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A POSTERIORI ERROR ESTIMATES OF FINITE ELEMENT SOLUTIONS OF PARAMETRIZED NONLINEAR EQUATIONS

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Nonlinear differential equations with parameters are called parametrized nonlinear equations. A posteriori error estimates of finite element solutions of second order parametrized strongly nonlinear equations in divergence form on one-dimensional bounded intervals are studied in this paper. In the previous paper by the authors, the finite element solutions are defined, and several a priori estimates are proved on regular branches and on branches around turning points. Using obtained a priori error estimates, several practical a posteriori error estimates which are asymptotically exact are obtained. Some numerical examples are given.

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A Posteriori Error Estimates of Finite Element Solutions of Parametrized Nonlinear Equations

Takuya Tsuchiya[†] Ivo Babuška[‡]

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Abstract. Nonlinear differential equations with parameters are called parametrized nonlinear equations. This paper studies a posteriori error estimates of finite element solutions of second order parametrized strongly nonlinear equations in divergence form on one-dimensional bounded intervals. In the previous paper by the authors, the finite element solutions are defined, and several a priori estimates are proved on regular branches and on branches around turning points. Using obtained a priori error estimates, we obtain several practical a posteriori error estimates which are asymptotically exact. Some numerical examples are given.

Key words. parametrized nonlinear equations, Fredholm operators, regular branches, turning points, finite element solutions, a priori and a posteriori error estimates

AMS(MOS) subject classifications. 65L10, 65L60

Abbreviated title. FE Solutions of Parametrized Nonlinear Equations

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1. Introduction.

Let X, Y be Banach spaces and $\Lambda \subset \mathbf{R}^n$ a bounded interval. Let $F : \Lambda \times X \to Y$ be a smooth operater. The nonlinear equation

$$(1.1) \qquad F(\lambda, u) = 0,$$

with parameters $\lambda \in \Lambda$ is called **parametrized nonlinear equations**.

In this paper we deal with the parametrized nonlinear equation $F : \Lambda \times H_0^1(J) \to H^{-1}(J)$ with one parameter $\lambda \in \Lambda \subset \mathbf{R}$ defined by

$$(1.2) \qquad F(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times H_0^1(J),$$

$$(1.3) \qquad < F(\lambda, u), v >:= \int_J [a(\lambda, x, u'(x))v'(x) + f(\lambda, x, u(x))v(x)]dx, \quad \forall v \in H_0^1(J),$$

where $J := (b, c) \subset \mathbf{R}$ is a bounded interval, $a, f : \Lambda \times J \times \mathbf{R} \to \mathbf{R}$ are sufficiently smooth functions, and $< \cdot, \cdot >$ is the duality pair of $H^{-1}(J)$ and $H_0^1(J)$. Since F is a second order differential operator in divergence form, finite element solutions of (1.2) are defined in a natural way.

In [TB1], the above problem was concerned, and a priori error estimates of finite element solutions were established. In this papar we try to develop a postriori error estimates of the finite element solutions of (1.2) and (1.3) using a priori estimates obtained in [TB1].

The basic idea is as follows. Suppose that we want to solve the nonlinear equation

$$(1.4) \qquad < K(x), v >= 0 \quad \text{for} \quad \forall v \in H_0^1(J),$$

where $K : H_0^1(J) \to H^{-1}(J)$ is a smooth nonlinear operator. Let $\mathring{S}_h \subset H_0^1(J)$ be a finite element space and $x_h \in \mathring{S}_h$ the finite element solution, that is,

$$(1.5) \qquad < K(x_h), v_h >= 0 \quad \text{for} \quad \forall v_h \in \mathring{S}_h.$$

Then, we consider the linearized equation

$$(1.6) \qquad < DK(x_h)\psi, v >= - < K(x_h), v > \quad \forall v \in H_0^1(J),$$

where $DK(x_h)$ is the Fréchet derivative at x_h which is assumed to be an isomorphism between $H_0^1(J)$ and $H^{-1}(J)$.

Let $x \in H_0^1(J)$ be the exact solution of (1.4). In Section 3 we will see that the magnitude $||\psi||$ represents the error $||x - x_h||$, that is,

$$||x - x_h|| \leq ||\psi||(1 + o(1)).$$

Of course, the exact solution $\psi \in H_0^1(J)$ of (1.6) should be approximated by a certain way in general. We will consider a finite element solution ψ_h of (1.6). We observe, however, that the finite element solution ψ_h of (1.6) over $\overset{\circ}{S}_h$ defined by

$$< DK(x_h)\psi_h, v_h >= - < K(x_h), v_h >, \quad \forall v_h \in \overset{\circ}{S}_h$$

is a zero function because of (1.5). Therefore, estimating the magnitude $||\psi||$ is equivalent to estimating $||\psi - \psi_h||$: the error of the finite element solution ψ_h.

Hence, if we have certain methodology for a posteriori error estimates for the linearized equation (1.6), we have a posteriori error estimates for the original nonlinear equation (1.4). In a short sentence, the principle obtained here is that

"If we have a posteriori estimates of linear equations,

we have a posteriori estimates of nonlinear equations."

Let (λ, u) be the exact solution of (1.2) and (λ_h, u_h) be the finite element solution corresponding to (λ, u). Usually, it is observed that the error $|\lambda - \lambda_h|$ is much smaller than the error $||u - u_h||$. In Section 4 we obtain elaborate error estimates of $|\lambda - \lambda_h|$ which verify the above observation.

In Section 5 practical aspects of our a posteriori estimates and some numerical examples are given. In the computation of our numerical examples the continuation program package PITCON (see [R]) developed by Rheinboldt and his colleagues is used.

This paper is a revision of a part of one of the authors Ph.D. dissertation [T].

2. Assumptions and A Priori Estimates.

In this section we sumerize the results obtained in [TB1]. Throughout this paper, we use same notation as in [TB1].

Here, we deal with the nonlinear operator $F : \Lambda \times W_0^{1,\infty} \rightarrow W^{-1,\infty}$ by, for $\lambda \in \Lambda$ and $u \in W_0^{1,\infty}$,

$$(2.1) \qquad < F(\lambda, u), v >:= \int_J [a(\lambda, x, u'(x))v'(x) + f(\lambda, x, u(x))v(x)]dx, \quad \forall v \in W_0^{1,1},$$

where $< \cdot, \cdot >$ is the duality pairing between $W^{-1,\infty}$ and $W_0^{1,1}$. Then, our problem is

Problem 2.1. *Solve the following equation: Find $\lambda \in \Lambda$ and $u \in W_0^{1,\infty}$ such that*

$$< F(\lambda, u), v >= 0, \quad \forall v \in W_0^{1,1}. \qquad \square$$

For F being well-defined and smooth we require several conditions to a and f. Let $\alpha = (\alpha_1, \alpha_2)$ be usual multiple index with respect to λ and y. That is, for $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha a(\lambda, x, y)$ means $\frac{\partial^{|\alpha|}}{\partial \lambda^{\alpha_1} \partial y^{\alpha_2}} a(\lambda, x, y)$.

Let $d \geq 1$ be an integer. For α, $|\alpha| \leq d$, we define the maps $\mathsf{A}^\alpha(u)$ and $\mathsf{F}^\alpha(u)$ for $u \in W_0^{1,\infty}$ by

$$\mathsf{A}^\alpha(u)(x) := D^\alpha a(\lambda, x, u'(x)), \tag{2.2}$$

$$\mathsf{F}^\alpha(u)(x) := D^\alpha f(\lambda, x, u(x)). \tag{2.3}$$

We then assume that

Assumption 2.2. *For all α, $|\alpha| \leq d$, we suppose that*

(1) *For almost all $x \in J$, $D^\alpha a(\lambda, x, y)$ and $D^\alpha f(\lambda, x, y)$ exist at all $(\lambda, y) \in \Lambda \times \mathbb{R}$, and they are Carathéodory continuous.*

(2) *The mapping A^α defined by (2.2) is a continuous operator from $W_0^{1,\infty}$ to L^∞, and the image $\mathsf{A}^\alpha(U) \subset L^\infty$ of any bounded subset $U \subset \Lambda \times W_0^{1,\infty}$ is bounded.*

(3) *The mapping F^α defined by (2.3) is a continuous operator from $W_0^{1,\infty}$ to L^1, and the image $\mathsf{F}^\alpha(U) \subset L^1$ of any bounded subset $U \subset \Lambda \times W_0^{1,\infty}$ is bounded.* $\square$

We define the subset $\mathcal{S} \subset \Lambda \times W_0^{1,\infty}$ by

$$\mathcal{S} := \{(\lambda, u) \in \Lambda \times W_0^{1,\infty} \mid a_y(\lambda, x, u'(x))^{-1} \in L^\infty\}. \tag{2.4}$$

Since the mapping $\Lambda \times W_0^{1,\infty} \ni (\lambda, u) \mapsto a_y(\lambda, x, u'(x)) \in L^\infty$ is continuous, we have

Lemma 2.3. *If a and f satisfy Assumption 2.2 with $d \geq 1$, $\mathcal{S}$ is an open set in $\Lambda \times W_0^{1,\infty}$.*
$\square$

From the standard theory of Fredholm operators, we obtain the following theorem:

Theorem 2.4. *Suppose that a and f satisfy Assumption 2.2 with $d \geq 1$. Then in $\mathcal{S}$, the operator $F : \mathcal{S} \to W^{-1,\infty}$ defined by (2.1) is a nonlinear Fredholm operator of index 1.* $\square$

We define the subset $\mathcal{R}(F, \mathcal{S}) \subset \mathcal{S}$ by

$$\mathcal{R}(F, \mathcal{S}) := \{(\lambda, u) \in \mathcal{S} | DF(\lambda, u) \text{ is onto}\}. \tag{2.5}$$

The elements of $\mathcal{R}(F, \mathcal{S})$ and $F(\mathcal{R}(F, \mathcal{S}))$ are called **regular points** and **regular values**, respectively. By Theorem 2.4, we can apply the Fink-Rheinboldt theory ([FR1],[FR2],[R]) to the operator F and obtain the following.

Theorem 2.5. *Suppose that a and f satisfy Assumption 2.2 with $d \geq 1$. Let $e \in F(\mathcal{R}(F, \mathcal{S}))$. Then*

$$\mathcal{M} = \mathcal{M}_e := \{(\lambda, u) \in \mathcal{R}(F, \mathcal{S}) \mid F(\lambda, u) = e\}$$

is a one-dimensional C^d-manifold without boundary. Moreover, for each $(\lambda, u) \in \mathcal{M}$, the tangent space $T_{(\lambda,u)}\mathcal{M}$ at (λ, u) is $\mathrm{Ker} DF(\lambda, u)$.

Therefore, if $0 \in F(\mathcal{R}(F, \mathcal{S}))$, the solutions of Problem 2.1 form a one-dimensional C^d-manifold without boundary in $\mathcal{R}(F, \mathcal{S})$. □

In the sequel of this paper we always assume that $0 \in F(\mathcal{R}(F, \mathcal{S}))$, that is, $\mathcal{M}_0 \neq \emptyset$.

For the regularigy of $(\lambda, u) \in \mathcal{M}_0$, we need additional assumptions. Let p^*, $2 \leq p^* \leq \infty$ be taken and fixed.

Assumption 2.6. *Under Assumption 2.2 with $d \geq 1$, we assume that*

(1) *For all $\lambda \in \Lambda$, the functions $a(\lambda, \cdot, \cdot)$, $a_y(\lambda, \cdot, \cdot) : J \times \mathbf{R} \to \mathbf{R}$ are continuous.*

(2) *For all $(\lambda, y) \in \Lambda \times \mathbf{R}$, there exist $a_x(\lambda, x, y)$ for almost all $x \in J$ and are Carathéodory continuous.*

(3) *The composition functions $f(\lambda, x, u(x))$, $a_x(\lambda, x, u'(x))$ are in L^{p^*} for any $(\lambda, u) \in \Lambda \times W_0^{1,\infty}$. Moreover, for any bounded subsets $K \subset \Lambda \times W_0^{1,\infty}$,*

$$\{f(\lambda, x, u(x)) \in L^{p^*} | (\lambda, u) \in K\}, \quad \{a_x(\lambda, x, u'(x)) \in L^{p^*} | (\lambda, u) \in K\}$$

are bounded in L^{p^*}. □

Theorem 2.7. *Under Assumption 2.2 and 2.6, we have $u \in W^{2,p^*}$ for all $(\lambda, u) \in \mathcal{M}_0$. Moreover, for all bounded closed subsets $\widetilde{\mathcal{M}} \subset \mathcal{M}_0$, there exists a constant $K(\widetilde{\mathcal{M}})$ such that $\sup_{(\lambda,u) \in \widetilde{\mathcal{M}}} \|u\|_{W^{2,p^*}} \leq K(\widetilde{\mathcal{M}})$.*

Let $\overset{\circ}{S}_h \subset H_0^1$ be a finite element space. We define the finite element solutions of Problem 2.1 by

Problem 2.8. *Find $\lambda_h \in \Lambda$ and $u_h \in \overset{\circ}{S}_h$ such that*

$$< F(\lambda_h, u_h), v_h >= 0, \quad \forall v_h \in \overset{\circ}{S}_h. \quad \square$$

Let $\alpha \in L^\infty$ be such that $\alpha(x) \geq \epsilon > 0$ for all $x \in J$ Let $(\cdot,\cdot)_\alpha$ be the inner product of H_0^1 defined by $(u,v)_\alpha := \int_J \alpha u'v'dx$ for $u, v \in H_0^1$. Define the isomorphism $T_\alpha \in \mathcal{L}(W^{-1,\infty}, W_0^{1,\infty})$ by $< \eta, v >= (T_\alpha \eta, v)_\alpha$, $\forall v \in W_0^{1,1}$ for $\eta \in W^{-1,\infty}$. Also, define the canonical projection $\Pi_h^\alpha : H_0^1 \to \overset{\circ}{S}_h$ by $(\psi - \Pi_h^\alpha \psi, v_h)_\alpha = 0$, $\forall v_h \in \overset{\circ}{S}_h$ for $\psi \in H_0^1$. Then, we observe that, for any $v_h \in \overset{\circ}{S}_h$ and any $v \in H_0^1$,

$$< F(\lambda_h, u_h), v_h >= 0 \Longleftrightarrow < T_\alpha^{-1}\Pi_h^\alpha T_\alpha F(\lambda_h, v_h), v >= 0.$$

Following the Fink-Rheinboldt theory we define $\overline{F_h^\alpha} : \Lambda \times W_0^{1,\infty} \to W^{-1,\infty}$ by

$$\overline{F_h^\alpha}(\lambda, u) := (I - P_h^\alpha)T_\alpha^{-1}u + P_h^\alpha F(\lambda, u),$$

where I is the identity of $W^{-1,\infty}$, and $P_h^\alpha := T_\alpha^{-1}\Pi_h^\alpha T_\alpha$.

Lemma 2.9 ([R,Lemma 5.1]). *The operator $\overline{F_h^\alpha}$ satisfies the following:*
(1) $\overline{F_h^\alpha}(\lambda, u) = 0$ *for some* $(\lambda, u) \in \Lambda \times H_0^1$ *if and only if* $(\lambda, u) \in \Lambda \times \overset{\circ}{S}_h$ *and* $F_h(\lambda, u) = 0$.
(2) $\overline{F_h^\alpha}$ *is a Fredholm operator of index 1 on* $\Lambda \times H_0^1$. $\square$

By Lemma 2.9, we have the following theorem as a consequence of the Fink-Rheinboldt theory.

Theorem 2.10. *Suppose that F is C^d mapping ($d \geq 1$). Then the set of the finite elements solutions of Problem 2.8,*

$$\mathcal{M}_h := \left\{(\lambda_h, u_h) \in \mathcal{R}(F_h, \Lambda \times H_0^1)) \,|\, F_h(\lambda_h, u_h) = 0\right\},$$

is a C^d manifold without boundary. $\square$

For a priori error estimates of the finite element solutions, we always assume the following.

Assumption 2.11. *We assume that*

(1) *Assumption 2.2 with d (i.e. F is a C^d Fredholm map).*

(2) $0 \in F(\mathcal{R}(F,\mathcal{S}))$ *(i.e. $\mathcal{M}_0 \neq \emptyset$).*

(3) *Assumption 2.6 (i.e. $u \in W^{2,p^*}$, $2 \leq p^* \leq \infty$ for any $(\lambda, u) \in \mathcal{M}_0$).*

(4) $\overset{\circ}{S}_h$ *is regular and* $\lim_{h\to 0} \inf_{v_h \in \overset{\circ}{S}_h} \|u - v_h\|_{H_0^1} = 0$, *for any $u \in H_0^1$.*

(5) *The triangulation of $\overset{\circ}{S}_h$ (in one dimensional case, the partition of J into small intervals) satisfies the inverse assumption* [C,p140]. □

In the sequel, we denote by $\widehat{\Pi}_h : W_0^{1,1} \to \overset{\circ}{S}_h$ the interpolant projection.

Theorem 2.12. *Suppose that Assumption 2.11 holds for $d \geq 2$. Also, suppose that $\widetilde{\mathcal{M}}_0 \subset \mathcal{M}_0$ is a compact regular branch, that is, there is a compact interval $\tilde{\Lambda} \subset \Lambda$ and C^2 map $\tilde{\Lambda} \ni \lambda \mapsto u(\lambda) \in W_0^{1,\infty}$ such that*

$$\widetilde{\mathcal{M}}_0 = \left\{(\lambda, u(\lambda)) \in \mathcal{M}_0 \mid D_u F(\lambda, u(\lambda)) \text{ is an isomorphism for } \forall\lambda \in \tilde{\Lambda}\right\}.$$

Then, there exists the corresponding finite element solution branch $\widetilde{\mathcal{M}}_h \subset \mathcal{M}_h$ which is parametrized by the same $\lambda \in \tilde{\Lambda}$ and

$$\begin{aligned}
\|\widehat{\Pi}_h u(\lambda) - u_h(\lambda)\|_{H_0^1} &\leq K_0 h^{\frac{1}{2}+\eta}, \\
\|u(\lambda) - u_h(\lambda)\|_{H_0^1} &\leq K_1 \|u(\lambda) - \widehat{\Pi}_h u(\lambda)\|_{H_0^1}, \\
\|u(\lambda) - u_h(\lambda)\|_{W_0^{1,\infty}} &\leq K_2 h^{\eta}
\end{aligned}$$

for all $\lambda \in \tilde{\Lambda}$, $u(\lambda) \in \widetilde{\mathcal{M}}_0$, $u_h(\lambda) \in \widetilde{\mathcal{M}}_h$, and η with $0 < \eta < \frac{1}{2}$. Here, $K_0, K_1, K_2 > 0$ are constants independent of h and λ.

Moreover, we have

$$\widetilde{\mathcal{M}}_h \subset \mathcal{R}(F,\mathcal{S}). \qquad \square$$

Theorem 2.13. *Suppose that Assumption 2.11 holds for $d \geq 2$. Let $\widetilde{\mathcal{M}}_0 \subset \mathcal{M}_0$ be a connected compact subset with the following properties:*

(1) *$D_\lambda F(\lambda, u) \neq 0$ for any $(\lambda, u) \in \widetilde{\mathcal{M}}_0$.*

(2) *There exist $x_0 \in J$ such that $DG(\lambda, u)$ defined by, for given $\gamma \in \mathbf{R}$,*

$$G(\lambda, u) := (u(x_0) - \gamma, F(\lambda, u)), \quad (\lambda, u) \in \mathcal{S} \tag{2.6}$$

$$DG(\lambda, u)(t, \psi) = (\psi(x_0), DF(\lambda, u)(t, \psi)), \quad t \in \mathbf{R}, \psi \in W_0^{1,1}, \tag{2.7}$$

is an isomorphism at all $(\lambda, u) \in \widetilde{\mathcal{M}}_0$.

Then $\widetilde{\mathcal{M}}_0$ *is parametrized by* $\gamma = u(x_0)$. *We assume without loss of generality that the above* x_0 *is a nodal point of* $\overset{\circ}{S}_h$ *for all sufficiently small* $h > 0$.

Then there exists the corresponding finite element solution branch $\widetilde{\mathcal{M}}_h \subset \mathcal{M}_h$ *which is parametrized by the same* γ, *that is,* $u_h(\gamma)(x_0) = \gamma$ *and* $F_h(\lambda_h(\gamma), u_h(\gamma)) = 0$ *for any* γ.

Moreover, we have

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|\widehat{\Pi}_h u(\gamma) - u_h(\gamma)\|_{H_0^1} \leq K_3 h^{\frac{1}{2}+\eta},$$

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|u(\gamma) - u_h(\gamma)\|_{H_0^1} \leq K_4 \|u(\gamma) - \widehat{\Pi}_h u(\gamma)\|_{H_0^1},$$

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|u(\gamma) - u_h(\gamma)\|_{W_0^{1,\infty}} \leq K_5 h^{\eta},$$

$$\widetilde{\mathcal{M}}_h \subset \mathcal{R}(F, \mathcal{S}),$$

for all $\gamma = u(x_0), (\lambda(\gamma), u(\gamma)) \in \widetilde{\mathcal{M}}_0, (\lambda_h(\gamma), u_h(\gamma)) \in \widetilde{\mathcal{M}}_h$, *and* η *with* $0 < \eta < \frac{1}{2}$. *Here,* K_3, K_4, K_5 *are positive constants independent of* h *and* γ. □

3. A Posteriori Error Estimates.

In this section we consider a posteriori error estimates. Before going into our problem, we observe an error estimate in a general Banach space setting.

Let X and Y be Banach spaces and $V \subset X$ open. We consider a generic C^2 mapping $K : V \to Y$ such that $DK(x) \in \mathcal{L}(X, Y)$ is an isomorphism at each $x \in V$, and $D^2K(x)$ is bounded on bounded subsets in V.

Suppose that we are considering the equation

$$K(t) = 0, \qquad t \in V. \tag{3.1}$$

Let $t_{EX} \in V$ be an exact solution, i.e. $K(t_{EX}) = 0$, and $t_{AP} \in V$ an approximate solution, i.e. $K(t_{AP}) \approx 0$. Note that, since $DK(t_{EX})$ is an isomorphism, $t_{EX} \in V$ is isolated, that is, there is no other solution of (3.1) in the small enough neighborhood of t_{EX}.

From elementary calculus on Banach spaces, we have

$$0 = K(t_{AP}) + DK(t_{AP})z + \frac{1}{2}\left(\int_0^1 (1-s)^2 D^2K(t_{AP} + sz)ds\right)(z, z), \tag{3.2}$$

where $z := t_{EX} - t_{AP}$. Let us consider the following linearized equation:

$$0 = K(t_{AP}) + DK(t_{AP})\tilde{z}, \qquad \tilde{z} \in V. \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$DK(t_{AP})(z-\tilde{z}) = -\frac{1}{2}\left(\int_0^1 (1-s)^2 D^2K(t_{AP}+sz)ds\right)(z,z), \tag{3.4}$$

$$z-\tilde{z} = -\frac{1}{2}DK(t_{AP})^{-1}\left[\left(\int_0^1 (1-s)^2 D^2K(t_{AP}+sz)ds\right)(z,z)\right], \tag{3.5}$$

and

$$\|z-\tilde{z}\|_X \le \frac{1}{2}\|DK(t_{AP})^{-1}\|_{\mathcal{L}(Y,X)} \int_0^1 \|D^2K(t_{AP}+sz)\|_{\mathcal{L}(X\times X,Y)}ds\|z\|_X^2.$$

Since $D^2K \in \mathcal{L}(X\times X, Y)$ is bounded on bounded subsets, there is a constant M such that $\|z-\tilde{z}\|_X \le M\|z\|_X^2$, and we obtain

$$\|z\|_X \le \|\tilde{z}\|_X(1+O(\|\tilde{z}\|_X)). \tag{3.6}$$

By the argument in [TB1,Section 4], we know that, at each $(\lambda, u) \in \mathcal{M}_0$, we have either

Case 1: $\mathrm{Ker}D_uF(\lambda,u) = \{0\}$ and $D_\lambda F(\lambda,u) \in \mathrm{Im}D_uF(\lambda,u)$, or

Case 2: $\dim\mathrm{Ker}D_uF(\lambda,u) = 1$ and $D_\lambda F(\lambda,u) \notin \mathrm{Im}D_uF(\lambda,u)$.

Now, let us suppose that we are in Case 1. To apply (3.6) we set up the following

$X = W_0^{1,\infty}$, $Y = W^{-1,\infty}$,

$X \supset V = \mathcal{R}(F,\mathcal{S})$ defined by (2.5),

$K(u) = F(\lambda,u)$ for given and fixed $\lambda \in \Lambda$.

From (3.6), we have

$$\|u-u_h\|_{W_0^{1,\infty}} \le \|U\|_{W_0^{1,\infty}}(1+O(\|U\|_{W_0^{1,\infty}})),$$

where $(\lambda,u) \in \mathcal{M}_0$, $(\lambda,u_h) \in \mathcal{M}_h$ and U is the exact solution of the linearized equation

$${}_{\infty}< D_uF(\lambda,u_h)U, v >_1 = -{}_{\infty}< F(\lambda,u_h), v >_1, \quad \forall v \in W_0^{1,1}. \tag{3.7}$$

By Theorem 2.12, we have $\lim_{h\to 0} F(\lambda,u_h) = 0$ and $\lim_{h\to 0}\|U\|_{W_0^{1,\infty}} = 0$. Therefore, we obtain

Theorem 3.1. *Suppose that $(\lambda,u) \in \mathcal{M}_0$ is on a regular branch. Then we have, under Assumption 2.11 with $d \ge 2$,*

$$\|u-u_h\|_{W_0^{1,\infty}} \le \|U\|_{W_0^{1,\infty}}(1+o(1)). \qquad \square$$

We next consider an a posteriori error estimate in H_0^1-norm. From (3.5), we have

$$z-U = -\frac{1}{2}D_uF(\lambda,u_h)^{-1}\left[\left(\int_0^1 (1-s)^2 D_{uu}^2F(\lambda,u_h+sz)ds\right)(z,z)\right].$$

where $z := u - u_h$. Recall that $D_u F(\lambda, u) \in \mathcal{L}(H_0^1, H^{-1})$ is an isomorphism, and

$$ {}_\infty< \left(\int_0^1 (1-s)^2 D_{uu}^2 F(\lambda, u_h + sz)ds\right)(z,z), v >_1 $$
$$ = \int_J (\alpha_1(x) z'^2 v' + \beta_1(x) z^2 v) dx, \quad \forall v \in W_0^{1,1}, $$

where $\alpha_1(x) := \int_0^1 (1-s)^2 a_{yy}(\lambda, x, u_h' + sz')ds$ and $\beta_1(x) := \int_0^1 (1-s)^2 f_{yy}(\lambda, x, u_h + sz)ds$. Hence, we get

$$ \left\| \left(\int_0^1 (1-s)^2 D_{uu}^2 F(\lambda, u_h + sz)ds\right)(z,z) \right\|_{H^{-1}} \le \|\alpha_1\|_{L^\infty} \|z'^2\|_{L^2} + \|\beta_1\|_{L^1} \|z\|_{L^\infty}^2 $$
$$ \le C_1 \|z\|_{W_0^{1,\infty}} \|z\|_{H_0^1}, $$

and

$$ \|z - U\|_{H_0^1} \le C_2 \|z\|_{W_0^{1,\infty}} \|z\|_{H_0^1}. $$

Therefore, by Theorem (2.12), we obtain $\|z\|_{H_0^1} \le \|U\|_{H_0^1}(1 + C_3 h^\eta)$ for any η with $0 < \eta < \frac{1}{2}$. and

Theorem 3.2. *Suppose that $(\lambda, u) \in \mathcal{M}_0$ is on a regular branch. Then we have, under Assumption 2.11 with $d \ge 2$,*

$$ \|u - u_h\|_{H_0^1} \le \|U\|_{H_0^1}(1 + o(1)). \qquad \square $$

Next, let $(\lambda, u) \in \mathcal{M}_0$ such that $D_\lambda F(\lambda, u) \ne 0$. Then, by Theorem 2.13, there exists a nodal point $x_0 \in J$ of $\overset{\circ}{S}_h$ such that the Fréchet derivative $DG(\lambda, u)$ of the mapping $G(\lambda, u) := (u(x_0) - \gamma, F(\lambda, u))$ is an isomorphism of $\mathbf{R} \times W_0^{1,\infty}$ to $\mathbf{R} \times W^{-1,\infty}$.

We consider the following problem.

Problem 3.3. *For given $\gamma \in \mathbf{R}$, find $u \in W_0^{1,\infty}$ and $\lambda \in \Lambda$ such that*

$$ {}_\infty< F(\lambda, u), v >_1 = 0, \quad \forall v \in W_0^{1,1}, \quad \textit{and} \quad u(x_0) = \gamma. \quad \square $$

Problem 3.3 corresponds to the equation $G(\lambda, u) = (0, 0)$. Naturally, we define the finite element solution for Problem 3.3 by

Problem 3.3$_{FE}$. *For given $\gamma \in \mathbf{R}$, find $u_h \in \overset{\circ}{S}_h$ and $\lambda_h \in \Lambda$ such that*

$$ < F(\lambda_h, u_h), v_h > = 0, \quad \forall v_h \in \overset{\circ}{S}_h, \quad \textit{and} \quad u_h(x_0) = \gamma. \quad \square $$

Since $DG(\lambda, u)$ is an isomorphism, we can apply (3.6) to Problem 3.3 and 3.3_{FE}. The linearized equation is

$$(3.8) \quad (U(x_0), \theta D_\lambda F(\lambda_h, u_h) + D_u F(\lambda_h, u_h)U) = (0, -F(\lambda_h, u_h)), \quad \theta \in \mathbf{R}, \ U \in W_0^{1,\infty}.$$

It follows from Theorem 2.13 that $\lim_{h \to 0} F(\lambda_h, u_h) = 0$ and $\lim_{h \to 0}(||U||_{W_0^{1,\infty}} + |\theta|) = 0$. We set

$X = \mathbf{R} \times W_0^{1,\infty}$, $Y = \mathbf{R} \times W^{-1,\infty}$,

$K(\lambda, u) = G(\lambda, u)$ for given $\gamma \in \mathbf{R}$

By (3.6), we have

Theorem 3.4. *Suppose that $(\lambda, u) \in \mathcal{M}_0$ satisfies $D_\lambda F(\lambda, u) \neq 0$. Then we have, under Assumption 2.11 with $d \geq 2$,*

$$|\lambda - \lambda_h| + ||u - u_h||_{W_0^{1,\infty}} \leq \left(|\theta| + ||U||_{W_0^{1,\infty}}\right)(1 + o(1)). \qquad \square$$

We can get an a posteriori error estimate in H_0^1-norm. Rewriting (3.4) in the above setting, we have

$$(3.9) \qquad DG(\lambda_h, u_h)(t - \theta, z - U) = -\frac{1}{2}\left(\int_0^1 (1-s)^2 D^2 G(\lambda_h + st, u_h + sz) ds\right)(t, z)^2,$$

where $t := \lambda - \lambda_h$, $z := u - u_h$, and

$$(3.10) \qquad D^2 G(\lambda, u) = (0, D^2 F(\lambda, u)).$$

Since

$$\begin{aligned} < D^2 F(\lambda, u)(t, z)^2, v > &= t^2 \int_J [a_{\lambda\lambda}(\lambda, x, u')v' + f_{\lambda\lambda}(\lambda, x, u)v] dx \\ &+ 2t \int_J [a_{\lambda y}(\lambda, x, u')z'v' + f_{\lambda y}(\lambda, x, u)zv] dx \\ &+ \int_J [a_{yy}(\lambda, x, u')z'^2 v' + f_{yy}(\lambda, x, u)z^2 v] dx, \end{aligned}$$

we easily obtain

$$(3.11) \qquad ||D^2 F(\lambda_h + st, u_h + sz)(t, z)^2||_{H^{-1}} \leq A|t|^2 + B|t| ||z||_{H_0^1} + C||z||_{W_0^{1,\infty}} ||z||_{H_0^1}$$

for any $s \in [0, 1]$, where A, B, C are constants independent of h.

Therefore, from (3.9)-(3.11), there exists a constant M such that

$$|t - \theta| + ||z - U||_{H_0^1} \leq M(|\theta|^2 + (|\theta| + ||z||_{W_0^{1,\infty}})||z||_{H_0^1}),$$

and we obtain

Theorem 3.5. *Suppose that* $(\lambda, u) \in \mathcal{M}_0$ *satisfies* $D_\lambda F(\lambda, u) \neq 0$. *Then we have, under Assumption 2.11 with* $d \geq 2$,

$$|\lambda - \lambda_h| + \|u - u_h\|_{H_0^1} \leq \left(|\theta| + \|U\|_{H_0^1}\right)(1 + o(1)). \qquad \square$$

Now, to get another a posteriori estimate, we consider the following auxiliary equation: find $W_\eta \in H_0^1$ such that

$$\begin{cases} -(\alpha_h(x)W_\eta')' + \beta_h(x)W_\eta = -F_h - \eta K_h & \text{on } J - \{x_0\}, \\ W_\eta(x_0) = 0, \end{cases} \tag{3.12}$$

where $\alpha_h(x) := a_y(\lambda_h, x, u_h'(x))$, $\beta_h(x) := f_y(\lambda_h, x, u_h(x))$, and

$$\begin{aligned} F_h &:= -a(\lambda_h, x, u_h'(x))' + f(\lambda_h, x, u_h(x)), \\ K_h &:= -a_\lambda(\lambda_h, x, u_h'(x))' + f_\lambda(\lambda_h, x, u_h(x)). \end{aligned}$$

Since (3.12) is equivalent to $DG(\lambda_h, u_h)(0, W_\eta) = (0, -F_h - \eta K_h 0)$ we see that, for sufficiently small $h > 0$, (3.12) has a unique solution W_η for $\eta \in \mathbb{R}$ which is sufficiently close to θ (see the proof of Lemma 4.3).

Note that, even if $u_h'(x)$ is not continuous,

$$\Phi_\eta(x) := \alpha_h(x)W_\eta'(x) + \eta a_\lambda(\lambda_h, x, u_h'(x)) + a(\lambda_h, x, u_h'(x))$$

is continuous on $J - \{x_0\}$. Then, we define the 'jump' $J(\eta)$ at $x = x_0$ by

$$J(\eta) := \lim_{x \to x_0-} \Phi_\eta(x) - \lim_{x \to x_0+} \Phi_\eta(x). \tag{3.13}$$

From (3.8) and (3.12), we clearly have $J(\theta) = 0$. Let $\tilde{U} := W_0$. Then, we claim that

Lemma 3.6. *We have*

$$\begin{aligned} |\theta| + \|U\|_{W_0^{1,\infty}} &\leq \|\tilde{U}\|_{W_0^{1,\infty}}(1 + o(1)), \\ |\theta| + \|U\|_{H_0^1} &\leq \|\tilde{U}\|_{H_0^1}(1 + o(1)). \end{aligned}$$

Proof. Integrating (3.12) by part, we have

$$< D_u F(\lambda_h, u_h)\tilde{U}, v >= - < F(\lambda_h, u_h), v > + v(x_0)J(0), \tag{3.14}$$

for any $v \in H_0^1$. Taking $\psi \in \mathring{S}_h$ with $\psi(x_0) = 1$ and fixing it, we obtain

$$< D_u F(\lambda_h, u_h)\tilde{U}, \psi >= J(0). \tag{3.15}$$

Let $z \in H_0^1$ be the function which satisfies $z(x_0) = 0$ and

$$< D_u F(\lambda_h, u_h)z, v > = < D_u F(\lambda_h, u_h)\psi, v >, \quad \forall v \in H_0^1[x_0],$$

where $H_0^1[x_0] := \{v \in H_0^1 \mid v(x_0) = 0\}$. Since $\tilde{U}(x_0) = 0$, we have

$$< D_u F(\lambda_h, u_h)z, \tilde{U} > = < D_u F(\lambda_h, u_h)\psi, \tilde{U} > . \tag{3.16}$$

It follows from (3.14) that

$$< D_u F(\lambda_h, u_h)\tilde{U}, w_h > = - < F(\lambda_h, u_h), w_h > = 0, \quad \forall w_h \in \mathring{S}_h[x_0], \tag{3.17}$$

where $\mathring{S}_h[x_0] := \left\{ v_h \in \mathring{S}_h \mid v_h(x_0) = 0 \right\}$.

Since $D_u F(\lambda_h, u_h)$ is self-adjoint, we obtain

$$< D_u F(\lambda_h, u_h)(z - w_h), \tilde{U} > = < D_u F(\lambda_h, u_h)\psi, \tilde{U} > \tag{3.18}$$

by subtracting (3.17) from (3.16). Combining (3.15) and (3.18), we obtain

$$|J(0)| = | < D_u F(\lambda_h, u_h)(z - w_h), \tilde{U} > |, \quad \forall w_h \in \mathring{S}_h[x_0].$$

Now, letting w_h be the finite element solution of z, we get

$$|J(0)| \leq C_1 \|z - w_h\|_{H_0^1} \|\tilde{U}\|_{W_0^{1,\infty}}, \qquad \lim_{h \to 0} \|z - w_h\|_{H_0^1} = 0,$$

where C_1 is a constant independent of h.

Since (3.12) is a linear equation with respect to W_η, the implicit mapping $\eta \mapsto W_\eta$ defined by (3.12) is C^∞ and therefore there exists the derivative of W_η with respect to η. We denote it by $\partial_\eta W_\eta$. We see that $\partial_\eta W_\eta \in H_0^1$, and it satisfies

$$\begin{cases} -(\alpha_h(x)(\partial_\eta W_\eta)')' + \beta_h(x)\partial_\eta W_\eta = -K_h & \text{on } J - \{x_0\}, \\ \partial_\eta W_\eta(x_0) = 0, \end{cases} \tag{3.19}$$

Note that integrating (3.19) by part shows us that $J(\eta)$ is differentiable and we have

$$< D_\lambda F(\lambda_h, u_h) + D_u F(\lambda_h, u_h)\partial_\eta W_\eta, v > = v(x_0)J'(\eta), \quad \forall v \in H_0^1.$$

We remark that $J'(\eta) \neq 0$. If $J'(\eta) = 0$, we would have

$$(\partial_\eta W_\eta(x_0), D_\lambda F(\lambda_h, u_h) + D_u F(\lambda_h, u_h)\partial_\eta W_\eta) = (0, 0).$$

Since $DG(\lambda_h, u_h)$ is an isomorphism, we obtain the contradiction $(1, \partial_\eta W_\eta) = (0, 0)$.

Therefore, we obtain $-J(0) = J(\theta) - J(0) = \theta J'(\xi\theta)$, $0 < \xi < 1$, and

$$|\theta| \leq |J'(\xi\theta)|^{-1}|J(0)| \leq C_2||z - w_h||_{H_0^1}||\tilde{U}||_{W_0^{1,\infty}}. \tag{3.20}$$

On the other hand, we have

$$||U - \tilde{U}||_{W_0^{1,\infty}} \leq C_3|\theta|, \tag{3.21}$$

because $U = W_\theta$, $\tilde{U} = W_0$, and the mapping $\eta \mapsto W_\eta$ is C^∞. Combining (3.20) and (3.21), we conclude that

$$\begin{aligned} |\theta| + ||U||_{W_0^{1,\infty}} &\leq |\theta| + ||U - \tilde{U}||_{W_0^{1,\infty}} + ||\tilde{U}||_{W_0^{1,\infty}} \\ &\leq ||\tilde{U}||_{W_0^{1,\infty}}(1 + o(1)). \end{aligned}$$

The second inequality is proved by the same manner. □

Theorem 3.7. *Suppose that $(\lambda, u) \in \mathcal{M}_0$ satisfies $D_\lambda F(\lambda, u) \neq 0$. Then we have, under Assumption 2.11 with $d \geq 2$,*

$$\begin{aligned} |\lambda - \lambda_h| + ||u - u_h||_{W_0^{1,\infty}} &\leq ||\tilde{U}||_{W_0^{1,\infty}}(1 + o(1)), \\ |\lambda - \lambda_h| + ||u - u_h||_{H_0^1} &\leq ||\tilde{U}||_{H_0^1}(1 + o(1)), \end{aligned}$$

where $\tilde{U} \in H_0^1$ is the exact solution of the following equation:

$$\begin{cases} \displaystyle\int_J [a_y(\lambda_h, x, u_h')\tilde{U}'v' + f_y(\lambda_h, x, u_h)\tilde{U}v]dx = - < F(\lambda_h, u_h), v >, & \forall v \in H_0^1[x_0]. \\ \tilde{U}(x_0) = 0. & \square \end{cases} \tag{3.22}$$

Remark 3.8. From Theorem 3.1, 3.2, 3.4, 3.5, and 3.7, a posteriori estimates of the error $||u - u_h||$ and $|\lambda - \lambda_h| + ||u - u_h||$ are reduced to estimates of $||U||$, $|\theta|$, and $||\tilde{U}||$, respectively. We will compute the finite element solutions U_h, θ_h, $\tilde{U}_h$ of the equations (3.7), (3.8), and (3.22) and see $||U_h||$, $|\theta_h|$, $||\tilde{U}_h||$ instead of $||U||$, $|\theta|$, $||\tilde{U}||$.

We must, however, notice that the right-hand sides of those linearized equations (3.7), (3.8), and (3.22) are the terms $F(\lambda, u_h)$ and $F(\lambda_h, u_h)$. By the definition, those terms vanish for $v_h \in \mathring{S}_h$, that is, $< F(\lambda, u_h), v_h > = < F(\lambda_h, u_h), v_h > = 0$ for any $v_h \in \mathring{S}_h$. Hence, the finite element solutions of (3.7), (3.8), and (3.22) over $\mathring{S}_h$ would be just the zero functions and they would be useless to estimate $||U||$, $|\theta|$, and $||\tilde{U}||$.

We use the different finite element space $\tilde{S}_h$ such that $\overset{\circ}{S}_h \subset \tilde{S}_h$ to avoid this difficulty. That is, to compute the finite element solutions of (3.7), (3.8), and (3.22), we use the refined mesh or higher order polynomials on each finite element. Then, we will obtain nonzero finite element solutions U_{h_1}, θ_{h_1}, $\tilde{U}_{h_1}$, and $\|U_{h_1}\|$, $|\theta_{h_1}|$, $\|\tilde{U}_{h_1}\|$ indicate the error $\|u-u_h\|$ and $|\lambda-\lambda_h|+\|u-u_h\|$, respectively.

Note that, in the computation of the $\|U_{h_1}\|$, $|\theta_{h_1}|$, $\|\tilde{U}_{h_1}\|$, we do not need to solve the entire problem. The estimations of $\|U_{h_1}\|$, $|\theta_{h_1}|$, $\|\tilde{U}_{h_1}\|$ are done by an element-by-element approach (see [BR]).

The details of the practical computation will be presented in Section 5. □

4. Elaborate Error Estimates of $|\lambda - \lambda_h|$.

Sometimes, one may want to estimate only the error $|\lambda-\lambda_h|$. Usually, it is observed that $|\lambda-\lambda_h|$ is much smaller than $\|u-u_h\|_{H_0^1}$. In this section we develop two elaborate error estimates of $|\lambda - \lambda_h|$.

Let $(\lambda, u) \in \mathcal{M}_0$ be such that (λ, u) is around a turning point or on a 'steep slope', that is, $D_\lambda F(\lambda, u) \neq 0$. Let the nodal point $x_0 \in J$ of $\overset{\circ}{S}_h$ be taken so that $DG(\lambda, u) \in \mathcal{L}(\mathbf{R} \times W_0^{1,\infty}, \mathbf{R} \times W^{-1,\infty})$ is an isomorphism (see Theorem 2.13). Let $\gamma := u(x_0)$. Then, (λ, u) is a solution of the following problem:

Problem 4.1. *Find $u \in W_0^{1,\infty}$ and $\lambda \in \Lambda$ such that*

$$
(4.1)\qquad \begin{cases} \displaystyle\int_J [a(\lambda, x, u'(x))v'(x) + f(\lambda, x, u(x))v(x)]dx = 0, & \forall v \in W_0^{1,1}, \\ u(x_0) = \gamma. \end{cases}
$$
□

By Theorem 2.13, it is guaranteed that, for sufficiently small $h > 0$, there exists a locally unique solution $(\lambda_h, u_h) \in \mathcal{M}_h$ of the following problem around $(\lambda, u) \in \mathcal{M}_0$.

Problem 4.1$_{FE}$. *Find $u \in \overset{\circ}{S}_h$ and $\lambda_h \in \Lambda$ such that*

$$
(4.2)\qquad \begin{cases} \displaystyle\int_J [a(\lambda_h, x, u_h'(x))v_h'(x) + f(\lambda_h, x, u_h(x))v_h(x)]dx = 0, & \forall v_h \in \overset{\circ}{S}_h, \\ u_h(x_0) = \gamma. \end{cases}
$$
□

To estimate the error $|\lambda - \lambda_h|$ we introduce the following auxiliary equation.

Problem 4.2. *Find $\tilde{u} \in W_0^{1,\infty}$ such that*

$$(4.3) \qquad \begin{cases} -(a(\lambda_h, x, \tilde{u}'(x)))' + f(\lambda_h, x, \tilde{u}(x)) = 0 & \text{on } J - \{x_0\}, \\ \tilde{u}(x_0) = \gamma. \end{cases} \qquad \Box$$

On the existence of the solution $\tilde{u}$ of Problem 4.2, we prove the following lemma.

Lemma 4.3. *Suppose that Assumption 2.11 holds for $d \geq 2$. Let $(\lambda_0, u_0) \in \mathcal{M}_0$ be such that $D_\lambda F(\lambda_0, u_0) \neq 0$. By Thorem 2.13 we can take a nodal point $x_0 \in J$ of $\overset{\circ}{S}_h$ such that $DG(\lambda_0, u_0)$ defined by (2.7) is an isomorphism.*

Then, there exist $\varepsilon > 0$ and a unique C^2 map $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \ni \tilde{\lambda} \mapsto w(\tilde{\lambda}) \in W_0^{1,\infty}$ which satisfies

$$(4.4) \qquad -(a(\tilde{\lambda}, x, w(\tilde{\lambda})'(x)))' + f(\tilde{\lambda}, x, w(\tilde{\lambda})(x)) = 0, \qquad w(\tilde{\lambda})(x_0) = \gamma,$$

where $\gamma := u_0(x_0)$.

Proof. Let $W_0^{1,1}[x_0] := \{v \in W_0^{1,1} \,|\, v(x_0) = 0\}$. First, we note that $w(\tilde{\lambda})$ satisfies (4.4) if and only if it is the solution of the following equation:

$$\begin{cases} \int_J [a(\tilde{\lambda}, x, w(\tilde{\lambda})'(x))v'(x) + f(\tilde{\lambda}, x, w(\tilde{\lambda})(x))v(x)]dx = 0, & \forall v \in W_0^{1,1}[x_0], \\ w(\tilde{\lambda})(x_0) = \gamma. \end{cases}$$

Let $(W_0^{1,1}[x_0])^*$ be the dual space of $W_0^{1,1}[x_0]$. We define the mapping $\tilde{G} : \Lambda \times W_0^{1,\infty} \to \mathbf{R} \times (W_0^{1,1}[x_0])^*$ by $\tilde{G}(\lambda, w) := \left(w(x_0) - \gamma, F(\lambda, w)\Big|_{W_0^{1,1}[x_0]}\right)$. Then we have, for $\psi \in W_0^{1,\infty}$,

$$(4.5) \qquad D_w\tilde{G}(\lambda, w)\psi = \left(\psi(x_0), (D_u F(\lambda, w)\psi)\Big|_{W_0^{1,1}[x_0]}\right).$$

If we show that $D_w\tilde{G}(\lambda_0, u_0) \in \mathcal{L}(W_0^{1,\infty}, \mathbf{R} \times (W_0^{1,1}[x_0])^*)$ is an isomorphism, Lemma 4.3 is proved by the implicit function theorem.

Recall that $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbf{R} \times W_0^{1,\infty}, \mathbf{R} \times W^{-1,\infty})$ is an isomorphism. Thus, the mapping $DG(\lambda_0, u_0)\Big|_{\{0\} \times W_0^{1,\infty}}$ is an isomorphism of $\{0\} \times W_0^{1,\infty}$ into its image. From (2.7), we have

$$(4.6) \qquad \left(DG(\lambda_0, u_0)\Big|_{\{0\} \times W_0^{1,\infty}}\right)\psi = (\psi(x_0), D_u F(\lambda_0, u_0)\psi).$$

By (4.5) and (4.6), we conclude that $D_w\tilde{G}(\lambda_0, u_0)$ is an isomorphism, and this completes the proof. $\Box$

For $w(\tilde{\lambda})$ defined by (4.4), we define the 'jump' $J(\tilde{\lambda})$ of $w(\tilde{\lambda})$ at $x = x_0$ by

$$J(\tilde{\lambda}) := \lim_{x \to x_0 -} a(\tilde{\lambda}, x, w(\tilde{\lambda})'(x)) - \lim_{x \to x_0 +} a(\tilde{\lambda}, x, w(\tilde{\lambda})'(x)).$$

From (4.4), we have

$$(4.7) \qquad < F(\tilde{\lambda}, w(\tilde{\lambda})), v >= \int_J [a(\tilde{\lambda}, x, w(\tilde{\lambda})')v' + f(\tilde{\lambda}, x, w(\tilde{\lambda}))v]dx = v(x_0)J(\tilde{\lambda}),$$

for any $v \in \overset{\circ}{W}{}^{1,1}_0$. Let $\tilde{u} := w(\lambda_h)$. Then, $\tilde{u}$ is the solution of Problem 4.2, and we have

$$(4.8) \qquad < F(\lambda_h, \tilde{u}), \psi >= \int_J [a(\lambda_h, x, \tilde{u}')\psi' + f(\lambda, x, \tilde{u})\psi]dx = J(\lambda_h),$$

for any $\psi \in W_0^{1,1}$ with $\psi(x_0) = 1$.

We take $\psi \in \overset{\circ}{S}_h$ with $\psi(x_0) = 1$ and fix it. Since $(\lambda_h, u_h) \in \mathcal{M}_h$, we have

$$(4.9) \qquad \int_J [a(\lambda_h, x, u_h'(x))\psi' + f(\lambda_h, x, u_h(x))\psi]dx = 0.$$

From (4.8) and (4.9) we get

$$(4.10) \qquad J(\lambda_h) = \sum_{i=0}^{2} \int_J [\alpha_i(x)(\tilde{u}' - u_h')^{i+1}\psi' + \beta_i(x)(\tilde{u} - u_h)^{i+1}\psi]dx,$$

where

$$(4.11) \qquad \begin{cases} \alpha_0(x) := a_y(\lambda_h, x, u_h'(x)), \quad \beta_0(x) := f_y(\lambda_h, x, u_h(x)), \\ \alpha_1(x) := \frac{1}{2}a_{yy}(\lambda_h, x, u_h'(x)), \quad \beta_1(x) := \frac{1}{2}f_{yy}(\lambda_h, x, u_h(x)), \\ \alpha_2(x) := \frac{1}{6}a_{yyy}(\lambda_h, x, u_h'(x) + \epsilon_1(\tilde{u}'(x) - u_h'(x))), \quad 0 < \epsilon_1 < 1, \\ \beta_2(x) := \frac{1}{6}f_{yyy}(\lambda_h, x, u_h(x) + \epsilon_2(\tilde{u}(x) - u_h(x))), \quad 0 < \epsilon_2 < 1, \end{cases}$$

From the proof of Lemma 4.3 and $\lim_{h \to 0} \|u - u_h\|_{\overset{\circ}{W}{}^{1,\infty}} = 0$, there exists a unique $z \in W_0^{1,\infty}$ such that $z(x_0) = 0$ and

$$(4.12) \qquad \int_J [\alpha_0(x)z'v' + \beta_0(x)zv]dx = \int_J [\alpha_0(x)\psi'v' + \beta_0(x)\psi v]dx, \quad \forall v \in W_0^{1,1}[x_0].$$

Note that $\tilde{u}(x_0) = u_h(x_0) = \gamma$. Hence, $\tilde{u} - u_h = 0$ at $x = x_0$. Thus, we obtain

$$(4.13) \qquad \begin{aligned} &\int_J [\alpha_0(x)z'(\tilde{u} - u_h)' + \beta_0(x)z(\tilde{u} - u_h)]dx \\ &\qquad = \int_J [\alpha_0(x)\psi'(\tilde{u} - u_h)' + \beta_0(x)\psi(\tilde{u} - u_h)]dx. \end{aligned}$$

On the other hand, for any $w \in \overset{\circ}{S}_h$ with $w(x_0) = 0$, we have

$$\int_J [a(\lambda_h, x, \tilde{u}'(x))w' + f(\lambda_h, x, \tilde{u}(x))w]dx = 0,$$

$$\int_J [a(\lambda_h, x, u_h'(x))w' + f(\lambda_h, x, u_h(x))w]dx = 0.$$

Therefore, we obtain

$$(4.14) \qquad \sum_{i=0}^{2} \int_J [\alpha_i(x)(\tilde{u}' - u_h')^{i+1}w' + \beta_i(x)(\tilde{u} - u_h)^{i+1}w]dx = 0.$$

From (4.10), (4.13) and (4.14), it follows that

$$|J(\lambda_h)| \le \left|\int_J [\alpha_0(x)(z' - w')(\bar{u}' - u_h') + \beta_0(x)(z - w)(\bar{u} - u_h)]dx\right| + \left|\sum_{i=1}^{2} \int_J [\alpha_i(x)(\bar{u}' - u_h')^{i+1}(\psi' - w') + \beta_i(x)(\bar{u} - u_h)^{i+1}(\psi - w)]dx\right|. \tag{4.15}$$

Now, let $z_h \in \overset{\circ}{S}_h$ be the finite element solution of z and plug it into (4.15). Then, we obtain the following higher order error estimate of $|J(\lambda_h)|$:

$$|J(\lambda_h)| \le \Big(| < D_u F(\lambda_h, u_h)(\bar{u} - u_h), z - z_h) > | + \frac{1}{2}| < D_{uu}^2 F(\lambda_h, u_h)(\bar{u} - u_h)^2, \psi - z_h > |\Big)(1 + o(1)). \tag{4.16}$$

Let us now see the relationship between $|\lambda - \lambda_h|$ and $|J(\lambda_h)|$. By Lemma 4.3, the solution $w(\tilde{\lambda})$ of (4.4) is differentiable with respect to $\tilde{\lambda}$. Differentiating (4.7) with respect to $\tilde{\lambda}$, we show that the function $\tilde{\lambda} \mapsto J(\tilde{\lambda})$ is C^2 and satisfies

$$v(x_0)J'(\tilde{\lambda}) =< D_\lambda F(\tilde{\lambda}, w(\tilde{\lambda})) + D_u F(\tilde{\lambda}, w(\tilde{\lambda}))(\partial_\lambda w(\tilde{\lambda})), v >, \quad \forall v \in W_0^{1,1}. \tag{4.17}$$

where $\partial_\lambda w(\tilde{\lambda}) \in W_0^{1,\infty}$ is the derivative of $w(\tilde{\lambda})$ with respect to $\tilde{\lambda}$.

Recall that $(\lambda, u) \in \mathcal{M}_0$, $D_\lambda F(\lambda, u) \ne 0$, $u = w(\lambda)$, and $J(\lambda) = 0$. We claim that $J'(\lambda) \ne 0$. If $J'(\lambda) = 0$, we would have $D_\lambda F(\lambda, u) + D_u F(\lambda, u)(\partial_\lambda u) = 0$, and $(\partial_\lambda u)(x_0) = 0$. Since $DG(\lambda, u)$ is an isomorphism, we have a contradiction that $(0, 0) = (1, \partial_\lambda u)$. Since $\lim_{h \to 0} |\lambda - \lambda_h| = 0$, we conclude that $J'(\lambda_h) \ne 0$ for sufficiently small $h > 0$.

Thus, we have $-J(\lambda_h) = J(\lambda) - J(\lambda_h) = (\lambda - \lambda_h)J'(\lambda_h + \xi(\lambda - \lambda_h))$, $0 < \xi < 1$, and

$$|\lambda - \lambda_h| = |J'(\lambda_h + \xi(\lambda - \lambda_h))|^{-1} |J(\lambda_h)|, \tag{4.18}$$

for sufficiently small $h > 0$.

We would like to replace the term $|J'(\lambda_h + \xi(\lambda - \lambda_h))|^{-1}$ by some computable one. We first note that

$$\begin{aligned} J'(\lambda_h + \xi(\lambda - \lambda_h)) &= J'(\lambda_h) + \xi(\lambda - \lambda_h)J''(\lambda_h + \mu(\lambda - \lambda_h)) \\ &= J'(\lambda_h)(1 + o(1)), \qquad (0 < \mu < 1). \end{aligned}$$

Hence, we just have to approximate $J'(\lambda_h)$ instead of $J'(\lambda_h + \xi(\lambda - \lambda_h))$.

Again, take $\psi \in \overset{\circ}{S}_h$ with $\psi(x_0) = 1$ and fix it. From (4.17), we have

$$J'(\lambda_h) =< D_\lambda F(\lambda_h, \bar{u}) + D_u F(\lambda_h, \bar{u})(\partial_\lambda \bar{u}), \psi > . \tag{4.19}$$

With (4.19) in our mind, we define the 'approximate jump' $J_h'(\lambda_h)$ at $x = x_0$ by

$$(4.20) \qquad J_h'(\lambda_h) := < D_\lambda F(\lambda_h, u_h) + D_u F(\lambda_h, u_h)(\partial_\lambda u_h), \psi >$$

where $\partial_\lambda u_h \in \overset{\circ}{S}_h[x_0]$ is the finite element solution of the following equation:

$$(4.21) \qquad < D_u F(\lambda_h, u_h)(\partial_\lambda u_h), v_h > = - < D_\lambda F(\lambda_h, u_h), v_h >, \quad \forall v_h \in \overset{\circ}{S}_h[x_0].$$

Now, we would like to estimate $|J'(\lambda_h) - J_h'(\lambda_h)|$. First, we note that $\|\partial_\lambda \tilde{u} - \partial_\lambda u_h\|_{H_0^1} = o(1)$, because $\partial_\lambda \tilde{u} \in W_0^{1,\infty}$ satisfies $(\partial_\lambda \tilde{u})(x_0) = 0$, and

$$< D_u F(\lambda_h, \tilde{u})(\partial_\lambda \tilde{u}), v > = - < D_\lambda F(\lambda_h, \tilde{u}), v >, \quad \forall v \in H_0^1[x_0].$$

Subtracting (4.20) from (4.19), we see

$$\begin{aligned} J'(\lambda_h) - J_h'(\lambda_h) \quad = \quad & < D_u F(\lambda_h, u_h)(\partial_\lambda \tilde{u} - \partial_\lambda u_h), \psi > + < D^2_{\lambda u} F(\lambda_h, u_h)(\tilde{u} - u_h), \psi > \\ & + < D^2_{uu} F(\lambda_h, u_h)(\partial_\lambda \tilde{u}, \tilde{u} - u_h), \psi > + \text{ higher order terms.} \end{aligned}$$

Therefore, we obtain

$$(4.22) \qquad |J'(\lambda_h) - J_h'(\lambda_h)| = o(1).$$

Finally, we replace the term $\tilde{u} - u_h$ in (4.16) by $u - u_h$. Since $u = w(\lambda)$, $\tilde{u} = w(\lambda_h)$, and the mapping $(\lambda - \varepsilon, \lambda + \varepsilon) \ni \tilde{\lambda} \mapsto w(\tilde{\lambda}) \in W_0^{1,\infty}$ is C^2 class, we have $\|u - \tilde{u}\|_{W_0^{1,\infty}} \leq C_0 |\lambda - \lambda_h|$ with a constant C_0 independent of h. Therefore, we immediately obtain that

$$(4.23) \qquad \|\tilde{u} - u_h\|_{H_0^1} \leq \|u - u_h\|_{H_0^1} + C\|u - \tilde{u}\|_{W_0^{1,\infty}} = \|u - u_h\|_{H_0^1} + CC_0|\lambda - \lambda_h|.$$

Combining (4.16), (4.18), (4.22), and (4.23), we obtain the first elaborate error estimate of $|\lambda - \lambda_h|$.

Theorem 4.4. *Suppose that Assumption 2.11 holds for $d \geq 3$. Let $(\lambda, u) \in \mathcal{M}$ be such that $D_\lambda F(\lambda, u) \neq 0$. By Theorem 2.13 we can take a nodal point $x_0 \in J$ such that $DG(\lambda, u)$ defined by (2.7) is an isomorphism. Let $(\lambda_h, u_h) \in \mathcal{M}_h$ be the finite element solution corresponding to (λ, u) with $u(x_0) = u_h(x_0)$. Then, we have the following estimate of $|\lambda - \lambda_h|$:*

$$\begin{aligned} (4.24) \qquad |\lambda - \lambda_h| \leq |J_h'(\lambda_h)|^{-1} \Big(& | < D_u F(\lambda_h, u_h)(u - u_h), z - z_h) > | \\ & + \frac{1}{2} | < D^2_{uu} F(\lambda_h, u_h)(u - u_h)^2, \psi - z_h > | \Big)(1 + o(1)), \end{aligned}$$

where $z \in H_0^1$ and $z_h \in \overset{\circ}{S}_h$ are, respectively, the exact and finite element solution of (4.12) for appropriate $\psi \in \overset{\bullet}{S}_h$ with $\psi(x_0) = 1$, and $J_h'(\lambda_h)$ is the 'approximate jump' defined by (4.20) and (4.21). □

Now, let us consider the second elaborate error estimate of $|\lambda - \lambda_h|$.

Again, we consider Problem 4.1 and 4.1_{FE}. From (4.1) and (4.2), we have

$$
\begin{aligned}
(4.25) \qquad 0 &= (\lambda - \lambda_h) < D_\lambda F^h, v_h > + < D_u F^h(u - u_h), v_h > \\
&+ \frac{1}{2}(\lambda - \lambda_h)^2 < D^2_{\lambda\lambda} F^h, v_h > + (\lambda - \lambda_h) < D^2_{\lambda u} F^h(u - u_h), v_h > \\
&+ \frac{1}{2} < D^2_{uu} F^h(u - u_h)^2, v_h > + \text{ higher order terms}, \quad \forall v_h \in \mathring{S}_h,
\end{aligned}
$$

where $D_\lambda F^h := D_\lambda F(\lambda_h, u_h)$, $D_u F^h := D_u F(\lambda_h, u_h)$, etc.

We introduce the following auxiliary equation: Find $\eta \in \mathbf{R}$ and $z \in H^1_0$ such that

$$
(4.26) \qquad \begin{cases} \int_J [\alpha_0(x) z' v' + \beta_0(x) z v] dx = \eta < \delta_{x_0}, v >, \quad \forall v \in H^1_0, \\ < D_\lambda F^h, z >= 1, \end{cases}
$$

where δ_{x_0} is Dirac's delta at x_0, and α_0, β_0 are defined by (4.11). For the existence of the solution of (4.26), we show the following lemma.

Lemma 4.5. *For sufficiently small* $h > 0$, (4.26) *has a unique solution* $(\eta, z) \in \mathbb{R} \times H^1_0$.

Proof. First, suppose that we have Case 1, that is, $D_u F^h := D_u F(\lambda_h, u_h) \in \mathcal{L}(H^1_0, H^{-1})$ is an isomorphism. Then, we have $< D_\lambda F^h, (D_u F^h)^{-1}(\delta_{x_0}) >=< \delta_{x_0}, (D_u F^h)^{-1}(D_\lambda F^h) > \neq 0$, because of the way of taking the nodal point $x_0 \in J$ (see the proof of [TB1,Lemma 8.1]). Therefore, (4.26) has the unique solution $\eta :=< D_\lambda F^h, (D_u F^h)^{-1}(\delta_{x_0}) >^{-1}$, $z := \eta (D_u F^h)^{-1}(\delta_{x_0})$.

Next, suppose that we have Case 2, that is, $\mathrm{Ker} D_u F^h = \mathrm{span}\{\psi\}$ and $D_\lambda F^h \notin \mathrm{Im} D_u F^h$. There exists $(\theta, \phi) \in \mathbf{R} \times H^1_0$ such that $\theta D_\lambda F^h + (D_u F^h)\phi = \delta_{x_0}$, and $\theta \in \mathbb{R}$ is determined uniquely. We check that $\delta_{x_0} \notin \mathrm{Im} D_u F^h$, and hence $\theta \neq 0$. If $\delta_{x_0} \in \mathrm{Im} D_u F^h$, there would exist $w \in H^1_0$ such that $\delta_{x_0} = D_u F^h w$. Thus, from [TB1,Lemma 8.1], we obtain an contradiction $0 \neq < \delta_{x_0}, \psi >=< D_u F^h w, \psi >=< D_u F^h \psi, w >= 0$.

Therefore, in this case, (4.26) has the unique solution $\eta := 0$, $z := \dfrac{\theta}{\psi(x_0)} \psi$ because

$$
< D_\lambda F^h, z >=< D_\lambda F^h, z > + \theta^{-1} < D_u F^h \phi, z >= \theta^{-1} < \delta_{x_0}, z >= 1. \quad \square
$$

Now, let us set $v := u - u_h$ in (4.26). Since $< \delta_{x_0}, u - u_h >= 0$, we obtain

$$
(4.27) \qquad < D_u F^h z, u - u_h, >= \int_J [\alpha_0(x) z'(u' - u'_h) + \beta_0(x) z (u - u_h)] dx = 0.
$$

Since $D_u F^h$ is self-adjoint, it follows from (4.27) that

$$-< D_u F^h(u-u_h), v_h >=< D_u F^h(u-u_h), z-v_h >, \quad \forall v_h \in \overset{\circ}{S}_h. \tag{4.28}$$

From (4.25), (4.28), and plugging the finite element solution z_h of the equation (4.26) into v_h in (4.25), we obtain

$$\begin{aligned}(\lambda-\lambda_h) < D_\lambda F^h, -z_h > \; = \; & - < D_u F^h(u-u_h), z-z_h > \\ & + \frac{1}{2}(\lambda-\lambda_h)^2 < D^2_{\lambda\lambda} F^h, z_h > + (\lambda-\lambda_h) < D^2_{\lambda u} F^h(u-u_h), z_h > \\ & + \frac{1}{2} < D^2_{uu} F^h(u-u_h)^2, z_h > + \text{ higher order terms,}\end{aligned}$$

and we have proved the following theorem.

Theorem 4.6. *Suppose that Assumption 2.11 holds for $d \geq 3$. Let $(\lambda, u) \in \mathcal{M}$ and $D_\lambda F(\lambda, u) \neq 0$. By Theorem 2.13 we can take a nodal point $x_0 \in J$ so that $DG(\lambda, u)$ defined by* (2.7) *is an isomorphism. Let $(\lambda_h, u_h) \in \mathcal{M}_h$ be the finite element solution corresponding to (λ, u) with $u(x_0) = u_h(x_0)$.*

Then, we have the elaborate error estimate

$$|\lambda-\lambda_h| \leq \Big(| < D_u F(\lambda_h, u_h)(u-u_h), z-z_h) > | \\ + \frac{1}{2}| < D^2_{uu} F(\lambda_h, u_h)(u-u_h)^2, z_h > |\Big)(1+o(1)), \tag{4.29}$$

where z and z_h are the exact and finite element solutions of the equation (4.26), *respectively.* □

Remark 4.7. By Theorem 4.4 and 4.6, a priori error estimates of $|\lambda - \lambda_h|$ are obtained. Since we have a posteriori error estimates for $||u-u_h||_{H^1_0}$ and $||z-z_h||_{H^1_0}$, and all terms in (4.24) and (4.29) are computable, those estimates are a posteriori error estimates as well. The detail of practical computation is given in Section 11. □

5. Numerical Examples.

In Section 5 we present several numerical examples and discuss some points for implementation of the a posteriori error estimates presented in this paper. Our first example is the following simple one:

Example 5.1.

$$\begin{cases} u''(x) + \lambda u(x) = \sin(\pi x) \;\; in \; J := (0,1), \\ \qquad u(0) = u(1) = 0. \end{cases} \tag{5.1}$$
□

The exact solution of (5.1) is $u(x) = \dfrac{\sin(\pi x)}{\lambda - \pi^2}$ around $\lambda = \pi^2$.

Let Δ_{15} be the uniform mesh of $J = (0,1)$ with 15 elements. Let $\overset{\circ}{S}_h \subset H_0^1$ be the finite element space of piecewise quadratic functions over Δ_{15}. We use PITCON to compute the finite element solutions $(\lambda_h, u_h) \in \mathbf{R} \times \overset{\circ}{S}_h$ of (5.1). Table 5.1 shows a part of the output of PITCON.

As λ_h is getting close to π^2, u is getting bigger and the 'slope' is getting steeper. In Table 5.1 the term $|\lambda - \lambda_h|$ stands for the error between the exact and computed λ. Hence, $|\lambda - \lambda_h| = 0$ means that λ is the continuation parameter at that step. We see that while $\lambda \leq 7.13828$, λ is the continuation parameter. When λ becomes greater than that point, $u_h(x_0)$, $x_0 := 7h$ or $x_0 := 8h$ $(h = 1/15)$ is taken as the continuation parameter.

Table 5.1: The output of PITCON: Example 5.1.

λ_h	$\|\lambda - \lambda_h\|$	H_0^1-error	$W_0^{1,\infty}$-error
0.0	0.0	3.67717D−4	1.15846D−3
1.32361	0.0	4.24670D−4	1.33773D−3
3.17042	0.0	5.41741D−4	1.70611D−3
5.16503	0.0	7.71425D−4	2.42834D−3
7.13828	0.0	1.32875D−3	4.17803D−3
7.83691	2.02465D−5	1.80565D−3	5.64451D−3
8.85394	2.29519D−5	3.59614D−3	1.12780D−2
9.38086	2.43528D−5	7.44952D−3	2.34080D−2
9.50916	2.46938D−5	1.00926D−2	3.17268D−2
9.68198	2.51532D−5	1.93655D−2	6.08957D−2
9.75356	2.53434D−5	3.12921D−2	9.83730D−2

In Table 5.1 'H_0^1-error' stands for $\|u-u_h\|_{H_0^1}$ if λ is the continuation parameter, and $|\lambda-\lambda_h| + \|u - u_h\|_{H_0^1}$ if λ is not the continuation parameter. Also, '$W_0^{1,\infty}$-error' stands for $\|u - u_h\|_{W_0^{1,\infty}}$ if λ is the continuation parameter, and $|\lambda - \lambda_h| + \|u - u_h\|_{W_0^{1,\infty}}$ if λ is not the continuation parameter.

Now, we discuss how we estimate those errors. If λ is the continuation parameter, we surely can use a usual method to estimate $\|u - u_h\|_{H_0^1}$. We however use the method presented in this paper to estimate $\|u - u_h\|_{H_0^1}$ here to show how we implement our a posteriori error estimates.

Let $u_h \in \overset{\circ}{S}_h$ be the piecewise quadratic finite element solution of (5.1) over Δ_{15}. We then introduce another finite element space $\tilde{S}_h \in H_0^1$ of piecewise polynomials of degree 4 over Δ_{15}.

We compute the finite element solution $\tilde{u}_h \in \tilde{S}_h$ of the following equation:

$$(5.2) \qquad \int_J(-\tilde{u}_h'\tilde{v}_h' + \lambda\tilde{u}_h\tilde{v}_h)dx = \int_J(u_h'\tilde{v}_h' - \lambda u_h\tilde{v}_h + \sin(\pi x)\tilde{v}_h)dx, \quad \forall\tilde{v}_h \in \tilde{S}_h.$$

Then, by Theorem 3.1, 3.2, and Remark 3.8, $||\tilde{u}_h||_{H_0^1}$ and $||\tilde{u}_h||_{W_0^{1,\infty}}$ indicate $||u - u_h||_{H_0^1}$ and $||u - u_h||_{W_0^{1,\infty}}$, respectively.

Note that, as stated in Remark 3.8, we do not need to solve the full system of (5.2). Instead of that we just solve tiny equations defined on each element because we can assume that the values of the finite element solutions of (5.2) at the nodal points are zero (see [BR]).

As mentioned before, if λ is not the continuation parameter, $x_0 = 7h$ or $x_0 = 8h$ ($h = 1/15$) was taken by PITCON so that $u_h(x_0)$ is the continuation parameter at that step. In that case, we use the methods presented by Theorem 3.4, 3.5, and 3.7. We first approximate the exact solution $(\theta, U_1) \in \mathbf{R} \times H_0^1$ of the equation

$$(5.3) \quad \int_J(-U_1'v' + \lambda_h U_1 v)dx + \theta\int_J u_h v dx = \int_J(u_h'v' - \lambda_h u_h v + \sin(\pi x)v)dx, \quad \forall v \in H_0^1,$$

with $U_1(x_0) = 0$ (see (3.8)). We also approximate the exact solution $U_2 \in H_0^1$ of the equation

$$(5.4) \qquad \int_J(-U_2'v' + \lambda_h U_2 v)dx = \int_J(u_h'v' - \lambda_h u_h v + \sin(\pi x)v)dx,$$

with $U_2(x_0) = 0$ for any $v \in H_0^1$ with $v(x_0) = 0$ (see (3.22)).

Let $(\theta_h, U_{1h}) \in \mathbf{R} \times \tilde{S}_h$ and $U_{2h} \in \tilde{S}_h$ be the finite element solutions of (5.3) and (5.4), respectively. By Theorem 3.4, 3.5, 3.7, and Remark 3.8, we have the following estimates:

$$(5.5) \qquad \begin{cases} |\lambda - \lambda_h| + ||u - u_h||_{H_0^1} \le (|\theta_h| + ||U_{1h}||_{H_0^1})(1 + o(1)), \\ |\lambda - \lambda_h| + ||u - u_h||_{W_0^{1,\infty}} \le (|\theta_h| + ||U_{1h}||_{W_0^{1,\infty}})(1 + o(1)), \end{cases}$$

$$(5.6) \qquad \begin{cases} |\lambda - \lambda_h| + ||u - u_h||_{H_0^1} \le ||U_{2h}||_{H_0^1}(1 + o(1)), \\ |\lambda - \lambda_h| + ||u - u_h||_{W_0^{1,\infty}} \le ||U_{2h}||_{W_0^{1,\infty}}(1 + o(1)). \end{cases}$$

Again, to solve (5.3) and (5.4), we take the element-by-element approach, that is, instead of computing full systems, we solve tiny equations on each element. Note that in the computation of (5.3), the value θ_h is determined at first by solving an equation defined by (5.3) on the two elements which contain x_0 as a nodal points. Then the rest of U_{1h} is computed using the obtained θ_h.

In Table 5.2, we summarize the results of computation. In Table 5.2, 'λ_h', 'H_0^1-error' and

Table 5.2: The estimated errors of $|\lambda - \lambda_h| + ||u - u_h||$: Example 5.1.

λ_h	H_0^1-error	H_0^1-est$^{(1)}$	H_0^1-est$^{(2)}$	$W_0^{1,\infty}$-error	$W_0^{1,\infty}$-est$^{(1)}$	$W_0^{1,\infty}$-est$^{(2)}$
0.0	3.67717D-4	3.67729D-4		1.15846D-3	1.15905D-3	
1.32361	4.24670D-4	4.24684D-4		1.33773D-3	1.33856D-3	
3.17042	5.41741D-4	5.41758D-4		1.70611D-3	1.70757D-3	
5.16503	7.71425D-4	7.71448D-4		2.42834D-3	2.43154D-3	
7.13828	1.32875D-3	1.32877D-3		4.17803D-3	4.18819D-3	
7.83691	1.80565D-3	1.80641D-3	1.78547D-3	5.64451D-3	5.64856D-3	5.62764D-3
8.85394	3.59614D-3	3.59692D-3	3.57331D-3	1.12780D-2	1.12864D-2	1.12628D-2
9.38086	7.44952D-3	7.45042D-3	7.42542D-3	2.34080D-2	2.34293D-2	2.34043D-2
9.50916	1.00926D-2	1.00936D-2	1.00683D-2	3.17268D-2	3.17597D-2	3.17343D-2
9.68198	1.93655D-2	1.93667D-2	1.93409D-2	6.08957D-2	6.09867D-2	6.09608D-2
9.75356	3.12921D-2	3.12935D-2	3.12675D-2	9.83730D-2	9.85787D-2	9.85525D-2

'$W_0^{1,\infty}$-error' are exactly same to those in Table 5.1. For λ_h less than 7.5, 'H_0^1-est$^{(1)}$' and '$W_0^{1,\infty}$-est$^{(1)}$' stand for $||\tilde{u}_h||_{H_0^1}$ and $||\tilde{u}_h||_{W_0^{1,\infty}}$, respectively, where $\tilde{u}_h \in \tilde{S}_h$ is the finite element solution of (5.2).

For λ_h greater than 7.5, 'H_0^1-est$^{(1)}$' and '$W_0^{1,\infty}$-est$^{(1)}$' are the estimated errors given by (5.5). Also, 'H_0^1-est$^{(2)}$' and '$W_0^{1,\infty}$-est$^{(2)}$' are the estimated errors given by (5.6). We see that the estimated error H_0^1-est$^{(1)}$ and $W_0^{1,\infty}$-est$^{(1)}$ match very well to the corresponding exact errors. H_0^1-est$^{(2)}$ and $W_0^{1,\infty}$-est$^{(2)}$ are slightly underestimated. We however notice that those estimated errors are very close to the exact errors $||u - u_h||_{H_0^1}$ and $||u - u_h||_{W_0^{1,\infty}}$, respectively.

Now, let us turn into a posteriori error estimates of $|\lambda - \lambda_h|$. Suppose that λ is not the continuation parameter. In that case, as stated before, either $x_0 := 7h$ or $x_0 := 8h$ $(h := 1/15)$ is taken by PITCON. Let $\psi \in \overset{\circ}{S}_h$ be a piecewise linear function such that $\psi(x_0) = 1$, and $\psi(x_i) = 0$ if $x_i \neq x_0$, where x_i are nodal points of Δ_{15}.

Then, we consider the following equation (see (4.12)): $z(x_0) = 0$ and

$$(5.7) \qquad \int_J (-z'v' + \lambda_h z v)dx = \int_J (-\psi' v' + \lambda_h \psi v)dx, \quad \forall v \in H_0^1 \text{ with } v(x_0) = 0.$$

Let $z_h \in \overset{\circ}{S}_h$ be the finite element solution of (5.7) on Δ_{15}. Of course, we can estimate the error $||z - z_h||_{H_0^1}$ by a usual method (see [BR]).

Next, let $w_h \in \overset{\circ}{S}_h$ be the finite element solution of the following equation (see (4.21)): $w_h(x_0) = 0$ and

$$\int_J (-w_h' v_h' + \lambda_h w_h v_h)dx = -\int_J u_h v_h dx, \quad \forall v_h \in \overset{\circ}{S}_h \text{ with } v_h(x_0) = 0.$$

We then compute the 'approximate jump' $J_h'(\lambda_h)$ by (see (4.20))

$$J_h'(\lambda_h) := \int_J (-w_h'\psi' + \lambda_h w_h \psi + u_h \psi) dx.$$

By Theorem 4.4, the error $|\lambda - \lambda_h|$ is estimated by

$$|\lambda - \lambda_h| \leq |J_h'(\lambda_h)|^{-1} \|z - z_h\|_{H_0^1} \|u - u_h\|_{H_0^1} (1 + o(1)). \tag{5.8}$$

On the second method of the a posteriori error estimates of $|\lambda - \lambda_h|$, we consider the following auxiliary equation (see (4.26)):

$$\int_J u_h z dx = 1, \quad \text{and} \quad \int_J (-z'v' + \lambda_h z v) dx = \eta v(x_0), \quad \forall v \in H_0^1. \tag{5.9}$$

Let $(\eta_h, z_h) \in \mathbf{R} \times \overset{\circ}{S}_h$ be the finite element solution of (5.9). To estimate the error $\|z - z_h\|_{H_0^1}$ we consider the finite element solution $(\tilde{\eta}_h, \tilde{z}_h) \in \mathbf{R} \times \tilde{S}_h$ of the following equation: $\int_J u_h \tilde{z}_h dx = 0$ and

$$\int_J (-\tilde{z}_h' \tilde{v}_h' + \lambda_h \tilde{z}_h \tilde{v}_h) dx - \tilde{\eta}_h \tilde{v}_h(x_0) = \eta_h \tilde{v}_h(x_0) + \int_J (z_h' \tilde{v}_h' - \lambda_h z_h \tilde{v}_h) dx, \quad \forall \tilde{v}_h \in \tilde{S}_h.$$

Then, we have the estimate $\|z - z_h\|_{H_0^1} \leq (|\tilde{\eta}_h| + \|\tilde{z}_h\|_{H^1})(1 + o(1))$. By Theorem 4.6, we have the error estimate

$$|\lambda - \lambda_h| \leq \|u - u_h\|_{H_0^1} \|z - z_h\|_{H_0^1} (1 + o(1)). \tag{5.10}$$

In Table 5.3 we summarize the results of computation.

Table 5.3: The estimated errors of $|\lambda - \lambda_h|$: Example 5.1.

λ_h	$\|\lambda - \lambda_h\|$	$\|\lambda - \lambda_h\|$-est$^{(1)}$	$\|\lambda - \lambda_h\|$-est$^{(2)}$
7.83691	2.02465D−5	2.09092D−5	2.09092D−5
8.85394	2.29519D−5	2.36144D−5	2.36145D−5
9.38086	2.43528D−5	2.50152D−5	2.50153D−5
9.50916	2.46938D−5	2.53562D−5	2.53563D−5
9.68198	2.51532D−5	2.58155D−5	2.58156D−5
9.75356	2.53434D−5	2.60058D−5	2.60058D−5

In Table 5.3, λ_h and $|\lambda - \lambda_h|$ are same to those in Table 5.1. The term '$|\lambda - \lambda_h|$-est$^{(1)}$' stands for the estimated error by the first method (5.8). Also, '$|\lambda - \lambda_h|$-est$^{(2)}$' stands for the estimated error by the second method (5.10).

Now, let us consider the second example:

Example 5.2.

$$(5.11) \qquad \begin{cases} u''(x) = \lambda e^{u(x)} & in\ J := (0,1), \\ u(0) = u(1) = 0. \end{cases} \qquad \square$$

The exact solution is, for $\lambda \geq 0$,

$$\lambda = 2\mu^2 \cos^{-2}\left(\tfrac{\mu}{2}\right), \quad u(x) = \ln\Big(\cos^2\left(\tfrac{\mu}{2}\right)\cos^{-2}\Big(\mu(x-\tfrac{1}{2})\Big)\Big),$$

with $0 \leq \mu < \pi$, and, for $\lambda \leq 0$,

$$\lambda = -2\mu^2 \cosh^{-2}\left(\tfrac{\mu}{2}\right), \quad u(x) = \ln\Big(\cosh^2\left(\tfrac{\mu}{2}\right)\cosh^{-2}\Big(\mu(x-\tfrac{1}{2})\Big)\Big),$$

with $0 \leq \mu < \infty$. The exact solution has a turning point around $\lambda = -3.5138307...$ ($\mu =$ 2.39935728...).

Again, we use the uniform mesh Δ_{15} and the piecewise quadratic finite element space $\overset{\circ}{S}_h$. In Table 5.4 we show a part of output of PITCON for the equation (5.11). In Table 5.4, the meaning of the each column is exactly same to that of Table 5.1. Since the exact solution manifold has the turning point, the continuation parameter was changed three times in this case.

Table 5.4: The output of PITCON: Example 5.2.

λ_h	$\lvert\lambda - \lambda_h\rvert$	H_0^1-error	$W_0^{1,\infty}$-error
8.40542	0.0	1.63866D−3	7.49987D−3
5.44008	0.0	8.22946D−4	3.57924D−3
2.49599	0.0	2.20383D−4	8.91786D−4
0.55874	0.0	1.37612D−5	5.19053D−5
-0.31518	9.05067D−10	4.98056D−6	1.79948D−5
-2.65152	1.03592D−6	6.32418D−4	1.85565D−3
-3.20484	2.62501D−6	1.27568D−3	3.55051D−3
-3.51384	6.80290D−6	2.80848D−3	7.86645D−3
-3.16924	1.36428D−5	5.66465D−3	1.69629D−2
-2.79878	1.72583D−5	7.69351D−3	2.41313D−2
-2.40918	0.0	9.92705D−3	3.22005D−2
-2.01181	0.0	1.26025D−2	4.28233D−2
-1.65256	2.38990D−5	1.56130D−2	5.56118D−2
-0.81887	2.38261D−5	2.73707D−2	1.07902D−1
-0.35494	1.92353D−5	4.41004D−2	1.82042D−1

Now, let us discuss how we estimate the errors. As before, if λ is the continuation parameter, we consider the following equation:

$$\int_J (\tilde{u}_h' \tilde{v}_h' + \lambda e^{u_h} \tilde{u}_h \tilde{v}_h) dx = -\int_J (u_h' \tilde{v}_h' + \lambda e^{u_h} \tilde{v}_h) dx, \quad \forall \tilde{v}_h \in \tilde{S}_h.$$

Then, by Theorem 3.1, 3.2, and Remark 3.8, $\|\tilde{u}_h\|_{H_0^1}$ and $\|\tilde{u}_h\|_{W_0^{1,\infty}}$ indicate $\|u - u_h\|_{H_0^1}$ and $\|u - u_h\|_{W_0^{1,\infty}}$, respectively.

If λ is not the continuation parameter $u_h(x_0)$, $x_0 := 7h$ or $x_0 := 8h$ $(h = 1/15)$ is taken by PITCON as the continuation parameter again. Let $(\theta_h, U_{1h}) \in \mathbf{R} \times \tilde{S}_h$ and $U_{2h} \in \tilde{S}_h$ be the finite element solutions of the following equations:

$$\int_J (U_{1h}'\bar{v}_h' + \lambda_h e^{u_h} U_{1h}\bar{v}_h)dx + \theta_h \int_J e^{u_h}\bar{v}_h dx = -\int_J (u_h'\bar{v}_h' + \lambda_h e^{u_h}\bar{v}_h)dx, \quad \forall \bar{v}_h \in \tilde{S}_h,$$

with $U_{1h}(x_0) = 0$ (see (3.8)), and

$$\int_J (U_{2h}'\tilde{v}_h' + \lambda_h e^{u_h} U_{2h}\tilde{v}_h)dx = -\int_J (u_h'\tilde{v}_h' + \lambda_h e^{u_h}\tilde{v}_h)dx, \quad \forall \tilde{v}_h \in \tilde{S}_h \text{ with } \tilde{v}_h(x_0) = 0,$$

with $U_{2h}(x_0) = 0$ (see (3.22)). Then, by Theorem 3.4, 3.5, 3.7, and Remark 3.8, we have the estimates (5.5) and (5.6).

Table 5.5: The estimated errors of $|\lambda - \lambda_h| + \|u - u_h\|$: Example 5.2.

λ_h	H_0^1-error	H_0^1-est$^{(1)}$	H_0^1-est$^{(2)}$	$W_0^{1,\infty}$-error	$W_0^{1,\infty}$-est$^{(1)}$	$W_0^{1,\infty}$-est$^{(2)}$
8.40542	1.63866D−3	1.63864D−3		7.49987D−3	7.51304D−3	
5.44008	8.22946D−4	8.22950D−4		3.57924D−3	3.58542D−3	
2.49599	2.20383D−4	2.20387D−4		8.91786D−4	8.93190D−4	
0.55874	1.37612D−5	1.37615D−5		5.19053D−5	5.19762D−5	
−0.31518	4.98056D−6	4.98073D−6	4.97980D−6	1.79948D−5	1.80164D−5	1.80155D−5
−2.65152	6.32418D−4	6.32790D−4	6.31403D−4	1.85565D−3	1.85655D−3	1.85520D−3
−3.20484	1.27568D−3	1.27705D−3	1.27309D−3	3.55051D−3	3.55423D−3	3.55014D−3
−3.51384	2.80848D−3	2.81408D−3	2.80173D−3	7.86645D−3	7.87953D−3	7.86662D−3
−3.16924	5.66465D−3	5.68141D−3	5.65102D−3	1.69629D−2	1.70139D−2	1.69809D−2
−2.79878	7.69351D−3	7.71847D−3	7.67615D−3	2.41313D−2	2.42075D−2	2.41704D−2
−2.40918	9.92705D−3	9.92680D−3		3.22005D−2	3.22601D−2	
−2.01181	1.26025D−2	1.26020D−2		4.28233D−2	4.29783D−2	
−1.65256	1.56130D−2	1.56603D−2	1.55881D−2	5.56118D−2	5.59372D−2	5.58390D−2
−0.81887	2.73707D−2	2.74236D−2	2.73431D−2	1.07902D−1	1.08632D−1	1.08644D−1
−0.35494	4.41004D−2	4.41342D−2	4.40703D−2	1.82042D−1	1.85414D−1	1.85044D−1

In Table 5.5, we summarize the results of computation. The meaning of the each column in Table 5.5 is same to that of Table 5.2.

To estimates of $|\lambda - \lambda_h|$, we consider the finite element solution $z_h \in \mathring{S}_h$ of the following equation (see (4.12)): $z_h(x_0) = 0$ and

$$\int_J (z_h'v_h' + \lambda_h e^{u_h} z_h v_h)dx = \int_J (\psi' v_h' + \lambda_h e^{u_h}\psi v_h)dx, \quad \forall v_h \in \mathring{S}_h \text{ with } v_h(x_0) = 0.$$

where $\psi \in \overset{\circ}{S}_h$ is defined as before.

Next, let $w_h \in \overset{\circ}{S}_h$ be the finite element solution of the following equation (see (4.21)): $w_h(x_0) = 0$ and

$$\int_J (w_h' v_h' + \lambda_h e^{u_h} w_h v_h) dx = -\int_J e^{u_h} v_h dx, \quad \forall v_h \in \overset{\circ}{S}_h \text{ with } v_h(x_0) = 0.$$

We then compute the 'approximate jump' $J_h'(\lambda_h)$ by (see (4.20))

$$J_h'(\lambda_h) := \int_J (w_h' \psi' + \lambda_h e^{u_h} w_h \psi + e^{u_h} \psi) dx.$$

By Theorem 4.4, the error $|\lambda - \lambda_h|$ is estimated by (5.8) as before.

For the second method of a posteriori error estimates of $|\lambda - \lambda_h|$, we consider the finite element solution $(\eta_h, z_h) \in \mathbf{R} \times \overset{\circ}{S}_h$ of the following equation (see (4.26)):

$$\int_J u_h z_h dx = 1, \quad \text{and} \quad \int_J (z_h' v_h' + \lambda_h e^{u_h} z_h v_h) dx = \eta_h v_h(x_0), \quad \forall v \in \overset{\circ}{S}_h.$$

Let $(\tilde{\eta}_h, \tilde{z}_h) \in \mathbf{R} \times \tilde{S}_h$ be the finite solution of the following: $\int_J e^{u_h} \tilde{z}_h dx = 0$ and

$$\int_J (\tilde{z}_h' \tilde{v}_h' + \lambda_h e^{u_h} \tilde{z}_h \tilde{v}_h) dx - \tilde{\eta}_h \tilde{v}_h(x_0) = \eta_h \tilde{v}_h(x_0) - \int_J (z_h' \tilde{v}_h' + \lambda_h e^{u_h} z_h \tilde{v}_h) dx, \quad \forall \tilde{v}_h \in \tilde{S}_h.$$

Then, we have the estimate $\|z - z_h\|_{H_0^1} \le (|\tilde{\eta}_h| + \|\tilde{z}_h\|_{H_0^1})(1 + o(1))$.

By Theorem 4.6, we obtain the estimate (5.10).

Table 5.6: The estimated errors of $|\lambda - \lambda_h|$: Example 5.2.

λ_h	$\lvert\lambda - \lambda_h\rvert$	$\lvert\lambda - \lambda_h\rvert$-est$^{(1)}$	$\lvert\lambda - \lambda_h\rvert$-est$^{(2)}$
-0.31518	9.05067D-10	9.07245D-10	9.07245D-10
-2.65152	1.03592D-6	1.03587D-6	1.03587D-6
-3.20484	2.62501D-6	2.62472D-6	2.62472D-6
-3.51384	6.80290D-6	6.80112D-6	6.80114D-6
-3.16924	1.36428D-5	1.36350D-5	1.36351D-5
-2.79878	1.72583D-5	1.72446D-5	1.72447D-5
-1.65256	2.38990D-5	2.38564D-5	2.38567D-5
-0.81887	2.38261D-5	2.37475D-5	2.37480D-5
-0.35494	1.92353D-5	1.91313D-5	1.91315D-5

In Table 5.6 we summarize the results of computation. In Table 5.6, the meaning of the each column is same to that of Table 5.3.

The last example is the following one which is strongly nonlinear (see [R,p17]):

Example 5.3.

$$(5.12)\qquad \begin{cases} \frac{d}{dx}A(u') + B(\lambda, u) = 0, & \textit{in } J := (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where the functions $A(y)$, $B(\lambda, y)$ are defined as follows:

$$A(y) := \arctan(y/2), \qquad B(\lambda, y) := a(\lambda, y)(1 + \xi(-b(\lambda, y)) + b(\lambda, y)\eta(a(\lambda, y)),$$

$$a(\lambda, y) := \lambda \sin(y) + \nu \cos(y), \qquad b(\lambda, y) := \lambda \cos(y) - \nu \sin(y),$$

$$\xi(t) := \begin{cases} t/(1-t), & \textit{for } t < 0, \\ t + 2t^2, & \textit{for } t \geq 0, \end{cases} \qquad \eta(t) := \arctan(t/2). \qquad \square$$

We set $\nu := 1.0$ in the computation.

The exact solution of (5.12) is not known. According to the output of PITCON, as is shown in [R], the solution branch of (5.12) is 'S-shaped' and has two turning points.

Again, we use the uniform mesh Δ_{15} and the piecewise quadratic finite element space.

We explain how the errors are estimated. As before, if λ is the continuation parameter, we consider the following equation:

$$\int_J (-A_y(u_h')\tilde{u}_h'\tilde{v}_h' + B_y(\lambda, u_h)\tilde{u}_h\tilde{v}_h)dx = -\int_J (-A(u_h')\tilde{v}_h' + B(\lambda, u_h)\tilde{v}_h)dx, \quad \forall \tilde{v}_h \in \tilde{S}_h.$$

By Theorem 3.1, 3.2, and Remark 3.8, $\|\tilde{u}_h\|_{H_0^1}$ and $\|\tilde{u}_h\|_{W_0^{1,\infty}}$ indicate $\|u - u_h\|_{H_0^1}$ and $\|u - u_h\|_{W_0^{1,\infty}}$, respectively.

If λ is not the continuation parameter $u_h(x_0)$, $x_0 := 7h$ ($h = 1/15$) is taken by PITCON as the continuation parameter. Let $(\theta_h, U_{1h}) \in \mathbf{R} \times \tilde{S}_h$ and $U_{2h} \in \tilde{S}_h$ be the finite element solutions of the following equations:

$$\int_J (-A_y(u_h')U_{1h}'\tilde{v}_h' + B_y(\lambda_h, u_h)U_{1h}\tilde{v}_h)dx + \theta_h \int_J B_\lambda(\lambda_h, u_h)\tilde{v}_h dx$$
$$= -\int_J (-A(u_h')\tilde{v}_h' + B(\lambda_h, u_h)\tilde{v}_h)dx, \quad \forall \tilde{v}_h \in \tilde{S}_h,$$

with $U_{1h}(x_0) = 0$ (see (3.8)), and

$$\int_J (-A_y(u_h')U_{2h}'\tilde{v}_h' + B_y(\lambda_h, u_h)U_{2h}\tilde{v}_h)dx$$
$$= -\int_J (-A(u_h')\tilde{v}_h' + B(\lambda_h, u_h)\tilde{v}_h)dx, \quad \forall \tilde{v}_h \in \tilde{S}_h \text{ with } \tilde{v}_h(x_0) = 0,$$

with $U_{2h}(x_0) = 0$ (see (3.22)). Then, by Theorem 3.4, 3.5, 3.7, and Remark 3.8, we have the estimates (5.5) and (5.6).

To estimates of $|\lambda - \lambda_h|$, we consider the finite element solution $z_h \in \overset{\circ}{S}_h$ of the following equation (see (4.12)): $z_h(x_0) = 0$ and

$$\int_J (-A_y(u_h')z_h'v_h' + B_y(\lambda_h, u_h)z_h v_h)dx = \int_J (-A_y(u_h')\psi'v_h' + B_y(\lambda_h, u_h)\psi v_h)dx,$$

for all $v_h \in \overset{\circ}{S}_h$ with $v_h(x_0) = 0$, where $\psi \in \overset{\circ}{S}_h$ is defined as before.

Next, let $w_h \in \overset{\circ}{S}_h$ be the finite element solution of the following equation (see (4.21)): $w_h(x_0) = 0$ and

$$\int_J (-A_y(u_h')w_h'v_h' + B_y(\lambda_h, u_h)w_h v_h)dx = -\int_J B_\lambda(\lambda_h, u_h)v_h dx, \quad \forall v_h \in \overset{\circ}{S}_h \text{ with } v_h(x_0) = 0.$$

We then compute the 'approximate jump' $J_h'(\lambda_h)$ by (see (4.20))

$$J_h'(\lambda_h) := \int_J (-A_y(u_h')w_h'\psi' + B_y(\lambda_h, u_h)w_h\psi + B_\lambda(\lambda_h, u_h)\psi)dx.$$

By Theorem 4.4, the error $|\lambda - \lambda_h|$ is estimated by

$$(5.13) \qquad |\lambda - \lambda_h| \leq |J_h'(\lambda_h)|^{-1}(C_1\|z - z_h\|_{H_0^1}\|u - u_h\|_{H_0^1} + C_2\|u - u_h\|_{H_0^1}^2)(1 + o(1)).$$

where $C_1 := \|A_y(u_h')\|_{L^\infty}$ and $C_2 := \frac{1}{2}\|A_{yy}(u_h')(\psi' - z_h')\|_{L^\infty}$.

For the second method of a posteriori error estimates of $|\lambda - \lambda_h|$, we consider the finite element solution $(\eta_h, z_h) \in \mathbf{R} \times \overset{\circ}{S}_h$ of the following equation (see (4.26)):

$$\int_J B_\lambda(\lambda_h, u_h)z_h dx = 1, \quad \text{and}$$
$$\int_J (-A_y(u_h')z_h'v_h' + B_y(\lambda_h, u_h)z_h v_h)dx = \eta_h v_h(x_0), \quad \forall v \in \overset{\circ}{S}_h.$$

Let $(\tilde{\eta}_h, \tilde{z}_h) \in \mathbf{R} \times \tilde{S}_h$ be the finite solution of the following: $\int_J B_\lambda(\lambda_h, u_h)\tilde{z}_h dx = 0$ and

$$\int_J (-A_y(u_h')\tilde{z}_h'\tilde{v}_h' + B_y(\lambda_h, u_h)\tilde{z}_h\tilde{v}_h)dx - \tilde{\eta}_h\tilde{v}_h(x_0)$$
$$= \eta_h\tilde{v}_h(x_0) - \int_J (-A_y(u_h')z_h'\tilde{v}_h' + B_y(\lambda_h, u_h)z_h\tilde{v}_h)dx, \quad \forall \tilde{v}_h \in \tilde{S}_h.$$

Then, we have the estimate $\|z - z_h\|_{H_0^1} \leq (|\tilde{\eta}_h| + \|\tilde{z}_h\|_{H_0^1})(1 + o(1))$.

By Theorem 4.6, we obtain the estimate

$$(5.14) \qquad |\lambda - \lambda_h| \leq (C_1\|u - u_h\|_{H_0^1}\|z - z_h\|_{H_0^1} + C_2\|u - u_h\|_{H_0^1}^2)(1 + o(1)),$$

where $C_1 := \|A_y(u_h')\|_{L^\infty}$ and $C_2 := \frac{1}{2}\|A_{yy}(u_h')z_h'\|_{L^\infty}$.

Table 5.7: The estimated errors of $|\lambda - \lambda_h| + ||u - u_h||$ and $|\lambda - \lambda_h|$: Example 5.3.

λ_h	H_0^1-est$^{(1)}$	H_0^1-est$^{(2)}$	$W_0^{1,\infty}$-est$^{(1)}$	$W_0^{1,\infty}$-est$^{(2)}$	$\lvert\lambda - \lambda_h\rvert$-est$^{(1)}$	$\lvert\lambda - \lambda_h\rvert$-est$^{(2)}$
0.19521	1.14416D–4		4.57789D–4			
0.57696	1.09716D–4		3.80008D–4			
0.97248	1.33223D–4		4.14384D–4			
1.14096	1.56647D–4	1.56434D–4	4.84460D–4	4.84272D–4	2.13860D–7	2.13860D–7
1.79178	4.69535D–4	4.68635D–4	1.60921D–3	1.60849D–3	9.00308D–7	9.00307D–7
2.46764	2.51702D–3	2.50907D–3	1.05407D–2	1.05376D–2	7.97236D–6	7.97234D–6
2.99606	1.24576D–2	1.23968D–2	5.72413D–2	5.72317D–2	6.19293D–5	6.19290D–5
3.24514	4.20951D–2	4.18066D–2	2.55047D–1	2.55964D–1	3.17004D–4	3.17003D–4
3.08956	5.04095D–2		3.20935D–1			
1.95627	1.58211D–2		7.29373D–2			
1.59592	1.19638D–2	1.19317D–2	5.39609D–2	5.41812D–2	3.37883D–5	3.37878D–5
1.19898	1.17190D–2	1.16916D–2	5.79405D–2	5.69798D–2	2.60112D–5	2.60120D–5
0.99403	3.20333D–2	3.20370D–2	1.91055D–1	1.91055D–1	4.70059D–5	4.70191D–5
1.05847	1.06782D–1	1.06848D–1	7.26805D–1	7.27113D–1	1.97460D–4	1.97328D–4
1.29395	7.73202D–1	7.89621D–1	6.10406D0	6.43073D0	1.12810D–3	1.11374D–3

In Table 5.7, we summarize the results of computation. According to PITCON, turning points are at $\lambda = 3.24513871$ and $\lambda = 0.99403398$. The meaning of the each column in Table 5.7 is same as before.

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The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

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