quad u_t = \frac{\partial}{\partial x} \left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}} \right) + \lambda A^* \left(\frac{Au - u_0}{u_0^2} \right) + \mu A^* \left(\frac{1}{u_0} \right).$$

We follow the gradient-projection technique, solving for both parameters with very successful results, as we shall see in section VI below.

5. Theoretical justification.

This work is joint with Professor Pierre-Louis Lions. We begin with analysis of the minimization problem (2.6a) subject to linear constraints.

We first need to formulate the problem in $BV(\Omega)$ – we assume that Ω is smooth enough so that extensions to $BV(\mathbb{R}^2)$ are possible, a fact that is true as long as Ω is smooth or if Ω is a rectangle... We denote by $\int_{\Omega} |Du|$ the seminorm on BV that coincides with $\int_{\Omega} \sqrt{u_x^2 + u_y^2} dx dy$ when u is smooth (or $W^{1,1} \dots$). Then, we consider the following minimization problem:

$$(5.1) \quad \inf \left\{ \int_{\Omega} |Du| / u \in BV(\Omega), \int_{\Omega} |Au - u_0|^2 = \sigma^2, \int_{\Omega} (u - u_0) = 0 \right\}.$$

Proposition 5.1: *The minimization problem (5.1) admits at least one solution if A is compact from $L^2(\Omega)$ into $L^2(\Omega)$.*

Proof. We first assume that because of (2.8) $A1 = 1$ and thus by standard functional analysis arguments, $\int_{\Omega} |Du| + \|Au\|_{L^2(\Omega)}$ is a norm on $BV(\Omega)$ which is equivalent to the usual norm.

Then, this implies that minimizing sequences of (5.1) are bounded in $BV(\Omega)$ and thus in $L^2(\Omega)$ by Sobolev imbedding. Therefore, if $\{u_n\}$ denotes an arbitrary minimizing sequence of (5.1), we may assume without loss of generality that $\{u_n\}$ converges weakly to some u in $BV(\Omega)$ (weak-* convergence) and in $L^2(\Omega)$. Therefore, we recover at the limit (2.5b) and (2.5c) follows from the assumption that A is compact so that Au_n converges to Au in $L^2(\Omega)$. We conclude then easily that u is a minimum since we have by weak convergence

$$\int_{\Omega} |Du| \leq \liminf_n \int_{\Omega} |Du_n|. \quad \square$$

Remark 5.1: Notice that the assumption made in Proposition (5.1) on A excludes the obviously interesting case when A is the identity operator, in which case (5.1) turns out to be a highly non-trivial minimization problem related to isoperimetric inequalities and geometrical problems. Any discussion of this would be too technical for the main purpose of this report.

We next study equations of the form (2.9a) namely

$$(5.2a) \quad u_t = \sum_{i=1}^2 \frac{\partial}{\partial x_i} \{a_i(\nabla u)\} - \lambda A^*(Au - u_0) \text{ for } x \in \Omega, t > 0$$

where $\lambda = \lambda(t)$ is a Lagrange multiplier associated to the constraint (2.5c), A is a bounded linear operator from $L^2(\Omega)$ into $L^2(\Omega)$ and A^* denotes its adjoint. Finally, $a_i(p) = \frac{\partial}{\partial p_i} a(p)$ ($i = 1, 2$) where a is smooth (say, $C^2(R^2)$ with bounded derivatives up to order 2), a is spherically symmetric (rotational invariance). Of course, the model introduced in this work corresponds to $a(p) = |p|$. However, this choice induces such singularities that a mathematical analysis does not seem to be possible. This is why we shall study some model cases involving slightly regularized variants of this choice. In fact, numerically, scaling (2.9a) also induces some regularizations of a quite similar to the ones we make below – and, thus, our analysis covers situations that are quite realistic from the numerical viewpoint.

We shall therefore assume that there exists $\nu \in (0, 1)$ such that

$$(5.3) \quad (\nu \delta_{ij}) \leq \left(\frac{\partial^2 a}{\partial p_i \partial p_j}(p) \right) \leq \left(\frac{1}{\nu} \delta_{ij} \right) \text{ for all } p \in \mathbb{R}^2$$

in the sense of symmetric matrices.

If we go back to our real choice of a , namely $a(p) = |p|$, we see that (5.3) does not hold for p near zero and for $|p| \rightarrow \infty$. The singularity at $p = 0$ induces mathematical and numerical difficulties. In practice we truncate $\frac{\partial a}{\partial p}$ near $p = 0$. The assumption for p large can be relaxed by proving some upper bounds with a rather technical argument (contained in the proof below). We prefer to skip this technical argument in order to avoid confusing the main issue which concern the imposition of constraints.

Finally, we observe that enforcing (2.5c) while scaling (5.2a) amounts to requiring that

$$(5.2b) \quad \lambda = A[u] A^*(Au - u_0) dx \|A^*(Au - u_0)\|_{L^2(\Omega)}^{-2}, \|Au - u_0\|_{L^2(\Omega)} = \sigma.$$

where $A[u] = \sum_{i=1}^2 \frac{\partial}{\partial x_i} (a_i(\nabla u))$.

We also prescribe an initial condition

$$(5.2c) \quad u|_{t=0} = u^0 \text{ in } \Omega$$

and boundary condition

$$(5.2d) \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

Of course, we need to assume that u^0 satisfies

$$(5.4) \quad \|Au^0 - u_0\|_{L^2(\Omega)} = \sigma.$$

Remark 5.2: The existence of some u^0 satisfying (5.4) is not obvious and depends very much on the properties of A . This existence is certainly ensured by the following assumption

$$(5.5) \quad R(A)(\text{range of } A) \text{ is dense in } L^2(\Omega).$$

Indeed, this condition implies that there exist $u^1, u^2 \in L^2(\Omega)$ (or as smooth as we wish by density and continuity of A) satisfying

$$(5.6) \quad \|Au^1 - u_0\|_{L^2(\Omega)} < \sigma, \quad \|Au^2 - u_0\|_{L^2(\Omega)} > \sigma.$$

And it is enough to take $u^0 = \theta u^1 + (1 - \theta)u^2$ for some convenient $\theta \in (0, 1)$. $\square$

In order to illustrate clearly the mathematical difficulties and results associated with the system (5.2a-d) we begin with the model case where $a(p) = \frac{1}{2}|p|^2$ so that $a_i(\nabla u) = \nabla u$ and $A[u]$ reduces to the classical linear operator $A[u] = \Delta u$ (the Laplace operator). In that case, we can prove the

Theorem 5.1: *Let A satisfy (5.5), let $u^0 \in H^1(\Omega)$ satisfy (5.4), let $u_0 \in L^2(\Omega)$ and let $a(p) = \frac{1}{2}|p|^2$ on $\mathbb{R}^2$. Then, there exists a unique solution u of (5.2a-d) satisfying: $u \in C([0, \infty); H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$, $u_t \in L^2(0, T; L^2(\Omega))$, $\lambda \in L^2(0, T)$ for all $T \in (0, \infty)$.*

Remark 5.3: If A is a convolution operator i.e. $Au = k * u$ then (5.5) holds if and only if the Fourier transform $\hat{k}$ of k satisfies

$$(5.7) \quad \text{meas } \{\xi \in \mathbb{R}^2 / \hat{k}(\xi) = 0\} = 0.$$

Remark 5.4: If A and A^* map boundedly $L^p(\Omega)$ into $L^p(\Omega)$ for $2 \leq p < \infty$ then the proof below also show that $u \in C([0, \infty); W^{1,p}(\Omega)) \cap L^p(0, T; W^{2,p}(\Omega))$, $u_t \in L^p(0, T; L^p(\Omega))$, $\lambda \in L^p(0, T)$ for all $T \in (0, \infty)$ if $u^0 \in W^{1,p}(\Omega)$, $u_0 \in L^p(\Omega)$.

Remark 5.5: The proof also applies to different type of constraints that can even be nonquadratic constraints. Let us only mention a few possibilities for which the same result as the one above holds. In the case of multiplicative noise as in the previous section, we can replace (5.2b) by

$$(5.8) \quad \int_{\Omega} \left(\frac{u}{u_0} \right)^2 dx = \sigma^2 > 0$$

assuming for instance that $\frac{1}{u_0^2} \in L^\infty(\Omega)$. Also, we might want to enforce local constraints on a finite partition (or subpartition) of Ω , that in practice can be obtained by a segmentation algorithm. In that case, we consider $\omega_1, \dots, \omega_m$ ($m \geq 1$) measurable sets in Ω such that $\text{meas}[\omega_i \cap \omega_j] = 0$ for all $1 \leq i \neq j \leq m$ and we replace (5.2b)

$$(5.9) \quad \int_{\omega_i} |Au - u_0|^2 dx = \sigma_i^2 > 0.$$

Then, Theorem (5.1) still holds for the corresponding condition equation that involves now m different Lagrange multipliers $\lambda_i = \int_{\omega_i} A[u] A^*(Au - u_0) dx$. $\square$

Proof of Theorem 5.1:

Step 1: General a priori estimates.

Here, we list some general consequences of the fact that the evolution equation we are considering is a gradient flow (of a constrained functional). Indeed, multiplying (5.2a) by u_t and using (5.2b), we deduce

$$(5.10) \quad \int_{\Omega} |u_t|^2 dx + \frac{d}{dt} \int_{\Omega} a(\nabla u) dx = 0 \text{ for } t \geq 0.$$

Hence, u_t is bounded in $L^2(0, \infty; L^2(\Omega))$ and ∇u is bounded in $L^\infty(0, \infty; L^2(\Omega))$. In particular, $u = \int_0^t u_t ds + u^0$ is bounded in $L^\infty(0, T; L^2(\Omega))$ for all $T \in (0, \infty)$ and thus u is bounded in $C([0, T]; H^1(\Omega))$ for all $T \in (0, \infty)$.

Step 2: A lower bound

We want to show that $\|A^*(Au - u_0)\|_{L^2(\Omega)}$ is bounded from below uniformly on $[0, T]$ (for all $T \in (0, \infty)$) and that the lower bound depends only on T and on the H^1 norm of u^0 . Indeed, if this were not the case, in view of the estimates shown in Step 1 and in view of (5.2b), this would yield the existence of a sequence $\{u_j\}_{j \geq 1}$ such that u_j is bounded in $H^1(\Omega)$ and

$$(5.11) \quad \|Au_j - u_0\|_{L^2(\Omega)} = \sigma, \quad \|A^*(Au_j - u_0)\|_{L^2(\Omega)} \xrightarrow{j} 0.$$

Without loss of generality, we may assume, extracting a subsequence if necessary, that u_j converges weakly in $H^1(\Omega)$ to some u and thus by the Rellich-Kondrakov theorem, u_j converges strongly in $L^2(\Omega)$ to some u . Since A and A^* are bounded from $L^2(\Omega)$ into $L^2(\Omega)$, (5.11) then implies

$$(5.12) \quad \|Au - u_0\|_{L^2(\Omega)} = \sigma, \quad A^*(Au - u_0) = 0.$$

In other words, $Au - u_0$ belongs to the kernel of A^* . But (5.5) implies that this kernel is trivial (reduces to $\{0\}$) therefore $Au = u_0$ and we reach a contradiction with the first statement in (5.12).

Step 3: $L_t^2(H_\lambda^2)$ estimates.

We multiply (5.2a) by $-\Delta u$ and we find

$$(5.13) \quad \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla u|^2 dx + \int_{\Omega} (-\Delta u)^2 dx = \left(\int_{\Omega} (-\Delta u) [A^*(Au - u_0)] dx \right)^2 (\|A^*(Au - u_0)\|_{L^2(\Omega)})^{-2}.$$

We then fix $T \in (0, \infty)$, use elliptic regularity and Steps 1 and 2 to deduce

$$(5.14) \quad \|u\|_{L^2(0,T,H^2(\Omega))}^2 \leq C_0 \left(1 + \int_0^T \left| \int_{\Omega} (-\Delta u) \{A^*(Au - u_0)\} dx \right|^2 dt \right)$$

where C_0 depends only on T and on H^1 bounds of u^0 .

Next, we observe that since $\{u(t)/t \in [0, T]\}$ is bounded in $H^1(\Omega)$, by Rellich-Kondrakov theorem, $\{u(t)/t \in [0, T]\}$ is relatively compact in $L^2(\Omega)$ and thus since A and A^* are bounded from $L^2(\Omega)$ into $L^2(\Omega)$, $\{A^*(Au(t) - u_0)/t \in [0, T]\}$ is relatively compact in $L^2(\Omega)$. This implies that we can decompose $A^*(Au(t) - u_0)$ as follows for all $\varepsilon > 0$

$$(5.15a) \quad A^*(Au(t) - u_0) = f(t) + g(t) \quad \forall t \in [0, T]$$

$$(5.15b) \quad \|f(t)\|_{L^2(\Omega)} \leq \varepsilon, \quad \|g(t)\|_{H^1(\Omega)} \leq C(\varepsilon), \quad \forall t \in [0, T]$$

for some $C(\varepsilon)$ that depends only on ε, T and H^1 bounds of u^0 .

Therefore, we have for all $t \in [0, T]$

$$\begin{aligned} \left| \int_{\Omega} (-\Delta u) \{A^*(Au - u_0)\} dx \right| &\leq \varepsilon \|u\|_{H^2(\Omega)} + \left| \int_{\Omega} (-\Delta u) g(t) dx \right| \\ &\leq \varepsilon \|u\|_{H^2(\Omega)} + \left| \int_{\Omega} \nabla u \cdot \nabla g(t) dx \right|. \end{aligned}$$

Since u satisfies (5.2d), and thus finally

$$|\int_{\Omega} (-\Delta u) \{A^*(Au - u_0)\} dx| \leq \varepsilon \|u\|_{H^2(\Omega)} + C(\varepsilon) \|u\|_{H^1(\Omega)}.$$

Hence, if we input this bound in (5.14) and use the bound shown in Step 1, we deduce

$$\|u\|_{L^2(0,T;H^2(\Omega))}^2 \leq C'(\varepsilon) + 2C_0\varepsilon \|u\|_{L^2(0,T;H^2(\Omega))}^2$$

and we conclude by choosing $\varepsilon = \frac{1}{4C_0}$.

Step 4: Uniqueness.

We consider two solutions u, v of (5.2a-d) and denote by λ, μ the corresponding Lagrange multipliers. Obviously, we have, for $u - v = w$

$$w_t - \Delta w = \lambda A^*Aw = (\lambda - \mu)A^*(Av - u_0).$$

Multiplying this equation by w and $-\Delta w$, integrating by parts and summing up, we find easily for all $T_0(0, \infty)$

(5.16)

$$\begin{aligned} \|w(t)\|_{H^1(\Omega)}^2 + \|w\|_{L^2(0,t;H^2(\Omega))}^2 &\leq C_1 \left| \int_0^t \lambda(s) ds \int_{\Omega} (A^*Aw)(-\Delta w + w) dx \right| + \\ &+ \int_0^t |\lambda - \mu| ds \left| \int_{\Omega} \{A^*(Av - u_0)\}(-\Delta w + w) dx \right| \text{ for all } t \in [0, T] \end{aligned}$$

for some positive constant C_1 depending only on T .

Using the same argument as in Step 3, we deduce the following bounds for all $\varepsilon > 0$

$$(5.17a) \quad \left| \int_{\Omega} (A^*Aw)(-\Delta w + w) dx \right| \leq \varepsilon \|w\|_{H^2(\Omega)} \|w\|_{H^1(\Omega)} + C(\varepsilon) \|w\|_{H^1(\Omega)}^2$$

$$(5.17b) \quad \left| \int_{\Omega} \{A^*(Av - u_0)\}(-\Delta w + w) dx \right| \leq \varepsilon \|w\|_{H^2(\Omega)} + C(\varepsilon) \|w\|_{H^1(\Omega)}$$

where $C(\varepsilon)$ denotes various positive constants depending only on ε and T . Inserting these bounds in (5.16) we find

$$\begin{aligned} \|w(t)\|_{H^1(\Omega)}^2 + \|w\|_{L^2(0,t;H^2(\Omega))}^2 &\leq C_1 \int_0^t |\lambda| \{ \varepsilon \|w\|_{H^2(\Omega)} \|w\|_{H^1(\Omega)} + C(\varepsilon) \|w\|_{H^1(\Omega)}^2 \} ds + \\ &+ C_1 \int_0^t |\lambda - \mu| \{ \varepsilon \|w\|_{H^2(\Omega)} + C(\varepsilon) \|w\|_{H^1(\Omega)} \} ds. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we deduce easily
(5.18)

$$\begin{aligned} \|w(t)\|_{H^1(\Omega)}^2 + \|w\|_{L^2(0,t;H^2(\Omega))}^2 &\leq \varepsilon \|w\|_{L^2(0,t;H^2(\Omega))}^2 + C(\varepsilon) \int_0^t a(s) \|w(s)\|_{H^1(\Omega)}^2 ds + \\ &+ C_1 \int_0^t |\lambda - \mu| \{ \varepsilon \|w\|_{H^2(\Omega)} + C(\varepsilon) \|w\|_{H^1(\Omega)} \} ds \end{aligned}$$

where $a \geq 0$, $a \in L^1(0, T)$ (for all $T \in (0, \infty)$).

$$(5.19) \quad \frac{d}{dt} \|w\|_{L^2(\Omega)}^2 + \nu \|w\|_{H^1(\Omega)}^2 \leq C(1 + |\lambda|) \|w\|_{L^2(\Omega)}^2$$

Next, we estimate $\lambda - \mu$. In view of Step 2, we have

$$\begin{aligned} |\lambda - \mu| &\leq C_2 [\|w\|_{L^2(\Omega)} | \int_{\Omega} (-\Delta u) \{A^* Au - u_0\} dx | + | \int_{\Omega} (-\Delta u) A^* A w dx | + \\ &+ | \int_{\Omega} (-\Delta w) \{A^* A v - u_0\} dx | \end{aligned}$$

for some $C_2 \geq 0$ which depends only on T . But this yields immediately

$$|\lambda - \mu| \leq b \|w\|_{L^2(\Omega)} + C_3 \|w\|_{H^2(\Omega)}, \text{ for some } b \in L^2(0, T).$$

Going back to (5.18), we obtain for all $t \in [0, T]$.

$$\begin{aligned} \|w(t)\|_{H^1(\Omega)}^2 + \|w\|_{L^2(0,t;H^2(\Omega))}^2 &\leq \varepsilon (1 + C_1 C_3) \|w\|_{L^2(0,t;H^2(\Omega))}^2 + \\ &C(\varepsilon) \int_0^t a(s) \|w(s)\|_{H^1(\Omega)}^2 ds + C_1 C_3 C(\varepsilon) \int_0^t \|w\|_{H^1(\Omega)} \|w\|_{H^2(\Omega)} ds + \\ &+ C_1 \varepsilon \int_0^t b(s) \|w\|_{L^2(\Omega)} \|w\|_{H^2(\Omega)} ds + C_1 C(\varepsilon) \int_0^t b(s) \|w\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)} ds. \end{aligned}$$

Using the Cauchy-Schwarz inequality, this yields:

$$\begin{aligned} \|w(t)\|_{H^1(\Omega)}^2 + \|w\|_{L^2(0,t;H^2(\Omega))}^2 &\leq \varepsilon (2 + C_1 C_3) \|w\|_{L^2(0,t;H^2(\Omega))}^2 + \\ &\int_0^t c(s) \|w(s)\|_{H^1(\Omega)}^2 ds \end{aligned}$$

where $c \geq 0$, $c \in L^1(0, T)$.

We then choose $\varepsilon = (2 + C_1 C_3)^{-1}$ and we conclude using Gronwall's inequality.

Conclusion: We conclude the proof of Theorem 5.1 here since only the existence part has not been completed. But this part is a straightforward consequence of solving approximate problems, proving the same a priori bounds uniformly for the approximate solution and passing to the limit using these bounds. We do not want to give all the details of such a tedious argument. Let us only mention a few possible approximations like a penalty method (penalizing the constraint), implicit time discretization (solving each stationary problem, for each iteration, by a minimization problem similar to the ones solved in Proposition 5.1), or splitting methods similar to the numerical method presented in the following section (where we solve first the equation without constraints on a time interval of length Δt and we then project back to the obtained solution on the constraints manifold by a simple affine rule). For all these approximation methods, one can adapt the a priori estimates shown above. But we certainly do not want to do so here in order to avoid confusing the main issues in this paper. $\square$

We now turn to a nonlinear equation (5.2a) where $a(p)$ satisfies the conditions mentioned in the beginning of this section and (5.3) in particular.

Theorem 5.2. *Let $a(p)$ satisfy (5.3), let $u^0 \in H^1(\Omega)$ satisfy (5.4), let $u_0 \in L^2(\Omega)$ and let A satisfy (5.5). Then,*

i) *there exists a solution u of (5.2a-d) satisfying: $u \in C([0, \infty); H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$, $u_t \in L^2(0, T; L^2(\Omega))$, $\lambda \in L^2(0, T)$ for all $T \in (0, \infty)$.*

ii) *If $u_0 \in H^1(\Omega)$ and A, A^* are bounded from $H^1(\Omega)$ into $H^1(\Omega)$, then the solution is unique and $\lambda \in L^\infty(0, T)$ for all $T \in (0, \infty)$.*

Remark 5.6. The analogues of Remarks 5.4 and (5.5) hold here. In particular, using this extra regularity, one can show the uniqueness of solutions by an argument quite similar to the one given in the proof of Theorem 3.1 and which does not use a regularity assumption on u_0 like in part ii) of Theorem 3.2 above. However, this argument relies upon a regularity result which is too technical to be detailed here.

Remark 5.7. If $\Omega = \mathbb{R}^2$ and A is a convolution operator then A, A^* are bounded operators from $H^1(\Omega)$ into $H^1(\Omega)$ since they are bounded from $L^2(\Omega)$ into $L^2(\Omega)$ and they commute with differentiation.

Proof of Theorem 5.2. We only explain the modifications that have to be made in the proof of Theorem 5.1. In particular, Steps 1 and 2 are identical. However, Step 3 has to be modified substantially. The final result being the same, these facts and the uniqueness argument shown below allow us to complete the proof of part i).

The $L_t^2(H_x^2)$ bound follows from multiplying (5.2a) by $-\Delta u$ and making some

observations. First of all, $\mathcal{A}[u] = \sum_{i,j=1}^2 \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$ where $\alpha_{ij} = \frac{\partial^2 a}{\partial p_i \partial p_j}(\nabla u)$ and thus $|\mathcal{A}[u]| \leq C|D^2 u|$. Next, we recall an adaptation of a famous inequality due to H.O. Cordes [16] (see also A.I. Koselev [17]) shown in P.L. Lions [18] in the case of Neumann boundary conditions. There exist $\alpha > 0$, $C \geq 0$ such that for all $u \in H^2(\Omega)$ satisfying (3.1d)

$$(5.19) \quad \int_{\Omega} \mathcal{A}[u] \Delta u dx \geq \alpha \|u\|_{H^2(\Omega)}^2 - C \|u\|_{L^2(\Omega)}^2.$$

This inequality allows us to adapt easily the rest of the proof made in Step 3.

We conclude with the proof of the uniqueness statement (part ii) above. Let u, v be two solutions of (5.2a-d) and let λ, μ be the corresponding Lagrange multipliers. We denote by $w = u - v$ and multiply by w the equation satisfied by w . We then find in view of (5.3)

$$(5.20) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2 dx + \nu \int_{\Omega} |\nabla w|^2 dx \leq C \{ |\lambda| \int_{\Omega} w^2 dx + |\lambda - \mu| \left(\int_{\Omega} w^2 dx \right)^{\frac{1}{2}} \}.$$

Next, we observe that λ can be written as

$$(5.21) \quad \lambda = - \left\{ \int_{\Omega} \sum_{i=1}^2 a_i(\nabla u) \frac{\partial}{\partial x_i} \{ A^*(Au - u_0) \} dx \right\} \|A^*(Au - u_0)\|_{L^2(\Omega)}^{-2}$$

and a similar expression holds for μ with u replaced by v .

From these expressions we deduce using the assumptions made about a , A , A^* and u_0

$$(5.22) \quad |\lambda - \mu| \leq C \|w\|_{H^1(\Omega)}$$

(recall that u, v are bounded in $L^\infty(0, T, H^1(\Omega))$ for all $T \in (0, \infty)$).

Inserting this bound in (5.20) we finally deduce

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2(\Omega)}^2 + \nu \|w\|_{H^1(\Omega)}^2 \leq C \{ |\lambda| \|w\|_{L^2(\Omega)}^2 + \|w\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)} \}.$$

Hence, we have for all $t \in [0, T]$, by the Cauchy-Schwarz inequality

$$(5.23) \quad \frac{d}{dt} \|w\|_{L^2(\Omega)}^2 + \nu \|w\|_{H^1(\Omega)}^2 \leq C(1 + |\lambda|) \|w\|_{L^2(\Omega)}^2$$

and the uniqueness follows from Gronwall's inequality.

Let us finally observe that the fact that $\lambda \in L^\infty(0, T)$ (for all $T \in (0, \infty)$) is straightforward in view of (5.21) since u is bounded in $H^1(\Omega)$ and $\|A^*(Au - u_0)\|_{L^2(\Omega)}$ is bounded from below (Step 2). $\square$

6. Results.

Page P1 shows an extremely noisy image (Picture 2) with various standard methods which we compare with our TV based method. Page P2 shows the result of edge detection applied to the images on P1. Notice picture P12 – the TV denoised image admits excellent edge detection.

Pages P3 and P4 show restoration when the noise is multiplicative as described in section 4. The only information we use concerns the mean and standard deviation of the noise. No other method can handle these situations.

Page P5 shows the “tank” restoration and pages P6-7 show the “chemical plant” restoration.

On page P8, we begin our deblurring/denoising demonstration. We use various types of blur with various amounts of additive white noise. Our restorations are shown on the right. On page P9 we show the results of other methods applied to the motion blurred image, on page P10 we do it for the Gaussian blur, and on page P11 we do it for the out of focus blur.

We believe that our restoration technique, supported by this contract, represents a break through in the area.

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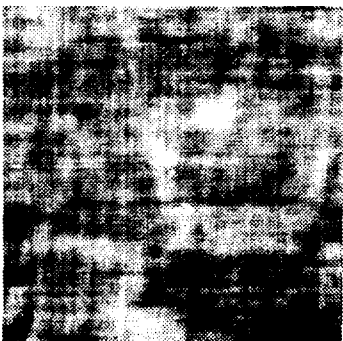
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Pict. 1
Original image



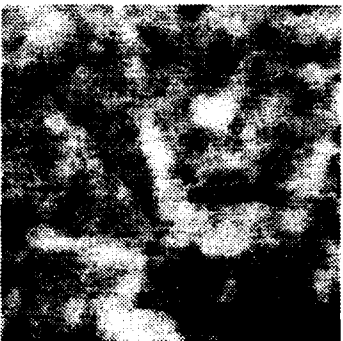
Pict. 2
Noisy image with
SNR 1: 4



Pict. 3
Image denoised by Wiener
filter with true power
spectrum



Pict. 4
Image denoised by Wiener
filter with estimated power
spectrum



Pict. 5
Image denoised by
circular median
filter



Pict. 6
Image denoised by
Cognitech's T.V based
methods



Pict. 7

Edge detection of original
clean image



Pict. 8

Edge detection of
noisy image



Pict. 9

Edge detection of image denoised
by Wiener filter with true power
spectrum



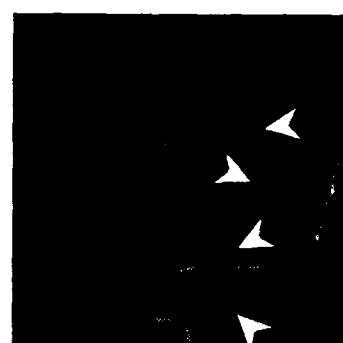
Pict. 10

Edge detection of image denoised
by Wiener filter with estimated
power spectrum



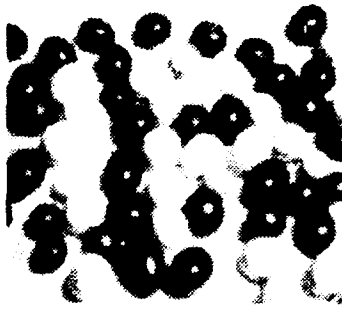
Pict. 11

Edge detection of image denoised
by circular median filter



Pict. 12

Edge detection of image denoised
by Cognitech's T.V based
methods



a) Original



b) Multiplicative noise
with $\sigma = 0.3$



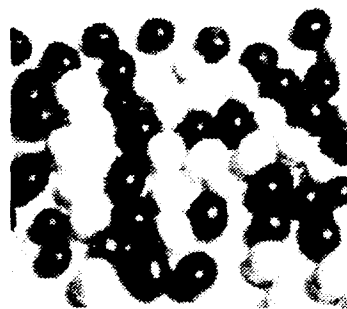
c) Restoration of "b"



d) Multiplicative noise
with $\sigma = 0.4$



e) Restoration of "d"



a) Original



b) Multiplicative noise
with $\sigma = 0.1$



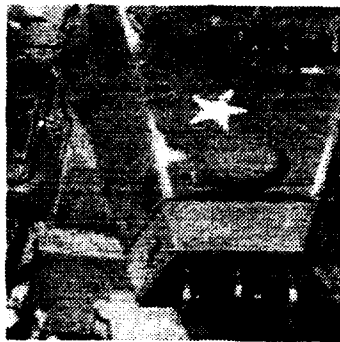
c) Restoration of "b"



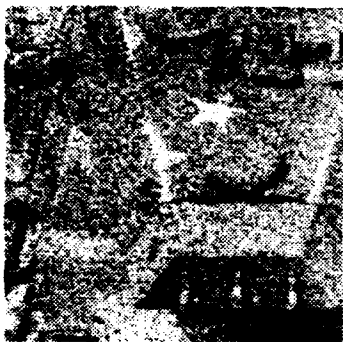
d) Multiplicative noise
with $\sigma = 0.25$



e) Restoration of "d"



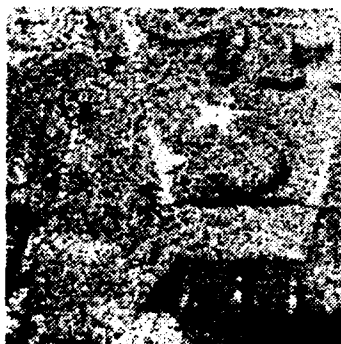
a) Original



b) Multiplicative noise
with $\sigma = 0.2$



c) Restoration of "b"



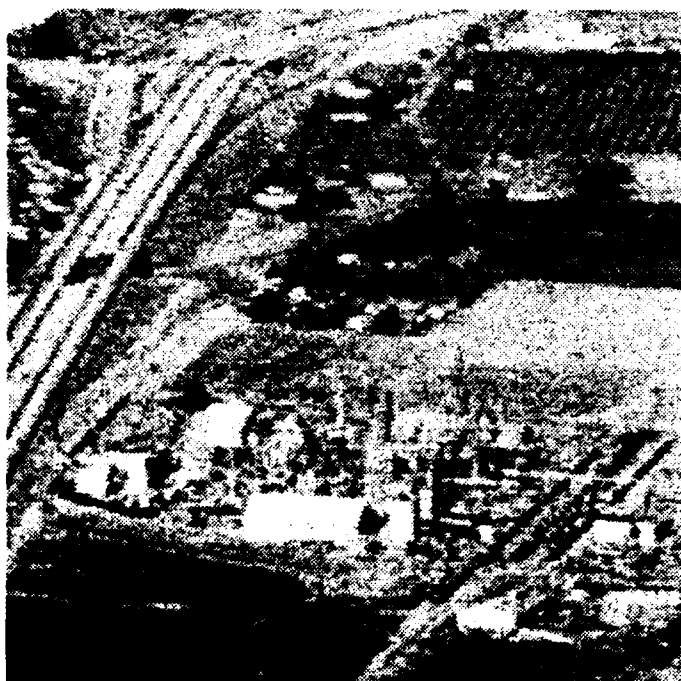
d) Multiplicative noise
with $\sigma = 0.25$



e) Restoration of "d"



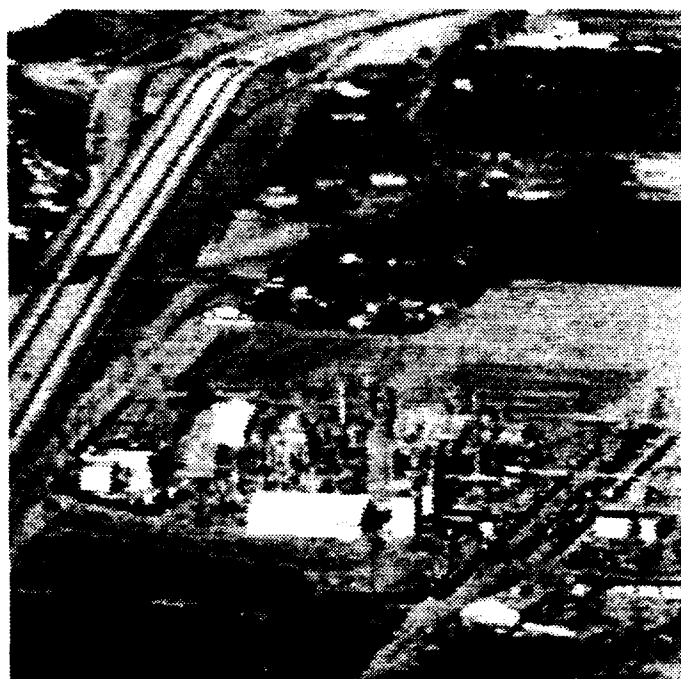
a) Original



b) Multiplicative noise with
 $\sigma = 0.2$



c) Restoration of "b"



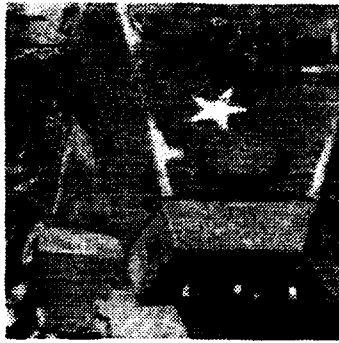
a) Original



b) Multiplicative noise with
 $\sigma = 0.25$



c) Restoration of "b"



a) Original



b) Motion blur by 11 pixels
plus noise with $\sigma = 5$.
(SNR = 4.2)



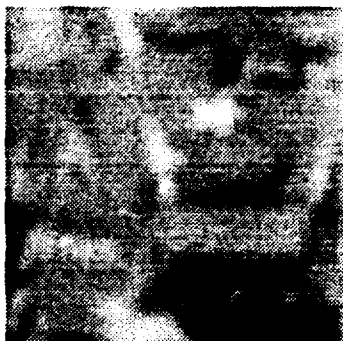
c) Restoration of "b"



d) Gaussian blur with gaussian
variance 7.3 plus noise with
 $\sigma = 7$. (SNR = 2.9)



e) Restoration of "d"



f) Defocus blur with variance
3.5 plus noise with $\sigma = 5$.
(SNR = 4.1)



g) Restoration of "f"

Motion deblurring plus denoising



a) Constrained least
- square filter



b) Geometrical mean
filter



c) Pseudo - inverse
filter



d) Wiener filter

Gaussian deblurring plus denoising



a) Constraned least -
square filter



b) Geometrical mean
filter



c) Pseudo - inverse
filter



d) Wiener filter

Defocus deblurring plus denoising



a) Constrained least -
square filter



b) Geometrical mean
filter



c) Pseudo - inverse
filter



d) Wiener filter