(13)

PLANAR REGULAR ONE-WELL-COVERED GRAPHS

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Abstract

An independent set in a graph is a subset of vertices with the property that no two of the vertices are joined by an edge, and a maximum independent set in a graph is an independent set of the largest possible size. A graph is called well-covered if every independent set that is maximal with respect to set inclusion is also a maximum independent set. If G is a well-covered graph and G - v is also well-covered for all vertices v in G, then we say G is 1-well-covered. By making use of a characterization of cubic well-covered graphs, it is straightforward to determination all cubic 1-well-covered graphs. Since there is no known characterization of k-regular well-covered graphs for $k \geq 4$, it is more difficult to determine the k-regular 1-well-covered graphs for $k \geq 4$. The main result in this regard is the determination of all 3-connected 4-regular planar 1-well-covered graphs.

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Introduction

A set of points in a graph is independent if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph G is called the independence number of G and is denoted by $\alpha(G)$. A set of independent points which attains the maximum size is referred to as a maximum independent set. A set S of independent points in a graph is maximal (with respect to set inclusion) if the addition to S of any other point in the graph destroys the independence. In general, a maximal independent set in a graph is not necessarily maximum.

In a 1970 paper, Plummer [12] introduced the notion of considering graphs in which every maximal independent set is also maximum; he called a graph having this property a well-covered graph. Equivalently, a well-covered graph is one in which every independent set can be extended to a maximum independent set. Sankaranarayana and Stewart [15] and, independently, Chvátal and Slater [3], have shown that determining if a given graph G is not well-covered is an NP-complete problem. Hence, determining if a graph is well-covered is in the class of problems referred to as co-NP-complete. What is not known is whether or not well-covered is an NP-complete property.

The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. The subclasses covered include cubic well-covered graphs ([1], [2] and [14]), well-covered graphs whose independence number is exactly one-half the size of the graph ([16], [4], [5]), well-covered graphs with girth at least five [6], well-covered graphs without 4-cycles and 5-cycles [7], and products of well-covered graphs [18].

Staples ([16] and [17]) introduced two subclasses of well-covered graphs which she called 1-well-covered and W_2. A well-covered graph is 1-well-covered if and only if the deletion of any point from the graph leaves a graph which is also well-covered. A well-covered graph G is in the class W_2 if and only if any two disjoint independent sets in G can be extended to two disjoint maximum independent sets. Some other results for graphs in W_2 were obtained in [11].

In this paper, we primarily consider 1-well-covered planar regular graphs. Campbell characterized the cubic planar well-covered graphs in [1]; however, the technique he employed becomes very cumbersome when applied to planar 4-regular or 5-regular well-covered graphs. For this reason, we focus on the one-well-covered graphs. The primary result is stated in Theorem 13.

Preliminary Results

Staples [16] proved an equivalency between two seemingly different subclasses of well-covered graphs, which we state as the following theorem.

Theorem 1. Suppose G is well-covered. Then G is 1-well-covered if and only if $G \in W_2$.

Since we will appeal mostly to the notion of extending two disjoint independent sets to disjoint maximum independent sets, henceforth we use the W_2 nomenclature instead of referring to 1-well-covered graphs.

Consider a graph G which is not complete and point v in G. By deleting v and its neighbors, we obtain a subgraph of G. Specifically, we define the subgraph $G_v = G\text{-}N[v]$. Campbell [1] proved the following very useful necessary condition for a graph to be well-covered.

Theorem 2. If a graph G is well-covered and is not complete, then G_v is well-covered for all v in G. Moreover, $\alpha(G_v) = \alpha(G) - 1$.

We prove in Theorem 3 that we have a similar *necessary* condition for a well-covered graph to be in W_2.

Theorem 3. If a graph G is in W_2 and G is not complete, then G_v is in W_2 for all v in G.

Proof. Let v be a point in G. Since G is not complete, then $G_v \neq \emptyset$. By Theorem 2, graph G_v is well-covered and $\alpha(G_v) = \alpha(G) - 1$. Suppose I_1 and I_2 are disjoint independent sets in G_v. Then $I_1 \cup \{v\}$ is an independent set in G, as is $I_2 \cup \{v\}$. Since G is in W_2, there exists maximum independent set $J_1 \supseteq I_1 \cup \{v\}$ such that $J_1 \cap I_2 = \emptyset$. Since $I_2 \cup \{v\}$ and J_1-v are disjoint independent sets in G, then there exists maximum independent set $J_2 \supseteq I_2 \cup \{v\}$ such that $J_2 \cap (J_1\text{-}v) = \emptyset$. Hence, J_2-v and J_1-v are disjoint independent sets in G_v. Since $|J_i| = \alpha(G)$, then $|J_i\text{-}v| = \alpha(G) - 1$, for i = 1, 2. Thus, J_1-v contains I_1, J_2-v contains I_2, and J_1-v and J_2-v are disjoint maximum independent sets in G_v. So any two disjoint independent sets in G_v can be extended to disjoint maximum independent sets in G_v. By definition of the class W_2, we conclude that $G_v \in W_2$. □

The next lemma will play a significant role for us. We will use it to eliminate many graphs from consideration as possible W_2 graphs.

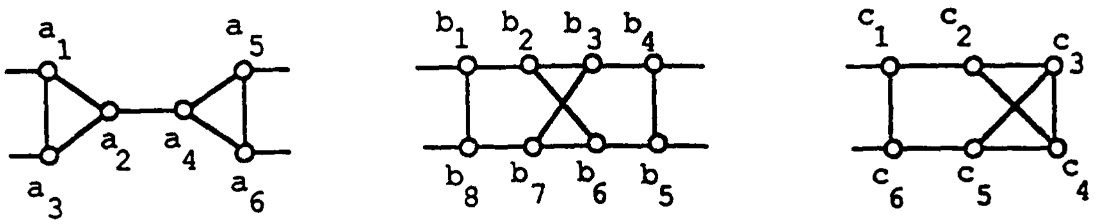
Lemma 4. Suppose G contains an independent set S and point $v \notin S$ such that (i) $S \cup \{v\}$ is independent, and (ii) if $y \in N(v)$, then $y \sim x$ for some $x \in S$ (that is, S dominates N(v)). Then G is not in W_2.

Proof. If G is not well-covered, then G is not in W_2. If G is well-covered, then from conditions (i) and (ii), we have that $S \cap N(v) = \emptyset$ and S dominates N(v). Thus, S and {v} are disjoint independent sets in G which don't extend to disjoint maximum independent sets in G. Therefore, G is not in W_2. □

For graphs drawn in the plane, we say two faces are adjacent if they share a line. If a face F contains point v, we say F is incident to v. The size of a face is the number of points it contains. We refer to the order and sizes of the faces incident to a point v as the face configuration at v. To reduce the number of face configurations considered, we will use the theory of Euler contributions. Lebesgue [8] developed the theory of Euler contributions for planar graphs and Ore [9] and Ore and Plummer [10] used the theory to study plane graph colorings. The Euler contribution of a point v, $\phi(v)$, is defined as the quantity $\phi(v) = 1 - (1/2)\deg(v) + \Sigma(1/x_i)$, where the sum is taken over all faces F_i incident to v and x_i is the size of F_i. If $|F(G)|$ denotes the number of faces in the plane graph G, then it follows that $\Sigma_v \phi(v) = |V(G)| - |E(G)| + |F(G)|$. Here the sum is taken over all points v in G. Since Euler's formula for plane graphs says $|V(G)| - |E(G)| + |F(G)| = 2$, then we have $\Sigma_v \phi(v) = 2$. Thus, $\phi(v)$ must be positive for some v in G. If $\phi(v) > 0$, we say v is a point with positive Euler contribution.

Cubic W_2 Graphs

Consider the three graph fragments given in Figure 1. Note that fragments A and B each have four semi-lines and fragment C has two semi-lines.



A B C

Figure 1

Let *W* be the family of cubic graphs obtained from fragments A, B and C by placing any number of the fragments in a cycle or path configuration and then joining the left-hand semi-lines of one fragment to the right-hand semi-lines of the fragment on its left. Since crossing the lines joining one fragment to another gives a graph which is isomorphic to the graph obtained without crossing the lines, then we can assume the lines do not cross.

Building on the work of Campbell [1], Royle and Ellingham [14] proved that, with a few small exceptions, all cubic well-covered graphs belong to *W*. We state their result in Theorem 5.

Theorem 5: All cubic well-covered graphs, except for the 6 graphs in Figure 2, belong to *W*. Moreover, all graphs in *W* are well-covered.

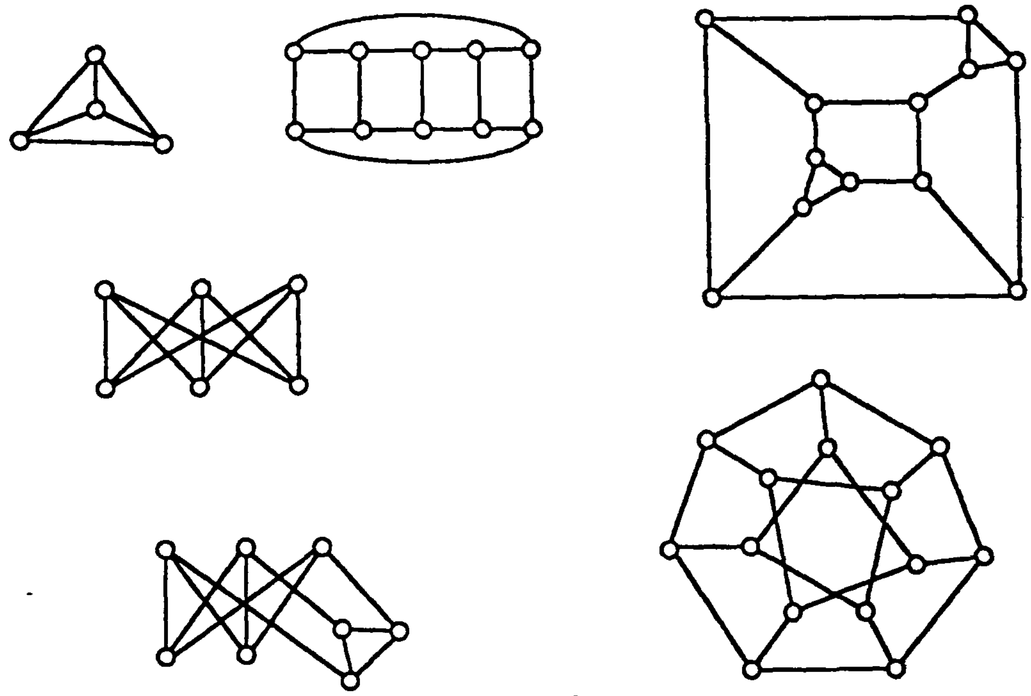


Figure 2

Using the characterization of cubic well-covered graphs given in Theorem 5, in the next theorem we determine all of the cubic W_2 graphs.

Theorem 6. The only cubic W_2 graphs are K_4 and the triangular prism.

Proof. Of the 6 exceptional cubic graphs given in Figure 2, only K_4 is a W_2 graph. For each of the other five graphs, it is straightforward to find two disjoint independent sets which don't extend to disjoint maximum independent sets in G. We omit the details.

Suppose G is a graph in the family *W*. Then G is obtained by connecting fragments A, B and C in paths or cycles.

Case 1. Suppose G contains fragment A. If $a_1 \sim a_5$ and $a_3 \sim a_6$, then G is the triangular prism. It is easily verified that the triangular prism is a W_2 graph.

Suppose $|V(G)| > 6$. Without loss of generality, let $x \sim a_5$ and $y \sim a_6$, where x and y are not in the original A fragment. Then $x \sim y$ and $\{y,a_2\}$ is independent. Thus, $\{y,a_2\}$ and $\{a_5\}$ don't extend to disjoint maximum independent sets in G. So $G \notin W_2$.

Case 2. Suppose G contains fragment B. If $b_1 \sim b_4$ and $b_5 \sim b_8$, then $\{b_3,b_5\}$ and $\{b_1\}$ don't extend to disjoint maximum independent sets in G. So $G \notin W_2$.

Suppose $|V(G)| > 8$. Without loss of generality, let $x \sim b_4$ and $y \sim b_5$, where x and y are not in the original B fragment. Then $x \sim y$ and $\{y,b_2\}$ is independent. Thus, $\{y,b_2\}$ and $\{b_4\}$ don't extend to disjoint maximum independent sets in G. So $G \notin W_2$.

Case 3. Suppose G contains fragment C. Then $|V(G)| > 6$. Let $x \sim c_1$ and $y \sim c_6$ such that x and y are not in the original C fragment. Then $x \sim y$ and $\{y,c_3\}$ is independent. Thus, $\{y,c_3\}$ and $\{c_1\}$ don't extend to disjoint maximum independent sets in G. So $G \notin W_2$.

Therefore, K_4 and the triangular prism are the only cubic W_2 graphs. □

4-regular Planar W_2 Graphs

We now turn our attention to 4-regular W_2 graphs. Since no characterization of 4-regular well-covered graphs is known (unlike the situation for cubic well-covered graphs), we focus most of our efforts on only the planar 3-connected 4-regular W_2 graphs. But first we show in Theorem 7 that no 4-regular W_2 graph has a cutpoint.

Theorem 7. Suppose G is 4-regular and in W_2. Then G is 2-connected.

Proof. Assume to the contrary that G has a cutpoint v. Since G is 4-regular, then G-v must have exactly two components, say G_1 and G_2, each containing two neighbors of v. Let $N(v) \cap G_1 = \{a_1,b_1\}$ and $N(v) \cap G_2 = \{a_2,b_2\}$. Define A_1, A_2, B_1 and B_2 as follows: $A_i = (N(a_i) \cap G_i) - \{b_i\}$, $B_i = (N(b_i) \cap G_i) - \{a_i\}$, for i = 1, 2. Let $y_1 \in B_1$.

Case 1. Suppose there exist points $u_1 \in A_1$, $y_1 \in B_1$, $u_2 \in A_2$ and $y_2 \in B_2$ such that u_1 is not adjacent to y_1 (possibly $u_1 = y_1$) and u_2 is not adjacent to y_2 (possibly $u_2 = y_2$). Then $\{u_1,u_2,y_1,y_2\}$ is independent and so $\{u_1,u_2,y_1,y_2\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, a contradiction since G is in W_2.

Case 2. So either every $u_1 \in A_1$ is adjacent to every $y_1 \in B_1$, or every $u_2 \in A_2$ is adjacent to every $y_2 \in B_2$. Without loss of generality, assume every $u_1 \in A_1$ is adjacent to every $y_1 \in B_1$. Let $z \in A_1$. Note that z is not adjacent to b_1. Thus, $\{u_1,a_2\}$ and $\{b_1\}$ are disjoint independent sets in G which don't extend to disjoint maximum independent sets in G, a contradiction since G is in W_2.

Therefore, G cannot have a cutpoint. □

The following four lemmas will be helpful in determining the 3-connected 4-regular planar W_2 graphs.

Lemma 8. Suppose G is 3-connected 4-regular and planar. Suppose v is a point in G with face configuration (3,3,x,y), x, y $\geq$ 3, where two triangles incident to v share a line. If two triangles at v are u_1u_2v and u_2u_3v, then u_1 is not adjacent to u_3.

Proof. Assume to the contrary that $u_1 \sim u_3$. Let u_4 be the fourth neighbor of v (see Figure 3). If u_1 has its fourth neighbor on one side of triangle u_1u_3v and u_3 has its fourth neighbor on the other side of triangle u_1u_3v, then either $\{v,u_1\}$ or $\{v,u_3\}$ is a cutset of G. This contradicts the 3-connected assumption. Thus, u_1 and u_3 each have their fourth neighbor on the same side of triangle u_1u_3v, and so either v or u_2 is a cutpoint for G. This again contradicts the 3-connected assumption. □

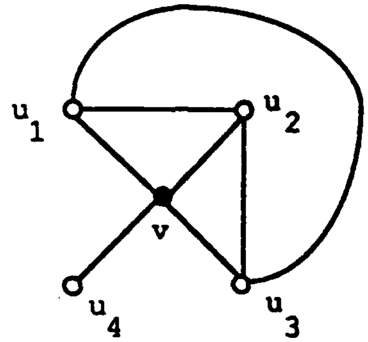


Figure 3

The next three lemmas are fairly obvious; hence, we omit proofs. Lemma 11 says that two faces in a 3-connected planar graph which are incident to the same point either have only that point in common or they are adjacent faces at the point and share only a line.

Lemma 9. Suppose G is 3-connected 4-regular and planar. Suppose $F_4 = vu_4...u_1$ is an n-face at v, $n \geq 3$, and $F_1 = vu_1u_2$ is a triangular face at v such that F_4 and F_1 share the line vu_1. If $x \in F_4$ such that $x \notin \{v,u_1\}$, then x is not adjacent to u_2.

Lemma 10. Suppose G is 3-connected and planar. Suppose x and y are non-consecutive points on a face of G. Then x is not adjacent to y.

Lemma 11. Suppose G is planar and 3-connected. Suppose v is a point of G with incident faces $F_1, F_2, \ldots, F_n$.

(i) If F_i and F_j share a line xv ($i \neq j$), then $F_i \cap F_j = xv$.

(ii) If F_i and F_j do not share a line of the form xv, for any $x \in N(v)$, then $F_i \cap F_j = \{v\}$.

In the following lemmas, we will repeatedly use Lemma 4. In particular, if S and v are an independent set and point, respectively, which satisfy the hypotheses of Lemma 4, we will say that S and {v} don't extend to disjoint maximum independent sets in G. If G is assumed to be a W_2 graph, then we will have a contradiction.

For the next lemma only, we don't require G to be planar.

Lemma 12.1. Suppose G is 3-connected 4-regular and in W_2. If G has a 4-wheel configuration at a point, then G is K_5.

Proof. Assume v is a point in G with $N(v) = \{u_1,u_2,u_3,u_4\}$, and triangles u_1u_2v, u_2u_3v, u_3u_4v and u_4u_1v forming a 4-wheel configuration at v.

Suppose $u_1 \sim u_3$. If u_2 is not adjacent to u_4, then $\{u_2,u_4\}$ is a cutset for G. So $u_2 \sim u_4$. It follows that G is K_5.

Suppose u_1 is not adjacent to u_3. Let x be the fourth neighbor of u_3. If $x \sim u_1$, then $\{u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So x is not adjacent to u_1.

Suppose $x \sim u_2$ and $x \sim u_4$. Then $\{x,u_1\}$ is a cutset for G since x is not adjacent to u_1. So we can assume either x is not adjacent to u_2 or x is not adjacent to u_4. Without loss of generality, assume x is not adjacent to u_2. Since G is 4-regular, there is a point y such that $y \sim x$ and y is not adjacent to u_1. Then $\{y,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

Hence, u_1 must be adjacent to u_3, and so G must be K_5. $\square$

We will attack the problem of finding all 3-connected 4-regular planar W_2 graphs using the theory of Euler contributions. In each of the next ten lemmas, we consider a particular face configuration at a point v. Afterwards, the result which we pursue will follow easily. We will implicity use Lemma 11 in each of these ten lemmas.

Lemma 12.2. Suppose G is 3-connected 4-regular planar and in W_2. If G has a point v with face configuration (3,3,3,4), then G is the graph given in Figure 4.

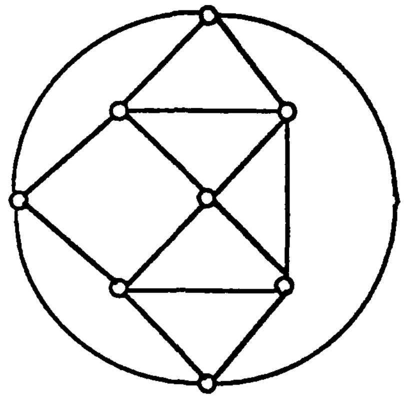


Figure 4

Proof. Suppose v has face configuration (3,3,3,4) with $N(v) = \{u_1,u_2,u_3,u_4\}$ and the 4-face at v is u_1vu_4x (see Figure 5).

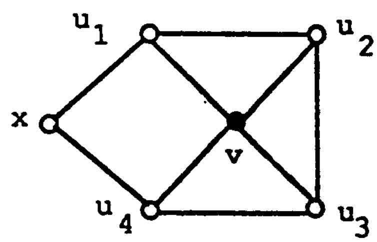


Figure 5

From Lemma 8, u_1 is not adjacent to u_3 and u_2 is not adjacent to u_4. From Lemma 9, x is not adjacent to u_2 and x is not adjacent to u_3. From Lemma 10, u_1 and u_4 are not adjacent.

Let z be the fourth neighbor of u_2. From above, $z \notin \{x,u_4\}$. Let $\{w\} = N(u_4) - \{x,v,u_3\}$.

Case 1. Suppose $z \sim u_4$. Since x is adjacent to neither u_2 nor u_3, then there exists a point $s \sim x$ such that $s \neq z$. Then $\{s,u_2\}$ is independent and so $\{s,u_2\}$ and $\{u_4\}$ do not extend to disjoint maximum independent sets in G, a contradiction. Thus z is not adjacent to u_4.

Case 2. Suppose $z \sim u_3$.

Case 2.1. If x and z are not adjacent, then $\{x,z\}$ and $\{v\}$ do not extend to disjoint maximum independent sets in G. So $x \sim z$.

Case 2.2. If $z \sim u_1$, then $\{x,u_4\}$ is a cutset for G. So z and u_1 are not adjacent.

Let $m \sim u_1$ such that $m \notin \{x,v,u_2\}$. Since G is planar, m and w are not adjacent (see Figure 6). If $z \sim m$, then $\{x,u_4\}$ is a cutset. So z and m are not adjacent. If $z \sim w$, then $\{x,w\}$ is a cutset. So z and w are not adjacent. But then $\{z,w,m\}$ is independent and so $\{z,w,m\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, a contradiction.

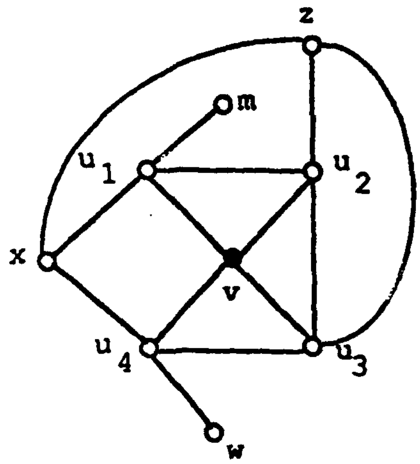


Figure 6

Thus, z and u_3 are not adjacent.

Case 3. Suppose $x \sim z$.

Case 3.1. Suppose z and u_1 are not adjacent. Let $y \in (N(u_1) - \{x,v,u_2\})$, and let $Y = N(y) - u_1$.

Case 3.1.1. Suppose there exists $p \in Y$ such that p is not adjacent to z. Then $\{p,z,u_4\}$ is independent and so $\{p,z,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G.

Case 3.1.2. Thus, $p \in Y$ implies $p \sim z$. If $y \sim z$, then $\{z,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So y and z are not adjacent. But then $\{z,v\}$ and $\{y\}$ don't extend to disjoint maximum independent sets in G.

Thus, $x \sim z$ implies $z \sim u_1$. See Figure 7.

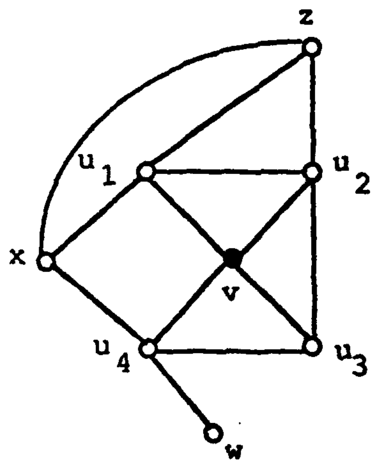


Figure 7

Case 3.2. Suppose w and u_3 are not adjacent. Let $y \sim u_3$, $y \notin \{v,u_2,u_4\}$. From above, $y \notin \{x,z\}$.

Case 3.2.1. If $y \sim w$, then $\{w,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So y and w are not adjacent.

Case 3.2.2. Suppose $z \sim y$. Let $\{a,b\} = N(y)-\{z,u_3\}$. If $w \sim a$ and $w \sim b$, then $\{w,u_2\}$ and $\{y\}$ don't extend to disjoint maximum independent sets in G. So, without loss of generality, assume w is not adjacent to a. If $a = x$ (that is, $x \sim y$), then $\{y,u_4\}$ is a cutset. So $a \neq x$ and $\{w,a,u_1\}$ is independent. But then $\{w,a,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

Hence, z and y are not adjacent.

Case 3.2.3. Suppose $z \sim w$. Then $\{w,u_3\}$ is a cutset. So z and w are not adjacent.

Hence, $\{z,w,y\}$ is independent and so $\{z,w,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Thus, $\underline{x \sim z \text{ implies } w \sim u_3}$.

Case 3.3. If z and w are not adjacent, then $\{w,z\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $z \sim w$.

Thus, $\underline{x \sim z \text{ implies } z \sim w}$.

Case 3.4. If x and w are not adjacent, then $\{x,w\}$ is a cutset. So $x \sim w$.

Thus, $\underline{x \sim z \text{ implies } x \sim w}$. See Figure 8.

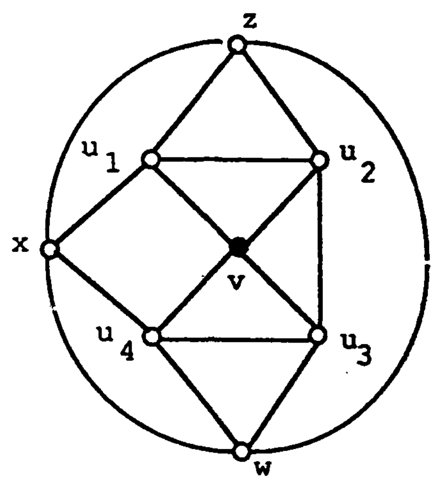


Figure 8

Consequently, if $x \sim z$ then G must be the graph given in Figure 4.

Now, recall from earlier that the following sets are independent: $\{x,u_2\}$, $\{x,u_3\}$, $\{z,u_3\}$, $\{z,u_4\}$, $\{u_2,u_4\}$, $\{u_1,u_3\}$, $\{u_1,u_4\}$. Thus there exists $y \sim u_3$ such that $y \notin \{x,z,v,u_1,u_2,u_4\}$. Since z and u_4 are not adjacent, it follows by symmetry that y and u_1 are not adjacent.

Case 4. If $x \sim y$, then by symmetry and the argument given in Case 3 for $x \sim z$, the only W_2 graph which can result is the graph obtained in Case 3.

Case 5. So we assume x is not adjacent to z and y is not adjacent to x.

If y and z are not adjacent, then $\{x,y,z\}$ is independent and so $\{x,y,z\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $y \sim z$.

Suppose $y \sim u_4$. Since y is not adjacent to u_1, then there exists $w \sim y$ such that $w \notin \{x,z,v,u_1,u_2,u_3,u_4\}$. If $w \sim x$, then $\{w,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So w and x are not adjacent.

Since G is 4-regular, there exist points s and t such that s and t are neighbors of x and $\{s,t\} \cap \{v,y,z,u_1,u_2,u_3,u_4\} = \emptyset$. Suppose w and s are not adjacent. Then $\{w,s,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So $w \sim s$ and, similarly, $w \sim t$ (see Figure 9). But then $\{v,w\}$ and $\{x\}$ don't extend to disjoint maximum independent sets in G.

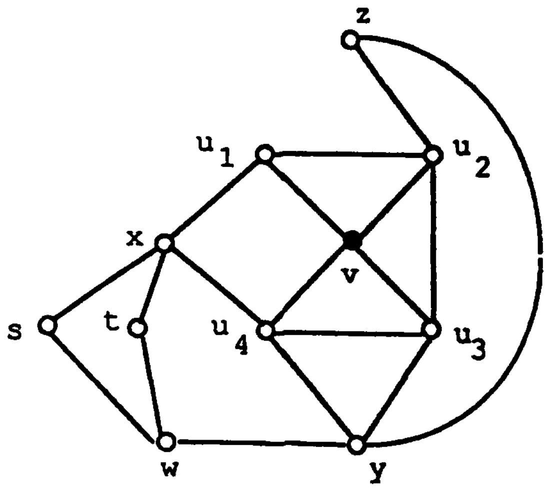


Figure 9

Hence, y and u_4 are not adjacent. By symmetry, z and u_1 are not adjacent. Thus there exists $m \sim u_1$ such that $m \notin \{x,y,z,v,u_1,u_2,u_3,u_4\}$. If $m \sim u_4$, then $\{z,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So m and u_4 are not adjacent.

Suppose $m \sim y$. Then there exists a point $n \sim u_4$ such that $\{n,z,u_1\}$ is independent, where $n \notin \{x,v,u_3\}$. But then $\{n,z,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So m and y are not adjacent (see Figure 10).

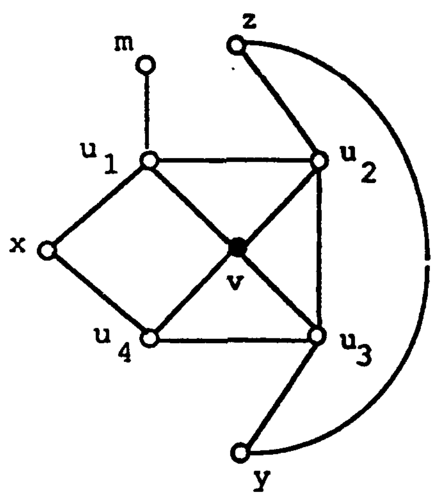


Figure 10

From above, we see that $\{m,y,u_4\}$ is independent. Then $\{m,y,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G.

Therefore, the graph shown in Figure 2.5 is the only 3-connected 4-regular planar W_2 graph with the (3,3,3,4) face configuration. □

Lemma 12.3. Suppose G is 3-connected 4-regular planar and in W_2. If v is a point in G, then v cannot have face configuration (3,3,3,5).

Proof. Assume to the contrary that v has face configuration (3,3,3,5). Let $N(v) = \{u_1,u_2,u_3,u_4\}$ and the 5-face at v be abu_4vu_1. From Lemma 8, u_1 is not adjacent to u_3 and u_2 is not adjacent to u_4. From Lemma 9, a is not adjacent to u_2, a is not adjacent to u_3, b is not adjacent to u_2, and b is not adjacent to u_3. From Lemma 10, a is not adjacent to u_4, u_1 is not adjacent to u_4, and b is not adjacent to u_1.

Thus, there exists $x \sim u_4$ such that $x \notin \{a,b,v,u_1,u_2,u_3\}$. By symmetry, there exists $y \sim u_1$ such that $y \notin \{a,b,v,u_2,u_3,u_4\}$ (we do not exclude the possibility that $y = x$).

Case 1. Suppose $a \sim x$. Then $\{a,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So <u>a is not adjacent to x</u>. By symmetry, <u>y is not adjacent to b</u>.

Let $\{p\} = N(u_2) - \{v,u_1,u_3\}$.

Case 2. If $p = x$ (that is, $x \sim u_2$) or $p \sim a$, then $\{a,u_4\}$and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So <u>$p \neq x$</u> and <u>p and a are not adjacent</u>.

Case 3. Suppose $p \sim x$.

Case 3.1. Suppose $p \sim u_3$. If $x \sim u_1$, then $\{p,t,u_1\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G, where $t \sim b$ such that $t \notin \{a,u_4\}$. So x is not adjacent to u_1. Thus $\{x,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

Hence, p is not adjacent to u_3.

Case 3.2. Suppose $x \sim u_3$.

Case 3.2.1. If $x \sim b$ or $x \sim u_1$, then $\{b,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So x is adjacent to neither b nor u_1.

Thus, there exists $z \sim x$ such that $z \notin \{a,b,u_1,u_3,u_4,p\}$.

Case 3.2.2. If z is not adjacent to a, then $\{a,z,u_2\}$ is independent and so $\{a,z,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So $z \sim a$.

Case 3.2.3. If $z \sim b$, then $\{z,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So z is not adjacent to b.

Case 3.2.4. If z is not adjacent to u_1, then $\{b,z,u_1\}$ is independent and so $\{b,z,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So $z \sim u_1$. But then $\{p,z\}$ is a cutset for G.

Thus, x is not adjacent to u_3. So there exists $w \sim u_3$ and $m \sim w$ such that $w \notin \{v,u_2,u_4\}$ and $\{w,m\} \cap \{p,x\} = \emptyset$ (see Figure 11). But then $\{b,m,u_1\}$ is independent and so $\{b,m,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

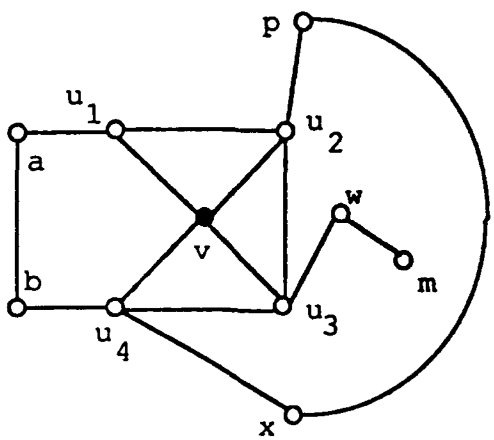


Figure 11

Hence, <u>p is not adjacent to x</u>. Thus $\{p,x,a\}$ is independent. By symmetry, there exists $q \sim u_3$ such that $q \notin \{v,u_2,u_4,a,y\}$ and <u>q is not adjacent to y</u>.

If any member of $\{p,x,a\}$ is adjacent to u_3, then $\{p,x,a\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $q \notin \{a,p,x\}$.

Suppose $x \sim u_1$ (that is, $x = y$). Then $\{p,t,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G, where $t \sim a$ such that $t \notin \{b,u_1\}$. Thus, x is not adjacent to u_1; hence, <u>$x \neq y$</u>. See Figure 12.

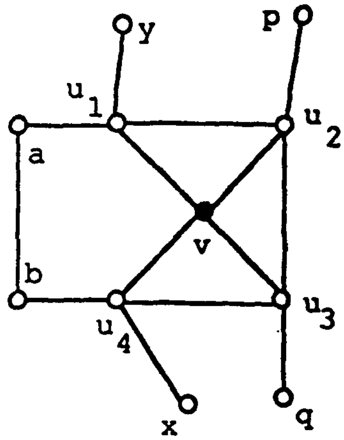


Figure 12

Suppose $p \sim q$. Then $\{q,y,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So p and q are not adjacent. Suppose $q \sim x$. Then $\{x,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So q is not adjacent to x and, by symmetry, p is not adjacent to y. If $q \sim a$, then $\{x,y,p,q\}$ is an independent set. Thus, $\{x,y,p,q\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So q is not adjacent to a, and it follows that $\{a,x,p,q\}$ is independent. But then $\{a,x,p,q\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Therefore, the face configuration (3,3,3,5) cannot occur. □

Lemma 12.4. Suppose G is 3-connected 4-regular planar and in W_2. If v is a point in G, then v cannot have face configuration (3,3,3,n), $n \geq 6$.

Proof. Assume to the contrary that v has face configuration (3,3,3,n), $n \geq 6$. Let $N(v) = \{u_1,u_2,u_3,u_4\}$, and let the n-face at v be $u_3cb_2\ldots bau_4v$. From Lemma 8, u_1 is not adjacent to u_3 and u_2 is not adjacent to u_4. From Lemma 9, a is not adjacent to u_1, u_1 and b are not adjacent, c is not adjacent to u_1, a is not adjacent to u_2, b is not adjacent to u_2, c is not adjacent to u_2, and u_1 and b_2 are not adjacent. From Lemma 10, a is not adjacent to c, a is not adjacent to u_3, and c is not adjacent to u_4.

Now let $s \sim u_2$ such that $s \notin \{v,u_1,u_3\}$.

Case 1. Suppose $s \sim c$. Then $\{c,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. Thus, s is not adjacent to c.

Case 2. Suppose $s \sim a$.

Case 2.1. If $s \sim u_4$, then $\{c,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So s is not adjacent to u_4.

Let $w \sim u_4$ such that $w \notin \{a,v,u_1\}$.

Case 2.2. If $w \sim a$, $w \sim s$ and $w \sim u_1$, then $\{a,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. Thus there exists $t \sim w$ such that $t \notin \{a,s,u_1,u_4\}$. But then $\{b,t,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G.

Hence, s is not adjacent to a.

Case 3. If $s \sim u_1$, then $\{a,s,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So s and u_1 are not adjacent.

Let $t \sim u_1$, where $t \notin \{v,u_2,u_4\}$; by symmetry with s, t is adjacent to neither a nor c.

Case 4. Suppose $s \sim t$.

Case 4.1. Suppose $s \sim u_3$.

Case 4.1.1. Suppose $t \sim u_4$. Let $\{w\} = N(t) - \{s,u_1,u_4\}$. If $a \sim w$, then $\{a,u_2\}$ and $\{t\}$ don't extend to disjoint maximum independent sets in G. So a is not adjacent to w.

Let $N(a) - \{b,u_4\} = \{y_1,y_2\}$. If $w \sim b$, $w \sim y_1$ and $w \sim y_2$, then $\{w,v\}$ and $\{a\}$ don't extend to disjoint maximum independent sets in G. Thus there exists some $x \sim a$, $x \neq$

u_4, such that x is not adjacent to w (see Figure 13). But then $\{x,w,u_2\}$ is independent and so $\{x,w,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G.

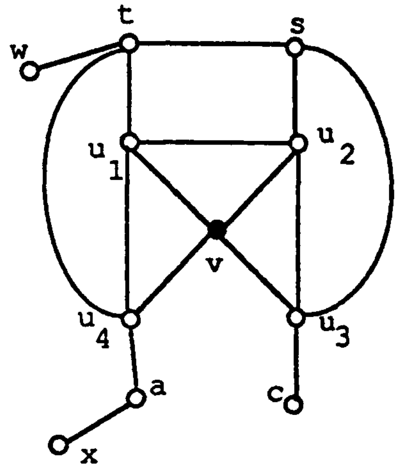


Figure 13

Case 4.1.2. So t is not adjacent to u_4. Then $\{t,u_4,c\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G.

Case 4.2. Hence, s is not adjacent to u_3. It follows that $\{a,s,u_3\}$ is independent. Hence, $\{a,s,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G.

Thus, s is not adjacent to t. Then $\{s,t,a,c\}$ is independent and $\{s,t,a,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Therefore, the face configuration $(3,3,3,n)$, $n \geq 6$, cannot occur. $\square$

Lemma 12.5. Suppose G is 3-connected 4-regular planar and in W_2. If v is a point in G, then v cannot have face configuration (3,3,4,4).

Proof. Assume to the contrary that v has face configuration (3,3,4,4). Let $N(v) = \{u_1,u_2,u_3,u_4\}$.

Case 1. Suppose the cyclic order of the faces at v is (3,4,3,4), with faces u_1u_2v, u_2bu_3v, u_3u_4v and u_4au_1v. By Lemma 9, a is not adjacent to u_2, a is not adjacent to u_3, b is not adjacent to u_1, and b is not adjacent to u_4.

If a is not adjacent to b, then $\{a,b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $a \sim b$. Thus there exists $x \sim u_1$, $y \sim u_2$, $s \sim u_3$ and $t \sim u_4$ such that $\{x,y,s,t\} \cap \{a,b,v,u_1,u_2,u_3,u_4\} = \emptyset$.

If $x = y$ and $s = t$, then $\{x,s\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So either $x \neq y$ or $s \neq t$. Without loss of generality, assume $x \neq y$. Suppose $x \sim y$. Then $\{y,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So x is not adjacent to y.

If $s = t$, then $\{s,x,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $s \neq t$. Since $s \neq t$ and by symmetry with x and y, it follows that s is not adjacent to t. But then $\{s,t,x,y\}$ is independent since G is planar, and so $\{s,t,x,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G (see Figure 14).

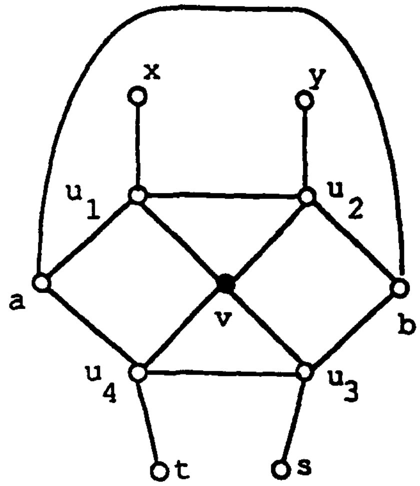


Figure 14

Thus the cyclic face order (3,4,3,4) cannot occur.

Case 2. Suppose the cyclic face order is (3,3,4,4), with faces u_1u_2v, u_2u_3v, u_3bu_4v and u_4au_1v. By Lemma 8, <u>u_1 is not adjacent to u_3</u>. By Lemma 9, <u>a is not adjacent to u_2</u>, <u>b is not adjacent to u_2</u>, and <u>u_2 is not adjacent to u_4</u>. By Lemma 10, <u>u_1 is not adjacent to u_4</u> and <u>u_3 is not adjacent to u_4</u>.

Case 2.1. Suppose $a \sim b$. Then there exists $z \sim u_4$ and $w \sim z$ such that $\{w,z\} \cap \{a,b\} = \emptyset$. Since G is planar, $\{u_1,u_3,w\}$ is independent; hence, $\{u_1,u_3,w\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. Thus, <u>a is not adjacent to b</u>.

Let <u>$y \sim u_2$</u> such that $y \notin \{v,u_1,u_3\}$.

Case 2.2. Suppose $y \sim a$.

Case 2.2.1. Suppose $y \sim u_1$ (see Figure 15).

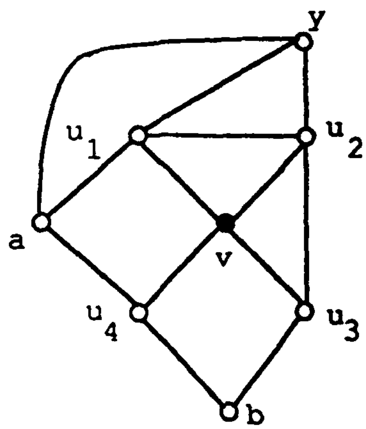


Figure 15

Thus, either we have a (3,3,3,4) face configuration at u_1, or there is a point inside triangle yau_1 or inside triangle yu_2u_1. From Lemma 12.2, point u_1 cannot have a (3,3,3,4) face configuration. If there is a point inside triangle yau_1, then $\{y,a\}$ is a cutset, contradicting 3-connectedness. If there is a point inside triangle yu_1u_2, then y is a cutpoint, contradicting 3-connectedness.

Case 2.2.2. Hence, y and u_1 are not adjacent (we are still assuming that $y \sim a$). Since y is not adjacent to u_1, there exists $z \sim u_1$ such that $z \notin \{a,b,v,y,u_1,u_2,u_3,u_4\}$, and $w \sim z$ such that $w \notin \{a,y\}$. Then $\{w,u_3,u_4\}$ is independent and so $\{w,u_3,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G.

Hence, y is not adjacent to a and, by symmetry, y is not adjacent to b. It follows that {a,b,y} is independent and so {a,b,y} and {v} don't extend to disjoint maximum independent sets in G.

Thus, the cyclic face order (3,3,4,4) cannot occur. From Cases 1 and 2, we conclude that the face configuration (3,3,4,4) cannot occur. □

Lemma 12.6. Suppose G is 3-connected 4-regular planar and in W_2. If v is a point in G, then v cannot have face configuration (3,3,4,5).

Proof. Assume to the contrary that v has face configuration (3,3,4,5). Let N(v) = $\{u_1,u_2,u_3,u_4\}$.

Case 1. Suppose the cyclic order of the faces at v is (3,3,4,5), with faces u_1u_2v, u_2u_3v, u_3cu_4v and u_4bau_1v. By Lemma 8, u_1 is not adjacent to u_3. By Lemma 9, u_2 is not adjacent to u_4, u_2 is not adjacent to a, u_2 is not adjacent to b, and u_2 is not adjacent to c. By Lemma 10, u_1 is not adjacent to u_4, u_3 is not adjacent to u_4, and a is not adjacent to u_4.

Case 1.1. Suppose a ~ c.

Case 1.1.1. Suppose c ~ u_1. Then $\{u_2,u_3\}$ is a cutset for G. So c is not adjacent to u_1.

Thus, there exists x ~ u_1 such that $x \notin \{a,b,c,v,u_2,u_3,u_4\}$.

Case 1.1.2. If x ~ u_3, then $\{x,u_2\}$ is a cutset for G. So x is not adjacent to u_3.

Case 1.1.3. Suppose c ~ x. Let m ~ u_3 such that $m \notin \{v,c,u_2\}$. Then $\{b,m,u_1\}$ and {c} don't extend to disjoint maximum independent sets in G. So c is not adjacent to x.

Case 1.1.4. If x ~ u_2, then {c,x} and {v} don't extend to disjoint maximum independent sets in G. So u_2 is not adjacent to x.

Thus, there exists y ~ u_2 such that $y \notin \{a,b,c,v,x,u_1,u_3,u_4\}$.

Case 1.1.5. If c ~ y, then $\{b,u_2\}$ and {c} don't extend. So c is not adjacent to y.

Case 1.1.6. If x is not adjacent to y, then {c,x,y} and {v} don't extend to disjoint maximum independent sets in G. So x ~ y.

Case 1.1.7. Suppose y ~ u_3. If y ~ a, then x is a cutpoint for G. So y is not adjacent to a. Thus, there exists z ~ y such that $z \notin \{a,b,c,v,x,u_1,u_2,u_3,u_4\}$. But then $\{z,u_1,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G (see Figure 16).

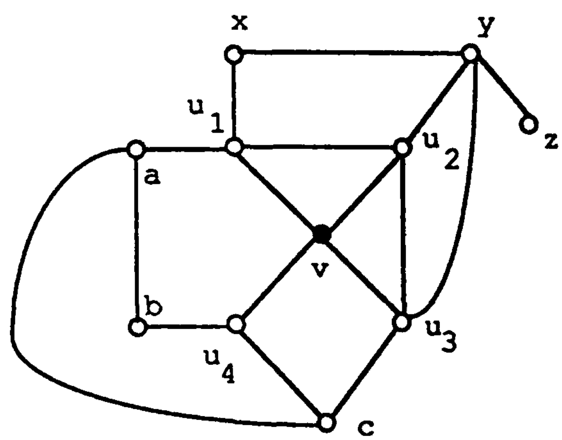


Figure 16

Hence, y is not adjacent to u_3. Since G is planar, $\{b,y,u_3\}$ is independent; so $\{b,y,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G.

Therefore, a is not adjacent to c. Let y ~ u_2 such that $y \notin \{a,b,c,v,u_1,u_3,u_4\}$.

Case 1.2. Suppose a ~ y.

Case 1.2.1. Suppose y ~ u_1. If y is not adjacent to c, then {y,c} and {v} don't extend to disjoint maximum independent sets in G. So y ~ c and $\{c,u_3\}$ is a cutset for G.

Thus, y is not adjacent to u_1. Let $x \sim u_1$ such that $x \notin \{a,v,u_2\}$.

Case 1.2.2. Suppose y is not adjacent to x. If y is not adjacent to c, then $\{x,y,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $y \sim c$. Then either $\{u_1,a\}$ or $\{u_3,c\}$ is a cutset for G (see Figure 17).

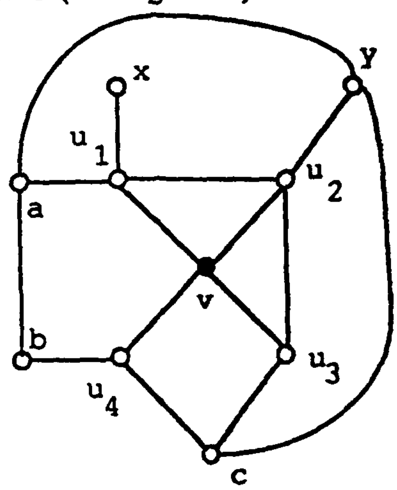


Figure 17

Thus $y \sim x$. Since G is 4-regular, y is not adjacent to at least one of u_3 or u_4. Then either $\{y,u_3\}$ and $\{u_1\}$ or $\{y,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G.

Hence, a is not adjacent to y.

Case 1.3. Suppose $y \sim c$.

Case 1.3.1. Suppose $y \sim u_3$ (see Figure 18).

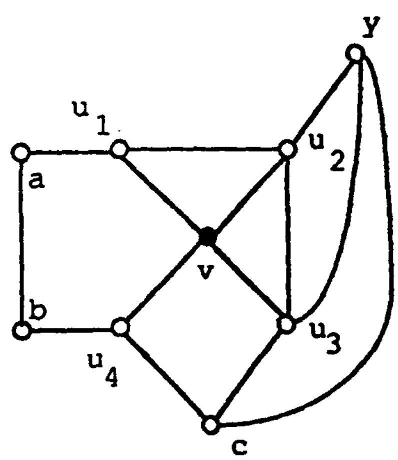


Figure 18

Either we have a (3,3,3,4) face configuration at u_3, or we have a point inside triangle yu_2u_3 or inside triangle ycu_3. From Lemma 12.2, we cannot have a (3,3,3,4) face configuration at u_3. If there is a point inside triangle yu_2u_3, then y is a cutpoint, contradicting 3-connectedness. If there is a point inside triangle ycu_3, then $\{y,c\}$ is a cutset, contradicting 3-connectedness.

Case 1.3.2. So y is not adjacent to u_3. Let $m \sim u_3$ such that $m \notin \{v,c,u_2\}$ and let $z \sim m$ such that $z \notin \{c,y\}$ (see Figure 19). Then $\{z,u_1,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

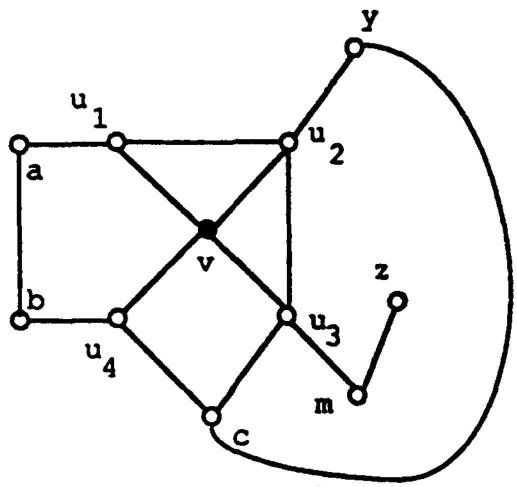


Figure 19

Hence, y is not adjacent to c. It follows that $\{a,y,c\}$ is independent and so $\{a,y,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Thus, the cyclic face configuration (3,3,4,5) cannot occur.

Case 2. Assume the cyclic face configuration is (3,4,3,5), with faces u_1u_2v, u_2cu_3v, u_3u_4v and u_4bau_1v. By Lemma 9, a is not adjacent to u_2, b is not adjacent to u_2, u_4 is not adjacent to u_2, c is not adjacent to u_1, c is not adjacent to u_4, a is not adjacent to u_3, b is not adjacent to u_3, and u_1 is not adjacent to u_3. By Lemma 10, u_4 is not adjacent to a, u_1 is not adjacent to b, and u_1 is not adjacent to u_4. So there exists $y \sim u_4$ such that $y \notin \{a,b,c,v,u_1,u_2,u_3\}$.

Case 2.1. Suppose $a \sim c$. Let $x \sim u_1$ such that $x \notin \{a,v,u_2\}$. If $c \sim x$ or $c \sim b$, then $\{u_1,u_4\}$ and $\{c\}$ don't extend to disjoint maximum independent sets in G. So c is adjacent to neither x nor b. Thus, $\{b,c,x\}$ is independent since G is planar. It follows that $\{b,c,x\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Thus, a is not adjacent to c and, by symmetry, b is not adjacent to c.

Case 2.2. Suppose $y \sim u_1$. Let $z \sim b$ such that $z \notin \{a,y,u_4\}$. Then $\{c,z,u_1\}$ is independent and so $\{c,z,u_1\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to u_1.

Thus, there exists $x \sim u_1$ such that $x \notin \{a,b,c,v,y,u_2,u_3,u_4\}$.

Case 2.3. Suppose $y \sim c$.

Case 2.3.1. If $x \sim c$, then $\{u_1,u_4\}$ and $\{c\}$ don't extend to disjoint maximum independent sets in G. So x is not adjacent to c.

Case 2.3.2. Suppose $x \sim b$. If $b \sim u_2$, then $\{a,x\}$ is a cutset for G. So b is not adjacent to u_2.

Case 2.3.2.1. Suppose $y \sim u_2$. Let $t \sim c$ such that $t \notin \{y,u_2,u_3\}$. Then $\{a,t,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to u_2.

Case 2.3.2.2. Suppose $x \sim u_2$ (see Figure 20).

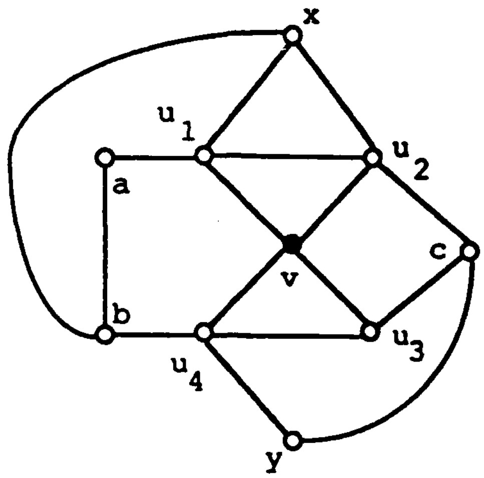


Figure 20

(i) Suppose $y \sim u_3$. If $y \sim x$, then $\{a,b\}$ is a cutset for G. So y is not adjacent to x. But then $\{x,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

(ii) Thus, y is not adjacent to u_3. So there exists $z \sim u_3$ and $w \sim z$ such that $z \notin \{c,v,y,u_4\}$ and $w \notin \{c,y\}$. Then $\{w,b,u_2\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

So x is not adjacent to u_2. Thus, there exists $d \sim u_2$ such that $d \notin \{a,b,c,v,x,y,u_1,u_3,u_4\}$ (see Figure 21).

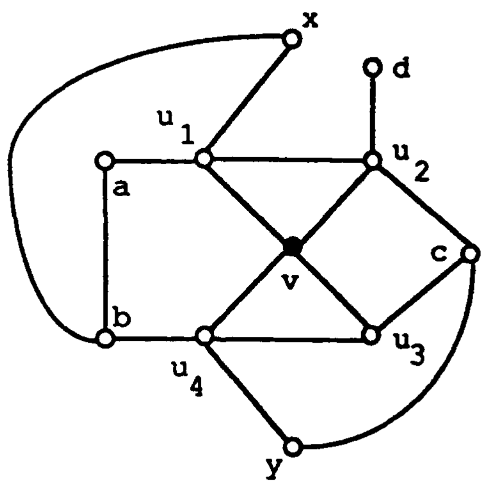


Figure 21

Case 2.3.2.3. If b is not adjacent to d, then $\{b,d,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So $b \sim d$. Then $\{a,x\}$ is a cutset for G.

Thus, x is not adjacent to b. It follows that $\{b,x,c\}$ is independent and so $\{b,x,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Hence, <u>v is not adjacent to c</u> and, by symmetry, <u>x is not adjacent to c</u>.

Case 2.4. Suppose $a \sim y$. Then $\{b,x,c\}$ is independent since G is planar. Hence, $\{b,x,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. Thus, <u>a is not adjacent to v</u>.

So $\{a,c,y\}$ is independent; thus, $\{a,c,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Hence, the cyclic face configuration (3,4,3,5) cannot occur. It follows that G cannot have a point with face configuration (3,3,4,5). $\square$

Lemma 12.7. Suppose G is 3-connected 4-regular planar and in W_2. Then G cannot have a point with face configuration (3,3,4,n), $n \geq 6$.

Proof. Assume to the contrary that v has face configuration (3,3,4,n), $n \geq 6$. Let $N(v) = \{u_1,u_2,u_3,u_4\}$.

Case 1. Assume the cyclic face configuration is (3,4,3,n). Let the faces at v be u_1u_2v, u_2bu_3v, u_3u_4v and $u_4de\ldots acu_1v$ (e = a when n = 6). By Lemma 10, u_1 is adjacent to neither u_4 nor d, and c is adjacent to neither u_4 nor d. By Lemma 9, b is not adjacent to u_1, b is not adjacent to u_4, u_2 is not adjacent to u_4, and u_1 is not adjacent to u_3.

Suppose $b \sim c$. Let $y \sim u_1$ such that $y \notin \{c,v,u_2\}$.

If $b \sim d$, then $\{u_1,u_4\}$ and $\{b\}$ don't extend to disjoint maximum independent sets in G. So b is not adjacent to d. If $b \sim y$, then $\{b,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So b is not adjacent to y. Since b is not adjacent to y, then $\{b,d,y\}$ is independent and so $\{b,d,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Thus, b is not adjacent to c and, by symmetry, b is not adjacent to d. It follows that $\{b,c,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Hence, the cyclic face configuration (3,4,3,n), $n \geq 6$, is not possible.

Case 2. Assume the cyclic face configuration is (3,3,4,n), $n \geq 6$, with faces u_1u_2v, u_2u_3v, u_3bu_4v and $u_4de\ldots acu_1v$ (e = a when n = 6). By Lemma 8, u_1 is not adjacent to u_3. By Lemma 10, u_1 is adjacent to neither u_4 nor d, c is adjacent to neither u_4 nor d and u_3 is not adjacent to u_4. By Lemma 9, b is not adjacent to u_2 and x is not adjacent to u_2, for any x in the n-face at v, $x \notin \{v,u_1\}$. So there exists $s \sim u_2$ such that $s \notin \{a,b,c,d,v,u_1,u_3,u_4\}$ and s is not on the n-face at v.

Case 2.1. Suppose $b \sim d$. Let $y \sim u_4$ such that $y \notin \{b,d,v\}$, and $w \sim y$ such that $w \notin \{b,d\}$. If e is not adjacent to u_3, then $\{e,w,u_3\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So $e \sim u_3$. Then $\{d,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

Thus, b is not adjacent to d.

Case 2.2. Suppose $b \sim c$.

Case 2.2.1. If $b \sim u_1$, then $\{a,v\}$ and $\{b\}$ don't extend to disjoint maximum independent sets in G. So b is not adjacent to u_1.

Case 2.2.2. If $c \sim u_3$, then $\{u_1,u_2\}$ is a cutset for G. So c is not adjacent to u_3.

Thus, there exist points x and t such that $x \sim u_3$, $t \sim u_1$ and $\{x,t\} \cap \{b,c,v,u_1,u_2,u_3\} = \emptyset$. If x = t, then $\{u_1,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So $x \neq t$.

Case 2.2.3. Suppose $t \sim b$. Then $\{u_1,x,d\}$ is independent since $x \neq t$ and G is planar. It follows that $\{u_1,x,d\}$ and $\{b\}$ don't extend to disjoint maximum independent sets in G. So t is not adjacent to b.

Case 2.2.4. Suppose $t \sim u_2$. Then $\{t,b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So t is not adjacent to u_2.

Since G is 4-regular, there exists $z \sim t$ such that $z \notin \{c,x\}$ (see Figure 22). Thus z is not adjacent to u_3, and so $\{a,z,u_3\}$ is independent. It follows that $\{a,z,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G.

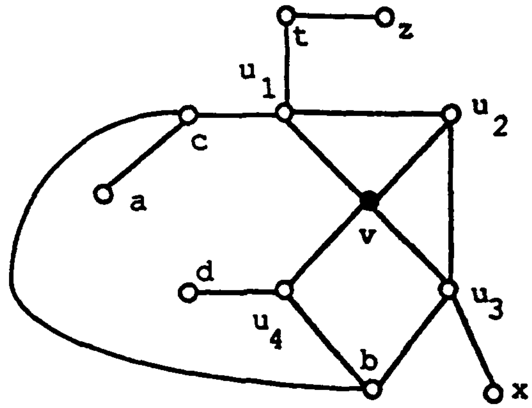


Figure 22

Therefore, b is not adjacent to c.

Case 2.3. Suppose $s \sim c$.

Case 2.3.1. If $s \sim b$, then either $\{c,u_1\}$ or $\{b,u_3\}$ must be a cutset of G. So s is not adjacent to b.

Case 2.3.2. If $s \sim u_1$, then $\{s,b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So s is not adjacent to u_1.

Let $w \sim u_1$ such that $w \notin \{v,c,u_2\}$.

Case 2.3.3. If w is not adjacent to s, then $\{w,s,b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $w \sim s$.

Case 2.3.4. If $s \sim u_3$, then $\{b,u_1\}$ and $\{s\}$ don't extend to disjoint maximum independent sets in G. So s is not adjacent to u_3; hence, $\{s,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G (see Figure 23).

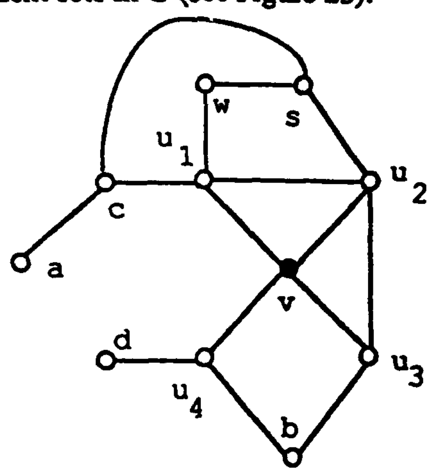


Figure 23

Thus, s is not adjacent to c.

Case 2.4. Suppose $s \sim b$.

Case 2.4.1. Suppose $s \sim u_3$. If s is not adjacent to d, then $\{s,c,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $s \sim d$. Then there exist $t \sim u_4$ such that $t \notin \{v,b,d,s\}$, and $z \sim t$ such that $z \notin \{b,d\}$. It follows that $\{e,z,u_3\}$ is independent and so $\{e,z,u_3\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G.

Hence, s is not adjacent to u_3. So there exists $w \sim u_3$ such that $w \notin \{s,b,v,u_2\}$.

Case 2.4.2. Suppose $s \sim u_4$. If s is not adjacent to w, then $\{s,w,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $s \sim w$. Then $\{d,u_3\}$ and $\{s\}$ don't extend to disjoint maximum independent sets in G.

So s is not adjacent to u_4.

Case 2.4.3. Suppose $s \sim w$. Then $\{s,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So s is not adjacent to w.

Let $W = N(w) - u_3$.

Case 2.4.4. Suppose $b \sim w$. Suppose $s \sim x$ for some $x \in W-b$. Then $\{v,x\}$ and $\{b\}$ don't extend to disjoint maximum independent sets in G. Let $x \in W-b$. Then x is not adjacent to s and so $\{s,x,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

So b is not adjacent to w. Since G is 4-regular, there exists $y \in W$ such that y is not adjacent to s. But then $\{y,s,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G (see Figure 24).

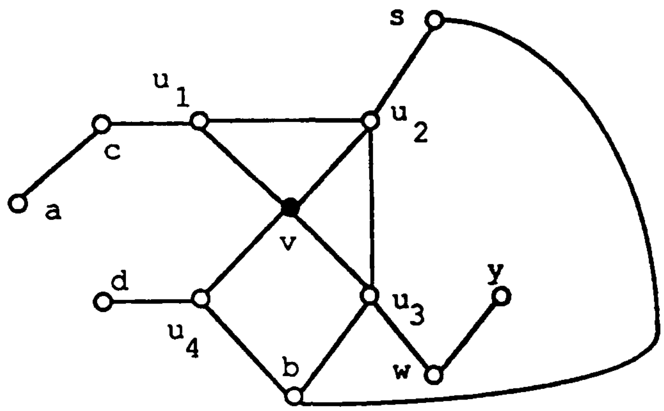


Figure 24

Hence, <u>s is not adjacent to b</u>. It follows that $\{s,b,c\}$ is independent and so $\{s,b,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

So the cyclic face configuration $(3,3,4,n)$, $n \geq 6$, cannot occur. Thus, the face configuration $(3,3,4,n)$, $n \geq 6$, cannot occur. □

<u>Lemma 12.8</u>. Suppose G is 3-connected 4-regular planar and in W_2. If v is a point in G, then v cannot have face configuration (3,3,5,5).

<u>Proof</u>. Assume to the contrary that v has face configuration (3,3,5,5), with $N(v) = \{u_1,u_2,u_3,u_4\}$.

Case 1. Assume the cyclic face configuration at v is (3,5,3,5), with faces u_1u_2v, u_2cdu_3v, u_3u_4v and u_4bau_1v. By Lemma 9, <u>a is not adjacent to u_2</u>, <u>a is not adjacent to u_3</u>, <u>b is not adjacent to u_2</u>, <u>b is not adjacent to u_3</u>, <u>c is not adjacent to u_1</u>, <u>c is not adjacent to u_4</u>, <u>d is not adjacent to u_1</u>, <u>d is not adjacent to u_4</u>, <u>u_1 is not adjacent to u_3</u>, and <u>u_2 is not adjacent to u_4</u>. By Lemma 10, <u>a is not adjacent to u_4</u>, <u>b is not adjacent to u_1</u>, <u>c is not adjacent to u_3</u>, <u>d is not adjacent u_2</u>, <u>u_1 is not adjacent to u_4</u>, and <u>u_2 is not adjacent to u_3</u>.

Hence, there exists <u>$x \sim u_1$</u> such that $x \notin \{a,b,c,d,v,u_2,u_3,u_4\}$.

Case 1.1. Suppose $a \sim c$.

Case 1.1.1. Suppose $x \sim u_2$. If b is not adjacent to d, then $\{x,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $b \sim d$.

Let $s \sim u_3$ and $t \sim u_4$ such that $s \notin \{d,v,u_4,b\}$ and $t \notin \{b,v,u_3,d\}$.

Case 1.1.1.1. If $s = t$, then $\{s,x\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $s \neq t$.

Case 1.1.1.2. If s is not adjacent to t, then $\{s,t,x\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $s \sim t$. But then $\{a,s,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G.

Case 1.1.2. Thus x is not adjacent to u_2. Let $y \sim u_2$ such that $y \notin \{v,c,u_1\}$. If x is not adjacent to y, then we can proceed as in Case 1.1.1 to obtain a contradiction. So $x \sim y$ (see Figure 25).

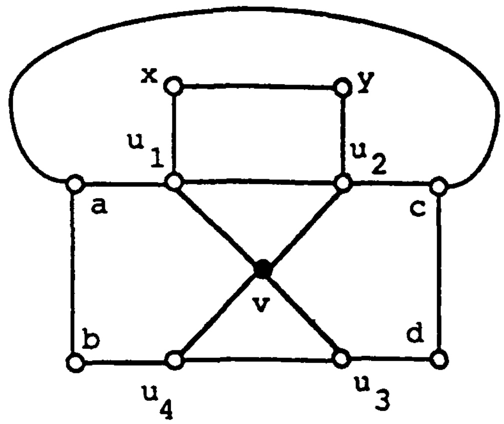


Figure 25

Case 1.1.2.1. Suppose $x \sim a$. If $x \sim c$, then y is a cutpoint for G. So x is not adjacent to c. Thus, there exists $z \sim x$ such that $z \notin \{a,y,u_1,c\}$. Then $\{z,u_2,u_4\}$ is independent and so $\{z,u_2,u_4\}$ and $\{a\}$ don't extend to disjoint maximum independent sets in G.

Case 1.1.2.2. So x is not adjacent to a. Since G is 4-regular, a is not adjacent to at least one of u_3 or u_4. Then either $\{a,x,u_4\}$ and $\{u_2\}$ or $\{a,x,u_3\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G.

Thus, a is not adjacent to c. By symmetry, b is not adjacent to d.

Case 1.2. Suppose $b \sim c$.

Case 1.2.1. If $b \sim x$, then $\{a,x\}$ is a cutset for G. So b is not adjacent to x.

Case 1.2.2. If $x \sim u_2$, then $\{x,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So x is not adjacent to u_2.

Let $y \sim u_2$ such that $y \notin \{v,c,u_1\}$.

Case 1.2.3. If $y \sim x$, then $\{x,d,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to x.

Case 1.2.4. If y is not adjacent to b, then $\{x,y,b,d\}$ is independent and so $\{x,y,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $y \sim b$. Then $\{b,x,u_3\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G.

Hence, b is not adjacent to c and, by symmetry, a is not adjacent to d.

If x is adjacent to any member of $\{b,c,u_3\}$, then $\{b,c,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So x is adjacent to no member of $\{b,c,u_3\}$. Thus, there exists $z \sim u_3$ such that $z \notin \{a,b,c,d,v,x,u_1,u_2,u_4\}$. By symmetry with x, it follows that z is adjacent to neither b nor c.

If z is not adjacent to x, then $\{z,x,b,c\}$ is independent and so $\{z,x,b,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $z \sim x$.

Suppose $x \sim u_2$. If $x \sim d$, then $\{c,d\}$ is a cutset for G. So x is not adjacent to d. Then $\{x,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So x is not adjacent to u_2 and, by symmetry, z is not adjacent to u_4.

Suppose $x \sim u_4$. There exists $t \sim a$ such that $t \notin \{b,x,u_1\}$ and $\{t,c,u_4\}$ is independent. Then $\{t,c,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So x is not adjacent to u_4 and, by symmetry, z is not adjacent to u_2.

Hence, there exist points p and q such that $p \sim u_2$, $q \sim u_4$ and $\{p,q\} \cap \{a,b,c,d,v,x,z,u_1,u_2,u_3,u_4\} = \varnothing$. Since $z \sim x$ from above and G is planar, then $p \neq q$ and p is not adjacent to q. See Figure 26.

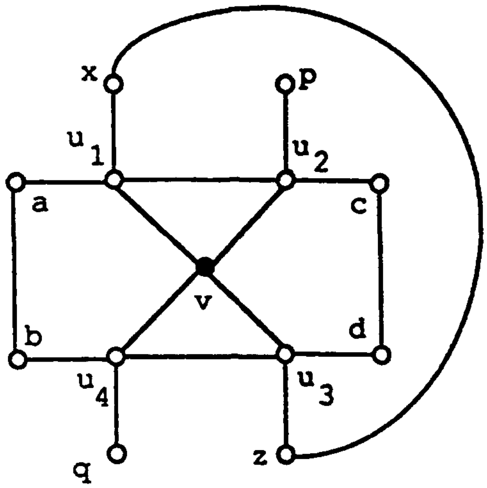


Figure 26

If $p \sim d$, then $\{d,u_4,a\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So p is not adjacent to d and, by symmetry, q is not adjacent to a. Thus, $\{a,d,p,q\}$ is independent; it follows that $\{a,d,p,q\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Hence, the cyclic face configuration (3,5,3,5) cannot occur.

Case 2. Assume the cyclic face configuration at v is (3,3,5,5), with faces u_1u_2v, u_2u_3v, u_3dcu_4v and u_4bau_1v. By Lemma 8, u_1 is not adjacent to u_3. By Lemma 9, u_2 is not adjacent to u_4, a is not adjacent to u_2, b is not adjacent to u_2, c is not adjacent to u_2, and d is not adjacent to u_2. By Lemma 10, u_1 is not adjacent to u_4, u_3 is not adjacent to u_4, d is not adjacent to u_4, a is not adjacent to u_4, b is not adjacent to u_1, and c is not adjacent to u_3.

Thus, there exists $w \sim u_4$ such that $w \notin \{a,b,c,d,v,u_1,u_2,u_4\}$.

Case 2.1. If $d \sim u_1$, then $\{b,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So d is not adjacent to u_1 and, by symmetry, a is not adjacent to u_3.

Case 2.2. Suppose $w \sim a$.

Case 2.2.1. If $a \sim c$, then $\{a,u_3\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So a is not adjacent to c.

Case 2.2.2. Suppose $c \sim u_1$. Since a is not adjacent to c, there exists $s \sim a$ such that $s \notin \{b,w,u_1,c\}$. But then $\{s,u_3,u_4\}$ is independent and so $\{s,u_3,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So c is not adjacent to u_1.

Case 2.2.3 If $w \sim u_3$, then $\{a,d,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So w is not adjacent to u_3.

Let $t \sim u_3$ such that $t \notin \{v,d,u_2\}$.

Case 2.2.4. If $c \sim t$, then $\{c,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So c is not adjacent to t.

Thus, there exists $z \sim c$ such that $z \notin \{d,u_4,a,u_1,u_2,w\}$ and z is not adjacent to u_3 (since G is 4-regular). See Figure 27.

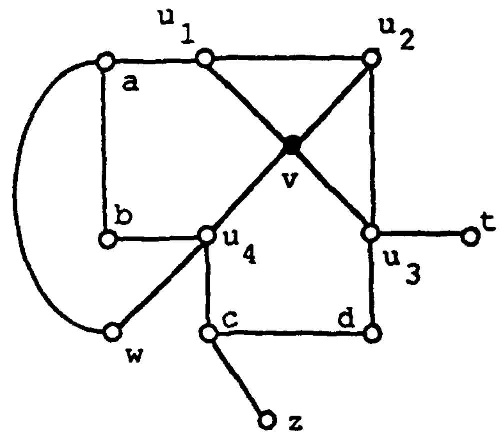


Figure 27

Case 2.2.5. If $z \sim a$, then $\{b,w\}$ is a cutset for G. So z is not adjacent to a. Then $\{a,z,u_3\}$ is independent and so $\{a,z,u_3\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G.

Hence, w is not adjacent to a. By symmetry, w is not adjacent to d.

Case 2.3. Suppose $a \sim d$. Then there exists $y \sim u_2$ such that $y \notin \{a,b,c,d,v,w,u_1,u_3,u_4\}$.

Case 2.3.1. Suppose $a \sim y$. Then $\{y,u_1\}$ is a cutset for G. So a is not adjacent to y and, by symmetry, d is not adjacent to y.

Case 2.3.2. If $y \sim u_3$, then $\{a,w,y\}$ is independent and so $\{a,w,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to u_3 and, by symmetry, y is not adjacent to u_1.

Thus, there exists $s \sim u_3$ such that $s \notin \{a,b,c,d,w,v,y,u_1,u_2,u_4\}$.

Case 2.3.3. If $y \sim s$, then $\{y,c,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to s.

Case 2.3.4. If a is not adjacent to s, then $\{a,y,w,s\}$ is independent and so $\{a,y,w,s\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $a \sim s$.

Case 2.3.5. If $s \sim u_1$, then $\{b,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So s is not adjacent to u_1.

So there exists $x \sim u_1$ such that $x \notin \{a,v,u_2,y,s\}$ (see Figure 28).

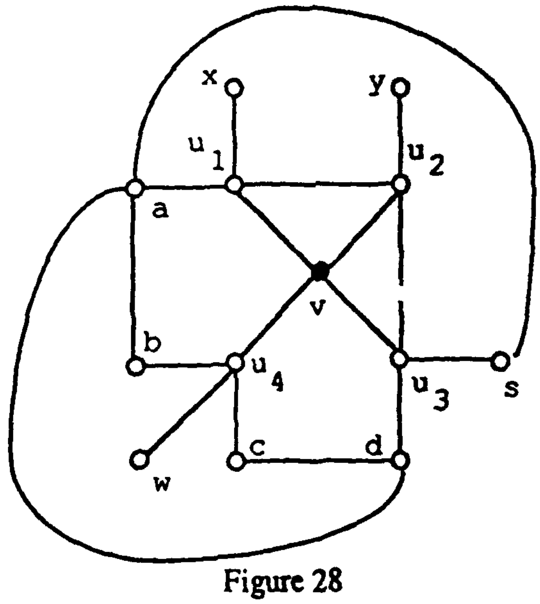


Figure 28

Case 2.3.6. If $y \sim x$, then $\{y,u_3,b\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to x; it follows that $\{x,y,w,d\}$ is independent. Thus, $\{x,y,w,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Hence, a is not adjacent to d.

Case 2.4. If $w \sim u_2$, then $\{a,d,w\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So w is not adjacent to u_2.

Thus, there exists $y \sim u_2$ such that $y \notin \{a,b,c,d,v,w,u_1,u_3,u_4\}$.

Case 2.5. If $a \sim y$, then $\{a,d,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So a is not adjacent to y. By symmetry, d is not adjacent to y.

Case 2.6. Suppose $y \sim w$.

Case 2.6.1. Suppose $y \sim u_1$. If $y \sim b$, then $\{a,u_3,u_4\}$ and $\{y\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to b. Then $\{b,d,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Thus, y is not adjacent to u_1 and, by symmetry, y is not adjacent to u_3.

Case 2.6.2. If $y \sim c$, then $\{a,y,u_3\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to c and, by symmetry, y is not adjacent to b.

Case 2.6.3. Consequently, y has two neighbors z_1 and z_2 such that $\{z_1,z_2\} \cap \{a,b,c,d,w,v,u_1,u_2,u_3,u_4\} = \emptyset$. If $a \sim z_1$ and $a \sim z_2$, then $\{u_1,z_i\}$ is a cutset for G, for some i. If $d \sim z_1$ and $d \sim z_2$, then $\{u_3,z_i\}$ is a cutset for G, for some i. If z_i is adjacent to neither a nor d, for some i, then $\{z_i,a,d,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. Thus, without loss of generality, we can assume $z_1 \sim a$ and $z_2 \sim d$ (see Figure 29).

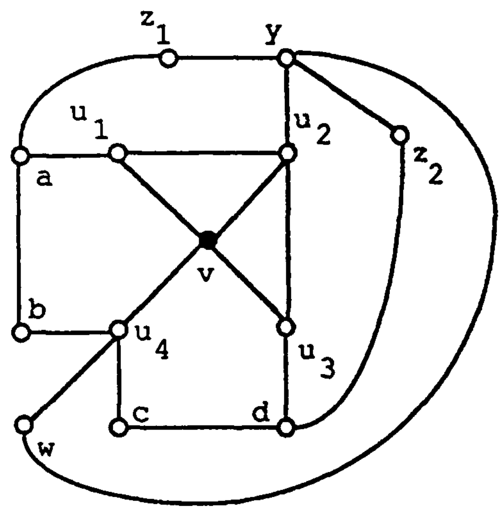


Figure 29

If $z_1 \sim u_1$, then $\{b,y,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So z_1 is not adjacent to u_1.

Thus, there exist $x \sim u_1$ and $t \sim x$ such that $x \notin \{a,v,y,u_2,z_1\}$ and $t \notin \{a,z_1\}$. But then $\{t,b,u_3\}$ is independent and so $\{t,b,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G.

Hence, y is not adjacent to w; thus, the set $\{a,y,d,w\}$ is independent. It follows that $\{a,y,d,w\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

So the cyclic face configuration (3,3,5,5) cannot occur. Therefore, the face configuration (3,3,5,5) cannot occur. □

Lemma 12.9. Suppose G is 3-connected 4-regular planar and in W_2. If v is a point in G, then v cannot have face configuration (3,3,5,n), for n = 6 or 7.

Proof. Assume to the contrary that v has face configuration (3,3,5,n), n = 6 or 7. Let $N(v) = \{u_1,u_2,u_3,u_4\}$.

Case 1. Suppose the cyclic face configuration is (3,5,3,n), with faces u_1u_2v, u_2abu_3v, u_3u_4v and u_4defcu_1v (e = f for the n = 6 case). By Lemma 9, a is not adjacent to u_1, b is not adjacent to u_1, a is not adjacent to u_4, b is not adjacent to u_4, c is not adjacent to u_2, d is not adjacent to u_2, e is not adjacent to u_2, f is not adjacent to u_2, c is not adjacent to u_3, d is not adjacent to u_3, e is not adjacent to u_3, f is not adjacent to u_3, u_2 is not adjacent to u_4, and u_3 is not adjacent to u_1. By Lemma 10, u_1 is not adjacent to u_4, u_2 is not adjacent to u_3, c is not adjacent to d, a is not adjacent to u_3, b is not adjacent to u_2, c is not adjacent to u_4 and d is not adjacent to u_1.

Thus, there exists $x \sim u_2$ such that $x \notin \{a,b,c,d,e,f,v,u_1,u_3,u_4\}$.

Case 1.1. Suppose $a \sim c$. Then there exists $z \sim u_1$ such that $z \notin \{a,b,c,d,e,f,v,u_2,u_3,u_4\}$.

Case 1.1.1. If $a \sim z$, then $\{a,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So a is not adjacent to z.

Case 1.1.2. If $z \sim c$ and $z \sim u_2$, then $\{z,a\}$ is a cutset for G. So z is not adjacent to at least one of c and u_2.

Since G is 4-regular, there exist points s and t adjacent to z such that $\{s,t\} \cap \{a,c,u_2\} = \emptyset$. Now either a is not adjacent to t or a is not adjacent to s. Say a is not adjacent to t. Then $\{a,t,u_3\}$ is independent and so $\{a,t,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G.

Thus, a is not adjacent to c. By symmetry, b is not adjacent to d.

Case 1.2. Suppose $x \sim u_3$. Let $t \sim b$ such that $t \notin \{a,x,u_3\}$. Then $\{t,d,u_2\}$ is independent since G is planar. So $\{t,d,u_2\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

Thus, x is not adjacent to u_3. Let $y \sim u_3$ such that $y \notin \{a,b,c,d,e,f,v,x,u_1,u_2,u_4\}$ (see Figure 30).

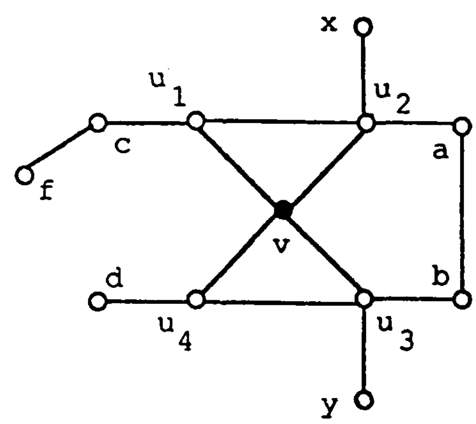


Figure 30

Case 1.3. Suppose $x \sim c$. If b is not adjacent to c, then $\{b,c,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G; so $b \sim c$. Let $w \sim f$ such that $w \notin \{c,y\}$. Then $\{u_2,u_3,w\}$ and $\{c\}$ don't extend to disjoint maximum independent sets in G.

Hence, x is not adjacent to c. By symmetry, y is not adjacent to d.

Case 1.4. Suppose $b \sim x$. If b is not adjacent to c, then $\{b,c,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So $b \sim c$ and $\{a,x\}$ is a cutset for G.

Thus, b is not adjacent to x and, by symmetry, a is not adjacent to y.

Case 1.5. Suppose $d \sim x$. If $a \sim d$, then $\{c,d,u_3\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So a is not adjacent to d. Then $\{a,c,d,y\}$ is independent and so $\{a,c,d,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Thus, d is not adjacent to x and, by symmetry, c is not adjacent to y.

If $x \sim y$, then {a,c,d,y} is independent. So {a,c,d,y} and {v} don't extend to disjoint maximum independent sets in G. Thus, x is not adjacent to y and it follows that {c,d,x,y} is independent. Hence, {c,d,x,y} and {v} don't extend to disjoint maximum independent sets in G.

Thus, the cyclic face configuration (3,5,3,n), n = 6 or 7, is not possible.

Case 2. Suppose the cyclic face configuration is (3,3,5,n), with faces u_1u_2v, u_2u_3v, u_3cdu_4v and u_4befau_1v (e = f when n = 6). By Lemma 8, u_1 is not adjacent to u_3. By Lemma 9, a is not adjacent to u_2, b is not adjacent to u_2, c is not adjacent to u_2, d is not adjacent to u_2, e is not adjacent to u_2, f is not adjacent to u_2, and u_2 is not adjacent to u_4. By Lemma 10, a is not adjacent to u_4, c is not adjacent to u_4, u_1 is not adjacent to u_4, u_3 is not adjacent to u_4, a is not adjacent to b, b is not adjacent to u_1, and d is not adjacent to u_3.

Thus, there exists $y \sim u_2$ such that $y \notin \{a,b,c,d,e,f,v,u_1,u_3,u_4\}$.

Case 2.1. If $a \sim u_3$, then $\{d,u_1\}$ is independent and so $\{d,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So a is not adjacent to u_3 and, by symmetry, c is not adjacent to u_1.

Case 2.2. Suppose $b \sim c$. Since c is not adjacent to u_4, then there exists $w \sim u_4$ such that $w \notin \{b,c,d,v\}$. If $w \sim c$, then {w,d} is a cutset for G. So w is not adjacent to c, and there exists $s \sim w$ such that $s \notin \{b,c,d,u_4\}$.

Case 2.2.1. If c is not adjacent to s, then $\{c,s,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So $c \sim s$ (see Figure 31).

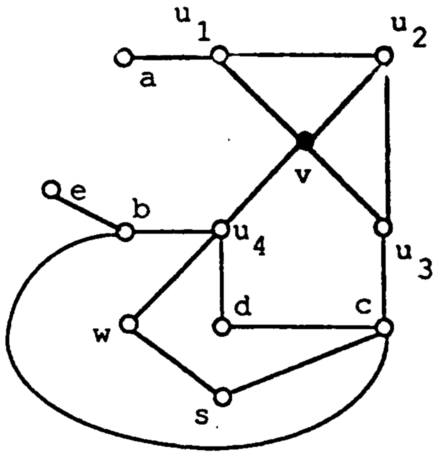


Figure 31

Case 2.2.2. If $w \sim b$, then let $t \sim e$ such that $t \neq b$. Then {s,v,t} and {b} don't extend to disjoint maximum independent sets in G. So w is not adjacent to b.

Thus, there exists $z \sim w$ such that $z \notin \{b,c,d,s,u_4\}$. Then $\{c,z,u_2\}$ is independent and so $\{c,z,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G.

Therefore, b is not adjacent to c.

Case 2.3. Suppose $a \sim c$.

Case 2.3.1. Suppose y is not adjacent to u_1. Let $x \sim u_1$ such that $x \notin \{a,b,c,d,e,f,v,y,u_2,u_3,u_4\}$. If $y \sim c$, then $\{c,x,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to c. If $y \sim x$, then $\{y,f,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to x. If $x \sim c$, then $\{y,c,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. So x is not adjacent to c.

Thus, {x,y,b,c} is independent and so {x,y,b,c} and {v} don't extend to disjoint maximum independent sets in G. So $y \sim u_1$.

Case 2.3.2. If $y \sim u_3$, then $\{y,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to u_3.

Case 2.3.3. If $y \sim c$, then $\{y,u_3\}$ is a cutset for G. So y is not adjacent to c.

Thus, $\{y,b,c\}$ is independent and so $\{y,b,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Hence, a is not adjacent to c.

Case 2.4. Suppose $b \sim u_3$. Then there exists $t \sim c$ such that $t \notin \{d,u_3\}$, t is not adjacent to u_4, and $\{t,u_1,u_4\}$ is independent. Thus, $\{t,u_1,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

So b is not adjacent to u_3.

Case 2.5. If $a \sim y$ or $c \sim y$, then $\{a,c,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G. So y is adjacent to neither a nor c.

Case 2.6. Suppose $b \sim y$. If $y \sim u_4$, then $\{a,c,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to u_4 and there exists $w \sim u_4$ such that $w \notin \{b,c,d,v,y,u_3\}$.

Case 2.6.1. Suppose $y \sim w$. If $y \sim u_1$, then $\{a,u_3,u_4\}$ and $\{y\}$ don't extend to disjoint maximum independent sets in G. So y is not adjacent to u_1. But then $\{c,y,u_1\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G.

Case 2.6.2. So y is not adjacent to w (see Figure 32). If $w \sim c$, then $\{c,e,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. So w is not adjacent to c. Then $\{a,c,w,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

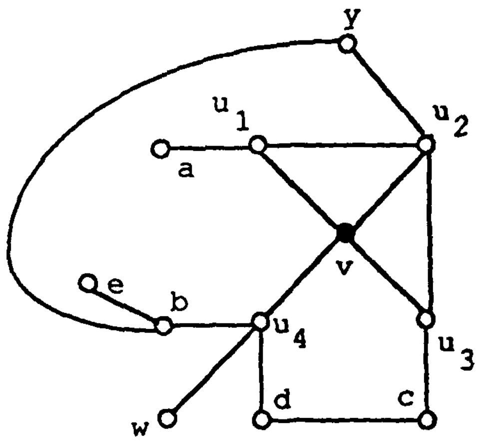


Figure 32

Hence, b is not adjacent to y; it follows that $\{a,b,c,y\}$ is independent and so $\{a,b,c,y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Thus, the cyclic face configuration (3,3,5,n), n = 6 or 7, cannot occur. Therefore, the face configuration (3,3,5,n), n = 6 or 7, cannot occur. □

Lemma 12.10. Suppose G is 3-connected 4-regular planar and in W_2. If v is a point in G, then v cannot have face configuration (3,4,4,4).

Proof. Assume to the contrary that v has face configuration (3,4,4,4). Let $N(v) = \{u_1,u_2,u_3,u_4\}$. Assume the faces at v are u_1u_2v, u_2bu_3v, u_3du_4v and u_4au_1v. By Lemma 9, a is not adjacent u_2 and b is not adjacent to u_1.

If a is not adjacent to b, then $\{a,b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $a \sim b$.

Let $x \sim u_1$, $x \notin \{a,v,u_2\}$, and $y \sim u_2$, $y \notin \{b,v,u_1\}$. If $x = y$, then $\{x,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $x \neq y$. If x is not adjacent to y, then $\{x,y,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So x ~

y. Since G is planar, $\{x,u_3\}$ is independent. Thus, $\{x,u_3\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G.

Therefore, the face configuration (3,4,4,4) cannot occur. □

Lemma 12.11. Suppose G is 3-connected 4-regular planar and in W_2. If v is a point in G, then v cannot have face configuration (3,4,4,5).

Proof. Assume to the contrary that v has face configuration (3,4,4,5). Let $N(v) = \{u_1,u_2,u_3,u_4\}$.

Case 1. Suppose the cyclic order of the faces is (3,4,5,4). Let the faces be u_1u_2v, u_2bu_3v, u_3cdu_4v and u_4au_1v.

By Lemma 9, a is not adjacent to u_2 and b is not adjacent to u_1. By Lemma 10, u_3 is not adjacent to u_4, d is not adjacent to u_3, and c is not adjacent to u_4.

If a is not adjacent to b, then $\{a,b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $a \sim b$. Thus, there exist $x \sim u_1$ and $y \sim u_2$ such that $\{x,y\} \cap \{a,b,c,d,v,u_1,u_2,u_3,u_4\} = \emptyset$.

If $a \sim u_3$, then $\{y,a\}$ is independent and so $\{y,a\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So a is not adjacent to u_3. Thus, there exists $w \sim u_3$ such that $w \notin \{a,b,c,d,v,x,y,u_1,u_2,u_4\}$.

If $w \sim d$, then $\{d,u_2\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So w is not adjacent to d. If $x = y$, then $\{w,x,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $x \neq y$. If x is not adjacent to y, then $\{d,x,y,w\}$ is independent since G is planar. Then $\{d,x,y,w\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $x \sim y$. But then $\{x,u_3\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in G (see Figure 33).

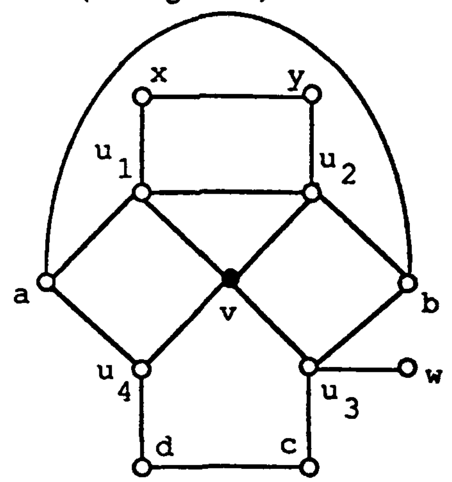


Figure 33

Thus, the cyclic face order (3,4,5,4) cannot occur.

Case 2. Suppose the cyclic order of the faces is (3,4,4,5). Let the faces be u_1u_2v, u_2bcu_3v, u_3du_4v and u_4au_1v. By Lemma 9, u_1 is not adjacent to u_3 and a is not adjacent to u_2. By Lemma 10, b is not adjacent to u_3.

Suppose $a \sim d$. Then there exist points z and w such that $w \sim u_4$, $z \sim w$, $w \notin \{a,d,v\}$ and $z \notin \{a,d\}$. Since u_1 is not adjacent to u_3, then $\{z,u_1,u_3\}$ is independent. Thus, $\{z,u_1,u_3\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in G. Hence, a is not adjacent to d.

Suppose $a \sim b$. Let $x \sim u_2$ such that $x \notin \{b,v,u_1\}$. If $x \sim a$, then $\{d,u_2\}$ and $\{a\}$ don't extend to disjoint maximum independent sets in G. So x is not adjacent to a. But then $\{a,d,x\}$ is independent and so $\{a,d,x\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. Thus, a is not adjacent to b.

Suppose $b \sim d$. Let $z \sim u_3$ such that $z \notin \{c,d,v\}$. From above, $b \neq z$. If $b \sim z$, then $\{b,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G. So b is not adjacent to z. Thus $\{a,b,z\}$ is independent and so $\{a,b,z\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Hence, b is not adjacent to d. So $\{a,b,d\}$ is independent. It follows that $\{a,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G.

Thus, the cyclic face order (3,4,4,5) cannot occur. Therefore, the face configuration (3,4,4,5) cannot occur. □

Now we are ready to state the main result of this paper in Theorem 13. In particular, there is only one 3-connected 4-regular planar W_2 graph.

Theorem 13. Suppose G is 3-connected 4-regular planar and in W_2. Then G is isomorphic to the graph in Figure 4.

Proof. Since G is 4-regular, then the Euler contribution for any point u in G is given by $\phi(u) = 1 - \deg(u)/2 + \Sigma(1/x_i) = -1 + \Sigma(1/x_i)$, where the sum is taken over all faces F_i incident with u and x_i is the size of face F_i. From the discussion earlier, we know that G must have a point with *positive* Euler contribution. Let v be a point in G with $\phi(v) > 0$. Then $\Sigma(1/x_i) > 1$, where the sum is taken over the four faces F_1, F_2, F_3, F_4 incident with v and x_i is the size of F_i, i = 1, 2, 3, or 4. The only solutions to the Diophantine inequality $\Sigma(1/x_i) > 1$ are:

(a) (3,3,3,n), for $n \geq 3$;
(b) (3,3,4,n), for $4 \leq n \leq 11$;
(c) (3,3,5,n), for $5 \leq n \leq 7$;
and (d) (3,4,4,n), for $4 \leq n \leq 5$.

Thus, v must have one of the face configurations given in (a)-(d). By Lemmas 12.1 - 12.11, it follows that G must be the graph given in Figure 4. □

Open Questions

Some questions related to the content of this paper remain open. They include the following:

(1) Are there any exactly 2-connected planar 4-regular 1-well-covered graphs?

(2) What are the planar 5-regular 1-well-covered graphs? The author conjectures that there are no such graphs (although there are known nonplanar 5-regular 1-well-covered graphs).

(3) Can the 4-regular 1-well-covered graphs be characterized? (In a computer search on all regular graphs with at most 13 points, Royle [13] found that there are only nine 4-regular 1-well-covered graphs.)

REFERENCES

1. S. R. Campbell, Some results on planar well-covered graphs, *Ph.D. Dissertation,* Vanderbilt University, 1987.

2. S. R. Campbell and M. D. Plummer, On well-covered 3-polytopes, *Ars Combin.* 25-A, 1988, 215-242.

3. V. Chvátal and P. J. Slater, A note on well-covered graphs, preprint, 1991.

4. N. Dean and J. Zito, Well-covered graphs and extendability, preprint, 1990.

5. O. Favaron, Very well covered graphs, *Discrete Math.* 42, 1982, 177-187.

6. A. Finbow, B. Hartnell, and R. Nowakowski, A characterization of well-covered graphs of girth 5 or greater, to appear.

7. A. Finbow, B. Hartnell, and R. Nowakowski, A characterization of well-covered graphs which contain neither 4- nor 5-cycles, preprint, 1990.

8. H. Lebesgue, Quelques conséquences simples de la formule d'Euler, *Jour. de Math.* 9, 1940, 27-43.

9. O. Ore, *The four-color problem*, Academic Press, New York, 1967, 54-61.

10. O. Ore and M. D. Plummer, Cyclic coloration of plane graphs, *Recent Progress in Combinatorics*, Ed.: W. T. Tutte, Academic Press, New York, 1969, 287-293.

11. M. Pinter, W_2 graphs and strongly well-covered graphs: two well-covered graph subclasses, *Ph.D. Dissertation*, Vanderbilt University, 1991.

12. M. D. Plummer, Some covering concepts in graphs, *J. Combinatorial Theory* 8, 1970, 91-98.

13. G. F. Royle, Private communication to the author, 1991.

14. G. F. Royle and M. N. Ellingham, A characterization of well-covered cubic graphs, preprint, 1991.

15. R. S. Sankaranarayana and L. K. Stewart, Complexity results for well-covered graphs, Technical Report TR 90-21, The University of Alberta, Edmonton, Alberta, Canada, August 1990.

16. J. W. Staples, On some subclasses of well-covered graphs, *Ph.D. Dissertation*, Vanderbilt University, 1975.

17. J. W. Staples, On some subclasses of well-covered graphs, *J. Graph Theory* 3, 1979, 197-204.

18. J. Topp and L. Volkmann, On the well-coveredness of products of graphs, to appear.