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AD-A266 607

# REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704 0188

of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this tions for reducing this burden, to Washington Headquarters Services, Directorate for information Operations and Reports, 1215 Jefferson 2202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

| blank) | 2. REPORT DATE | 3. REPORT TYPE AND DATES COVERED |
|---|---|---|
| | 20 April 1993 | Final 15 May 86 - 14 Sep 89 |

4. TITLE AND SUBTITLE

Nonparametric Statistics: Weighted Rankings Analysis

5. FUNDING NUMBERS

DAAL03-86-G-0084

6. AUTHOR(S)

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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

U. S. Army Research Office
P. O. Box 12221
RTP, NC 27709

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

ARO 23369.6-MA

DTIC
SELECTE
JUL 09 1993
S B D

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

We examine the credibility hypothesis associated with the method of Weighted Rankings Analysis. We consider a multivariate test for homogeneity based on all nearest neighbors. We extend exact tables for Spearman's footrule by using a Markov chain structure over the permutation group of 1,...,ñ . We prove the asymptotic permutational normality of certain weighted measures of correlation.

93 7 08 084

93-15502

14. SUBJECT TERMS

Weighted Rankings Analysis, Weighted Rank Correlation, Testing Equality of Two Distributions based on nearest neighors

15. NUMBER OF PAGES
16

16. PRICE CODE

| 17. SECURITY CLASSIFICATION OF REPORT | 18. SECURITY CLASSIFICATION OF THIS PAGE | 19. SECURITY CLASSIFICATION OF ABSTRACT | 20. LIMITATION OF ABSTRACT |
|---|---|---|---|
| Unclassified | Unclassified | Unclassified | SAR |

NSN 7540-01-280-5500

Research Grant on Nonparametric Statistics

# WEIGHTED RANKINGS ANALYSIS

FINAL REPORT

by

Ibrahim A. Salama

April 20, 1993

US ARMY RESEARCH OFFICE

Grant Number: DAALO–86–G–0084

NORTH CAROLINA CENTRAL UNIVERSITY

This report contains a summary of the main results obtained by research supported by thus grant.

1. Justification of the basis for weighted rankings analysis

The method of weighted rankings analysis was introduced in Quade (1972, 1979). A brief description is as follows: Let $X_{ij}$ be the yield of the j–th treatment in the i–th block, for i = 1,...,n and j = 1,...,m and let $R_{ij}$ be the within–block rank of $X_{ij}$. Let $D_i$ be a measure of apparent variability for the i–th block, $Q_i$ be the corresponding rank. Let $t_1,...,t_m$ be any constants, and let $s_1,...,s_n$ be constants such that $0 \le s_1 \le ... \le s_n$. Then a statistic of the form

$$C_n = \frac{\sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} S_{Q_i} t_{R_{ij}} \right\}^2}{\sum_{i=1}^{n} S_i^2 \sum_{j=1}^{m} t_j^2}$$

may be used for testing the hypothesis of no treatment effects.

The method of weighted rankings is based on the idea that if some blocks appear more variable than others, then they are perhaps better referred to as more discriminating. Hence it seems intuitively reasonable that these blocks receive greater weight in the analysis.

In this research we examine this credability hypothesis and provide objective evidence showing how the more variable blocks do indeed better reflect any true treatment effects. The n–block two treatment exponential case was considered, and the foundation is established to extend the result to the more general cases (see Appendix I).

2. A nonparametric multivariate test for homogeneity based on all nearest neighbors

Let $\{X_{ij}: i = 1,2; j = 1,...,n_i\}$ be two independent random samples of observations in the Euclidean space $R^d$, where $X_{ij}$ has distribution function $F_i$. The

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problem under consideration is to test the hypothesis $H_0 : F_1 \equiv F_2$, against the completely general alternative $H_a : F_1 \neq F_2$.

Let $R(i,j;i',j')$ be the rank of observation $X_{i'j'}$ with respect to nearness to $X_{ij}$; we assume that there will be no ties. Then we define $X_{i'j'}$ as "the" k–th nearest neighbor of $X_{ij}$ if $R(i,j;i',j') = k$, and as "a" k–nearest neighbor if $R(i,j;i',j') \leq k$. Interest in statistical procedures based on such nearest neighbors has grown as high–speed computers have made the application of these techniques practicable, since the idea of making inferences about an object based on nearby objects appears to be a fundamental mechanism of human perception. A review of early work using nearest–neighbor approaches to our problem may be found in Schilling (1986).

Schilling's own approach is as follows. Let

$$I_{ij}(r) = I\{\text{r–th nearest neighbor of } X_{ij} \text{ is in sample i}\}$$

$$\text{for } i = 1, 2;\ j = 1,\ldots, n_i;$$

$$r = 1,\ldots,N-1\ (N = n_1+n_2),$$

where $I\{E\}$ is the indicator function of the event E. Count the number of k–nearest neighbors to $X_{ij}$ which are in the i–th (same) sample, viz.

$$C_{ijk} = \sum_{r=1}^{k} I_{ij}(r).$$

Summing these counts over all observations yields what may be called the "Schilling total", of order k:

$$T_k = \sum_{ij} C_{ijk}.$$

His test statistic for our problem is then

$$S_k = T_k/Nk,$$

which is "the proportion of all k–nearest neighbor comparisons in which a point and its neighbor are members of the same sample" (1986, p. 800), or "Schilling

proportion" of order k. One would expect $S_k$ to have a larger value under $H_a$ than under $H_0$ because of a lack of complete mixing of the two samples when the parent distributions are not identical; hence large values of $S_k$ may be considered significant. Schilling shows that the asymptotic distribution of $S_k$ under $H_0$ is normal, for every positive integer k, and that the test which rejects for large values of $S_k$ is consistent against the general alternative $H_a$.

Schilling's work suggests that the choice of order is not of great importance; nevertheless, it is arbitrary, and he gives no guidance for choosing it. To resolve this issue we propose to take as test statistic the sum of the Schilling totals, which is equivalent to a certain weighted average of the Schilling proportions:

$$W = \Sigma T_k = N \Sigma k S_k.$$

This statistic may have intuitive appeal in that it is equivalent to a sum of N Wilcoxon rank sums, and is also a linear combination of two U–statistics, as we have shown. We study some of the exact properties of W. For the asymptotic properties, we note that W is equivalent to a sum of Schilling proportions, each of which is asymptotically normal; but asymptotic normality of their joint distribution has not been shown. Similarly, we have shown that W is a sum of Wilcoxon statistics, each of which is asymptotically normal; but again asymptotic normality of their joint distribution has not been shown. We might also attempt to show this by means of U–statistics theory, but have not yet been able to work out the necessary conditions on the variance. Nevertheless, it seems reasonable to conjecture that W itself is asymptotically normal. Computer simulation was used to estimate the 95th percentile of W, then this value will be used as the critical value for a test. That is, the null hypothesis will be rejected if W is greater than this critical value. The empirical power of the test based on W was also calculated based on this critical value.

## 3. Exact tables for Spearman's footrule

Spearman (1904) based the measure of rank correlation as his "footrule" on

$$D(\sigma,\pi) = \sum_{i=1}^{n} |\sigma(i) - \pi(i)|,$$

where $\sigma$ and $\pi$ are any two permutations of $S_n$, the set of all permutations of the first n integers. Ury and Kleinecke (1979) provided the exact commulative distribution of D for n = 2(1)10. Franklin (1988) has extended the tabulation to n = 11(1)18. The difficulty in extending these tables is due to the exponential growth rate of time needed to generate all permutations of $S_n$. By relating D to a Markov chain on $S_n$, we extend the exact tabulation to n = 19(1)40. We also investigate the adequacy of approximation to the normal distribution.

## 4. The asymptotic permutational normality of certain weighted measures of correlation

Let be given observation $(X_i,Y_i)$, i = 1,...,n, on the continuous bivariate random variable (X,Y). Without loss of generality we may relabel the X's to have ranks 1,...,n: then let the corresponding ranks of the Y's be $R_1,...,R_n$. Define

$$T_j = \sum_{i=i}^{j} I\{R_i \le j\}$$

where I{E} is the indicator function of the event E, and let

$$T = \sum_{j=1}^{n} w_j T_j$$

where $w_1,...,w_n$ is a sequence of weights. Then T may be regarded as a weighted measure of rank correlation between the X's and Y's. This notion was introduced in Salama and Quade (1982), although they discussed only the special case where $w_j = 1/j$.

In this research, a martingale approach is used to establish the asymptotic normality of a class of weighted correlation statistics.

5. Spherical uniformity and the Cauchy distribution

To motivate the characterization problems (to be posed), we consider the following. Let Y be a stochastic (m–) vector for which $EY = X\beta$, where X is a known m×n (design) matrix and $\beta$ is an n–vector of unknown regression parameters. Then $\hat{\beta}$, the least squares estimator (LSE) of $\beta$, satisfies the equation: $(X'X)\hat{\beta}, = X'Y$. When X'X is not of full rank, additional constraints are imposed on $\beta$ in order that the LSE may be defined uniquely. In the absence of a physically natural form of such a (linear) constraint, often this choice is made rather arbitrarily. In some random effects or mixed models, when $\text{Rank}(X'X) = n-1$, an additional random constraint is taken as $c'\beta = 0$, where $\|c\| = 1$. In robust regression (cf. Huber (1981, p. 170)), when n is large, often c is chosen at random with respect to the invariant measure on the unit sphere $\|c\| = 1$. In the context of asymptotic normality of estimators, Huber (1981) assumed that $\text{Rank}(X'X) = n$, and we shall see that a different picture emerges when $\text{Rank}(X'X)$ is less than n. Suppose that $\text{Rank}(X'X) = n-1$ and let $c'\beta = 0$ be an additional constraint such that $\|c\| = 1$. Let $\hat{\beta}_c$ be the solution satisfying $(X'X)\hat{\beta}_c = X'Y$ and $c'\beta_c = 0$. Let v be the eigenvector corresponding to the null eigenvalue of X'X. Then $\hat{\beta}_c$ may be written as

$$\hat{\beta}_c = \hat{\beta}_0 + ((c'\hat{\beta}_0)/(c'v))v, \qquad (1)$$

where $\hat{\beta}_0$ is any particular solution of $(X'X)\beta = X'Y$ (and is therefore a random variable). Thus, given $\hat{\beta}_0$, we may write

$$(\hat{\beta}_c - \hat{\beta}_0) = v[(c'\hat{\beta}_0)/(c'v)], \qquad (2)$$

and its conditional distribution, given $\hat{\beta}_0$, is generated exclusively by the uniform distribution of c on the spherical surface. Note that v is nonstochastic (as X'X is), so

that the key factor on the right–hand side of (2) is the ratio $(c'\hat{\beta}_0)/(c'v)$, where $\hat{\beta}_0$ is held fixed. This leads us to the following problem.

For two arbitrary points b and v in $R^n$, let $L(t) = b + tv$, $t \in R$ be a given line. Let $S^{n-1} = \{c: c'c = 1\}$ be the unit sphere, and for every $c \in S^{n-1}$, let P(c) be the hyperplane defined by $Pc = 0$, $P \in R^n$. Let $L(t_c) = b + (t_c)v$ be the point at which L and P(c) intersect, for $c \in S^{n-1}$. Define then

$$X_n = X_n(c) = \text{sign}(t_c)\|L(t_c) - L(0)\| = \text{sign}(t_c)|t_c|\,\|v\|, \quad c \in S^{n-1}. \tag{3}$$

Assuming that c has a uniform distribution on the sphere $S^{n-1}$, we show that for every $n \geq 2$, $X_n$ has the same distribution as $aX + d$, where a and d are real numbers and X has the standard Cauchy distribution. Some other related characterization results are also considered in the same view.

6. Topological entropy of countable Markov chains

We consider a symbolic dynamical system $(X,\sigma)$ on a countable state space. We introduce a kind of topological entropy for such systems, denoted $h^*$, which coincides with usual topological entropy when X is compact. We use a pictorial approach, to classify a graph $\Gamma$ (or a chain) as transient, null recurrent, or positive recurrent. We show that given $0 \leq \alpha \leq \beta \leq \infty$, there is a chain whose $h^*$ entropy is $\beta$ and where Gurevic entropy is $\alpha$. We compute the topological entropies of some classes of chains, including larger chains built up from smaller ones by a new operation which we call the Cartesian sum.

List of publications under the grant

Salama, I.A. (1988). "Topological entropy and recurrence of countable chains." *Pacific Journal of Mathematics*, Vol. 134, No. 2, 325–341.

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# APPENDIX I

# REMARKS ON JUSTIFYING THE INTUITIVE BASIS FOR THE METHOD OF WEIGHTED RANKINGS

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## 1. INTRODUCTION

Let $X_{ij}$ be the observation of the j-th of m treatments in the i-th of n complete blocks, and consider the hypothesis of no treatment effects, specifically

$H_0$: $X_{i1}, \ldots, X_{im}$ are interchangeable for each i.

(By definition random variables are *interchangeable* if their joint distribution function is invariant under permutation; this implies that they have identical marginal distributions and equal -- but not necessarily zero -- correlations.) We assume throughout:

(I) Independent Blocks

For i=1, ..., n, the random vectors $\underset{\sim}{X}_i=(X_{i1}, \ldots, X_{im})'$ (the blocks), are mutually independent.

To simplify the exposition it is convenient also to assume:

(II) No Within-Blocks Ties

$P\{X_{ij}=X_{ij'}\}=0$ for $j\neq j'$

Thus with probability 1 there will be no ties within blocks.

The alternatives under consideration can be fairly general: however, we have particularly in mind that there may be *additive treatment effects*, as follows:

UNORDERED CASE

$H_a[u]$: There exist quantities $\tau_1, \ldots, \tau_m$ (treatment effects) not all equal to zero, such that for i=1, ..., n, $X_{i1}-\tau_1, \ldots, X_{im}-\tau_m$ are interchangeable.

ORDERED CASE

$H_a[o]$: The quantities $\tau_1, \ldots, \tau_m$ (as above) satisfy $\tau_1\leq\ldots\leq\tau_m$, with $\tau_1\neq\tau_m$.

Standard nonparametric procedures for attacking this problem are based on within-block rankings: for example, the tests of Friedman (1937) and Brown and Mood (1951) for $H_a[u]$; and Lyerly (1952), Page (1963) and Jonckheere (1954) for $H_a[o]$. Let $C_{ii'}$ be some measure of rank correlation between block i and block i'. Then

$$\overline{\overline{C}} = \sum_{i<i'} C_{ii'}/\binom{n}{2}$$

is the *average internal rank correlation*, and we reject $H_0$ in favor of $H_a[u]$ if $\overline{\overline{C}}$ is too big. (This is equivalent to Friedman's test if Spearman correlation is used.) Similarly let $C_i$ be the rank correlation between block i and the ordering given by the alternative. Then

$$\overline{C} = \sum_i C_i/n$$

is the *average external rank correlation*, and we reject $H_0$ in favor of $H_a[o]$ if $\overline{C}$ is too big. (This is equivalent to Page's test if Spearman correlation is used.)

The standard procedure is parametric two-way analysis of variance, but this adds two assumptions:

(III) Additive Block Effects

There exist quantities $\beta_1, \ldots, \beta_n$ (block effects) such that the random vectors $(X_{i1}-\beta_i, \ldots, X_{im}-\beta_1)'$ are identically distributed.

(IV) Normality

The $X_{ij}$'s are [jointly] normal.

By Assumption III, comparisons of observations are possible between blocks as well as within, so procedures which use only within-block comparisons waste information. A method of *weighted* within-block rankings, which makes use of Assumption (III) without requiring Assumption (IV), has been introduced by Quade (1972, 1979). The idea behind this method is that blocks in which the observations are more distinct are more likely to reflect any underlying

true ordering of the treatment effects. These blocks, which may be referred to as more *credible*, should receive greater weight in the analysis. (In practice, credibility is measured by *apparent* variability; but note that by Assumption III the *true* variability is the same in all blocks.)

To determine the weight for the i-th block, use some location-free statistic $D_i = D(X_{i1}, \ldots, X_{im})$ which measures the credibility of the block with respect to treatment ordering, and let $Q_i$ be the rank of $D_i$ among $D_1, \ldots, D_n$. Again for *simplicity of exposition, make the (unessential)* assumption:

(V) No Between-Block Ties

$$P\{D_i = D_{i'}\} = 0 \text{ for } i \neq i' ,$$

This assures that there will be no ties in the ranking of the blocks. Let $0 \leq b_1 \leq \ldots \leq b_n$, with $0 \neq b_n$, be a fixed set of block scores; and weight the i-th block proportionally to $b_{Q_i}$.

Then in testing against the unordered alternative we use the *weighted average internal rank correlation*

$$\bar{\bar{W}} = \Sigma\Sigma_{i \neq i'} b_{Q_i} b_{Q_{i'}} C_{ii'} \Big/ ((\Sigma b_i)^2 - \Sigma b_i^2) .$$

Against the ordered alternative we use the *weighted average external rank correlation*

$$\bar{W} = \sum_{i=1}^{n} b_{Q_i} C_i \Big/ \sum_{i=1}^{n} b_i ,$$

The purpose of this paper is to examine the notion on which these weighted rank correlation coefficients are based, which we call the *credibility hypothesis.*

## 2. THE CASE OF TWO TREATMENTS

With $m=2$ treatments, suppose the true ranking is (1, 2): i.e., $\tau_1 < \tau_2$. Let $R_{ij}$ be the rank of $X_{ij}$ within the i-th block, and consider

$$P\{(R_{i1}, R_{i2}) = (1, 2) | Q_i = k\} = \phi_k \text{ (say)}.$$

The intuitive notion is that

$$\phi_1 \leq \phi_2 \leq \ldots \leq \phi_n ,$$

where of course

$$\Sigma\phi_i/n = \phi \text{ (say)}$$

is the unconditional probability $P\{(R_{i1}, R_{i2}) = (1, 2)\}$. With two treatments we may let $D_i = X_{2i} - X_{1i}$. Then

$$(R_{i1}, R_{i2}) = (1, 2) \Leftrightarrow D_i > 0$$

and

$$Q_i = k \Leftrightarrow |D_i| \text{ has rank } k \text{ among } |D_1|, \ldots, |D_n|.$$

*In the very special case of 2 treatments and 2 blocks, let us define*

$$\phi = P\{D_1 > 0\}$$

and

$$\theta = P\{D_1 + D_2 > 0\}.$$

Then

$$\phi_1 = P\{D_1 > 0 \,||D_1| < |D_2|\} = \phi - (\theta - \phi) = 2\phi - \theta$$

$$\phi_2 = P\{D_1 > 0 \,||D_1| > |D_2|\} = \phi + (\theta - \phi) = \theta$$

and the credibility hypothesis holds if $\theta > \phi$. Suppose $(X_{i1} - \tau_1, X_{i2} - \tau_2)$ are IID normal with variance $\sigma^2$; then it is easily seen that

$$\phi = \Phi\left(\frac{\tau_2 - \tau_1}{\sigma\sqrt{2}}\right), \quad \theta = \Phi\left(\frac{\tau_2 - \tau_1}{\sigma}\right)$$

where $\Phi$ is the standard normal distribution function, and $\theta > \phi$ if $\tau_1 < \tau_2$. On the other hand, suppose $(X_{i1} - \tau_1, X_{i2} - \tau_2)$ are IID Cauchy; then $\theta = \phi$, and the credibility hypothesis does not hold. Note, by the way, that with only two treatments the weighted rankings procedures are equivalent to signed-rank procedures, which are thoroughly discussed in Chapter 3 of Pratt and Gibbons (1981).

If we do not limit ourselves to additive treatment effects, we may consider the interesting special case where X and Y are exponentially distributed, with parameters (say) $\lambda$ and $\mu$, respectively. Then it is easily shown that $\phi = \lambda/(\lambda + \mu)$, $\theta = \phi^2(3 - 2\phi)$, and $\theta > \phi$ if and only if $\lambda < \mu$.

## 3. THE GENERAL CASE

We now turn our attention to the general case where we have m treatments and n blocks.

Write

$$(R_{i1}, \ldots, R_{im}) = \underset{\sim}{R}_i ,$$

and consider

$$P\{\underset{\sim}{R}_i = \underset{\sim}{r} | Q_i = k\}$$

$$= nP\{\underset{\sim}{R}_i = \underset{\sim}{r}, Q_i = k\} \text{ since } P\{Q_i = k\} = 1/n$$

$$= nP\left\{\underset{\sim}{R}_i = \underset{\sim}{r}, \underset{D's}{k-1} < D_i < \underset{D's}{n-k}\right\}$$

$$= nP\{\underset{\sim}{R}_i = \underset{\sim}{r}, D_{(k-1)} < D_i < D_{(k)}\}$$

where $D_{(j)}$ is the j-th order statistic from a sample of $(n-1)$ values of D [that is, all values except $D_i$].

Theorem:

Let $g_{k-1,k}$ be the joint density function of $D_{(k-1)}$ and $D_{(k)}$. Then

$$P\{\underset{\sim}{R}_i = \underset{\sim}{r} | Q_i = k\}$$

$$= n \int_0^{\infty} \int_0^{b} P\{\underset{\sim}{R}_i = \underset{\sim}{r}, a < D_i < b\} g_{k-1,k}(a, b) da db.$$

Proof:

$$P\{\underset{\sim}{R}_i = \underset{\sim}{r} | Q_i = k\}$$

$$= nP\{\underset{\sim}{R}_i = \underset{\sim}{r}, Q_i = k\}$$

$$= nP\Big\{\underset{\sim}{R}_i = \underset{\sim}{r},$$

$$D_{i_1} < \ldots < D_{i_{k-1}} < D_i < D_{ik} < \ldots < D_{i_{n-1}}\Big\}$$

where $(i_1, \ldots, i_{n-1})$ is a permutation of $(1, \ldots, i-1, i+1, \ldots, n)$. This in turn is equal to

$$nP\{\underset{\sim}{R}_i = \underset{\sim}{r}, d_{(k-1)} < D_i < d_{(k)}\}$$

where $d_{(j)}$ is the j-th order statistic of a sample $(n-1)$ observations on D, and thence equal to the integral of the Theorem. QED

This form allows the possibility of actual computation. The density g is well-known given the distribution of D. And the probability expression in the integrand depends only on the m-dimensional joint distribution of the observations within a single block. However, it does involve working with distributions of order statistics from heterogeneous distributions.

In the special cases of $k=1$ and $k=n$ we have

$$P\{\underset{\sim}{R}_i) = \underset{\sim}{r} | Q_i = 1\}$$

$$= n \int_0^{\infty} P\{\underset{\sim}{R}_i = \underset{\sim}{r}, 0 \le D_i < t\} f_{(1)}(t) dt$$

and

$$P\{\underset{\sim}{R}_i = \underset{\sim}{r} | Q_i = n\}$$

$$= n \int_0^{\infty} P\{\underset{\sim}{R}_i = \underset{\sim}{r}, t \le D_i < \infty\} f_{(n-1)}(t) dt ,$$

where $f_{(k)}$ is the density function of $D_{(k)}$.

## 4. AN EXPONENTIAL SPECIAL CASE

In this section we consider an application of the preceding theorem to a simple and analytically tractable case. We consider two treatments realized by the random variables X and Y, which we assume independent with probability density functions $f_x(x) = e^{-x}$, $f_y(y) = \lambda e^{-\lambda y}$, and $0 \le X, Y < \infty$. For the n blocks we have independent observations $\underset{\sim}{Z}_i = (X_i, Y_i)$, $i=1, \ldots, n$. Let $D_i = |X_i - Y_i|$, $i=1, \ldots, n$. Assuming that $\lambda > 1$, the "correct" ordering of (X, Y) is given by the permutation $\sigma = (2, 1)$. [The "correct" ordering means $P(X>Y) > P(Y>X)$]. (In this notation the range of the observations in the i-th block is less than that of the $(i+1)^{st}$ block.)

$$\text{Since } \mathrm{Corr}(R_{(i)}, \sigma) = \begin{cases} 1 \text{ if } R_{(i),1} = 2 \\ -1 \text{ if } R_{(i),1} = 1 \end{cases}$$

it follows that

$$E(\mathrm{Corr}(R_{(i)}, \sigma)) = P\{R_{(i),1} = 2\} - P\{R_{(i),1} = 1\}$$

$$= P\{R_{(i),1} = 2\} - [1 - P\{R_{(i),1} = 2\}]$$

$$= 2P\{R_{(i),1} = 2\} - 1 .$$

Thus, we compute

$$P_k = P\{R(X_i) = 2 | R(D_i) = k\}, \; k = 1, \ldots, n+1$$

and show that $\{P_k\}$ is strictly monotone increasing in k. The following lemma will be used in showing the result.

Lemma:

Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics for a sample of size n from $H(x)$, $0 < x < 1$. Let $d_k = E(X_{(k+1)}) - E(X_{(k)})$. If $h(x)$ is monotone increasing (decreasing), then $\{d_k\}$ is monotone decreasing (increasing) in k.

Proof:

$$E(X_{(k)})$$

$$= \int_0^1 x \frac{n!}{(k-1)!(n-k)!} H^{k-1}(x)h(x)[1-H(x)]^{n-k}dx$$
$$= \sum_{i=k}^{n}\binom{n}{i} H^i(x)[1-H(x)]^{n-i}x\Big|_0^1$$

$$-\int_0^\infty \sum_{i=k}^{n}\binom{n}{i} H^i(x)[1-H(x)]^{n-i}dx$$

$$= 1 - \int_0^1 \sum_{i=k}^{n}\binom{n}{i} H^i(x)[1-H(x)]^{n-i}dx,$$

So

$$d_k = E(X_{(k+1)}) - E(X_{(k)})$$

$$= \int_0^1 \binom{n}{k} H^k(x)[1-H(x)]^{n-k}dx.$$

Let $y=H(x)$, so $x=H^{-1}(y)$, and set $\mu(y)=dx/dy$. Then

$$d_k = \int_0^1 \mu(y)\binom{n}{k} y^k(1-y)^{n-k}dy$$

$$= \int_0^1 \mu(y)L_k(y)dy \quad \text{(say)}.$$

Clearly, $L_k(0)=L_k(1)=0$, $L_k$ is unimodal, and there exists a $y^*$ such that $0<y^*<1$ with $L_k(y^*)=L_{k+1}(y^*)$. Now if $h(x)$ is monotone increasing, then $\mu(y)$ is monotone decreasing and

$$d_{k+1} = \int_{y^*}^1 \mu(y)[L_{k+1}(y)-L_k(y)]dy$$

$$-\int_0^{y^*} \mu(y)[L_k(y)-L_{k+1}(y)]dy$$

$$< \mu(y^*)\int_{y^*}^1 \{L_{k+1}(y)-L_k(y)dy$$

$$-\mu(y^*)\int_0^{y^*}[L_k(y)-L_{k+1}(y)]dy$$

$$= \mu(y^*)\int_0^1 [L_{k+1}(y)-L_k(y)]dy$$

$$= 0.$$

$$\left[\text{Note } \int_0^1 L_{k+1}(y)dy = \int_0^1 L_k(y)dy = \frac{1}{n+1}.\right]$$

Hence $\{d_k\}$ in monotone increasing in k. The case where $h(x)$ is monotone decreasing is similar.

Theorem:

Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the observations corresponding to a design with two treatments and n blocks. Assume that X and Y are independent, with density functions $f_x(x) = e^{-x}$ and $f_y(y) = \lambda e^{-\lambda y}$, $0 < X,Y < \infty$. Let $P_k = P\{R(X_i) = 2|R(D_i) = k\}$, where $D_i=|X_i-Y_i|$. If $\lambda>1$, then $\{P_k\}$ is monotone in k; that is, $P_1<P_2<\ldots<P_n$.

Proof:

Let $D(X, Y)=|X-Y|$, then

$$G(t)=P(D\le t)=1-\frac{\lambda}{1+\lambda}e^{-t}-\frac{1}{1+\lambda}e^{-\lambda t},$$

$$g(t)=\frac{\lambda}{1+\lambda}(e^{-t}+e^{-\lambda t}),\ 0\le t<\infty.$$

We also have

$$P\{R(X)=2,\ 0\le D<t\}=\frac{\lambda}{1+\lambda}(1-e^{-t}),$$

$$P\{R(X)=2,\ t_1\le D<t_2\}=\frac{\lambda}{1+\lambda}(e^{-t_1}-e^{-t_2}),$$

and

$$P\{R(X)=2,\ t\le D<\infty\}=\frac{\lambda}{1+\lambda}e^{-t}.$$

By the preceding theorem, for $k=2, \ldots, n$ we have

$$P_k = P\{R(X)=2|R(D)=k\}$$

$$= n\int_0^\infty \int_0^{t_2} P\{R(X)=2,\ t_1\le D<t_2\}$$

$$g_{(k-1,k)}(t_2,t_1)dt_1dt$$

$$= n\int_0^\infty \int_0^{t_2} \left(\frac{\lambda}{1+\lambda}\right)(e^{-t_1}-e^{-t_2})g_{(k-1,k)}(t_2, t_1)dt_1dt_2$$

$$= \frac{n}{1+\lambda}\int_0^\infty \int_0^{t_2} (e^{-t_1}-e^{-t_2})\frac{(n-1)!}{(k-2)!(n-1-k)!}$$

$$[G(T_1)]^{k-2}g(t_1)g(t_2)[1-G(t_2)]^{n-1-k}dt_1dt_2$$

$$= \frac{n\lambda}{1+\lambda}\int_0^\infty e^{-t}g_{(k-1)}(t)dt - \frac{n}{1+\lambda}\int_0^\infty e^{-t}g_{(k)}(t)dt.$$

Now,

$$\int_0^\infty e^{-t}g_{(k)}(t)dt =$$

$$\int_0^\infty e^{-t}\frac{(n-1)!}{(k-1)!(n-1-k)!}[G(t)]^{k-1}g(t)[1-G(t)]^{n-1-k}dt$$

$$=\int_0^\infty e^{-t_1}\frac{(n-1)!}{(k-1)!(n-1-k)!}\left[1-\frac{\lambda}{1+\lambda}e^{-t}-\frac{\lambda}{1+\lambda}e^{-dt}\right]^{k-1}$$

$$*\left(\frac{\lambda}{1+\lambda}\right)(e^{-t}+e^{-\lambda t}\left\{\left[\frac{\lambda}{1+\lambda}e^{-t}+\frac{\lambda}{1+\lambda}e^{-\lambda t}\right]^{n-1-k}dt\right.$$

$$=\int_0^\infty X\frac{(n-1)!}{(k-1)!(n-1-k)!}\left[1-\frac{\lambda}{1+\lambda}X-\frac{1}{1+\lambda}X^{\lambda}\right]^{k-1}$$

$$*\left(\frac{\lambda}{1+\lambda}\right)(X+X^{\lambda})\left[\frac{\lambda}{1+\lambda}X+\frac{1}{1+\lambda}X^{\lambda}\right]^{n-1-k}\left(\frac{1}{X}\right)dx.$$

(via the transformation $X=e^{-t}$)

$$=\int_0^1 X\frac{(n-1)!}{(k-1)!(n-1-k)!}\left[1-\frac{\lambda}{1+\lambda}X-\frac{1}{1+\lambda}X^{\lambda}\right]^{k-1}$$

$$\left(\frac{\lambda}{1+\lambda}\right)(1+X^{\lambda-1})\left[\frac{\lambda}{1+\lambda}X+\frac{1}{1+\lambda}X^{\lambda}\right]dx$$

$$=\int_0^1 X\frac{(n-1)!}{(k-1)!(n-1-k)!}H^{n-1-k}(x)h(x)[1-H(x)]^{k-1}dx,$$

where H(x) is the distribution function given by

$$H(x)=\frac{\lambda}{1+\lambda}X+\frac{1}{1+\lambda}X^{\lambda},\ 0\leq X\leq 1.$$

Thus we have

$$\int_0^\infty e^{-t}g_{(k)}(t)dt=E(X_{(n-k)}),$$

where $X_{(k)}$ is the k-th order statistic of a sample of size n-1 from H(x). Now

$$P_k=\frac{(n+1)\lambda}{1+\lambda}\left[E(X_{(n-k+1)}-E(X_{(n-k)})\right]$$

and for $\lambda>1$, h(x) is strictly monotone increasing, hence $P_n>P_{n-1}\ldots>P_1$.

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