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MICROMECHANICS OF FATIGUE

GRANT N° AFOSR-89-0329

Final scientific report

June 1992

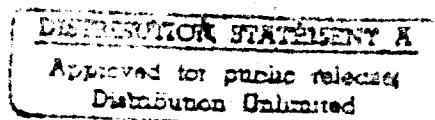
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Jean LEMAITRE, René BILLARDON and Marie-Pierre VIDONNE

Rapport Interne n° 134

LABORATOIRE DE MECANIQUE ET TECHNOLOGIE

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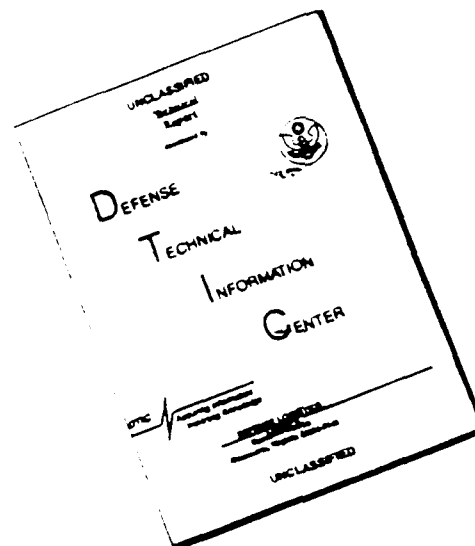
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ABSTRACT

In section 1., brittle and ductile isotropic damage mechanisms are studied from a meso-mechanical viewpoint. Relationships between crack density and void volume fraction defined at meso-scale on one hand, and a scalar internal variable characterizing damage on the other hand, are given.

In section 2., a general form for the evolution law for this damage variable is derived. A threshold which defines the onset of this evolution is derived from thermodynamical considerations.

In section 3., it is proposed to relate the ultimate stage of continuum damage evolution, i.e. the local failure, to localization phenomena. The corresponding criteria are studied in details.

In Section 5, a post processor is fully described which allows the calculation of the crack initiation conditions from the history of strain components taken as the output of a finite element calculation. It is based upon damage mechanics using coupled strain damage constitutive equations for linear isotropic elasticity, perfect plasticity and a unified kinetic law of damage evolution. The localization of damage allows this coupling to be considered only for the damaging point for which the input strain history is taken from a classical structure calculation in elasticity or elastoplasticity. The listing of the code, a "friendly" code, with less than 600 FORTRAN instructions is given and some examples show its ability to model ductile failure in one or multi dimension, brittle failure, low and high cycle fatigue with the non-linear accumulation, and, multiaxial fatigue.

In section 6, application is performed on the Alpha-Two-Titanium Aluminide Alloy. With a first serie of strain controlled fatigue test the locally coupled model is first identified and then checked on a second serie of stress controlled fatigue tests. The good agreement allows to use this locally coupled method to predict fatigue crack initiation on brittle or quasibrittle materials.

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1. MICROMECHANICS OF DAMAGE

Micromechanics consists in deriving the behaviour of materials at meso-scale from the study of specific mechanisms at micro-scale. The micro-mechanisms must be precisely defined from physical observations for both the geometries and the kinematics. Their mechanical modelling is performed with elementary usual constitutive equations established at meso- or macro-scale for strain, crack growth and fracture. When compared to the direct analysis of the macroscopic properties, the additional power of this approach comes from a better modelling of the possible interactions between different mechanisms and from the homogenization that bridges the gap between micro- and meso-scales.

When using classical continuum thermodynamics concepts, the effects at meso-scale of material degradation at micro-scale are characterized by an internal variable called damage. Hereafter, the effects of the material degradation are assumed to be isotropic at meso-scale and the damage variable to be a scalar denoted by D . In this section, definitions for this scalar variable are derived from the study of two different micro-mechanisms.

1.1. Brittle isotropic damage

1.1.1. Microcracks and scalar damage variable

The main mechanism of brittle damage is the nucleation, growth and coalescence of microcracks up to the initiation at meso-scale of a crack. Hereafter, a relationship between the micro-crack pattern and the damage variable D is established.

Let us consider a Representative Volume Element at meso-scale as a cube of dimension $(l \times l \times l)$. This RVE is assumed to be constituted at microscale of cubic cells of dimension $(d \times d \times d)$ in which may lie a microcrack of any area s_i and any orientation (see Fig. 1). The number of cells is $m = l^3/d^3$ and the number of cracks $n \leq m$.

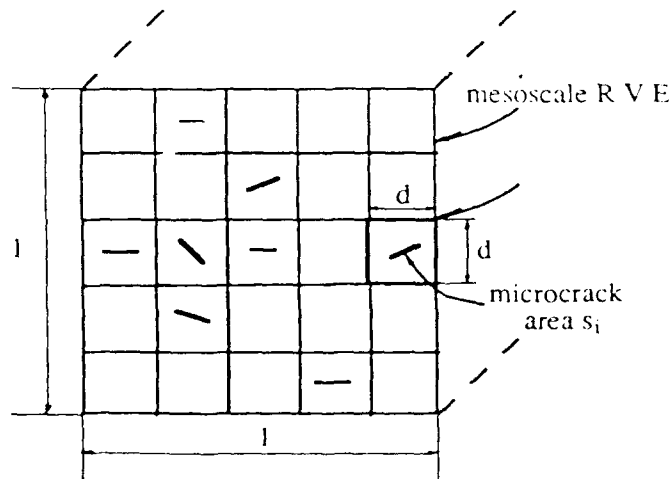


Fig. 1. Micro- and meso-models for brittle damage

The geometry being defined, the modelling consists in writing the balance of the dissipated energy calculated by classical fracture mechanics one one hand, and calculated by continuum damage mechanics on the other hand.

For a cracked cell i , subjected to a given state of stress, if G_i denotes the strain energy release rate corresponding to a crack of area s_i , D_i the equivalent damage of the cell and Y_i the strain energy density release rate, the balance of dissipated energy can be written as :

$$G_i \dot{s}_i = Y_i \dot{D}_i d^3$$

For the n cracked cells of the meso-cube, the previous relation becomes :

$$\sum_{i=1}^n G_i \dot{s}_i = \sum_{i=1}^n Y_i \dot{D}_i d^3$$

Assuming that brittle growth of microcracks occurs at $G = G_c = \text{constant}$, corresponding to $Y = Y_c = \text{constant}$, it can be written that :

$$G_c \sum_{i=1}^n \dot{s}_i = Y_c d^3 \sum_{i=1}^n \dot{D}_i$$

Furthermore, if it is assumed that $s_i = 0$ corresponds to $D_i = 0$, integration of the previous relation yields :

$$G_c \sum_{i=1}^n s_i = Y_c d^3 \sum_{i=1}^n D_i$$

The simplest homogenization consists in defining the damage D at macro-scale by the mean value of the damages D_i of all the $1^3/d^3$ micro-cells, so that

$$D = \frac{1}{n} \sum_{i=1}^n D_i = \frac{1}{n} \sum_{i=1}^n D_i = \frac{d^3}{1^3} \sum_{i=1}^n D_i$$

or
$$D = \frac{G_c}{Y_c 1^3} \sum_{i=1}^n s_i$$

The term $\frac{G_c}{Y_c 1^3}$ may be evaluated from a rupture criterion defining the initiation of a meso crack. Because of the localization of the damage phenomenon, it can be assumed that the meso crack initiation occurs when only part of the flat volume ($1 \times 1 \times d$) is micro-cracked, the other microcracks being neglected.

In other words, if it is assumed that the mesocrack initiates, i.e. $D = D_c$ at meso-scale, when

$$\sum_{i=1}^n s_i = k 1^2$$

then,

$$D_c = \frac{G_c}{Y_c 1^3} k 1^2 \quad \frac{G_c}{Y_c 1^3} = \frac{D_c}{k 1^2}$$

Hence,

$$D = \frac{\sum_{i=1}^n s_i}{l^2} \frac{D_c}{k}$$

In this case, the damage variable D appears as the micro-cracks surface density ($\sum_{i=1}^n s_i / l^2$) corrected by a factor (here D_c/k).

If the following simplest fracture criterion is considered

$$\sum_{i=1}^n s_i = l^2 \rightarrow k = 1 \rightarrow D_c = 1 \quad \text{then, } D = \frac{\sum_{i=1}^n s_i}{l^2}$$

By the way, this calculation gives an order of magnitude of a characteristic length which allows for the matching between fracture mechanics and damage mechanics namely l , the size of the Representation Volume Element. Since,

$$\frac{D}{\sum_{i=1}^n s_i} = \frac{G_c}{Y_c l^3} = \frac{D_c}{k l^2}$$

$$l = k D_c \frac{G_c}{Y_c}$$

Hence, for the simple fracture criterion $k = 1$, $D_c = 1$,

$$l \equiv \frac{G_c}{Y_c}$$

For most metallic materials

light alloys

$$0.005 \leq G_c \leq$$

$$2 \leq Y_c \leq$$

$$0.0025 \leq l \leq$$

Steel and high alloys

$$0.05 \text{ MPa m}$$

$$10 \text{ MPa}$$

$$0.005 \text{ m}$$

and for concrete in tension $G_c \equiv 3 \cdot 10^{-5} \text{ MPa.m}$, $Y_c \equiv 1.5 \cdot 10^{-4} \text{ MPa}$, so that $l \equiv 2 \cdot 10^{-1}$.

This shows that the size of the physical RVE must be of the order of the millimeter for metals and of the order of the decimeter for concrete.

1.1.2. Brittle damage growth

As a specific example, let us derive the kinetic damage evolution law at meso-scale which corresponds to the fatigue microcracks growth at micro-scale of Fig.1. For simplicity sake, the analysis is restricted to a two dimensional problem, for which e denotes the uniform thickness of the whole RVE.

With $D_c = k$, it has been established that :

$$D = \frac{\sum_{i=1}^n s_i}{l^2} \quad \text{and} \quad \dot{D} = \frac{\sum_{i=1}^n \dot{s}_i}{l^2}$$

and, for each cracked cell, if $2a$ denotes the crack length, the following relations hold :

$$K_I = (EG_I)^{1/2} \quad \dot{K}_I = \frac{E^{1/2}}{2} G_I^{-1/2} \dot{G}_I \quad \dot{s}_i = 2e \dot{a}_i$$

The surface growth rate $\dot{s}_i$ of each crack can be expressed as a function of the strain energy release rate G_I of the corresponding cell, by means of the Paris' law of fatigue crack growth. If N denotes the number of cycles of loading in mode I, and $K_{M1} = K_{M1\max}$ the amplitude of the stress intensity factor (with $K_{\min} = 0$), then

$$\frac{\delta a}{\delta N} = C K_{M1}^\eta$$

where C and η are two material constants, with $\eta \cong 4$ for many metallic materials.

If one assumes that this Paris' law corresponds to the integration over one cycle of :

$$\dot{a} = \eta C K^{\eta-1} \dot{K} ,$$

then

$$\dot{s}_i = C e E^{\eta-2} G_I^{\eta-1} \dot{G}_I$$

A relationship between G_I and Y_I can be found through their definition from the elastic energy. If w denotes the elastic strain energy density and W_I the elastic strain energy of the elementary cell, then :

$$G_I = - \frac{\partial W_I}{\partial s_i} , \quad \text{whereas} \quad Y_I = - \frac{\partial w_i}{\partial D_i}$$

Since

$$W_i = w_i d^2 e,$$

$$G_i = - \frac{\partial(w_i d^2 e)}{\partial D} \frac{dD}{ds}$$

If

$$D_i \equiv \frac{s_i}{d_i^2} = \frac{2 a_i e}{d_i e}, \quad G_i = Y_i d_i, \quad \dot{G}_i = \dot{Y}_i d_i$$

then

$$\dot{s}_i = \eta C e E^{\eta/2} d^{\eta/2} Y_i^{\frac{\eta-1}{2}} \dot{Y}_i$$

Hence, the damage rate is :

$$\dot{D}_i = \frac{\sum \dot{s}_i}{l^2} = \frac{\eta C E^{\eta/2} d^{\eta/2}}{l^2} \sum_1^n e Y_i^{\frac{\eta-1}{2}} \dot{Y}_i$$

Assuming that all the n cracked cells have the same strain energy density release rate

$Y_i = Y_n$, the homogenized strain energy density release rate for the meso-RVE is :

$$Y = n Y_n \quad \text{and} \quad \dot{Y} = n \dot{Y}_n$$

Consequently

$$\sum_1^n e_i Y_i^{\frac{\eta-1}{2}} \dot{Y}_i = n e Y_n^{\frac{\eta-1}{2}} \dot{Y}_n = e n^{1-\frac{\eta}{2}} Y_i^{\frac{\eta-1}{2}} \dot{Y}$$

$$\dot{D} = \frac{\eta C E^{\eta/2} d^{\eta/2} e}{l^2 n^{\frac{\eta-1}{2}}} Y_i^{\frac{\eta-1}{2}} \dot{Y}$$

In this example, the damage rate is an increasing function of the strain energy density release rate ; for most materials since $\eta \cong 4$, the damage rate is quasi-proportional to the strain energy density release rate. The damage rate is also proportional to the rate $\dot{Y}$. This will be used in section 2.1 as a guideline to derive a general kinetic law for damage evolution.

1.2. Ductile isotropic damage

1.2.1. Microcavities and scalar damage variable

The main mechanism of ductile damage is the nucleation, the growth and the coalescence of microcavities by large local plastic deformations. Hereafter, a relationship between the density of micro voids and the damage variable D is established.

Let us consider again a Representative Volume Element at meso-scale as a cube of dimension $(l * l * l)$. This RVE is assumed to be constituted at microscale of cubic cells of dimension $(d * d * d)$ in which may lie a void of volume d^3 (Fig.2.).

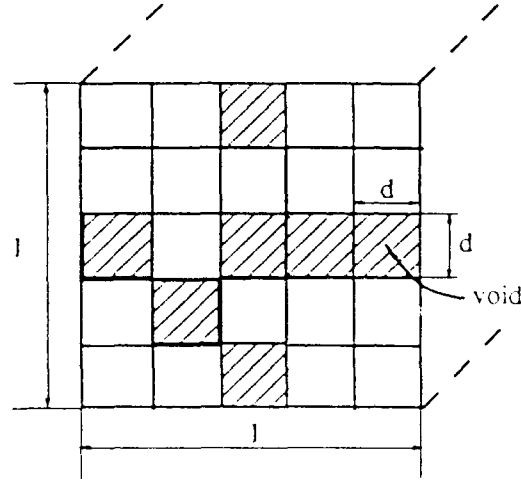


Fig. 2. Micro-meso element for ductile damage

On this very simple geometry, the modelling consists in writing the balance of the dissipated energy calculated from the growth of the cavities on one hand, and calculated by continuum damage mechanics on the other hand.

For the geometrical model under consideration, the porosity P can be defined as :

$$P = 1 - \frac{\rho}{\rho_0} = \frac{n d^3}{l^3}$$

where ρ and ρ_0 are the current and initial porosity respectively, and n the number of cavities.

According to Gurson's model, the porosity P at meso-scale is equal to the hydrostatic part $\epsilon_H^p = \epsilon_{kk}^p$ of the plastic strain due to the growth of voids.

Here, this assumption leads to the following equality written at meso-scale :

$$\dot{P} = \dot{\epsilon}_H^p$$

At meso-scale, given an homogenized stress σ_{ij} and a plastic strain rate $\dot{\epsilon}_{ij}^p$, the total power dissipated is

$$\mathcal{P} = \sigma_{ij} \dot{\epsilon}_{ij}^p$$

This can be split in two parts by means of the deviatoric and hydrostatic quantities i.e.

$$\begin{aligned} \mathcal{P} &= (\sigma_{ij}^D + \sigma_H \delta_{ij}) (\dot{\epsilon}_{ij}^{pD} + \dot{\epsilon}_H^p \delta_{ij}) \\ &= \sigma_{ij}^D \dot{\epsilon}_{ij}^{pD} + 3 \sigma_H \dot{\epsilon}_H^p + 0 \end{aligned}$$

The first term is the power dissipated in pure plasticity by slips. The second term which corresponds to the irreversible change of volume may be interpreted as the power dissipated in the RVE to increase the material discontinuities by growth of the cavities. This latter part must balance the damage dissipation i.e. :

$$3 \sigma_H \dot{\epsilon}_H^p l^3 = Y \dot{D} l^3$$

so that

$$\dot{D} = \frac{3 \sigma_H}{Y} \dot{P}$$

Assuming for simplicity, proportional loading, perfect plasticity, $\frac{3 \sigma_H}{Y} = \text{const}$, and the initial condition $P = 0 \rightarrow D = 0$, the integration yields :

$$D = \frac{3 \sigma_H}{Y} n \frac{d^3}{l^3}$$

As for brittle damage, because of the localization of the damage phenomenon, it can be assumed that the meso-crack initiation occurs when a set of cavities occupies only part of the flat volume ($l * l * d$), the other cavities in the RVE being neglected.

In other words, the critical value of the porosity corresponding to $D = 1$ is assumed as

$$P_c = n \frac{d^3}{l^3} = k \frac{l^2 d}{l^3} = k \frac{d}{l}$$

This allows for the calculation of the term $\frac{3 \sigma_H}{Y}$ in the damage equation :

$$D = 1 = \frac{3 \sigma_H}{Y} k \frac{d}{l}$$

Hence,

$$\boxed{D = \frac{n d^2}{k l^2}}$$

1.2.2. Ductile damage growth

As a specific example, let us derive the kinetic damage evolution law at meso-scale which corresponds to the voids growth at micro-scale. For simplicity sake, the analysis is restricted to the particular case $k = 1$.

The kinetic law for damage evolution can be directly derived from the expression for D established in the previous section, so that

$$\dot{D} = \frac{d^2}{l^2} \dot{n} + 2n \frac{d}{l^2} \dot{d}$$

The first term accounts for the increase of the number of cavities and $\dot{n}$ denotes the number of cavities nucleated per second. The second term accounts for the cavity growth. In the Gurson model, the porosity rate is also the sum of two terms accounting for nucleation and growth.

a) Damage growth by nucleation of cavities

To model nucleation, Tvergaard proposed the following kinetic law for porosity

$$\dot{P} = A \dot{\sigma}_{eq} + B \dot{\sigma}_H$$

where A and B are material parameters.

Assuming for simplicity sake a sudden nucleation of cavities of a fixed size d

$$\dot{P} = \dot{n} \frac{d^3}{l^3}$$

and

$$\dot{D} = \dot{P} \frac{1}{d} = \frac{1}{d} (A \dot{\sigma}_{eq} + B \dot{\sigma}_H)$$

so that,

$$\dot{D} = \frac{1}{d} \dot{\sigma}_{eq} \left(A + B \frac{\dot{\sigma}_H}{\dot{\sigma}_{eq}} \right)$$

Damage can be expressed as a function of the accumulated plastic strain rate $\dot{p}$ written in terms of the plastic tangent modulus E_T . Assuming proportional loading, i.e.

$$\frac{\dot{\sigma}_H}{\dot{\sigma}_{eq}} = \frac{\sigma_H}{\sigma_{eq}}$$

$$\dot{p} = \frac{2}{3} \dot{\epsilon}_{ij}^p \dot{\epsilon}_{ij}^p = \frac{\dot{\sigma}_{eq}}{E_T}$$

so that,

$$\dot{D} = \frac{1}{d} E_T \left(A + B \frac{\sigma_H}{\sigma_{eq}} \right) \dot{p}$$

b) *Damage growth by enlargement of a fixed number n of cavities.*

The problem of void growth has received much attention in the past 20 years. Essential results are the McClintock and Rice & Tracey analyses which derive the rate of growth of a cylindrical or spherical cavity of volume V in a perfectly plastic infinite body as a function of the accumulated plastic strain rate $\dot{p}$ and triaxiality ratio σ_H/σ_{eq} , i.e.

$$\dot{V} = 0.85 V \dot{p} \exp\left(\frac{3}{2} \frac{\sigma_H}{\sigma_{eq}}\right)$$

Taking

$$V = d^3$$

leads to

$$3 d^2 \dot{d} = 0.85 d^3 \dot{p} \exp\left(\frac{3}{2} \frac{\sigma_H}{\sigma_{eq}}\right)$$

Since

$$\dot{D} = 2 n \frac{d \dot{d}}{l^2}, \quad D = n \frac{d^2}{l^2}, \quad \dot{D} = 2 D \frac{\dot{d}}{d}$$

$$\boxed{\dot{D} = 0.57 D \dot{p} \exp\left(\frac{3}{2} \frac{\sigma_H}{\sigma_{eq}}\right)}$$

c) *Conclusions*

In this example, the damage rate by nucleation and growth of voids appears as

- proportional to the accumulated plastic strain rate,
- an increasing function of the triaxiality ratio $\frac{\sigma_H}{\sigma_{eq}}$,
- through E_T or D , a function of the current state of the material.

This will be used in section 2.1 as a guideline to derive a general kinetic law for damage evolution.

2. GENERAL PROPERTIES AND FORMULATION

2.1. A general form for continuum damage evolution law

a) General formalism

Within the framework of classical continuum thermodynamics, the thermo-mechanical state of a material is described by the following set of independent state variables :

$$\mathbb{V} \equiv (\varepsilon, T, \mathbf{V}) \quad \text{with} \quad \mathbf{V} \equiv (D, \varepsilon^p, \mathbf{V}^p),$$

where ε , ε^p denote the total and plastic strain tensors respectively, T the temperature, $\mathbf{V}$ the set of the internal variables, and D the damage variable. For simplicity sake, hereafter only isothermal situations will be considered.

The reversible behaviour is described by the Helmholtz specific free energy

$$\Psi = \Psi(\varepsilon, \mathbf{V}),$$

chosen as

$$\Psi = \Psi^e(\varepsilon - \varepsilon^p, D) + \Psi^p(\varepsilon^p, \mathbf{V}^p)$$

with

$$\Psi^e = \frac{1}{2} (\varepsilon - \varepsilon^p) : (1 - D) \Xi_0 : (\varepsilon - \varepsilon^p)$$

where Ξ_0 denotes the elasticity matrix of the undamaged material. The thermodynamic forces

$$\mathbf{A} \equiv (\boldsymbol{\sigma}, \mathbf{A})$$

are defined from the following state laws :

$$\boldsymbol{\sigma} = \rho \frac{\partial \Psi}{\partial \varepsilon}, \quad \mathbf{A} \equiv (-Y, -\boldsymbol{\sigma}, \mathbf{A}^p) = \rho \frac{\partial \Psi}{\partial \mathbf{V}},$$

where ρ denotes the mass density, $\boldsymbol{\sigma}$ the stress tensor, and Y the strain energy density release rate or damage energy release rate.

The irreversible behaviour is described by a dissipation potential

$$\Phi = \Phi(\boldsymbol{\sigma}, \mathbf{A}; \mathbb{V}),$$

from which the following evolution laws are derived :

$$\dot{\varepsilon}^p = - \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}, \quad \dot{\mathbf{V}} = - \frac{\partial \Phi}{\partial \mathbf{A}}$$

In particular, the damage evolution law can be chosen such that

$$\dot{D} = \lambda \frac{\partial F_D}{\partial Y}$$

b) damage vs (micro-)plasticity

In section 1., it has been established that damage is always related to some irreversible strain either at micro- or meso-level. This property can be taken into account in the evolution law for the damage variable by assuming that the factor $\dot{\lambda}$ is proportional to the accumulated plastic strain so that

$$\dot{D} = \frac{\partial F_D}{\partial Y} \dot{p}$$

The irreversible nature of damage is directly taken into account by the fact that the variable $\dot{p}$ is always positive or null.

In most materials, a certain amount of plasticity must be accumulated before damage at meso-level appears. In metals, this corresponds to the accumulation of micro-stresses in the vicinity of initial defects, of dislocations, ..., prior to the nucleation of micro-cracks or micro-voids. To model this phenomenon, since the damage evolution is governed by the accumulated plastic strain rate, it is natural to introduce a threshold p_D on the variable p , such that

$$\dot{D} = \frac{\partial F_D}{\partial Y} \dot{p} \quad \text{if } p \geq p_D$$

$$\dot{D} = 0 \quad \text{if } p < p_D$$

or

$$\dot{D} = \frac{\partial F_D}{\partial Y} \dot{p} H(p - p_D)$$

where $H()$ is the Heavyside step function.

In monotonic loading, p_D can be identified as the uniaxial damage threshold ϵ_D^p , whereas for fatigue or creep processes p_D is a function of the applied stress, as it will be discussed in section 2.2.

c) driving force for damage

From the thermodynamical analysis, it has been deduced that the driving force for damage is the strain energy density release rate Y . Hence, F_D must be a function of Y :

$$F_D = F_D(Y, \dots)$$

d) influence of the triaxiality ratio

Another important feature of fracture mechanisms is the influence of the triaxiality ratio (σ_H/σ_{eq}), where it is recalled that σ_H denotes the hydrostatic stress and σ_{eq} the Von Mises' equivalent stress. This effect is directly taken into account through the damage energy release rate Y which is a function of the triaxiality factor R_V :

$$Y = \frac{\sigma_{eq}^2 R_V}{2 E (1 - D)^2}$$

with

$$R_V = \frac{2}{3} (1 + \nu) + 3 (1 - 2\nu) \left(\frac{\sigma_H}{\sigma_{eq}} \right)^2$$

e) a general form

In order to choose the proper and simplest expression for F_D , let us recall the kinetic damage laws obtained by micromechanics for particular mechanisms in section 1. :

Brittle damage by fatigue growth of micro-cracks :

$$\dot{D} = \frac{\eta C E^{\eta/2} d^{\eta/2} e^{\frac{\eta-1}{2}}}{l^2 n^{\frac{\eta-1}{2}}} Y^{\frac{\eta-1}{2}} \dot{Y}$$

η being of the order of 4

$$\dot{D} = (\text{const}) * Y \dot{Y}$$

Although no plasticity has been introduced in the analysis, it always exists at micro-scale at the crack tips of the micro-cracks and it is possible, at least formally to relate $\dot{Y}$ to $\dot{p}$ through a plasticity constitutive equation :

$$Y = Y(\sigma_{eq}) \rightarrow \dot{Y}(\dot{\sigma}_{eq})$$

and

$$\sigma_{eq}(p) \rightarrow \dot{\sigma}_{eq}(\dot{p})$$

Ductile damage by nucleation of micro-cavities :

$$\dot{D} = \frac{1}{d} E_T \left(A + B \frac{\sigma_H}{\sigma_{eq}} \right) \dot{p}$$

Ductile damage by enlargement of micro-cavities :

$$\dot{D} = 0.57 D \dot{p} \exp\left(\frac{3}{2} \frac{\sigma_H}{\sigma_{eq}}\right)$$

The qualitative conclusion which can be drawn from these three results is that the damage rate $\dot{D}$ can be considered as proportional to Y , which is a function of (σ_H/σ_{eq}) , and $\dot{p}$:

$$\dot{D} \sim Y \dot{p}$$

or

$$F_D \sim Y^2$$

As in any realistic constitutive equation, a material dependent scale factor, such as (const), $\frac{1}{d} E_T$, or 0.57, must be introduced. Let us denote by S this material constant so

that

$$F_D \sim \frac{Y^2}{S}$$

Finally, according to the qualitative properties listed above, the damage potential is naturally written as :

$$F_D(Y; (p, D)) = \frac{Y^2}{2S} H(p - p_D)$$

where the factor 2 has been introduced to compensate for the factor (1/2) coming from the derivation. Hence, the proposed general continuum damage evolution law is the following :

$$\boxed{\dot{D} = \frac{Y}{S} \dot{p} H(p - p_D)}$$

where two material dependent parameters are introduced, viz. S and p_D which characterize the energetic resistance against the damage process and the damage threshold, respectively. The effects of the temperature T are taken into account through the variation of these coefficients with T and through the accumulated plastic strain rate $\dot{p}$ which is also a function of T .

Several important properties, though not directly introduced in the formulation, are also naturally exhibited by this general evolution law, i.e. :

- the non linear accumulation of damage,
- the effect of mean stress in fatigue,
- the non linear interaction of different kinds of damage.

2.2. Damage threshold

Under monotonic loading, the damage threshold p_D can be identified with the uniaxial damage threshold ϵ_D^p , whereas in the case of fatigue or creep loadings it is a function of the applied stress. It corresponds to the critical level of plasticity which induces the nucleation of microcracks without any consequence on the mechanical properties and can be related to the energy stored in the material.

Experiments in fatigue have shown that the total plastic strain energy dissipated may reach tremendous values before failure although the stored energy remains constant at microcrack initiation. This stored energy is the result of microstress concentrations which develop in the neighbourhood of dislocation networks in metals and of inhomogeneities in other materials. For a unit volume, it is equal to the difference between the total plastic strain energy ($\int_0^t \sigma_{ij} \dot{\epsilon}_{ij}^p dt$) and the energy dissipated in heat as given by the Clausius-Duhem inequality of the second principle of thermodynamics.

For instance, in the case of a material exhibiting kinematic X and isotropic R strain hardenings and no damage, under an isothermal transformation the rate of energy dissipated in heat is :

$$\dot{\Theta} = \sigma_{ij} \dot{\epsilon}_{ij}^p - R \dot{p} - X_{ij} \dot{\alpha}_{ij} \geq 0$$

This expression may be calculated from :

- the potential of dissipation e.g.

$$F = (\sigma^D - X)_{eq} - R - \sigma_y - \frac{3}{4 X_\infty} X_{ij} X_{ij}$$

- its associated normality flow rule

$$\dot{\epsilon}_{ij}^p = \frac{\partial F}{\partial \sigma_{ij}} \dot{\lambda}, \quad \dot{p} = - \frac{\partial F}{\partial R} \dot{\lambda} = \dot{\lambda}, \quad \dot{\alpha}_{ij} = - \frac{\partial F}{\partial X_{ij}} \dot{\lambda}$$

- and the yield criterion

$$f = (\sigma^D - X)_{eq} - R - \sigma_y = 0$$

so that

$$\dot{\Theta} = \left(\sigma_y + \frac{3}{2 X_\infty} X_{ij} X_{ij} \right) \dot{p}$$

Hence, the stored energy W_s as a function of time t is :

$$W_s(t) = \int_0^t \sigma_{ij} \dot{\epsilon}_{ij}^p dt - \int_0^t \left(\sigma_y + \frac{3}{2 X_\infty} X_{ij} X_{ij} \right) \dot{p} dt$$

This formula can be simplified if the following assumptions are made :

- the effect of the kinematic hardening is neglected :

$$W_s = \int_0^p (\sigma_{eq} - \sigma_y) dp$$

- the variation of σ_{eq} is neglected as for a quasi perfectly plastic material

$$W_s = [\text{Sup}(\sigma_{eq}) - \sigma_y] p$$

If this stored energy is considered as a constant for the damage threshold p_D , its value can be identified in the one-dimensional monotonic case used as a reference with $p_D = \epsilon_{pD}$.

In the particular case considered above, and with the crude approximations made, the onset for the damage process, or damage threshold, corresponds in the case of a monotonic loading to the ultimate stress σ_u :

$$[\text{Sup}(\sigma_{eq}) - \sigma_y] p_D = (\sigma_u - \sigma_y) \epsilon_{pD}$$

so that

$$p_D = \epsilon_{pD} \frac{\sigma_u - \sigma_y}{\text{Sup}(\sigma_{eq}) - \sigma_y}$$

As a summary, in the particular case considered in this section, the whole set of equations that governs the damage evolution is :

$$\dot{D} = \frac{Y}{S} \dot{p} H(p - \epsilon_{pD} \frac{\sigma_u - \sigma_y}{\text{Sup}(\sigma_{eq}) - \sigma_y})$$

3. LOCAL FAILURE CRITERIA

The ultimate stage of continuum damage evolution corresponds to meso-crack initiation. As a first approximation, this critical damage state can be characterized by a critical value D_c of the damage variable, for instance $D_c = 1$.

In fact, the value of D_c is not solely material dependent. As discussed below, it also depends on the local stress state.

It is proposed to relate the conditions for the meso-crack initiation to the conditions for the localization of the deformation in the material. In this section, these latter conditions are studied in details, inside, at the boundaries or at the interfaces of rate-independent solids for both the linear and the non-linear case. Physical interpretations of these conditions are also given.

3.1. Introduction

Since the pioneering works of Hadamard, Hill, Mandel and Rice, the localization of the deformation in rate-independent materials is treated as the bifurcation of the rate problem.

Here, we give a general view of the various bifurcation and localization phenomena for possibly heterogeneous solids made of rate-independent materials. Under the small strain assumption, the behaviour of these materials is described by the following piece-wise linear rate constitutive laws :

$$(1) \quad \begin{aligned} \dot{\Sigma} &= \dot{\mathbb{L}} : \dot{\Sigma}(\mathbf{v}) \quad \text{with} \\ \dot{\mathbb{L}} &= \dot{\mathbb{E}} \quad \text{when} \quad f < 0, \quad \text{or} \quad f = 0 \text{ and } \dot{\Sigma} : \dot{\Sigma}(\mathbf{v}) < 0 \\ \dot{\mathbb{L}} &= \dot{\mathbb{H}} \quad \text{when} \quad f = 0 \text{ and } \dot{\Sigma} : \dot{\Sigma}(\mathbf{v}) \geq 0 \end{aligned}$$

where $\dot{\Sigma}$ and $\dot{\Sigma}(\mathbf{v}) = \dot{\Sigma}$ respectively denote the stress and strain rates, $\mathbf{v}$ the velocity, and f the yield function.

The general class of materials modelled by relations (1) includes for instance the elasto-plastic damageable solids the behaviour of which is described by the constitutive equations discussed in section 2.1.a.

It is shown that, in general, different types of localization phenomena may occur, depending on the failure of one of the three conditions which are described in section 3.2. Their physical interpretation is the following :

- the ellipticity condition is very classical. Its failure is the condition for localization given by Rice and linked to the appearance of deformation modes involving discontinuities of the velocity gradient. It has also been related to stationary acceleration waves ;

- the boundary complementing condition governs instabilities at the boundary of the solid. Its failure leads to deformation mode localized at the boundary and is related to stationary surface waves (for instance Rayleigh waves) ;

- the interfacial complementing condition governs instabilities at interfaces. Its failure leads to deformation modes localized at each side of the interface and is related to stationary interfacial waves (Stonely waves).

3.2. Rate problem analysis : the linear case

Let us consider for instance the body sketched in Fig.3.

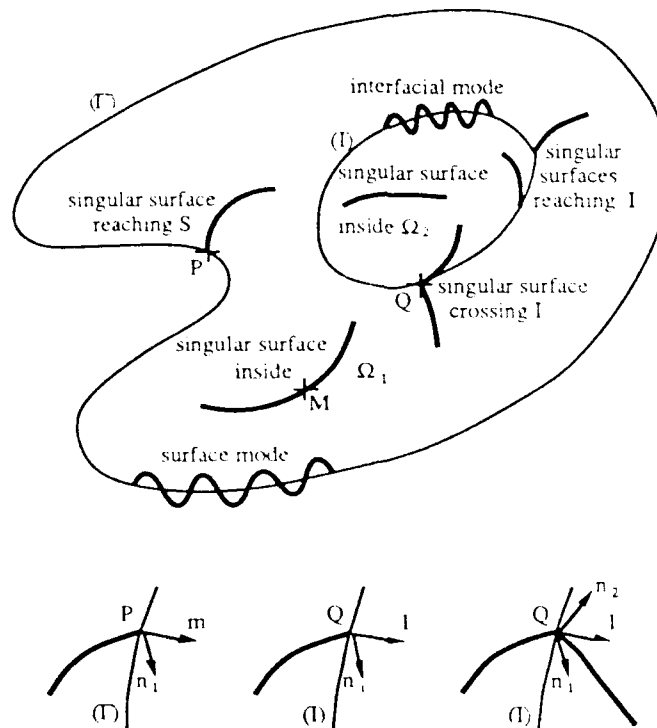


Fig. 3. Different types of localization modes

Qualitative results can be exhibited from the analysis of the rate problem for the so-called linear comparison solid (see Hill). In this case, this linear problem is well-posed if and only if the following conditions are met :

- *the ellipticity condition* : the rate equilibrium equations must be elliptic in the closure of the body Ω , i.e.

$$\det(\mathbf{n} \cdot \mathbb{E} \cdot \mathbf{n}) \neq 0 \quad \text{for any vector } \mathbf{n} \neq \mathbf{0}, \text{ and any point } M \in \Omega.$$

- *the boundary complementing condition* : this relation between the coefficients of the field and boundary operators must be satisfied at every point P belonging to the boundary Γ where the boundary conditions are formally written as $\mathbb{E}(\mathbf{v}) = \mathbf{g}$. This condition is easily phrased in terms of an associated problem on a half space defined by $z > 0$. It requires for every vector $\mathbf{k} = (k_1, k_2, 0) \neq \mathbf{0}$, that the only solution to the rate equilibrium equations with constant coefficients (equal to those of the operator at point P), in the form

$$\mathbf{v}(x, y, z) = \mathbf{w}(z) \exp[i(k_1 x + k_2 y)]$$

with bounded $\mathbf{w}$ and satisfying the homogeneous boundary conditions $\mathbb{E}(\mathbf{v}) = \mathbf{0}$, is the identically zero solution $\mathbf{v} \equiv \mathbf{0}$.

- *the interfacial complementing condition* : this relation between the coefficients of the field operators in Ω_1 and Ω_2 must be satisfied at every point Q of the interface I between Ω_1 and Ω_2 . This condition is again easily phrased in terms of an associated problem on the whole space divided by the plane interface $z = 0$. It requires for every vector $\mathbf{k} = (k_1, k_2, 0) \neq \mathbf{0}$, that the only solution to the rate equilibrium equations with constant coefficients (equal to those of the operators at point Q, in Ω_1 for $z < 0$ and in Ω_2 for $z > 0$), in the form

$$(\mathbf{v}_1(x, y, z), \mathbf{v}_2(x, y, z)) = (\mathbf{w}_1(z), \mathbf{w}_2(z)) \exp[i(k_1 x + k_2 y)]$$

with bounded $(\mathbf{w}_1, \mathbf{w}_2)$ and satisfying the continuity requirements (continuity of the velocity and the traction rates) across the interface $z = 0$, is the identically zero solution $(\mathbf{v}_1(z), \mathbf{v}_2(z)) \equiv (\mathbf{0}, \mathbf{0})$; (where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the solutions, respectively for $z < 0$ and for $z > 0$).

When these three conditions are fulfilled, the rate boundary problem admits a finite number of linearly independent solutions, which depend continuously on the data, and which constitute diffuse modes of deformation.

Remarks :

- these three conditions are local, and this is particularly important when considering their numerical implementation ;
- the above-given results remain valid for an arbitrary number of non-intersecting interfaces, an interfacial condition being written for each interface ;
- the failure of these conditions can be interpreted as localization criteria as recalled in section 3.1. These localization criteria can also be used as indicators of the local failure of the material ;
- both boundary and interfacial complementing conditions fail in the elliptic regime of the equilibrium equations, or at the latest, when the ellipticity condition fails. Thus, localized modes of deformation at the boundary or at the interface generally occur before the onset of so-called shear banding modes.

3.3. The non-linear case : some results

Although the complete analysis of the non-linear problem is not yet available, some results can be given for the possibility of emergence of deformation modes involving jumps of the velocity gradient for the bi-linear rate constitutive laws (1).

The necessary and sufficient conditions for the onset of such modes inside the body have been given by Borré & Maier who extended the results given by Rice, and Rudnicki & Rice for so-called continuous and discontinuous localizations. We have amplified these results by seeking necessary and sufficient conditions for which a discontinuity surface for the velocity gradient appears at, or reaches the boundary of the solid. These conditions are given below for the constitutive laws (1) with

$$\dot{\mathbf{E}} = \mathbf{E} - \frac{(\mathbf{E} : \mathbf{a}) \otimes (\mathbf{b} : \mathbf{E})}{h}$$

where it is assumed that $h > 0$, and $\mathbf{E}$ is strictly positive definite.

At a point P of the boundary Γ where only surface traction rates $\dot{\mathbf{F}}$ are applied, the necessary and sufficient conditions for continuous localization [i.e. the material is in loading ($L = H$) on each side of the singular surface] are

- (2a)
- i) there exists $\dot{\mathbf{E}}_0$ such that $\mathbf{m} \cdot \mathbf{H} : \dot{\mathbf{E}}_0 = \dot{\mathbf{F}}$
 - ii) $\det(\mathbf{n} \cdot \mathbf{H} \cdot \mathbf{n}) = 0$
 - iii) $(\mathbf{m} \cdot \mathbf{E} \cdot \mathbf{n}) \cdot (\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})^{-1} \cdot (\mathbf{n} \cdot \mathbf{E} : \mathbf{a}) = \mathbf{m} \cdot \mathbf{E} : \mathbf{a}$

4. A CRACK INITIATION POST PROCESSOR

4.1. Introduction

The constitutive equations for elasto-plasticity and damage at microscale developed during the first year of the grant (see the first annual report, LEMAITRE-BILLARDON, Mai 1990) were used to write a post processor allowing the determination of crack initiation even when the damage is highly localized as in high cycle fatigue.

4.2. Locally Coupled Analysis of Damage in Structures

In most applications, the damage is very localized in such a way that the damaged material occupies a volume small in comparison to the macro-scale of the structural component and even to the mesoscale of the RVE. This is due to the high sensitivity of damage to stress concentrations at the macroscale and to defects at the microscale. This allows us to consider that the effect of the damage on the state of stress and strain occurs only in very small damaged regions. In other words, the coupling between damage and strains may be neglected everywhere in the structure except in the RVE(s) where the damage develops. This is the principle of the locally coupled analysis [Lienard, 1989; Lemaitre, 1990] where the procedure may be split into the following two steps as shown in Figure 4.

- A classical structure calculation in elasticity or elastoplasticity by the Finite Element Method (FEM) or any other method to obtain the fields of strain and stress.
- A local analysis at the critical point(s) only dealing with the elasto-plastic constitutive equations coupled with the kinetic law of damage evolution, that is a set of differential equations.

This method is much more simple and saves a lot of computer time in comparison to the fully coupled analysis which takes into account the coupling between damage and strain in the whole structure. The fully coupled method must be used when the damage is not localized but diffused in a large region as in some areas of ductile and creep failures

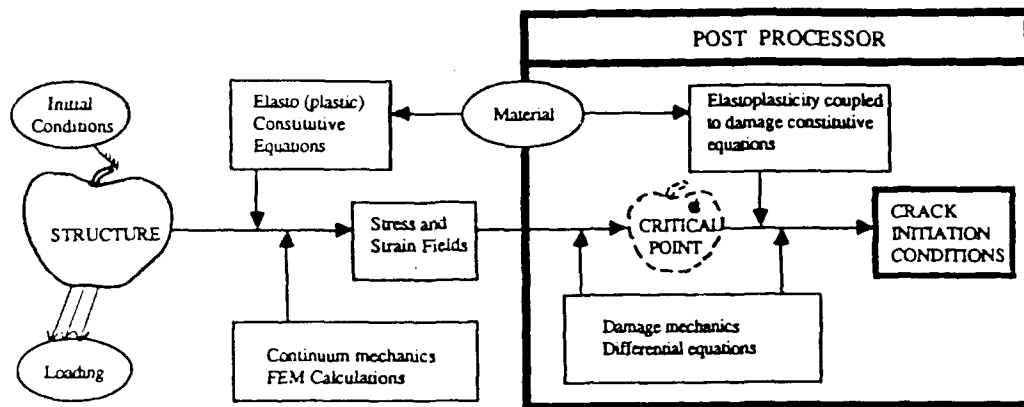


Fig. 4 Locally coupled analysis of crack initiation.

The locally coupled analysis is most suitable for those cases of brittle or fatigue failures when the structure remains elastic everywhere except in the critical point(s).

If the damage localizes, this is of course because stress concentrations may occur but also, because some weakness in strength always exists at the microscale. Let us consider a micro-mechanics model of a RVE at mesoscale made of a matrix containing a micro-element weaker by its yield stress, all other material characteristics being the same in the inclusion and in the matrix (Figure 5).

- The behavior of the matrix is elastic or elasto-plastic with the following characteristics of the material: yield stress σ_y , ultimate stress σ_u and fatigue limit σ_f ($\sigma_f < \sigma_y$).

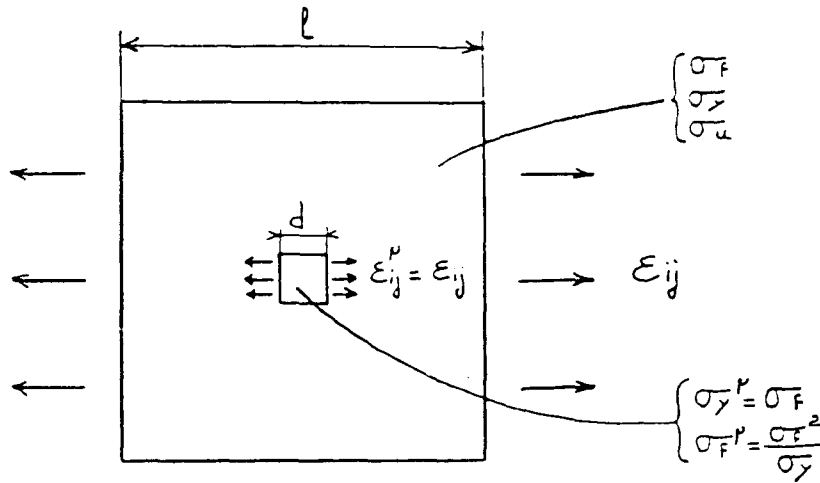


Fig. 5 Two scale Representative Volume Element

• The behavior of the inclusion is elastic-perfectly plastic and can suffer damage. Its weakness is due to a plastic threshold which is taken to be lower than σ_y and equal to the fatigue limit σ_f if not known by another consideration

$$\sigma_s^\mu \geq \sigma_y^\mu = \sigma_f$$

Below the fatigue limit σ_f no damage should occur. The fatigue limit of the inclusion is supposed to be reduced in the same proportion:

$$\sigma_f^\mu = \sigma_f \frac{\sigma_f}{\sigma_y}$$

The second assumption which makes the calculation simple is the Lin-Taylor's strain compatibility hypothesis [Taylor, 1938] which states that the state of strain at micro-scale is equal to the state of strain imposed at macroscale.

Then, there is no boundary value problem to be considered. Only the set of coupled constitutive equations must be solved for the given history of the mesostrains at the critical

point(s).

The determination of this critical point is the bridge between the F.E.M. calculation and the post processor. If the loading is proportional, that is constant principal directions of the stresses, this is the point M^* where at any time the damage equivalent stress is maximum

$$\sigma^*_{(M^*)} = \text{Max}(\sigma^*_{(M)}) \rightarrow M^*$$

If the loading is non-proportional, the damage equivalent stress may vary differently at different points as functions of the time like parameter defining the history. A small σ^* for a long time may be more damaging than a large σ^* for a short time! There is no rigorous way to select the critical point(s) but an "intelligent" look at the evolution of σ^* as function of the space and the time may restrict the number of the dangerous areas to a few points at which the calculation may be performed.

Another point of interest is the mesocrack initiation criterion. The result of the calculation is a crack initiation at micro-scale that is a crack of the size of the inclusion; its area is $\delta A = d^2$. This corresponds to a value of the strain energy release rate at mesoscale G that may be calculated from the variation of the strain energy W at constant stress

$$G = -\frac{1}{2} \frac{\delta W}{\delta A}$$

But $-\frac{\delta W}{2}$ is also the energy dissipated in the inclusion by the damaging process at constant stress

$$G_{(\delta A)} = \frac{d^3 \int_0^{D_c} Y dD}{\delta A}$$

Assuming a constant strain energy density release rate $Y = Y_c$ yields

$$G(\delta A) = Y_c D_c d$$

Another interesting relation comes from the matching between damage mechanics and fracture mechanics. Writing the equality between the energy dissipated by damage at micro-crack initiation and the energy dissipated to create the same crack through a brittle fracture mechanics process:

$$Y_c D_c d^3 = G_c d^2, \text{ where } G_c \text{ is the toughness of the material. It yields}$$

$$Y_c D_c d = G_c$$

Comparing the two relations for $Y_c D_c d$ shows that

$$G(\delta A) = G_c \rightarrow \text{the instability of the meso RVE}$$

Then, the criterion of micro-crack initiation is also the criterion for a brittle crack instability at mesoscale.

4.3. Constitutive Equations

4.3.1. *Formulation*

The following hypotheses are made:

- small deformation
- uncoupled partitioning of the total strain in elastic strain and plastic strain

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p$$

- linear isotropic elasticity

perfect plasticity. The latter hypothesis is justified by the fact that in most cases damage develops when the accumulated plastic strain is large enough to consider the strain hardening as saturated. Nevertheless, the code described in section 4.3 allows one to consider stepwise perfect plasticity in order to take into consideration high values of strain hardening and the cyclic stress strain curve for multilevel fatigue processes.

The threshold of plasticity σ_s is bounded by the fatigue limit and the ultimate stress

$$\sigma_f \leq \sigma_s \leq \sigma_u$$

The plastic constitutive equations are derived in the most classical way from the yield function in which the kinetic coupling with damage obeys to the strain equivalence principle applied on the plastic yield function:

$$f = \bar{\sigma}_{eq} - \sigma_s = 0 \quad \text{with} \quad \bar{\sigma}_{eq} = \frac{\sigma_{eq}}{1-D}$$

The plastic strain rate derives from f through the normality rule of standard materials [Lemaitre and Chaboche, 1990].

$$\dot{\epsilon}_{ij}^p = \frac{\partial f}{\partial \sigma_{ij}} \dot{\lambda} = \frac{3}{2} \frac{\sigma_{ij}^D}{\sigma_{eq}} \frac{\dot{\lambda}}{1-D} \quad \text{if} \quad \begin{cases} f = 0 \\ \dot{f} = 0 \end{cases}$$

where $\dot{\lambda}$ is the plastic multiplier. This shows that

$$\frac{\dot{\lambda}}{1-D} = \left(\frac{2}{3} \dot{\epsilon}_{ij}^p \dot{\epsilon}_{ij}^p \right)^{1/2} = \dot{p}, \quad \text{where } p \text{ is the accumulated plastic strain.}$$

The kinetic law of damage evolution, valid for any kind of ductile, brittle, low or high cycle fatigue damage derives from the potential of dissipation F in the framework of the State Kinetic Coupling theory [Marquis and Lemaitre, 1989]

$$F = f + \frac{Y^2}{2S(1-D)} \quad Y = \frac{\sigma_{eq}^2 R_v}{2E(1-D)^2} = \frac{\sigma_s^2 R_v}{2E}$$

S is a damage strength, a constant characteristic of each material

$$\dot{D} = \frac{\partial F}{\partial Y} \dot{\lambda} = \frac{Y}{S} \frac{\dot{\lambda}}{1-D}, \text{ or } \dot{D} = \frac{Y}{S} \dot{p} \text{ if } p \geq p_D$$

mesocrack initiation if $D = D_c$ as demonstrated in section 2.

- p_D is a damage threshold corresponding to microcrack nucleation, it is related to the energy stored in the material. Consider the material of the inclusion having a plastic threshold stress σ_s and a fatigue limit $\sigma_f^u = \frac{\sigma_f^2}{\sigma_y}$ considered as the "physical yield stress"

(Figure 6).

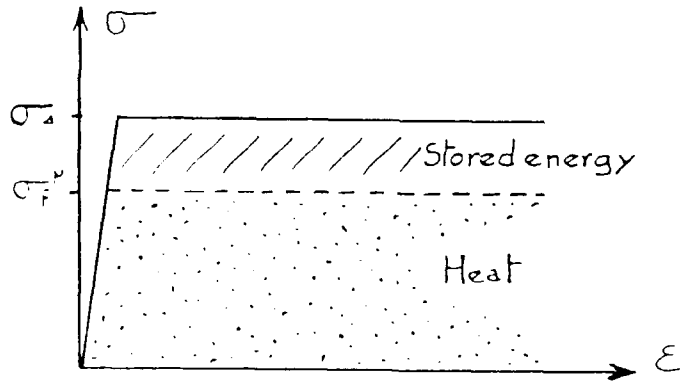


Fig. 6. Modelling a stress-strain curve.

The second principle of thermodynamics states that in the absence of any damage, the

energy dissipated in heat is $\sigma_f^\mu p$. Then the stored energy density is

$$w_s = \int \sigma_{ij} d\epsilon_{ij}^p - \sigma_f^\mu p$$

From the perfectly plastic constitutive equation

$$w_s = (\sigma_s - \sigma_f^\mu) p$$

Considering that the threshold p_D is governed by a critical value of this energy identified from the unidimensional reference tension test having a damage threshold ϵp_D , a plastic threshold equal to the ultimate stress σ_u and a fatigue limit σ_f

$$(\sigma_s - \sigma_f^\mu) p_D = (\sigma_u - \sigma_f) \epsilon p_D$$

and
$$p_D = \epsilon p_D \frac{\sigma_u - \sigma_f}{\sigma_s - \sigma_f^\mu} \quad \text{with} \quad \sigma_f^\mu = \frac{\sigma_f^2}{\sigma_y}$$

If piecewise perfect plasticity of several thresholds σ_{si} is considered

$$w_s = \sum \sigma_{si} (p_i - p_{i-1}) - \sigma_f^\mu p$$

and p_D is defined by

$$\sum_{p=0}^{p_D} \sigma_{si} (p_i - p_{i-1}) - \sigma_f^\mu p_D = (\sigma_u - \sigma_f) \epsilon p_D$$

The critical value of damage at mesocrack initiation D_c corresponds to an instability [Billardon and Doghri, 1989]. Nevertheless, it can be related to a certain amount of energy dissipated in the damaging process

$$\int_0^{D_c} Y dD = \text{constant at failure}$$

Assuming for that calculation a proportional loading for which $R_V = \text{const}$

$$\int_0^{D_c} Y dD = \int_0^{D_c} \frac{\sigma_s^2 R_V}{2E} dD = \frac{\sigma_s^2 R_V}{2E} D_c$$

Taking again the pure tension test as a reference which gives the critical value of the damage at mesocrack initiation D_{1c} related to the stress to rupture σ_R and the tensile decohesive ultimate stress σ_u together with $R_V = 1$ gives the rupture criterion:

$$\frac{\sigma_s^2 R_V}{2E} D_c = \frac{\sigma_u^2}{2E} D_{1c}, \text{ i.e., } D_c = D_{1c} \frac{\sigma_u^2}{\sigma_s^2 R_V}$$

Collecting all the equations gives the set of constitutive equations to be solved by a numerical analysis

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p$$

$$\epsilon_{ij}^e = \frac{1+\nu}{E} \frac{\sigma_{ij}}{1-D} - \frac{\nu}{E} \frac{\sigma_{kk}}{1-D} \delta_{ij}$$

$$\dot{\epsilon}_{ij}^p = \begin{cases} \frac{3}{2} \frac{\tilde{\sigma}_{ij}^D}{\sigma_s} \dot{p} & \dots & \text{if } f = 0 \\ 0 & \dots & \text{if } f < 0 \end{cases} \left\{ f = \frac{\sigma_{eq}}{1-D} - \sigma_s \right.$$

$$\dot{D} = \begin{cases} \frac{\sigma_s^2}{2ES} R_v \dot{p} & \dots \text{ if } p \geq p_D \\ 0 & \dots \text{ if } p < p_D \end{cases} \quad p_D = \epsilon_{pD} \frac{\sigma_u - \sigma_f}{\sigma_s - \frac{\sigma_f^2}{\sigma_y}}$$

$$\text{Mesocrack initiation if } D = D_{lc} \frac{\sigma_u^2}{\sigma_s^2 R_v} \leq 1$$

As the plastic strain rate $\dot{\epsilon}_{ij}^p$ is governed by $\dot{p}$, and $\dot{p}$ related to the damage rate $\dot{D}$, in this model it is the damage which governs the plastic strain.

Note also that the perfect plasticity hypothesis allows one to write R_v only as a function of the total strain, since

$$\sigma_H = \frac{E(1-D)}{1-2\nu} \epsilon_H^e, \quad \epsilon_H^e = \epsilon_H \text{ as } \epsilon_H = \frac{1}{3} (\epsilon_{kk}^e + \epsilon_{kk}^p) \text{ and } \epsilon_{kk}^p = 0$$

$$\sigma_{eq} = \sigma_s (1 - D)$$

$$\text{then } \frac{\sigma_H}{\sigma_{eq}} = \frac{E}{1-2\nu} \frac{\epsilon_H}{\sigma_s}$$

4.3.2. Identification of the Material Parameters

In order to be able to perform numerical calculations, it is an important task to determine the material coefficients in each case of application. This is always difficult because one is never sure that the handbook or the test results used correspond exactly to the material considered in the application. Those coefficients are

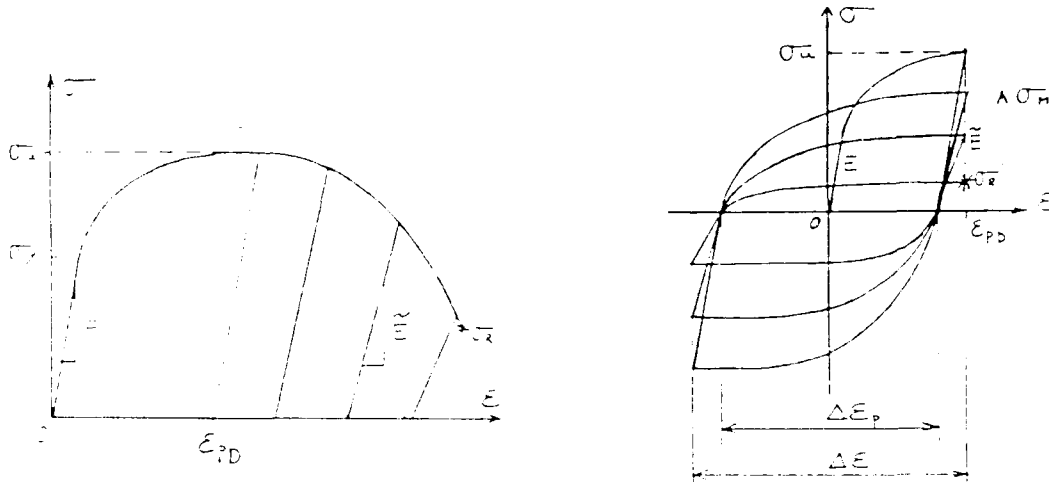
E and ν for elasticity

σ_f , σ_y , σ_u and the choice of σ_s for plasticity

S , ϵ_{pD} , D_{lc} for damage

They can be easily obtained from a tensile stress strain curve with unloadings or from a very low fatigue test at constant strain amplitude ($N_R \leq 10$) as those of Figure 7.

A classical experimental procedure with strains measured by strain gauges permits the determination of the Young's modulus E , the Poisson's ratio ν and also the damaged elasticity modulus $\tilde{E}$ as defined on the graphs of Fig. 4 and as a function of p



Tensile stress-strain curve

Very low cycle fatigue test ($\Delta\epsilon(\sigma_u)$)

Fig. 7. Identification of elasto-plastic and damage parameters.

- * $p = \epsilon_p$ on the first graph
 - * $p = 2N \Delta\epsilon_p$ on the second graph since $\Delta\epsilon_e = \frac{2\sigma_M}{E(1-D)} = \frac{2\sigma_u}{E} = \text{const}$
- N is the number of cycles

The yield stress σ_y and the ultimate stress σ_u are also of a classical procedure. The fatigue limit σ_f is the stress for which the number of cycles to failure N_R in a high cycle fatigue test is very large: $N_R \cong 10^6$ to 10^7 cycles. Usually

$$\frac{1}{2}\sigma_y \leq \sigma_f \leq \frac{2}{3}\sigma_y$$

The damaged parameters are deduced from the measurement of $\tilde{E}$ as the law of elasticity allows to write [Lemaitre and Dufailly, 1987]

$$D = 1 - \frac{\tilde{E}}{E}$$

Then, the damage D is plotted versus the accumulated plastic strain p as in **Figure 8** from which it is easy to deduce:

- * the damage threshold ϵ_{pD} , defined by $\epsilon_p < \epsilon_{pD} \rightarrow \dot{D} = 0$
- * the damage strength S as $S = \frac{\sigma_s^2}{2E} \frac{\delta p}{\delta D}$

since in one dimension $R_V = 1$ and $\dot{D} = \frac{\sigma_s^2}{2ES} \dot{p}$

- * the critical value of the damage D_{1c} defined by $D_{1c} = \text{Max}(D)$.

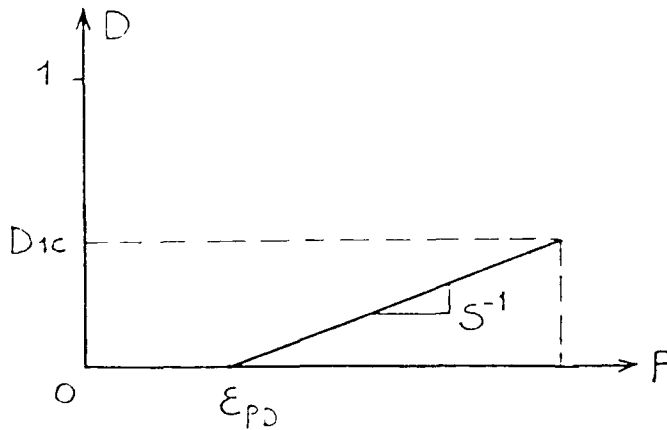


Figure 8. Identification of damaged parameters.

The plastic yield threshold σ_s is a matter of choice regarding the type of accuracy needed in the calculation.

- * take $\sigma_s = \sigma_f$ for the highest bound on the time-like loading parameter to crack initiation.

- * take $\sigma_s = \sigma_u$ for the lowest bound on the time-like loading parameter to crack initiation.

4.4. Numerical Procedure

4.4.1. Method of Integration

The method used is a strain driven algorithm: at each "time" (or time-like parameter) increment, and for a given total strain increment, one knows the values of the stresses and the other material model variables at the beginning of the increment (t_n) and they have to be updated at the end of the increment (t_{n+1}). We show here that the general numerical procedure [Benallal, Billardon, Doghri, 1988] becomes particularly simple and efficient because analytically derived in a closed form.

In the following, the subscript (n+1) will be omitted and all the variables which do not contain the subscript (n+1) are computed at t_{n+1} . Tensorial notations will be used for convenience.

We first assume that the increment is entirely elastic. So, we have

$$\bar{\sigma} = \lambda \text{tr} \varepsilon \mathbf{1} + 2\mu(\varepsilon - \varepsilon_n^p)$$

where λ and μ are the Lamé coefficients

$$\lambda = \frac{Ev}{(1-2\nu)(1+\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}$$

and $\mathbf{1}$ is the second order identity tensor.

All the other "plastic" variables are equal to their values at t_n . If this "elastic

predictor" satisfies the yield condition $f \leq 0$, then the assumption is valid and the computation for this time increment ends. If $f > 0$, this elastic state is "corrected" as explained in the following in order to find the plastic solution.

The rate relations of Section 3 are discretized in an incremental form corresponding to a fully implicit integration scheme which presents the advantage of unconditional stability [Ortiz and Popov, 1985]. Then the solution at t_{n+1} has to satisfy the following relations:

$$f = \bar{\sigma}_{eq} - \sigma_s = 0$$

$$\bar{\sigma} = \lambda \text{tr} \epsilon \mathbf{1} + 2\mu(\epsilon - \epsilon_n^p - \Delta \epsilon^p)$$

$$\Delta \epsilon^p = N \Delta p$$

$$\Delta D = \frac{Y}{s} \Delta p$$

where $\Delta(\bullet) = (\bullet)_{n+1} - (\bullet)_n$ and $N = \frac{3}{2} \frac{\bar{\sigma}^D}{\bar{\sigma}_{eq}}$

If we replace $\Delta \epsilon^p$ by its expression in the second equation, we see that the problem is reduced to the first 2 equations for the 2 unknowns: $\bar{\sigma}$ and p .

Let's find $\bar{\sigma}$ and p which satisfy:

$$f = \bar{\sigma}_{eq} - \sigma_s = 0$$

$$h = \bar{\sigma} - \lambda \text{tr} \epsilon \mathbf{1} - 2\mu(\epsilon - \epsilon_n^p) + 2\mu N \Delta p = 0$$

This nonlinear system is solved iteratively by a Newton method. For each iteration (s) we have

$$f + \frac{\partial f}{\partial \bar{\sigma}} : C_{\bar{\sigma}} = 0$$

$$h + \frac{\partial h}{\partial \bar{\sigma}} : C_{\bar{\sigma}} + \frac{\partial h}{\partial p} C_p = 0$$

where f , h and their partial derivatives are taken at t_{n+1} and at the iteration (s) . The "corrections" $C_{\tilde{\sigma}}$ and C_p are defined by

$$C_{\tilde{\sigma}} = (\tilde{\sigma})_{s+1} - (\tilde{\sigma})_s \quad \text{and} \quad C_p = (p)_{s+1} - (p)_s$$

The starting iteration $(s=0)$ corresponds to the elastic predictor.

The set of equations for f and h may be rewritten as

$$f + N : C_{\tilde{\sigma}} = 0$$

$$h + \left[\mathbb{I} + 2\mu\Delta p \frac{\partial N}{\partial \tilde{\sigma}} \right] : C_{\tilde{\sigma}} + 2\mu N C_p = 0$$

where $\mathbb{I}$ is the fourth order identity tensor and

$$\frac{\partial N}{\partial \tilde{\sigma}} = \frac{1}{\tilde{\sigma}_{eq}} \left[\frac{3}{2} (\mathbb{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1}) - N \otimes N \right]$$

Since $N : N = \frac{3}{2}$ and $N \cdot \frac{\partial N}{\partial \tilde{\sigma}} = 0$, the system gives explicitly the correction for p :

$$C_p = \frac{f - N : h}{3\mu}$$

we shall prove that the correction for $\tilde{\sigma}$ can be found explicitly also. First let's prove that $C_{\tilde{\sigma}}$ is a deviatoric tensor.

From the second equation of the system and using the expression of $\frac{\partial N}{\partial \tilde{\sigma}}$ one obtains

$$\text{tr}(C_{\tilde{\sigma}}) = -\text{tr } h$$

Using the definitions of h and $C_{\tilde{\sigma}}$,

$$\text{tr}[(\bar{\sigma})_{s+1}] = (3\lambda + 2\mu)\text{tr} \varepsilon = \text{tr}[(\bar{\sigma})_0] = \text{constant}$$

These relations being true for any iteration (s), then $\text{tr}(C_{\bar{\sigma}}) = 0$ and the proof is achieved.

Using this last result together with the expressions of $\frac{\partial N}{\partial \bar{\sigma}}$ and C_p yields for the second equation of the system:

$$A:C_{\bar{\sigma}} = -h - \frac{2}{3}(f - N:h)N$$

where
$$A = \left(1 + \frac{3\mu}{\bar{\sigma}_{eq}} \Delta p\right) \Pi - \frac{2\mu}{\bar{\sigma}_{eq}} \Delta p N \otimes N$$

One can also check the following result: the tensor A is invertable if, and only if, $\bar{\sigma}_{eq} \neq 0$, and in this case:

$$A^{-1} = \frac{1}{1 + \frac{3\mu}{\bar{\sigma}_{eq}} \Delta p} \left[\Pi + \frac{2\mu}{\bar{\sigma}_{eq}} \Delta p N \otimes N \right]$$

Using this expression gives explicitly the correction for $\bar{\sigma}$

$$C_{\bar{\sigma}} = -\frac{2}{3}(f - N:h)N - \frac{1}{1 + \frac{3\mu}{\bar{\sigma}_{eq}} \Delta p} \left[h + \frac{2\mu}{\bar{\sigma}_{eq}} \Delta p (N:h) N \right]$$

To summarize, this method is a fully implicit scheme with the advantage of an explicit scheme: no linear system to be solved and the unknown are updated explicitly.

Once p and $\bar{\sigma}$ are found, ε^p and D are calculated from their discretized constitutive equations and the stress components are given by $\sigma_{ij} = (1 - D)\bar{\sigma}_{ij}$.

Let us remark that the method described above can be applied to the case of nonlinear

isotropic and kinematic hardenings, with or without damage.

4.4.2. Jump in Cycles Procedure

For periodic loadings of fatigue with large number of cycles, the computation step by step in time becomes prohibitive and the complete computation is out of a reasonable occupation of any computer. For that reason, a simplified method has been proposed [Billardon, 1989] and a slightly different version is used in this work. It allows one to "jump" large numbers of cycles for a given approximation to the final solution.

(a) Before any damage growth (i.e., when $p < p_D$)

(i) the calculation is performed until a stabilized cycle N_s is reached. Let δp be the increment of p over this cycle. We assume that during a number $\overline{\Delta N}$ of cycles, p remains linear versus N with the slope $\frac{\delta p}{\delta N}$. This number $\overline{\Delta N}$ is given by

$$\overline{\Delta N} = \frac{\overline{\Delta p}}{\delta p}$$

where $\overline{\Delta p}$ is a given value which determines the accuracy on accumulated plastic strain.

(ii) a jump $\overline{\Delta N}$ of cycles is performed and p is updated as
 $p(N_s + \overline{\Delta N}) = p(N_s) + \overline{\Delta N} \cdot \delta p$.

Go to (i)

(b) Following damage growth (i.e. when $p \geq p_D$)

(i) the calculation is performed with a constant value of the damage ($\dot{D} = 0$) until a stabilized cycle N_s is reached. Then a fully coupled elasto-plastic and damage computation is performed for the next cycle. Let δp and δD be resp. the increments of p and D for this cycle. We assume that during a number $\overline{\Delta N}$ of cycles, p and D remain linear versus N

with resp. the slopes $\frac{\delta p}{\delta N}$ and $\frac{\delta D}{\delta N}$. This number $\overline{\Delta N}$ is given by

$$\overline{\Delta N} = \min \left\{ \frac{\overline{\Delta p}}{\delta p}, \frac{\overline{\Delta D}}{\delta D} \right\}$$

where $\overline{\Delta D}$ is a given value which determines the accuracy on the damage.

(ii) A jump $\overline{\Delta N}$ of cycles is performed and p and D are updated as

$$p(N_s + 1 + \overline{\Delta N}) = p(N_s + 1) + \overline{\Delta N} \cdot \delta p$$

$$D(N_s + 1 + \overline{\Delta N}) = D(N_s + 1) + \overline{\Delta N} \cdot \delta D$$

Go to (i)

The choice of $\overline{\Delta p}$ and $\overline{\Delta D}$ is of a first importance. For this work we chose

$$\overline{\Delta D} = \frac{D_{1c}}{50} \text{ and } \overline{\Delta p} = \frac{S}{Y(\overline{\Delta \epsilon})} \overline{\Delta D}, \text{ where } \overline{\Delta \epsilon} \text{ is the strain amplitude.}$$

This method gives good results but as any heuristic method, sometimes it fails and the values of $\overline{\Delta D}$ or $\overline{\Delta p}$ must be changed.

4.4.3. The Post Processor DAMAGE 90

A computer program called DAMAGE 90 has been written as a "friendly" code on the basis of the method developed. The input data are the material parameters and the history of the total strain components ϵ_{ij} . DAMAGE 90 gives as output the evolution of the damage D, the accumulated plastic strain p, and the stresses σ_{ij} up to crack initiation. DAMAGE 90 distinguishes between two loading cases:

(a) **General Loading History.** In this case, the history is defined by the values of

ϵ_{ij} at given "time" values. DAMAGE 90 assumes a linear history between two consecutive given values.

(b) Piecewise Periodic History. In this case, the loading is defined by blocks of cycles. In each block, each total strain ϵ_{ij} varies linearly between two given "peak" values. DAMAGE 90 asks the user if he wishes to perform a complete calculation, or a simplified one using the jump in cycles procedure described in Section 4.2 if the number of cycles is large.

DAMAGE 90 is written in FORTRAN 77 as available on a CONVEX computer, and contains near 600 FORTRAN instructions. It is designed to be used as a post processor after a FEM code. However, it may be used in an interactive way. In all the examples of Section 5, the average computing time on the CONVEX machine was between 1 and 15 sec for each run.

The questions asked by DAMAGE 90 for a case corresponding to an example of section 5.4 are listed in Appendix 1 together with the results. Appendix 2 is the complete Fortran listing of the program DAMAGE 90.

4.5. Examples

The following examples were performed with the "DAMAGE 90" code in order to point out the main properties of the constitutive equations used together with the locally coupled method.

4.5.1. Case of Ductile Damage

The material data are those of a stainless steel

$$E = 200,000 \text{ MPa}, \quad \nu = 0.3$$

$$\sigma_f = 200 \text{ MPa}, \quad \sigma_y = 300 \text{ MPa}, \quad \sigma_u \doteq 500 \text{ MPa}$$

$$S = 0.06 \text{ MPa}, \quad \epsilon_{pD} = 10\%, \quad D_{1c} = 0.99$$

Pure tension at microscale

The pure tension case at micro-scale is obtained by giving $p_D = \epsilon p_D = 10\%$,
 $D_C = D_{1C} = 0.99$ and ϵ_{11} as the main input. DAMAGE 90 computes the other strains

$$\epsilon_{22} = \epsilon_{33} = -\frac{1}{2}\epsilon_{11} + \left(\frac{1}{2} - \nu\right) \frac{\sigma_{11}}{E(1-D)}$$

which represents the elasto-plastic contraction resulting in the incompressible plastic flow hypothesis $\left(\epsilon_{22}^p = \epsilon_{33}^p = -\frac{1}{2}\epsilon_{11}^p\right)$.

For a better representation of the large strain hardening of that material, the piecewise perfect plasticity procedure is used with the following data:

$\epsilon_{11}\%$	0	0.25	1.5	5
σ_s MPa	200	300	400	500

The results are those of **Figure 9** where the strain softening and the damage are linear functions of the strain as it is introduced in the constitutive equations

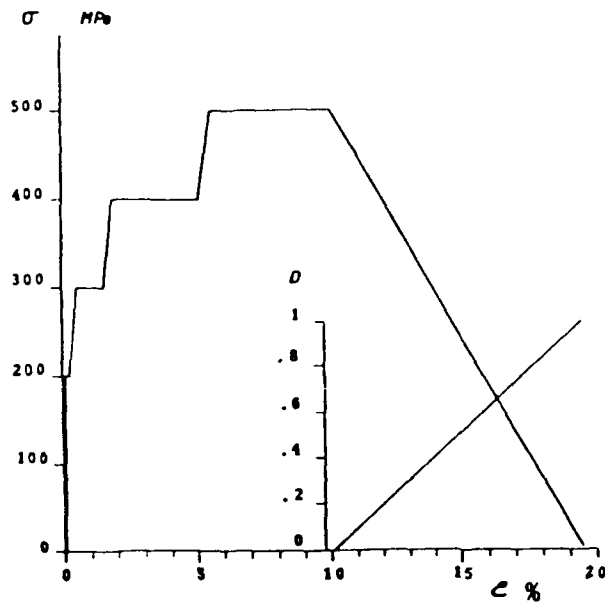


Fig. 9. Elastoplasticity and ductile damage in pure tension at microscale.

Plane Strain at meso (or micro) scale

Those cases of two dimensional loadings were performed in pure perfect plasticity with a plasticity threshold $\sigma_s = \sigma_u = 500$ MPa.

The influence of the triaxiality (ratio σ_H/σ_{eq}) upon the accumulated plastic strain to rupture p_R was obtained by calculations for which in each of them σ_H/σ_{eq} is fixed to a certain value. As from the constitutive equations, $\frac{\sigma_H}{\sigma_{eq}} = \frac{E}{1-2\nu} \frac{\epsilon_H}{\sigma_s}$, this fixes the value of $\epsilon_{11} + \epsilon_{22} = 3\epsilon_H$. Then, for each point ϵ_{11} was chosen and ϵ_{22} taken as $\epsilon_{22} = 3\epsilon_H - \epsilon_{11}$.

The points of the limit curve in strains corresponding also to rupture were obtained with proportional loading in strain histories: $\epsilon_{22} = \alpha \epsilon_{11}$, $\epsilon_{33} = 0$

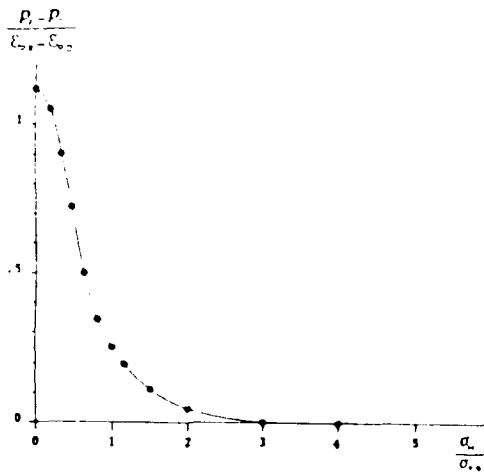


Fig. 10. Effect of the triaxiality upon the strain to rupture.
 ϵ_{PR} is the strain to rupture in pure tension $\sigma_H/\sigma_{eq} = 1/3$,
 $\epsilon_{PR} = 19.6\%$

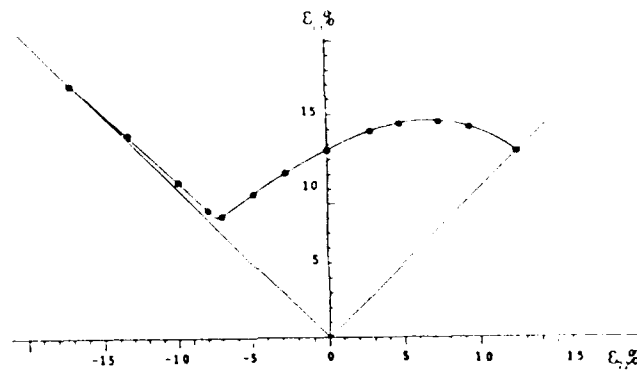


Fig. 11. Crack initiation limit curve in plane strains

the results are plotted in Figures 10 and 11 where are obtained the classical strong effect of the triaxiality which makes the rupture more and more brittle $(PR - PD) \rightarrow 0$ [McClintock, 1968] and the classical "S" shape of the limit curve of metal forming [Cordebois, 1983] here in plane strains.

4.5.2 Case of Brittle Damage

The material data are those of a ceramic

$$E = 400,000 \text{ MPa} \quad \nu = 0.2$$

$$\sigma_f = 200 \text{ MPa}, \quad \sigma_y = 250 \text{ MPa}, \quad \sigma_u = 300 \text{ MPa}$$

$$s = 0.00012 \text{ MPa}, \quad \epsilon_{pD} = 0, \quad D_{1c} = 0.05$$

An elastic tension case is considered at meso-scale, ϵ_{11} is the main input and the two other principal strain components are imposed as $\epsilon_{22} = \epsilon_{33} = -\nu \epsilon_{11}$.

At microscale this induces a pure tension in the elastic range but a three-dimensional state of stress occurs as soon as the plastic threshold is reached due to the difference of the elasto-plastic contraction of the inclusion and the elastic contraction imposed by the matrix.

The plastic threshold is taken as $\sigma_s = 200 \text{ MPa}$.

The result are those of Figure 12 for the behavior at mesoscale where no appreciable plastic strain appears and of Figure 13 at microscale where the effect of damage is sensitive.

4.5.3. Low and High Cycle Fatigue

In the damage model, there is no difference in the equation between low and high cycle fatigue, which obey, this is the hypothesis, to the same energetic mechanisms. Furthermore, written as a damage rate equation it is valid even when a cycle cannot be defined.

The material data are those of an aluminum alloy

$$E = 72,000 \text{ MPa}, \quad \nu = 0.32$$

$$\sigma_f = 303 \text{ MPa}, \quad \sigma_y = 306 \text{ MPa}, \quad \sigma_u = 500 \text{ MPa}$$

$$s = 6 \text{ MPa}, \quad \epsilon_{pD} = 10\%, \quad D_{1c} = 0.99$$

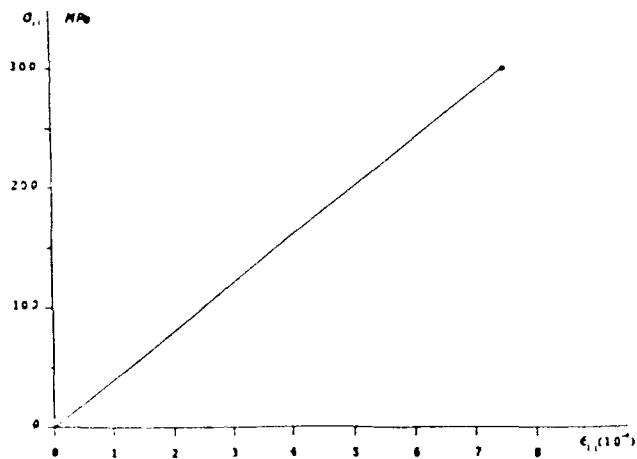


Fig. 12. Stress strain and brittle damage at mesoscale in pure tension

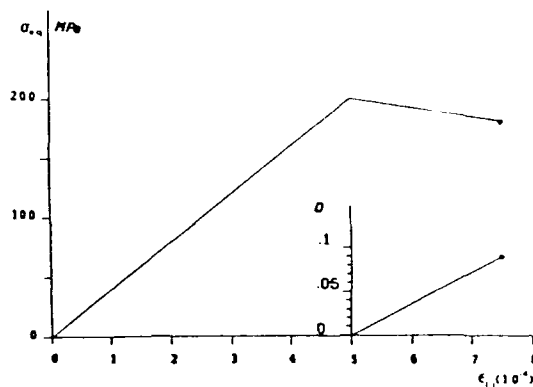


Fig. 13. Stress-strain and damage at microscale when pure tension is applied at mesoscale

As for the example of Section 5.1, pure tension cases at microscale are considered by giving as input data ϵ_{11} , the other strains being computed by DAMAGE 90.

In order to take into account the cyclic strain hardening the following plastic thresholds taken from a cyclic stress strain curve are considered for the different constant strain amplitudes imposed

$\epsilon_{11}\%$	± 0.425	± 0.43	± 0.47	± 1	$+ 3.5$	± 4.5
σ_s MPa	303	305	308	370	440	460

The number of cycles to rupture obtained are given as a function of the strain amplitude imposed in Figure 14 on which the number of cycles to damage initiation N_0 ($p = p_D$) is also reported. As shown by experiments, the ratio N_0/N_R increases as N_R increases.

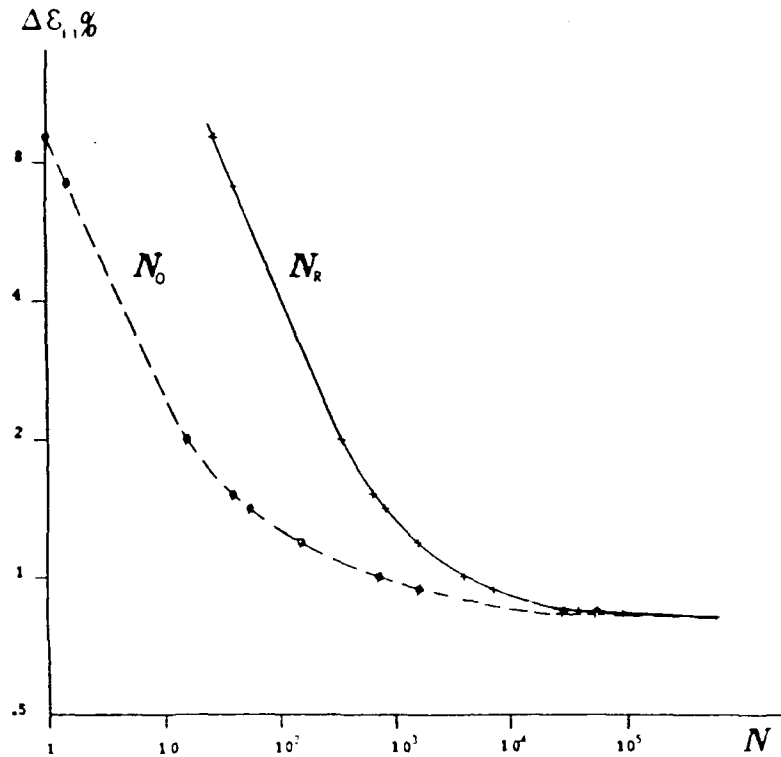


Fig. 14. Fatigue rupture curve in pure tension at microscale.

The details of the results corresponding to the case $\Delta\epsilon_{11} = 7\%$ are given in the graphs of Figure 15 where are reported the evolution of the strain ϵ_{11} and the stress-strain loops $(\sigma_{11}, \epsilon_{11})$ as a function of the number of cycles, the equivalent stress σ_{eq} and the accumulated plastic strain p together with the damage D .

It is interesting to compare with the case of a pure elastic tension case at mesoscale inducing a three-dimensional state of stress at microscale. The same input are used as for the case of Figure 12 except that the strains applied are

$$\epsilon_{11} = \pm 3.5\%, \quad \epsilon_{22} = \epsilon_{33} = -\nu \epsilon_{11}.$$

The four similar graphs are shown in Figure 16.

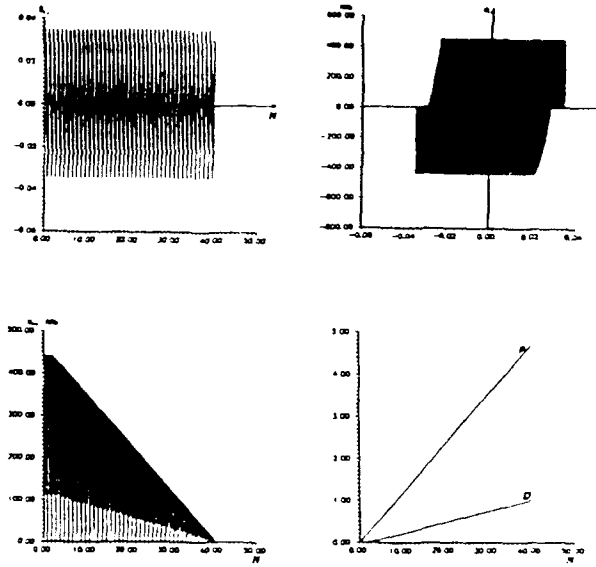


Fig. 15 . Results of a very low cycle fatigue in pure tension at microscale $\Delta\epsilon_{11} = 7\%$, $N_R = 40$ cycles.

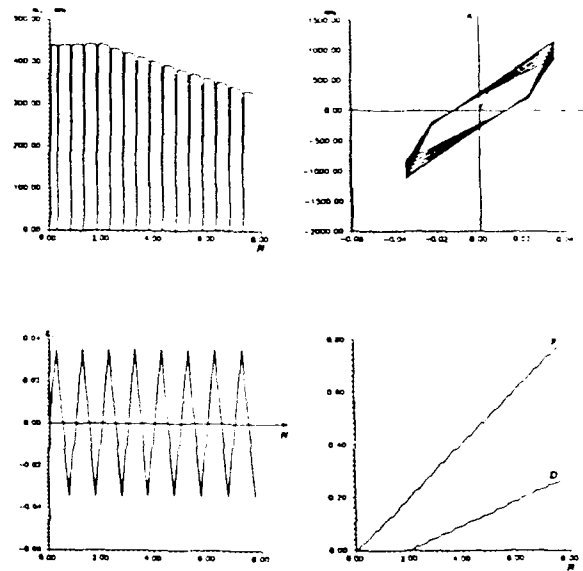


Fig. 16. Results of a very low cycle fatigue in pure tension at mesoscale inducing three-dimensional fatigue at microscale $\Delta\epsilon_{11} = 7\%$, $N_R = 8$

4.5.4. Nonlinear Accumulation in Fatigue

An important feature of fatigue damage is the nonlinearity of the accumulation of damages due to loadings of different amplitudes. If a two level loading history is considered with N_1 cycles at the strain amplitude $\Delta\epsilon_1$ and N_2 cycles at the strain amplitude $\Delta\epsilon_2$ with $N_1 + N_2 = N_R$ the number of cycles to rupture, in opposition with the Palmgreen-Miner rule, generally

$$\frac{N_1}{N_{R1}(\Delta\epsilon_1)} + \frac{N_2}{N_{R2}(\Delta\epsilon_2)} < 1 \text{ if } \Delta\epsilon_1 > \Delta\epsilon_2$$

$$\frac{N_1}{N_{R1}(\Delta\epsilon_1)} + \frac{N_2}{N_{R2}(\Delta\epsilon_2)} > 1 \text{ if } \Delta\epsilon_1 < \Delta\epsilon_2$$

N_{R1} and N_{R2} being the number of cycles to failure corresponding to the constant amplitude cases of fatigue respectively at $\Delta\epsilon_1$ and $\Delta\epsilon_2$.

This effect was studied with the same material data as those of the example 5.3 on a one-dimensional case at mesoscale $\epsilon_{22} = \epsilon_{33} = -\nu \epsilon_{11}$.

$\epsilon_{11} = \pm 0.47\%$ for the highest level $N_R = 7720$ cycles

$\epsilon_{11} = \pm 0.425\%$ for the lowest level $N_R = 109570$ cycles

The results are given in the accumulation diagram of Figure 17, they are in good qualitative agreement with experimental results

4.5.5. Multiaxial Fatigue

Two cases of multiaxial fatigue were studied: biaxial fatigue and fatigue in tension and shear. Both were performed with the material data of examples 5.3 and 5.4. The game played with the code, DAMAGE 90 consisted of finding the state of strains which produces a given number of cycles to failure. This was done by repeated trial from many calculations.

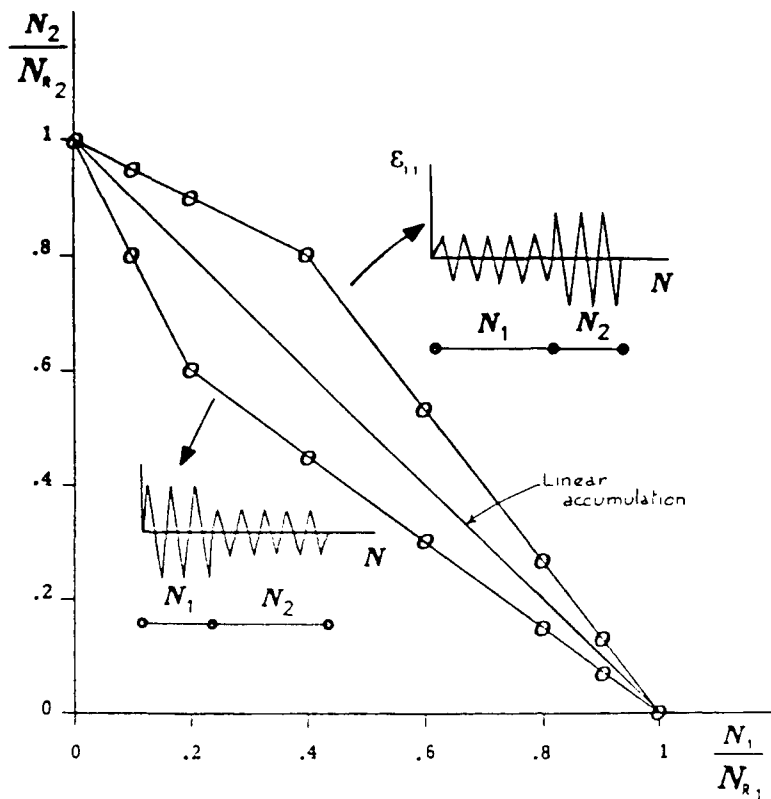


Fig. 17. Accumulation diagram for two level tests in tension at mesoscale.

Biaxial Fatigue in plane strain at meso (or micro) scale

A plane state of strain is considered with the following in phase strains

$$\epsilon_{11} = \pm x \quad \epsilon_{22} = \pm y \quad \epsilon_{33} = 0$$

Figure 18 shows contours of cycles to failure corresponding to $N_R = 7800$ cycles.

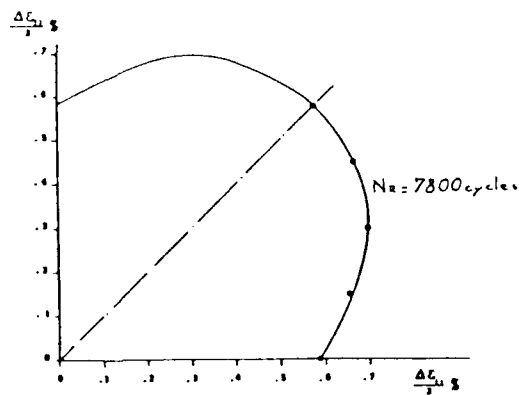


Fig. 18. Biaxial fatigue in plane strain.

Fatigue in Tension and Shear strains imposed at meso (or micro) scale

This is the same type of calculation with

$$\epsilon_{11} = \pm x \quad \epsilon_{12} = \pm y \quad \text{all other components} = 0$$

Figure 19 summarizes the results for the same number of cycles to failure.

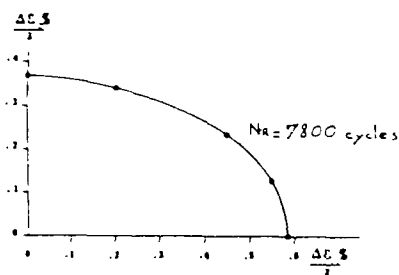


Fig. 19 Fatigue in tension and shear strains imposed.


```

ENERG1=(SIGU-SIGP)*EPD
ENERG2=0.
ENERG3=0.
TOLERR=2.5D-2
ILOOK0=0
COUPL='y'
IPASS=0
ONE=1.D+00
TWO=2.D+00
THREE=3.D+00
BACK=CSQRT(TWO)
NU1=11
NU2=12
FILE1='direct.out'
FILE2='shear.out'
OPEN(UNIT=NU1,NAME=FILE1,STATUS='unknown')
OPEN(UNIT=NU2,NAME=FILE2,STATUS='unknown')
COMMENT=' TIME'
IF((CYCLIC.EQ.'y').OR.(CYCLIC.EQ.'X'))COMMENT=' CYCLE'
WRITE(NU1,312)COMMENT
WRITE(NU2,312)COMMENT
WRITE(NU1,315)COMMENT
312 FORMAT(2X,A6,21X,'STRAINS',27X,'STRESSES',16X,'DAMAGE',
. 9X,'p',8X,'MISES',2X,'DAM.EQ.STRESS')
WRITE(NU1,313)
WRITE(NU2,314)
313 FORMAT(19X,'11',9X,'22',9X,'33',10X,'11',9X,'22',9X,'33',
. 11X,'D',23X,'SIG EQ',5X,'SIG')
314 FORMAT(19X,'12',9X,'13',9X,'23',10X,'12',9X,'13',9X,'23',
. 11X,'D',23X,'SIG EQ',5X,'SIG')
315 FORMAT(5X,A6,10X,'DAMAGE',8X,'p',5X,'MISES EQ.STRESS',
. 2X,'DAMAGE EQ.STRESS')
C*** IF THE LOADING IS NOT CYCLIC
IF((CYCLIC.EQ.'n').OR.(CYCLIC.EQ.'X'))THEN
JUMP='n'
TIME=TIME(1)
DTCN=DTIME/10000.
TYM=TIME(NGTV)
IGTV=1
CALL OUTPUT(NU1,NU2,TIME,STRAN,STRS,STATEV,NTENS,NSTATV,
. STRSB,STAR)
DO WHILE (TIME+DTIME.LE.TFIN)
DO IST=1,MAXI
DSTRAN(IST)=DTIME*(NIST(IST,IGTV+1)-NIST(IST,IGTV))
. / (TIME(IGTV+1)-TIME(IGTV))
END DO
CONVER='y'
DO IST=1,NSTATV
STATEV(IST)=STATEV(IST)
END DO
SIGS=SYIEL(IGTV)
IF(SIGS.LE.SIGF.OR.SIGS.GT.SIGU)STOP'*** PLEASE REVIEW THE
. VALUES OF SIGF SIGU AND SIGS .'
CALL INTEGR(STRAN,DSTRAN,NTENS,NSTATV,UNLAXI,COUPL,
. CONVER,STRS,STATEV,STRSB,STAR,RNU)
C*** IF NO CONVERGENCE, DIVIDE TIME INC. BY 2
IF(CONVER.EQ.'y'.AND.STATEV(11).GT.0.)THEN
ENERG2=ENERG1+(SIGS-SIGF*(SIGF/SIGU))*STATEV(11)
IF(ANSDC.EQ.'n')DCRIT=DIC*((SIGU/SIGS)**2)/RNU
IF(ILOOK0.EQ.0.AND.(ANSPD.EQ.'y'.OR.ANSPD.EQ.'X'))
. AND.STATEV(2).GT.1.05*SNPD.OR.
. ILOOK0.EQ.0.AND.(ANSPD.EQ.'n'.OR.ANSPD.EQ.'X').AND.
. ENRG2.GT.1.05*ENERG1.OR.
. STATEV(3).GT.1.05*DCRIT)THEN
CONVER='n'
END IF
END IF
IF(CONVER.EQ.'n')THEN
IPASS=IPASS+1
DTIME=DTIME/2.
C*** IF CONVERGENCE
ELSE
IF((ANSDC.EQ.'n'.OR.ANSDC.EQ.'X')
. AND.DCRIT.GT.0.99D0)DCRIT=0.99D0
IPASS=0
DO IST=1,NSTATV
STATEV(IST)=STATEV(IST)
END DO
DO IST=1,6
STRAN(IST)=STRAN(IST)+DSTRAN(IST)
END DO
TIME=TIME+DTIME
CALL OUTPUT(NU1,NU2,TIME,STRAN,STRS,STATEV,NTENS,NSTATV,
. STRSB,STAR)

```

```

IF(STATEV(3).GE.DCRIT)THEN
WRITE(NU1,*)' '
WRITE(NU1,*)'*** CRACK INITIATION.'
GO TO 308
END IF
IF(STATEV(3).GE.1.1)STOP'*** DAMAGE CANNOT EXCEED 1.'
C*** FIND p0
IF(ILOOK0.EQ.0.AND.STATEV(11).GT.0.)THEN
ENERG1=ENERG2
IF((ANSPD.EQ.'n'.OR.ANSPD.EQ.'X')
. AND.ENERG1.GE.ENERG1)THEN
SNPD=STATEV(2)
ILOOK0=1
ELSE IF((ANSPD.EQ.'y').OR.(ANSPD.EQ.'X').AND.
. STATEV(2).GE.SNPD)THEN
ILOOK0=1
END IF
END IF
IF(DABS(TIME-TIME(IGTV+1)).LE.DTCN)THEN
IGTV=IGTV+1
END IF
END IF
DTIME=DTIME*1.1
IF(TIME+DTIME.GT.TIME(IGTV+1)+DTCN.AND.IGTV+1.LE.NGTV)THEN
DTIME=TIME(IGTV+1)-TIME
END IF
IF(IPASS.EQ.13.OR.DTIME.LE.DTCN)THEN
WRITE(NU1,*)'*** NO CONVERGENCE'
WRITE(NU1,*)'DTIME=',DTIME,'IPASS=',IPASS
GO TO 308
END IF
END DO
END IF
C*** IF THE LOADING IS CYCLIC
IF((CYCLIC.EQ.'y').OR.(CYCLIC.EQ.'X'))THEN
TPER=1.
DTCN=TPER/40000.
TYM=0.
ICYCLE=1
NRCYC=ICYCLE
DTCN=0.
DO I=1,6
SIGI(I)=0.
END DO
SNPI=SNPD
DIL=00
IF((JUMP.EQ.'y').OR.(JUMP.EQ.'X'))THEN
COUPL='n'
OPEN(UNIT=80,NAME='fatigue.out',STATUS='unknown')
WRITE(80,*)'REAL TIME TIME D p'
WRITE(80,*)'.....'
WRITE(80,*)'BEGINNING OF THE CYCLE'
WRITE(80,*)'CYCLE (REAL) =',NRCYC
WRITE(80,*)'CYCLE (MACHINE) =',ICYCLE
WRITE(80,400)RTIME,TIME,DIL,SNPI
END IF
CALL OUTPUT(NU1,NU2,TIME,STRAN,STRS,STATEV,NTENS,NSTATV,
. STRSB,STAR)
IBL=1
DTIME=TPER/INCUSE(IBL)
DO IST=1,6
IF(EPSB(IST,IBL)*EPSS(IST,IBL).LT.0.)THEN
PERDIV(IST)=TPER/4.
ELSE
PERDIV(IST)=TPER/2.
END IF
IPER(IST)=1
ISLOPE(IST)=1
END DO
NRCYSUP=0
DO WHILE (NRCYC.LE.NCYCLE)
DO IST=1,MAXI
IF(TIME+DTIME.LE.PERDIV(IST))THEN
DSTRAN(IST)=EPSB(IST,IBL)*DTIME/PERDIV(IST)
ELSE
XNU= EPSB(IST,IBL)-EPSB(IST,IBL)
DSTRAN(IST)=ISLOPE(IST)*XNU*2.*DTIME/TPER
END IF
END DO
CONVER='y'
DO IST=1,NSTATV
STATEV(IST)=STATEV(IST)
END DO
SIGS=SYIEL(IBL)

```

```

EEL=SIGS/1000.
EPSPC=SIGS/10/10000.
IF (SIGS.LT.SIGF.OR.SIGS.GT.SIGU) STOP*** PLEASE REVIEW THE
VALUES OF SIGF, SIGU and SIGS.
CALL INTEG (STRAN, DSTRAN, WTEMS, NSTATV, UNLAXI, COUPL,
CONVER, STRES, STATEV, STRSS, STARS, RNU)
C*** IF NO CONVERGENCE, DIVIDE TIME INC. BY 2
IF ((ANSDC.EQ.'N'.OR.ANSDC.EQ.'N') .AND.
STATEV(11).GT.0.)
DCRIT=DIC*((SIGU/SIGS)**2)/RNU
IF (CONVER.EQ.'N') THEN
IPASS=IPASS+1
UTIME=UTIME/2.
C*** IF CONVERGENCE
ELSE
IF ((ANSDC.EQ.'N'.OR.ANSDC.EQ.'N')
.AND.DCRIT.GT.0.99DO) DCRIT=0.99DO
IPASS=0
DO IST=1, NSTATV
STATEVI(IST)=STATEV(IST)
END DO
DO IST=1, 6
STRAN(IST)=STRAN(IST)+DSTRAN(IST)
END DO
TIME=TIME+UTIME
KTIME=TIME+DTIME
CALL OUTPUT (NUL, RUT, KTIME, STRAN, STRES, STATEV, WTEMS, NSTATV,
STRSS, STARS)
IF (STATEV(3).GE.DCRIT) THEN
WRITE (MS, *) ' '
WRITE (MS, *) '*** CRACK INITIATION.'
GO TO 308
END IF
IF (STATEV(3).GE.1.1) STOP*** DAMAGE CANNOT EXCEED 1.
DO IST=1, 6
IF (DABS (TIME-IPER (IST))*PERDIV (IST)).LE.DDGN) THEN
IPER (IST)=IPER (IST)+1
UTIME=1./DDGUSE (IBL)
END IF
IF (ISLOPE (IST).GT.0.AND.DABS (STRAN (IST)-EPSS (IST, IBL)).LE.
EPSPC.OR.ISLOPE (IST).LT.0.AND.DABS (STRAN (IST)-EPSS (IST, IBL))
.LE.EPSPCN) THEN
ISLOPE (IST)=ISLOPE (IST)
END IF
END DO
C*** END OF A CYCLE
IF (DABS (TIME-TPER)*ICICLE).LE.DDGN) THEN
IDELT=0
DO I=1, 6
SIGFI (I)=STRES (I)
END DO
SOPF=STATEV (2)
DHF=STATEV (3)
DELTP=SOPF-SOP1
DELTD=DHF-DHI
C*** FIND PD
IF (FLOOK0.EQ.0) THEN
ENERG3=ENERG0+(SIGS-SIGF*(SIGF/SIGY))*
DELTP*(NRCYCL-NCYSUM)
IF ((ANSPO.EQ.'N'.OR.ANSPO.EQ.'N')
.AND.ENERG3.GE.ENERG1) THEN
SOPD=SOPF
NRCYCL=NRCYCL-NCYSUM
FLOOK0=1
ELSE IF ((ANSPO.EQ.'Y'.OR.ANSPO.EQ.'Y').AND.
STATEV (2).GE.SOPD) THEN
NRCYCL=NRCYCL-NCYSUM
FLOOK0=1
END IF
END IF
C*** JUMP IN CYCLES PROCEDURE
IF ((JUMP.EQ.'Y').OR.(JUMP.EQ.'Y')) THEN
WRITE (80, *) 'END OF THE CYCLE'
WRITE (80, 400) RTIME, TIME, DHF, SOPF
DPMAX=DCRIT/50.
IF ((ANSDC.EQ.'N').OR.(ANSDC.EQ.'N'))
DPMAX=DIC/50.
IF ((UNLAXI.EQ.'Y').OR.(UNLAXI.EQ.'Y')) THEN
YYY=SIGS*SIGS/2./E0
ELSE
TRADEPS=0.
DO IST=1, 3
TRADEPS=TRADEPS+EPSS (IST, IBL)-EPSS (IST, IBL)
END DO

```

```

YTL=(1.+XNU)*SIGS*SIGS/3./E0
YT2=E0*TRADEPS*TRADEPS/(1.-2.*XNU)/6.
YTL=YTL+YT2
END IF
DPMAX=50.*DPMAX/YYY
DPMAX=DPMAX
PSLOPE=NRCYCL*(IBL)
IF (DPMAX.GT.PSLOPE) DPMAX=PSLOPE
WRITE (80, *) 'FLOOK0=', FLOOK0, 'NRCYCL=', NRCYCL, 'PD=', SOPD
IF (COUPL.EQ.'N'.OR.STATEV (2).LT.SOPD) THEN
STAB='Y'
DO KK=1, 6
IF (DABS (SIGFI (KK)-SIGFI (KK)).GT.DHI) STAB='D'
END DO
WRITE (80, *) 'ENERG1=', ENERG1, 'ENERG3=', ENERG3
IF (STAB.EQ.'Y') THEN
WRITE (80, *) 'STABILIZED CYCLE'
IF (STATEV (2).LT.SOPD) THEN
C*** JUMP OF CYCLES BEFORE DAMAGE GROWTH
IDELT=IDELT+(DPMAX/DELTP)
IF (NRCYCL+IDELT.GT.NCYSUM+NRCYCL (IBL))
IDELT=NCYSUM+NRCYCL (IBL)-NRCYCL
ENERG2=ENERG0+(SIGS-SIGF)*DELTP*(NRCYCL+IDELT-NCYSUM)
IF (ENERG2.GT.ENERG1*1.05)
IDELT=(ENERG1-ENERG3)/(SIGS-SIGF)/DELTP
WRITE (80, *) '*** JUMP OF 'IDELT, ' CYCLES'
WRITE (80, *) 'DPMAX=', IDELT*DELTP, 'YTL=', YTL
ELSE
WRITE (80, *) 'COUPLED COMPUTATION FOR NEXT CYCLE'
COUPL='Y'
END IF
END IF
ELSE
C*** JUMP OF CYCLES AFTER DAMAGE GROWTH
DPMAX=DPMAX
DSLOPE=(NRCYCL (IBL)-NRCYCL0)*DELTD
IF (DPMAX.GT.DSLOPE) DPMAX=DSLOPE
NRCYCL0=0
IDELT=IDELT+(DPMAX/DELTD)
IF (NRCYCL+IDELT.GT.NCYSUM+NRCYCL (IBL))
IDELT=NCYSUM+NRCYCL (IBL)-NRCYCL
IF (DHF+DELTD*IDELT.GT.DCRIT*1.05)
IDELT=IDELT+((DCRIT-DHF)/DELTD)
IF (IDELT.GT.IDELT (DPMAX/DELTP)) IDELT=IDELT (DPMAX/DELTP)
WRITE (80, *) '*** JUMP OF 'IDELT, ' CYCLES'
WRITE (80, *) 'DPMAX=', IDELT*DELTD, 'DPMAX=', IDELT*DELTD
COUPL='N'
END IF
IF (DELTP.GT.DPMAX) WRITE (80, *) '*** PROBLEM :
DPMAX IS TOO SMALL.'
IF (DELTD.GT.DPMAX) WRITE (80, *) '*** PROBLEM :
DPMAX IS TOO SMALL.'
IF (DELTD.GT.DPMAX.OR.DELTP.GT.DPMAX) IDELT=0
IF (IDELT.LT.0.) STOP*** PROBLEM : NEGATIVE JUMP OF CYCLES.
END IF
DO J=1, 6
SIGFI (J)=SIGFI (J)
END DO
SOP1=SOPF+DELTP*IDELT
DHI=DHF+DELTD*IDELT
STATEVI (2)=SOP1
STATEVI (3)=DHI
DXTIME=DXTIME+TPER*IDELT
NRCYCL=NRCYCL+IDELT
KTIME=TIME+DXTIME
IF ((JUMP.EQ.'Y').OR.(JUMP.EQ.'Y')) THEN
WRITE (80, *) ' '
WRITE (80, *) 'BEGINNING OF THE CYCLE'
WRITE (80, *) 'CYCLE (REAL) =', NRCYCL+1
WRITE (80, *) 'CYCLE (MACHINE) =', ICYCLE+1
WRITE (80, 400) RTIME, TIME, DHI, SOP1
END IF
IF (NRCYCL.EQ.NCYSUM+NRCYCL (IBL)) THEN
ENERG0=ENERG0+(SIGS-SIGF*(SIGF/SIGY))*DELTP*(NRCYCL-NCYSUM)
NCYSUM=NRCYCL+NRCYCL (IBL)
IBL=IBL+1
END IF
ICYCLE=ICYCLE+1
NRCYCL=NRCYCL+1
END IF
FORMAT (1X, 4 (E12.6, ' '))
END IF
DO IST=1, 6
IF (TIME+DTIME.GT.IPER (IST))*PERDIV (IST)+DDGN) THEN

```

```

      DTIME=(IPER(IST)*PERRDIV(IST))-TIME
    END IF
  END DO
  IF (IPASS.EQ.1).OR.(DTIME.LT.DTIMEC) THEN
    WRITE(MS,*)'*** NO CONVERGENCE'
    WRITE(MS,*)'DTIME=',DTIME,'IPASS=',IPASS
    GO TO 308
  END IF
  END DO
  END IF
308 WRITE(MS,*)'*** THE JOB IS ENDED. YOUR RESULTS FILES ARE : '
  WRITE(MS,*)' '
  IF ((JUMP.EQ.'n').OR.(JUMP.EQ.'N')) WRITE(MS,*)FILE1,FILE2
  IF ((JUMP.EQ.'y').OR.(JUMP.EQ.'Y'))
    .WRITE(MS,*)FILE1,FILE2,'fatigue.out'
  STOP
  END
C.....
SUBROUTINE INTEGR(STRAN,DSTRAN,STRES,ESTASV,INLAXI,COUPL,
&
CONVER,STRS,STATEV,STRSB,STAR,NTJ)
C.....
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 STATEV(NTJ*7),STRAN(NTJ*5),DSTRAN(NTJ*5),
&
STRN(6),TD(6,6),AS(6),
&
STRS(6),ESTRS(6),ESTRS0(6),CESTRS(6),KESTRS(6),
&
N(6),PSTRN(6),DPSTRN(6)
REAL*8 LAMDA,NU
LOGICAL CRITERE
CHARACTER*1 CONVER,COUPL,UNLAXI
COMMON/UTQ1/EO,ENU,SIGS,SDPD,SO
COMMON/UTQ2/CNE,TWO,THREE
C
KN=COUPL/TWO
RAC2=SDPD/(TWO)
TOLRES=SIGS*1.D-6
NMISS=NTJ*5
C*** IDENTITY MATRIX
DO I=1,NMISS-1
  DO J=I+1,NMISS
    ID(I,J)=0.
    ID(J,I)=0.
  END DO
  ID(I,I)=ONE
END DO
ID(NMISS,NMISS)=CNE
C*** IDENTITY VECTOR
DO I=1,NMISS
  IF (I.LE.3) AS(I)=CNE
  IF (I.GE.4) AS(I)=0.
END DO
C*** LAME COEFF.
KX=EO/(CNE+ENU)/TWO
LAMDA=EO*ENU/(CNE+TWO*ENU)/(CNE+ENU)
C
SDP=STATEV(1)
SDP=STATEV(2)
D=STATEV(3)
DO I=1,NMISS
  PSTRN(I)=STATEV(I+4)
END DO
C*** TOTAL STRAIN
DO I=1,NMISS
  STRN(I)=STRAN(I)+CSTRAN(I)
  IF (I.GE.4) STRN(I)=STRN(I)*RAC2
END DO
C*** ELASTIC PREDICTOR
DENOM=0.
IF ((UNLAXI.EQ.'y').OR.(UNLAXI.EQ.'Y')) THEN
  STRN(2)=XNU*STRN(1)-(0.5-XNU)*PSTRN(1)
  STRN(3)=STRN(2)
END IF
C....Trace of strain tensor
TRPS=STRN(1)+STRN(2)+STRN(3)
C....Effective stress
DO I=1,NMISS
  ESTAS(I)=LAMDA*TRPS*AS(I)+2.*XNU*(STRN(I)-PSTRN(I))
END DO
CALL INVAR(ESTAS,ESTRSH,ESTRSB,NMISS,
&
SIGN=1.
IF (ESTRS(1).LT.0.) SIGN=-1.
C*** TEST THE FIELD CONDITION
P=ESTRSB-SIGS
IF (P.LE.0.) THEN
  GO TO 300

```

```

ELSE
C*** IF THE ELASTIC PREDICTOR DOES NOT VERIFY THE YIELD CONDITION
DO I=1,NMISS
  N(I)=KN*(ESTRS(I)-ESTRSH*AS(I))/ESTRSB
END DO
C*** Corrections over the elastic predictor
C....Correction over p
DENOM=1.*XNU
CSNP=P/DENOM
C....Correction over effective stress
DO I=1,NMISS
  CESTRS(I)=2.*P*AS(I)/3.
END DO
C*** LOOP ON THE PLASTIC CORRECTIONS
NFOIS=0
CRITERE=.TRUE.
DO WHILE(CRITERE)
  NFOIS=NFOIS+1
  DIV1=ESTRSB
  NTEST=2
  IF (NFOIS.EQ.1) NTEST=1
C*** UPDATE THE STATE
C....Update inc. of p
DSNP=DSNP+CSNP
C....Update effective stress
DO I=1,NMISS
  ESTRS(I)=ESTRS(I)+CESTRS(I)
END DO
CALL INVAR(ESTRS,ESTRSH,ESTRSB,NMISS)
DO I=1,NMISS
  N(I)=KN*(ESTRS(I)-ESTRSH*AS(I))/ESTRSB
END DO
C
  IF (NTEST.EQ.2) THEN
    DIV2=ESTRSB
    REL=(DIV2-DIV1)/DIV1
  END IF
C....If we converge too slowly, or we diverge, exit. The main
C....program will propose a smaller "time" increment.
  IF (NFOIS.EQ.50).OR.
&
  (NTEST.EQ.2.AND.REL.GE.15.D-02)) THEN
    CONVER='n'
    GO TO 300
  END IF
C*** Compute residual functions
P=ESTRSB-SIGS
IF ((UNLAXI.EQ.'y').OR.(UNLAXI.EQ.'Y')) THEN
  STRN(2)=0.5*STRN(1)+(0.5-XNU)*SIGN*SIGS/EO
  STRN(3)=STRN(2)
  TRPS=STRN(1)+STRN(2)+STRN(3)
END IF
DO I=1,NMISS
  KESTRS(I)=ESTRS(I)-LAMDA*TRPS*AS(I)-
&
  2.*XNU*(STRN(I)-PSTRN(I)-DSNP*AS(I))
END DO
C....Max. residuals
RESTRS=0.
DO I=1,NMISS
  IF (DABS(KESTRS(I)).GT.RESTRS) RESTRS=DABS(KESTRS(I))
END DO
C*** ITERATIVE TEST ON THE YIELD CONDITION
IF (DABS(P).LT.TOLRES.AND.RESTRS.LT.TOLRES) THEN
  CRITERE=.FALSE.
ELSE
C*** PLASTIC CORRECTIONS
CALL VIRAVI(N,KESTRS,XKES,NMISS)
C....Correction over p
DENOM=1.*XNU
CSNP=(P-XKES)/DENOM
C....Correction over effective stress
DENOM=DENOM+DSNP/ESTRSB*(1.)
DO I=1,NMISS
  CESTRS(I)=(2./3.)*(XKES-DENOM*P)*N(I)-KESTRS(I)
  CESTRS(I)=CESTRS(I)/DENOM
END DO
C*** END OF THE ITERATIVE TEST ON THE YIELD CONDITION
END IF
C*** END OF THE LOOP ON THE PLASTIC CORRECTIONS
END DO
C*** END OF TEST ON THE YIELD CONDITION
END IF
300 CONTINUE
IF (CONVER.EQ.'y') THEN
  IF ((UNLAXI.EQ.'y').OR.(UNLAXI.EQ.'Y')) THEN

```

```

        DSTRAN(2)=STRN(2)-STRAN(2)
        DSTRAN(3)=DSTRAN(2)
    END IF
C....Plastic strain increment
    DO I=1,NLSS
        DPSTRN(I)=H(I)*DSMP
    END DO
C....Damage increment
    DD=0.
    CALL DAMAGE(TREPS,ESTRSB,Y,RNU)
    IF(DSMP.GT.0. .AND. SMP>DSMP.GE.SMPD.AND.
        COUPL.DQ.'Y') THEN
        DD=Y*DSMP/50
    END IF
    STAR=(ONE-D-DD)*DSQR(2.*E0*Y)
C....Plastic multiplier increment
    DSMP=(ONE-D-DD)*DSMP
C....Compute stresses
    DO I=1,NLSS
        STAS(I)=ESTRS(I)*(ONE-D-DD)
    END DO
    STRSB=ESTRSB*(ONE-D-DD)
C*** STORE STATEV AT THE END OF THE INCREMENT
    STATEV(1)=SMP+DSMP
    STATEV(2)=SMP+DSMP
    STATEV(3)=D+DD
    STATEV(4)=DD
    DO I=1,NLSS
        STATEV(I+4)=PSTRN(I)+DPSTRN(I)
    END DO
    STATEV(11)=DSMP
END IF
RETURN
END
C.....
SUBROUTINE INVAR(V,TN,TB,NLSS)
C.....
C... 1st and 2nd stress invariants
    IMPLICIT REAL*8(A-H,O-Z)
    REAL*8 V(6)
    COMEN/TTIG1/CNE,TWO,TREE
C
    VM=(V(1)+V(2)+V(3))/THREE
    CONST=0.
    DO I=1,NLSS
        IF(I.LE.3) CONST=CONST+(V(I)-VM)*(V(I)-VM)
        IF(I.GE.4) CONST=CONST+V(I)*V(I)
    END DO
    VB=DSQR(THREE*CONST/TWO)
    RETURN
END
C.....
SUBROUTINE DAMAGE(TREPS,ESTRSB,Y,RNU)
C.....
    IMPLICIT REAL*8(A-H,O-Z)
    COMEN/TTIG1/EO,XNU,SIGS,SMPD,50
C
    TERM1=ESTRSB*ESTRSB*(1.+XNU)/EO
    TERM2=EO*TREPS*TREPS/(1.-2.*XNU)/2.
    Y=(TERM1+TERM2)/3.
    RNU=2.*E0*Y/ESTRSB/ESTRSB
    RETURN
END
C.....
SUBROUTINE VTRM(V,T,VTV1,NLSS)
C.....
C... Inner product of 2 symmetric 2nd order tensors
    IMPLICIT REAL*8 (A-H,O-Z)
    REAL*8 V(6),VT(6)
C
    VTV1=0.
    DO I=1,NLSS
        VTV1=VTV1+V(I)*VT(I)
    END DO
    RETURN
END
C.....
SUBROUTINE OUTPUT(NU1,NU2,TIME,STRAN,STAS,STATEV,
    & NTENS,NSTATEV,STRSB,STAR)
C.....
    IMPLICIT REAL*8(A-H,O-Z)
    REAL*8 STRAN(NTENS),STAS(NTENS),STATEV(NSTATEV)
C
    WRITE(NU1,130)TIME,(STRAN(I),I=1,3),(STAS(I),I=1,3),

```

```

        STATEV(3),STATEV(2),STRSB,STAR
    WRITE(NU2,130)TIME,(STRAN(I),I=4,6),(STAS(I),I=4,6),
    & STATEV(3),STATEV(2),STRSB,STAR
    WRITE(90,444)TIME,STATEV(3),STATEV(2),STRSB,STAR
130 FORMAT(1X,E10.4,' ',',',3(E10.4,' ','),',3X,3(E10.4,' ','),
    & ' ',',',3(E10.4,' ','),',2(E10.4,' ','),
    & ' ',',',3X,E10.4,3X,E10.4,3X,E10.4,3X,E10.4,4X,E10.4)
444 FORMAT(3X,E10.4,5X,E10.4,3X,E10.4,3X,E10.4,3X,E10.4,4X,E10.4)
    RETURN
END

```

**5. APPLICATION TO THE ALPHA TWO TITANIUM
ALUMINIDE ALLOY**

The material was provide by AFOSR from the Materials Laboratory WRDC/MLLN. Wright Patterson AFB in order to apply the micromechanics model to a practical case interesting for applications.

5.1. Material

- Composition (Ti3 Al) Ti -24 Al- 11 Nb

Ti	Al	Nb	Fe	O	N
bal	13.9	21.7	0.073	0.065	0.012

- Heat treatment

Solution 2100°F/1HR/VACUM Forced fan. Cool with argon

Aging 1400°F/1HR/VACUM forced fan air cool

- Mechanical properties.

Figure 20 gives the yield stress and the ultimate stress as a function of the tempe rature.

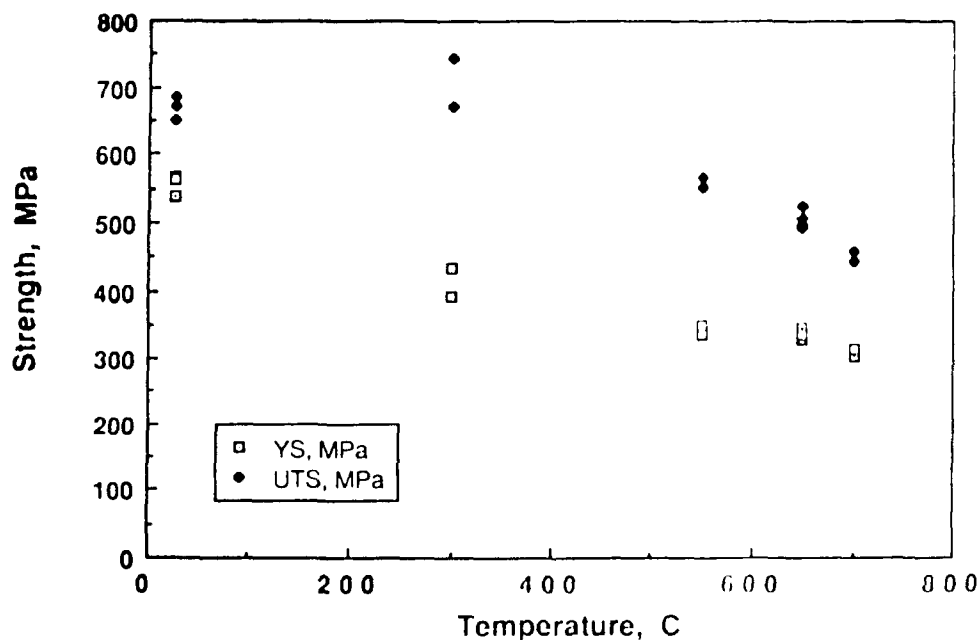


Fig. 20 Yield stress and ultimate stress of the Alpha-Two. Titanium Aluminide Alloy.

Figure 21 gives the Young Modulus also as a function of the temperature

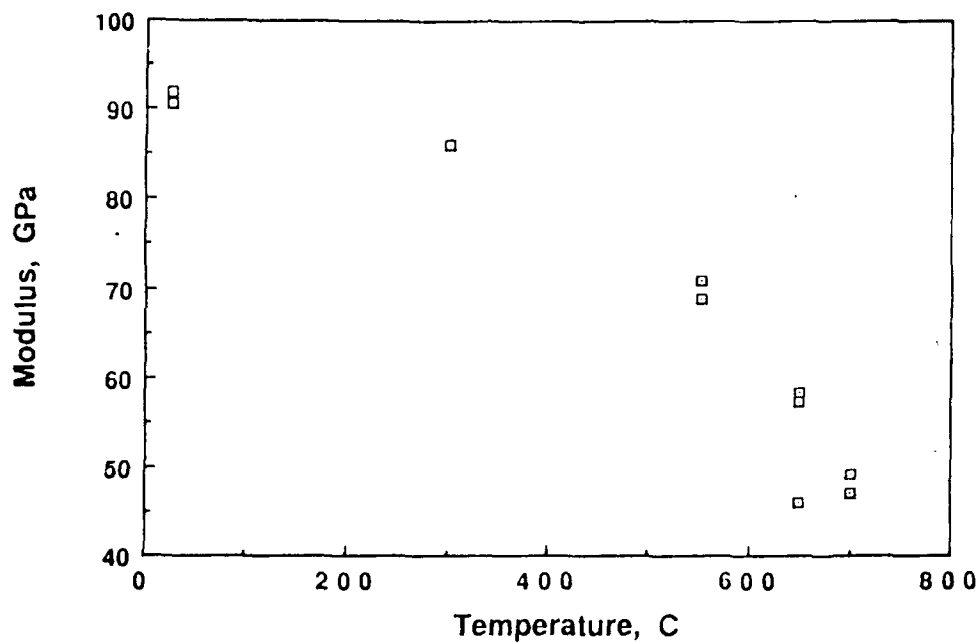


Fig. 21 Young Modulus of the Alpha-Two Titanium Alumide Alloy

5.2. Tensile specimens

The location of the tensile specimens as taken in the piece of material is indicated in figure 22.

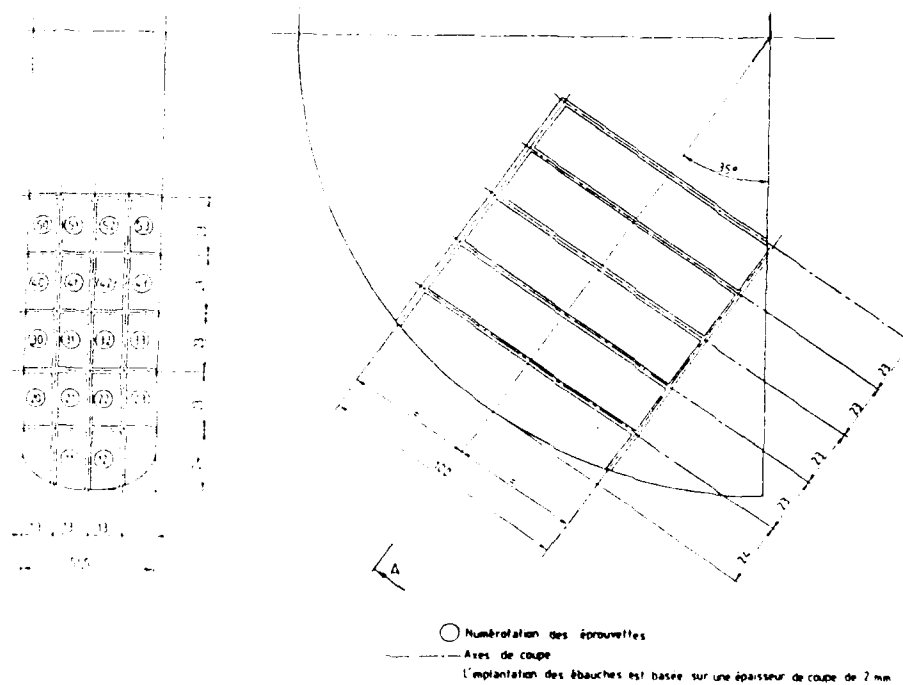
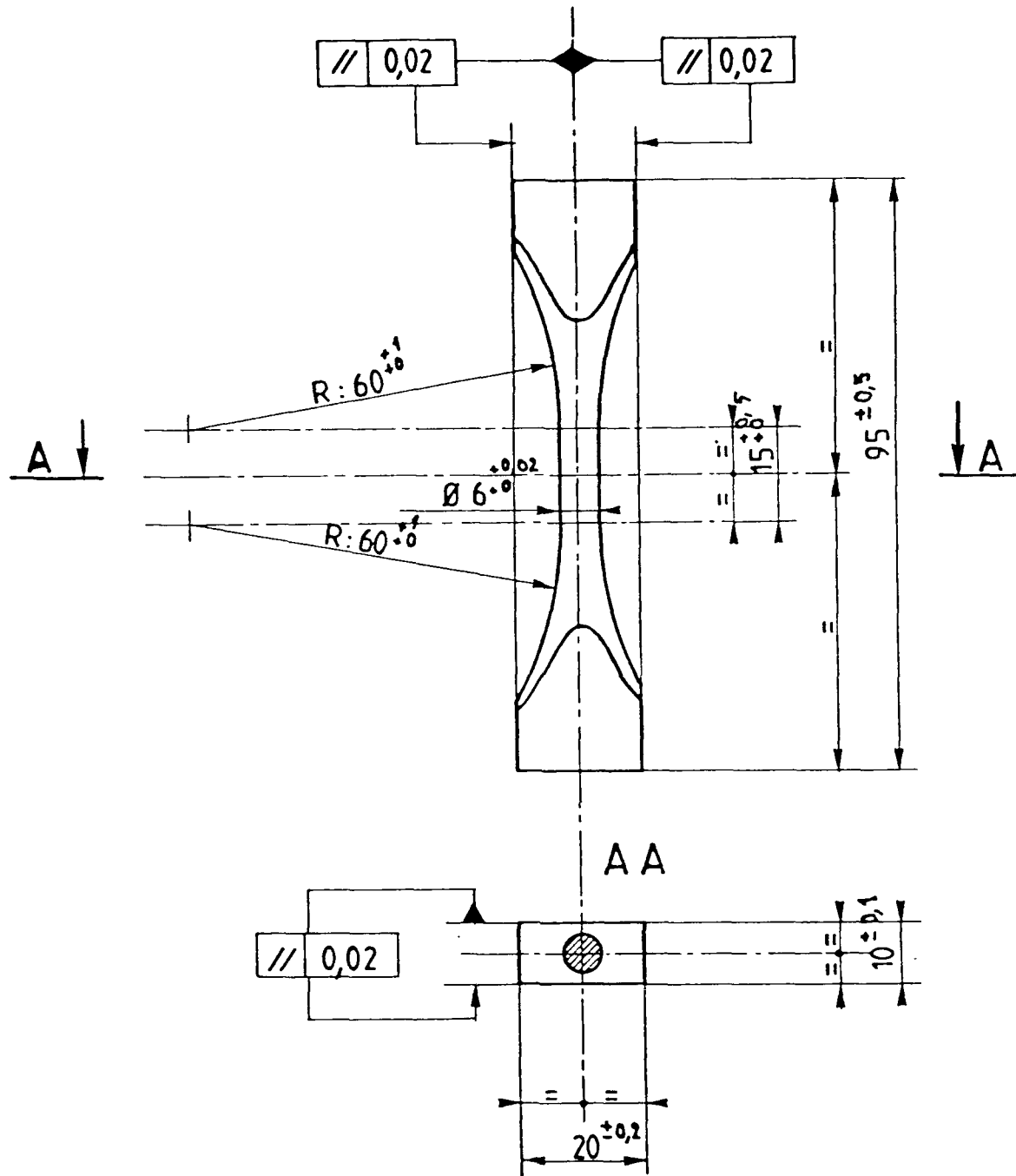


Fig. 22 Localization of the tensile specimens

The drawing of specimens is indicated in figure 23.

The drawing of specimens is indicated in figure 23 below



Matière :		
Février 90	Echelle : 1	DUFALLY
EPROUVETTE CYLINDRIQUE		

5.3. Identification tests

The 3 basic tests needed for the identification of the model are described in the following figures.

Test n°1 is a classical tensile test strain controlled $\dot{\epsilon} = 10^{-5} \text{ s}^{-1}$ on specimen n° 41

Young modulus	90 000 MPa
Failure stress	523
Failure strain	1.59 %

The figure 24 shows a typical non ductile type behavior without any softening prior to failure.

Test n°2 is a strain controlled cyclic test ($\dot{\epsilon} = 10^{-5} \text{ s}^{-1}$, $\epsilon = \pm 0.96 \%$) on specimen n° 40

Number of cycles to failure 52.

The figures 25 a, b, c and d show a large cyclic hardening without any softening due to damage.

Test n°3 is a cyclic test on specimen n° 20 ($\dot{\epsilon} = 10^{-5} \text{ s}^{-1}$, $\epsilon = \pm 1.35 \%$) for wich the failure occurs after a single cycle (Figure 26).

The conclusion of these tests from the point of view of damage is a typical brittle behavior for which the damage mechanics must be applied only through the locally coupled approach as explained in section 4 of this report.

The crack initiation occurs at microlevel and then propagate very fast to induce macroscopic failure

identification of the model

Using the DAMAGE 90 post processor in an iterative process of identification the following set of materials parameters has been obtained from the tree basic test results.

$$\dot{D} = \frac{\sigma_s^2}{2ES} \dot{p} \quad \text{if } p > p_D = \epsilon p_D \frac{\sigma_u - \sigma_f}{\sigma_s - \sigma_f}$$

$D = D_c \rightarrow$ crack initiation

$$E = 90\,000 \text{ MPa}$$

$$\nu = .3$$

$$S = 3$$

$$\epsilon p_D = 0.01$$

$$\sigma_f = 315 \text{ MPa}$$

$$\sigma_y = 320 \text{ MPa}$$

$$\sigma_u = 525 \text{ MPa}$$

$$D_c = 0,2$$

TEST N° I

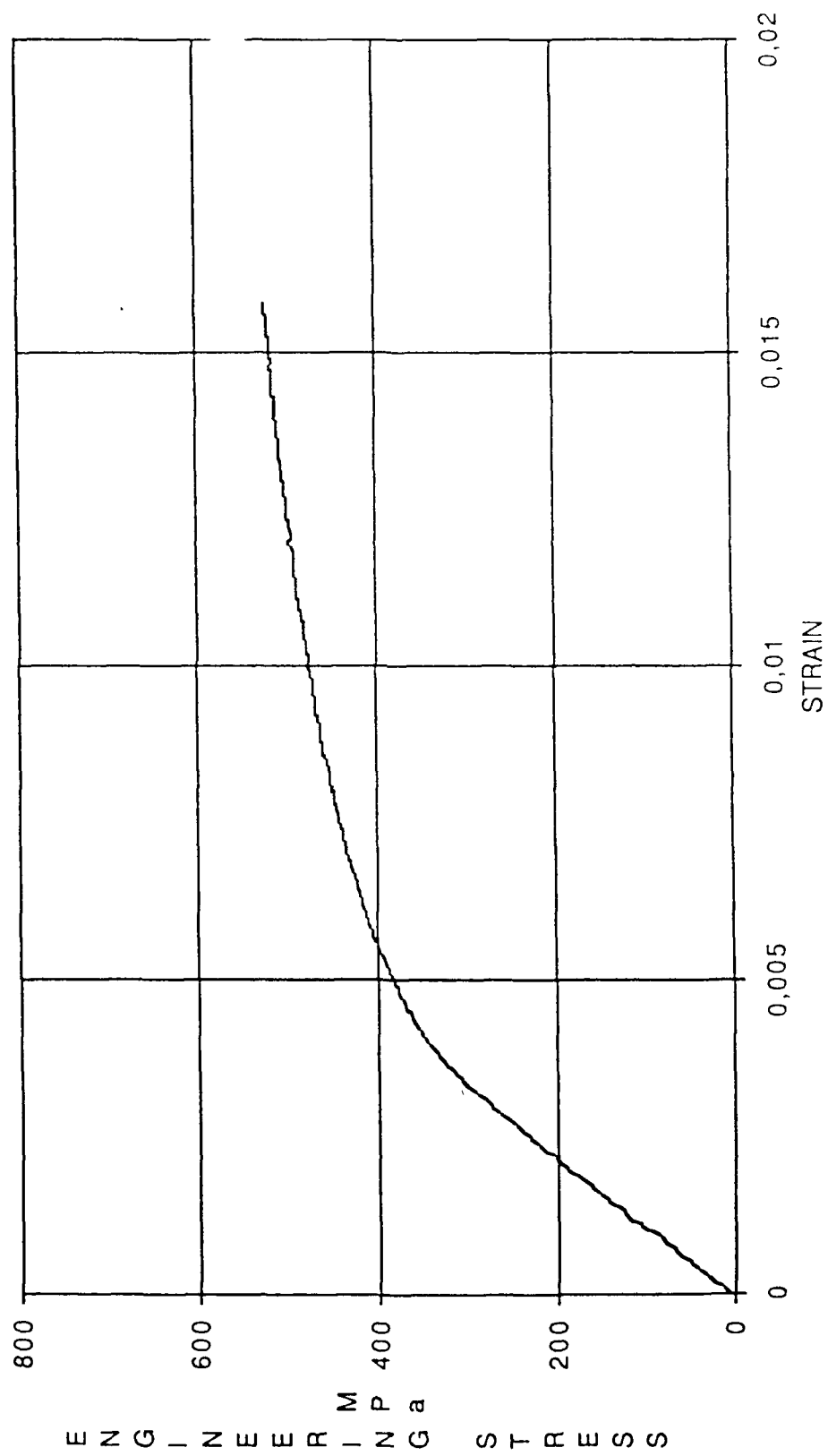


Figure 24

TEST N° 2 1st cycle

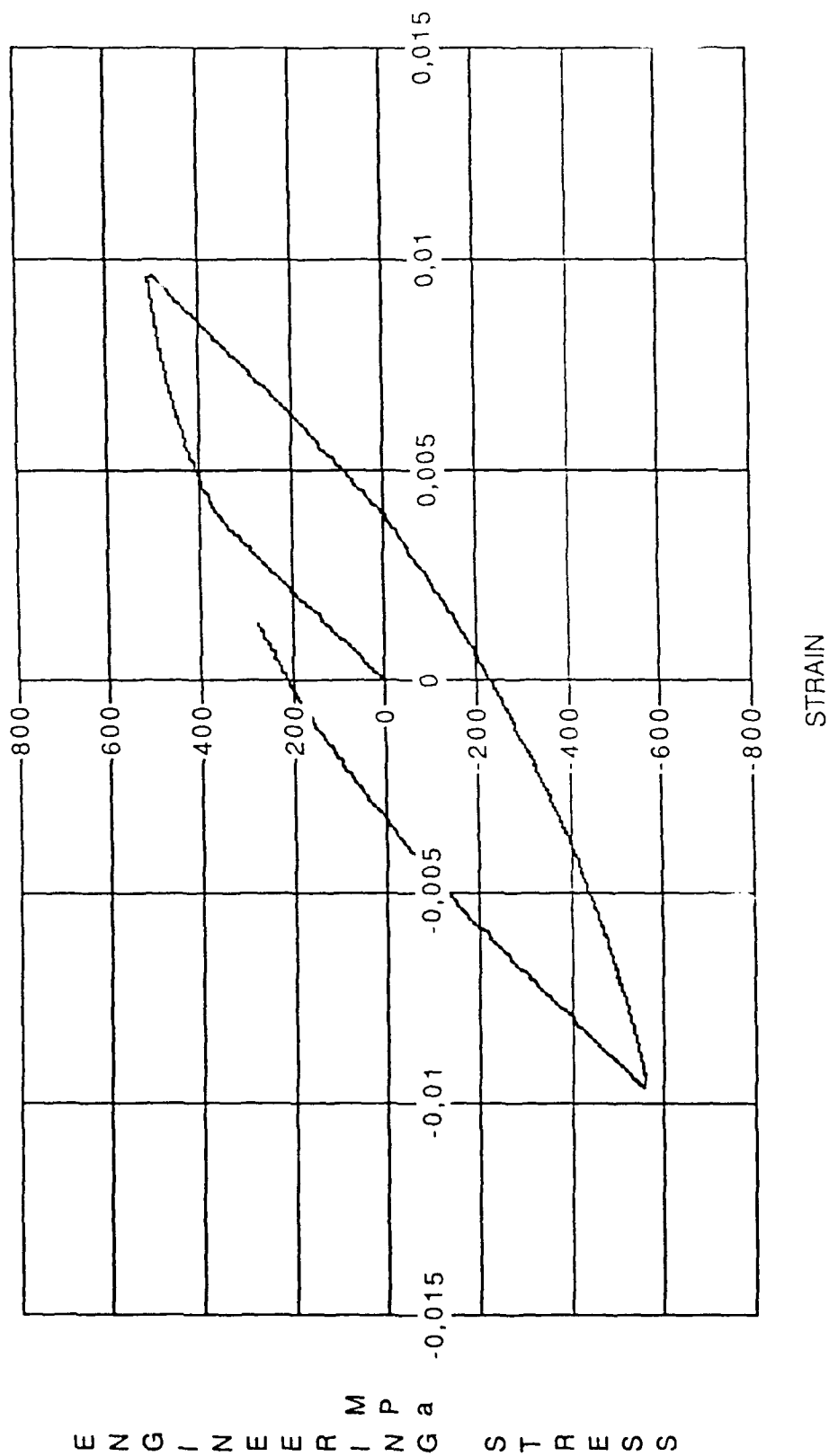


Figure 25 (a)

TEST N° 2 10th cycle

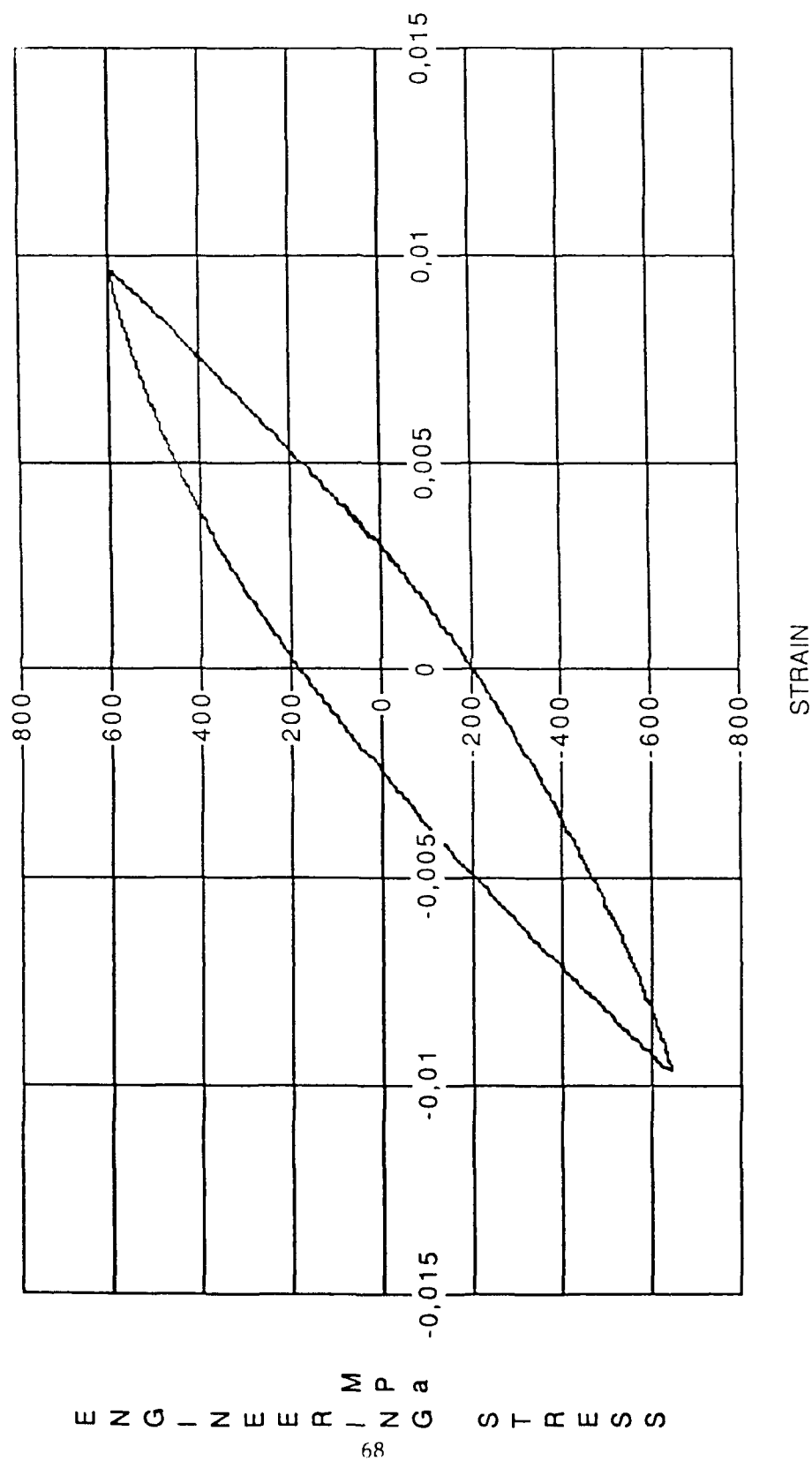


Figure 25 (b)

TEST N° 2 32th cycle

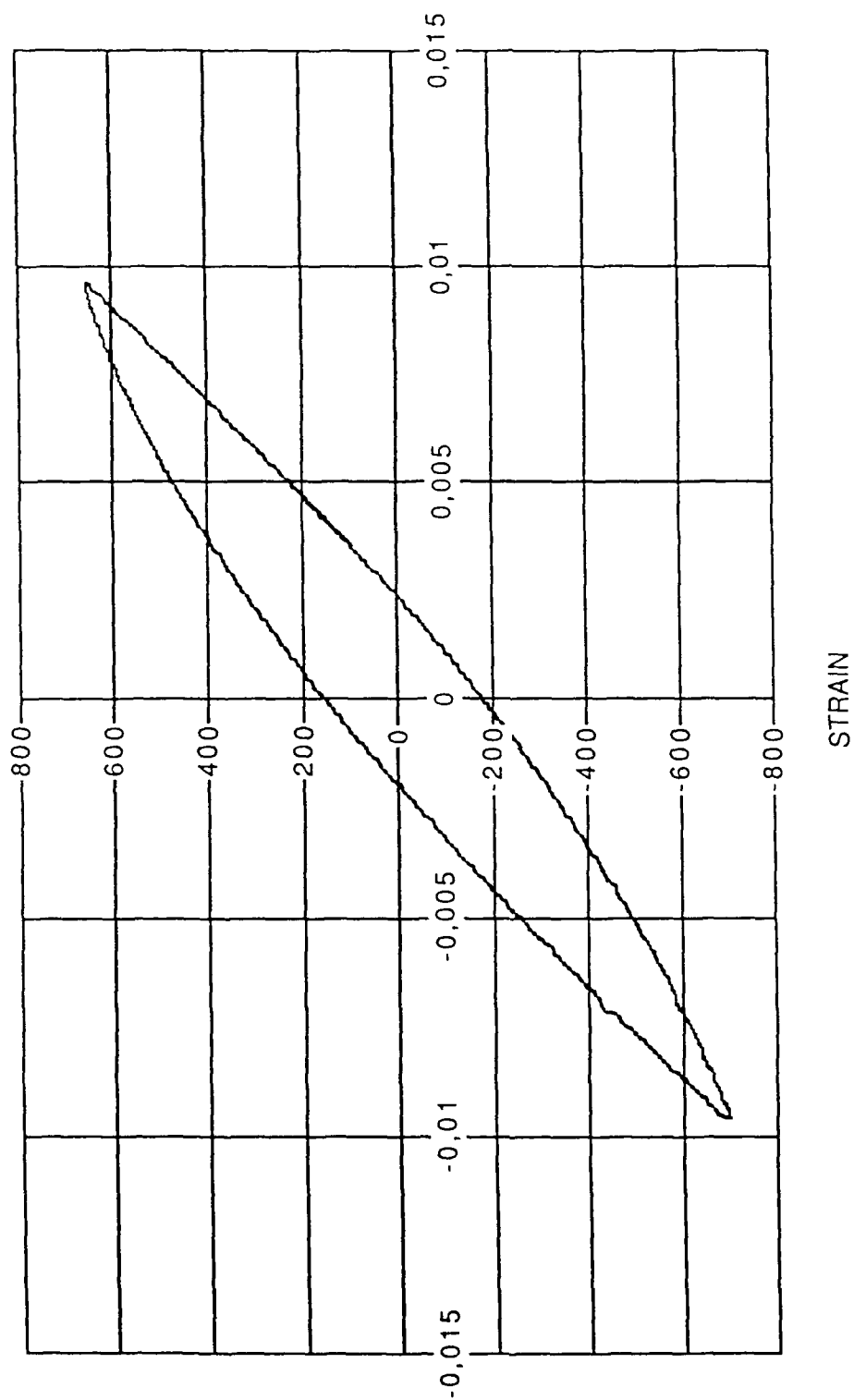


Figure 25 (c)

ENGINEERING STRESS

TEST N° 2 50th cycle

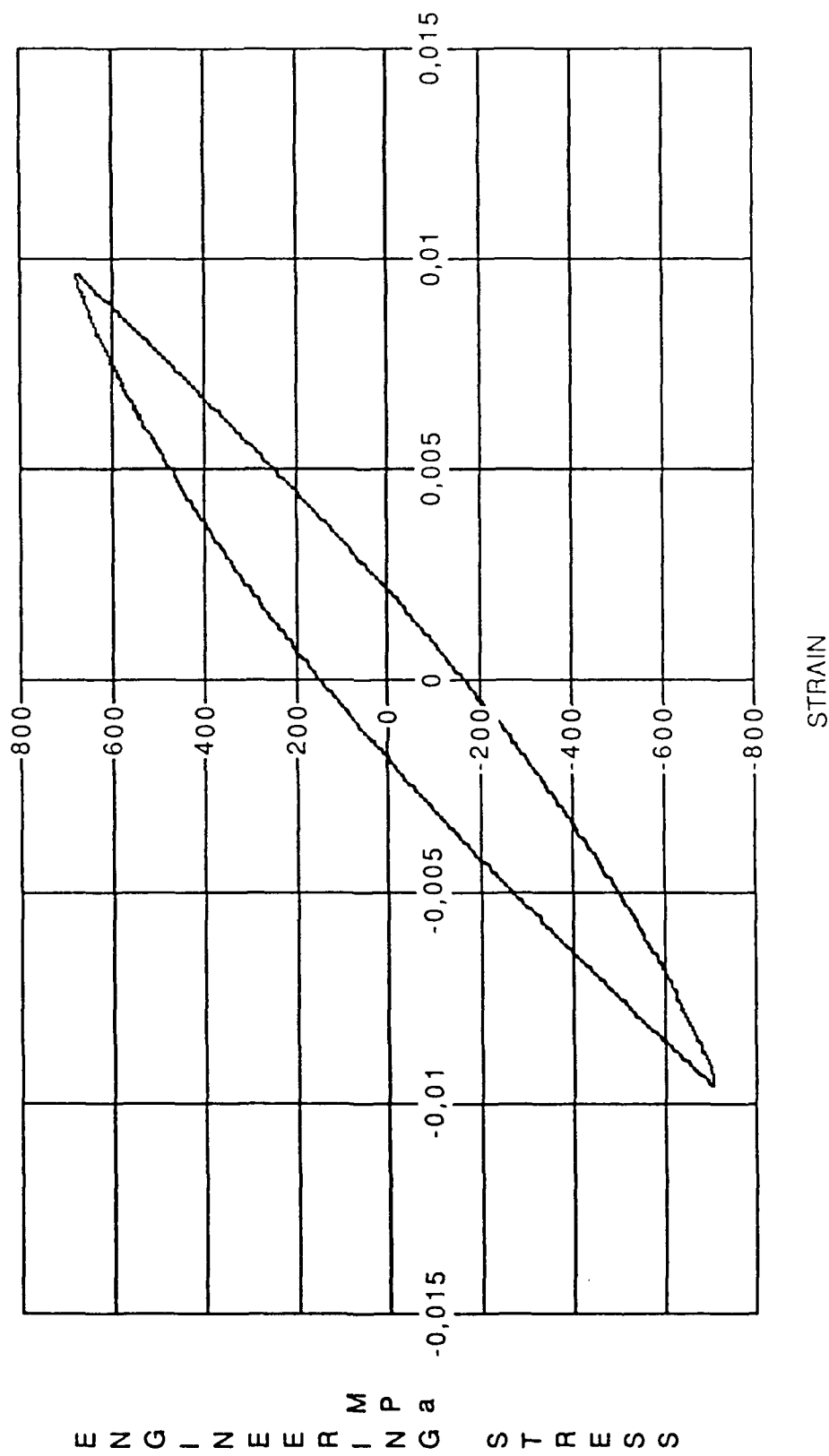


Figure 25 (d)

TEST N° 3

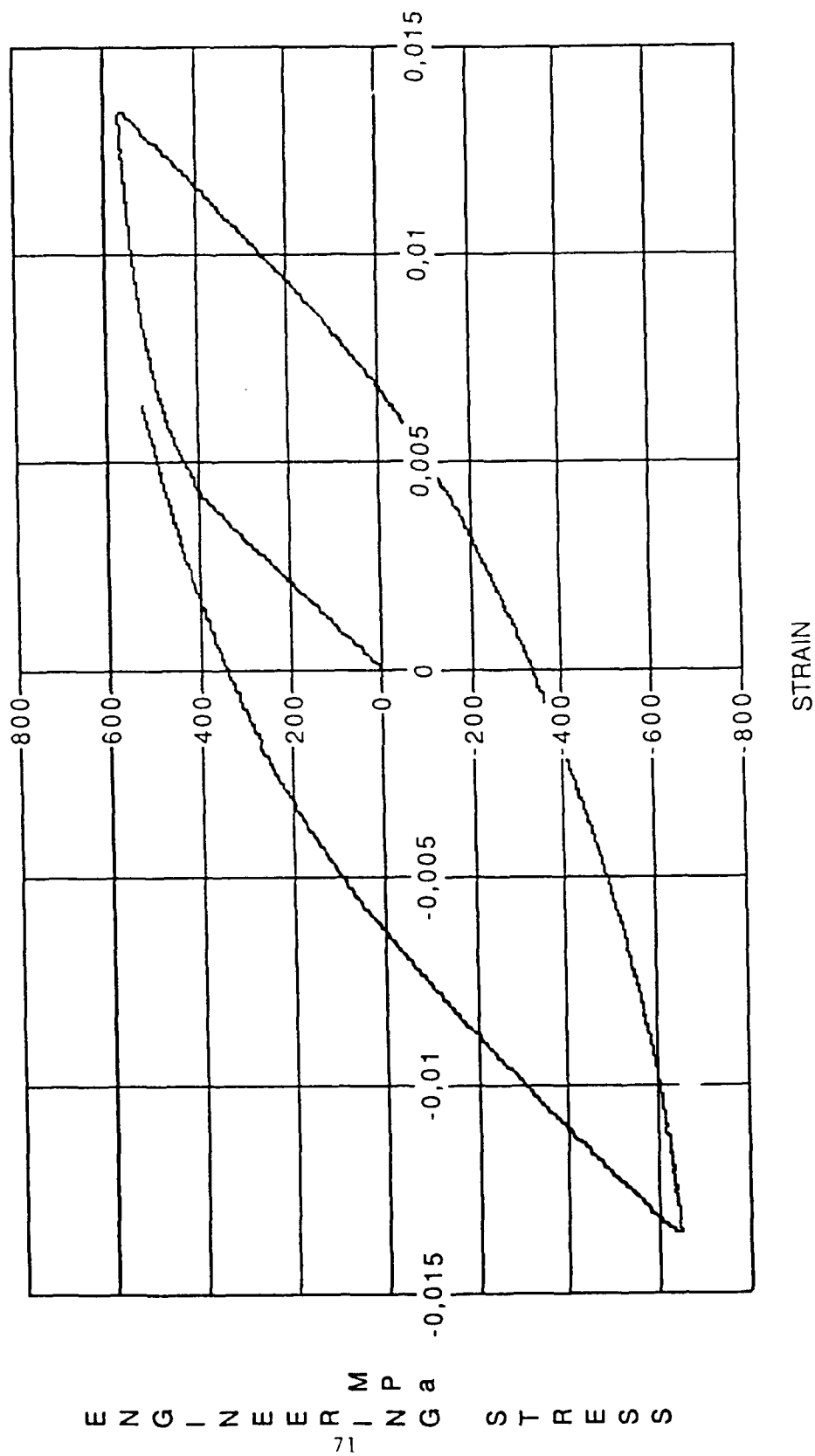


Figure 26

5.4. Vérification tests

In order to check the applicability of the model to the Alpha-Two Titanium Aluminide Alloy, some stress controlled tests have been performed.

The main features of these tests are reported in the table fig. 27 and the figures 28, 29, 30, 31, give the shapes of the stress-strain loops for different number of cycles.

Figures 32 indicates the type of brittle failure by pictures taken with an electron beam microscope.

Those stress controlled tests have been also calculated using the Post-Processor DAMAGE 90. The comparison between the calculated number of cycles to failure and the test results is given in the chart of figure 33.

The agreement is within in order of magnitude of the usual discrepancy.

test n°	4	5	6	7
controlled stress MPa	±424,4	±459,8	±502,2	±509,3
number of cycles to rupture	2400	1040	749	57
strain max %	0,52	0,61	1,04	1,33
strain min %	-0,49	-0,73	-0,6	-0,62
stabilized strain max %	0,5	0,54	0,74	0,96
stabilized strain min %	-0,38	-0,51	-0,38	-0,27
accumulated plastic strain at rupture Pr %	490	84,7	80,4	24,9

Figure 27 : Table of verification tests

Fig. 28 : Stress-controlled test: 424,4 MPa

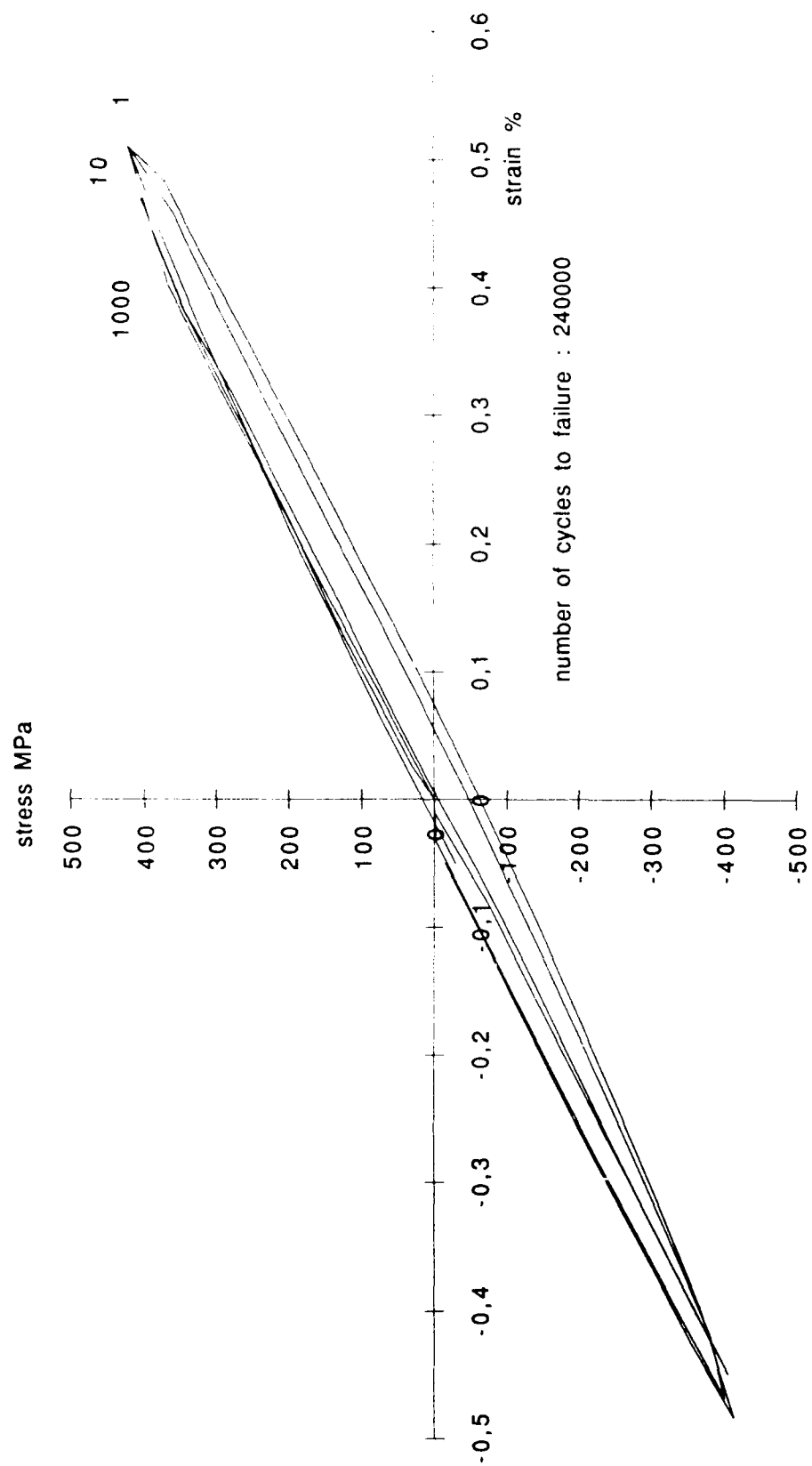
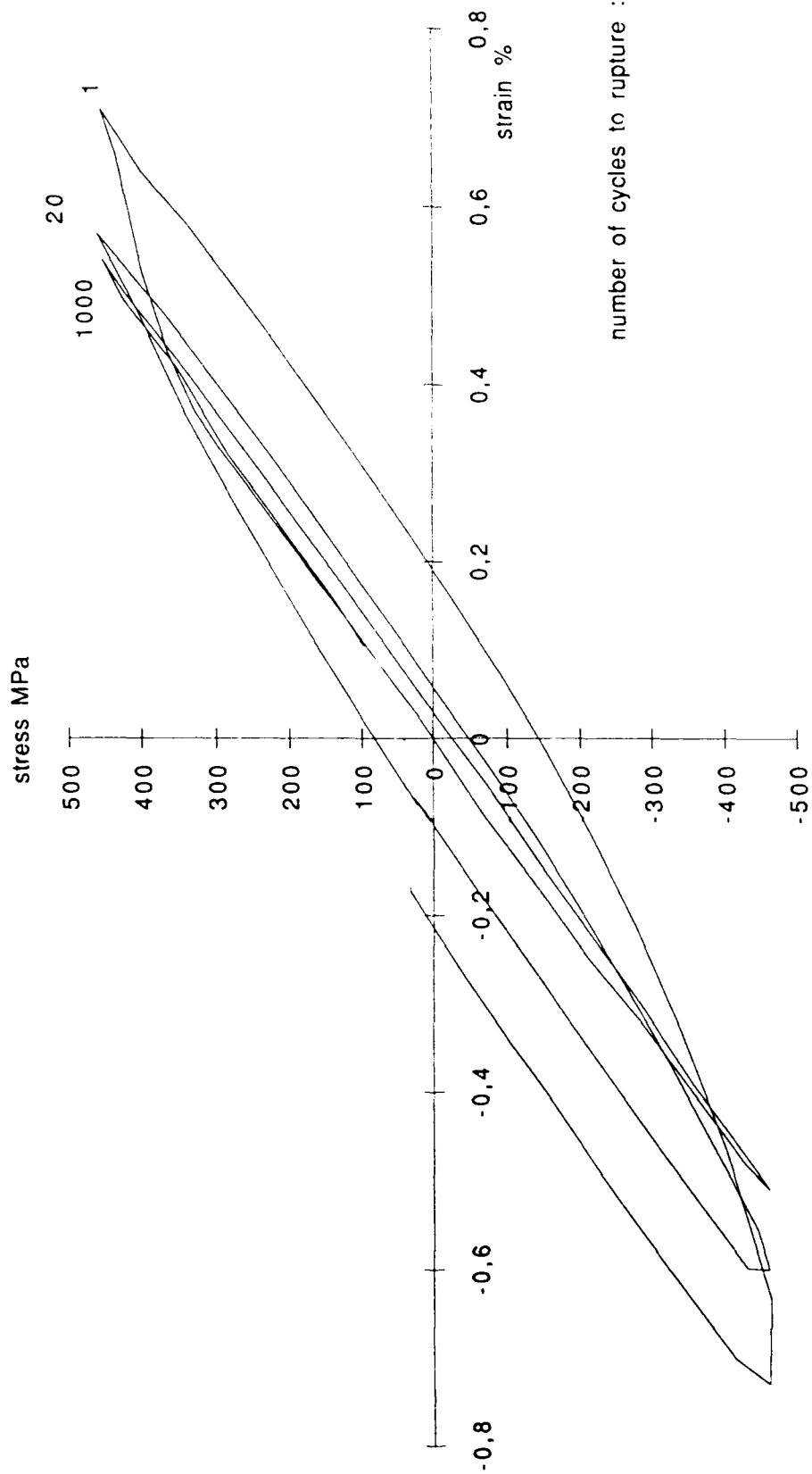


Fig. 29 : Stress-controlled test $\pm 459,8$ MPa



number of cycles to rupture : 1040

Fig. 30 : Stress-controlled test 502,2 MPa

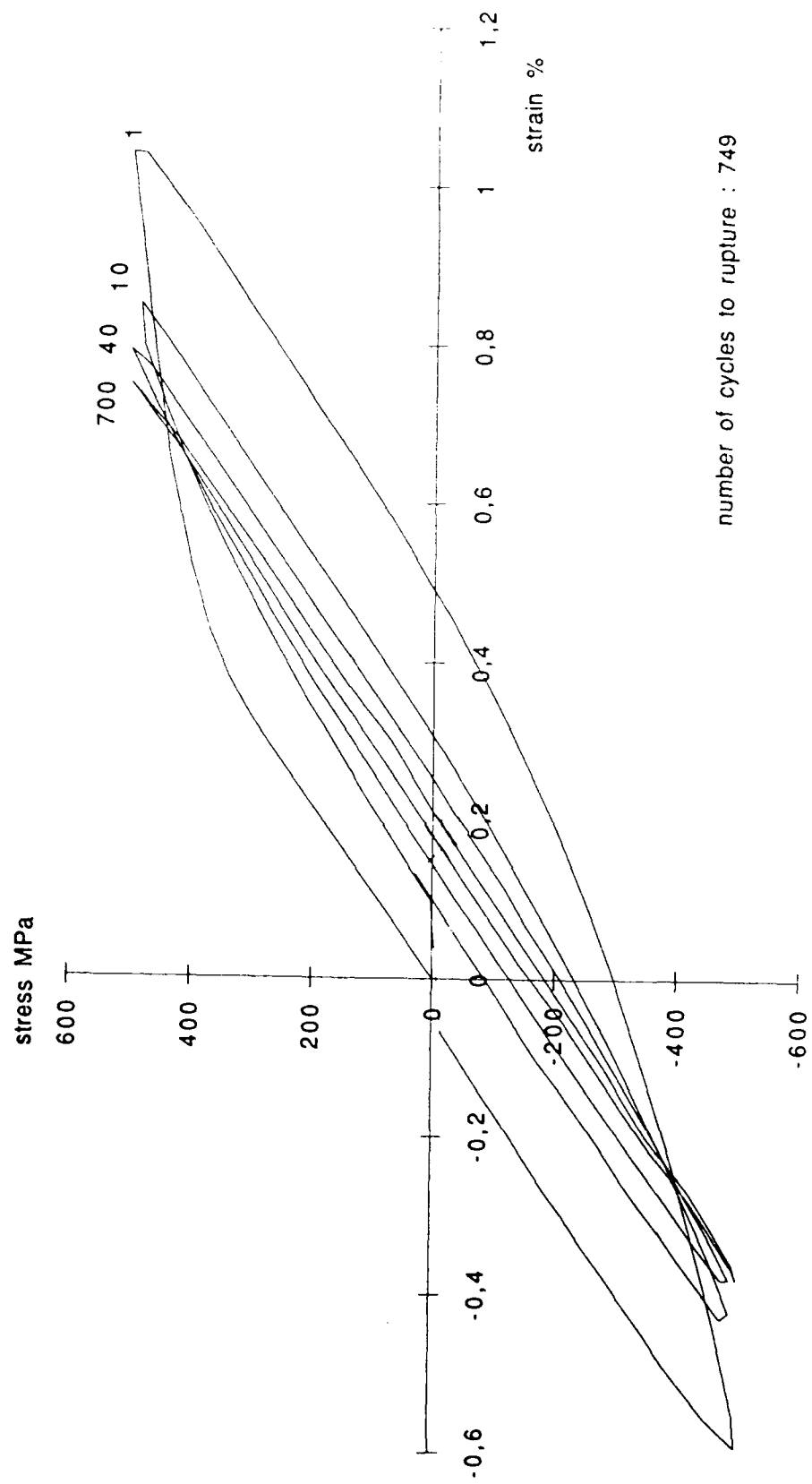
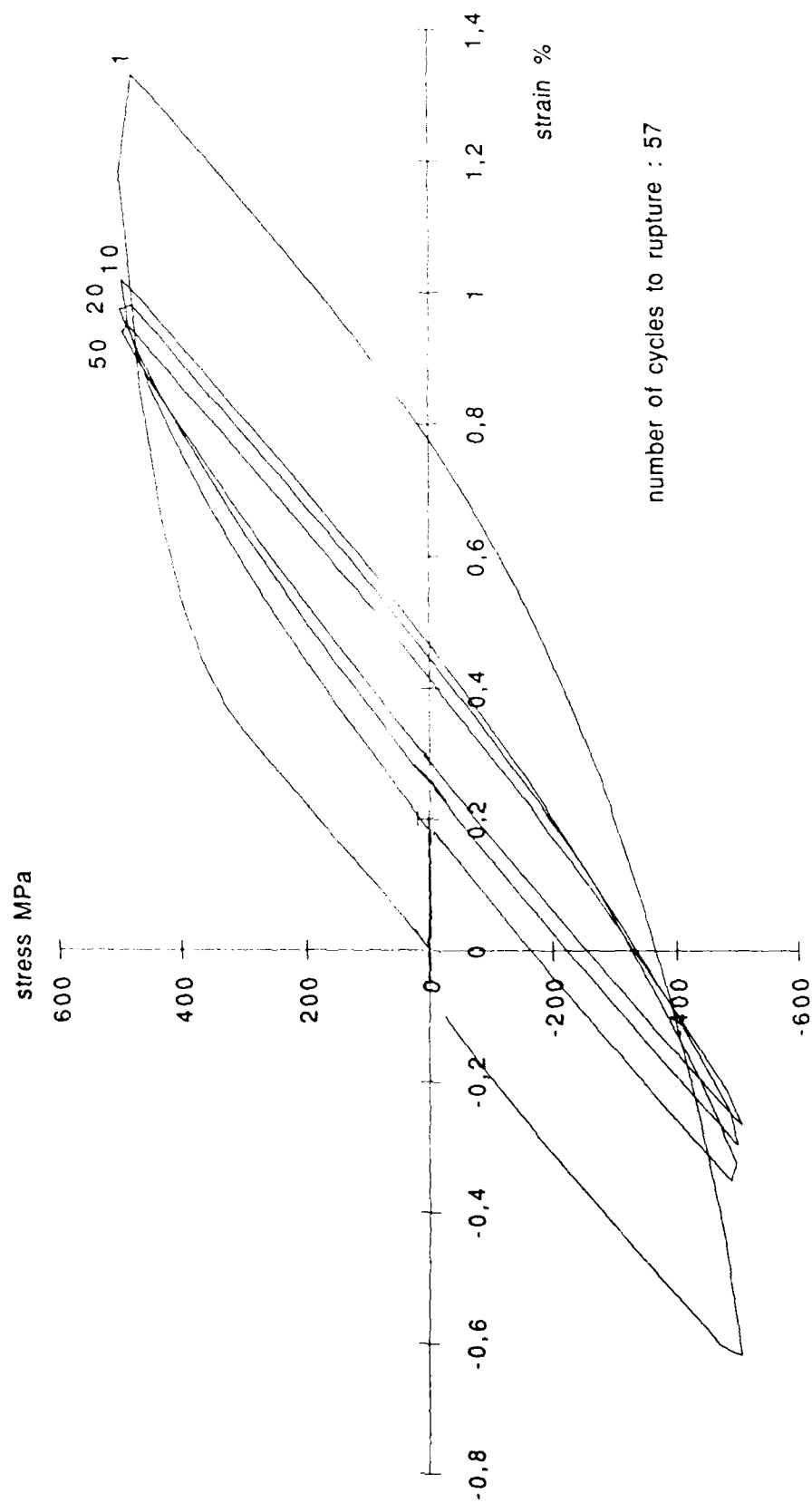
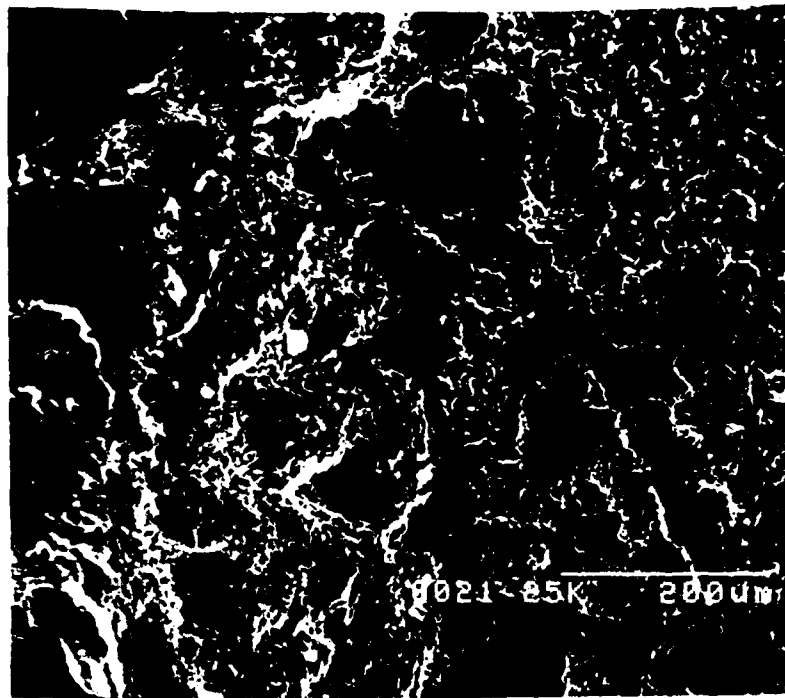
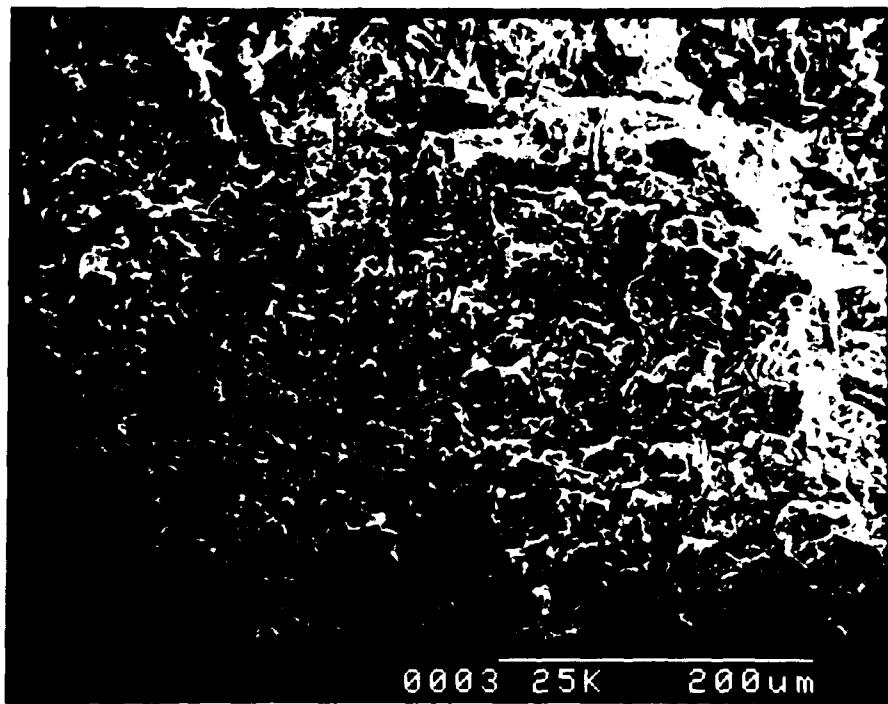


Fig. 31 : Stress-controlled test $\pm 509,3$ MPa





test n°4



test n°7

Fig. 32 : Micrograph pictures of fracture surface

Test n°	4	5	6	7
Controlled stress MPa	± 424.4	± 459.8	± 502.2	± 509.3
Number of cycles to rupture from tests	24 000	1040	749	57
Number of cycles to rupture calculated	25 070	2100	110	56

Fig. 33 : Stress controlled tests results