ON CONSTRUCTING SOME STRONGLY WELL-COVERED GRAPHS

by

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Abstract

A graph is well-covered if every maximal independent set is a maximum independent set. If a well-covered graph G has the additional property that G-e is also well-covered for every line e in G, then we say the graph is strongly well-covered. We exhibit a construction which produces strongly well-covered graphs with arbitrarily large (even) independence number. The construction is in terms of a lexicographic graph product.

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INTRODUCTION

A set of points in a graph is independent if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph G is called the independence number of G and is denoted by $\alpha(G)$. A set of independent points which attains the maximum size is referred to as a maximum independent set. A set S of independent points in a graph is maximal (with respect to set inclusion) if the addition to S of any other point in the graph destroys the independence. In general, a maximal independent set in a graph is not necessarily maximum.

In a 1970 paper, Plummer [13] introduced the notion of considering graphs in which every maximal independent set is also maximum; he called a graph having this property a well-covered graph. The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. Campbell [2] characterized all cubic well-covered graphs with connectivity at most two, and Campbell and Plummer [3] proved that there are only four 3-connected cubic planar well-covered graphs. Royle and Ellingham [16] have recently completed the picture for cubic well-covered graphs by determining all 3-connected cubic well-covered graphs.

For a well-covered graph with no isolated points, the independence number is at most one-half the size of the graph. Well-covered graphs whose independence number is exactly one-half the size of the graph are called very well-covered graphs. The subclass of very well-covered graphs was characterized by Staples [17] and includes all well-covered trees and all well-covered bipartite graphs. Independently, Ravindra [14] characterized bipartite well-covered graphs and Favaron [6] characterized the very well-covered graphs. Recently, Dean and Zito [4] characterized the very well-covered graphs as a subset of a more general (than well-covered) class of graphs.

A set S of points in a graph dominates a set V of points if every point in V-S is adjacent to at least one point of S. Finbow and Hartnell [7] and Finbow, Hartnell, and Nowakowski [8] studied well-covered graphs relative to the concept of dominating sets. Finbow, Hartnell, and Nowakowski have also obtained a characterization of well-covered graphs with girth at least five [9].

A well-covered graph is 1-well-covered if and only if the deletion of any point from the graph leaves a graph which is also well-covered. A well-covered graph is strongly well-covered if and only if the deletion of any line from the graph leaves a graph which is also well-covered. A well-covered graph is in the class W_2 if and only if any two disjoint independent sets in the graph can be extended to disjoint maximum independent sets. Staples [18] showed that a well-covered graph is 1-well-covered if and only if it is in W_2. For the remainder of this paper, we use the W_2 nomenclature instead of referring to 1-well-covered graphs.

The class of well-covered graphs contains all complete graphs and all complete bipartite graphs of the form $K_{n,n}$. The only cycles which are well-covered are C_3, C_4, C_5, and C_7. We note that all complete graphs (except K_1) are also in W_2, but no complete bipartite graphs (except $K_{1,1}$) are in W_2. The cycles C_3 and C_5 are the only cycles in W_2. Also note that the only complete graphs which are strongly well-covered are K_1 and K_2, the only complete bipartite graphs which are strongly well-covered are $K_{1,1}$ and $K_{2,2}$, and C_4 is the only strongly well-covered cycle.

In [12], we show that a strongly well-covered graph with more than four points has minimum degree at least four and is 3-connected. Also, we show that all strongly well-covered graphs other than K_1 and K_2 have girth at most four, where the girth of a graph is the size of a smallest cycle in the graph and a graph with no cycles has infinite girth. In this paper we construct strongly well-covered graphs with triangles and strongly well-covered graphs with girth four.

PRELIMINARY RESULTS

Unless otherwise stated, we assume all graphs are connected. Note that a disconnected graph is a W_2 graph (strongly well-covered graph) if and only if each of its components is a W_2 graph (strongly well-covered graph). For notation and terminology not defined here, see [1].

For a point v in a graph G, let $N[v] = N(v) \cup \{v\}$. Define G_v to be the graph induced by G-N[v]. In other words, G_v is the graph that remains after deleting v and all of its neighbors. In [12], the author shows that if G is a strongly well-covered graph and G is not complete, then for all points v in G, the graph G_v cannot contain a component which is a line. Campbell and Plummer [3] proved the following very useful necessary condition for a graph to be well-covered. We will use this later to verify a construction.

Theorem 1. If a graph G is well-covered and is not complete, then G_v is well-covered for all v in G. Moreover, $\alpha(G_v) = \alpha(G) - 1$.

Recall from earlier that if G is a W_2 graph, then for all points v the graph G-v is well-covered (since a W_2 graph is 1-well-covered). On the other hand, we show in [12] that strongly well-covered is a *sufficient* condition for G to have the property that for all points v the graph G-v is not well-covered. We state this here as Theorem 2. As a consequence, K_2 is the only strongly well-covered graph which is also a W_2 graph .

Theorem 2. If G ($G \neq K_1$ or K_2) is strongly well-covered, then for all points v in G the graph G-v is not well-covered .

Next, we state the characterization of the strongly well-covered graphs with independence number two, as given in [12]. This characterization will be quite helpful in building strongly well-covered graphs with independence number larger that two.

Theorem 3. Suppose G is well-covered with $\alpha(G) = 2$. Then G is strongly well-covered if and only if G is ($|V(G)|$ - 2) - regular.

If $G \neq K_2$ is well-covered and e = uv is a line in G, consider maximal independent sets in the graph G-e. Suppose J is a maximal independent set in G-e which does not contain at least one endpoint of e (that is, $J \cap \{u,v\} \neq \{u,v\}$). Then it follows that J is a maximal independent set in G. Since G is well-covered, then $|J| = \alpha(G)$. Thus, every maximal independent set in G-e which does not contain at least one endpoint of e has size $\alpha(G)$. Consequently, to show that G-e is well-covered it suffices to show that every maximal independent set in the graph G-e which contains both endpoints of e has size $\alpha(G)$.

A CONSTRUCTION

For our construction, we use a product of well-covered graphs. Suppose H is a graph with n points and $\{G_i\}$, i = 1, . . . , n, is a family of disjoint graphs. Associate one member of $\{G_i\}$ with each point of H. We assume $V(H) = \{v_1, \ldots, v_n\}$ and G_i is associated with v_i, for all i. We define the lexicographic product graph of H and $\{G_i\}$, denoted $H\circ(G_1,\ldots,G_n)$, as follows: $V(H\circ(G_1, \ldots, G_n)) = V(G_1) \cup \ldots \cup V(G_n)$ and $E(H\circ(G_1,\ldots, G_n)) = E(G_1) \cup \ldots \cup E(G_n) \cup \{xy\colon\ x\in V(G_i), y\in V(G_j) \text{ and } v_i \sim v_j \text{ in } H\}$.

If every member of the family $\{G_i\}$ is the same graph G, then the lexicographic product consists of replacing each point of H with a copy of the graph G and joining the copies as indicated above. In this special case, we denote the lexicographic product by $H \circ G$.

Topp and Volkmann [19] considered several different types of products of well-covered graphs. In particular for the lexicographic product of well-covered graphs, they proved a theorem which implies the following theorem.

Theorem 4. If H is well-covered and $\{G_i\}$, $i = 1, \ldots, |V(H)|$, is a family of well-covered graphs with $\alpha(G_i) = \alpha(G_j)$ for all i and j, then $H \circ (G_1, \ldots, G_{|V(H)|})$ is well-covered. Moreover, $\alpha(H \circ (G_1, \ldots, G_{|V(H)|})) = \alpha(H)\alpha(G_1)$.

In the next theorem, we give an additional condition on a well-covered graph H which is *sufficient* to obtain a strongly well-covered lexicographic product graph.

Theorem 5. Suppose H is a well-covered graph with the following additional property: if $e = uv$ is a line in H, then H_{uv} is a well-covered graph and $\alpha(H_{uv}) = \alpha(H) - 1$, where H_{uv} is defined to be the graph $H - (N[u] \cup N[v])$.

Let $|V(H)| = n$ and $\{G_i\}$, $i = 1, \ldots, n$, be a family of strongly well-covered graphs with $\alpha(G_i) = 2$ for all i and each G_i is connected or $2K_1$. Let $L = H \circ (G_1, \ldots, G_n)$. Then L is strongly well-covered, and $\alpha(L) = 2\alpha(H)$.

Proof. By Theorem 4, the lexicographic product graph L is well-covered and $\alpha(L) = 2\alpha(H)$. Let $V(H) = \{u_1, u_2, \ldots, u_n\}$. Note the following about the structure of the lexicographic product graph: V(L) is the union of $V(G_i)$, for $i = 1, \ldots, n$. If $u_i \sim u_j$ in H, then $x \sim y$ in L for all $x \in V(G_i)$, for all $y \in V(G_j)$. If u_i is not adjacent to u_j in H ($i \neq j$), then x is not adjacent to y in L for all $x \in V(G_i)$, for all $y \in V(G_j)$. Also, $a \sim b$ in G_i if and only if $a \sim b$ in L, for all a and b in $V(G_i)$.

We proceed to show that L is *strongly* well-covered. Suppose e is a line in L. Then either e corresponds to a line in H, or e corresponds to a line in some G_j.

Case 1. Suppose $e = xy$ corresponds to a line in G_j, for some j. Since G_j is strongly well-covered with $\alpha(G_j) = 2$, then $\{x,y\}$ is a maximum independent set in the graph G_j-e. We consider the graph L-e.

To this end, consider the graph $H_{u_j} = H - N[u_j]$ (a subgraph of H). By Theorem 1, graph H_{u_j} is well-covered and $\alpha(H_{u_j}) = \alpha(H) - 1$. Let S_j be the subgraph of L corresponding to the components of H_{u_j}. Observe that S_j is a lexicographic product graph itself. Then since H_{u_j} is well-covered, by Theorem 4 the graph S_j is well-covered and $\alpha(S_j) = 2\alpha(H_{u_j}) = 2(\alpha(H) - 1)$.

Suppose J is a maximal independent set in L-e such that $J \supseteq \{x,y\}$. Since $\{x,y\}$ is a maximum independent set in G_j-e, then $J' = J-\{x,y\}$ must be contained in S_j. Since S_j is well-covered, each component of S_j is well-covered and it follows that $|J'| = \alpha(S_j) = 2(\alpha(H) - 1)$. Thus, $|J| = 2\alpha(H)$. So a maximal independent set in L-e which contains the endpoints of e has size $2\alpha(H)$. Thus, every maximal independent set in L-e has size $2\alpha(H)$ and hence is a maximum independent set in L-e. Therefore, L-e is well-covered.

Case 2. Suppose e corresponds to the line u_iu_j in H. Say $e = xy$, where $x \in V(G_i)$ and $y \in V(G_j)$.

By hypothesis, $H_{u_iu_j}$ is well-covered and $\alpha(H_{u_iu_j}) = \alpha(H) - 1$. Suppose $J \supseteq \{x,y\}$ is maximal independent in L-e. Let S_{ij} be the subgraph of L corresponding to $H_{u_iu_j}$. Observe that S_{ij} is a lexicographic product graph itself. Since $H_{u_iu_j}$ is well-covered, then by Theorem 4 the graph S_{ij} is well-covered with $\alpha(S_{ij}) = 2\alpha(H_{u_iu_j}) = 2\,(\alpha(H) - 1)$. Let $J' = J-\{x,y\}$. Then J' is contained in S_{ij} and is maximal independent in S_{ij}. Thus, $|J'| = 2\,(\alpha(H) - 1)$, and so $|J| = 2\alpha(H)$. Hence, a maximal independent set in L-e which contains $\{x,y\}$ necessarily has size $2\alpha(H)$. Since L is well-covered, then every maximal independent set in L-e has size $2\alpha(H)$. Thus, L-e is well-covered.

From Cases 1 and 2, we conclude that L-e is well-covered for all lines e in L. Therefore, L is strongly well-covered. []

Note in Theorem 5 that G_i is allowed to be disconnected. In this case, G_i must be $2K_1$ since $\alpha(G_i) = 2$, the graphs K_1 and K_2 are the only complete strongly well-covered graphs, and from above, for every point v in G, the graph G_v cannot contain a component which is a line.

Although the condition in Theorem 5 is very restrictive, there are well-covered graphs which satisfy the condition and, hence, lead to the construction of infinite families of strongly well-covered graphs. We now give five such infinite families based on the five well-covered graphs shown in Figure 1.

Corollary 6. Suppose H is one of the five graphs in Figure 1 and $\{G_i\}$, i = 1, ..., $|V(H)|$, is a family of strongly well-covered graphs with $\alpha(G_i) = 2$ and each G_i is connected or $2K_1$. Then $H\circ(G_1, ..., G_{|V(H)|})$ is strongly well-covered.

Proof. If H is one of the five graphs in Figure 1, it can be shown that H is well-covered, and for any line uv in H, the graph $H_{uv} = H - (N[u] \cup N[v])$ is well-covered with $\alpha(H_{uv}) = \alpha(H) - 1$. By Theorem 5, it follows that $H\circ(G_1, ..., G_{|V(H)|})$ is strongly well-covered. []

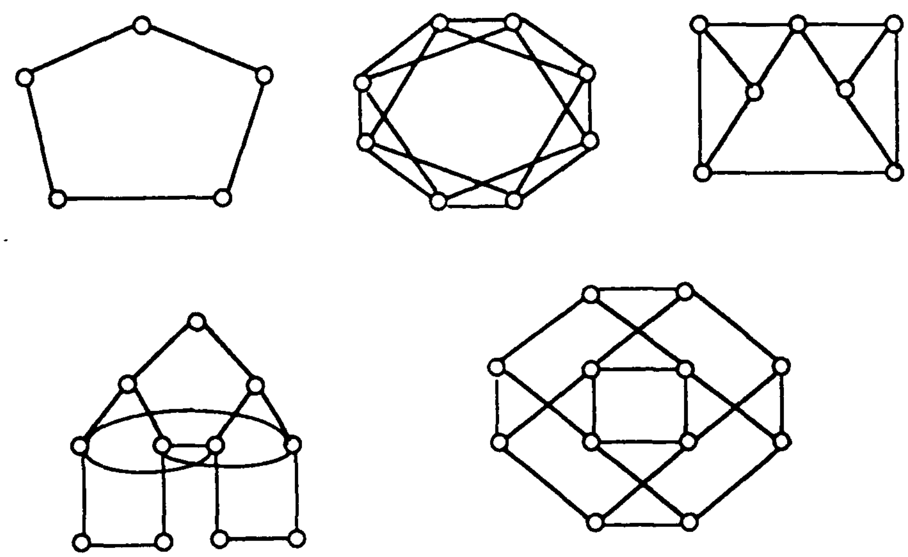

Figure 1

We stated earlier that a strongly well-covered graph has girth at most four. From the following corollary, we are assured of the existence of strongly well-covered graphs with girth exactly four.

Corollary 7. If H is a triangle-free well-covered graph which satisfies the conditions in Theorem 5, then $H \circ 2K_1$ is a girth 4 strongly well-covered graph.

Proof. If H is triangle-free, then $H \circ 2K_1$ is also triangle-free. Clearly $H \circ 2K_1$ has 4-cycles. The result then follows immediately from Theorem 5. []

For example, the graph in Figure 2 is $C_5 \circ 2K_1$. This graph was found by Royle [15] with the aid of a computer, and independently by the author.

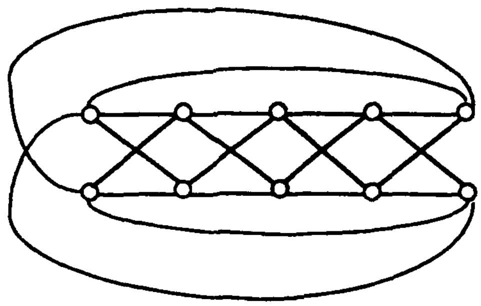

Figure 2

STRONGLY WELL-COVERED GRAPHS VIA W_2 GRAPHS OF GIRTH FOUR

From the graphs given in Figure 1, we can construct strongly well-covered graphs with $\alpha \leq 8$. In order to construct strongly well-covered graphs with arbitrarily large independence number, we turn to the family of W_2 graphs of girth 4. First, we prove the following lemma about W_2 graphs of girth 4, which will allow us to use Theorem 5 to construct families of strongly well-covered graphs.

Lemma 8. Suppose H is a W_2 graph of girth 4 and e = uv is a line in H. Let H_{uv} be the graph H - (N[u] $\cup$ N[v]). Then H_{uv} is well-covered and $\alpha(H_{uv}) = \alpha(H) - 1$.

Proof. Suppose e = uv is a line in H. Let U = N(u) - v and V = N(v) - u. Since H has no triangles, then $U \cap V = \emptyset$.

Suppose J is a maximal independent set in the graph H_{uv}. Clearly $|J| < \alpha(H)$. We wish to show that $|J| = \alpha(H) - 1$. We assume to the contrary that $|J| < \alpha(H) - 1$.

If J dominates V, then $J \cup \{u\}$ is maximal independent in H. Since $|J \cup \{u\}| < \alpha(H)$ and H is well-covered, we have a contradiction. Thus, J does not dominate V. Hence, there exists a point y such that $y \in V$ and J does not dominate y (see Figure 3).

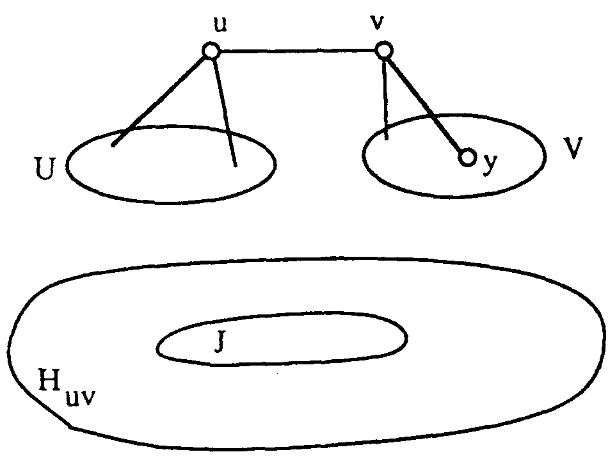

Figure 3

Note that N(y) - v is contained in $V(H_{uv}) \cup U$, since H has no triangles. Therefore, $(J \cup \{u\}) \cap N(y) = \emptyset$, $J \cup \{u\}$ is independent, and $J \cup \{u\}$ dominates N(y). It follows that $J \cup \{u\}$ and $\{y\}$ are disjoint independent sets in H which cannot be extended to disjoint maximum independent sets in H, and so H is not in W_2. This contradicts our hypothesis.

Thus, $|J| = \alpha(H) - 1$. Therefore, every maximal independent set in H_{uv} has size $\alpha(H) - 1$. It follows that H_{uv} is well-covered and $\alpha(H_{uv}) = \alpha(H) - 1$. []

In [10] and [11], the author presents constructions which yield W_2 graphs of girth four with arbitrarily large independence number. Based on the W_2 graphs obtained from these constructions, we show in the following theorem that we can construct infinite families of strongly well-covered graphs with arbitrarily large (even) independence number.

Theorem 9. Suppose H is a W_2 graph of girth 4 with n points, and $\{G_i\}$, i = 1,..., n, is a family of strongly well-covered graphs with $\alpha(G_i) = 2$ and G_i is connected or $2K_1$, for all i. Then the lexicographic product graph $H\circ(G_1, ..., G_n)$ is a strongly well-covered graph, and $\alpha(H\circ(G_1, ..., G_n)) = 2\alpha(H)$.

Proof. From Lemma 8, if e = uv is a line in H, then the graph H_{uv} is well-covered and $\alpha(H_{uv}) = \alpha(G) - 1$. Thus, the graph H satisfies the additional condition required of a well-covered graph in Theorem 5. It follows by Theorem 5 that $H\circ(G_1, ..., G_n)$ is strongly well-covered and $\alpha(H\circ(G_1, ..., G_n)) = 2\alpha(H)$. []

Recall that a W_2 graph H has the property that for all points v in H, the graph H-v is well-covered, and a strongly well-covered G has the property that for all points v in G, the graph G-v is *not* well-covered. Given this disparity between the two types of well-covered graphs, it is perhaps surprising that the lexicographic product of a W_2 graph and a family of strongly well-covered graphs as produced in Theorem 9 will yield a strongly well-covered graph.

If H is a W_2 graph of girth 4, then $H\circ 2K_1$ is strongly well-covered by Theorem 9. Clearly, $H\circ 2K_1$ has girth 4. Since there are infinitely many W_2 graphs of girth 4, it follows that there are infinitely many *girth 4* strongly well-covered graphs.

The graphs given in Figure 4 are the strongly well-covered lexicographic product graphs $H_1\circ 2K_1$ and $H_2\circ 2K_1$, where H_1 and H_2 are planar W_2 graphs of girth 4 with eight points and eleven points, respectively (see [10] for a discussion of planar W_2 graphs of girth 4). Each of these graphs has points with degree four. Hence, the lower bound of four for the minimum degree in a strongly well-covered graph (mentioned above) is sharp.

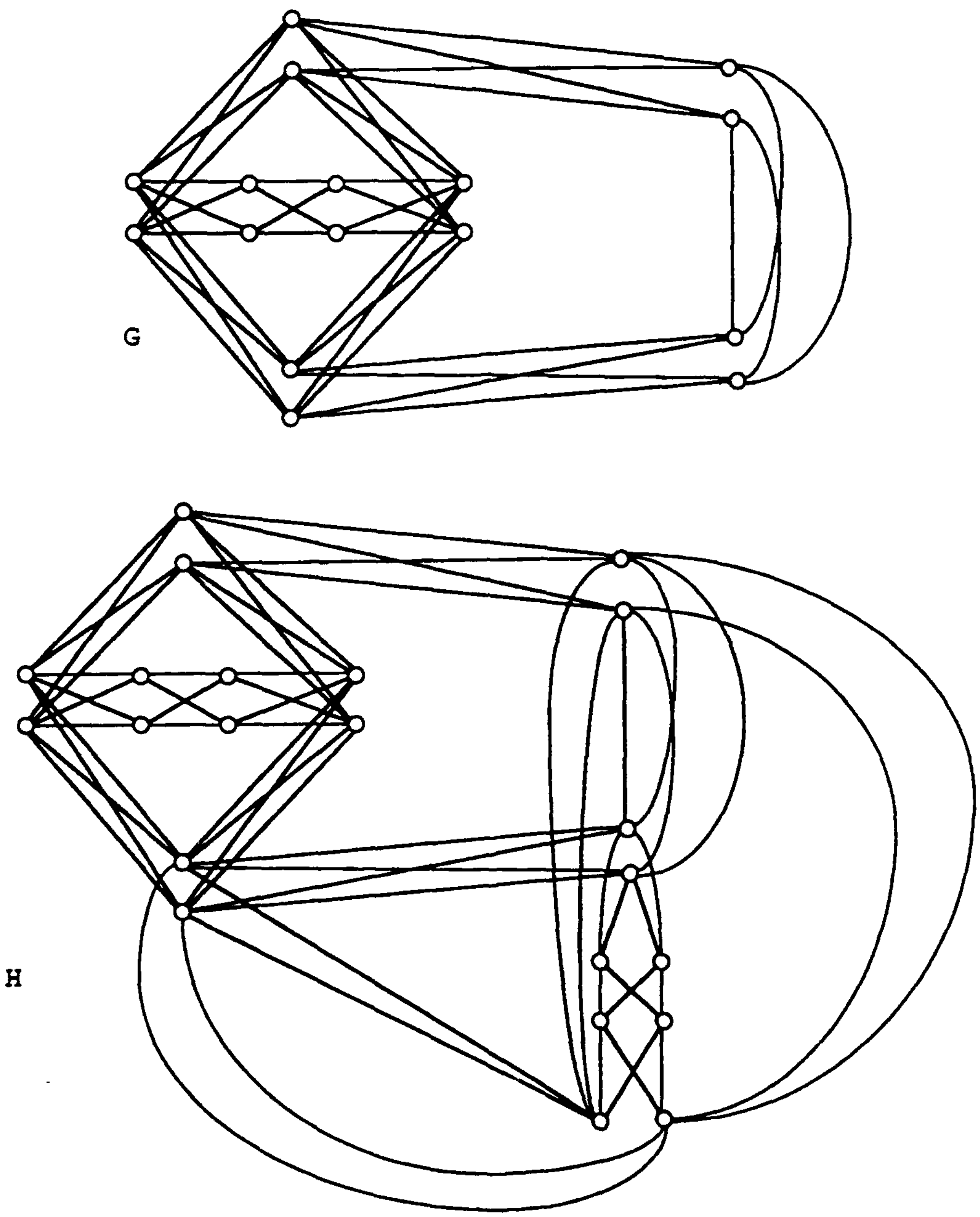

Figure 4

A line in a graph G is a critical line if its removal increases the independence number. A line-critical graph is a graph with only critical lines. Staples proved in [17] that a triangle-free W_2 graph is line-critical.

In searching for well-covered graphs H such that $H\circ(G_1, ..., G_{|V(H)|})$ is strongly well-covered, for an appropriate family of graphs $\{G_i\}$, we discovered the following necessary condition on H.

Theorem 10. Suppose $\{G_i\}$, $i = 1, ... ,n$, is a family of strongly well-covered graphs with $\alpha(G_i) = 2$, for all i. If H is a well-covered graph on n points and $H\circ(G_1,...,G_n)$ is strongly well-covered, then H is line-critical.

Proof. Assume to the contrary that $e = uv$ is not a critical line in H. Thus, $\alpha(H\text{-}e) = \alpha(H)$. Let $L = H\circ(G_1, ..., G_n)$. Let $e' = u_iv_j$ be a line in L corresponding to the line e in H, with $u_i \in V(G_i)$, $v_j \in V(G_j)$ $(i \neq j)$. Since $\alpha(H\text{-}e) = \alpha(H)$, then there exists a maximal independent set J in H-e which contains $\{u,v\}$ such that $|J| \leq \alpha(H)$. So $J\text{-}\{u,v\}$ dominates H_{uv} and is contained in $V(H_{uv})$. For $x \in J\text{-}\{u,v\}$, we have $x \in V(G_m)$ for some m, $m \notin \{i,j\}$. Since $\alpha(G_m) = 2$, there exists maximum independent set $I_x \supset \{x\}$ in G_m with $|I_x| = 2$. Let $I = \bigcup\{I_x\colon\ x \in J\text{-}\{u,v\}\ \}$. So I is in V(L). Since $|J\text{-}\{u,v\}| \leq \alpha(H) - 2$ and $|I_x| = 2$, then $|I| \leq 2\,(\alpha(H) - 2) = 2\alpha(H) - 4$. But then $I \cup \{u_i,v_j\}$ is maximal independent in L-e', and $|\, I \cup \{u_i,v_j\}\, | \leq 2\alpha(H) - 2 < 2\alpha(H)$. Since $\alpha(L) = 2\alpha(H)$ by Theorem 4 and L is assumed to be strongly well-covered, we have a contradiction. []

However, if H is line-critical, then $H\circ(G_1, ..., G_{|V(H)|})$ is not necessarily strongly well-covered. In fact, being line-critical and in W_2 are not *sufficient* conditions to ensure that $H\circ(G_1, ..., G_{|V(H)|})$ is strongly well-covered. If H is the line-critical W_2 graph in Figure 5, then $H\circ 2K_1$ is not strongly well-covered. Note that the graph $H_{uv} = H\text{-}(N[u] \cup N[v])$ is not well-covered.

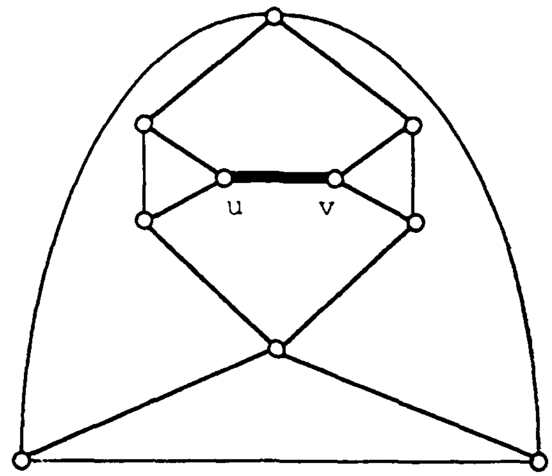

Figure 5

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