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TRADING SECURITIES USING TRAILING STOPS

by

Peter W. Glynn and Donald L. Iglehart

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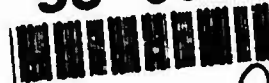
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1. Introduction

A common trading maxim is to "cut your losses and let your profits run." To implement this policy traders often use what is called a trailing stop. Suppose a trader buys a security for \$100 in hopes that it will appreciate in price. At the time of purchase he enters in the market a stop order to sell at \$95, say. In this case the stop order becomes a market order if the security trades at \$95 or lower. The market can gap down through the \$95 level, but the sale price is normally close to \$95. In placing this stop order, the trader is limiting his loss to approximately \$5. Now as the security trades, the stop order is moved to \$5 below the maximum price attained by the security. Since the maximum price is never less than \$100, the stop order can never be less than \$95. The trade will be "stopped out" whenever the price of the security suffers a \$5 "draw down" from its peak price. Being stopped out is essentially guaranteed. This procedure has also been called "trading the line." The questions we address in this paper involve the distribution and moments of the gain to the trader in following this strategy as well as the duration of the trade. Also of interest is the optimal value of the distance to the stop, \$5 in the above example. This model can of course be inverted to handle short selling.

We shall study both discrete and continuous time versions of the price process. In the discrete time version we assume that the price process is a random walk: the sum of independent, identically distributed (i.i.d.) random variables (r.v.'s). Since we have in mind the case of an appreciating security, we shall be principally interested in random walks with positive one-period expected price change. It turns out that this problem is closely related to the behavior of the waiting time process for the GI/G/1 queue with traffic intensity less than one. For the special case of a simple random walk (price changes of +1 with probability p and -1 with probability q) explicit formulas can be derived for the quantities of interest. Partial results can also be obtained for more general random walks.

In the continuous time version we assume the price process is given by a Brownian motion process with mean price change per unit time of μ and variance of price change per unit time of σ^2 . With this price process, explicit formulas can be derived for the quantities

of interest. A geometric Brownian motion price process can also be handled if the stop criterion is changed from a fixed drop (\$5) to a percentage loss.

This paper is organized as follows. Section 2 contains the analysis of the discrete time model and Section 3 of the Brownian model. Section 4 is devoted to optimizing the distance to the stop, \$5 in our example.

2. Discrete Time Model

In this model the security only changes price at a discrete set of equally spaced time points, for example every week, day, or five minutes. Assume that the security is purchased at time 0 at price $P_0 = 0$. Let Y_k denote the price change in the k^{th} time period, where the r.v.'s $\{Y_k: k \geq 1\}$ are i.i.d. We are principally interested in the case where $EY_1 \equiv \nu > 0$. The price at the end of period n is $P_n \equiv Y_1 + \dots + Y_n, n \geq 1$. The maximum price achieved by the security up to time n is given by $M_n \equiv \max\{P_k: 0 \leq k \leq n\}$. The trading strategy to be used is a trailing stop $L(>0)$ units lower than the maximum price reached by the security up to the current period. Thus the trade will be stopped out at the end of period n if and only if $M_k - L < P_k$ for $k = 0, 1, \dots, n-1$, and $M_n - L \geq P_n$. We let $T(L) \equiv \inf\{n > 0: P_n \leq M - n - L\}$, the number of periods before the trade terminates. Each time the price process reaches a new high, the stop is raised by the difference between the new high and the previous high. Let R_n denote the amount the stop has been raised up to the end of period n . The drawdown of the trade at the end of period n is given by $D_n \equiv M_n - P_n$, the amount the trade has lost relative to its best level to-date.

As we mentioned above, this model is closely related to the general single-server queueing model, GI/G/1. To display this relationship, let $X_k \equiv -Y_k (k \geq 1)$, $S_0 \equiv 0$, and $S_n \equiv X_1 + \dots + X_n, n \geq 1$. Observe that

$$D_n = M_n - P_n = S_n - \min\{S_k: 0 \leq k \leq n\}, \quad n \geq 1.$$

In queueing parlance D_n represents the waiting time (in queue) or delay time for customer n , where the X_k 's are differences of service and interarrival times. Hence $T(L)$ is the first time the waiting time process equals or exceeds level L . The assumption that $\nu > 0$ implies that $EX_1 \equiv \mu < 0$ and that the traffic intensity ρ is less than 1.

The process $\{D_n: \geq 0\}$ is a regenerative process and is conveniently analyzed by decomposing its sample paths into i.i.d. regenerative cycles. Each such cycle begins at a random time point n for which $D_n = 0$. Since $\rho < 1$, there will be an infinite number of such times which we denote by $0 = N_0 < N_1 < N_2 < \dots$. Let $n_k = N_k - N_{k-1}, k \geq 1$. Each N_k corresponds to a period where the process $\{S_n: n \geq 0\}$ makes or ties an all time low. This in turn means that the price process $\{P_n: n \geq 0\}$ makes or ties an all time high. Every time the price makes a new high we raise our stop. The amount by which the stop is raised is $-(S_{N_k} - S_{N_{k-1}})$ which is the length of the k^{th} descending ladder height in random walk theory, or in queueing theory the length of the k^{th} idle period.

To continue the analysis, let $N(L) \equiv \sup\{k \geq 0: N_k \leq T(L)\}$, the number of completed busy periods before time $T(L)$. Define $M_k^+ \equiv \max\{D_j: N_{k-1} \leq j < N_k\}$, the

maximum waiting time in the k^{th} busy period. Since the regenerative (or busy) cycles are i.i.d., the r.v. $N(L)$ has a geometric distribution given by

$$P\{N(L) = k\} = (\gamma(L))^k \delta(L), \quad k = 0, 1, \dots,$$

where $\gamma(L) \equiv P\{M_1^+ < L\}$, and $\delta(L) \equiv 1 - \gamma(L)$. The amount the stop is raised before the trade terminates is

$$R_{T(L)} = i_1 + \dots + i_{N(L)}$$

and the total gain on the trade is $G(L) = R_{T(L)} - D_{T(L)}$. In the Bernoulli case, $D_{T(L)} = L$. However, with more general distributions for the X_i 's, $D_{T(L)}$ may jump over L . In the latter case, $D_{T(L)}$ will typically be close to L . Our goal is to compute the distributions and moments for $T(L)$ and $G(L)$.

Example 1. Consider the Bernoulli case for which $Y_i = +1$ with probability p and $Y_i = -1$ with probability $q \equiv 1 - p$. Converting to the queueing interpretation, $X_i = +1$ with probability q and $X_i = -1$ with probability p . Let $\tau_k = \inf\{n > 0 : S_n = k\}$, $0 \leq k \leq L$, and $P_x\{\cdot\}$ denote the conditional probability of some event given the random walk $\{S_n : n \geq 0\}$ starts in state x at time 0. The following probability is well known; cf., Ross (1983), p. 116.

$$\alpha(L) \equiv P_1\{\tau_L < \tau_0\} = \begin{cases} 1/L & , \quad p = 1/2 \\ \frac{1 - (p/q)}{1 - (p/q)^L} & , \quad p \neq 1/2 \end{cases}$$

To analyze the busy period structure for this queue we note that three types of busy periods can occur before time $T(L)$. First, we have B_1 busy periods in which just one customer is served, $n_k = 1$ and $i_k = 1$. In this case the stop would be raised one unit in the security problem and the trade is not stopped out. Second, we have B_2 busy periods in which $n_k > 1$ and $M_k^+ < L$. Here the trade is not stopped out and the stop is not raised, $i_k = 0$. Finally, there is one final busy period for which $n_k > 1$ and $M_k^+ = L$; the trade is stopped out and the stop is not raised. The total gain on the trade $G(L) = B_1 - L = -S_{T(L)}$. Let $\beta(L) = 1 - \alpha(L)$. First we compute the distribution of B_1 :

$$\begin{aligned}
P\{B_1 = l\} &= q\alpha(L) \sum_{k=l}^{\infty} \binom{k}{k-l} p^l (q\beta(L))^{k-l}, l \geq 0 \\
&= q\alpha(L) p^l \sum_{m=0}^{\infty} \binom{m+1}{m} (q\beta(L))^m \\
&= q\alpha(L) p^l (1 - q\beta(L))^{-l-1} \\
&= \frac{q\alpha(L)}{1 - q\beta(L)} \cdot \left(\frac{q}{1 - q\beta(L)} \right)^l.
\end{aligned}$$

So B_1 has a geometric distribution with mean and variance given by

$$E\{B_1\} = \frac{p}{q\alpha(L)}$$

and

$$\sigma^2\{B_1\} = \frac{p(1 - q\beta(L))}{(q\alpha(L))^2}.$$

We then have immediately the mean and variance of the gain $G(L)$:

$$E\{G(L)\} = \frac{p}{q\alpha(L)} - L$$

and

$$\sigma^2\{G(L)\} = \frac{p(1 - q\beta(L))}{(q\alpha(L))^2}.$$

The fact that $M(T(L))$ has a geometric distribution is contained in Problems 15 and 16 on p. 79 of Karlin and Taylor (1975).

To compute the mean of $T(L)$ we use the fact that $G(L) = -S_{T(L)}$ and Wald's equation to conclude for $p \neq q$ that

$$(2.1) \quad E\{T(L)\} = E\{G(L)\}/(p - q).$$

For $p = q = 1/2$,

$$E\{G(L)\} = 0,$$

$$\sigma^2\{G(L)\} = L + L^2$$

and

$$E\{T(L)\} = (L + L^2)/2;$$

the latter expression is obtained by applying L'Hospital's rule to (2.1) as $p/q \rightarrow 1$.

Example 2. This example treats the trading problem associated with the M/M/1 queue. For this queue the interarrival times are exponential with parameter λ and the service times are exponential with parameter μ , with $\lambda < \mu$. The density of the X_i 's is a two-sided exponential given by

$$f_{X_i}(x) = \begin{cases} \frac{\lambda\mu}{\lambda + \mu} e^{-\mu x}, & x \geq 0 \\ \frac{\lambda\mu}{\lambda + \mu} e^{\lambda x}, & x < 0. \end{cases}$$

For this queue the function $\gamma(L) \equiv P\{M_1^+ < L\}$ is known and given by

$$\gamma(L) = \frac{1 - \rho e^{-\kappa L}}{1 - \rho^2 e^{-\kappa L}},$$

where $\rho \equiv \lambda/\mu$ and $\kappa = \mu - \lambda$; see Iglehart and Stone (1985) for this result. In this queue the idle times, i_k , are i.i.d. exponential with parameter λ . For this example we can represent the gain as

$$G_1(L) = i_1 + i_2 + \cdots + i_{N(L)} - (L + c(L)),$$

where $e(L)$ is the amount of the overshoot of L at the time the process $\{D_n: \geq 0\}$ first exceeds L . For this queue the ascending ladder heights, H_n^+ , have a defective density $\lambda e^{-\mu x}$ for $x > 0$; cf., Feller (1971), p. 193. The value of the defect is $1 - p$, which we arbitrarily assign to the value 0.

The defect is the probability that the downward trending random walk has reached its maximum. The conditional distribution of H_1^+ , given $H_1^+ > 0$, is exponential with parameter μ . Hence the overshoot random variable is the excess over L in a renewal process with interrenewal times which are exponential with parameter μ . The loss of memory property of the exponential leads to the fact that $e(L)$ is also exponential with parameter μ . Since the random variable $i_1 + \dots + i_{N(L)}$ is determined by the first $N(L)$ busy periods and $e(L)$ is determined by busy period $N(L) + 1$, $i_1 + \dots + i_{N(L)}$ and $e(L)$ are conditionally independent given $N(L)$. Also the random variable $N(L)$ is independent of the idle times i_1, i_2, \dots . Now we can easily compute the mean and variance of $G(L)$.

$$\begin{aligned} E\{G(L)\} &= \lambda^{-1} E\{N(L)\} - (L + \mu^{-1}) \\ &= \lambda^{-1} \gamma(L)/\delta(L) - (L + \mu^{-1}) \end{aligned}$$

and

$$\begin{aligned} \sigma^2\{G(L)\} &= E\{N(L)\} \sigma^2\{i_1\} + (E\{i_1\})^2 \sigma^2\{N(L)\} + \mu^{-2} \\ &= \gamma(L)/\gamma^2 \delta(L) + \gamma(L)/\gamma^2 \delta^2(L) + \mu^{-2} \end{aligned}$$

Observe also that the stop is raised $N(L)$ times with $E\{N(L)\} = \gamma(L)/\delta(L)$ and $\sigma^2\{N(L)\} = \gamma(L)/\delta^2(L)$. As in Example 1, we can use the fact that $G(L) = -S_{T(L)}$ and Wald's equation to conclude that

$$\begin{aligned} E\{T(L)\} &= -E\{G(L)\}/E\{X_1\} \\ &= \gamma \mu E\{G(L)\}/(\mu - \lambda). \end{aligned}$$

As expression for $\gamma(L)$ is available for the M/G/1 queue (Cohen (1969, p. 606) and for the M/PH/1 queue (Asmussen and Perry (1992)). With some effort the computations for M/M/1 given above could be obtained for M/G/1 and M/PH/1.

Example 3. For the general GI/G/1 queue and corresponding trading model, we can obtain a limit theorem for $L \rightarrow \infty$. Recall that

$$G(L) = (i_1 + i_2 + \dots + i_{N(L)}) - (L + e(L)).$$

We need to know the behavior of $\delta(L) = P\{M_1^+ \geq L\}$ as $L \rightarrow \infty$. However, we know under certain regularity conditions on the moment generating function of X_1 that

$$\delta(L) \sim [(1 - E\{e^{\kappa S_{n_1}}\}) / \kappa \mu_\kappa E\{n_1\}] e^{-\kappa L}$$

see Iglehart (1972), Lemma 1. It is easy to see that as $L \rightarrow \infty$, $\delta(L) \rightarrow 0$, $\delta(L)e(L) \Rightarrow 0$, and $\delta(L)N(L) \Rightarrow \exp(1)$, an exponential random variable with parameter 1. Now write

$$\begin{aligned} \delta(L)G(L) &= \left(\frac{i_1 + i_2 + \dots + i_{N(L)}}{N(L)} \right) \delta(L)N(L) \\ &\quad - (\delta(L)L + \delta(L)e(L)) \\ &\Rightarrow E\{i_1\} \cdot \exp(1). \end{aligned}$$

This concludes our discussion of the discrete model, and we turn to the continuous time version involving Brownian motion.

3. Continuous Time Model

We now turn to a continuous time price process formed from standard Brownian motion, $\{B(t): t \geq 0\}$, with $B(0)=0$. For finite μ and $\sigma^2 > 0$, we take as our price process $P(t) \equiv \mu t + \sigma B(t)$, $t \geq 0$. Note that $P(0) = 0$. As for the discrete model, the trader will be stopped out of the trade whenever the price drops below the maximum price attained by L units. To continue the analysis we define the following processes:

$$M(t) \equiv \sup\{P(s): 0 \leq s \leq t\}$$

$$T(L) \equiv \inf\{t \geq 0: P(t) \leq M(t) - L\},$$

$$G(L) \equiv P(T(L)) = M(T(L)) - L$$

$$X(t) \equiv \sigma B(t) - \mu t,$$

$$m(t) \equiv \inf\{X(s): 0 \leq s \leq t\}, \text{ and,}$$

$$D(t) \equiv M(t) - P(t).$$

As in our discrete model, $D(t)$ denotes the drawdown at time t . Again it is convenient to use the connection with queuing theory. Note that

$$D(t) = \sup\{\mu s + \sigma B(s) - \mu t - \sigma B(t): 0 \leq s \leq t\}$$

$$\begin{aligned} & \stackrel{\mathcal{D}}{=} \sup\{\mu s - \sigma B(s): 0 \leq s \leq t\} - \mu t + \sigma B(t) \\ & = X(t) - m(t) \equiv Z(t) \end{aligned}$$

where $\stackrel{\mathcal{D}}{=}$ denotes two processes with the same distribution. [The process $Z \equiv \{X(t): t \geq 0\}$ is what Harrison (1985), p. xii, refers to as regulated Brownian motion.]

Now let $I(t) = \sup\{X(s)^-: 0 \leq s \leq t\}$, where $x^- = \max(0, -x)$. The quantity $I(t)$ represents the local time in the interval $[0, t]$ corresponding to $Z=0$. In our security model $I(t)$ also represents the amount the stop has been raised in the interval $[0, t]$, that is locked in profit. We also have the representation

$$Z(t) = X(t) + I(t), \quad t \geq 0;$$

see Harrison (1985), p. 18. From above we see that

$$T(L) \stackrel{\mathcal{D}}{=} \inf\{t \geq 0: Z(t) \geq L\}.$$

Since L will remain a fixed positive number throughout this section, we shall drop the L in $T(L)$ and $G(L)$. Since Z is a Markov process, we can let $P_z\{\cdot\} \equiv P\{\cdot \mid Z(0) = z\}$ and $E_z\{\cdot\} \equiv E\{\cdot \mid Z(0) = z\}$. Our first task is to find the Laplace transform of T . Harrison (1985), page 95, problem 3 and 4, discusses this transform. We derive the transform here as the method of proof will be used later.

PROPOSITION 1. For $\lambda > 0$, let u be a twice continuously differentiable function on $[0, L]$ satisfying the differential equation

$$\frac{\sigma}{2}u'' - \mu u' - \lambda u = 0$$

with $\lambda > 0$ and boundary conditions $u'(0) = 0$ and $u(L) = 1$. Then,

$$u(x) = E_x\{\exp(-\lambda T)\}, \quad 0 \leq x \leq L$$

is the Laplace transform of T for $Z(0) = x$.

Proof. Let $v(t, Z(t)) \equiv \exp(-\lambda t)u(Z(t))$. Then using the product rule for differentiation and Itô's formula we obtain

$$\begin{aligned} dv(t, Z(t)) &= -\lambda \exp(-\lambda t)u(Z(t))dt \\ &\quad + \exp(-\lambda t)[\sigma dB(t) - \mu dt + dI(t)] \cdot u'(Z(t)) \\ &\quad + \exp(-\lambda t) \frac{\sigma^2}{2} dt u''(Z(t)) \end{aligned}$$

or in integrated form

$$\begin{aligned} v(t, Z(t)) - v(0, Z(0)) &= \int_0^t \exp(-\lambda s) \left[\frac{\sigma^2}{2} u''(Z(s)) - \mu u'(Z(s)) - \lambda u(Z(s)) \right] ds \\ &\quad + \sigma \int_0^t \exp(-\lambda s) u'(Z(s)) dB(s) \\ &\quad + \int_0^t \exp(-\lambda s) u'(Z(s)) dI(s). \end{aligned}$$

The first term is zero, since $(\sigma^2/2)u'' - \mu u' - \lambda u = 0$, and the last term is

$$u'(0) \int_0^t \exp(-\lambda s) dI(s),$$

since I increases only when $Z(\cdot) = 0$. However, because $u'(0) = 0$ this term also vanishes. We are left with

$$v(t, Z(t)) - v(0, Z(0)) = \sigma \int_0^t \exp(-\lambda s) u'(Z(s)) dB(s).$$

The integral on the right-hand side above is a martingale, since it is an Itô integral with respect to standard Brownian motion, $\{B(t): t \geq 0\}$. We wish to apply the optimal stopping theorem to this martingale. Since u is twice continuously differentiable on $[0, L]$, $u'(Z(s))$ is bounded for $0 \leq s \leq T$. Thus

$$\begin{aligned} E_x \left\{ \int_0^{t \wedge T} (\exp(-\lambda s))^2 u'(X(s))^2 ds \right\} \\ \leq K E_x \left\{ \int_0^\infty \exp(-2\lambda s) ds \right\} < \infty. \end{aligned}$$

Hence we can apply the optional stopping theorem to the right-hand side of (3.1) and conclude that

$$\begin{aligned} E_x \{v(T, Z(T))\} &= v(0, x) \\ &= u(x). \end{aligned}$$

So,

$$u(x) = E_x \{ \exp(-\lambda T) u(Z(T)) \}.$$

But $Z(T) = L$ and $u(L) = 1$, thus

$$u(x) = E_x \{ \exp(-\lambda T) \}, \quad 0 \leq x \leq L.$$

■

We proceed now to solve the differential equation given in Proposition 1. A trial solution of the form $u(x) = \exp(rx)$ leads to the equation

$$\left[\frac{\sigma^2}{2} r^2 - \mu r - \lambda \right] u(x) = 0.$$

Setting the bracketed term equal to 0 yields two distinct roots

$$r_1 = \frac{\mu}{\sigma^2} + \frac{\sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \quad \text{and}$$

$$r_2 = \frac{\mu}{\sigma^2} - \frac{\sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}$$

Thus our solution u becomes

$$u(x) = A \exp(r_1 x) + B \exp(r_2 x).$$

To fix the boundary conditions, note that

$$u'(0) = r_1 A + r_2 B = 0$$

$$u'(L) = \exp(r_1 L)A + \exp(r_2 L)B = 1$$

Solving this pair of simultaneous equations, we have

$$A = \frac{r_2}{r_1 \exp(r_2 L) - r_2 \exp(r_1 L)} \quad \text{and}$$

$$B = \frac{r_1}{r_1 \exp(r_2 L) - r_2 \exp(r_1 L)}.$$

Finally, the Laplace transform of T for initial state $x = 0$, $u(0)$, is

$$(3.2) \quad E_0\{\exp(-\lambda T)\} = \frac{r_1 - r_2}{r_1 \exp(r_2 L) - r_2 \exp(r_1 L)}.$$

To compute $E_0\{T\}$ we can either differentiate the Laplace transform of T or use the fact that

$$(3.3) \quad P(T) = M(T) - L = G(L)$$

Since $\{P(t): t \geq 0\}$ is assumed to be a (μ, σ^2) Brownian motion, we have for every $t > 0$

$$\begin{aligned} E_0\{P(T \wedge t)\} &= \mu E_0\{T \wedge t\} + \sigma E_0\{B(T \wedge t)\} \\ &= \mu E_0\{T \wedge t\}. \end{aligned}$$

We let $t \rightarrow +\infty$, and note that by monotone convergence $E_0\{T \wedge t\} \rightarrow E_0\{T\}$. Next use the inequalities

$$\begin{aligned} (3.4) \quad E_0\{|P(T \wedge t)|\} &= E_0\{|X(T \wedge t)|\} \\ &\leq E_0\{|Z(T \wedge t)|\} + E_0\{I(T \wedge t)\} \\ &\leq L + E_0\{M(T)\} < \infty \end{aligned}$$

plus dominated convergence to conclude from (3.3) and (3.4) that

$$\mu E_0\{T\} = E_0\{G(L)\}.$$

Finally, we have

$$E_0\{T(L)\} = \frac{L}{\mu\beta}[\exp(\beta) - 1] - \frac{L}{\mu}$$

Next we seek the Laplace transform for $M(T)$, which is equal to the gain $G(L)$ except for a factor of $-L$. Note that for initial state $Z(0) = 0$, $M(T) = I(T)$.

PROPOSITION 2. For $\lambda > 0$, let u be a twice continuously differentiable function on $[0, L]$ satisfying the differential equation

$$\frac{\sigma^2}{2}u'' - \mu u' = 0$$

with boundary conditions $u'(0) - \lambda u(0) = 0$ and $u(L) = 1$. Then, for $0 \leq x \leq L$,

$$u(x) = E_x\{\exp(-\lambda I(T))\},$$

and thus

$$u(0) = E_0\{\exp(-\lambda M(T))\}.$$

Proof. Let $v(t, Z(t)) = \exp(-\lambda I(t))u(Z(t))$. Then following the procedure used in Proposition 1 we obtain

$$\begin{aligned} dv(t, Z(t)) &= \exp(-\lambda I(t))(-\lambda dI(t)) u(Z(t)) \\ &\quad + \exp(-\lambda I(t))du(Z(t)) \\ &= -\lambda u(Z(t))\exp(-\lambda I(t))dI(t) \\ &\quad + \exp(-\lambda I(t))[\sigma dB(t) - \mu dt + dI(t)] \cdot u'(Z(t)) \\ &\quad + \exp(-\lambda I(t))\frac{\sigma^2}{2}dt u''(Z(t)) \end{aligned}$$

or in integrated form

$$\begin{aligned} v(t, Z(T)) - v(0, Z(0)) &= \int_0^t \exp(-\lambda I(s))\left[\frac{\sigma^2}{2}u''(Z(s)) - \mu u'(Z(s))\right]ds \\ &\quad + \sigma \int_0^t \exp(-\lambda I(s))u'(Z(s))dB(s) \\ &\quad + \int_0^t \exp(-\lambda I(s))[u'(Z(s)) - \lambda u(Z(s))]dI(s). \end{aligned}$$

The first term is zero since $(\sigma^2)/2u'' - \mu u' = 0$, and the last term is zero since I only increases when $Z(\cdot) = 0$ and $u'(0) - \lambda u(0) = 0$. This leaves

$$v(t, Z(t)) - v(0, Z(0)) = \sigma \int_0^t \exp(-\lambda I(s)) u'(Z(s)) dB(s),$$

a martingale. From here the argument follows that used in Proposition 1. ■

To solve the differential equation given in Proposition 2 we again try a solution of the form $u(x) = \exp(rx)$.

Then

$$\frac{\sigma^2 r^2}{2} - \mu r = 0,$$

so $r = 0$ and $r = 2\mu/\sigma^2$. Therefore,

$$u(x) = A + B \exp(2\mu x/\sigma^2).$$

Since $u'(0) - \lambda u(0) = 0$ and $u(L) = 1$, we have

$$\frac{2B\mu}{\sigma^2} - \lambda(A + B) = 0 \quad \text{and}$$

$$A + B \exp(2\mu L/\sigma^2) = 1.$$

Solving for A and B yields,

$$A = \frac{2\mu/\sigma^2 - \lambda}{\lambda(\exp(2\mu L/\sigma^2) - 1) + 2\mu/\sigma^2} \quad \text{and}$$

$$B = \frac{\lambda}{\lambda(\exp(2\mu L/\sigma^2) - 1) + 2\mu/\sigma^2}.$$

Hence,

$$\begin{aligned} u(0) &= E_0\{\exp(-\lambda M(T))\} \\ &= \frac{2\mu[\sigma^2(\exp(2\mu L/\sigma^2) - 1)]^{-1}}{2\mu[\sigma^2(\exp(2\mu/\sigma^2) - 1)]^{-1} + \lambda}. \end{aligned}$$

This Laplace transform reveals that $M(T)$ is exponentially distributed with parameter $\beta/[\exp(\beta) - 1]L$, where $\beta = 2\mu L/\sigma^2$. This result is known in the literature. Karatzas and Shreve (1987), equation (4.16) on p. 429, established the result for $\mu = 0$ as an application of the Feynman-Kac formula. Athreya and Weerasinghe (1989), Theorem 2, extended the result to more general reflecting diffusions by a method different from the one used here. The proof given here is different and included for the sake of completeness. The mean and variance of the gain, $G(L) \equiv M(T) - L$, are given by

$$E\{G(L)\} = L[(\exp(\beta) - 1)/\beta - 1]$$

and

$$\sigma^2\{G(L)\} = L^2[\exp(\beta) - 1]^2/\beta^2.$$

Now we turn to two ways of computing discounted rewards. First, suppose that the reward from a trade is earned at the end of the trade, namely, at time T . Then the discounted reward, $R_1(L)$, is given by

$$R_1(L) = \exp(-\rho T)[M(T) - L],$$

where $\rho > 0$ is the discount factor. On the other hand, if the reward is earned continuously in time as the stop is raised, the discounted reward, $R_2(L)$, is given by

$$R_2(L) = \int_0^T \exp(-\rho s) dM(s) - L \exp(-\rho T).$$

We would like to obtain the expected value of both $R_1(L)$ and $R_2(L)$. The next two propositions are directed to that end. [Note that from Proposition 1 we already have the value of $E_0\{\exp(-\rho T)\}$].

PROPOSITION 3. For $\rho, \beta > 0$, let u be a twice continuously differentiable function on $[0, L]$ satisfying

$$\frac{\sigma^2}{2}u'' - \mu u' - \rho u = 0$$

with boundary conditions $u'(0) - \beta u(0) = 0$ and $u(L) = 1$. Then, for $0 \leq x \leq L$,

$$u(x) = E_x \{ \exp(-\rho T - \beta I(T)) \}.$$

Proof. Let $v(t, I(t), Z(t)) = \exp(-\rho t - \beta I(t))u(Z(t))$. Then using Itô's formula results in

$$\begin{aligned} dv(t, I(t), Z(t)) &= (-\rho dt - \beta dI(t)) \exp(-\rho t - \beta I(t)) u(Z(t)) \\ &+ \exp(-\rho t - \beta I(t)) [\sigma dB(t) - \mu dt + dI(t)] u'(Z(t)) \\ &+ \exp(-\rho t - \beta I(t)) \frac{\sigma^2}{2} dt u''(Z(t)) \end{aligned}$$

or in integrated form (after using the boundary conditions)

$$\begin{aligned} v(t, I(t), Z(t)) &= u(Z(0)) \\ &+ \sigma \int_0^t \exp(-\rho s - \beta I(s)) u'(Z(s)) dB(s). \end{aligned}$$

Again using the optimal sampling theorem and $u(L) = 1$, we obtain the desired conclusion. ■

We solve the differential equation in Proposition 3 using the same method applied above and find that

$$u(x) = A \exp(r_1 x) + B \exp(r_2 x),$$

where

$$(3.5) \quad r_1 = \frac{\mu}{\sigma^2} + \frac{\sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2},$$

$$(3.6) \quad r_2 = \frac{\mu}{\sigma^2} - \frac{\sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2},$$

$$A = -(r_2 - \beta)/[(r_1 - \beta)\exp(r_2 L) - (r_2 - \beta)\exp(r_1 L)], \text{ and}$$

$$B = (r_1 - \beta)/[(r_1 - \beta)\exp(r_2 L) - (r_2 - \beta)\exp(r_1 L)].$$

Since $u(0) = A + B$,

$$E_0\{\exp(-\rho T - \beta M(T))\} = A + B.$$

Differentiating with respect to β and setting $\beta = 0$ yields an expression for $E_0\{M(T)\exp(-\rho T)\}$. This together with the expression for $E_0\{\exp(-\rho T)\}$ gives the value of $E_0\{R_1(L)\}$.

PROPOSITION 4. For $\rho > 0$, let u be a twice continuously differentiable function on $[0, L]$ satisfying

$$\frac{\sigma^2}{2}u'' - \mu u' - \rho u = 0$$

with boundary conditions $u'(0) = -1$ and $u(L) = 0$. Then, for $0 \leq x \leq L$,

$$u(x) = E_x\left\{\int_0^T \exp(-\rho s) dI(s)\right\}.$$

Proof. We proceed as in Proposition 1 by setting $v(t, Z(t)) \equiv \exp(-\rho t)u(Z(t))$. From the proof of Proposition 1, we see that

$$\begin{aligned} v(t, Z(t)) &= v(0, Z(0)) + \sigma \int_0^t \exp(-\rho s) u'(Z(s)) dB(s) \\ &\quad + u'(0) \int_0^t \exp(-\lambda s) dI(s). \end{aligned}$$

Now applying optimal sampling as in Proposition 1, we see that

$$E_x\{e^{-\rho T} u(Z(T))\} = u(x) - E_x\left\{\int_0^T \exp(-\rho s) dI(s)\right\}.$$

Since $Z(T) = L$ and $u(L) = 0$, we have the desired conclusion. \blacksquare

Solving the differential equation in Proposition 4 we find that

$$u(x) = A \exp(r_1 x) + B \exp(r_2 x),$$

where r_1 and r_2 are defined in (3.5) and (3.6),

$$A = [r_2 \exp((r_1 - r_2)L) - r_1]^{-1}, \text{ and}$$

$$B = -A \exp((r_1 - r_2)L).$$

Since $u(0) = A + B$, we have

$$E_0 \left\{ \int_0^T \exp(-\rho s) dM(s) \right\} = \frac{\exp(r_1 L) - \exp(r_2 L)}{r_1 \exp(r_2 L) - r_2 \exp(r_1 L)}.$$

This expression together with $E_0 \{ \exp(-\rho T) \}$ from (3.1) gives the value of $E_0 \{ R_2(L) \}$.

4. Optimizing the Stop Parameter

In the previous sections we have treated L , the distance from the current price to the stop, as a fixed parameter. Here we explore various criteria that could be used to select an optimal value of L . In this section we consider only the continuous time model of Section 3.

Let $\alpha \equiv 2\mu/\sigma^2$ and $\gamma \equiv \alpha/(\exp(\alpha L) - 1)$. Recall that the gain, $G(L)$, is an exponential random variable with parameter γ , minus the constant L . So its mean for $\mu \neq 0$ is given by

$$(4.1) \quad E\{G(L)\} = [(\exp(\alpha L) - 1)/\alpha] - L$$

For $\mu=0$, it is easy to show that the function satisfying the conditions of Proposition 2 is $u(x) = (1 + \lambda x)/(1 + \lambda L)$. In this case the exponential parameter $\gamma = 1/L$.

When $\mu > 0$ (and hence $\alpha > 0$), equation (4.1) shows that $E\{G(L)\}$ grows exponentially in L . Thus if the criterion being optimized is to maximize $E\{G(L)\}$ the obvious solution is to select $L^* = \infty$. That is, to effectively not use a stop. If the criterion is to maximize $E\{G(L)\}/\sigma\{G(L)\}$, the conclusion is the same; namely, $L^* = \infty$. Of course, when $L^* = \infty$, the trade in progress never ends.

What motivation might the trader have for selecting a finite L^* , $0 \leq L^* < \infty$? We suggest several:

- (4.2) a utility function which places an extremely heavy penalty on any losses;
- (4.3) a concern that the unknown parameter α might be negative; and
- (4.4) a desire to end the game in finite time.

As an example of where (4.2) comes into play, assume that $\mu > 0$ and we are maximizing the expected utility of the gain for a utility function given by

$$u(x) = \begin{cases} \exp(-\delta x) + 1 & , x \leq 0 \\ cx & , x > 0, \end{cases}$$

where c and δ are two positive parameters. Then

$$\begin{aligned} E\{u(G(L))\} &= \int_0^L [-e^{-\delta(y-L)} + 1] \gamma \exp(-\gamma y) dy \\ &+ c \int_L^\infty (y-L) \gamma \exp(-\gamma y) dy. \\ (4.5) \quad &= \frac{-\gamma e^{\delta L}}{\gamma + \delta} (1 - e^{-(\delta + \gamma)L}) \\ &+ 1 - e^{-\gamma L} + c\gamma^{-1} e^{-\gamma L}. \end{aligned}$$

Observe that for $\alpha > 0$, $\gamma(L) \sim \alpha \exp(-\alpha L)$ as $L \rightarrow \infty$. Using this fact in (4.5) shows that the $\lim_{L \rightarrow \infty} E\{u(G(L))\} = -\infty$, provided $\delta > 2\alpha$. Since $E\{u(G(L))\}$ is continuous in L and positive for small L , the value L^* that maximizes $E\{u(G(L))\}$ must be finite, if the penalty for losses is sufficiently large, namely, $\delta > 2\alpha$.

As for (4.3), consider the case where α is unknown but where α has a prior density f . Suppose we wish to maximize the expected gain

$$\begin{aligned} g(L) &= \int_{-\infty}^{\infty} E_\alpha\{G(L)\} f(\alpha) d\alpha \\ &= \int_{-\infty}^{\infty} \frac{(e^{\alpha L} - 1)}{\alpha} f(\alpha) d\alpha - L. \end{aligned}$$

If the mean of the density f is negative and f places positive mass on $(0, \infty)$, it is easy to show that the optimal $L^* = \infty$. Of course, if the mean of f is positive $L^* = \infty$. If f has support on $(-\infty, 0]$, then $L^* = 0$. To obtain an L^* such that $0 < L^* < \infty$, one needs to introduce a utility function that dampens the expected utility when α is positive and place some conditions on the density f . This computation would probably have to be carried out numerically.

Finally, consider (4.4), where the trader wishes to end the game in finite time. Here the expected discounted rewards, $E\{R_1(L)\}$ and $E\{R_2(L)\}$, can be used as an optimization criterion. Both of these criteria depend on the discount factor ρ , and thus the optimal $L^*(\rho)$ will depend on ρ . The computation of $L^*(\rho)$ would have to be carried out numerically. Using the Cauchy-Schwarz inequality, the exponential distribution of $M(T)$, and the Laplace transform of T it is easy to show that $E\{M(T)\exp(-\rho T)\} \rightarrow 0$ as $L \rightarrow \infty$ for sufficiently large ρ . This shows that $L^*(\rho)$ for criteria $E\{R_1(L)\}$ must be finite for sufficiently large ρ . The same conclusion holds for criteria $E\{R_2(L)\}$ using the explicit expression for $E\{R_2(L)\}$.

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A common trading maxim is to "cut your losses and let your profits run." To implement this policy, traders often use what is called a trailing stop. Suppose a trader buys a security for \$100 in hopes that it will appreciate in price. At the time of purchase he enters in the market a stop order to sell at \$95, say. the stop order becomes a market order, if the security trades at \$95. In placing this stop order, the trader is limiting his loss to approximately \$5. As the security trades, the stop order is moved to \$5 below the maximum price attained by the security. Since the maximum price is never less than \$100, the stop order can never be less than \$95. The trade will be "stopped out" whenever the price of the security suffers a \$5 "draw down" from its peak price. Being stopped out is essentially guaranteed. This procedure has been called "trading the line." In the paper we discuss the distribution and moments of the gain to the trader in following this strategy as well as the duration of the trade. Both discrete time random walks and continuous time Brownian motion and geometric Brownian motion price processes are treated. The discrete time model is closely related to the GI/G/1 queue. Also of interest is the optimal value of the distance to the stop, \$5 in the above example. This model can, of course, be inverted to handle short selling.