AD-A262 448

(2)

REPORT DOCUMENTATION PAGE — Form Approved OMB No 0704-0188

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE Dec. 1992	3. REPORT TYPE AND DATES COVERED Final Report, 4/1/90 -- 7/31/92
4. TITLE AND SUBTITLE Mechanics in Material Space (u)		5. FUNDING NUMBERS 169J147
6. AUTHOR(S) Professor George Herrmann		AFOSR-TR- 93 0491
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Stanford University Applied Mechanics Division Mechanical Engineering Department Stanford, CA 94305-4040		8. PERFORMING ORGANIZATION REPORT NUMBER SPO NO: 7065 (2DJB409)
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AFOSR/NA Bolling Air Force Base Washington, D.C. 20332-6448		10. SPONSORING/MONITORING AGENCY REPORT NUMBER
11. SUPPLEMENTARY NOTES Work performed under Grant No. AFOSR-90-0195		
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release, distribution unlimited		12b. DISTRIBUTION CODE

DTIC ELECTE APR 0 1 1993 S B D

13. ABSTRACT (Maximum 200 words)

Considerable advances have been achieved during the period reported. Conservation laws and path-independent integrals in non-homogeneous plane elastostatics have been established. Further, conservation laws for non-homogeneous bars and beams of variable cross-section, as well as for non-homogeneous plates have been constructed. These laws should permit a more direct and simple analysis of cracks and other defects in these structural elements.

A significant breakthrough came in our success of constructing conservation laws for dissipative systems. Until now, it was possible to construct conservation laws only for systems which had a Lagrangian. We have now succeeded in establishing a general procedure, which we call the "Neutral Action" method, which allows the construction of conservation laws (and path-independent integrals) for systems with damping, for which no systematic procedures existed before. Thus we have generalized in an important and practical way the classical, celebrated theorem of Noether to a large class of technically more realistic systems.

14. SUBJECT TERMS 93 3 31 077			15. NUMBER OF PAGES 15 + Appendix
			16. PRICE CODE
17. SECURITY CLASSIFICATION OF REPORT unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT unclassified	20. LIMITATION OF ABSTRACT

NSN 7540-01-280-5500 — Standard Form 298 (Rev 2-89) Prescribed by ANSI Std Z39-18

Reproduced From Best Available Copy

93-06658 27PY

20001101207

FINAL TECHNICAL REPORT

For the period

April 1, 1990 through July 31, 1992

of work preformed under Grant No. AFOSR-90-0195

for Research in the Field of

MECHANICS IN MATERIAL SPACE

Submitted by

George Herrmann
Principal Investigator

Division of Applied Mechanics
Department of Mechanical Engineering
Stanford University
Stanford CA 94305-4040

DTIC QUALITY INSPECTED 5

December 1992

Accession For
NTIS GRA&I ☑
DTIC TAB ☐
Unannounced ☐
Justification

By
Distribution/

Availability Codes

Dist A-1 | Avail and/or Special

I. INTRODUCTION

During the period covered by this Technical Report, considerable advances have been made in many areas of research under the general title of "Mechanics in Material Space". These advances are reported under three main headings.

Under the heading of "Conservation laws obtained using Noether's First Theorem", results for the various non-homogeneous systems considered are obtained via Noether's first theorem[1], combined with Lie's group theory[2]. The essence of this approach is as outlined in Section 3 of the proposal that lead to Grant No. 90-0195. This methodology for constructing conservation laws is applicable only to Lagrangian systems, and we had applied it to four different Lagrangian systems under the sub-headings of plane elastostatics, bars, beams and plates.

Under the heading of "Conservation laws obtained using Neutral Action Method", results for the various dissipative systems, as well as for one Lagrangian system, are obtained via the Neutral Action method. This new methodology for the construction of conservation laws has been established with the support from Grant No. 90-0195, and this achievement has been reported in Section 7 of the progress report submitted for the period April 1, 1990 through March 31, 1991. Details on this new methodology, which is applicable to dissipative as well as to Lagrangian systems, can be found in the published brief note[3] attached as an Appendix. Using the Neutral Action method, results obtained are reported under five sub-headings: linear viscoelasticity, non-homogeneous beams, systems under initial stress, Sezawa beam and fluid mechanics.

Under the heading of "Results on General Theory", the results presented dealt with the comparison of the two methodologies for constructing conservation laws, Noether's first theorem and the Neutral Action method. The connection between the Neutral Action method and the symmetries of the governing equations for any system of interest was also established.

Until the discovery of the Neutral Action method, there existed no systematic procedure for constructing conservation laws valid for dissipative systems. Since conservation laws are of great value in the analysis of fracture and defects in these systems, most of our efforts during this reporting year have been concentrated in developing conservation laws for dissipative systems using this new methodology. Results in this area are presented in Section III.

Results presented in Section II and Section IV are follow-ups on the results reported in the progress report for 1990-1991. Any advances in the areas covered within these sections, as well as the current status of the relevant manuscripts are described in these sections.

During the reporting period a Ph.D. dissertation was completed under the title, "Conservation laws in non-homogeneous and dissipative systems" by Nelly Y. Chien, who graduated in June 1992.

II. CONSERVATION LAWS OBTAINED USING NOETHER'S FIRST THEOREM

1) Conservation laws (and path-independent integrals) in non-homogeneous plane elastostatics

As mentioned in the last progress report, previous research by Eshelby[4], Sanders[5], Rice[6], Günther[7], Knowles and Sternberg[8] relating to path-independent integrals (**J**, **L** and M) plays a prominent role in the study of fracture mechanics of homogeneous elastic bodies.

The aforementioned studies are applicable only to homogeneous elastic bodies, in order to analyze fracture and defects in non-homogeneous bodies with continuously varying elastic properties, our study has concentrated in developing conservation laws in such non-homogeneous materials. Results obtained can be applied to a novel advanced material called functionally gradient material (FGM)[9]. These materials are expected to be used in light-weight structures such as aircraft and to solve problems in the thermal-protection systems of aerospace vehicles.

Our mathematical apparatus, used in constructing conservation laws for elastic bodies with continuously varying material properties, relied on Noether's first theorem combined with Lie's group theory. Using a novel application of this apparatus, we managed to relate the condition of the invariance of the Lagrangian in Noether's theorem to a condition relating the elastic moduli. In particular, we extended the classical **J**-integral to materials for which Poisson's ratio is constant, while the Young's modulus varies exponentially or as a power law. This **J**-integral is path-independent and characterizes the crack propagation in such materials.

However, we would like to point out that our work has not been exhaustive, since we restricted it to the so-called geometric symmetries. It is expected that by including generalized symmetries the class of non-homogeneous materials, for which we can derive conservation laws, will be enlarged.

A manuscript [M1]* covering this work has been written up during the reporting period and will be submitted to an appropriate journal for publication.

2) Conservation laws for non-homogeneous bars

As mentioned in the last progress report, using Noether's first theorem combined with Lie's group theory, conservation laws were obtained for smoothly non-homogeneous bars. The non-homogeneity can be due to variable cross-section or to bars formed from non-homogeneous materials. These conservation laws will prove useful in studying

* Numbers in bracket preceded by letter M refer to the list of manuscripts at the end of this report.

concentrated defects, such as jump discontinuities and cracks in smoothly non-homogeneous bars.

A manuscript [M2] covering this work has been written up during this reporting period and has been accepted for publication in *Acta Mechanica.*

3) Conservation laws for non-homogeneous beams

As mentioned in the last progress report, conservation laws for smoothly non-homogeneous Bernoulli-Euler beams as well as Timoshenko beams, has been established based on Noether's first theorem combined with Lie's group theory. The two beam theories differ in that the Timoshenko's theory of flexural motions in an elastic beam takes rotatory inertia and transverse-shear deformation into account and contains two independent variables instead of the one transverse displacement of the Bernoulli-Euler theory. Both statics and dynamics of these beam theories are considered in the derivation of conservation laws.

Conservation laws are found to exist only for special classes of non-homogeneous beams. These admissible non-homogeneities are expressible in the form a differential equation. Given any non-homogeneity of a beam, one only needs to introduce the material dependence into the differential equation to determine the conservation laws applicable, if any.

As previously reported, a total of six conservation laws are found for the static of a Bernoulli-Euler beam, and a total of eight for the dynamics of a Bernoulli-Euler beam. These laws are obtained assuming that the beam is loaded only at its end.

During this reporting period, conservation laws are obtained for the statics and dynamics of a non-homogeneous Bernoulli-Euler beam under a continuously distributed loading. As the non-homogeneities of a beam admitting conservation laws are expressible in the form of a differential equation, the distributed loading on a beam that admits conservation laws is also found to be expressible in the form of a differential equation. A total of two conservation laws are obtained for the static case, and a total of four for the dynamic case.

For the non-homogeneous Timoshenko beam, it has been noted in the last progress report that four conservation laws have been obtained for the static case, and six for the dynamic case. During this reporting period, two more conservation laws were obtained for the static case, and one additional for the dynamic case. Also, the class of non-homogeneities admitting conservation laws has been enlarged.

However, we would like to point out that our work completed to date has not been exhaustive, since we restricted it also to the so-called geometric symmetries. It is expected that by including generalized symmetries more conservation laws will be obtained, and the class of non-homogeneous beams admitting conservation laws will also be enlarged.

A manuscript [M3] covering these results has been accepted for publication in the International Journal of Solids and Structures.

4) Conservation laws for non-homogeneous plates

As mentioned in the last progress report, conservation laws were obtained for smoothly non-homogeneous Mindlin plate using Noether's first theorem combined with Lie's group theory. Mindlin's plate bending theory is one that includes the effects of transverse-shear deformation and rotatory inertia. As in the case for non-homogeneous beams, the non-homogeneity of the plate admitting conservation laws is expressible in the form of a set of differential equations. Given any non-homogeneity of the plate, and using this set of differential equations, the applicable conservation laws, if any, can be readily determined.

It has been noted in the last progress report that a total of four conservation laws are found for the statics of a Mindlin plate, and a total of six for the dynamics. During this reporting period, two additional conservation laws were obtained for the static case and an additional three were obtained for the dynamics. Also, the class of non-homogeneous plates admitting conservation laws has been enlarged. These laws are obtained assuming that the beam is loaded only at its ends.

Conservation laws were also obtained for the statics and dynamics of a non-homogeneous Mindlin plate under a continuous distributed loading during this reporting period. Similar to case of a Bernoulli-Euler beam with loading, the distributed loading on a plate is also found to be expressible in the form of a differential equation. A total of six conservation laws were obtained for the statics case and a total of four for dynamics.

Also established during the reporting period were some analogues of the conservation laws relating to the **J**, **L** and M integrals in elasticity. These conservation laws expressed in the form of path-independent integrals are obtained for the statics of a Mindlin plate with and without distributed loading. For dynamics with and without loading, the corresponding conservation laws are expressible as balance laws, where path-integrals are balanced by the rate of change of volume integrals. These results are applicable in the analysis of fracture and defects of non-homogeneous plates similar to the **J**, **L** and M integral in elasticity.

However, we would like to point out that our work completed to date has not been exhaustive, since we restricted it again to the so-called geometric symmetries. It is expected that by including generalized symmetries more conservation laws will be obtained, and the class of non-homogeneous plates admitting conservation laws will also be enlarged.

A manuscript [M4] covering these results has been written up during this reporting period and will be submitted to an appropriate journal for publication in the near future.

III. CONSERVATION LAWS OBTAINED USING THE NEUTRAL ACTION METHOD

1) Conservation laws in linear viscoelasticity

As mentioned in the last progress report, conservation laws were obtained for one- and two-dimensional linear viscoelasticity (Voigt model) using the Neutral Action method.

A conservation law for the one-dimensional case reported previously, is that whose time current can be any function of the total stress and the space current is the negative derivative of this function times some derivative of the displacement. If one chooses the unknown function as proportional to the stress squared, one can obtain the results that shows that the dissipation of the elastic stress is being equal to the rate of work done due to tractions minus the energy dissipation.

In addition to the conservation law mentioned above, two additional conservation laws valid for one-dimensional linear viscoelasticity were obtained during this reporting period. One of these laws is quite trivial in that it expresses the fact that any function of the constant stress is also a conserved quantity. The other conservation law expresses a relation between stress, displacement and velocity that can be obtained by integrating the constitutive equation of the system while taking into account the equilibrium equation.

For the two-dimensional problem, a conservation law that relates the dissipation of elastic energy similar to that in the one-dimensional case was previously obtained only for two special cases, namely, a special pure dilatation case and a second case without the dilatation terms. However, during this reporting period, such a law for the complete problem was finally obtained.

In addition to the conservation laws which describe the dissipation of energy, a conservation law without a time current is also available for two-dimensional linear viscoelasticity as reported previously. Without a time current, this law provides a path-independent integral that might be useful in the analysis of cracks and other defects in linear viscoelastic bodies.

However, we would like to point out again that our work completed to date has not been exhaustive. By allowing the characteristic of conservation law in the Neutral Action method to depend on more and higher order derivatives of the dependent variables of the system, more conservation laws should be obtainable.

A manuscript [M5] covering these results has been written up during this reporting period and accepted for publication in ZAMP. A marked proof of this paper is attached in the Appendix.

2) Conservation laws for non-homogeneous beams

During this reporting period, conservation laws were obtained for the statics and dynamics of non-homogeneous Bernoulli-Euler beams, with and without a distributed loading using the Neutral Action method.

Since the basic building block for construction of conservation laws by the Neutral Action method are the governing equations of the system of interest, this method is applicable to dissipative systems without a Lagrangian function as well as to Lagrangian systems governed by the associated Euler-Lagrange equations.

Even though the problem of non-homogeneous Bernoulli-Euler beams has been treated previously based on Noether's first theorem combined with Lie's group theory limited to geometric symmetries, in order to compare the two methodologies for constructing conservation laws, we applied the Neutral Action method to the same system.

The resulting conservation laws by the Neutral Action method are shown to not only encompass all previous results derived for the same system using Noether's first theorem combined with Lie's group theory utilizing geometric symmetries, but they are also more numerous and are applicable to a wider range of non-homogeneities and loading.

A total of eleven conservation laws are obtained for the static case with and without loading, and a total of fourteen for the dynamic case with and without loading.

The fact that conservation laws derived by the Neutral Action method encompass all previous results obtained using Noether's first theorem combined with Lie's group theory limited to geometric symmetries, is consistent with the comparison between the two methodologies. It has been noted in Section IX of the last progress report that, for Lagrangian systems, the requirement for existence of conservation laws by the NA method and by Noether's first theorem are mathematically identical. However, as this comparison is based on the general form of Noether's first theorem admitting an extension by Bessel-Hagen[10] (divergence symmetries), it is expected that the conservation laws obtained here for non-homogeneous Bernoulli-Euler beams using the NA method to be more general and to encompass all previous results reported in Section III-3.

Also, for a conservation law obtained, the Neutral Action method of constructing conservation laws is shown to be more efficient than the classical method of Noether's first theorem with its extension by Bessel-Hagen. While the condition for existence of conservation laws by the classical methods requires the use of three unknown functions, the Neutral Action method uses only one unknown function to arrive at the same conservation law.

However, we would like to point out that our work completed to date has not been exhaustive. By again allowing the characteristic of conservation law in the Neutral Action method to depend on more and higher order derivatives of the dependent variables of the system, more conservation laws will be obtained, and the class of non-homogeneous beams admitting conservation laws will also be enlarged.

A manuscript [M3] covering these results has been written up and will be published in the International Journal of Solids and Structures.

3) Conser   n laws for systems under initial stress

During this reporting period, conservation laws were sought using the Neutral Action method for linear elastic systems under initial stress. Since structural components are frequently prestressed or exhibit some distribution of residual stresses, a conservation law that relates to the energy release rate will be extremely useful in the analysis of fracture and defects for such components.

Up until the present, no conservation law was yet derived. The difficulties lie in the generality of the system at hand (the initial stress state being unspecified) and the determination of the characteristic functions for conservation laws which must satisfy approximately twenty partial differential equations simultaneously. Nevertheless, it is expected that results will be obtained in the near future for some special cases such as systems under hydrostatic or uniaxial stress.

4) Conservation laws for Sezawa Beam

During this reporting period, conservation laws for the Sezawa Beam were derived using the Neutral Action method. This beam theory is a modification of the Bernoulli-Euler beam theory which incorporates the effects of internal damping. Results obtained indicated that conservation laws for this system exist only in physical space. As no conservation laws were found in materials space, a balance law that relates to the energy release rate is unavailable.

5) Conservation laws for fluid mechanics

During this reporting period, conservation laws were derived using the Neutral Action method for 2-dimensional Navier-Stokes equations for an incompressible fluid with constant fluid density, constant viscosity and in the absence of body forces. Results obtained will be most useful in the numerical analysis of the system, as well as in providing insights into the behavior of the fluid, both mathematically and physically.

Result obtained so far are still limited. In the general case, a total of four conservation laws were obtained. These laws do not contain energy terms and they relate only to the incompressibility of the system. Also, conservation laws were obtained for cases with special pressure fields, such as linear fields and fields which satisfy the Laplace equation. The conservation laws in these special cases show a balance of displacement gradients, pressure gradients and strain rates. The physical interpretation of these laws is yet to be investigated. The search for conservation laws in this area is far from complete. Efforts will be undertaken to obtain additional conservation laws and to interpret their physical significance, hopefully on the level of energies.

IV. RESULTS ON GENERAL THEORY

Conservation laws and symmetries

As reported in the last progress report, the Neutral Action method for constructing conservation laws has a relation to the concept of symmetries. For each characteristic of a conservation law by the Neutral Action method, there exists in the adjoint field a corresponding characteristic for the symmetries of the governing equations for the system of interest. However, the converse is not true.

Also mentioned in the last progress report is that the Neutral Action method of constructing conservation laws, when applied to Lagrangian systems, is identical to the method by Noether's first theorem with Bessel-Hagen's extension. The conditions for existence of conservation laws of both methodologies can be transformed into identical form.

During this reporting period, it has been further shown that, within Lagrangian systems, both conservation laws derived by the Neutral Action method and by Noether's first theorem with Bessel-Hagen's extension have a one-to-one correspondence to the variational symmetries of the Lagrangian functional. This is an additional supporting fact for the equivalence of both methods when applied to Lagrangian systems.

A manuscript [M6] covering these results has been written up and submitted to an appropriate journal for publication.

References

1. E. Noether, "Invariante Variationsprobleme", *Nachr. König. Gessel. Wissen. Göttingen, Math.-Phys. Kl.*, pp. 235-257, 1918 (See *Transport Theory and Stat. Phys.* **1**, pp. 186-207 for an English translation.)

2. P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.

3. Honein, T., Chien, N., and Herrmann, G., "On Conservation Laws of Dissipative Systems," *Physics Letters A*, **155**, pp. 223-224, 1991.

4. J.D. Eshelby, "The Force on an Elastic Singularity," *Philosophical Transactions Royal Society*, A **244**, pp. 87-112, 1951.

5. J.L. Sanders, "On the Griffith-Irwin Fracture Theory," *J. Applied Mechanics*, **27**, pp. 325-353, 1960.

6. J.R. Rice, "A Path-Independent Integral and the Approximate Analysis of Strain Concentrations by Notches and Cracks," *J. Applied Mechanics*, **35**, pp. 379-386, 1968.

7. W. Günther, "Über einige Randintegrale der Elastomechnik," *Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft*, **XIV**, Verlag Friedr. Vieweg & Sohn, Braunschweig, pp. 53-72, 1962.

8. J.K. Knowles and E. Sternberg, "On a Class of Conservation Laws in Linearized and Finite Elastostatics," *Archives for Rational Mechanics and Analysis*, **44**, Springer-Verlag, Germany, pp. 187-211, 1972.

9. Yamanouchi *et al.*, ed., *Proceedings of the First International Symposium of Functionally Gradient Materials*, Sendai, Japan, 1990.

10. E. Bessel-Hagen, "Über die Erhaltungssätze der Electrodynamik", *Math. Ann.* **84**, pp. 258-276, 1921.

List of Manuscripts Prepared during the Reporting Period

M1) T. Honein and G. Herrmann, Conservation Laws (and Path-Independent Integrals) in Non-Homogeneous Plain Elastostatics.

M2) T. Honein and G. Herrmann, Conservation Laws for Non-Homogeneous Bars.

M3) N. Chien, T. Honein, and G. Herrmann, Conservation Laws for Non-Homogeneous Bernoulli-Euler Beams.

M4) N. Chien, T. Honein, and G. Herrmann, Conservation Laws for Non-Homogeneous Mindlin Plates.

M5) N. Chien, T. Honein, and G. Herrmann, Conservation Laws for Linear Viscoelasticity.

M6) N. Chien, T. Honein, and G. Herrmann, Dissipative Systems, Conservation Laws and Symmetries.

Appendix

On conservation laws for dissipative systems

T. Honein, N. Chien and G. Herrmann
Division of Applied Mechanics, Stanford University, Stanford, CA 94305-4040, USA

Received 2 January 1991; accepted for publication 12 March 1991
Communicated by J.P. Vigier

In this brief note, a new methodology is advanced with the aim of establishing conservation laws for dissipative systems described by partial differential equations. This methodology extends Noether's celebrated first theorem, which is applicable only to systems governed by equations derivable variationally from a Lagrangian. Several simple, yet non-trivial, examples serve to illustrate the proposed procedure.

The purpose of this brief note is to advance and illustrate a procedure for constructing conservation laws (i.e., divergence-free expressions) for dissipative systems described by partial differential equations. Such expressions are most useful for a variety of reasons. Whereas for non-dissipative systems, which arise from a variational principle, Noether's first theorem [1] is available to establish conservation laws in a systematic fashion (cf. a very complete presentation and discussion by Olver [2]), no corresponding methodology was known and exploited for dissipative systems, since these might not be related to a variational principle and thus Noether's procedure becomes inapplicable.

To keep this note short, reference again has to be made to ref. [2] for all details.

In a non-dissipative system, let L be a Lagrangian and $E(L)=0$ the associated Euler–Lagrange differential equation. Based on Noether's theorem it can be shown that if one finds a "characteristic" Q of a conservation law (having made use of the underlying symmetry), then the associated conservation law may be expressed as

$$\mathrm{Div}\, P \equiv P^i_i = QE(L) = 0,$$

where P^i $(i=1, 2, ..., n)$ is an n-tuple of functions determined by Q, n is the number of independent variables and the subscript indicates total differentiation.

Now let a dissipative system be given by the partial differential equation

$$\Delta(u)=0,$$

where u is the dependent variable.

Even though this equation may not be derivable from a variational principle, still we set

$$f\Delta(u)=P^i_i,$$

but now f is considered not to be pre-determined as Q was, but has to be found from the above equation. We note that since $f\Delta(u)$ is required to be divergence-free, it has to be a null-Lagrangian, and this is the condition which determines f. Thus we set

$$\mathscr{L}=f\Delta(u)$$

and require $\delta\mathscr{L}=0$ identically in u in order to find f.

The generalization of the foregoing procedure to a system of partial differential equations is straightforward.

As a first illustration consider the diffusion equation

$$\Delta(u)=u_t-\alpha u_{xx}=0.$$

The condition $\delta\mathscr{L}=0$ leads to

$$f_t+\alpha f_{xx}=0,$$

which is, in this instance, the adjoint equation. The associated conservation law is given by

$$P^x=\alpha(f_x u-fu_x)\,,\quad P^t=fu\,.$$

As a second illustration consider the nonlinear wave equation

$$\Delta(u)=u_t-uu_x=0\,.$$

The condition $\delta\mathcal{L}=0$ leads here to

$$f_t-uf_x=0$$

and to a class of conservation laws with

$$P^x=-C\left(t\frac{u^{n+2}}{n+2}+x\frac{u^{n+1}}{n+1}\right),$$

$$P^t=C\left(t\frac{u^{n+1}}{n+1}+x\frac{u^{n}}{n}\right),$$

where n is any real number except -2, -1, 0, and C is an arbitrary constant.

Lastly, consider linear viscoelasticity in the form

$$\Delta(u)=Yu_{xx}+\eta u_{xxt}=0\,.$$

Here Y is Young's modulus, η the viscosity of a Voigt element. The stress σ is related to the displacement u by

$$\sigma=Yu_x+\eta u_{xt}\,.$$

One possible conservation law may be obtained as

$$P^t=g(\sigma)\,,$$

$$P^x=-g'\int\sigma_t\,\mathrm{d}x=-g'(Yu_t+\eta u_{tt})\,,$$

where $g(\sigma)$ is arbitrary.

If we choose $g(\sigma)=\sigma^2/2Y$, with

$$P^t_t+P^x_x=0$$

and with

$$\sigma^e=Y\epsilon\,,$$

we may obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sigma^{e2}}{2Y}=\sigma\dot\epsilon-\eta\dot\epsilon^2\,,$$

which has an immediate physical interpretation and is a useful result.

It is quite remarkable that the methodology advanced in this note has not been exploited earlier. Indeed, the tools to establish our procedure are all available (cf. ref. [2], proposition 5.33) [#1] but they have been used to construct a machinery whose purpose is different from the one herein presented.

It is seen that the procedure to construct conservation laws advanced here, has led to novel results. Further aspects of the methodology presented here will be discussed in forthcoming papers.

This work was performed with the support of the AF Office of Scientific Research and the U.S. Department of Energy. This support is gratefully acknowledged. The authors would like also to thank Professor P. Olver for many valuable discussions.

[#1] It is relevant to note here that we became aware of proposition 5.33 after our work was completed.

References

[1] E. Noether, Invariante Variationsprobleme, Nachr. K. Ges. Wiss. Göttingen, Math.-Phys. Kl. (1918) pp. 235–257 [Engl. transl.: Transp. Theory Stat. Phys. 1 (1971) 186].

[2] P.J. Olver, Applications of Lie groups to differential equations (Springer, Berlin, 1986).

Fri-Sep

Appendix

3 -09

Z angew Math Phys 44 (1993)
(ZAMP)

0044-2275/93/030405-17 $ 1.50 + 0.20
© 1993 Birkhäuser Verlag, Basel

ZAMP 92011

Conservation laws for linear viscoelasticity

By N. Chien, T. Honein, and G. Herrmann, Division of Applied Mechanics, Stanford University, Stanford, CA 94305-4040, USA

1. Introduction

Since the introduction of the J, L, and M integrals in fracture mechanics, the importance of path-independent integrals has been widely recognized. These path-independent integrals are conservation laws of a divergence-free form in material space. Classically, conservation laws are obtainable through Noether's first theorem [2], with extension of Bessel-Hagen [1]. However, Noether's approach presupposes the existence of a Lagrangian function for the mechanical systems considered, and since systems with damping and various other dissipative mechanisms do not possess Lagrangian functions, Noether's theorem cannot be applied to dissipative systems.

entitled

In a recent brief note ~~titled~~ "On Conservation Laws for Dissipative Systems" [4], a new approach for constructing conservation laws was proposed. Given a system governed by a set of differential equations, the proposed procedure, termed the ~~"Neutral Action (NA) Method"~~ in [5], allows one to systematically construct the divergence-free quantities applicable to the system considered.

"Neutral Action (NA)" method
• move quotation mark
• small m

It is the purpose of this present contribution to derive conservation laws valid for one- and two-dimensional linear viscoelasticity (Voigt model) using the proposed method. The results presented are not exhaustive, but as a limited set, these laws should prove useful in the analysis of defects and fracture in linear viscoelastic material.

Neutral Action

2. The ~~natural action~~ (NA) method

Given any system with m independent variables x^i $(i = 1, 2, \ldots, m)$, n dependent variables u^k $(k = 1, 2, \ldots, n)$, the governing equations can be represented by

$$\Delta^j(x^i, u^k, u^k_j) = 0. \tag{1}$$

28. Sep. 1992

BC

Approved
Bon à tirer
Gut zum Druck

Datel 22 Oct 92

Unterschrift Signature

Using the multi-index notation introduced by Olver [3], u^k_J in equation (1) represents all possible pth order partial derivatives of u^k,

$$u^k_J \equiv \frac{\partial^p u^k}{\partial x^{j_1}\, \partial x^{j_2} \cdots \partial x^{j_p}},$$

with $J = (j_1, j_2, \ldots, j_p)$ as an unordered p-tuple of integers, $1 \le j_\alpha \le m$ indicating which derivatives are being taken, and $\# J = p$ indication how many derivatives are being taken.

indicating

For any system governed by a set of differential equations as in equation (1), the "Neutral Action (NA) Method" proposed in [4] states that it is possible to construct conservation laws valid for the system in the form of

"Neutral Action (NA)" method

$$f^i \Delta^i = D_i P^i = 0, \tag{2}$$

$f^j \Delta^j$ change $i \to j$

if

$$E^k(f^i \Delta^i) \equiv 0, \tag{3}$$

comma ; $f^j = f^j(x^i, u^k, u^k_J)$ change $j \to i$

insert for all dependent variables u^k.

Here $f^i = f^i(x^j, u^k, u^k_J)$ are the unknown characteristics of conservation laws, and E^k is the Euler operator defined as

$$E^k(L) = (-D)_J \frac{\partial L}{\partial u^k_J}, \quad 0 \le \# J \le p, \tag{4}$$

with

$$D_J \equiv D_{j_1} D_{j_2} \cdots D_{j_p}$$

representing all possible pth order total derivatives, and

$$(-D)_J = D_J \quad \text{for } \# J = \text{even},$$
$$(-D)_J = -D_J \quad \text{for } \# J = \text{odd}.$$

insert → In this paper, repeated dummy indices indicate summation.

Since our objective is to construct some divergence-free expressions out of $f^i \Delta^i$, and since the Euler operator acting on any total divergence always gives a null result by calculus of variation, it follows that equation (3) is a requirement for existence of conservation laws. Equation (3) also implies that $f^i \Delta^i$ is a null Lagrangian whose action integral,

$f^j \Delta^j$

the

$$A = \int_\Omega f^i \Delta^i \, dV, \tag{5}$$

$f^j \Delta^j$

has vanishing variation for any dependent variables u^k, $\delta A = 0$. In other words, in order to construct conservation laws for any system (dissipative or Lagrangian) governed by a set of differential equations $\Delta^i = 0$ by the NA method, we try to construct a product of $f^i \Delta^i$ whose action integral does not change variationally. Hence the name "Neutral Action Method" given to this procedure.

i.e. $\delta A = 0$

change i to j

, $\Delta^j = 0$,

$f^j \Delta^j$

"Neutral Action" method

In practice, given any set of differential equations, one only needs to solve equation (3) for the unknown characteristics $\boxed{f^i}$ and then proceed to construct the conserved currents P^i valid for the system governed by this set of differential equations.

f^j

3. One dimensional linear viscoelasticity

Conservation laws derived for 1-D linear viscoelasticity were given as an example in the brief note "On Conservation Laws for Dissipative Systems" [4] to illustrate the construction of conservation laws via the NA method. For completeness of the present paper, these results will be presented again in this section with some additional details.

The governing equation for 1-D linear viscoelasticity (Voigt model) is given by

$$\Delta = Yu_{xx} + \eta u_{xxt} = 0, \tag{6}$$

are

is

is

where Y is ~~the~~ Young's modulus, η the viscosity coefficient of a Voigt element, x and t the spatial coordinate and time, and u the displacement for the system. Subscripts ~~in this paper~~ indicate differentiation.

The stress σ of this system is related to the displacement u by

$$\sigma = Yu_x + \eta u_{xt}. \tag{7}$$

here partial

If one $\boxed{\text{transform}}$ the variables in this system such that

transforms

$$\phi = Yu + \eta u_t, \tag{8}$$

the governing equation of 1-D linear viscoelasticity can be written as

$$\Delta = \phi_{xx} = 0. \tag{9}$$

Assuming the characteristic of conservation laws for the system to be

$$f = f(\phi, \phi_x, \phi_t) \tag{10}$$

the condition for existence of conservation laws, equation (3), by the NA method will require that

$\equiv$ change $= \to \equiv$

$$E(f\Delta) = \frac{\partial f}{\partial \phi}\phi_{xx} - D_x\left(\frac{\partial f}{\partial \phi_x}\phi_{xx}\right) - D_t\left(\frac{\partial f}{\partial \phi_t}\phi_{xx}\right) + D_{xx}(f) \boxed{=} 0. \tag{11}$$

Since the only unknown in the above equation is the characteristic f which depends on ϕ, ϕ_x, and ϕ_t, it follows that all coefficients of second and higher order derivatives of ϕ in this equation must be set equal to zero independently. The resulting set of equations is as follows:

coefficient	equation
$\phi_{xx}\phi_{tt}$	$\dfrac{\partial^2 f}{\partial \phi_t^2} = 0$
ϕ_{xt}^2	$\dfrac{\partial^2 f}{\partial \phi_t^2} = 0$
ϕ_{xt}	$2\,\dfrac{\partial^2 f}{\partial \phi\, \partial \phi_t}\,\phi_x = 0$
ϕ_{xx}	$2\,\dfrac{\partial f}{\partial \phi} - \dfrac{\partial^2 f}{\partial \phi\, \partial \phi_t}\,\phi_t + \dfrac{\partial^2 f}{\partial \phi\, \partial \phi_x}\,\phi_x = 0$
remaining	$\dfrac{\partial^2 f}{\partial \phi^2}\,\phi_x^2 = 0.$ (12)

* move this block 1" over

After some mathematical manipulations, the solution of the above set of equations is found to be

$$f = f^1(\phi_x)\phi_t + f^2(\phi_x) + C\phi\phi_x^{-2}. \tag{13}$$

Since f^1 and f^2 are arbitrary functions of ϕ_x, it is permissible to rename them as

$$\begin{aligned} f^1(\phi_x) &= -g''(\phi_x), \\ f^2(\phi_x) &= h'(\phi_x), \end{aligned} \tag{14}$$

with $(\ )'$ denoting differentiation with respect to the argument. Defining f^1 and f^2 as such will allow the conservation laws derived to have a simpler appearance.

With

$$f = -g''(\phi_x)\phi_t + h'(\phi_x) + C\phi\phi_x^{-2}, \tag{15}$$

and

$$\Delta = \phi_{xx}, \tag{16}$$

it is possible to construct divergence-free expressions, $D_x P^x + D_t P^t = 0$, out of the product of f and Δ. The resulting conservation law is

$$\begin{aligned} P^x &= -g'(\phi_x)\phi_t + h(\phi_x) - C\left(\frac{\phi}{\phi_x} - x\right), \\ P^t &= g(\phi_x). \end{aligned} \tag{17}$$

Since the stress, $\sigma = \phi_x$, of this system is constant with respect to x, the conservation law pertaining to only $h(\phi_x) \neq 0$,

$$P^x = h(\phi_x) = A_1(t), \tag{18}$$

implies that any function of the constant stress is a constant with respect to x, where $A_1(t)$ is an arbitrary function of time.

With $\sigma = \phi_x = \boxed{\text{const}}$, the conservation law corresponding to $C \neq 0$ can be rewritten as
constant only

$$P^x = \phi - x\phi_x = A_2(t), \tag{19}$$

where $A_2(t)$ is an arbitrary function in t. In terms of stress and displacement,

$$P^x = Yu + \eta u_t - x\sigma = A_2(t). \tag{20}$$

This law expresses a relation between stress, displacement and $\boxed{\text{strain-rate}}$ velocity which can be obtained by integrating the constitutive equation of the system while taking into account the equilibrium equation.

In terms of stress, the divergence-free expression that pertains to only $g(\phi_x) \neq 0$ can be writtten as

$$\begin{aligned} P^x &= -g'(\sigma) \int \sigma_t \, dx, \\ P^t &= g(\sigma), \end{aligned} \tag{21}$$

where g is any function of stress.
denotes
If one chooses $g(\sigma) = \sigma^2/2Y$, and $\boxed{\text{denoting}}$ the elastic component of stress as $\sigma^e = \boxed{Eu}$, and the strain rate $\boxed{\text{given by}}$ $\varepsilon_t = u_{xt}$, this particular conservation law can be written as
YU_x then as

$$D_t\left(\frac{\sigma^{e2}}{2Y}\right) = \sigma\varepsilon_t - \eta\varepsilon_t^2, \tag{22}$$

whose physical interpretation is that the dissipation of elastic energy $(\sigma^{e2}/2Y)$ is equal to the rate of work done due to tractions $(\sigma\varepsilon_t)$ minus the energy dissipation $(\eta\varepsilon_t^2)$.

Additional conservation laws for 1-D linear viscoelasticity can be obtained by assuming different dependence of stress in the function $g(\sigma)$.

4. Two dimensional linear viscoelasticity

Denoting the normal stresses in the x and y direction as σ^{xx} and σ^{yy}, the shear stress as σ^{xy}, the equilibrium condition for any 2-D continuum mechanics problem is given by

$$\begin{aligned} \Delta^1 &= \sigma^{xx}_x + \sigma^{xy}_y = 0, \\ \Delta^2 &= \sigma^{yy}_y + \sigma^{xy}_x = 0. \end{aligned} \tag{23}$$

based on the Voigt model

For linear viscoelasticity, the stress components are related to the normal strains in the x and y direction (ε^{xx}, ε^{yy}) and the shear strain (ε^{xy}) by

c/(x3)

$$
\begin{aligned}
\sigma^{xx} &= (\lambda + 2\mu)\varepsilon^{xx} + \lambda\varepsilon^{yy} + (\alpha + 2\beta)\varepsilon_t^{xx} + \alpha\varepsilon_t^{yy},\\
\sigma^{yy} &= (\lambda + 2\mu)\varepsilon^{yy} + \lambda\varepsilon^{xx} + (\alpha + 2\beta)\varepsilon_t^{yy} + \alpha\varepsilon_t^{xx},\\
\sigma^{xy} &= 2\mu\varepsilon^{xy} + 2\beta\varepsilon_t^{xy},
\end{aligned}
\tag{24}
$$

where λ and μ are the Lamé constants, α and β the viscosity coefficients of the system.

With u and v being the displacements of the system in the x and y directions, the strain components are given by

c/(x2)

$$
\begin{aligned}
\varepsilon^{xx} &= u_x,\\
\varepsilon^{yy} &= v_y,\\
\varepsilon^{xy} &= \tfrac{1}{2}(u_y + v_x).
\end{aligned}
\tag{25}
$$

Combining the above equations, the two governing equations for 2-D linear viscoelasticity in terms of displacements are

c/ c/

$$
\begin{aligned}
\Delta^1 &= (\lambda + 2\mu)u_{xx} + \mu u_{yy} + (\lambda + \mu)v_{xy}\\
&\quad + (\alpha + 2\beta)u_{xxt} + \beta u_{yyt} + (\alpha + \beta)v_{xyt},\\
\Delta^2 &= (\lambda + 2\mu)v_{yy} + \mu v_{xx} + (\lambda + \mu)u_{xy}\\
&\quad + (\alpha + 2\beta)v_{yyt} + \beta v_{xxt} + (\alpha + \beta)u_{xyt}.
\end{aligned}
\tag{26}
$$

Given the governing equations for the system, the condition for existence of conservation laws by the NA method, as given in equation (3), is

$$\boxed{E(f^1\Delta^1 + f^2\Delta^2) = 0.} \to E^5\,(f^1\Delta^1 + f^2\Delta^2) \equiv 0. \tag{27}$$

Δ^j's

of

f^j's

Since Δ"s contain third derivatives in u and v, in order to evaluate this condition ~~for~~ existence, one needs to compute the third total derivative of f"s. If one assumes some characteristics that depend on x, y, t, u, v and also on derivatives of u and v, calculating the third total derivatives of f"s would be virtually impossible in the absence of advanced computing devices with large memory capacity. Due to this difficulty in evaluating equation (27), the general solution for the characteristics is yet unknown. However, particular solutions may be found heuristically. f^j's

By restricting the dependence of f"s to x, y and t only, the condition for existence of conservation law by the NA method, equation (27), requires that

f^j's

c/

$$
\begin{aligned}
f^1_{xx} + f^1_{yy} &= 0,\\
f^2_{xx} + f^2_{yy} &= 0,\\
f^1_x + f^2_y &= \boxed{C_3(t)},
\end{aligned}
\tag{28}
$$

$A_3(t)$

where $C_3(t)$ is an arbitrary function of t. The corresponding conservation law is $A_2(t)$

$$
\begin{aligned}
P^x &= f^1\sigma^{xx} + f^2\sigma^{xy} - f^1_x[(\lambda+2\mu)u + (\alpha+2\beta)u_t] \\
&\quad - f^1_y[\mu v + \beta v_t] - f^2_x[\mu v + \beta v_t] - f^2_y[\lambda u + \alpha u_t] \\
P^y &= f^1\sigma^{xy} + f^2\sigma^{yy} - f^1_x[\lambda v + \alpha v_t] - f^1_y[\mu u + \beta u_t] \\
&\quad - f^2_x[\mu u + \beta u_t] - f^2_y[(\lambda+2\mu)v + (\alpha+2\beta)v_t], \\
P^t &= 0. \qquad (29)
\end{aligned}
$$

move punctuation

where f^1 and f^2 are functions satisfying equation (28).

If f^1, f^2 are taken to be constant, the above conservation law expresses the condition of equilibrium for the system. Due to the absence of a time current, equation (29) provides path-independent integrals in the material space, which might be useful in numerical analysis of cracks and defects for 2-D viscoelastic material.

delete comma

to the

Conservation law that relates dissipation of elastic energy for 2-D linear viscoelasticity can be constructed if one considered a special case where the Lamé constants (λ, μ) and viscosity coefficients (α, β) are related by

A conservation

considers

$$\mu\alpha = \lambda\beta, \qquad (30)$$

which implies that

$$\gamma = \frac{\mu}{\beta} = \frac{\lambda}{\alpha} \qquad (31)$$

add

One solution for existence of conservation laws in this special case is given by

$$
\begin{aligned}
f^1 &= A[\gamma u_t + u_{tt}], \\
f^2 &= A[\gamma v_t + v_{tt}],
\end{aligned} \qquad (32)
$$

and the corresponding conservation law being reads

add missing open bracket

$$
\begin{aligned}
P^x &= A[\gamma u_t + u_{tt})\sigma^{xx} + (\gamma v_t + v_{tt})\sigma^{xy}] \\
P^y &= A[(\gamma v_t + v_{tt})\sigma^{yy} + (\gamma u_t + u_{tt})\sigma^{xy}], \\
P^t &= -\frac{A}{2}[(\alpha+2\beta)[(\gamma u_x + u_{xt})^2 + (\gamma v_y + v_{yt})^2] \\
&\quad + \beta[(\gamma u_y + u_{yt})^2 + (\gamma v_x + v_{xt})^2] \\
&\quad + 2\alpha(\gamma v_y + v_{yt})(\gamma u_x + u_{xt}) + 2\beta(\gamma v_x + v_{xt})(\gamma u_y + u_{yt})].
\end{aligned} \qquad (33)
$$

delete

indentation for new paragraph

Denoting the elastic energy of the system as

$$W^e = W|_{\alpha=\beta=0}, \qquad (34)$$

where W is the strain energy density given by

$$W = \tfrac{1}{2}\sigma^{ij}\varepsilon^{ij}, \tag{35}$$

law

the conservation $\boxed{\text{laws}}$ for this special case of $\mu\alpha = \lambda\beta$ can be written as

$$\frac{D}{Dt}[W^e] = \sigma^{xx}\varepsilon_t^{xx} + \sigma^{yy}\varepsilon_t^{yy} + 2\sigma^{xy}\varepsilon_t^{xy} - \boxed{(\alpha + 2\theta}(\varepsilon_t^{xx2} + \varepsilon_t^{yy2})$$

(α + 2β)

$$- 2\alpha\varepsilon_t^{xx}\varepsilon_t^{yy} - 4\beta\varepsilon_t^{xy2}, \tag{36}$$

which expresses the balance between dissipation of elastic energy, rate of work done due to tractions and the total energy dissipated for $\boxed{\text{a}}$ special 2-D viscoelasticity problem.

this

The balance law for the special 2-D viscoelasticity problem where $\mu\alpha = \lambda\beta$, equation (36), can be verified to hold also in the general case without any restraint on the Lamé constants and viscosity coefficients. The conservation law that would yield this result is found to be

$$\begin{aligned} P^x &= A[u_t\sigma^{xx} + v_t\sigma^{xy} + u\sigma_t^{xx} + v\sigma_t^{xy}] \\ P^y &= A[v_t\sigma^{yy} + u_t\sigma^{xy} + v\sigma_t^{yy} + u\sigma_t^{xy}] \\ P^t &= -A[2W^e + (\alpha + 2\beta)(u_x u_{xt} + v_y v_{yt}) + \alpha(v_y u_{xt} + u_x v_{yt}) \\ &\quad + \beta(u_y u_{yt} + v_x v_{xt} + v_x u_{yt} + u_y v_{xt})]. \end{aligned} \tag{37}$$

method

Within the framework of the Neutral Action $\boxed{\text{Method}}$, equation (37) can be obtained if one $\boxed{\text{modified}}$ the condition for existence of conservation $\boxed{\text{law}}$ as given by equation (27) to

modifies

laws

3

$$\boxed{E(f^i\Delta^i + g^i\Delta_t^i) \equiv 0} \tag{38}$$

In which case, the corresponding characteristics are

$E^k(f^i\Delta^i + g^i\Delta_t^i) \equiv 0.$

$$\begin{aligned} f^1 &= u_t, \quad g^1 = u \\ f^2 &= v_t, \quad g^2 = v. \end{aligned} \tag{39}$$

5. Conclusions

In this present contribution, conservation laws are presented for one- and two-dimensional linear viscoelasticity. These divergence-free expressions are obtained using the Neutral Action (NA) $\boxed{\text{Method}}$ applicable for constructing conservation laws for dissipative systems [4].

method

For 1-D linear viscoelasticity, the non-trivial conservation laws obtained took a very general form. The conserved current in time can be any function of the total stress, and the conserved current in space is the negative derivative of this function times some derivative of the displacement. If one between

need spaces

chooses the unknown function to be proportional to the stress squared, the resulting conservation law shows balance of the dissipation of elastic energy and the strain rate of the system. Obviously, more conservation laws can be constructed by assuming different dependences of stress in the unknown function.

rate of work done

In 2-D linear viscoelasticity, the conservation laws obtained are not as general as in the 1-D case. For the general and a special case of the 2-D problem, conservation laws are constructed to show the balance between the dissipation of energy and rate of work done parallel to the 1-D ~~example~~. problem Also, it is found that conservation laws without the conserved time current do exist which provide path-independent integrals that might be useful in analysis of cracks and other defects in linear viscoelastic material.

exist. These laws

The conservation laws derived here are not exhaustive. If one allows the characteristics of conservation laws to be functions of higher order derivatives of the dependent variables, more conservation laws can be obtained. In lack of an advanced computing device with large memory capacity which is necessary to solve for high dependence characteristics, the conservation laws presented here are limited. However, as a limited set, these laws are non-trivial results for viscoelasticity and they serve as examples on application of the Neutral Action Method.

method

Acknowledgements

This work was performed with the support of the U.S. Department of Energy and the Air Force Office of Scientific Research. This support is gratefully acknowledged.

References

[1] E. Bessel-Hagen, *Über die Erhaltungssätze der Electrodynamik*. Math. Ann. *84*, 258–276 (1921).
[2] E. Noether, *Invariante Variationsprobleme*. Nachr. König. Ges. Wissen. Göttingen, Math.-Phys. Kl. pp. 235–257 (1918) (see Transport Theory and Stat. Phys. *1*, 186–207 (1971) for an English translation).
[3] P. J. Olver, *Applications of Life Groups to Differential Equations*. Springer, New York 1986.
[4] T. Honein, N. Chien and G. Herrmann, *On conservation laws for dissipative systems*. Phys. Lett. A. *155*, 223–224 (1991).
[5] N. Chien, T. Honein and G. Herrmann, *Dissipative Systems, Conservation Laws and Symmetries*, 1991, to be published.

?

Abstract

entitled

In the brief note titled "On Conservation Laws for Dissipative Systems" [4], a new method for constructing conservation laws was proposed. This method was termed the "Neutral Action (NA) Method" in [5]. For any system governed by a set of differential equations, the NA method offers a systematic approach for determination of conservation laws applicable to the system. It is the purpose of the present paper to establish conservation laws for one- and two-dimensional viscoelasticy (Voigt model) via the NA method. The conservation laws derived should prove useful in studies of fracture and defects in a viscoelastic material.

"Neutral Action (NA)" meth

(Received: February 17, 1992)

END

FILMED

DATE: 4-93

DTIC