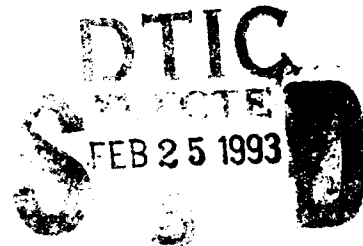




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Exact Detection Performance of Normalizer with Multiple-Pulse Frequency-Shift-Keyed Signals in a Partially-Correlated Fading Medium With Generalized Noncentral Chi-Squared Statistics

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PREFACE

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13. ABSTRACT (Maximum 200 words) The detection and false alarm probabilities, for a normalizer operating in a partially-correlated fading medium with an additive unknown noise level, are derived exactly in the form of series expansions which can be quickly and accurately computed. This new work is an extension of [1], where the characteristic function of the decision variable, for a system operating in a known background noise level, was derived exactly and evaluated for a variety of cases. These new results also replace the approximate analysis of the same normalizer system, that was conducted in [2], with an exact analysis here. The specific situation considered here is as follows. The transmitted signal consists of K pulses separated in time, frequency space so as to be nonoverlapping. In passing through the medium to the receiver, each signal pulse is subjected to fading. In particular, pulse pairs which are closely spaced in				
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13. ABSTRACT (continued)

time, frequency space can fade in a highly dependent fashion, while those pairs more widely separated can have relatively independent fading behavior; that is, the transmitted frequency-shift-keyed signal pulses undergo partially-correlated fading of a very general character that can contain both deterministic components as well as random components. Thus, for example, the amplitude-fading statistics are not limited to be Rayleigh.

Additive zero-mean Gaussian noise of unknown level, which is stationary over the total signal transmission time and which has a flat spectrum over the total signal bandwidth, is present at the input to the receiver, in addition to the fluctuating signal pulses (when present).

Receiver processing consists of matched filtering of each of the K time-delayed, frequency-shifted pulse locations, followed by squared-envelope detection, sampling, summation, and comparison of this decision variable with a variable threshold, for statement on signal presence or absence. The variable threshold is obtained by estimating the noise power in L time, frequency bins in which signal is known to be absent. The additional uncertainty, caused by this lack of knowledge of the additive noise level, requires additional input signal-to-noise ratio to maintain the same performance as for the fixed-threshold (known noise level) system. The exact quantitative evaluation of this degradation is the subject of this study.

14. SUBJECT TERMS (continued)

noncentral	chi-squared statistics
detection probability	false alarm probability
additive noise	unknown noise level
characteristic function	time, frequency space
matched filtering	squared-envelope detection
variable threshold	noise level estimation

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LIST OF SYMBOLS

K	number of signal pulses, figure 1
L	number of noise-only bins or samples, figure 1
t	time, figure 1
f	frequency, figure 1
γ	signal processor output, sum of K signal bins, (7)
ξ	characteristic function argument, (7), (A-1)
f_{γ}	characteristic function of γ , (7), (A-3)
Q_{mk}	auxiliary constant, (8)
e_{mk}	auxiliary constant, (8)
z	transformation variable, (11), (A-1)
β	scale factor in transformation, (11)
f_p	series coefficients, (12), (15), (A-7)
F	auxiliary constant, (13)
a_p	auxiliary constants, (14)
p_{γ}	probability density function of γ , (17)
u	fixed threshold, (18)
E_{γ}	exceedance distribution function of γ , (18)
$\text{Pr}(a)$	probability of event a , (20)
$\underline{E}(v,u)$	exceedance distribution function, (21)
v	scale factor, (22); threshold for normalizer, (25)
γ_0	sum of L noise-only bin outputs, (22)
p_{γ_0}	probability density function of γ_0 , (24)
$E_n(v)$	exceedance distribution function of normalizer, (25)
overbar	ensemble average, (1), (25), (C-1)
$T(m,n,x)$	auxiliary summation, (26)

$U(k,n,x)$	auxiliary function, (26)
${}_1F_0$	confluent hypergeometric function, (32)
$P_F(v)$	false alarm probability of normalizer output, (48)
E_1	average random received signal energy in one pulse
N_0	received noise one-sided spectral level
SNR	signal-to-noise ratio E_1/N_0 in dB, figure 2
$C(v,x)$	cumulative distribution function, (B-1)
$E(v,x)$	exceedance distribution function, (B-14)
$\underline{E}(n,x)$	exceedance distribution function, (B-17)
μ_n	n-th moment of ratio γ/γ_0 , (C-1)
Var	variance, (C-5)
$A(p)$	auxiliary array, (D-1)
$F(p)$	auxiliary array, (D-2)
$Be, B1$	auxiliary arrays, (D-3)
Cp, Bp	auxiliary arrays, (D-4)
Dp	scaled power, (D-5)

EXACT DETECTION PERFORMANCE OF NORMALIZER WITH MULTIPLE-PULSE
FREQUENCY-SHIFT-KEYED SIGNALS IN A PARTIALLY-CORRELATED FADING
MEDIUM WITH GENERALIZED NONCENTRAL CHI-SQUARED STATISTICS

INTRODUCTION

In a recent study [1], the detection performance capability of a multiple-pulse, frequency-shift-keyed system, with received signals subject to partially-correlated fading, was analyzed exactly in terms of the characteristic function of the system output decision variable. It was assumed there that the additive background noise level was known, so that a fixed threshold could be set for an arbitrarily specified false alarm probability. Then, the detection probability was evaluated exactly, as a function of the threshold level, the received signal-to-noise ratio, the number K of signal pulses, and the fading statistics.

Here, we extend these earlier results to cover the case where, instead, the background noise level is unknown and must be estimated on the basis of a finite number L of noise-only samples. Furthermore, the current analysis will also be exact, yielding new series expansions for the detection and false alarm probabilities. These new results replace the approximate analysis of the same normalizer system that was conducted in [2]. This normalizer possesses the important capability of constant false alarm rate; that is, a prescribed false alarm probability can be achieved, exactly, in the absence of knowledge of the actual noise background level.

The reader is referred to [1] for additional background, motivation, interpretations, and related references. For the sake of brevity, we will employ the same notation as in [1] and presume that the reader has complete familiarity with the earlier material and developments.

PROBLEM DEFINITION

BASIC SIGNAL PROCESSING

The problem of interest will be couched in a time, frequency setting, in which the individual transmitted signal pulses occupy disjoint nonoverlapping cells; however, the extension to other variables, such as space or angle, should be obvious.

A set of K signal pulses is transmitted at known time, frequency locations, as indicated in figure 1. These pulses may be abutting in time and/or frequency, or they may be widely separated in time, frequency space. The individual signal pulses need not have unity time-bandwidth product, although many practical applications would utilize that format.

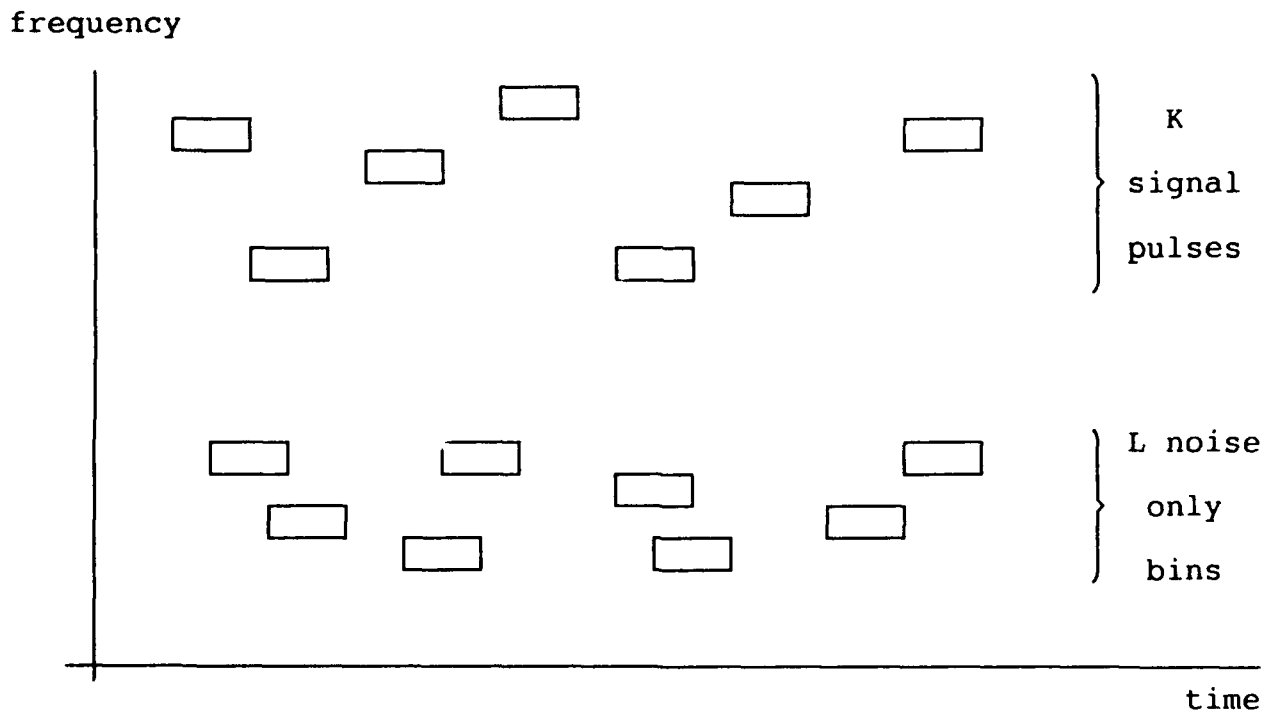


Figure 1. Time, Frequency Occupancy Diagram

At the receiver, the noisy received waveform is processed through K matched filters, which utilize knowledge of the particular time, frequency occupancy pattern of the signal pulses. Their squared envelope outputs are then appropriately sampled and summed for comparison with a scaled estimate of the background level obtained from the reference collection of L noise-only bins. The scale factor applied to the noise-level estimate yields the threshold value, which is selected to realize the desired false alarm probability. If the threshold is exceeded by the sum of K squared envelopes, a signal is declared present in that pattern of K pulses.

Depending on the time, frequency separation of the signal pulses in figure 1, the received signal strengths of the individual pulses, when signal is present, may fade considerably and with statistical dependence between them. The exact amount and effect of the fading depends on the distribution of the fading and on the $K \times K$ covariance matrix between the fading amplitudes applied to each signal pulse.

It is presumed, as in [1], that during the time extent of a single pulse, the fading of that particular pulse is essentially constant, resulting in a constant amplitude scaling and phase shift applied to each received pulse. The phase shifts of distinct pulses can have arbitrary statistics and dependencies; the particular phase shifts are irrelevant in the method of signal processing adopted here and in [1].

The choice of L , the number of noise-only pulses, must be based on a compromise. Larger L results in a more stable estimate of the background noise level, in which case the performance of the normalizer processor for figure 1 approaches that for the fixed-threshold processor in [1]. However, L cannot be chosen so large as to encounter nonstationary and/or nonwhite noise fields over the time,frequency extent depicted in figure 1.

For signal pulses consisting of bursts of pure tones, the receiver implementation of figure 1 can be accomplished by means of fast Fourier transforms. This case would utilize signal bursts and noise slots that have approximately unity time-bandwidth products, and would result in minimum consumption of time, frequency space in figure 1. However, the results presented below are not limited to this special case.

REVIEW OF PARAMETERS

Very heavy reliance will be made upon the methods and results presented for the fixed-threshold system analyzed in [1], and which the reader must be familiar with. However, it is very worthwhile at this point to review and summarize the most important and critical parameters that governed performance in [1], and which are relevant to the normalizer analysis here, as well. These parameters are listed below; their first appearance and detailed explanation can be found in [1; pages iv - viii] in the list of symbols there.

K	number of transmitted signal pulses; see figure 1 above
M	number of components in fading model; see [1; (45) - (47)]
E_{1m}	average random received signal energy in the m -th component of a single pulse, $1 \leq m \leq M$; see [1; (52)]
N_0	one-sided spectral level of received white noise (unknown)
$\underline{C}^{(m)}$	$K \times K$ normalized covariance matrix of signal amplitude fading; see [1; (127)]
$\underline{\Lambda}^{(m)}$	$K \times K$ eigenvalue matrix of $\underline{C}^{(m)}$
$\underline{\lambda}_k^{(m)}$	k -th eigenvalue of matrix $\underline{\Lambda}^{(m)}$, $1 \leq k \leq K$
D_{km}	deterministic received signal energy in the m -th component of the k -th pulse; see [1; (50)]
$\underline{v}_k^{(m)}$	k -th eigenvector of $\underline{C}^{(m)}$
$v_{kj}^{(m)}$	j -th component of vector $\underline{v}_k^{(m)}$, $1 \leq j \leq K$
$\underline{\epsilon}_k^{(m)}$	auxiliary constants; see [1; (137)]
ψ_m	fractional strengths of random components; see [1; (191)]

It is also worthwhile at this point to expound on the covariances of the fading model, since these will be used at a later stage of the development. The covariance function of the alternating component $g_m(t, f)$ in [1; (45) & (47)] is

$$R_{mn}(\tau, \nu) = \overline{g_m(t, f) g_n(t-\tau, f-\nu)} , \quad (1)$$

while the covariance of power-scaling variate $q_k = q(t_k, f_k)$ is, from [1; (43), (44), (54), (55)]

$$\tilde{R}_{kj} = \text{Cov}(q_k, q_j) = \text{Cov}(q(t_k, f_k), q(t_j, f_j)) . \quad (2)$$

The covariance coefficient between q_k and q_j is

$$\rho_{kj} = \frac{\tilde{R}_{kj}}{(\tilde{R}_{kk} \tilde{R}_{jj})^{1/2}} = \left[\frac{R_{11}(t_k - t_j, f_k - f_j)}{R_{11}(0, 0)} \right]^2 , \quad (3)$$

from [1; (56), (66) or (69)]. We also define the normalized covariance function

$$\text{Cov}(\tau, \nu) = \frac{R_{11}(\tau, \nu)}{R_{11}(0, 0)} \quad (4)$$

as in [1; (194)], from which there follows

$$\rho_{kj} = \text{Cov}^2(t_k - t_j, f_k - f_j) \quad (5)$$

as in [1; (56), (66) or (69)]. Finally, we define a special case of (4), for the case of equally time-spaced signal pulses without any frequency shifts, according to

$$\text{Cov}_1 = \text{Cov}(t_{k+1}-t_k, 0) = \frac{R_{11}(t_{k+1}-t_k, 0)}{R_{11}(0, 0)} \quad \text{for all } k, \quad (6)$$

since this quantity will be very useful later; this last quantity in (6) was denoted by Cov in [1] and all the figures therein.

EXCEEDANCE DISTRIBUTION FUNCTION OF NORMALIZER OUTPUT

FIXED-THRESHOLD PROCESSOR OUTPUT

Before we begin the analysis of the normalizer, we need to review and manipulate the results for the fixed-threshold case. The general result for the characteristic function of signal processor output γ , in the case of M uncorrelated components in fading model [1; (45)], is given by [1; (136)] as

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left(\prod_{m=1}^M \prod_{k=1}^K (1 - i2\xi Q_{mk}) \right)^{-\frac{1}{2}} \times \\ \times \exp \left(i\xi \sum_{m=1}^M \sum_{k=1}^K \frac{e_{mk}}{1 - i2\xi Q_{mk}} \right), \quad (7)$$

where

$$Q_{mk} = 1 + \frac{2E_{1m}}{N_0} \lambda_k^{(m)}, \quad e_{mk} = \varepsilon_k^{(m)2} \quad \text{for } 1 \leq m \leq M, \quad 1 \leq k \leq K. \quad (8)$$

This situation pertains to ignoring the L noise-only outputs in the lower portion of figure 1.

We will limit consideration here to the special case 6 delineated in [1; page 23]. Then, eigenvalues $\{\lambda_k\}$ and eigenvectors $\{V_k\}$ are not functions of component number m , and

$$Q_{mk} = 1 + \frac{2E_1}{N_0} \psi_m \lambda_k \quad (9)$$

from (8) and [1; (191)]. Also, from [1; (157)],

$$\varepsilon_k^{(m)} = \sum_{j=1}^K v_{kj} \left(\frac{2D_{jm}}{N_0} \right)^{\frac{1}{2}}. \quad (10)$$

We now want to perform a transformation of variable ξ in characteristic function (7). In particular, consider the use of the slightly more general transformation (relative to that used in [1; (D-3)]), where scale factor β is under our choice:

$$z = \frac{1}{1 - i\xi\beta} \quad (\beta > 0). \quad (11)$$

Substitution into (7), and expansion in a power series in z (according to appendix A), yields the characteristic function of the fixed-threshold processor output γ in the (modified but exact) form

$$f_Y(\xi) = F z^K \sum_{p=0}^{\infty} f_p z^p = F \sum_{p=0}^{\infty} f_p (1 - i\xi\beta)^{-K-p}, \quad (12)$$

where the constants are found according to

$$F = \left(\frac{\beta}{2} \right)^K \left(\prod_{m=1}^M \prod_{k=1}^K Q_{mk} \right)^{-\frac{1}{2}}, \quad a_0 = -\frac{1}{2} \sum_{m=1}^M \sum_{k=1}^K \frac{e_{mk}}{Q_{mk}}, \quad (13)$$

$$\begin{aligned} p a_p = & \frac{1}{2} \sum_{m=1}^M \sum_{k=1}^K \left(1 - \frac{\beta}{2Q_{mk}} \right)^{p-1} \left[1 - \frac{\beta}{2Q_{mk}} \left(1 - p \frac{e_{mk}}{Q_{mk}} \right) \right] - \\ & - K \left(\frac{M}{2} - 1 \right) \left(1 - \frac{\beta}{2} \right)^p \quad \text{for } p \geq 1, \end{aligned} \quad (14)$$

$$f_0 = \exp(a_0), \quad f_p = \frac{1}{p} \sum_{n=1}^p n a_n f_{p-n} \quad \text{for } p \geq 1. \quad (15)$$

For $\beta = 2$, (14) reduces to [1; (185) or (D-10)]. However, we want to choose β here so that the magnitudes of the coefficients $\{F f_p\}$ in (12) decay as rapidly as possible with p . On the other hand, some of the $\{f_p\}$ can then be negative.

Now, suppose we have found the characteristic function of any random variable γ in the series form (12), where K is integer. Then, the origin value of the characteristic function is

$$f_Y(0) = F \sum_{p=0}^{\infty} f_p = 1, \quad (16)$$

while the corresponding probability density function of γ is

$$p_Y(x) = F \sum_{p=0}^{\infty} f_p \frac{x^{K+p-1} \exp(-x/\beta)}{\beta^{K+p} (K+p-1)!} \quad \text{for } x > 0. \quad (17)$$

The exceedance distribution function of γ is then

$$\begin{aligned} E_Y(u) &= \int_u^{\infty} dx \, p_Y(x) = F \sum_{p=0}^{\infty} f_p \int_u^{\infty} dx \, \frac{x^{K+p-1} \exp(-x/\beta)}{\beta^{K+p} (K+p-1)!} = \\ &= F \sum_{p=0}^{\infty} f_p \, \underline{E}\left(K+p, \frac{u}{\beta}\right) = \end{aligned} \quad (18)$$

$$= 1 - F \sum_{p=0}^{\infty} f_p \left[1 - \underline{E}\left(K+p, \frac{u}{\beta}\right) \right] \quad \text{for } u > 0, \quad (19)$$

where we used (16). Physically, this exceedance distribution function is the probability that fixed-threshold processor output γ is greater than a constant value u ; that is,

$$E_{\gamma}(u) = \Pr(\gamma > u) . \quad (20)$$

Here, in (18), we used the following definition of the exceedance distribution function $\underline{E}$, which can be expressed in terms of incomplete gamma functions:

$$\underline{E}(v, u) = \int_u^{\infty} dx \frac{x^{v-1} \exp(-x)}{\Gamma(v)} = \frac{\Gamma(v, u)}{\Gamma(v)} \quad \text{for } u \geq 0, \quad v > 0 . \quad (21)$$

More details on recursions for these functions, when v is equal to an integer or a half-integer, are given in appendix B.

NORMALIZER OUTPUT

Now, let fixed threshold u in (20) be replaced by a variable threshold according to

$$u = v \gamma_0, \quad (22)$$

where v is a positive scaling constant, and γ_0 is an independent positive random variable given by the sum of the L independent noise-only bin outputs in figure 1. This latter quantity is proportional to an estimate of the noise background power level in which the normalizer system is operating. The characteristic function of γ_0 can be found from [1; (40)], by replacing K by L and by setting signal strength $\tilde{E}$ to zero:

$$f_{\gamma_0}(\xi) = (1 - i2\xi)^{-L}. \quad (23)$$

The corresponding probability density function of γ_0 is

$$p_{\gamma_0}(x) = \frac{x^{L-1} \exp(-x/2)}{2^L (L-1)!} \quad \text{for } x > 0. \quad (24)$$

We are now interested in the exceedance distribution function $E_n(v)$ of the normalizer output, namely ratio γ/γ_0 :

$$\begin{aligned} E_n(v) &\equiv \Pr\left(\frac{\gamma}{\gamma_0} > v\right) = \Pr(\gamma > v \gamma_0) = \overline{E_\gamma(v \gamma_0)}^{\gamma_0} = \\ &= \int dx p_{\gamma_0}(x) E_\gamma(v x) = \int_0^\infty dx \frac{x^{L-1} \exp(-x/2)}{2^L (L-1)!} F \sum_{p=0}^\infty f_p \underline{E}\left(k+p, \frac{v}{\beta} x\right) = \end{aligned}$$

$$\begin{aligned}
&= F \sum_{p=0}^{\infty} f_p \int_0^{\infty} dx \frac{x^{L-1} \exp(-x/2)}{2^L (L-1)!} \exp\left(-\frac{v}{\beta}x\right) \sum_{k=0}^{K-1+p} \frac{1}{k!} \left(\frac{v}{\beta}x\right)^k = \\
&= F \sum_{p=0}^{\infty} f_p \sum_{k=0}^{K-1+p} \frac{1}{k!} \left(\frac{v}{\beta}\right)^k \int_0^{\infty} dx \frac{x^{L-1+k}}{2^L (L-1)!} \exp\left(-x\left(\frac{1}{2} + \frac{v}{\beta}\right)\right) = \\
&= F \left(\frac{\beta}{\beta + 2v}\right)^L \sum_{p=0}^{\infty} f_p \sum_{k=0}^{K-1+p} \binom{L-1+k}{k} \left(\frac{2v}{\beta + 2v}\right)^k = \\
&= F \left(\frac{\beta}{\beta + 2v}\right)^L \sum_{p=0}^{\infty} f_p T\left(K-1+p, L-1, \frac{2v}{\beta + 2v}\right), \quad (25)
\end{aligned}$$

where we define the auxiliary functions

$$T(m, n, x) \equiv \sum_{k=0}^m \binom{n+k}{k} x^k = \sum_{k=0}^m (n+1)_k \frac{x^k}{k!} \equiv \sum_{k=0}^m U(k, n, x). \quad (26)$$

As a partial check on (25), we have origin value

$$E_n(0) = F \sum_{p=0}^{\infty} f_p T(K-1+p, L-1, 0) = F \sum_{p=0}^{\infty} f_p = 1, \quad (27)$$

using (16).

For rapid evaluation of functions T and U in (26), we have recursions

$$T(m, n, x) = T(m-1, n, x) + U(m, n, x) \quad \text{for } m \geq 1, \quad (28)$$

$$U(m, n, x) = U(m-1, n, x) \frac{n+m}{m} x \quad \text{for } m \geq 1, \quad (29)$$

along with starting values

$$T(0, n, x) = 1, \quad U(0, n, x) = 1. \quad (30)$$

A general result that will find frequent application is

$$(1-x)^{n+1} T(m,n,x) = 1 - x^{m+1} T(n,m,1-x) . \quad (31)$$

As a special case, we also have [3; 15.1.8]

$$T(\infty, n, x) = \sum_{k=0}^{\infty} (n+1)_k \frac{x^k}{k!} = {}_1F_0(n+1; x) = (1-x)^{-n-1} . \quad (32)$$

By means of this last relation, we have

$$T\left(\infty, L-1, \frac{2v}{\beta + 2v}\right) = \left(1 - \frac{2v}{\beta + 2v}\right)^{-L} = \left(\frac{\beta + 2v}{\beta}\right)^L , \quad (33)$$

which enables the normalizer exceedance distribution function in (25) to be expressed in the alternative form

$$\begin{aligned} E_n(v) &= F \left(\frac{\beta}{\beta+2v}\right)^L \sum_{p=0}^{\infty} f_p \left[T\left(K-1+p, L-1, \frac{2v}{\beta+2v}\right) - \left(\frac{\beta+2v}{\beta}\right)^L + \left(\frac{\beta+2v}{\beta}\right)^L \right] = \\ &= 1 - F \sum_{p=0}^{\infty} f_p \left[1 - \left(\frac{\beta}{\beta+2v}\right)^L T\left(K-1+p, L-1, \frac{2v}{\beta+2v}\right) \right] , \end{aligned} \quad (34)$$

where we used (16). This latter form is recommended over (25) because both f_p and the bracketed term, $[]$, in (34) go to zero as $p \rightarrow \infty$. The approach of f_p to zero follows from (16). It is also very important to notice that the bracketed term in (34) is not a function of the signal-to-noise ratio.

By means of (31), (34) can also be expressed as

$$E_n(v) = 1 - F \sum_{p=0}^{\infty} f_p \left(\frac{2v}{\beta+2v}\right)^{K+p} T\left(L-1, K-1+p, \frac{\beta}{\beta+2v}\right) . \quad (35)$$

Although both f_p and the succeeding power in (35) do tend to zero as $p \rightarrow \infty$, function T does not; of course, the product of the power term and T do tend to zero. Nevertheless, (34) will be more useful, since a simple recursion on p exists. (It has been numerically verified that (25), (34), and (35) are all independent of the particular value of the positive scaling constant β in transformation (11); we are now free to choose the value of β that results in the most rapid convergence of the various series.) Results (25), (34), and (35) are alternative expressions for the detection probability $P_D(v)$ of the normalizer output ratio γ/γ_0 , with threshold value v ; that is,

$$P_D(v) = \Pr(\gamma/\gamma_0 > v) \text{ for fixed } v.$$

The numerical evaluation of form (34) requires a difference in the bracketed term and thereby suffers a loss of accuracy. A more useful numerical alternative, using a sum of positive terms, is furnished by

$$E_n(v) = 1 - F \left(\frac{\beta}{\beta + 2v} \right)^L \sum_{p=0}^{\infty} f_p \sum_{k=K+p}^{\infty} \frac{(L)_k}{k!} \left(\frac{2v}{\beta + 2v} \right)^k. \quad (36)$$

This result follows upon use of (26) and (33). The inner sum on k in (36) obviously goes to zero as p increases, as does coefficient f_p . Recursions for the inner sum on k make this a rapid accurate alternative to (34); result (36) is programmed and used in this investigation.

CHECK ON EXCEEDANCE DISTRIBUTION (25)

Result (25) gave the normalizer exceedance distribution function

$$E_n(v) = \Pr(\gamma > v \gamma_0) = \Pr\left(\frac{\gamma}{\gamma_0} > v\right) . \quad (37)$$

If we now let the normalizer fixed threshold v be given by

$$v = \frac{u}{2L} , \quad (38)$$

then we find

$$E_n(v) = E_n\left(\frac{u}{2L}\right) = \Pr\left(\gamma > u \frac{\gamma_0}{2L}\right) . \quad (39)$$

But as $L \rightarrow \infty$, random variable $\gamma_0/(2L)$ tends to the value 1, since the mean of γ_0 is $2L$ and its standard deviation is $2\sqrt{L}$; see (24), for which the k -th cumulant is $L (k-1)! 2^k$. Thus, we should have

$$\lim_{L \rightarrow \infty} E_n\left(\frac{u}{2L}\right) = \Pr(\gamma > u) = E_\gamma(u) , \quad (40)$$

where the latter exceedance distribution was given by (18).

In order to confirm this result, substitute $v = u/(2L)$ into (25) to obtain

$$E_n\left(\frac{u}{2L}\right) = F\left(\frac{\beta}{\beta + u/L}\right)^L \sum_{p=0}^{\infty} f_p T\left(K-1+p, L-1, \frac{u/L}{\beta + u/L}\right) . \quad (41)$$

But

$$\left(\frac{\beta}{\beta + u/L}\right)^L = \left(\frac{1}{1 + u/(\beta L)}\right)^L \sim \exp\left(-\frac{u}{\beta}\right) \text{ as } L \rightarrow \infty , \quad (42)$$

while

$$\begin{aligned}
 T\left(m, L-1, \frac{u/L}{\beta + u/L}\right) &= \sum_{k=0}^m \binom{L-1+k}{k} \left(\frac{u/L}{\beta + u/L}\right)^k = \\
 &= \sum_{k=0}^m \frac{(L)_k}{k!} \frac{(u/\beta)^k}{(L + u/\beta)^k} \sim \sum_{k=0}^m \frac{(u/\beta)^k}{k!} \quad \text{as } L \rightarrow \infty. \quad (43)
 \end{aligned}$$

Therefore,

$$\lim_{L \rightarrow \infty} E_n\left(\frac{u}{2L}\right) = F \exp\left(-\frac{u}{\beta}\right) \sum_{p=0}^{\infty} f_p \sum_{k=0}^{K-1+p} \frac{(u/\beta)^k}{k!} = F \sum_{p=0}^{\infty} f_p \underline{E}\left(K+p, \frac{u}{\beta}\right) \quad (44)$$

in agreement with (18). Here, we used (B-17).

FALSE ALARM PROBABILITY

If signal is absent from the processor input, the earlier results for the normalizer output exceedance distribution function should reduce to the false alarm probability. The moments of normalizer output γ/γ_0 are presented in appendix C, when signal is absent. Also, in this case, we have $Q_{mk} = 0$, $e_{mk} = 0$, giving, from (13) - (14),

$$F = \left(\frac{\beta}{2}\right)^K, \quad a_0 = 0, \quad p a_p = K \left(1 - \frac{\beta}{2}\right)^p \quad \text{for } p \geq 1. \quad (45)$$

There follows (see (A-5) - (A-6))

$$\begin{aligned} \sum_{p=0}^{\infty} f_p z^p &= \exp\left(\sum_{p=0}^{\infty} a_p z^p\right) = \exp\left(K \sum_{p=1}^{\infty} \frac{1}{p} \left(1 - \frac{\beta}{2}\right)^p z^p\right) = \\ &= \exp\left(-K \ln\left[1 - \left(1 - \frac{\beta}{2}\right)z\right]\right) = \left[1 - \left(1 - \frac{\beta}{2}\right)z\right]^{-K} = \\ &= \sum_{p=0}^{\infty} \binom{-K}{p} \left(1 - \frac{\beta}{2}\right)^p (-z)^p = \sum_{p=0}^{\infty} \frac{(K)_p}{p!} \left(1 - \frac{\beta}{2}\right)^p z^p, \end{aligned} \quad (46)$$

from which we find coefficients

$$f_p = \frac{(K)_p}{p!} \left(1 - \frac{\beta}{2}\right)^p \quad \text{for } p \geq 0. \quad (47)$$

Then (25) reduces to the normalizer false alarm probability

$$P_F(v) = \left(\frac{\beta}{2}\right)^K \left(\frac{\beta}{\beta + 2v}\right)^L \sum_{p=0}^{\infty} \frac{(K)_p}{p!} \left(1 - \frac{\beta}{2}\right)^p T\left(K-1+p, L-1, \frac{2v}{\beta + 2v}\right). \quad (48)$$

Furthermore, since this result must be independent of β , we set $\beta = 2$ to get

$$P_F(v) = \frac{1}{(1+v)^L} T\left(K-1, L-1, \frac{v}{1+v}\right) =$$

$$= \frac{1}{(1+v)^L} \sum_{k=0}^{K-1} \frac{(L)_k}{k!} \left(\frac{v}{1+v}\right)^k = \quad (49)$$

$$= 1 - \left(\frac{v}{1+v}\right)^K \sum_{j=0}^{L-1} \frac{(K)_j}{j!} \frac{1}{(1+v)^j}, \quad (50)$$

where we used (31). This result, (50), agrees with [2; (22)].

The form in (49) utilizes a sum of all-positive terms and thereby retains significance for very small P_F .

GRAPHICAL RESULTS

The scenario to be considered here is the same as that used in [1], namely, a sequence of K signal pulses, equally-spaced in time but with no frequency shifts. Also, the covariance between the signal amplitude fading coefficients will be taken to be exponential in time separation. In this manner, we can make a direct comparison of the results for the fixed-threshold processor in [1] with the normalizer here, and numerically determine the degradation associated with utilizing too small a value for L , the number of noise-only bins. As in [1; page 68], the KM deterministic signal-to-noise ratios $\{D_{km}/N_O\}$ are all taken to be zero, and the M strengths $\{r^{(m)}\}$ are all 1, for the numerical examples investigated here.

Due to the multitude of parameters in this investigation, it is expedient to hold all but one of them fixed, and to let one parameter vary, thereby determining the effect of that parameter by itself. The first series of plots in figures 2, 3, 4, 5, 6 correspond to varying K , the number of signal pulses in figure 1, over the range $K = 2, 3, 4, 5, 6$. The curve for $K = 1$ has already been given in [2; figure 32]; that analysis was, in fact, exact for $K = 1, M = 2$.

The abscissa in all the plots is L , the number of noise-only bins that are used to estimate the background noise level, and which varies over the range from 1 to 100 in all cases. The asymptotic values of required signal-to-noise ratio at $L = \infty$ are available in [1]. The ordinate in the following plots is the

input signal-to-noise ratio (SNR) E_1/N_0 in dB, required in order to realize the specified performance levels in terms of detection and false alarm probabilities, where E_1 is the average random received signal energy in one pulse, and N_0 is the additive noise power density level. The curves are parameterized in terms of the false alarm probability, which is set to values 10^{-2} , 10^{-4} , 10^{-6} , and 10^{-8} . The particular parameter values that apply to each plot appear in the title for each figure. Parameter Cov_1 was defined and explained in (1) - (6); basically, it is the covariance between adjacent signal pulses.

For most rapid convergence of the series for the exceedance distribution function of the normalizer output, the value of scaling β in transformation (11) should be changed for each point computed on these plots. Some samples for the best β values are tabulated in appendix D, along with the particular program used for computing the results in all the figures here.

The first result in figure 2, for $K = 2$, replaces the approximate result presented in [2; figure 33]. It will be seen that approximately 1 dB less is required, according to the exact results of figure 2 here; thus, the approximate analysis in [2] was somewhat pessimistic in terms of the input signal-to-noise ratio required to realize a specified performance level. This same conclusion was also reached upon a comparison of the exact analysis of the fixed-threshold processor in [1] with the corresponding approximate analysis in [4].

As the number of pulses, K , increases from 2 to 6 in figures 2 through 6, the required value of per-pulse SNR, E_1/N_0 , decreases monotonically. However, the total SNR, summed over all K pulses, increases with K ; this same effect was observed and investigated quantitatively in [4], although by an approximate analysis.

The next series of plots, in figures 7 - 11, correspond to varying covariance coefficient Cov_1 in (6) over the range 0 to 1 in steps of $(n/4)^{1/2}$, for $n = 0, 1, 2, 3, 4$. The remaining parameters are held fixed at values $P_D = .5$, $K = 4$, $M = 2$. As expected, the required input signal-to-noise ratio increases monotonically with Cov_1 . The increases, as n makes the four changes from 0 to 4, are in the ranges .15 to .21 dB, .21 to .28 dB, .31 to .35 dB, and .36 to .45 dB, respectively. The total increases, as Cov_1 varies from 0 (figure 7) to 1 (figure 11) are in the range 1.1 to 1.2 dB. Thus, at least for this example, the cost of the fading covariance coefficient approaching unity is moderate.

The final series of plots, in figures 12 - 14, depict the variation of required input signal-to-noise ratio with M , the number of components in the fading model [1; (45) - (47)]. (The plot in figure 12 is identical to figure 4 above.) As M changes from 2 to 3, with other parameters fixed, the required input SNR decreases by approximately 1 dB, while the change of M from 3 to 4 allows for an additional decrease of .5 dB in required input SNR. Generally, larger M means a fading channel model subject to less severe deep fades; see [1; figures 2 - 8].

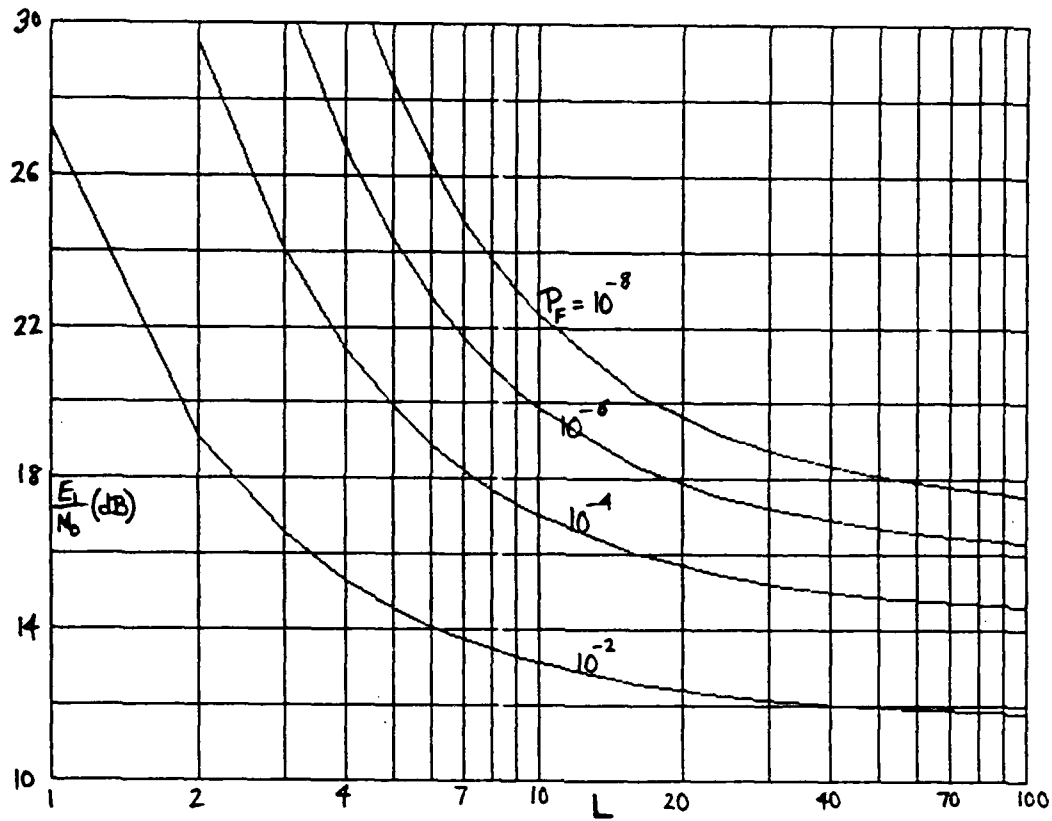


Figure 2. Required SNR for $P_D = .9$, $K = 2$, $M = 2$, $\text{Cov}_1 = \sqrt{.5}$

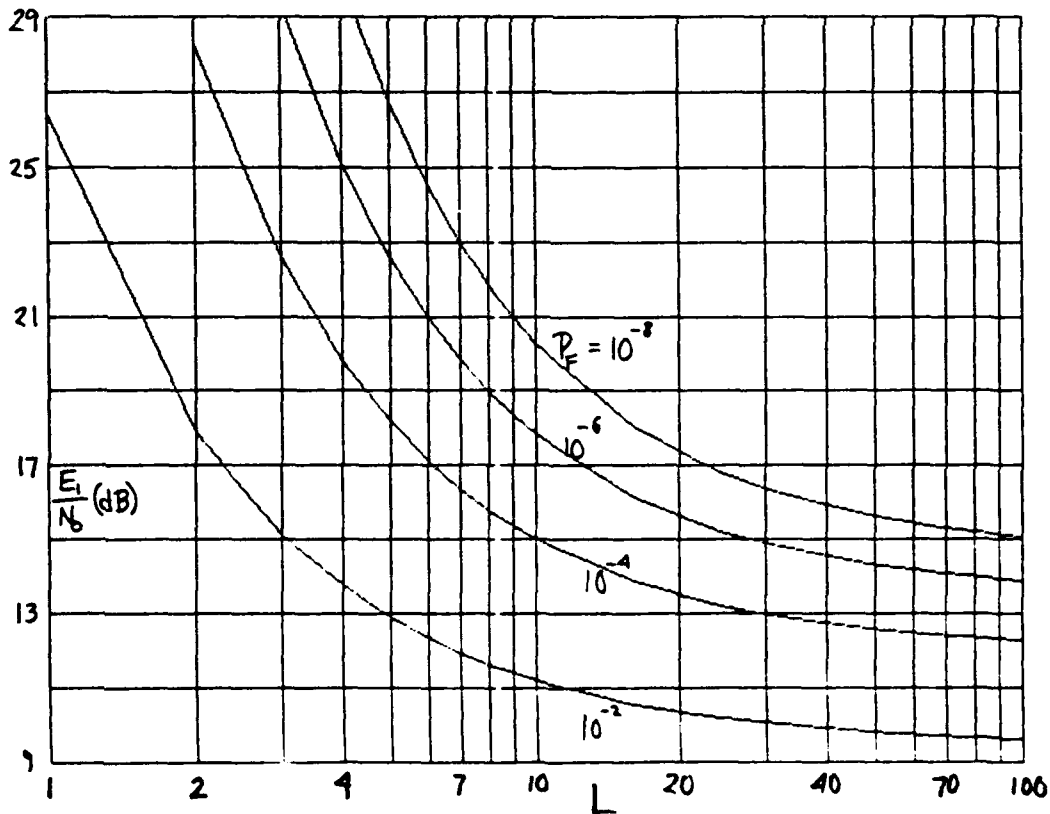


Figure 3. Required SNR for $P_D = .9$, $K = 3$, $M = 2$, $\text{Cov}_1 = \sqrt{.5}$

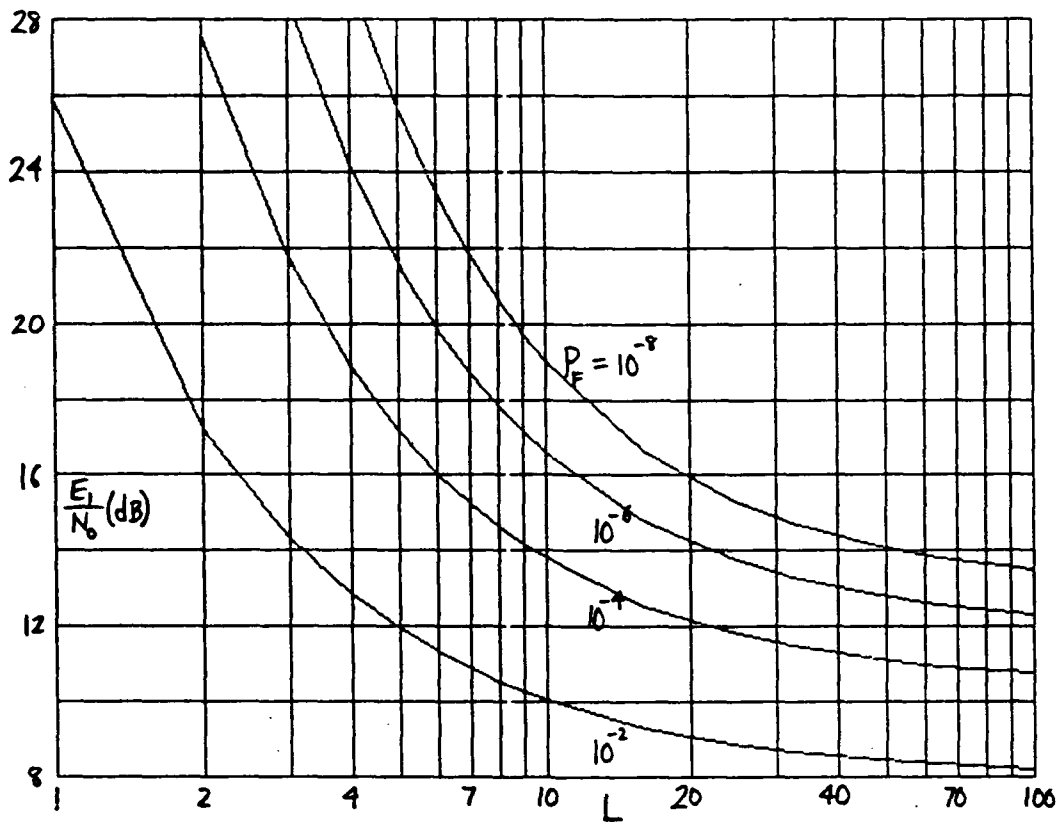


Figure 4. Required SNR for $P_D = .9$, $K = 4$, $M = 2$, $\text{Cov}_1 = \sqrt{.5}$

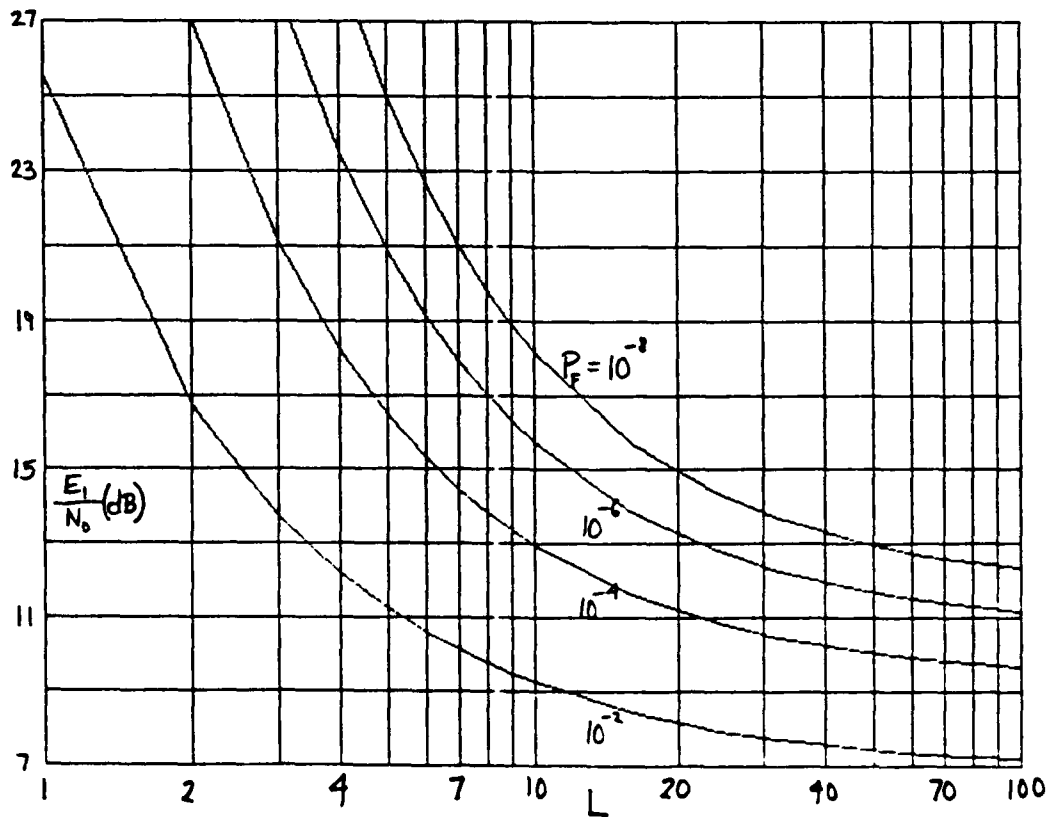


Figure 5. Required SNR for $P_D = .9$, $K = 5$, $M = 2$, $\text{Cov}_1 = \sqrt{.5}$

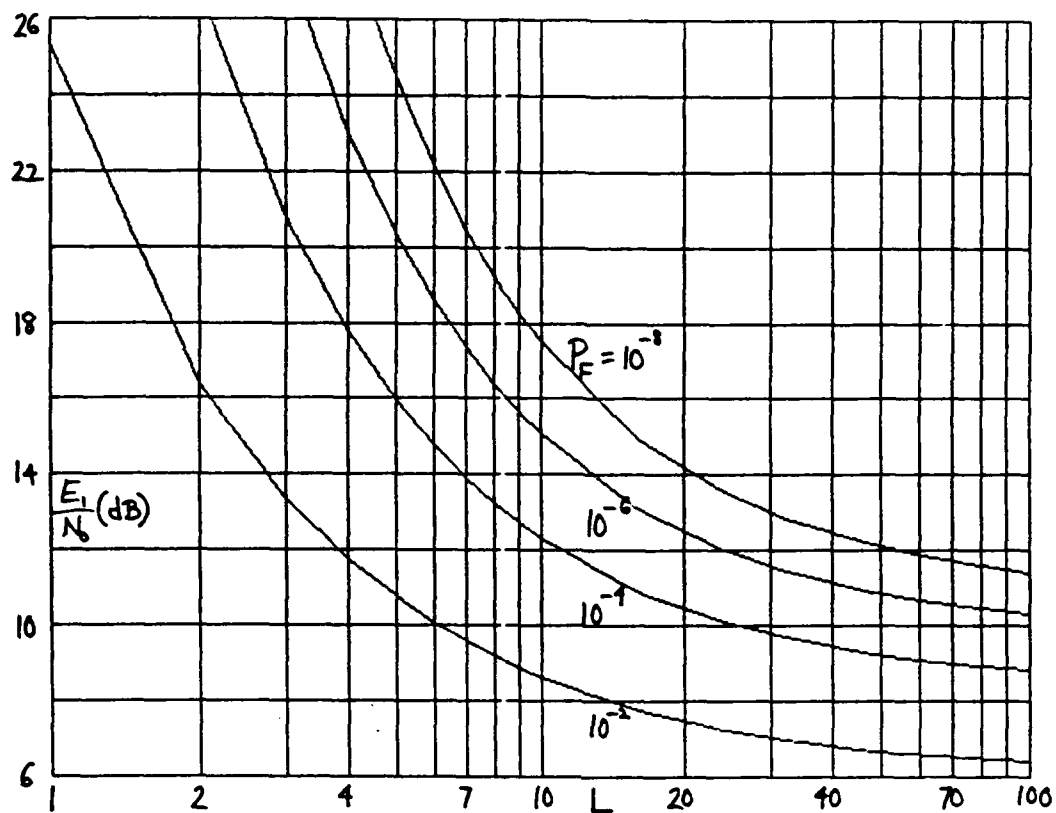


Figure 6. Required SNR for $P_D = .9$, $K = 6$, $M = 2$, $\text{Cov}_1 = \sqrt{.5}$

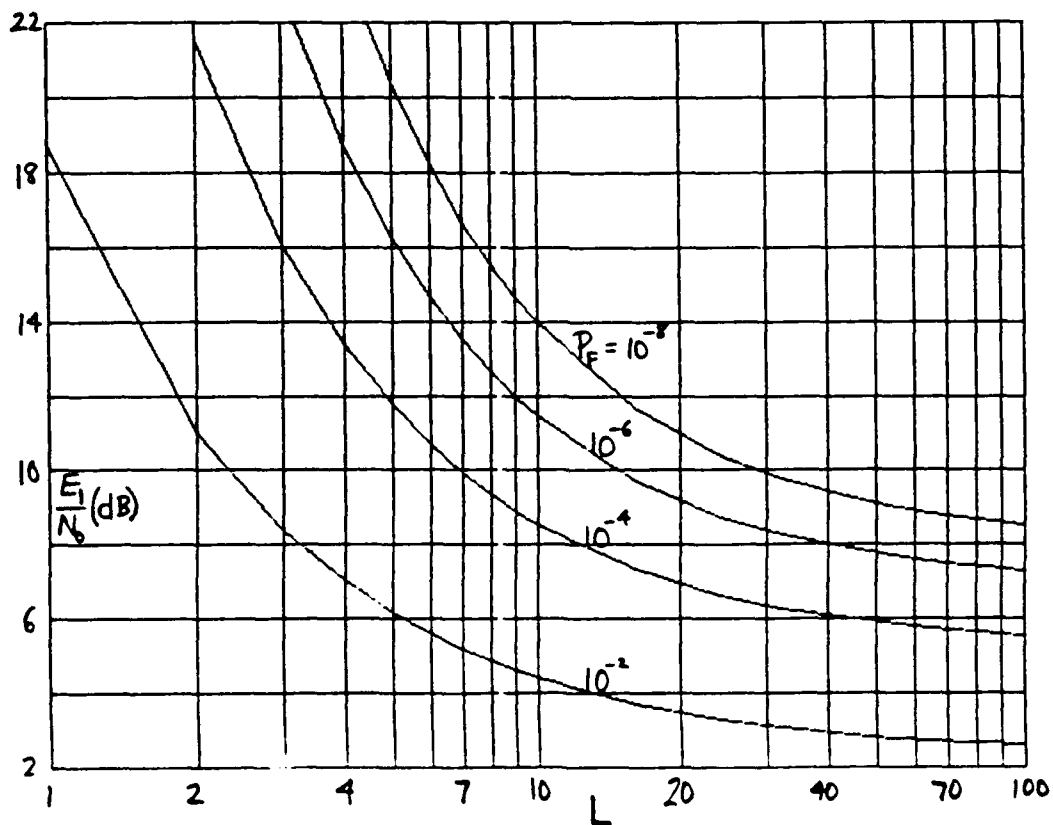


Figure 7. Required SNR for $P_D = .5$, $K = 4$, $M = 2$, $\text{Cov}_1 = 0$

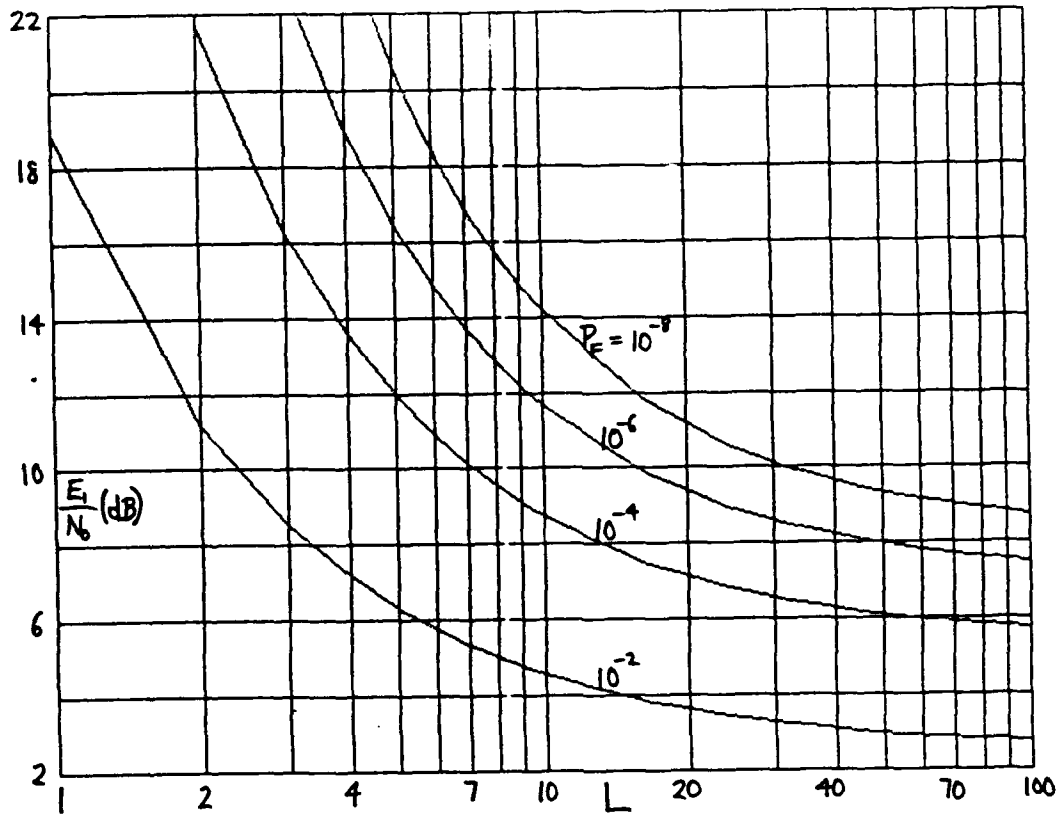


Figure 8. Required SNR for $P_D = .5$, $K = 4$, $M = 2$, $\text{Cov}_1 = \sqrt{.25}$

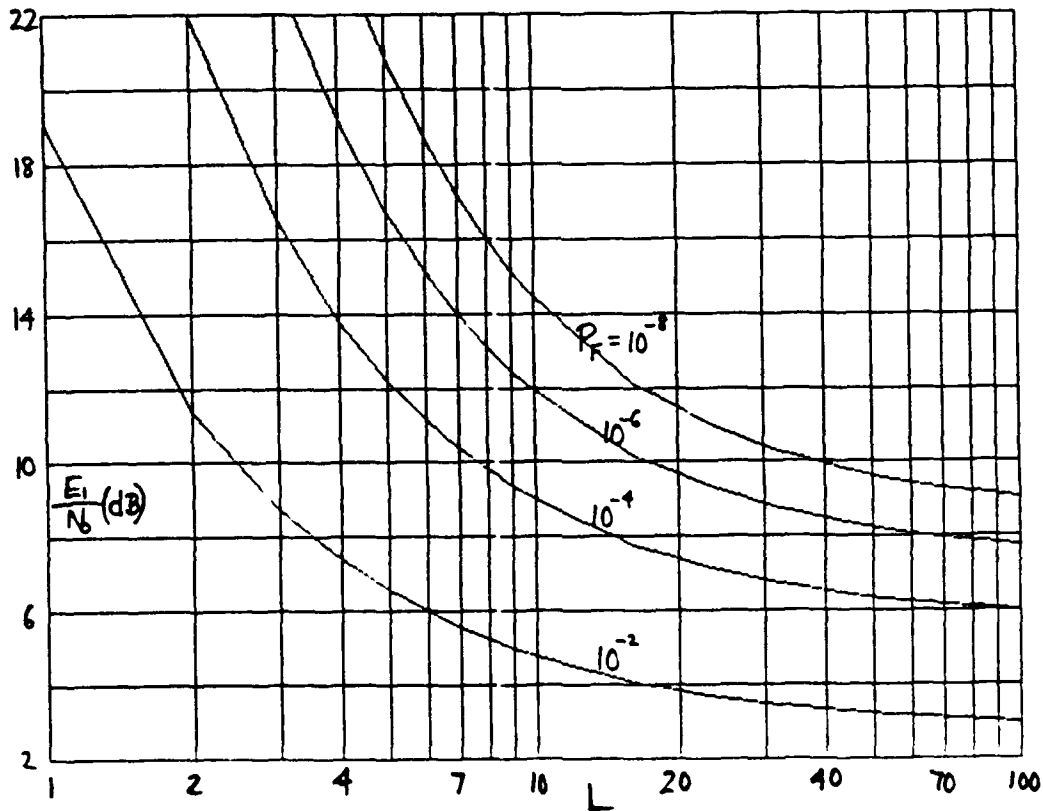


Figure 9. Required SNR for $P_D = .5$, $K = 4$, $M = 2$, $\text{Cov}_1 = \sqrt{.5}$

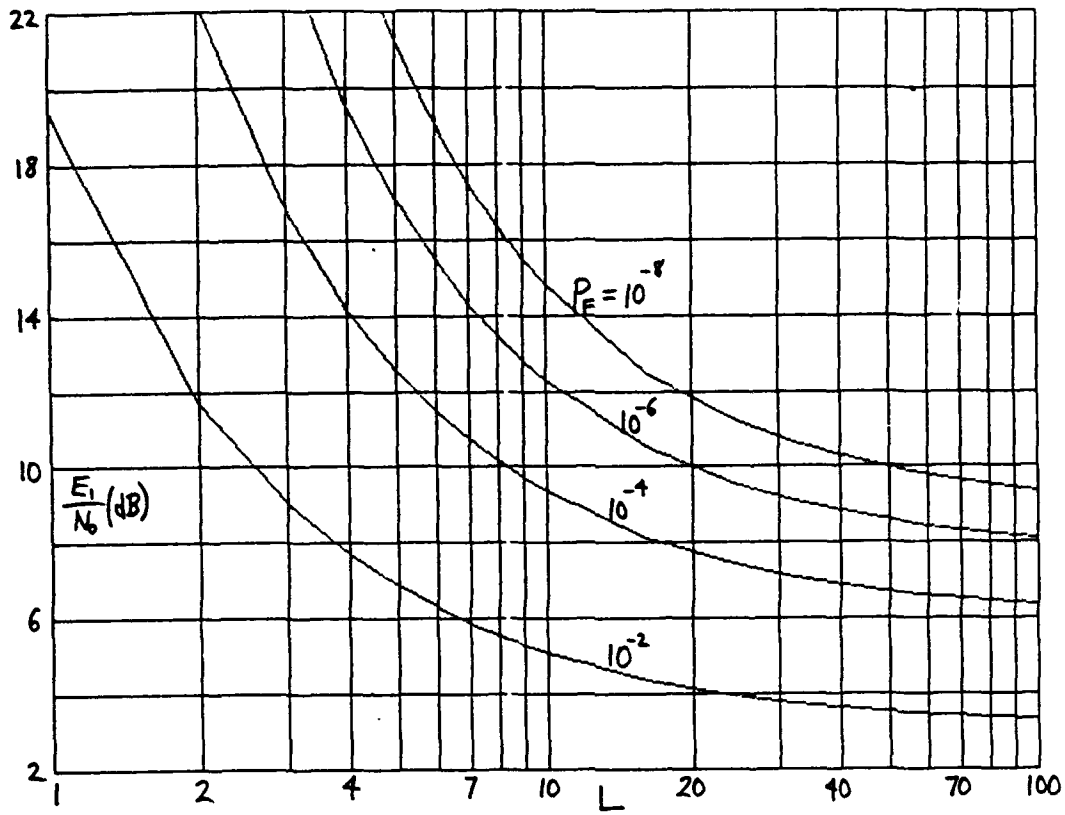


Figure 10. Required SNR for $P_D = .5$, $K = 4$, $M = 2$, $\text{Cov}_1 = \sqrt{.75}$

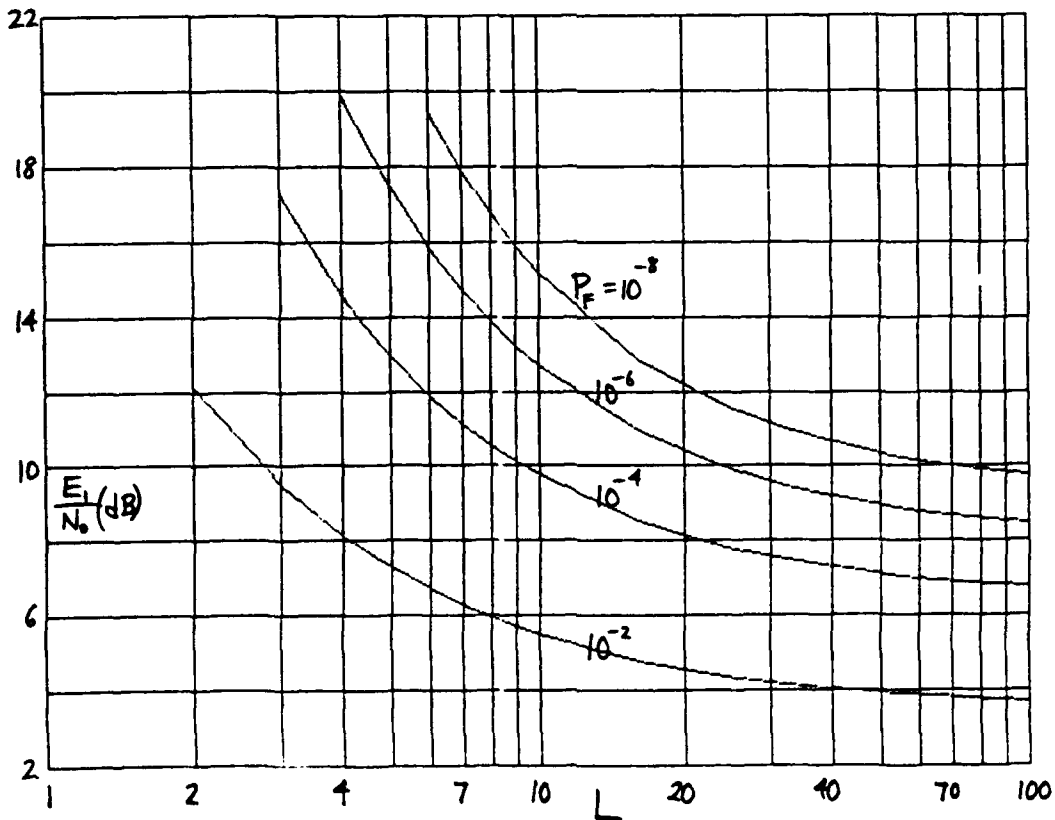


Figure 11. Required SNR for $P_D = .5$, $K = 4$, $M = 2$, $\text{Cov}_1 = 1$

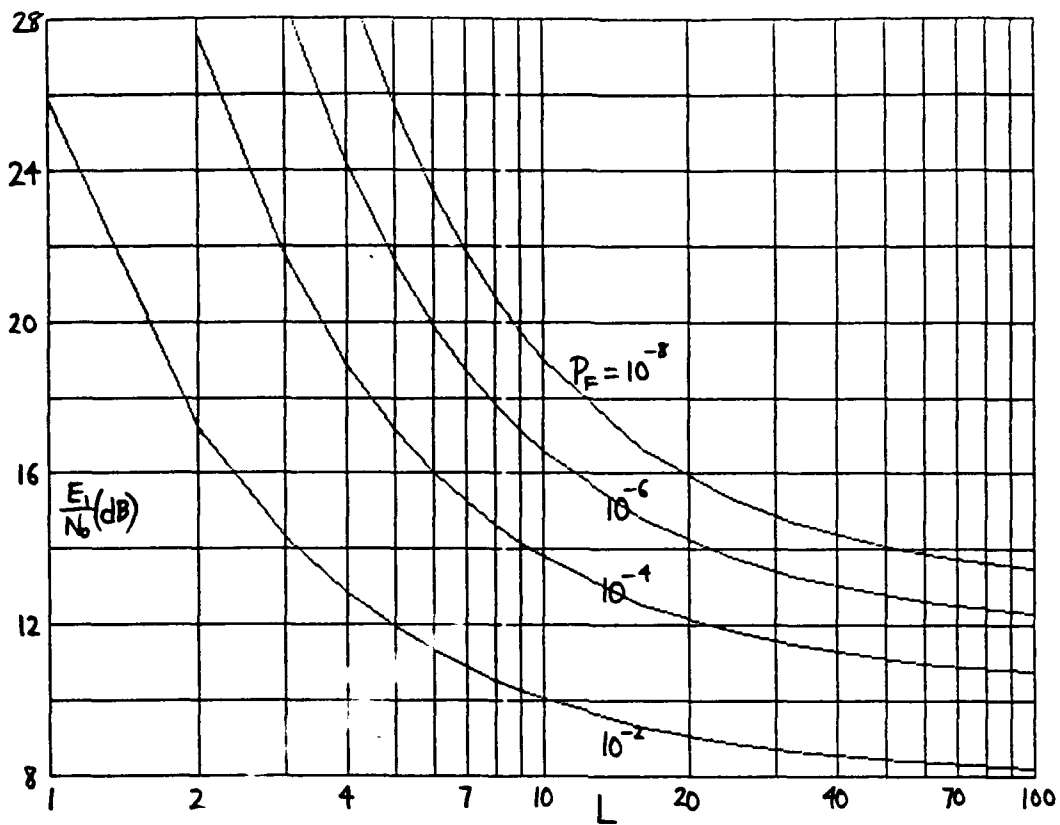


Figure 12. Required SNR for $P_D = .9$, $K = 4$, $M = 2$, $\text{Cov}_1 = \sqrt{.5}$

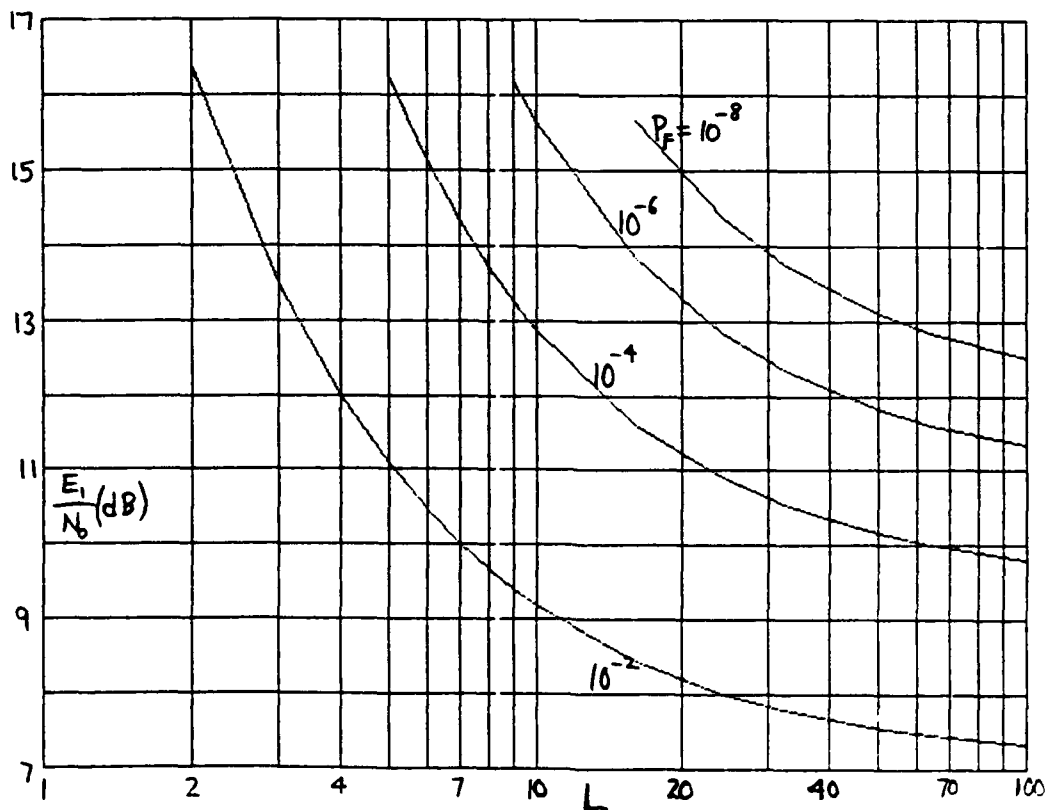


Figure 13. Required SNR for $P_D = .9$, $K = 4$, $M = 3$, $\text{Cov}_1 = \sqrt{.5}$

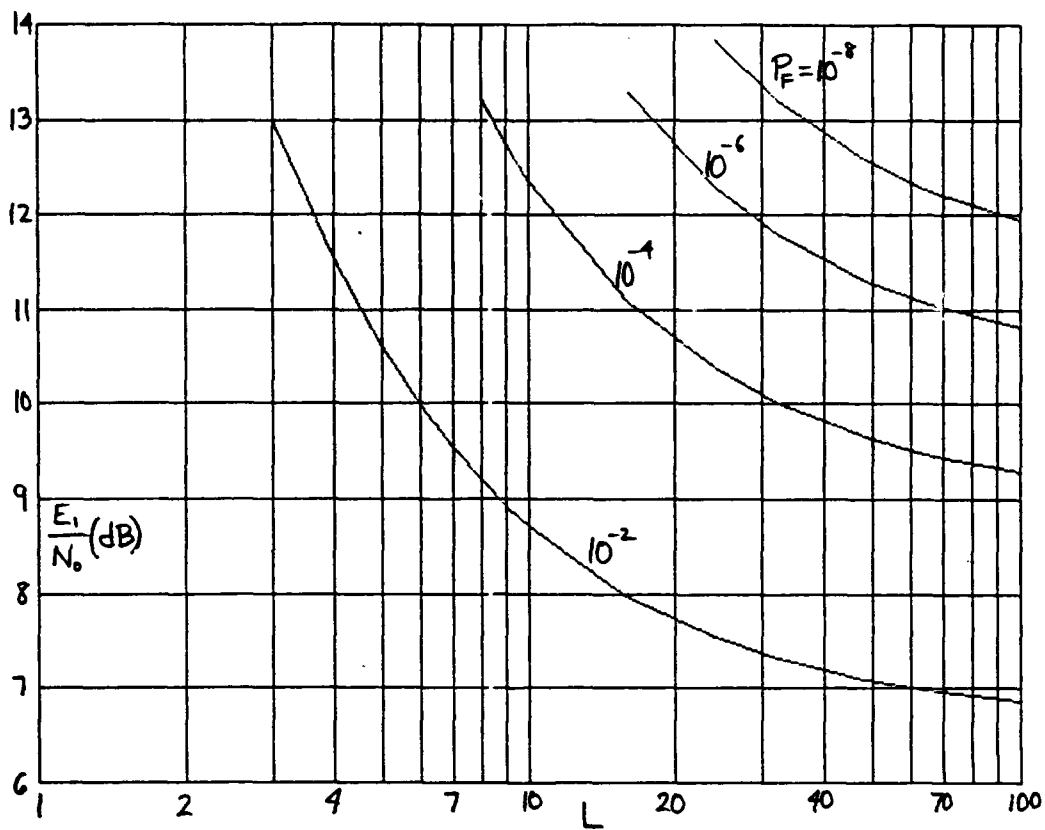


Figure 14. Required SNR for $P_D = .9$, $K = 4$, $M = 4$, $\text{Cov}_1 = \sqrt{.5}$

SUMMARY

The false alarm and detection probabilities, for a normalizer operating in a fading medium, have been derived exactly in the form of series expansions which are capable of efficient recursive numerical evaluation. These results allow for multiple signal pulses with an arbitrary covariance matrix between the amplitude fadings applied to each signal pulse. Lack of knowledge of the additive background noise level requires the use of a group of noise-only bins, from which an estimate of the noise level is made and used to establish a threshold for the normalizer, in order to realize a specified false alarm probability. The degradation associated with this noisy threshold has been evaluated quantitatively; a program for its evaluation in a specific environment is furnished in appendix D. Some examples of its use have been presented in the figures herein.

The expansion of the characteristic function in a series, as developed here in appendix A, will be extended in [6] to include a more general form of transformation than (11). This should allow for series with more rapid convergence properties than those derived here in (34) - (36), for example. In particular, extensions of the Hermite series and the generalized Laguerre series used in [5] will be developed.

APPENDIX A. SERIES EXPANSION OF CHARACTERISTIC FUNCTION (7)

The transformation of variable in (11) is given by

$$z = \frac{1}{1 - i\xi\beta}, \quad -i\xi = \frac{1 - z}{\beta z}, \quad (\text{A-1})$$

which yields

$$1 - i2\xi = \frac{1 - z(1 - \frac{1}{2}\beta)}{\frac{1}{2}\beta z}, \quad 1 - i2\xi Q = \frac{1 - z(1 - \frac{1}{2}\beta/Q)}{\frac{1}{2}\beta z/Q}. \quad (\text{A-2})$$

When these expressions are substituted into characteristic function (7) and simplified, there follows

$$\begin{aligned} f_Y(\xi) = F z^K & \frac{[1 - z(1 - \frac{1}{2}\beta)]^{K(\frac{1}{2}M-1)}}{\left(\prod_{m=1}^M \prod_{k=1}^K \left[1 - z \left(1 - \frac{1}{2}\beta/Q_{mk} \right) \right] \right)^{\frac{1}{2}}} \times \\ & \times \exp \left[- (1 - z) \sum_{m=1}^M \sum_{k=1}^K \frac{\frac{1}{2}e_{mk}/Q_{mk}}{1 - z \left(1 - \frac{1}{2}\beta/Q_{mk} \right)} \right], \end{aligned} \quad (\text{A-3})$$

where F is given by (13).

Except for the factor $F z^K$, we now take the natural logarithm of the right-hand side of (A-3) and expand in a power series in z . After considerable simplification and grouping of similar terms together, we find that this logarithm is given by

$$\sum_{p=0}^{\infty} a_p z^p, \quad (\text{A-4})$$

where the coefficients $\{a_p\}$ are given in (13) and (14).

Recalling the factor $F z^K$ in (A-3), we obtain the characteristic

function of the fixed-threshold processor output γ in the form

$$f_{\gamma}(\xi) = F z^K \exp\left(\sum_{p=0}^{\infty} a_p z^p\right). \quad (A-5)$$

We now employ [5; page 93] to obtain the desired series in powers of z according to

$$f_{\gamma}(\xi) = F z^K \sum_{p=0}^{\infty} f_p z^p, \quad (A-6)$$

where z is given by (A-1), and

$$f_0 = \exp(a_0), \quad f_p = \frac{1}{p} \sum_{n=1}^p n a_n f_{p-n} \quad \text{for } p \geq 1. \quad (A-7)$$

Expression (A-6) is the desired result. Numerous recursions can be developed from these equations that enable rapid efficient numerical evaluation of characteristic function f_{γ} . More will be said on this topic in the program for evaluation of the detection probability for the normalizer output; see appendix D.

APPENDIX B. INCOMPLETE GAMMA RELATIONS

Define the function

$$\underline{C}(v, x) = \int_0^x dy \frac{y^{v-1} \exp(-y)}{\Gamma(v)} \quad \text{for } x \geq 0, \quad v > 0. \quad (\text{B-1})$$

(This is $P(v, x)$ in [3; 6.5.1].) Then, from [3; 6.5.2 and 6.5.3],

$$\underline{C}(v, x) = \frac{\gamma(v, x)}{\Gamma(v)} = 1 - \frac{\Gamma(v, x)}{\Gamma(v)}, \quad (\text{B-2})$$

in terms of incomplete gamma functions. The function $\underline{C}(v, x)$ is a cumulative distribution function; that is,

$$\underline{C}(v, \infty) = 1. \quad (\text{B-3})$$

It is also useful to notice that

$$\underline{C}(\infty, x) = 0, \quad (\text{B-4})$$

since the unit area of the integrand of (B-1) moves progressively farther out in the positive direction as v increases. This can be seen from the fact that the density (integrand) in (B-1) has mean v and standard deviation $\sqrt{v}$.

Also, define function

$$p(v, x) = \frac{x^{v-1} \exp(-x)}{\Gamma(v)} \quad \text{for } x \geq 0, \quad v > 0. \quad (\text{B-5})$$

Then function (B-1) satisfies recurrence

$$\underline{C}(v, x) = \int_0^x dy \, p(v, y) = \underline{C}(v-1, x) - p(v, x) \quad \text{for } v > 1, \quad (\text{B-6})$$

while

$$p(v, x) = p(v-1, x) \frac{x}{v-1} \quad \text{for } v > 1. \quad (\text{B-7})$$

Convenient starting values for (B-6) and (B-7) are

$$\underline{C}(\frac{1}{2}, x) = 2 \Phi((2x)^{\frac{1}{2}}) - 1, \quad \underline{C}(1, x) = 1 - \exp(-x), \quad (\text{B-8})$$

$$p(\frac{1}{2}, x) = (\pi x)^{-\frac{1}{2}} \exp(-x), \quad p(1, x) = \exp(-x). \quad (\text{B-9})$$

As an example, with $v = n$,

$$p(n, x) = p(n-1, x) \frac{x}{n-1} \quad \text{for } n = 2, 3, 4, \dots \quad (\text{B-10})$$

$$\underline{C}(n, x) = \underline{C}(n-1, x) - p(n, x) \quad \text{for } n = 2, 3, 4, \dots \quad (\text{B-11})$$

along with the starting values for $p(1, x)$ and $\underline{C}(1, x)$ given in (B-9) and (B-8), respectively.

As a second example, with $v = n + \frac{1}{2}$,

$$p(n+\frac{1}{2}, x) = p(n-\frac{1}{2}, x) \frac{x}{n-\frac{1}{2}} \quad \text{for } n = 1, 2, 3, \dots \quad (\text{B-12})$$

$$\underline{C}(n+\frac{1}{2}, x) = \underline{C}(n-\frac{1}{2}, x) - p(n+\frac{1}{2}, x) \quad \text{for } n = 1, 2, 3, \dots \quad (\text{B-13})$$

along with the starting values for $p(\frac{1}{2}, x)$ and $\underline{C}(\frac{1}{2}, x)$ given in (B-9) and (B-8), respectively.

We also define exceedance distribution function

$$\begin{aligned}\underline{E}(\nu, x) &= 1 - \underline{C}(\nu, x) = \int_x^{\infty} dy \frac{y^{\nu-1} \exp(-y)}{\Gamma(\nu)} = \\ &= \Gamma(\nu, x) / \Gamma(\nu) \quad \text{for } x \geq 0, \quad \nu > 0.\end{aligned}\tag{B-14}$$

Then

$$\underline{E}(\nu, x) = \underline{E}(\nu-1, x) + p(\nu, x) \quad \text{for } \nu > 1,\tag{B-15}$$

with recursion (B-7) for $p(\nu, x)$, and starting values

$$\underline{E}(\frac{1}{2}, x) = 2 \Phi(-(2x)^{\frac{1}{2}}), \quad \underline{E}(1, x) = \exp(-x).\tag{B-16}$$

We also have closed form

$$\underline{E}(n, x) = \exp(-x) \sum_{k=0}^{n-1} x^k / k! \quad \text{for } n \geq 1.\tag{B-17}$$

APPENDIX C. MOMENTS OF NORMALIZER OUTPUT FOR SIGNAL ABSENT

For ease in determining the value of threshold v to use in (49), in order to realize specified false alarm probability P_F , it is helpful to know the moments of the normalizer output γ/γ_0 when signal is absent. Here, we will determine the n -th moment of ratio γ/γ_0 :

$$\mu_n \equiv \overline{(\gamma/\gamma_0)^n} = \overline{\gamma^n} \overline{\gamma_0^{-n}}, \quad (C-1)$$

where we have used the statistical independence of γ and γ_0 . By reference to (24), we find

$$\overline{\gamma_0^{-n}} = \int_0^{\infty} dx \, x^{-n} \frac{x^{L-1} \exp(-x/2)}{2^L (L-1)!} = \frac{(L-1-n)!}{2^n (L-1)!} \quad \text{for } L \geq n+1. \quad (C-2)$$

In a similar manner, for signal absent (replacing L by K),

$$\overline{\gamma^n} = \int_0^{\infty} dx \, x^n \frac{x^{K-1} \exp(-x/2)}{2^K (K-1)!} = \frac{(K-1+n)!}{(K-1)!} \frac{2^n}{2^K}. \quad (C-3)$$

The first two moments of ratio γ/γ_0 are then given by

$$\mu_1 = \frac{K}{L-1} \quad \text{for } L \geq 2, \quad \mu_2 = \frac{(K+1)K}{(L-1)(L-2)} \quad \text{for } L \geq 3. \quad (C-4)$$

The variance of the normalizer output is then given by

$$\text{Var}(\gamma/\gamma_0) = \mu_2 - \mu_1^2 = \frac{(L+K-1)K}{(L-1)^2 (L-2)} \quad \text{for } L \geq 3, \quad (C-5)$$

when signal is absent.

APPENDIX D. PROGRAM FOR NORMALIZER SIGNAL-TO-NOISE RATIO

The program in this appendix computes the false alarm probability by means of (49) and the detection probability by means of (36). Since exceedance distribution function value (36) is independent of β , a guess at β is required, preferably one that leads to a rapidly convergent series. Here, this choice has been made by trial and error and by observing and noting values for β that were advantageous for "nearby" parameter values. For the purpose of ease of programming and for interpretation of the recursions in the listing below, we have utilized the following notation:

$$A(p) = p a_p \quad \text{for } p \geq 1, \quad (D-1)$$

$$F(p) = F f_p \quad \text{for } p \geq 0, \quad (D-2)$$

$$Be(m,k) = \frac{\beta}{2Q_{mk}} \frac{e_{mk}}{Q_{mk}}, \quad Bl(m,k) = 1 - \frac{\beta}{2Q_{mk}}, \quad (D-3)$$

$$Cp(m,k) = Bl(m,k) + p Be(m,k), \quad Bp(m,k) = Bl(m,k)^{p-1}, \quad (D-4)$$

$$Dp = K \left(\frac{M}{2} - 1 \right) \left(1 - \frac{\beta}{2} \right)^p. \quad (D-5)$$

The program below allows for arbitrary values of the deterministic signal-to-noise ratios $\{D_{km}/N_o\}$; however, it was utilized here only for the special case of all $\{D_{km}/N_o\}$ equal to zero. Also, the program is written such that $\{D_{km}/N_o\}$ are held fixed while E_1/N_o is varied to meet the specified detection probability P_D .

```

10 ! NORMALIZER, NUWC-NPT TR 10275, 21 JANUARY 1993
20 Pf=.01 ! FALSE ALARM PROBABILITY
30 Pd=.9 ! DETECTION PROBABILITY
40 K=2 ! NUMBER OF SIGNAL PULSES
50 L=10 ! NUMBER OF NOISE-ONLY BINS
60 M=2 ! NUMBER OF FADING COMPONENTS
70 DATA 1.,1. ! POWER RATIOS r(m) FOR 1 <= m <= M
80 DATA 0.,0.,0.,0. ! Dkm/No FOR 1 <= k <= K, 1 <= m <= M
90 DATA 1.,2. ! TIMES tk (SEC) FOR 1 <= k <= K
100 DATA 0.,0. ! FREQUENCIES fk (HZ) FOR 1 <= k <= K
110 E1no0=20. ! E1/No STARTING VALUE (GUESS)
120 E1no1=E1no0*.1 ! E1/No INCREMENT
130 Ef=1.E-15 ! TOLERANCE ON Pf
140 Ed=1.E-10 ! TOLERANCE ON Pd
150 P=500 ! MAXIMUM NUMBER OF TERMS IN SUM (36)
160 OPTION BASE 1
170 COM DOUBLE K,L,M,P ! INTEGERS (NOT DOUBLE PRECISION)
180 DOUBLE Ms,Ks,Js,L1 ! INTEGERS
190 DIM Rs(5),Dn(10,5),Ts(10),Fs(10),Psi(5),Cbar(10,10)
200 DIM U(10,10),V(10,10),Eig(10),Sq(10,5)
210 COM V,B2,Prod(5,10),Es(5,10),E(0:500),B1(5,10)
220 COM Be(5,10),Bp(5,10),Cp(5,10),F(0:500),A(500)
230 REDIM Rs(M),Dn(K,M),Ts(K),Fs(K),Psi(M),Cbar(K,K)
240 REDIM U(K,K),V(K,K),Eig(K),Sq(K,M)
250 REDIM Prod(M,K),Es(M,K),E(0:P),B1(M,K)
260 REDIM Be(M,K),Bp(M,K),Cp(M,K),F(0:P),A(P)
270 PRINT
280 PRINT "Pf =";Pf;" K =";K;" L =";L;" M =";M
290 READ Rs(*),Dn(*),Ts(*),Fs(*) ! Dn(*) WAS FILLED IN THE ORDER:
300 S=0. ! Dn(1,1),Dn(1,2),Dn(1,3),...,Dn(k,m),...,Dn(K,M)
310 FOR Ms=1 TO M
320 S=S+Rs(Ms)
330 NEXT Ms
340 FOR Ms=1 TO M
350 Psi(Ms)=Rs(Ms)/S ! [1; (191)]
360 NEXT Ms
370 FOR Ks=1 TO K
380 FOR Js=1 TO K
390 Cov=FNCov(Ts(Ks)-Ts(Js),Fs(Ks)-Fs(Js))
400 Cbar(Ks,Js)=Cov ! NORMALIZED COVARIANCE MATRIX [1; (127)]
410 NEXT Js
420 NEXT Ks
430 MAT U=Cbar
440 CALL Svd(K,K,U(*),V(*),Eig(*)) ! OUTPUTS: U(*),V(*),Eig(*)
450 PRINT "EIGENVALUES:"
460 PRINT Eig(*)
470 FOR Ms=1 TO M
480 T=2.*Psi(Ms)
490 FOR Ks=1 TO K
500 Prod(Ms,Ks)=T*Eig(Ks)
510 Sq(Ks,Ms)=SQR(2.*Dn(Ks,Ms))
520 NEXT Ks
530 NEXT Ms
540 FOR Ks=1 TO K
550 FOR Ms=1 TO M
560 S=0.

```

```

570   FOR Js=1 TO K
580   S=S+V(Js,Ks)*Sq(Js,Ms)
590   NEXT Js
600   Es(Ms,Ks)=S*S
610   NEXT Ms
620   NEXT Ks
630   IF L<3 THEN 680
640   Mean=K/(L-1) ! (C-4)
650   Var=(L+K-1)*K/((L-1)^2*(L-2)) ! (C-5)
660   V0=Mean+SQR(Var)*3. ! THRESHOLD STARTING VALUE (GUESS)
670   GOTO 690
680   V0=200 ! THRESHOLD STARTING VALUE (GUESS)
690   V1=V0*.1 ! THRESHOLD INCREMENT
700   CALL Inversfunction1(-Pf,Ef,V0,V1,V)
710   Beta=23. ! SCALING FACTOR (GUESS)
720   PRINT "Threshold V =";V;" Beta =";Beta
730   B2=Beta/2.
740   X=V/(B2+V)
750   Tk=(B2/(B2+V))^L
760   L1=L-1
770   FOR Ks=1 TO K-1
780   Tk=Tk*X*(Ks+L1)/Ks
790   NEXT Ks
800   FOR Ks=K TO K+P
810   Tk=Tk*X*(Ks+L1)/Ks
820   E(Ks-K)=Tk
830   NEXT Ks
840   S=Tk
850 A: FOR Ks=K+P+1 TO 2000
860   Tk=Tk*X*(Ks+L1)/Ks
870   S=S+Tk
880   IF Tk<=S*1.E-15 THEN 920 ! ERROR TOLERANCE
890   NEXT Ks
900   PRINT "2000 TERMS ARE INSUFFICIENT IN LINE A:"
910   PAUSE
920   E(P)=S
930   FOR Ks=P-1 TO 0 STEP -1
940   E(Ks)=S+S+E(Ks)
950   NEXT Ks
960   CALL Inversfunction2(Pd,Ed,E1no0,E1no1,E1no)
970   Db=10.*LGT(E1no) ! SNR E1/No in dB
980   Y1=MIN(F(*))
990   Y2=MAX(F(*))
1000  PRINT "dB =";Db;" Y1 = ";Y1;" Y2 =";Y2
1010  GINIT
1020  PLOTTER IS "GRAPHICS"
1030  GRAPHICS ON
1040  WINDOW 0.,50.,Y1,Y2
1050  LINE TYPE 3
1060  GRID 10.,.1
1070  LINE TYPE 1
1080  FOR Js=0 TO P
1090  PLOT Js,F(Js) ! EXPANSION COEFFICIENTS F f(sub p)
1100  NEXT Js
1110  PENUP
1120  PAUSE
1130  END
1140  !

```

```

1150 DEF FNCov(Tau,Nu)      ! NORMALIZED COVARIANCE:
1160   Rho=.5                ! R11(Tau,Nu)/R11(0,0); SEE (4)
1170   A=-.5*LOG(Rho)
1180   B=0.
1190   Cov=EXP(-A*ABS(Tau)-B*ABS(Nu)) ! EXPONENTIAL
1200   RETURN Cov
1210 FNEED
1220 !
1230 DEF FNPf(V)             ! PROBABILITY OF FALSE ALARM
1240   COM DOUBLE K,L        ! INTEGERS
1250   DOUBLE Ks,L1          ! INTEGERS
1260   IF V<=0. THEN RETURN 1.
1270   V1=V/(1.+V)
1280   Pf=T=EXP(-L*LOG(1.+V))
1290   L1=L-1
1300   FOR Ks=1 TO K-1
1310     T=T*V1*(L1+Ks)/Ks
1320   Pf=Pf+T
1330   NEXT Ks
1340   RETURN -Pf            ! - TO YIELD INCREASING FUNCTION OF V
1350 FNEED
1360 !
1370 DEF FNPd(E1no)          ! PROBABILITY OF DETECTION
1380   COM DOUBLE K,L,M,P    ! INTEGERS
1390   COM V,B2,Prod(*),Es(*),E(*),B1(*),Be(*),Bp(*),Cp(*),F(*),A(*)
1400   DOUBLE Ms,Ks,Ps       ! INTEGERS
1410   Tol=1.E-8             ! RELATIVE ERROR OF SUM
1420   F=1.
1430   S=0.
1440   FOR Ms=1 TO M
1450     FOR Ks=1 TO K
1460       Q=1.+E1no*Prod(Ms,Ks)
1470       Eq=Es(Ms,Ks)/Q
1480       F=F*Q
1490       S=S+Eq
1500       Bq=B2/Q
1510       B1(Ms,Ks)=1.-Bq
1520       Be(Ms,Ks)=Bq*Eq
1530       Bp(Ms,Ks)=1.
1540       Cp(Ms,Ks)=1.-Bq
1550     NEXT Ks
1560   NEXT Ms
1570   F(0)=EXP(K*LOG(B2)-.5*S)/SQR(F)
1580   Pd=1.-F(0)*E(0)
1590   B21=1.-B2
1600   Dp=K*(M/2.-1.)
1610   T1=1.
1620 B: FOR Ps=1 TO P
1630   Dp=Dp*B21
1640   S=0.
1650   FOR Ms=1 TO M
1660     FOR Ks=1 TO K
1670       IF Ps=1 THEN 1690
1680       Bp(Ms,Ks)=T1=Bp(Ms,Ks)*B1(Ms,Ks)
1690       Cp(Ms,Ks)=T2=Cp(Ms,Ks)+Be(Ms,Ks)
1700       S=S+T1*T2
1710     NEXT Ks
1720   NEXT Ms

```

```

1730  A(Ps)=.5*S-Dp
1740  S=0.
1750  FOR Ms=1 TO Ps
1760  S=S+A(Ms)*F(Ps-Ms)
1770  NEXT Ms
1780  F(Ps)=F=S/Ps          ! APPENDIX A
1781  Do=Del
1790  Del=F*E(Ps)
1800  Pd=Pd-Del
1801  IF Ps=1 THEN 1820
1810  IF ABS(Do)+ABS(Del)<=Pd*Tol THEN 1850
1820  NEXT Ps
1830  PRINT P;"TERMS ARE INSUFFICIENT IN LINE B:"
1840  PAUSE
1850  FOR Ms=Ps+1 TO P
1860  F(Ms)=A(Ms)=0.
1870  NEXT Ms
1880  PRINT "Pd =";Pd;"      Ps =";Ps,"E1/No =";E1no
1890  RETURN Pd
1900  FNEND
1910  !
1920  SUB Inversfunction1(Desired,Error,X1,Del,X2)
1930  X2=X1+Del
1940  F1=FNPF(X1)          ! FALSE ALARM PROBABILITY
1950  F2=FNPF(X2)
1960  IF F2>=Desired THEN 2010
1970  X1=X2
1980  X2=X2+Del
1990  F1=F2
2000  GOTO 1950
2010  IF F1<=Desired THEN 2070
2020  X2=X1
2030  X1=X1-Del
2040  F2=F1
2050  F1=FNPF(X1)
2060  GOTO 2010
2070  Xa=X1
2080  Xb=X2
2090  IF (F2-Desired)<<(Desired-F1) THEN 2160
2100  T=X1
2110  X1=X2
2120  X2=T
2130  T=F1
2140  F1=F2
2150  F2=T
2160  IF ABS(F2-Desired)<Error THEN 2260
2170  IF F2=F1 THEN 2260
2180  T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
2190  T=MAX(T,Xa)
2200  T=MIN(T,Xb)
2210  X1=X2
2220  X2=T
2230  F1=F2
2240  F2=FNPF(X2)
2250  GOTO 2160
2260  SUBEND
2270  !

```

```

2280 SUB Inversfunction2(Desired,Error,X1,Del,X2)
2290 X2=X1+Del
2300 F1=FNPd(X1) ! DETECTION PROBABILITY
2310 F2=FNPd(X2)
2320 IF F2>=Desired THEN 2370
2330 X1=X2
2340 X2=X2+Del
2350 F1=F2
2360 GOTO 2310
2370 IF F1<=Desired THEN 2430
2380 X2=X1
2390 X1=X1-Del
2400 F2=F1
2410 F1=FNPd(X1)
2420 GOTO 2370
2430 Xa=X1
2440 Xb=X2
2450 IF (F2-Desired)<<(Desired-F1) THEN 2520
2460 T=X1
2470 X1=X2
2480 X2=T
2490 T=F1
2500 F1=F2
2510 F2=T
2520 IF ABS(F2-Desired)<Error THEN 2620
2530 IF F2=F1 THEN 2620
2540 T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
2550 T=MAX(T,Xa)
2560 T=MIN(T,Xb)
2570 X1=X2
2580 X2=T
2590 F1=F2
2600 F2=FNPd(X2)
2610 GOTO 2520
2620 SUBEND
2630 !
2640 SUB Sod(DOUBLE M,N,REAL A(*),V(*),W(*))
2650 ! THIS SUBROUTINE COMPUTES THE SINGULAR VALUE DECOMPOSITION
2660 ! OF AN ARBITRARY REAL MxN MATRIX A: A = U W Vt, M >= N.
2670 ! U IS MxN, V IS NxN, W IS NxN: W = DIAG(D(n)).
2680 ! NUMERICAL RECIPES, PAGES 60-64
2690 IF M>=N THEN 2720 ! A(*) IS OVER-WRITTEN
2700 PRINT "M<N IS DISALLOWED"
2710 PAUSE
2720 DOUBLE I,J,K,L,Ils,Nm,Jj ! INTEGERS (NOT DOUBLE PRECISION)
2730 G=Scale=Anorm=0.
2740 FOR I=1 TO N
2750 L=I+1
2760 Rv1(I)=Scale*G
2770 G=S=Scale=0.
2780 IF I>M THEN 3060
2790 FOR K=I TO M
2800 Scale=Scale+ABS(A(K,I))
2810 NEXT K
2820 IF Scale=0. THEN 3060
2830 FOR K=I TO M
2840 Aa=A(K,I)=A(K,I)/Scale
2850 S=S+Aa*Aa
2860 NEXT K

```

```

2870 F=A(I,I)
2880 G=-SQR(S)
2890 IF F<0. THEN G=-G
2900 H=F*G-S
2910 A(I,I)=F-G
2920 IF I=N THEN 3030
2930 FOR J=L TO N
2940 S=0.
2950 FOR K=I TO M
2960 S=S+A(K,I)*A(K,J)
2970 NEXT K
2980 F=S/H
2990 FOR K=I TO M
3000 A(K,J)=A(K,J)+F*A(K,I)
3010 NEXT K
3020 NEXT J
3030 FOR K=I TO M
3040 A(K,I)=A(K,I)*Scale
3050 NEXT K
3060 W(I)=Scale*G
3070 G=S/Scale=0.
3080 IF (I>M) OR (I=N) THEN 3380
3090 FOR K=L TO N
3100 Scale=Scale+ABS(A(I,K))
3110 NEXT K
3120 IF Scale=0. THEN 3380
3130 FOR K=L TO N
3140 Aa=A(I,K)=A(I,K)/Scale
3150 S=S+Aa*Aa
3160 NEXT K
3170 F=A(I,L)
3180 G=-SQR(S)
3190 IF F<0. THEN G=-G
3200 H=F*G-S
3210 A(I,L)=F-G
3220 FOR K=L TO N
3230 Rv1(K)=A(I,K)/H
3240 NEXT K
3250 IF I=M THEN 3350
3260 FOR J=L TO M
3270 S=0.
3280 FOR K=L TO N
3290 S=S+A(J,K)*A(I,K)
3300 NEXT K
3310 FOR K=L TO N
3320 A(J,K)=A(J,K)+S*Rv1(K)
3330 NEXT K
3340 NEXT J
3350 FOR K=L TO N
3360 A(I,K)=A(I,K)*Scale
3370 NEXT K
3380 Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
3390 NEXT I
3400 FOR I=N TO 1 STEP -1
3410 IF I>=N THEN 3580
3420 IF G=0. THEN 3550

```

```

3430   FOR J=L TO N
3440   V(J,I)=A(I,J)/A(I,L)/G
3450   NEXT J
3460   FOR J=L TO N
3470   S=0.
3480   FOR K=L TO N
3490   S=S+A(I,K)*V(K,J)
3500   NEXT K
3510   FOR K=L TO N
3520   V(K,J)=V(K,J)+S*V(K,I)
3530   NEXT K
3540   NEXT J
3550   FOR J=L TO N
3560   V(I,J)=V(I,J)+0.
3570   NEXT J
3580   V(I,I)=1.
3590   G=Rv1(I)
3600   L=I
3610   NEXT I
3620   FOR I=N TO 1 STEP -1
3630   L=I+1
3640   G=W(I)
3650   IF I>=N THEN 3690
3660   FOR J=L TO N
3670   A(I,J)=0.
3680   NEXT J
3690   IF G=0. THEN 3860
3700   G=1./G
3710   IF I=N THEN 3820
3720   FOR J=L TO N
3730   S=0.
3740   FOR K=L TO M
3750   S=S+A(K,I)*A(K,J)
3760   NEXT K
3770   F=S/A(I,I)*G
3780   FOR K=I TO M
3790   A(K,J)=A(K,J)+F*A(K,I)
3800   NEXT K
3810   NEXT J
3820   FOR J=I TO M
3830   A(J,I)=A(J,I)*G
3840   NEXT J
3850   GOTO 3890
3860   FOR J=I TO M
3870   A(J,I)=0.
3880   NEXT J
3890   A(I,I)=A(I,I)+1.
3900   NEXT I
3910   FOR K=N TO 1 STEP -1
3920   FOR Its=1 TO 30
3930   FOR L=K TO 1 STEP -1
3940   Nm=L-1
3950   IF (ABS(Rv1(L))+Anorm)=Anorm THEN 4170
3960   IF (ABS(W(Nm))+Pnorm)=Pnorm THEN 3980
3970   NEXT L
3980   C=0.
3990   S=1.

```

```

4000   FOR I=L TO K
4010   F=S*Rv1(I)
4020   Rv1(I)=C*Rv1(I)
4030   IF (ABS(F)+Anorm)=Anorm THEN 4170
4040   G=W(I)
4050   H=SQR(F*F+G*G)
4060   W(I)=H
4070   H=1./H
4080   C=G*H
4090   S=-F*H
4100   FOR J=1 TO M
4110   Y=A(J,Nm)
4120   Z=A(J,I)
4130   A(J,Nm)=Y*C+Z*S
4140   A(J,I)=-Y*S+Z*C
4150   NEXT J
4160   NEXT I
4170   Z=W(K)
4180   IF L<>K THEN 4250
4190   IF Z>=0. THEN 4240
4200   W(K)=-Z
4210   FOR J=1 TO N
4220   V(J,K)=-V(J,K)
4230   NEXT J
4240   GOTO 4780
4250   IF Its<30 THEN 4280
4260   PRINT "NO CONVERGENCE IN 30 ITERATIONS"
4270   PAUSE
4280   X=W(L)
4290   Nm=K-1
4300   Y=W(Nm)
4310   G=Rv1(Nm)
4320   H=Rv1(K)
4330   F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
4340   G=SQR(F*F+1.)
4350   Aa=ABS(G)
4360   IF F<0. THEN Aa=-Aa
4370   F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
4380   C=S=1.
4390   FOR J=L TO Nm
4400   I=J+1
4410   G=Rv1(I)
4420   Y=W(I)
4430   H=S*G
4440   G=C*G
4450   Z=SQR(F*F+H*H)
4460   Rv1(J)=Z
4470   C=F/Z
4480   S=H/Z
4490   F=X*C+G*S
4500   G=-X*S+G*C
4510   H=Y*S
4520   Y=Y*C
4530   FOR Jj=1 TO N
4540   X=V(Jj,J)
4550   Z=V(Jj,I)
4560   V(Jj,J)=X*C+Z*S
4570   V(Jj,I)=-X*S+Z*C
4580   NEXT Jj

```

```

4590     Z=SQR(F*F+H*H)
4600     W(J)=Z
4610     IF Z=0. THEN 4650
4620     Z=1./Z
4630     C=F*Z
4640     S=H*Z
4650     F=C*G+S*Y
4660     X=-S*G+C*Y
4670     FOR Jj=1 TO M
4680     Y=A(Jj,J)
4690     Z=A(Jj,I)
4700     A(Jj,J)=Y*C+Z*S
4710     A(Jj,I)=-Y*S+Z*C
4720     NEXT Jj
4730     NEXT J
4740     RV1(L)=0.
4750     RV1(K)=F
4760     W(K)=X
4770     NEXT Its
4780     NEXT K
4790     SUBEND

```

The values of threshold v and required input signal-to-noise ratio E_1/N_0 , that are yielded by the above program, are listed below, as a function of L , the number of noise-only bins. The values of scaling factor β are chosen to approximately minimize the number of terms, N_p , used in the p -summation in (36).

L	v	E_1/N_0 (dB)	β	N_p
1	198.5	27.263	535	14
2	15.98	19.123	85	12
3	6.099	16.566	48	10
4	3.503	15.317	36	10
5	2.398	14.577	31	10
6	1.804	14.086	28	10
7	1.439	13.738	26	10
8	1.193	13.477	25	9
9	1.018	13.274	24	9
10	.8861	13.113	23	9
16	.4962	12.569	21	8
24	.3115	12.267	20	8
32	.2267	12.117	19	8
48	.1467	11.966	18	8
64	.1084	11.891	18	8
100	.0683	11.810	18	8

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