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From August 1983 to December 1986, I attended Florida State University for an AFIT sponsored PhD in statistics. Before I could complete my dissertation, I was assigned to AFIT/DFMS. On October 21, 1992 I successfully defended my dissertation and graduated on December 12, 1992. I have attached one copy of my dissertation and two copies of the abstract as you requested in our telephone conversation of 28 July 1992.

*Donna C Herge*

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EFFECTS OF INSPECTION ERROR ON OPTIMAL INSPECTION  
POLICIES AND SOFTWARE FAULT DETECTION MODELS

Donna Carol Herge, Ph.D.  
Florida State University, 1992  
Major Professors: Frank Proschan, Ph.D.  
Jayaram Sethuraman, Ph.D.

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and an algorithm to compute optimal intervals is derived. Additionally, monotonicity and majorization results are obtained for an optimal inspection sequence with Type II error. The impact of fault-detection error on a software optimal release time model is shown. The effect of fault diversity on the Jelinski-Moranda model and how this relates to imperfect fault detection is demonstrated.

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THE FLORIDA STATE UNIVERSITY  
COLLEGE OF ARTS AND SCIENCES

EFFECTS OF INSPECTION ERROR ON OPTIMAL INSPECTION  
POLICIES AND SOFTWARE FAULT DETECTION MODELS

By

DONNA CAROL HERGE

A Dissertation submitted to the  
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The members of the committee approve the dissertation  
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Thomas: Your arrival delayed this work's progress, but  
just as surely inspired its successful completion.

John: You are the wind beneath my wings.

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## Abstract

Inspection policies are essential for many types of systems for which the status (functioning or failed) can be determined only by actual inspection. Two types of inspection error may occur. A functioning system may be incorrectly assessed as having failed or a failed system may be incorrectly assessed as functioning. These errors are designated as Types I and II respectively, and their impact on optimal inspection policies and software fault detection models is analyzed. For a periodic inspection model with Type I error, an optimal replacement age is obtained, then monotonicity and asymptotic properties of the long-run expected cost per unit of time are presented. Type I error is incorporated into a cumulative damage model. When the failure density is reverse rule of order 2, an algorithm to compute an optimal inspection sequence is derived, and it is proven that optimal intervals are increasing. Extending the optimal inspection sequence model of Barlow, Hunter, and Proschan to include Type II inspection error, it is proven that optimal intervals are nonincreasing for a  $PF_2$  density, and an algorithm to compute optimal intervals is derived. Additionally, monotonicity and majorization results are

obtained for an optimal inspection sequence with Type II error. The impact of fault-detection error on a software optimal release time model is shown. The effect of fault diversity on the Jelinski-Moranda model and how this relates to imperfect fault detection is demonstrated.

## 1. INTRODUCTION AND SUMMARY

### 1.1. Importance of Inspection Models.

Inspection policies are essential for many types of systems for which the status (functioning or failed) can be determined only by actual inspection. Safety and security systems are prime examples. Events such as the April 1986 nuclear reactor accident at Chernobyl and the December 1984 chemical leak at Bhopal remind us of the importance of safety devices in system design and of the need for optimal scheduling of inspection. Inspection policies also have applications in environmental control, medical screening, and inventory optimization.

An inspection error may be due to human error or malfunction of a detection device. A simple example is a fire extinguisher which is checked periodically. If the pressure gauge malfunctions, it may erroneously indicate the extinguisher is empty, and so the extinguisher is unnecessarily replaced.

The consequences of inspection error can be very costly. Failure to identify an O-ring problem resulted in the explosion of the space shuttle Challenger on January 28, 1986. Nine passengers were killed and 23 injured when

the failure of a cargo door latch ruptured the skin and a large section of the fuselage ripped away from the 19 year old United Airlines Boeing 747 enroute from Hawaii on February 25, 1989. Ironically the Federal Aviation Administration had ordered inspection of all 747 cargo door latches the previous year.

### 1.2 Imperfect Repair Models.

Recent work on maintenance models takes into account additional real-world factors such as "minimal" or "imperfect" repair action. Brown and Proschan (1983a) proposed several different imperfect maintenance and imperfect inspection models and (1983b) studied properties of the distribution of the time between perfect repairs. Fontenot and Proschan (1984) then developed optimal policies for several maintenance models based on the imperfect repair model of Brown and Proschan (1983b). Block, Borges, and Savits (1985) extended the results of Brown and Proschan (1983b) by allowing the probability of imperfect repair to depend on the age of the equipment. Chan and Sethuraman (1985) compared the expected number of failures by time  $t$  in several different minimal repair models. Nakagawa (1981) analyzed generalized models for determining the optimal number of minimal repairs before replacement.

### 1.3 Inspection Models.

Another real world factor, imperfect inspection, has not been considered as extensively in the literature. Many inspection models assume perfect inspection. Savage (1956) showed how to choose an optimal replacement procedure for a "preparedness" model, where equipment is kept in storage until an emergency occurs. Derman and Sacks (1960) solved the problem of choosing an optimal replacement rule for deteriorating equipment, where the amount of deterioration is observed periodically. Barlow, Hunter, and Proschan (1963) showed how to obtain optimum inspection schedules for a broad class of failure distributions. Taylor (1975), assuming a cumulative damage model for system failure, found an optimal replacement strategy that minimized the long-run expected cost per unit of time. Abdel-Hameed (1986) obtained an optimal periodic replacement policy for a system subject to shocks which are modeled by any counting process with jump size one. For systems which fail when cumulative continuous wear reaches a threshold, Park (1988) derived an optimal replacement policy under periodic inspections. Parmigiani (1990a) derived a Bayesian decision rule for optimal scheduling of inspections when the distribution of failure times is not fully known.

#### 1.4 Type I and II Inspection Errors.

Although perfect inspection is often assumed, that is, no error is made in assessing the status of the system, two types of inspection error may actually occur. A functioning system may be incorrectly assessed as having failed or a failed system may be incorrectly assessed as functioning. Designate these errors as Type I and Type II respectively, analogous to Type I and Type II errors in hypothesis testing.

#### 1.5. Imperfect Inspection Models.

Assuming Type II inspection error, Derman (1961) obtained the minimax inspection schedule for an unknown failure distribution. Weiss (1962) assumed Type II error in his periodic inspection model and obtained the asymptotic availability of a system with exponential failure distribution. Luss and Kander (1974) obtained optimal inspection policies for a system which continues to operate during inspection. They assumed that the failure of a system prior to an inspection is detected with probability one, but allowed for the occurrence of both Type I and Type II inspection errors during system checking. Kaio and Osaki (1984) introduced Type II error into Keller's (1974, 1982) models for near-optimum inspection schedules using an inspection density.

Assuming an exponential failure density, Sengupta

(1982) extended the model of Barlow, Hunter, and Proschan (1963) to obtain optimum inspection schedules with Type II inspection error. Parmigiani (1990d) further extended this model for any log-concave failure density with both Type I and Type II errors. A decision theoretic model for optimal screening ages for detection of chronic diseases was presented by Parmigiani (1990b,d). Herge, Proschan, and Sethuraman (1986) obtained an optimal replacement age under Type I inspection error. With Type II error, Parmigiani (1990c) obtained optimal one-test and two-test policies to avoid replacing a functioning unit.

#### 1.6 Imperfect Detection of Software Faults.

Corresponding to the problem of inspecting hardware systems for failure, we have the problem of detecting faults in software systems. In today's world, the number and complexity of processes being controlled by computers is growing rapidly. As the size and complexity of computer programs increases, the probability that the software will function error-free is near zero. Thus it is essential to be able to measure the probability that software will perform its intended function at a specified time in a given environment, i.e., to measure software reliability. Failure of the software to perform its intended function may be due to many factors ranging from errors in the program code (bugs) to incorrect specification of program

requirements. Goel (1985) gives an overview of many of the analytical models that have been developed to measure software reliability, and provides a critical analysis of their underlying assumptions. Most models make use of the number of faults in the software as a basis to measure reliability.

The Jelinski-Moranda (JM) model (1972) is one of the first and simplest models and, therefore, probably most used. It assumes the failure rate of the software is proportional to the number of faults remaining in the software after a period of testing. Let  $N$  be the initial number of unknown software faults, and  $t_i$  be a time between discovery of the  $(i-1)$ st and  $i$ th fault,  $i=1, \dots, N$ . Then the failure rate is  $r(t_i) = \phi[N-i+1]$ , where  $\phi > 0$  is a proportionality constant. The model further assumes that faults are independent and equally likely to cause software failures. It also assumes that detected faults are removed with certainty and no new faults are created. Also the occurrence of a fault is equivalent to the detection of the fault. This is analogous to the assumption of a perfect inspection process. These "perfect" assumptions are common to most software reliability models; however, they are rarely true in practice.

Goel and Okumoto (1978) present an imperfect debugging model which is an extension of the JM model, where  $p$  is the probability that a detected fault is corrected. Then the

failure rate at time  $t_i$  between the  $(i-1)$ st and  $i$ th failure is  $r(t_i) = [N-p(i-1)]\phi$ . The number of faults in the system is governed by a semi-Markov process with transition probabilities  $Q_{ij}(t) = p[1-\exp(-i\phi t)]$ . Bayesian prior estimates for  $N$  and  $\phi$  are gamma distributed, and for  $p$  a beta prior is used; see Dunn and Ullman (1982).

The Shooman model, like the JM model assumes the failure rate is proportional to the number of remaining faults in the program. Specifically  $r(t)=k[N/I - n(\tau)]$ , where  $t$  is operating time of the system,  $\tau$  is debugging time (independent of  $t$ ),  $N$  is the initial number of faults,  $I$  is total number of instructions in the program,  $n(\tau)$  is total number of faults corrected during  $\tau$  (normalized by  $I$ ), and  $k$  is a proportionality constant. This model also assumes perfect fault detection and correction. An extension of this model by Shooman and Natarajan (1977) allows for the introduction of new faults during the correction process.

One of the important applications of a software reliability model is in the decision process to determine when to end the testing phase of software development and to release the software for operational use. Okumoto and Goel (1980) present a procedure to minimize the cost of software in a release time model. In order to minimize the entire life-cycle cost of the software, based on the JM model, Bai and Yun (1988) determine the optimum number,  $n$ ,

of errors that need to be corrected before releasing a software system to the operational phase. The more errors that are corrected during the testing phase, the higher the reliability and the lower the operating cost of the software. But a prolonged testing phase results in increased cost of the software. To achieve an optimum  $n$ , the average gain is used as a criterion. Bai and Yun conclude that the results of their model for the optimum number of errors to release software, are approximately the same as those obtained by the Koch and Kubat (1983) model which determines the optimum time,  $T$ , to release software based on the average gain function. Again as in most models, a basic assumption is made that faults are detected and corrected with certainty, taking negligible time.

### 1.7 Software Fault Diversity.

In addition to the assumption of perfect fault detection, many software models, including the JM model, assume that each fault is equally likely to cause a software failure. But in reality some faults are more likely to occur than others. Boland, Proschan, and Tong (1987) introduced a discrete software reliability model based on the multinomial distribution and demonstrated the impact of fault diversity on time to detection of faults. If the software initially contains  $N$  faults, then the

initial state of the system can be described by a vector,  $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_N)$ , where  $\theta_i$  is the probability of detecting fault  $i$ . It is shown that the more diverse (in terms of majorization)  $\underline{\theta}$  is, the longer it takes (stochastically) to eliminate faults. The impact of fault diversity is also demonstrated for models of the Jelinski-Moranda type, but not specifically for the Jelinski-Moranda model.

### 1.8 Summary.

In this dissertation we analyze the impact of imperfect inspection on several models. Chapters 2 and 3 present Type I inspection error models. Chapters 4, 5, and 6 present Type II inspection error models.

In Chapter 2 we obtain an optimal replacement age in an imperfect inspection model and give an example for a device with a truncated normal life distribution. We give monotonicity and asymptotic properties for the long-run expected cost per unit of time. In Chapter 3 we extend the cumulative damage model of Derman and Sacks (1960) for the case in which the probability of perfect inspection is less than one.

Chapter 4 extends the Barlow, Hunter, and Proschan (1963) optimal inspection sequence model to include reverse rule two (RR2) failure densities. Chapter 5 then extends the Barlow, Hunter, and Proschan (1963) model to include Type II error, and monotonicity and majorization results

are presented.

Chapter 6 presents two software fault detection models. Section 6.2 explores the impact of imperfect fault detection on the Koch and Kubat (1983) model. Section 6.3 shows the effect of fault diversity of the JM model and shows how this relates to imperfect fault detection.

## 2. MODEL 1: OPTIMAL REPLACEMENT AGE WITH TYPE I INSPECTION ERROR

### 2.1. Formulation of the Model; Derivation of Solution.

Herge, Proschan, and Sethuraman (1986) formulate and analyze a periodic inspection model with Type I inspection error for a system which is subject to an age replacement policy. We describe the model and present the results in this chapter. A list of basic symbols follows.

$X$  = lifelength with distribution  $F$ ,  $\bar{F}(x) \equiv 1 - F(x)$ .

$k, 2k, \dots$  are scheduled inspection times.

$f_k \equiv f(k)$ ,  $\bar{F}_k \equiv \bar{F}(k)$ ,  $r_k \equiv f_k/\bar{F}_k$ .

$T$  = replacement age (positive integer multiple of  $k$ ).

$c_1$  = cost of unscheduled replacement.

$c_2$  = cost of scheduled (age) replacement ( $c_1 > c_2$ ).

$q_1$  = probability of Type I inspection error.

$p_1 = 1 - q_1$ .

$Y$  = observed lifelength of device with lifelength  $X$ .

$L_{p_1}(T)$  = long-run expected cost per unit of time.

A device has lifelength  $X$  with distribution function  $F$ , which is absolutely continuous with density  $f$  and  $F(0)=0$ . The device is installed at time zero and is inspected at successive times  $k, 2k, \dots$  to determine if it

is functioning or failed. We follow an age replacement policy which calls for the unit to be replaced by a new unit at age  $T$  (which is a positive integer multiple of  $k$ ) or at the first inspection following failure, whichever comes first. Replacement time is assumed negligible. When an error in inspection (Type I) occurs (with probability  $0 < q_1 \leq 1$ ), a functioning unit is replaced by a new unit. Define  $p_1 = 1 - q_1$ . On the other hand, a failed unit is replaced at inspection with probability one. This process is continued indefinitely. Clearly the process renews itself at times of replacement.

It is apparent that an age replacement policy is inappropriate if the underlying failure rate is decreasing. Thus we will assume that the failure rate  $r(t) = f(t) / [\bar{F}(t)]$  is increasing. The probability that the device survives until time  $t = ik$  is  $P[Y > ik] = \bar{F}_{ik} p_1^i$  for  $i = 0, 1, 2, \dots$  in the absence of planned replacement, since the device must survive to time  $ik$  and pass  $i$  inspections, where  $Y$  is the observed lifelength of the device. Let  $Z$  be the elapsed time between replacements; then  $Z = \min\{Y, T\}$ . Note that

$$E(Y) = k \sum_{i=0}^{\infty} \bar{F}_{ik} p_1^i .$$

Let  $c_1$  be the cost of an unscheduled replacement (resulting from actual or believed failure) and  $c_2$  be the cost of a scheduled (age) replacement. Assume  $c_1 > c_2$ . Our objective is to find the replacement age  $T$  which

minimizes  $L_{p_1}(T)$ , the long-run expected cost per unit of time. Theorem 2.1 gives a formula for  $L_{p_1}(T)$  and Theorem 2.2 describes the optimal value  $T^*$  of  $T$ .

Theorem 2.1.

$$L_{p_1}(T) = \frac{c_1 - (c_1 - c_2) \bar{F}_{T-k} p_1^{T/k-1}}{k \sum_{i=0}^{T/k-1} \bar{F}_{ik} p_1^i}.$$

Proof. The times of replacement are renewal times in a renewal-reward process whose interarrival distribution is that of  $Z = \min \{Y, T\}$ . From renewal theory, we have that  $L_{p_1}(T) = C(T)/D(T)$ , where  $C(T)$  is the expected cost per renewal cycle, and  $D(T)$  is the expected duration of a renewal cycle. Denote the distribution function of  $Y$  by  $G$ . Then for  $i=1, 2, \dots$

$$\bar{G}(t) = P[Y > t] = \bar{F}_{(i-1)k} p_1^{i-1}, \quad (i-1)k \leq t < ik.$$

Then  $D(T)$ , the expected length of a cycle is

$$\begin{aligned} D(T) = E(Z) &= \int_0^T \bar{G}(t) dt = \sum_{i=1}^{T/k} \int_{(i-1)k}^{ik} \bar{G}(t) dt \\ &= \sum_{i=1}^{T/k} \bar{F}_{(i-1)k} p_1^{i-1} \int_{(i-1)k}^{ik} dt = k \sum_{i=0}^{T/k-1} \bar{F}_{ik} p_1^i. \end{aligned}$$

With probability  $\bar{F}_{T-k} p_1^{T/k-1}$ , the device is replaced after time

$T-k$  and a cost of  $c_2$  is incurred. Otherwise, it is replaced at or before time  $T-k$  and a cost of  $c_1$  is incurred. Therefore the expected cost per cycle is

$$C(T) = c_1(1 - \bar{F}_{T-k} p_1^{\frac{T}{k}-1}) + c_2 \bar{F}_{T-k} p_1^{\frac{T}{k}-1}. \quad \parallel$$

**Theorem 2.2.** In addition to the assumptions about  $F$  given above, assume that  $r$  is differentiable. Then

a.  $L'_{p_1}(T) = \bar{F}_{T-k} p_1^{T/k-1} [k \sum_{i=0}^{T/k-1} \bar{F}_{ik} p_1^i]^{-2} H(T)$  , where

$$H(T) = (c_1 - c_2) [(kr_{T-k} - \ln p_1) \sum_{i=0}^{T/k-1} \bar{F}_{ik} p_1^i + \bar{F}_{T-k} p_1^{T/k-1}] - c_1 .$$

b. An optimal replacement age  $T^*$  exists ( $T^*$  may be infinite), and  $T^*=k \Leftrightarrow p_1 \leq \exp(c_2/(c_2-c_1) + kr_0)$ .

c.  $T^*$  is finite  $\Leftrightarrow r_\infty > c_1/(c_1-c_2) EY + (1/k) \ln p_1$ .

d.  $EY/k < c_1/c_2$  implies that  $T^* < \infty$ .

e. A lower bound for  $T^*$  is  $(c_2/c_1) EY$ .

**Proof.**

a. Differentiate.

b. Given  $c_1 > c_2$  note that  $H$  is increasing since  $r$  is

increasing and  $H'(T) = (c_1 - c_2) k r'_{T-k} \sum_{i=0}^{T/k-1} \bar{F}_{ik} p_1^i$ . Then  $T^* = k \Leftrightarrow L_{p_1}$  is increasing  $\Leftrightarrow H(k) \geq 0 \Leftrightarrow$

$$(c_1 - c_2) (kr_0 - \ln p_1 + 1) - c_1 \geq 0 \Leftrightarrow p_1 \leq \exp(c_2/(c_2 - c_1) + kr_0).$$

If  $p_1 > \exp(c_2/(c_2-c_1) + kr_0)$ , then  $H(k) < 0$ . Thus if we can find  $T_0 = \min\{T: H(T) \geq 0\}$ , then  $L_{p_1}'$  is negative on  $(0, T_0-k)$  and positive on  $(T_0, \infty)$ . So we choose  $T^*$  such that  $L(T^*) = \min\{L(T_0-k), L(T_0)\}$ . Otherwise, if there is no  $T$  such that  $H(T) \geq 0$ , then  $L$  is decreasing and  $T^* = \infty$ ; i.e., planned replacement does not occur.

c. The inequality involving  $r_\infty$  holds if and only if  $\lim_{T \rightarrow \infty} H(T) > 0$ . Since  $H(0) < 0$ , the equation  $T_0 = \min\{T: H(T) \geq 0\}$  has a finite solution.

d. Note that  $L_{p_1}(T) \leq L_{p_1}(k) = c_2/k$ . If  $T^* = \infty$ , then  $L_{p_1}(T^*) = c_1/EY \leq c_2/k$ . By the contrapositive,  $EY/k < c_1/c_2$  implies that  $T^* < \infty$ .

e. It is easy to show that  $L_{p_1}(T) \geq c_2/T$ . Thus  $L_{p_1}(T)$  must cross  $L_{p_1}(\infty) = c_1/EY$  (if it crosses at all) to the right of the value at which  $c_2/T$  crosses  $c_1/EY$ ; this value is  $T = (c_2/c_1)EY$ . Therefore, we should schedule replacement at an age greater than  $(c_2/c_1)EY$ . ||

## 2.2. Asymptotic and Monotonicity Properties of $L_{p_1}(T)$ .

If we let  $p_1=1$ , we get a desired asymptotic result for  $L_1(T)$  as the length of the inspection interval,  $k$ , goes to zero. In this case the model reduces to the simple age

replacement model for which the optimization problem was solved by Barlow and Proschan (1965), pp. 85-90. If  $\bar{F}$  is Riemann integrable, then since the denominator of  $L_1(T)$  is a Riemann sum, it converges to the integral form given in Barlow and Proschan (1965) p. 88, as follows:

$$\lim_{k \rightarrow 0} L_1(T) = \lim_{k \rightarrow 0} \frac{c_2 + (c_1 - c_2) F_{T-k}}{\sum_{i=0}^{T/k-1} k \bar{F}_{ik}} = \frac{c_2 + (c_1 - c_2) F_T}{T \int_0^T \bar{F}(x) dx}.$$

Next for  $0 < p_1 < 1$ , we provide asymptotic results for  $L_{p_1}(T)$  as the length of the inspection interval,  $k$ , goes to zero, and also as  $k$  goes to infinity.

Theorem 2.3. For  $0 < p_1 < 1$ ,  $\lim_{k \rightarrow 0} L_{p_1}(T) = \infty$ , and

$$\lim_{k \rightarrow \infty} L_{p_1}(T) = 0.$$

Proof. Case 1. For  $T = k$ ,  $L_{p_1}(k) = \frac{c_2}{k}$ , then  $\lim_{k \rightarrow 0} \frac{c_2}{k} = \infty$ ,

$$\text{and } \lim_{k \rightarrow \infty} \frac{c_2}{k} = 0.$$

Case 2. For  $T = \infty$ ,  $L_{p_1}(\infty) = \frac{c_1}{EY} = \frac{c_1}{k \sum_{i=0}^{\infty} \bar{F}_{ik} P_1^i}$ , then

$$\lim_{k \rightarrow 0} \frac{c_1}{k \sum_{i=0}^{\infty} \bar{F}_{ik} p_1^i} > \lim_{k \rightarrow 0} \frac{c_1}{k \sum_{i=0}^{\infty} p_1^i} = \lim_{k \rightarrow 0} \frac{c_1}{k \left( \frac{1}{1-p_1} \right)} = \infty,$$

and

$$\lim_{k \rightarrow \infty} \frac{c_1}{k \sum_{i=0}^{\infty} \bar{F}_{ik} p_1^i} = \lim_{k \rightarrow \infty} \frac{c_1}{k \left( 1 + \sum_{i=1}^{\infty} \bar{F}_{ik} p_1^i \right)} = 0.$$

Case 3. For  $k < T < \infty$ ,  $T = mk$  for a positive integer  $m$ , and  $L_{p_1}(mk) = \frac{c_1 - (c_1 - c_2) \bar{F}_{(m-1)k} p_1^{m-1}}{k \sum_{i=0}^{m-1} \bar{F}_{ik} p_1^i}$ , then

$$\lim_{k \rightarrow 0} \frac{c_1 - (c_1 - c_2) \bar{F}_{(m-1)k} p_1^{m-1}}{k \sum_{i=0}^{m-1} \bar{F}_{ik} p_1^i} > \lim_{k \rightarrow 0} \frac{c_1 - (c_1 - c_2) p_1^{m-1}}{k \sum_{i=0}^{m-1} p_1^i} = \lim_{k \rightarrow 0} \frac{c_1 - (c_1 - c_2) p_1^{m-1}}{k \left( \frac{1 - p_1^m}{1 - p_1} \right)} = \infty,$$

and

$$\lim_{k \rightarrow \infty} \frac{c_1 - (c_1 - c_2) \bar{F}_{(m-1)k} p_1^{m-1}}{k \sum_{i=0}^{m-1} \bar{F}_{ik} p_1^i} = \lim_{k \rightarrow \infty} \frac{c_1}{k \left( 1 + \sum_{i=1}^{m-1} \bar{F}_{ik} p_1^i \right)} = 0. \quad \parallel$$

At  $p_1 = 1$ ,  $\lim_{k \rightarrow 0} L_{p_1}(T)$  is discontinuous since

$$\lim_{k \rightarrow 0} L_1(T) = \frac{c_2 + (c_1 - c_2) F_T}{\int_0^T \bar{F}(x) dx} < \infty$$

while for  $0 < p_1 < 1$ ,  $\lim_{k \rightarrow 0} L_{p_1}(T) = \infty$ .

Theorem 2.3 shows that the more frequently you inspect, the greater the cost per unit of time. This logically results in the conclusion that it would be best not to inspect at all, but simply to perform a planned replacement at time  $T$ . However, there are situations in which a fixed  $k$  is reasonable and justifiable; e.g.: the government requires that a certain vital part of the airplane (like the recorder) is inspected every  $k$  hours, the fire alarm system in a home or factory is inspected every  $k$  hours, etc.

It is easily seen that  $L_{p_1}(T)$  increases as  $p_1 \downarrow 0$  as follows.

a. Note that  $\bar{F}_{ik} p_{1m}^i \leq \bar{F}_{ik} p_{1n}^i$  for  $p_{1m} < p_{1n}$ , for  $i = 0, 1, \dots, T/k - 1$ . Thus for  $T = k, 2k, \dots$

$$1 / \left( k \sum_{i=0}^{T/k-1} \bar{F}_{ik} p_{1m}^i \right) \geq 1 / \left( k \sum_{i=0}^{T/k-1} \bar{F}_{ik} p_{1n}^i \right).$$

When  $T > k$ , we have strict inequality above.

b. Note that for  $p_{1m} < p_{1n}$  and  $T = k, 2k, \dots$

$$\bar{F}_{T-k} p_{1m}^{\frac{T}{k}-1} \leq \bar{F}_{T-k} p_{1n}^{\frac{T}{k}-1}.$$

When  $T > k$  we have strict inequality. Thus

$$c_1 - (c_1 - c_2) \bar{F}_{T-k} p_{1m}^{\frac{T}{k}-1} \geq c_1 - (c_1 - c_2) \bar{F}_{T-k} p_{1n}^{\frac{T}{k}-1}.$$

From a. and b. we have  $L_{p_{im}}(T) = L_{p_{in}}(T)$  for  $T = k$  and

$L_{p_{im}}(T) > L_{p_{in}}(T)$  for  $p_{im} < p_{in}$  and  $T = 2k, 3k, \dots$ .

This is intuitively reasonable since more mistakes in inspection leads to an increase in unnecessary replacements. This can be seen in Figure 2.5 in the next section. Clearly when  $p_1 \downarrow 0$   $\lim L_{p_1}(T) = c_1/k = L_0(T)$ , thus  $L_0(T)$  is an upper bound for  $L_{p_1}(T)$  for  $0 \leq p_1 \leq 1$ .

Since  $L_{p_1}(T)$  is a decreasing function of  $p_1$  (see Figure 2.5), we obtain a sharp lower bound for  $L_{p_1}(T)$  by taking the limit as  $p_1$  increases to one. Thus

$$L_{p_1}(T) \geq \lim_{p_1 \uparrow 1} L_{p_1}(T) = \frac{c_1 - (c_1 - c_2) \bar{F}_{T-k}}{T/k - 1} = L_1(T).$$

$$k \sum_{i=0}^{T/k-1} \bar{F}_{ik}$$

We denote the optimal value of  $T$  for a particular  $p_1$  as  $T_{p_1}^*$ . Then the optimal long-run expected cost per unit of time is  $L_{p_1}(T_{p_1}^*) = \min_T L_{p_1}(T)$ . Since for all  $T$  and  $p_{im} < p_{in}$

we have  $L_{p_{im}}(T) \geq L_{p_{in}}(T)$ , it follows that

$$L_{p_{im}}(T_{p_{im}}^*) \geq L_{p_{im}}(T_{p_{in}}^*) \geq L_{p_{in}}(T_{p_{in}}^*).$$

### 2.3. Application to the Exponential Distribution.

Our model makes sense only for a failure rate which is

increasing. We see from Theorem 2.1 that the denominator of  $L_{p_1}(T)$  is a sum which can be explicitly solved for an exponential distribution  $F$ . Intuitively it is clear that the optimal policy is to replace only at failure times, because replacing an exponential component by a new component does not change the residual life. We show that Theorem 2.2 confirms that  $T^* = \infty$ . Let  $\bar{F}(x) = \exp(-\lambda x)$ ,  $\lambda > 0$ . Then  $EY = \sum_{i=0}^{\infty} (p_1 e^{-\lambda k})^i = \frac{k}{1 - p_1 e^{-\lambda k}}$ . Setting  $L'_{p_1}(T) = 0$  solving

for  $T$  we get

$$T = k + k \ln \left[ \frac{\left[ \frac{c_1}{c_1 - c_2} \right] (1 - p_1 e^{-\lambda k}) + \ln p_1 e^{-\lambda k}}{1 + p_1 e^{-\lambda k} (-1 + \ln p_1 e^{-\lambda k})} \right] / \ln p_1 e^{-\lambda k}.$$

Note that  $1 + p_1 e^{-\lambda k} (-1 + \ln p_1 e^{-\lambda k}) > 0$  since it reduces to the form  $\ln x < x - 1$ , which is true for  $x = p_1^{-1} e^{\lambda k} > 1$ . Thus for  $T$  to exist, we need  $\left[ \frac{c_1}{c_1 - c_2} \right] (1 - p_1 e^{-\lambda k}) + \ln p_1 e^{-\lambda k} > 0$ , which implies  $r_{\infty} = \lambda < \frac{c_1}{(c_1 - c_2) EY} + \frac{1}{k} \ln p_1$ . Thus from Theorem 2.2.c. we see that  $T^* = \infty$ .

#### 2.4. Application to the Truncated Normal Distribution.

An application of the model for a truncated normal distribution was chosen so that results can be compared to the asymptotic case treated in Example 1, p.90, of Barlow and Proschan (1965).

Example 2.1. A certain tube used in commercial airline communication equipment has a truncated normal failure distribution with a mean life of 9080 hours and a standard deviation of 3027. Suppose  $c_1 = \$1100$  and  $c_2 = \$100$ .

The density  $f(x)$  of the truncated normal distribution may be written as

$$f(x) = \begin{cases} \frac{1}{b\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right), & x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{where } \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ and } b = \frac{1}{\sigma} \int_0^{\infty} \varphi\left(\frac{x-\mu}{\sigma}\right) dx.$$

From  $L'(T) = 0$  we get

$$(kr_{T-k} - \ln p_1) \sum_{i=0}^{T-k-1} \bar{F}_{ik} p_1^i + \bar{F}_{T-k} p_1^{T-k} = \frac{c_1}{c_1 - c_2}. \quad (2.1)$$

Making the change of variable  $u = (T-k-\mu)/\sigma$ , we have from (2.1) for a truncated normal distribution  $F$ ,

$$\left(\frac{k}{\sigma} r(u) - \ln p_1\right) \sum_{i=0}^{\frac{\sigma u + \mu}{k}} p_1^i \int_{\frac{ik-\mu}{\sigma}}^{\infty} \varphi(v) dv + p_1^{\frac{\sigma u + \mu}{k}} \int_u^{\infty} \varphi(v) dv = \frac{bc_1}{c_1 - c_2}. \quad (2.2)$$

Define  $K(u)$  to be the left hand side of (2.2).

If  $\mu/\sigma \geq 3$ , then  $f(x)$  is very close to the density of a normal distribution with mean  $\mu$  and standard deviation  $\sigma$ . Thus, in Example 1, Barlow and Proschan estimate  $b$  by 1 and

graphically estimate  $u$  to be  $-1.5$ . To obtain more accurate results for comparison with our model, we compute  $b = .9987$  and  $K(u) = 1.0986$ . Then  $u = -1.63$ , and the optimal replacement age is 4146 hours with an associated minimum cost of  $L(4146) = \$.036$ .

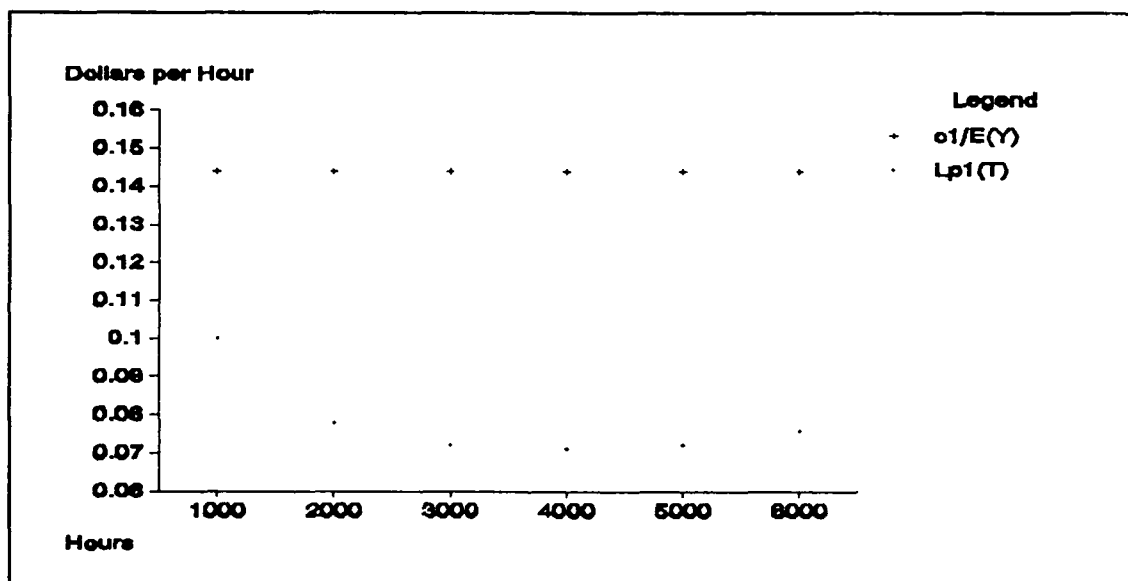
Next we observe the impact of Type I inspection error on this example. We obtain  $T^*$  for four cases: when the length of the inspection interval is  $k=1000$  and the probability of no error in inspecting a functioning unit is  $p_1=.95$ ; when  $k=500$  and  $p_1=.95$ ; when  $k=1000$  and  $p_1=.5$ ; and when  $k=500$  and  $p_1=.5$ . Note that for the truncated normal distribution  $r_\infty = \infty$ , so we know by Theorem 2.2.c. that  $T^*$  is finite.

Using (2.2) we will find the value of  $T$  such that  $K((T-k-\mu)/\sigma)$  is nearest to  $K(u) = bc_1/(c_1-c_2) = 1.0986$ . Tables 1 through 4 show the results of these computations. The corresponding graphs of  $L_{p_1}(T)$  are shown in Figures 1 through 4.

We see from Table 1 below that  $T^* = 4000$  is the optimal value, which gives us the minimum cost  $L_{.95}(T^*) = \$.0710$  as shown in Figure 1. We compute  $EY = 7629$ . Recall from the proof of Theorem 2.2.e. that  $c_1/EY = .144$  is the horizontal asymptote for  $L_{p_1}(T)$ . Also  $(c_2/c_1)EY = 694$  is a lower bound for  $T^*$ .

Table 1. Values of  $K(u)$  for  $k=1000$ ,  $p_1=.95$ 

T	u	$K(u)$
1000	-3.00	1.050
2000	-2.67	1.051
3000	-2.34	1.060
4000	-2.01	1.088
5000	-1.68	1.149

Figure 1. Values of  $L_{p_1}(T)$  for  $k=1000$ ,  $p_1=.95$ .

We see from Table 2 below that  $T^* = 4000$  is the optimal value and  $L_{p_1}(T^*) = \$.1271$  shown in Figure 2. We compute  $EY = 5979$ . Then  $(c_2/c_1)EY = 544$  is a lower bound for  $T^*$ . Notice in Figure 2 that  $c_2/k > c_1/EY$ ; thus the implication in Theorem 2.2.d. cannot be reversed.

Comparing the cases where  $k=500$  and  $k=1000$ , it is clear that more frequent inspections result in a higher optimal cost  $L_{p_1}(T^*)$ .

Table 2. Values of  $K(u)$  for  $k=500$ ,  $p_1=.95$

T	u	K(u)
3500	-2.01	1.0700
4000	-1.84	1.0916
4500	-1.65	1.1205
5000	-1.51	1.1605

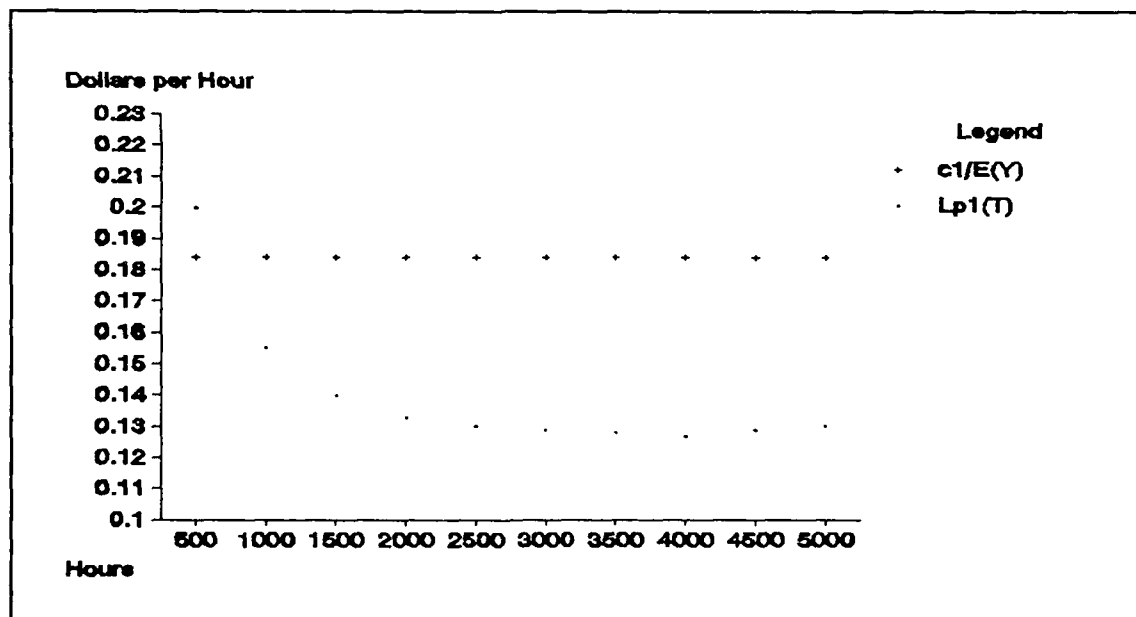


Figure 2. Values of  $L_{p_1}(T)$  for  $k=500$ ,  $p_1=.95$ .

We see from Table 3 below that the graph of  $K(u)$  does not cross  $bc_1/(c_1-c_2) = 1.0986$ . Thus there is no solution

to  $L'_{p_1}(T) = 0$ , so that  $L'_{p_1} > 0$ , and the cost is strictly increasing, as shown in Figure 3.

Table 3. Values of  $K(u)$  for  $k=1000$ ,  $p_1=.5$

T	u	K(u)
1000	-3.00	1.6923
2000	-2.67	1.5410
3000	-2.34	1.4710
4000	-2.01	1.4490
5000	-1.68	1.4590
6000	-1.35	1.4969

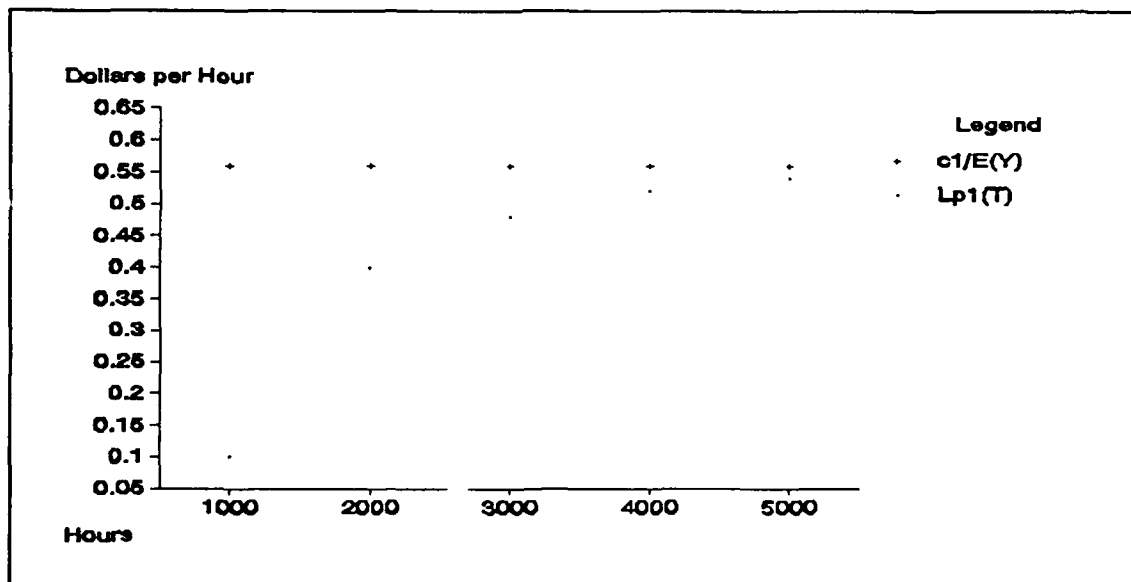


Figure 3. Values of  $L_{p_1}(T)$  for  $k=1000$ ,  $p_1=.5$ .

Then the minimum cost is  $\$.10 = L_{p_1}(1000)$  at  $T^* = k$ .

This is expected from Theorem 2.2.b. since  $.5 = p_1 \leq$

$$\exp(c_2/(c_2-c_1)+kr_0) = \exp(-.1 + 1000(.0000015)) \approx .9062.$$

Since  $EY = 1980$ ,  $(c_2/c_1)EY = 180$  is a lower bound for  $T^*$ .

As in the previous case where  $p_1=.5$ , we see in Table 4 below that  $K(u)$  does not cross  $bc_1/(c_1-c_2) = 1.0986$ , so that

Table 4. Values of  $K(u)$  for  $k=500$ ,  $p_1=.5$

T	u	K(u)	T	u	K(u)
500	-3.00	1.6915	3000	-2.17	1.4040
1000	-2.83	1.5384	3500	-2.01	1.4042
1500	-2.67	1.4630	4000	-1.84	1.4089
2000	-2.50	1.4260	4500	-1.68	1.4164
2500	-2.34	1.4096	5000		

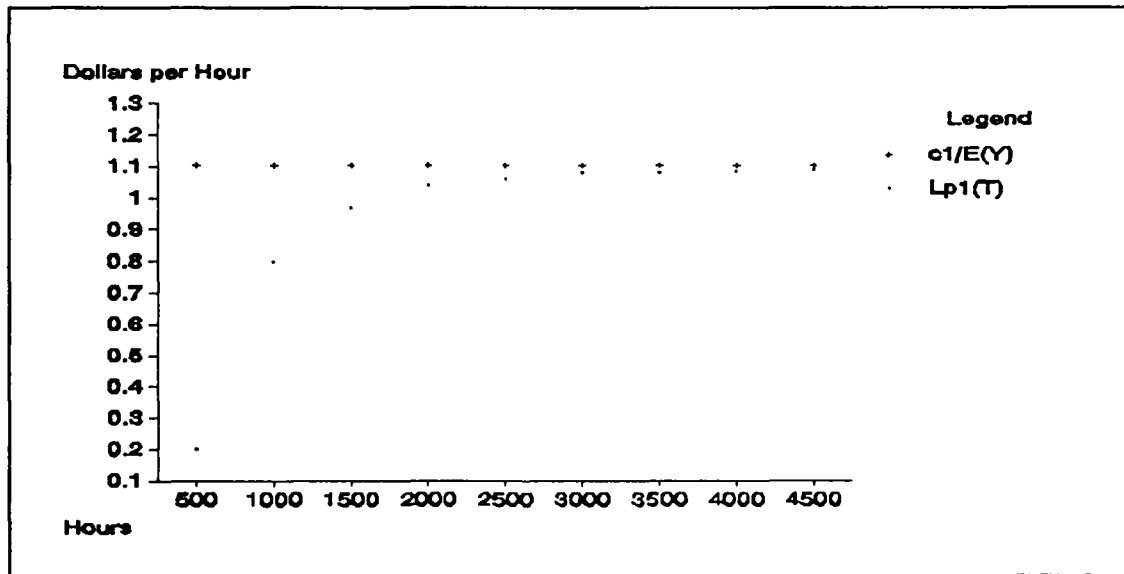


Figure 4. Values of  $L_{p_1}(T)$  for  $k=500$ ,  $p_1=.5$ .

cost is strictly increasing and  $T^* = k$  as expected from Theorem 2.2.b. since  $.5 = p_1 \leq \exp(c_2/(c_2-c_1) + kr_0) =$

$\exp(-.1 + 500(.0000051)) \approx .9055$ . A lower bound for  $T^*$  is  $(c_2/c_1)EY = 91$ .

We summarize the results of these cases in the Table 5 which follows.

Table 5. Summary of Examples

$p_1$	$k$	$(c_2/c_1)EY$	$T^*$	$EY$	$L_{p_1}(T)$
1.00	0	825	4146	9080	\$.036
.95	1000	694	4000	7629	\$.071
.95	500	544	4000	5979	\$.127
.50	1000	180	1000	1980	\$.100
.50	500	91	500	998	\$.200

Recall from Theorem 2.2.e. that  $(c_2/c_1)EY$  is a lower bound for  $T^*$ . It appears that  $EY$  is an upper bound for  $T^*$ . Note that for  $p=.95$  and  $k=1000$ , and  $p=.95$  and  $k=500$ , when  $T^* < \infty$  and  $T^* \neq k$ , we could have used the method in Example 1 p.90 of Barlow and Proschan (1965) to compute the optimal replacement age, then chosen the closest integer multiple of  $k$  to get the desired result.

Next, to observe asymptotic and monotonicity properties of  $L_{p_1}(T)$ , for  $k=1000$ , we compare  $L_{p_1}(T)$  for

values of  $p_1$  from .5 to 1.0 in Figure 5. For  $p_1 = .5, .7,$  and .9 we have  $T^* = k = 1000$ . Clearly  $L_{p_1}(T^*)$  and  $L_{p_1}(T)$  are decreasing in  $p_1$ . The upper bound is  $L_0(T)=1.1$ , and the sharp lower bound is  $L_1(T)$ . Horizontal asymptotes are labeled  $L_{p_1}(\text{inf})$ .

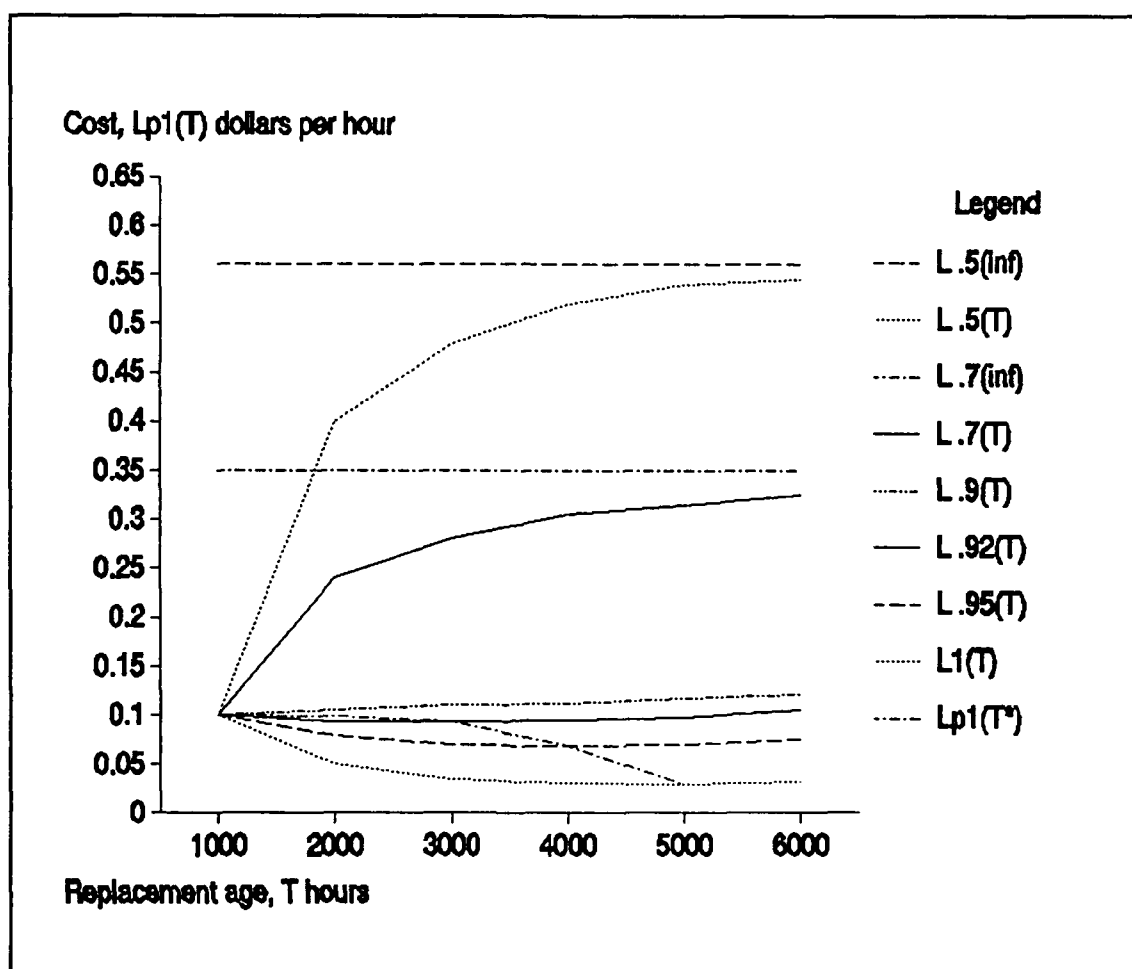


Figure 5. Comparison of  $L_{p_1}(T)$  for  $k=1000$ .

### 3. MODEL 2: CUMULATIVE DAMAGE ASSESSMENT WITH TYPE I INSPECTION ERROR

#### 3.1. Derman and Sacks Model and Stopping Lemma.

Derman and Sacks (1960) considered the problem of choosing an optimal replacement rule when the amount of deterioration of the equipment can be observed at each inspection time. The criterion for evaluating the effectiveness of the replacement rule is based on expected cost per unit of time per replacement cycle instead of long-run expected cost per unit of time. We will summarize their results and give an extension that incorporates the probability of a Type I inspection error.

Let  $Z$  be the lifelength of a device with distribution function  $G$ . Let  $c_1$  be the cost of replacement after failure and  $c_2$  be the cost of replacement before failure. At each inspection time  $t=1,2,\dots$  we observe the amount of deterioration  $X_t$  which occurred during the interval  $(t-1,t)$ . Assume that  $\{X_t\}$  is a sequence of i.i.d., nonnegative, random variables with common distribution function  $F$ . The equipment fails when the cumulative damage  $S_n = \sum_{i=1}^n X_i$  exceeds a given threshold  $L$ .

We restrict ourselves to the class  $\mathcal{J}$  of replacement

rules of the form  $N = \min(\nu, [Z] + 1)$ , where  $[Z]$  denotes the greatest integer less than  $Z$ , and  $\nu$  is an optional stopping rule based on the cumulative damage. We wish to find a rule  $N^* \in \mathcal{J}$  which minimizes the average cost per unit of time per replacement cycle. This is accomplished by using the following optimal optional stopping lemma. For the proof see Derman and Sacks (1960).

Lemma 3.1. Let  $\{\mathcal{F}_n, n=1,2,\dots\}$  be an increasing sequence of  $\sigma$ -fields of a sample space  $\Omega$ . Let  $\{Y_n\}$  be a sequence of random variables which are  $\mathcal{F}_n$  measurable, and such that  $EY_n$  exists and is finite for all  $n$ . Let  $\mathcal{C}$  be the class of all optional stopping rules such that  $EN < \infty$ . If there exists an optional stopping rule  $N^*$  with

- a.  $EN^* < \infty$ ,
- b.  $E\{Y_n \mid \mathcal{F}_{n-1}\} \leq Y_{n-1}$  when  $n \leq N^*(\omega)$   
 $\geq Y_{n-1}$  when  $n > N^*(\omega)$

for almost all  $\omega \in \Omega$ ; and if there is some  $M < \infty$  such that

- c. for all  $n$ ,  $E\{|Y_{n+1} - Y_n| \mid \mathcal{F}_n\} \leq M$ ,

then  $EY_{N^*} = \min_{N \in \mathcal{C}} EY_N$ . Thus  $N^*$  is an optimal stopping rule.

### 3.2. Application of Optimal Optional Stopping Lemma.

In order to apply the lemma to the replacement problem, let

$$Y_n = \begin{cases} c_2/n & \text{if } n < Z \\ \frac{c_1}{[Z]+1} & \text{if } n \geq Z. \end{cases}$$

Let  $N$  be the number of time periods until a replacement under our replacement rule. Then we wish to choose  $N$  to minimize  $EY_N$ .

Derman and Sacks show that if we choose  $n^*$  to be the smallest  $n$  such that

$$1 - F(L - S_{n-1}) \geq \frac{c_2}{(c_1 - c_2)(n-1)},$$

then  $N^* = \min(n^* - 1, [Z] + 1)$  is the optimal rule.

### 3.3. Extension of the Model for Type I Error.

We define  $q_1(S_n)$  as the probability of mistakenly replacing a functioning unit after observing the cumulative damage  $S_n$ , because we conclude that the threshold has been exceeded when actually it has not been exceeded. Since we are more likely to make such a mistake as the cumulative damage nears the threshold value, it makes sense for  $q_1(S_n)$  to be an increasing function of  $n$ . Then  $p_1(S_n) = 1 - q_1(S_n)$  is the probability of concluding that the cumulative damage does not exceed the threshold when it actually does not exceed the threshold. On the other hand, a failed unit is replaced at inspection with probability one.

Define  $W$  to be the observed lifelength of the

equipment so that  $P[W > i] = \bar{G}_i \prod_{j=0}^i p_1(S_j)$ , for  $i=0,1,2,\dots$ ,

where  $p_1(S_0)=1$ . Let

$$Y_n = \begin{cases} c_2/n & \text{if } W > n \\ \sum_{i=1}^{[W]} \frac{c_2}{i} \prod_{j=0}^{i-1} p_1(S_j) q_1(S_i) + \frac{c_1}{[W]+1} \prod_{i=0}^{[W]} p_1(S_i) & \text{if } W \leq n. \end{cases}$$

Since the  $Y_n$  are uniformly bounded, condition c. of the lemma holds. We see that if  $n-1 < W$ , then

$$\begin{aligned} E[Y_n | X_1, \dots, X_{n-1}, p_1(S_1), \dots, p_1(S_{n-1})] &= \frac{c_2}{n} F(L-S_{n-1}) + \frac{c_1}{n} [1-F(L-S_{n-1})] \\ &= \frac{c_2}{n} + \frac{(c_1-c_2)}{n} [1-F(L-S_{n-1})], \end{aligned}$$

and if  $n-1 \geq W$ , then

$$E[Y_n | X_1, \dots, X_{n-1}, p_1(S_1), \dots, p_1(S_{n-1})] = \sum_{i=1}^{[W]} \frac{c_2}{i} \prod_{j=0}^{i-1} p_1(S_j) q_1(S_i) + \frac{c_1}{[W]+1} \prod_{i=0}^{[W]} p_1(S_i).$$

Let  $\mathcal{F}_n$  be the  $\sigma$ -field generated by

$X_1, \dots, X_n, p_1(S_1), \dots, p_1(S_n)$ . Then

$$E(Y_n | \mathcal{F}_{n-1}) \begin{cases} < \frac{c_2}{n-1} = Y_{n-1} & \text{if } n-1 < W, 1-F(L-S_{n-1}) < \frac{c_2}{(c_1-c_2)(n-1)}, \\ \geq \frac{c_2}{n-1} = Y_{n-1} & \text{if } n-1 < W, 1-F(L-S_{n-1}) \geq \frac{c_2}{(c_1-c_2)(n-1)}, \\ = Y_{n-1} & \text{if } n-1 \geq W. \end{cases}$$

Let  $n^*$  be as defined above. Then, for all  $n \geq n^*$ , we have  $1-F(L-S_{n-1}) \geq c_2/(c_1-c_2)(n-1)$ , since  $1-F(L-S_{n-1})$  is

increasing in  $n$  and  $c_2/(c_1-c_2)(n-1)$  is decreasing in  $n$ . Define  $N^{**} = \min(n^*-1, [W]+1)$ . Then routine computation shows that condition b. of the lemma is satisfied. Also  $EN^{**} \leq 1 + EW < \infty$ ; thus condition a. is satisfied. Similarly all replacement rules in  $\zeta$  have finite expectations. Thus  $N^{**}$  is the optimal rule. Note that  $N^{**} \leq N^*$ , since there will be more frequent replacements when there is a chance of inspection error.

#### 4. MODEL 3: OPTIMAL INSPECTION SCHEDULE FOR A $RR_2$ DENSITY

##### 4.1. Formulation of the Model.

Barlow, Hunter, and Proschan (1963) presented a model and algorithm to obtain an optimum inspection schedule with results based on the assumption that the system is deteriorating, specifically that the lifelength density function,  $f$ , is a Pólya frequency function of order 2 ( $PF_2$ ). Before presenting the impact of Type II inspection error on this model in Chapter 5, this chapter presents results obtained under the assumption that the system is aging beneficially, specifically, that the lifelength density function  $f(x-y)$  is reverse rule 2 ( $RR_2$ ).

Consider a system for which the lifelength  $T$  has known failure distribution  $F$  with finite mean  $\mu_T$ . Suppose that system failure is discovered only by inspection and possibly only after some time has elapsed since failure. Assume that inspection takes negligible time, does not degrade the system, and system failure cannot occur during inspection. Each inspection costs  $c_1$ , so we do not wish to check too often. On the other hand, there is a cost  $c_2$  per unit of time elapsed between system failure and detection of failure, so that we wish to check often enough to detect

failure quickly. Assume an inspection always follows failure. Upon detection of failure, inspection ends.

Our goal is to determine an inspection policy which minimizes the total expected cost resulting from both inspection and failure.

As shown in Chapter 4, Section 2.4 of Barlow and Proschan (1965), any random schedule may be improved upon by a nonrandom schedule. Thus define a checking procedure by an increasing sequence of nonnegative numbers  $\{x_k\}_{k=0}^{\infty}$ , where  $x_0=0$  and the support of  $F$  is equal to  $[0, \lim_{n \rightarrow \infty} x_n)$ . For  $k=1,2,\dots$ , the  $k$ th check is performed at time  $x_k$  if the system has not failed by the  $(k-1)$ st check. If the system fails at time  $t$ , where  $x_k < t \leq x_{k+1}$ , then the total cost incurred is  $c_1(k+1) + c_2(x_{k+1}-t)$ . Hence the expected cost is

$$\begin{aligned} C &= \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_1(k+1) + c_2(x_{k+1}-t)] dF(t) \\ &= \sum_{k=0}^{\infty} [c_1(k+1) + c_2 x_{k+1}] [F(x_{k+1}) - F(x_k)] - c_2 \mu_T. \end{aligned} \quad (4.1)$$

When  $F$  is a continuous life distribution with a finite mean, then an optimum checking schedule must exist, as proven in Barlow, Hunter, and Proschan (1963). Next they show how the optimum checking schedule may be obtained if the failure density,  $f(x)$ , is PF<sub>2</sub>. We will present analogous results when  $f(x-y)$  is RR<sub>2</sub>. For convenience, the results from Barlow, Proschan, and Hunter (1963) are stated

here first without proofs.

#### 4.2. Results for a PF<sub>2</sub> Density.

From Barlow and Proschan (1981) p. 76, we have the following definition for a Pólya frequency function of order 2.

Definition 4.1. A function  $h(x)$ ,  $-\infty < x < \infty$ , is PF<sub>2</sub> if

(a)  $h(x) \geq 0$  for  $-\infty < x < \infty$ , and

(b) 
$$\begin{vmatrix} h(x_1 - y_1) & h(x_1 - y_2) \\ h(x_2 - y_1) & h(x_2 - y_2) \end{vmatrix} \geq 0$$

for all  $-\infty < x_1 < x_2 < \infty$  and  $-\infty < y_1 < y_2 < \infty$ . Equivalently,

(b')  $\ln h(x)$  is convex on  $(-\infty, \infty)$ , or

(b'') for fixed  $\Delta > 0$ ,  $h(x+\Delta)/h(x)$  is nonincreasing in

$x$  for  $a \leq x \leq b$ , where  $a = \inf_{h(y)>0} y$ ,  $b = \sup_{h(y)>0} y$ .

Thus it can be seen from Definition 4.1 that log-concave functions and PF<sub>2</sub> functions are the same. We define a "strictly PF<sub>2</sub>" function by using a strict inequality,  $>$ , in Definition 4.1 (b).

Assuming a failure density  $f$ , a necessary condition that a sequence  $\{x_k\}$  be a minimum-cost checking procedure is that  $\partial C / \partial x_k = c$  for all  $k$ . Hence using (4.1), we obtain for  $k=1, 2, \dots$ ,

$$\delta_k \equiv (x_{k+1} - x_k) = \frac{F(x_k) - F(x_{k+1})}{f(x_k)} - \frac{c_1}{c_2}. \quad (4.2)$$

When  $f(x_k)=0$ ,  $x_{k+1}-x_k=\infty$ ; thus no more checks are scheduled.

The sequence is determined recursively once we choose  $x_1$ .

The optimum checking policy is denoted  $\{x_k^*\}$ .

Theorem 4.1. If the failure density  $f$  is PF<sub>2</sub>, and  $f(x) > 0$  for  $x > 0$ , the optimum checking intervals are nonincreasing.

Theorem 4.2. Let  $f$  be strictly PF<sub>2</sub> with  $f(x) > 0$  for  $x > 0$ . If  $x_1 > x_1^*$ , then  $\delta_n > \delta_{n-1}$  for some positive integer  $n$ . If  $x_1 < x_1^*$ , then  $\delta_n < 0$  for some positive integer  $n$ .

Algorithm 4.1.

1. To balance the cost of a single check against the expected cost of undetected failure occurring before the first check, choose  $x_1$  to satisfy:

$$c_1 = c_2 \int_0^{x_1} (x_1 - t) dF(t) = c_2 \int_0^{x_1} F(t) dt.$$

2. Compute  $x_2, x_3, \dots$  from (4.2).

3. If  $\delta_k > \delta_{k-1}$ , reduce  $x_1$  and repeat. If  $\delta_k < 0$ , increase  $x_1$  and repeat.

Intuitively these results are not surprising. Since a

PF<sub>2</sub> density has an increasing failure rate distribution (IFR), we need to check more and more often as the system failure rate increases. Similarly, we will show that we need to check less and less frequently if the system failure rate decreases. Specifically, we will show that optimum checking intervals are nondecreasing if the failure density  $f(x-y)$  is RR<sub>2</sub>. Then we will present a procedure to derive the optimum schedule. Finally note the decreasing failure rate (DFR) Weibull, gamma, and exponential distributions all satisfy these conditions; thus for these distributions, nondecreasing checking intervals are optimal.

#### 4.3. Results for a RR<sub>2</sub> Density.

From Karlin (1968) p. 12, we have the following definition for a function which is reverse rule of order 2.

Definition 4.2. Let A and B be subsets of the real line. A function  $K(x,y)$  on  $A \times B$  is said to be reverse rule of order 2 (RR<sub>2</sub>) if

$$\begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) \\ K(x_2, y_1) & K(x_2, y_2) \end{vmatrix} \leq 0$$

for all  $x_1 < x_2$  in A and  $y_1 < y_2$  in B.

We are interested in the special case  $K(x,y) = h(x-y)$ . From the above definition, it is easy to verify for a RR<sub>2</sub>

function  $h(x-y)$ , that  $h(x)$  is log-convex. In other words, by reversing the inequality in Definition 4.1 (b) we obtain the definition of a log-convex function. If strict inequality,  $<$ , holds, then we have a "strictly log-convex" function.

Note that a log-convex function is not necessarily a density function. But if a density function  $f$  is log-convex with support on  $[0, \infty)$ , then the corresponding distribution  $F$  is DFR. The proof is analogous to that for corresponding conditions on a  $PF_2$  density in Barlow and Proschan (1981) p.77.

Part (b) of the definition gives us a very simple way to show when the Weibull and gamma distributions are DFR. The following example is exercise 10 on page 79 in Barlow and Proschan (1981).

Example 4.1. Let  $g$  be the gamma density:

$$g(t; \lambda, \alpha) = \frac{\lambda^\alpha t^{\alpha-1}}{\Gamma(\alpha)} \exp(-\lambda t) \quad \text{for } t \geq 0, \lambda > 0, \alpha > 0.$$

Then  $\ln g(t; \lambda, \alpha) = \ln[\lambda^\alpha / \Gamma(\alpha)] + (\alpha-1) \ln t - \lambda t$ .

Differentiating twice we get

$$\frac{d^2}{dt^2} \ln g(t; \lambda, \alpha) = -(\alpha-1)t^{-2} > 0 \quad \text{for } 0 < \alpha < 1 \\ < 0 \quad \text{for } \alpha > 1.$$

Thus  $g$  is log-convex and  $G$  is DFR for  $0 < \alpha \leq 1$ , while  $g$  is  $PF_2$  (log-concave) and  $G$  is IFR for  $\alpha \geq 1$ .

We need the following lemmas to prove that optimal checking intervals are nondecreasing.

Lemma 4.1. A density  $f$  is log-convex if and only if, for all  $\Delta$ ,  $[F(x+\Delta)-F(x)]/f(x)$  is nondecreasing in  $x$ .

The proof is analogous to that of Theorem 1, Appendix 1 of Barlow, Hunter, and Proschan (1963) where  $f$  is  $PF_2$ .

Equivalent to Lemma 4.1, a density  $f$  is log-convex if and only if for all  $\Delta \geq 0$ ,  $[F(x+\Delta)-F(x)]/f(x)$  is nondecreasing in  $x$ , and for all  $\Delta \geq 0$ ,  $[F(x)-F(x-\Delta)]/f(x)$  is nonincreasing in  $x$ .

Lemma 4.2. If  $f$  is log-convex, where  $x < y$ ,  $a \geq 1$ , and

$$\Delta \geq 0, \text{ then, } a \left[ \frac{F(y)-F(y-\Delta)}{f(y)} \right] < \frac{F(x)-F(x-a\Delta)}{f(x)}.$$

The proof is analogous to the proof of Theorem 3, Appendix 1 of Barlow, Hunter, and Proschan (1963) where  $f$  is  $PF_2$ .

Theorem 4.3. If the failure density  $f$  is log-convex, and  $f(x) > 0$  for  $x > 0$ , then optimum checking intervals are nondecreasing.

Proof. Let  $\{x_k^*\}$  denote the optimum checking policy and  $\{x_k\}$

another checking policy obtained from (4.2). Assume for some  $k$  and  $a > 1$  that  $\delta_{k-1}/\delta_{k-2} = 1/a < 1$ . We will show that this implies  $\delta_k/\delta_{k-1} \leq 1/a$ .

Note that

$$a\delta_k - \delta_{k-1} = a \left[ \frac{F(x_k) - F(x_{k-1})}{f(x_k)} \right] - \left[ \frac{F(x_{k-1}) - F(x_{k-2})}{f(x_{k-1})} \right] + \frac{c_1}{c_2}(1-a).$$

The right hand side is negative by Lemma 4.2 and the assumption that  $a\delta_{k-1} = \delta_{k-2}$ . Hence  $a\delta_{k-1} \leq \delta_{k-2}$  implies that  $a\delta_k \leq \delta_{k-1}$ . Thus we conclude that if for any  $k$ ,  $\delta_k < \delta_{k-1}$ , then  $\delta_n \rightarrow 0$  geometrically fast from some point on as  $n \rightarrow \infty$ .

Next we show  $\delta_k^* < \delta_{k-1}^*$  for some  $k$  contradicts the fact  $\{x_k^*\}$  is an optimum policy. Notice  $x_{n+1}^*$  is the optimum first checking point for the conditional density  $f(t+x_n^*)/\bar{F}(x_n^*)$ . Expected cost for the optimum first check is then,

$$c_1 + c_2 \int_{x_n^*}^{x_{n+1}^*} (x_{n+1}^* - t) \frac{f(t) dt}{\bar{F}(x_n^*)} = c_1 + c_2 \int_0^{\delta_n} (\delta_n - t) \frac{f(t+x_n^*) dt}{\bar{F}(x_n^*)} \rightarrow c_1$$

since  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . This is a contradiction because the expected cost for the optimum first check using the density  $f(t+x_n^*)/\bar{F}(x_n^*)$  is greater than  $c_1$  as follows since  $f' < 0$ :

$$c_1 + c_2 \int_{x_n^*}^{x_{n+1}^*} (x_{n+1}^* - t) \frac{f(t) dt}{\bar{F}(x_n^*)} \geq c_1 + c_2 \frac{f(x_{n+1}^*)}{\bar{F}(x_n^*)} \int_{x_n^*}^{x_{n+1}^*} (x_{n+1}^* - t) dt > c_1.$$

Hence optimum checking intervals are nondecreasing.  $\parallel$

Now since the exponential density is both log-concave and log-convex, applying Theorems 4.1 and 4.3 we obtain the following result, which is stated without proof in Barlow, Proschan, and Hunter (1963).

Corollary 4.1. If  $f(x) = \lambda \exp(-\lambda x)$ ,  $x \geq 0$ ,  $\lambda > 0$ , then optimum checking intervals are a constant  $\delta_i = m$ , where  $m$  is the unique solution of  $\exp(\lambda m) - 1 = \lambda(c_1/c_2 + m)$ .

To obtain an algorithm for computing an optimum checking sequence when  $f$  is log-convex we need the following lemma.

Lemma 4.3. Assume  $f$  is log-convex. Let  $\{x_k\}$  be generated by (4.2) with  $\delta_k > 0$  for  $k=1, 2, \dots, N$  where  $\delta_k = x_{k+1} - x_k$ .

$$\text{Then } \frac{d\delta_k}{dx_1} \leq \frac{f(x_1)}{f(x_k)} \frac{d\delta_1}{dx_1}.$$

The proof is by induction, analogous to the proof of Lemma 1 in Barlow, Hunter, and Proschan (1963).

Theorem 4.4. Let  $f$  be strictly log-convex with  $f(x) > 0$  for  $x > 0$ . If  $x_1 < x_1^*$ , then  $\delta_n < 0$  for some positive integer  $n$ .

The proof is analogous to the proof of Theorem 6 in Barlow, Hunter, and Proschan (1963).

Algorithm 4.2.

1. Choose  $x_1$  as in Algorithm 4.1.
2. Compute  $x_2, x_3, \dots$  recursively from (4.2).
3. If any  $\delta_k < 0$ , increase  $x_1$  and repeat above. If  $\delta_k < \delta_{k-1}$  for some  $k$ , reduce  $x_1$  and repeat above.
4. Repeat to desired level of accuracy.

Example 4.2. Suppose the time to failure has a Weibull density  $f(t) = \alpha \lambda^\alpha t^{\alpha-1} \exp[-(\lambda t)^\alpha]$  with  $\alpha=.5$ ,  $\lambda=.1$ ,  $c_1=10$ , and  $c_2=1$ . Thus  $f$  is DFR with a mean life of 20 and cost per inspection is 10 times the cost per unit of time for undetected failure. By step 1 we estimate  $x_1=17.676$  which gives us all  $\delta_k \geq 0$ . The next estimate  $x_1=8$  results in  $\delta_6 < 0$ . Repeating step 3 eventually yields  $8.546 < x_1^* < 8.547$ . Checking intervals obtained for  $x_1=17.676$ ,  $x_1=8$ , and  $x_1=8.547$  are contained in Tables 6 and 7 below.

Using (4.1) to compute the optimum cost we get  $C=27.746$  when  $x_1^*=8.547$ . To see the impact on cost if we do not follow the optimum sequence, suppose we just change  $x_2^*=26.663$  to  $x_2=20$ . Then the cost computed from (4.1) is 28.314. The cost obtained by using the sequence starting with  $x_1=17.676$  through  $x_5=9.957 \times 10^{46}$  is  $C=45.764$ .

Table 6. Optimum Checking Sequences:  $x_1=17.676$ ,  $x_1=8.000$ 

k	$x_k$	$\delta_k$	$x_k$	$\delta_k$
1	<u>17.676</u>	63.899	<u>8.000</u>	15.866
2	81.575	195.793	23.866	18.313
3	$1.340 \times 10^3$	$1.058 \times 10^3$	42.179	17.250
4	$1.250 \times 10^5$	$1.242 \times 10^5$	59.429	12.828
5	$9.957 \times 10^{46}$	$9.957 \times 10^{46}$	72.258	5.289
6	$9.957 \times 10^{46}$	0	77.547	-4.348

Table 7. Optimum Checking Sequence for  $x_1=8.547$ 

k	$x_k$	$\delta_k$	k	$x_k$	$\delta_k$
1	<u>8.547</u>	18.116	16	593.556	55.552
2	26.663	23.660	17	649.108	58.091
3	50.323	27.740	18	707.199	61.149
4	78.063	31.041	19	768.348	64.973
5	109.105	33.853	20	833.321	69.926
6	142.958	36.328	21	903.247	76.564
7	179.286	38.558	22	979.811	85.779
8	217.845	40.606	23	$1.07 \times 10^3$	99.089
9	258.451	42.519	24	$1.17 \times 10^3$	119.249
10	300.970	44.335	25	$1.28 \times 10^3$	151.883
11	345.305	46.090	26	$1.43 \times 10^3$	210.089
12	391.395	47.823	27	$1.65 \times 10^3$	331.771
13	439.218	49.576	28	$1.98 \times 10^3$	674.578
14	489.794	51.400	29	$2.65 \times 10^3$	$2.67 \times 10^3$
15	540.194	53.362	30	$5.32 \times 10^3$	$4.09 \times 10^5$

Next in Chapter 5, we modify the model by introducing the possibility of a Type II inspection error.

5. MODEL 4: OPTIMAL INSPECTION SCHEDULE  
WITH TYPE II INSPECTION ERROR

5.1. Formulation of the Model.

Consider a system for which the lifelength  $T$  has known failure distribution  $F$  with finite mean  $\mu_T$ . Model 4 is identical to Model 3 in the last chapter except that now we let  $p_2 < 1$  be the probability that failure is detected upon inspection and  $q_2 = 1 - p_2$  be the probability that we fail to detect a system failure at an inspection. If the system fails at time  $t$ , where  $x_k < t \leq x_{k+1}$ , then the probability the failure is detected at time  $x_{k+i}$  is  $p_2 q_2^{i-1}$ , and the cost incurred is then  $c_1(k+i) + c_2(x_{k+i}-t)$ . Hence the expected cost is

$$\begin{aligned}
 C_{p_2} &= \sum_{i=1}^{\infty} p_2 q_2^{i-1} \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+i}} [c_1(k+i) + c_2(x_{k+i}-t)] dF(t) \\
 &= -c_2 \mu_T + \sum_{k=0}^{\infty} \left[ c_1 k + \frac{c_1}{p_2} + c_2 \sum_{i=1}^{\infty} p_2 q_2^{i-1} x_{k+i} \right] [F(x_{k+1}) - F(x_k)].
 \end{aligned} \tag{5.1}$$

5.2. Existence of an Optimal Solution.

The proof of the existence of an optimal checking sequence follows the proof for  $p_2=1$  in Barlow, Hunter, and Proschan (1963) with some modifications. Given that a failure is detected at the  $j$ th inspection following failure

with probability  $p_2=1$  the expected cost is

$$C_j = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_1(k+j) + c_2(x_{k+j}-t)] dF(t).$$

It can be shown that an upper bound for the expected cost is  $c_1 j + 2\sqrt{c_1 c_2 \alpha j}$ , following the proof of Lemma, p.1080, Barlow, Hunter, and Proschan (1963).

Next we can show that  $L_j(x)$ , the infimum of expected cost over all policies given that failure is detected at the  $j$ th inspection and given that the component is of initial age  $x$ , is continuous. The proof follows the proof of Theorem 1, p. 1081, Barlow, Hunter, and Proschan (1963). Then the infimum of expected cost over all policies,  $p_2 \leq 1$ , given the component is of initial age  $x$  is

$$L_{p_2}(x) = \sum_{j=1}^{\infty} p_2 q_2^{j-1} L_j(x).$$

By the Weierstrass M-test,  $\sum_{j=1}^{\infty} p_2 q_2^{j-1} L_j(x)$  converges uniformly to  $L_{p_2}(x)$ , and since each  $p_2 q_2^{j-1} L_j(x)$  is continuous, then  $L_{p_2}(x)$  is continuous. The remainder of the proof follows the proof of Theorem 1, p. 1081, Barlow, Hunter, and Proschan (1963).

A necessary condition for a sequence  $\{x_k\}$  to be a minimum cost checking procedure is that  $\partial C_{p_2} / \partial x_k = 0$  for all  $k$ . Hence using (5.1), we obtain

$$\delta_k = x_{k+1} - x_k = N_k - q_2 N_{k+1} - \frac{c_1}{c_2},$$

$$\text{where } N_k \equiv \sum_{i=1}^k \frac{[F(x_i) - F(x_{i-1})] q_2^{k-i}}{f(x_k)}. \quad (5.2)$$

The sequence is determined recursively once we choose  $x_1$ .

### 5.3. Schedule for Systems with Finite Lifelength.

Realistically, no system lives forever without repair or replacement. If it is known that a system lifelength must be less than  $T < \infty$ , we can give necessary and sufficient conditions on the failure distribution  $F$  to ensure that an optimal checking policy consists of a single check at time  $T$ . Maintaining the same assumptions as given in the previous model, we assume a check always follows a failure.

Theorem 5.1. Let  $F(t)=1$  for  $t \geq T$ . If

$$F(t) \leq \frac{1}{P_2 \left[ 1 + \frac{c_2}{c_1} (T-t) \right]} \quad (5.3)$$

for  $0 \leq t \leq T$ , then the optimum policy consists of a single check at time  $T$ . Conversely, if

$$F(t) > \frac{1}{P_2 \left[ 1 + \frac{c_2}{c_1} (T-t) \right]} \quad (5.4)$$

for some  $0 \leq t \leq T$ , then the optimum policy will require in

addition to the check at time T one or more checks before time T.

Proof. If a single check is performed at time T, the expected cost is

$$D_1 = c_1 + c_2 \int_0^T (T-t) dF(t) = c_1 + c_2 \int_0^T F(t) dt.$$

If an additional check is performed at time x, then the expected cost is

$$D_2(x) = p_2 \int_0^x [c_1 + c_2(x-t)] dF(t) + q_2 \int_0^x [2c_1 + c_2(T-t)] dF(t) + \int_x^T [2c_1 + c_2(T-t)] dF(t).$$

Thus a single check at time T is preferable if for all  $0 \leq x \leq T$ ,  $D_2(x) - D_1 \geq 0$ ; or equivalently,

$$D_2(x) - D_1 = c_1[1 - p_2 F(x)] - c_2(T-x)p_2 F(x) \geq 0,$$

which gives us (5.3).

Now assuming a single check at T is preferred to checks at x and T, we show that a single check at T is preferable to a check at time T preceded by two checks at, say, times  $x_1$  and  $x_2$ . Define  $\underline{x}_k = (x_1, \dots, x_k)$ . Let  $D_3(\underline{x}_2)$  be the cost of checking at times  $x_1$ ,  $x_2$ , and T. Given that a check has occurred at time  $x_1$ , a single check before T, namely at time  $x_2$ , is warranted if and only if  $D_3(\underline{x}_2) - D_2(\underline{x}_1)$

$< 0$ , or equivalently,

$$D_3(\underline{x}_2) - D_2(\underline{x}_1) = c_1[1 - p_2 F(x_2) - p_2 q_2 F(x_1)] - c_2 p_2 (T - x_2) [F(x_2) - p_2 F(x_1)] < 0.$$

This gives us

$$\frac{F(x_2) - p_2 F(x_1)}{1 - p_2 F(x_1)} > \frac{1}{p_2 [1 + (T - x_2) c_2 / c_1]}$$

for some  $x_1 \leq x_2 \leq T$ . But the above implies

$$F(x_2) > 1 / p_2 [1 + (T - x_2) c_2 / c_1] \text{ since}$$

$[1 - p_2 F(x_1)] F(x_2) > F(x_2) - p_2 F(x_1)$ . But this contradicts our assumption that  $F(x) \leq 1 / p_2 [1 + (T - x) c_2 / c_1]$  for  $0 \leq x \leq T$ .

Similarly it is easy to show that (5.3) implies a single check at time  $T$  is preferable to  $k$  checks before  $T$  at times  $x_1, x_2, \dots, x_k$ . The expected cost of performing  $k+1$  checks at times  $x_1, x_2, \dots, x_k, T$  is

$$D_{k+1}(\underline{x}_k) = D_k(\underline{x}_{k-1}) + c_1(1 - M_{k+1}) - c_2 p_2 (T - x_k) [F(x_k) - M_k] \quad (5.5)$$

where  $M_1 = 0$ ,  $M_k = p_2 \sum_{i=1}^{k-1} F(x_{k-i}) q_2^{i-1}$  for  $k=2, 3, \dots$

Given that checks have occurred at times  $x_1, x_2, \dots, x_{k-1}$ , an additional check at some time  $x_k$ , where  $x_{k-1} < x_k < T$ , is warranted if and only if  $D_{k+1}(\underline{x}_k) - D_k(\underline{x}_{k-1}) < 0$ , or equivalently,

$$\frac{F(x_k) - M_k}{1 - M_k} > \frac{1}{p_2 [1 + (T - x_k) c_2 / c_1]}.$$

But the above inequality and the fact that  $F(x_{k-1}) > M_k$  imply

$$\frac{F(x_k) - F(x_{k-1})}{1 - F(x_{k-1})} > \frac{1}{p_2[1 + (T - x_k)c_2/c_1]},$$

which in turn implies  $F(x_k) > 1/p_2[1 + (T - x_k)c_2/c_1]$ ,

contradicting our initial assumption that a single check at  $T$  is preferred. Conversely if (5.3) is not true for some  $x$  in  $[0, T]$ , then checks at times  $x$  and  $T$  will yield a lower expected cost than does a single check at  $T$ . Thus the optimum checking policy will require one or more checks before  $T$ . ||

Inequality (5.4) suggests that we would prefer to conduct more checks than just a single check at  $T$  if the cost per inspection,  $c_1$ , is sufficiently small compared to the cost of failure,  $c_2$ , or if the probability of detecting a failure,  $p_2$ , is sufficiently large.

We can compute a checking sequence,  $x_1, x_2, \dots, x_n = T$  and the expected cost,  $D_n(x_{n-1})$ , recursively from (5.2) and (5.5) once we have chosen  $x_1$ . If our choice of  $x_1$  is greater than the unknown optimal value  $x_1^*$ , then the checking sequence computed from (5.2) will not contain an inspection time  $x_n = T$ , but rather will contain an  $x_n$  exceeding  $T$  for some  $n$ . On the other hand if our choice of  $x_1$  is less than the optimal  $x_1^*$ , the checking sequence computed from (5.2) will have  $x_k < T$  for all  $k$ , and for

some  $n$  we will obtain  $x_{n+1} < x_n$ . A reasonable first guess for  $x_1$  would be the same as in Algorithm 4.1 where we balance the expected cost of a failure occurring before  $x_1$  against the cost of a single inspection at  $x_1$ .

Example 5.1. Suppose it is known that a system will not function after time  $T=10$ . Assume the life distribution of the system can be represented by an exponential distribution truncated at  $T=10$  with parameters  $\lambda=.1$  and cost factors  $c_1=1$  and  $c_2=10$ . Then

$$\begin{aligned} f(x) &= .1e^{-.1x} / \int_0^{10} .1e^{-.1t} dt \quad \text{for } 0 \leq x \leq 10 \\ &= .1e^{-.1x} / (1 - e^{-1}). \end{aligned}$$

Let the probability of failure detection be  $p_2=.9$ . Our first guess for  $x_1$  is 1.146 which is too large. A second guess of  $x_1=1.0$  is too small. Continuing to choose values for  $x_1$  halfway between a value which is too small and a value which is too large, we can obtain an optimal value for  $x_1$  to whatever degree of accuracy is required to make the sequence end at  $T=10$ . The sequences obtained using  $x_1=1.146$ ,  $x_1=1.0$ , and  $x_1=1.0933$  and their associated costs are given in Table 8 on the next page.

Using Theorem 5.1 we can show when  $p_2=.9$  and  $x_1=1.09330$  that 17 inspections are preferred to 16, but 18 inspections are not preferred to 17. So the optimal checking schedule requires 17 inspections, resulting in an expected cost of

Table 8. Checking Schedules  $\{x_i\}$  and Costs  $\{D_i(x)\}$  for a Truncated Exponential  $\lambda=.1$ ,  $T=10$ ,  $c_1=1$ ,  $c_2=10$ , for  $p_2=.9$

i	$x_i$	$D_i$	$x_i$	$D_i$	$x_i$	$D_i$
1	<u>1.146</u>	59.198	<u>1.000</u>	59.198	<u>1.09330</u>	59.198
2	2.142	46.394	1.852	47.868	2.03695	46.917
3	3.109	36.435	2.661	38.937	2.94612	37.314
4	4.044	28.790	3.423	31.960	3.81743	29.889
5	4.944	23.017	4.134	26.563	4.64732	24.223
6	5.805	18.744	4.790	22.435	5.43207	19.967
7	6.624	15.664	5.387	19.317	6.16783	16.829
8	7.397	13.517	5.921	16.995	6.85067	14.568
9	8.120	12.091	6.389	15.295	7.47663	12.984
10	8.789	11.210	6.787	14.075	8.04179	11.913
11	9.400	10.730	7.112	13.226	8.54233	11.222
12	9.949	10.532	7.361	12.660	8.97461	10.803
13	10.432	10.522	7.532	12.314	9.33526	10.570
14	10.846	10.523	7.623	12.144	9.62127	10.457
15			7.633	12.124	9.83006	10.413
16			7.561	12.246	9.95956	10.402
17			7.407	12.520	10.00828	10.40197

10.40197 versus the cost of 59.198 if only a single inspection is performed at  $T=10$ . Under the optimal schedule for  $p_2=1$ , we obtain  $x_1=1.0859344$  and only 15 inspections are required with an expected cost of 9.436. Thus when  $p_2 < 1$  our initial inspection time,  $x_1=1.09330$ , is later than when  $p_2=1$ ,  $x_1=1.0859344$ . But we must inspect more frequently when  $p_2 < 1$ , resulting in a higher expected cost. The optimal inspection schedule and cost when  $p_2=1$  are given in Table 9 on the next page.

Table 9. Checking Schedules  $\{x_i\}$  and Costs  $\{D_i(x)\}$  for a Truncated Exponential  $\lambda=.1$ ,  $T=10$ ,  $c_1=1$ ,  $c_2=10$ , for  $p_2=1$

i	$x_i$	$D_i$	i	$x_i$	$D_i$
1	1.0859344	59.198	9	7.9760442	11.137
2	2.130250	45.524	10	8.5363479	10.315
3	3.1369001	35.121	11	9.0126459	9.839
4	4.0928928	27.317	12	9.4004691	9.590
5	4.9960732	21.559	13	9.8957602	9.479
6	5.8412965	17.397	14	9.9976199	9.442
7	6.6232679	14.462	15	10.0000002	9.436
8	7.336261	12.455	16		

#### 5.4. Optimal Checking Intervals for an Exponential Density when $0 < p_2 < 1$ .

Due to the memoryless property of the exponential distribution, one would expect optimal checking intervals to be constant. But Sengupta (1982) showed that the first checking interval is longer, solving the problem as a continuous-time Markov decision process. Denote the first checking interval by  $\delta_0=x_1$  and set  $\delta_i=m$  for  $i=1,2,\dots$ , then we can solve for  $m$  from equation (5.1) and  $\partial C_p/\partial x_1 = 0$  and  $\partial C_p/\partial x_2 = 0$ . Let  $F(t) = 1 - \exp(-\lambda t)$ ; then

$$\begin{aligned} \lambda m &= -\ln(q_2 + p_2 e^{-\lambda x_1}) \\ \lambda x_1 &= \ln \left[ 1 + \left( \frac{c_1}{c_2} + m \right) \frac{\lambda}{p_2} \right]. \end{aligned} \quad (5.6)$$

Solving for  $m$ , we get

$$\lambda m = \ln \left[ \frac{c_1}{c_2} + m + \frac{P_2}{\lambda} \right] - \ln \left[ q_2 \frac{c_1}{c_2} + q_2 m + \frac{P_2}{\lambda} \right]. \quad (5.7)$$

Thus our optimal checking sequence is  $x_1, x_1+m, x_1+2m, \dots$

and the expected cost is

$$C_{p_2} = -\frac{c_2}{\lambda} + c_2(x_1-m) + (c_1+c_2m) \left[ \frac{1}{P_2} + e^{-\lambda x_1} \left( \frac{e^{\lambda m}}{e^{\lambda m}-1} \right) \right]. \quad (5.8)$$

Example 5.2. Let  $F$  be an exponential distribution with  $\lambda=.01$ . Assume the probability of detecting failure at inspection is  $p_2=.9$ , the cost per unit of time for undetected failure is  $c_2=1$  and the cost per inspection is  $c_1=10$ . Then using (5.6) and (5.7) we obtain  $x_1=41.874$  and  $m=36.805$ . The expected cost from (5.8) is  $C_{p_2} = 57.075$ .

Table 10 below compares results for different values of  $p_2$  when  $\lambda=.01$ ,  $c_1=10$ , and  $c_2=1$ . Thus the smaller the probability of detecting a failure, the longer we wait to perform the first inspection, but then the remaining inspection intervals  $\delta_i=m$  are shorter for smaller  $p_2$ . This reflects the lack of confidence in the ability to detect a failure, and as  $p_2$  decreases, the expected cost  $C_{p_2}$  increases. These relationships are verified by finding the derivative with respect to  $p_2$  of  $C_{p_2}$ ,  $x_1$ , and  $m$ .

Table 10. Optimal Schedules and Costs for an Exponential Density  $\lambda=.01$ ,  $c_1=10$ ,  $c_2=1$

$p_2$	$x_1$	$m$	$Cp_2$
1.0	41.622	41.622	51.622
.9	41.874	36.805	57.075
.8	42.630	32.526	63.262
.7	43.936	28.619	70.487
.6	45.913	24.961	79.220
.5	48.799	21.452	90.251
.4	53.056	17.995	105.048
.3	59.663	14.480	126.781
.2	71.141	10.737	164.091
.1	97.138	6.416	254.881

5.5. Optimal Checking Intervals for a  $PF_2$  Density when  $0 < p_2 < 1$ .

For  $p_2=1$ , it is proven in Theorem 5, p. 1088 of Barlow, Hunter, and Proschan (1963) that optimal checking intervals are nonincreasing if the failure density  $f$  is  $PF_2$ .

For  $0 < p_2 < 1$ , the same conclusion, that optimal intervals are nonincreasing, is proven here when the failure density is  $PF_2$ . Parmigiani (1990d) formulated a model incorporating both Type I and Type II inspection errors. A Type I error occurs when an inspector mistakenly says that a system has failed when it is in fact functioning. A Type II error occurs when an inspection fails to detect a malfunctioning system. Parmigiani

(1990d) found that the problem of minimizing the long-run expected cost depends on the probability of a Type I error only as a function of the constant cost factors. Thus for simplicity of notation, we will use our Model 4, which uses only Type II error. First we present monotonicity properties of the model in Section 5.5.1. In Section 5.5.2, we prove that when the failure density is  $PF_2$ , optimal inspection intervals are nonincreasing. In Section 5.5.3, we obtain an algorithm for computing optimal intervals. Finally, we derive majorization results for an optimal inspection policy in Section 5.5.4.

Recall that in Model 4 a system has lifelength  $T$  with known failure distribution  $F$  with finite mean  $\mu_T$ . The probability of detecting the failure of the system upon inspection is a constant  $p_2$ . We denote a sequence of inspection times by  $\{x_k\}$  for  $k=0,1,\dots$  and  $x_0 \equiv 0$ . Failure of the system can only be detected at one of these inspection times. Let  $S$  be the time when failure is detected. Then

$$\begin{aligned}
 P(S = x_k) &= P(x_{k-1} < S \leq x_k) \\
 &= \sum_{i=0}^{k-1} p_2 q_2^{k-1-i} P(x_i < T \leq x_{i+1}) \\
 &= \sum_{i=0}^{k-1} p_2 q_2^{k-1-i} [F(x_{i+1}) - F(x_i)].
 \end{aligned} \tag{5.9}$$

Denote the failure distribution of  $S$  by  $G$ . Then

$$G(x_k) = P(S \leq x_k) = \sum_{j=1}^k \sum_{i=0}^{j-1} p_2 q_2^{j-1-i} [F(x_{i+1}) - F(x_i)]. \tag{5.10}$$

### 5.5.1 Monotonicity Properties.

It is reasonable to conjecture that as the probability of detecting a failure decreases, then the expected time to detect the failure will increase. We show that the expected time of failure detection,  $\mu_S$ , is a decreasing function of  $p_2$  as follows.

$$\mu_S = E(S) = \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} p_2 q_2^{i-1} x_{k+i} [F(x_{k+1}) - F(x_k)].$$

$$\frac{d\mu_S}{dp_2} = \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} x_{k+i} [F(x_{k+1}) - F(x_k)] q_2^{i-2} (1-p_2 i).$$

By definition  $x_{k+i} = x_k + \delta_k + \delta_{k+1} + \dots + \delta_{k+i-1}$ .

Let  $\delta_m = \max_{k \leq j \leq k+i-1} \delta_j$ . Then  $x_{k+i} < x_k + i\delta_m$ .

Applying this to  $d\mu_S/dp_2$  we get

$$\frac{d\mu_S}{dp_2} < \sum_{i=1}^{\infty} q_2^{i-2} (1-p_2 i) \sum_{k=0}^{\infty} (x_k + i\delta_m) [F(x_{k+1}) - F(x_k)]$$

$$= \sum_{k=0}^{\infty} [F(x_{k+1}) - F(x_k)] \left[ -\frac{\delta_m}{p_2^2} \right] < 0.$$

By a similar proof we can show that  $C_{p_2}$  is a decreasing function of  $p_2$ . Again it is reasonable that the long-run expected cost would decrease as the expected time to detect the failure decreases.

Another way to view the impact of imperfect inspection on the long-run expected cost  $C_{p_2}$  is that,  $C_{p_2}$  increases as

the difference between the expected time to failure-detection,  $\mu_S$ , and the expected time to failure,  $\mu_T$ , increases. This can be seen by writing

$$C_{p_2} = c_1 \sum_{k=0}^{\infty} k [F(x_{k+1}) - F(x_k)] + \frac{c_1}{p_2} + c_2 (\mu_S - \mu_T).$$

### 5.5.2. Optimal Intervals are Nonincreasing for a PF<sub>2</sub> Failure Density.

When  $p_2=1$ , Barlow, Hunter, and Proschan (1963) proved intervals are nonincreasing for a PF<sub>2</sub> density. The proof by contradiction showed that if inspection intervals began increasing at some point in time, then they continued to increase geometrically, resulting in the expected cost also increasing to infinity. But when  $0 < p_2 < 1$ , an increase in inspection intervals at some point in time does not result in intervals increasing to infinity. However, we will prove here that the intervals between expected detection times, conditional on time of failure, increase geometrically and this is sufficient to provide the proof by contradiction.

Given an inspection sequence  $\{x_k\}$ , the expected time of detection of failure,  $S$ , conditioned on the time of failure,  $T$ , occurring in an particular interval is

$$z_k \equiv E(S | x_{k-1} < T \leq x_k) = \sum_{i=0}^{\infty} p_2 q_2^i x_{k+i} \quad k=1, 2, \dots \quad (5.11)$$

The inspection times,  $x_k$ , and the conditional, expected detection times,  $z_k$ , are related by  $z_k = p_2 x_k + q_2 z_{k+1}$ ; equivalently,

$$x_k = \frac{1}{p_2} z_k - \frac{q_2}{p_2} z_{k+1}. \quad (5.12)$$

The optimal inspection sequence  $\{x_k\}$  minimizes the expected cost function (5.1) which by applying (5.11) can be written as follows

$$C_{p_2} = -c_2 \mu_T + \sum_{k=0}^{\infty} [c_1 k + \frac{c_1}{p_2} + c_2 z_{k+1}] [F(x_{k+1}) - F(x_k)]. \quad (5.13)$$

Furthermore applying (5.12) to the optimal inspection intervals  $\delta_k$  in (5.2) we obtain optimal intervals between conditional, expected detection times for  $k = 1, 2, \dots$

$$\gamma_k \equiv z_{k+1} - z_k = \sum_{i=1}^k p_2 q_2^{k-i} \frac{[F(x_i) - F(x_{i-1})]}{f(x_k)} - \frac{c_1}{c_2}. \quad (5.14)$$

Iteratively substituting the  $k$ th condition in the  $(k+1)$ st gives

$$\begin{aligned} \gamma_1 &= p_2 \frac{F(x_1)}{f(x_1)} - \frac{c_1}{c_2}, \quad \text{and for } k=2, 3, \dots \\ \gamma_k &= p_2 \frac{F(x_k) - F(x_{k-1})}{f(x_k)} + q_2 \frac{f(x_{k-1})}{f(x_k)} \left[ \gamma_{k-1} + \frac{c_1}{c_2} \right] - \frac{c_1}{c_2} \end{aligned} \quad (5.15)$$

Then from (5.12) and (5.14) we can write

$$\gamma_k = p_2 \delta_k + q_2 \gamma_{k+1} \quad k=1, 2, \dots \quad (5.16)$$

To prove that optimal inspection intervals,  $\delta_k$ , are nonincreasing for a  $PF_2$  density, we need the following facts and lemmas. We assume that  $f$  is  $PF_2$  and  $f'$  exists everywhere. Let  $m$  denote the mode of  $f$  which is unique by Barlow and Proschan (1965) p.230.

Clearly when  $f$  is a  $PF_2$  density and  $f'$  exists everywhere, then  $f^2(x) \geq f'(x)F(x)$ . As a consequence, we have

$$\begin{aligned} f'(x) > 0 &\Rightarrow f(x)/f'(x) \geq F(x)/f(x) \\ f'(x) < 0 &\Rightarrow f(x)/f'(x) \leq F(x)/f(x). \end{aligned}$$

Furthermore, we know when  $f$  is a  $PF_2$  density and  $f'$  exists everywhere, if  $x < y$ , then  $f'(x)/f(x) \geq f'(y)/f(y)$ . As a consequence of this fact, if  $x_1 < x_2 < m$  (mode of  $f$ ), or  $m < x_1 < x_2$  then,  $f(x_2)/f'(x_2) \geq f(x_1)/f'(x_1)$ . The inequality is reversed when  $x_1 < m$  and  $x_2 > m$ .

Lemma 5.1. Let  $\{x_k\}$  be an inspection sequence generated by (5.2). Then  $f_k/f_{k-1} > q_2^2$  for  $k=2,3,\dots$ .

Proof. By definition

$$\delta_k = \left[ \frac{F_{k-1} - p_2 F_{k-2} - \dots - p_2 q_2^{k-3} F_1}{f_{k-1}} \right] - q_2 \left[ \frac{F_k - p_2 F_{k-1} - \dots - p_2 q_2^{k-2} F_1}{f_k} \right] - \frac{c_1}{c_2}.$$

$$\text{Then } \frac{f_k}{f_{k-1}} = \frac{f_k \left[ \delta_{k-1} + \frac{c_1}{c_2} \right] + q_2 [F_k - p_2 F_{k-1} - \dots - p_2 q_2^{k-2} F_1]}{F_{k-1} - p_2 F_{k-2} - \dots - p_2 q_2^{k-3} F_1}.$$

Thus  $f_k/f_{k-1} > q_2^2$  if and only if

$$f_k \left[ \delta_{k-1} + \frac{c_1}{c_2} \right] + q_2 [F_k - p_2 F_{k-1} - \dots - p_2 q_2^{k-2} F_1] > q_2^2 [F_{k-1} - p_2 F_{k-2} - \dots - p_2 q_2^{k-3} F_1]$$

if and only if  $f_k \left[ \delta_{k-1} + \frac{c_1}{c_2} \right] > (q_2^2 + p_2 q_2) F_{k-1} - q_2 F_k = q_2 (F_{k-1} - F_k)$ .

The last inequality holds since the LHS is positive and the RHS is negative. ||

Using (5.15) and (5.16) we can write  $\gamma_k$  as a function of  $x_k$  and  $\delta_{k-1}$  as follows:

$$\gamma_k = \frac{p_2 \frac{[F(x_k) - F(x_k - \delta_{k-1})]}{f(x_k)} + q_2 \frac{f(x_k - \delta_{k-1})}{f(x_k)} \left[ p_2 \delta_{k-1} + \frac{c_1}{c_2} \right] - \frac{c_1}{c_2}}{1 - q_2^2 \frac{f(x_k - \delta_{k-1})}{f(x_k)}} \quad (5.17)$$

$$\equiv u(x_k, \delta_{k-1}), \quad k \geq 1.$$

**Lemma 5.2.** At the optimum, if  $x_k < m$  (the mode of  $f$ ), then  $\gamma_k < \gamma_{k-1}$ , for  $k > 1$ .

**Proof.** Using the notation  $F_k \equiv F(x_k)$  and  $f_k \equiv f(x_k)$ , from (5.17) we have

$$\gamma_k = \frac{p_2 [F_k - F_{k-1}] + q_2 f_{k-1} \left[ p_2 \delta_{k-1} + \frac{c_1}{c_2} \right] - \frac{c_1}{c_2} f_k}{f_k - q_2^2 f_{k-1}}.$$

For  $x_k < m$ , clearly  $F_k - F_{k-1} < f_k \delta_{k-1}$ , thus

$$\gamma_k < \frac{p_2 f_k \delta_{k-1} + q_2 f_{k-1} \left[ p_2 \delta_{k-1} + \frac{c_1}{c_2} \right] - \frac{c_1}{c_2} f_k}{f_k - q_2^2 f_{k-1}}.$$

Applying (5.16) to  $p_2 \delta_{k-1}$  on the RHS we get

$$\gamma_k < \frac{(\gamma_{k-1} - q_2 \gamma_k) (f_k + q_2 f_{k-1}) - \frac{c_1}{c_2} (f_k - q_2 f_{k-1})}{f_k - q_2^2 f_{k-1}}.$$

Since  $f_k > f_{k-1}$ , the last term in the numerator is positive

so that  $\gamma_k < (\gamma_{k-1} - q_2 \gamma_k) (f_k + q_2 f_{k-1}) / (f_k - q_2^2 f_{k-1})$ .

Rearranging we get  $\gamma_k < \gamma_{k-1} \left[ \frac{1}{1 + q_2} \right] \left[ 1 + q_2 \frac{f_{k-1}}{f_k} \right]$ . The desired

result follows from our assumption  $f_{k-1} < f_k$ .  $\parallel$

The following two lemmas show how  $\gamma_k$  increases geometrically.

Lemma 5.3. At optimum,  $\delta_k > \delta_{k-1}$  implies  $\gamma_{k+1} > \gamma_k$ ,  $k \geq 1$ .

Proof.  $\gamma_{k+1} - \gamma_k = u(x_{k+1}, \delta_k) - u(x_k, \delta_{k-1})$ . The result follows when we show  $\partial u / \partial x_k \geq 0$  and  $\partial u / \partial \delta_{k-1} > 0$ . For  $k=1$ ,  $\delta_0 = x_1$  and  $\gamma_1 = p_2 F_1 / f_1 - c_1 / c_2$ . From Barlow, Proschan, and Hunter (1963) p.1087,  $F(x)/f(x)$  is increasing in  $x$  for a PF<sub>2</sub> density; thus  $\partial u / \partial x_1 = \partial u / \partial \delta_0 > 0$ . Next we prove the results for  $k > 1$ .

From Barlow and Proschan (1965) p. 229, for fixed  $\delta_{k-1}$ ,  $[F(x_k) - F(x_k - \delta_{k-1})] / f(x_k)$  and  $f(x_k - \delta_{k-1}) / f(x_k)$  are nondecreasing

in  $x_k$  for a  $PF_2$  density  $f$ . Thus the denominator of  $u$  is nonincreasing in  $x_k$  and the numerator of  $u$  is nondecreasing, hence  $\partial u / \partial x_k \geq 0$ .

To show  $\partial u / \partial \delta_{k-1} > 0$  we must consider whether  $x_{k-1}$  is greater than the mode of  $f$  or not. In the first case assume  $f'(x_{k-1}) \leq 0$ . Taking the partial derivative,

$$\frac{\partial u}{\partial \delta_{k-1}} = \frac{[p_2(1+q_2)f_{k-1} - q_2 f'_{k-1} \left[ p_2 \delta_{k-1} + \frac{c_1}{c_2} \right]] [f_k - q_2^2 f_{k-1}]}{[f_k - q_2^2 f_{k-1}]^2} - \frac{[p_2(F_k - F_{k-1}) + q_2 f_{k-1} \left[ p_2 \delta_{k-1} + \frac{c_1}{c_2} \right] - \frac{c_1}{c_2} f_k] [q_2^2 f'_{k-1}]}{[f_k - q_2^2 f_{k-1}]^2}.$$

Simplifying we get

$$\frac{\partial u}{\partial \delta_{k-1}} = \frac{p_2(1+q_2)f_{k-1}}{f_k - q_2^2 f_{k-1}} - q_2 f'_{k-1} \frac{p_2 \delta_{k-1} + \frac{c_1}{c_2} + q_2 u(x_k, \delta_{k-1})}{f_k - q_2^2 f_{k-1}}. \quad (5.18)$$

The denominator is positive by Lemma 5.1, and by assumption  $f'_{k-1} \leq 0$ , thus

$$\frac{\partial u}{\partial \delta_{k-1}} \geq \frac{p_2(1+q_2)f_{k-1}}{f_k - q_2^2 f_{k-1}} > 0. \quad (5.19)$$

For the second case assume  $f'_{k-1} > 0$ . From (5.18)

$\partial u / \partial \delta_{k-1} \geq 0$  if and only if

$$p_2(1+q_2)f_{k-1} \geq q_2 f'_{k-1} \left[ p_2 \delta_{k-1} + \frac{c_1}{c_2} + q_2 u(x_k, \delta_{k-1}) \right].$$

Using (5.16), the above inequality is equivalent to

$$p_2(1+q_2) \frac{f_{k-1}}{f'_{k-1}} \geq q_2 \left( \gamma_{k-1} + \frac{c_1}{c_2} \right).$$

By Lemma 5.2,  $\gamma_{k-1} \leq \gamma_1$ ,  $k > 1$ . Since  $f$  is  $PF_2$  and  $f'$  exists everywhere, we have  $f_{k-1}/f'_{k-1} \geq f_1/f'_1 > F_1/f_1$ . Then applying (5.15) we get

$$q_2 \left[ \gamma_{k-1} + \frac{c_1}{c_2} \right] \leq q_2 \left[ \gamma_1 + \frac{c_1}{c_2} \right] = q_2 p_2 \frac{F_1}{f_1} < p_2 q_2 \frac{f_{k-1}}{f'_{k-1}} < p_2 (1+q_2) \frac{f_{k-1}}{f'_{k-1}}.$$

Therefore  $\partial u / \partial \delta_{k-1} > 0$ .  $\parallel$

Let  $y_0$  be such that  $f''(y) > 0$  for  $y > y_0 > m$ , (the mode of  $f$ ). Such  $y_0$  exists since  $f(y) > 0$  for  $y > 0$ .

Lemma 5.4. At the optimum, for  $a > 1$  and  $x_{k-1} > y_0$ ,

$\delta_k = a\delta_{k-1}$  implies  $\gamma_{k+1} > a\gamma_k$ ,  $k \geq 1$ .

Proof. From Lemma 5.3,  $\partial u / \partial \delta_{k-1} > 0$  and  $\partial u / \partial x_k \geq 0$ . We will prove that  $\partial^2 u / \partial \delta_{k-1}^2 > 0$  then the desired conclusion obtains from

$$\gamma_{k+1} = u(x_{k+1}, \delta_k) = u(x_{k+1}, a\delta_{k-1}) \geq u(x_k, a\delta_{k-1}) > a u(x_k, \delta_{k-1}) = a\gamma_k.$$

To prove  $\partial^2 u / \partial \delta_{k-1}^2 > 0$ , write

$$u(x_k, \delta_{k-1}) = v(x_k, \delta_{k-1}) + w(x_k, \delta_{k-1}) u(x_k, \delta_{k-1}),$$

where

$$v(x_k, \delta_{k-1}) = p_2 \frac{F(x_k) - F(x_k - \delta_{k-1})}{f(x_k)} + q_2 \frac{f(x_k - \delta_{k-1})}{f(x_k)} \left[ p_2 \delta_{k-1} + \frac{c_1}{c_2} \right] + \frac{c_1}{c_2}$$

$$\text{and } w(x_k, \delta_{k-1}) = q_2^2 \frac{f(x_k - \delta_{k-1})}{f(x_k)}.$$

Then taking partial derivatives,

$$\frac{\partial^2 u}{\partial \delta_{k-1}^2} = \frac{\partial^2 v}{\partial \delta_{k-1}^2} + \frac{\partial^2 w}{\partial \delta_{k-1}^2} u(x_k, \delta_{k-1}) + 2 \frac{\partial w}{\partial \delta_{k-1}} \frac{\partial u}{\partial \delta_{k-1}} + \frac{\partial^2 u}{\partial \delta_{k-1}^2} w(x_k, \delta_{k-1}).$$

Since  $1 - w(x_k, \delta_{k-1})$  is positive by Lemma 5.1,  $u(x_k, \delta_{k-1})$  is positive by assumption, and  $\partial u / \partial \delta_{k-1}$  is positive by the previous Lemma, it is sufficient to show that  $\partial^2 v / \partial \delta_{k-1}^2 > 0$ ,  $\partial^2 w / \partial \delta_{k-1}^2 > 0$  and  $\partial w / \partial \delta_{k-1} > 0$ . Now by assumption

$$\frac{\partial^2 v}{\partial \delta_{k-1}^2} = q_2 \frac{f''(x_k - \delta_{k-1})}{f(x_k)} \left[ p_2 \delta_{k-1} + \frac{c_1}{c_2} \right] - q_2 (3 - 2p_2) \frac{f'(x_k - \delta_{k-1})}{f(x_k)} > 0.$$

Likewise, as required,

$$\frac{\partial w}{\partial \delta_{k-1}} = -q_2^2 \frac{f'(x_k - \delta_{k-1})}{f(x_k)} > 0$$

$$\frac{\partial^2 w}{\partial \delta_{k-1}^2} = -q_2^2 \frac{f''(x_k - \delta_{k-1})}{f(x_k)} > 0$$

thus completing the proof.  $\parallel$

Next we present the theorem which proves that if the failure density is  $PF_2$  then optimum inspection intervals are nonincreasing.

**Theorem 5.2.** Let  $f$  be a  $PF_2$  failure density with  $f'$  existing everywhere. At the optimum,  $\delta_{i+1} \geq \delta_i$  for  $i \geq 1$ .

**Proof.** If  $\delta_{i+1} \geq a\delta_i$  for  $i \geq 1$ , then  $\gamma_{i+1} \geq a\gamma_i$  for  $i \geq 2$ , since from (5.11) and (5.16) we have

$$\gamma_{i+1} - a\gamma_i = \sum_{j=0}^{\infty} p_2 q_2^j (\delta_{j+i-1} - a\delta_{j+i}). \quad (5.20)$$

Consequently, if  $\gamma_i > a\gamma_{i+1}$  for some  $i$ , then there is a

positive integer  $k$  such that  $\delta_{i+k} > a\delta_{i+k-1}$ .

Suppose for a contradiction that  $\delta_i > \delta_{i-1}$ . By Lemma 5.3,  $\delta_i > \delta_{i-1}$  implies  $\gamma_{i+1} > \gamma$ . Hence from (5.20) for some  $j > i$ ,  $\delta_j > \delta_{j-1}$ . Thus, for some  $j$ ,  $\delta_j = a\delta_{j-1}$  with  $a > 1$  and  $x_{j-1} > y_0$  where  $y_0$  is such that  $f''(y) > 0$  for  $y > y_0 > m$ . It then follows from Lemma 5.4 that  $\gamma_{j+1} > a\gamma_j$ . From (5.16)

$$\gamma_{j+1} - a\gamma_j = p_2(\delta_{j+1} - a\delta_j) + q_2(\gamma_{j+2} - a\gamma_{j+1}),$$

hence either  $\gamma_{j+2} > a\gamma_{j+1}$  or  $\delta_{j+1} > a\delta_j$ ; but, from Lemma 5.4, the latter also implies that  $\gamma_{j+2} \geq a\gamma_{j+1}$ . Consequently  $\gamma_i$  increases at least geometrically resulting in  $\lim_{i \rightarrow \infty} \gamma_i = \infty$ .

Next we show that  $\lim_{i \rightarrow \infty} \gamma_i = \infty$  contradicts the fact that  $\{x_i^*\}$  is an optimum policy. Note that  $x_{n+1}^*$  is the first optimum checking point for the conditional density  $f(t+x_n^*)/\bar{F}(x_n^*)$ . Let  $\mu(x_n^*)$  denote the mean of this conditional density, which is finite by Barlow, Hunter, and Proschan (1963) p.1089. Hence the expected cost for the optimum first check using the conditional density is bounded above uniformly in  $n$ . But this cost is

$$\begin{aligned} & \frac{c_1}{p_2} + c_2 \int_{x_n^*}^{x_{n+1}^*} \sum_{i=0}^{\infty} p_2 q_2^{i-1} (x_{n+i}^* - t) \frac{f(t) dt}{\bar{F}(x_n^*)} \\ &= \frac{c_1}{p_2} + c_2 \int_{x_n^*}^{x_{n+1}^*} (z_{n+1}^* - t) \frac{f(t) dt}{\bar{F}(x_n^*)} \\ &= \frac{c_1}{p_2} + c_2 \int_0^{\delta_n} (\gamma_n + z_n^* - x_n^* - t) \frac{f(t+x_n^*) dt}{\bar{F}(x_n^*)} \\ &\rightarrow \frac{c_1}{p_2} + c_2 [\gamma_n + (z_n^* - x_n^*) - \mu(x_n^*)] \rightarrow \infty \text{ since } \gamma_n \rightarrow \infty, \end{aligned}$$

which gives the desired contradiction.  $\parallel$

### 5.5.3. Computation of Optimal Intervals.

The next theorem allows us to obtain an algorithm to compute an optimum inspection sequence; it requires the following lemma.

Lemma 5.5. At the optimum,  $d\gamma_{i-1}/dx_1 \geq 0$  and  $dx_i/dx_1 > 0$  for  $i > 1$ . Moreover, if  $x_{j-1} > m$ , then

$$\frac{d\gamma_{i-1}}{dx_1} \geq \eta^{i-j} \frac{d\gamma_{j-1}}{dx_1} \quad \text{for } i \geq j$$

where  $\eta = \min_{0 \leq n < k} \left\{ \frac{(1+q_2)f(x_{j+n-1})}{f(x_{j+n}) + q_2 f(x_{j+n-1})} \right\} > 1$ .

Proof. We prove the first part of the lemma,  $d\gamma_{i-1}/dx_1 \geq 0$  and  $dx_i/dx_1 > 0$  for  $i > 1$ , by induction.

Since  $f$  is  $PF_2$ , the ratio  $F(y)/f(y)$  is nondecreasing in  $y$  so that at the optimum,  $d\gamma_1/dx_1 \geq 0$ . For  $i \geq 2$ ,

$$\frac{d\gamma_i}{dx_1} = \frac{\partial \gamma_i}{\partial x_i} \frac{dx_i}{dx_1} + \frac{\partial \gamma_i}{\partial \delta_{i-1}} \frac{d\delta_{i-1}}{dx_1}. \quad (5.21)$$

Moreover,  $d\delta_{i-1}/dx_1 = dx_i/dx_1 - dx_{i-1}/dx_1$  and from (5.16),

$$\frac{d\gamma_i}{dx_1} = \frac{p_2}{q_2} \left( \frac{dx_{i-1}}{dx_1} - \frac{dx_i}{dx_1} + \frac{1}{p_2} \frac{d\gamma_{i-1}}{dx_1} \right).$$

Equating the above to (5.21) and solving for  $dx_i/dx_1$ , we get

$$\frac{dx_i}{dx_1} = \frac{\left[ \frac{p_2}{q_2} + \frac{\partial \gamma_i}{\partial \delta_{i-1}} \right] \frac{dx_{i-1}}{dx_1} + \frac{1}{q_2} \frac{d\gamma_{i-1}}{dx_1}}{\frac{p_2}{q_2} + \frac{\partial \gamma_i}{\partial x_i} + \frac{\partial \gamma_i}{\partial \delta_{i-1}}}. \quad (5.22)$$

At the optimum  $\gamma_i = u(x_i, \delta_{i-1})$ ; from Lemma 5.3  $\partial \gamma_2 / \partial x_2 \geq 0$  and  $\partial \gamma_2 / \partial \delta_1 > 0$ . Hence  $dx_2 / dx_1 > 0$ .

Next we assume  $dx_{i-1} / dx_1 > 0$  and  $d\gamma_{i-2} / dx_1 \geq 0$ . If  $d\delta_{i-2} / dx_1 \geq 0$ , then by (5.21)  $d\gamma_{i-1} / dx_1 \geq 0$ ; if instead  $d\delta_{i-2} / dx_1 < 0$ , then by (5.16)  $d\gamma_{i-1} / dx_1 \geq 0$  again. Also by (5.22)  $dx_i / dx_1 > 0$ . Hence, by induction, for  $i > 1$ ,  $d\gamma_{i-1} / dx_1 \geq 0$  and  $dx_i / dx_1 > 0$ .

To prove the second part of the lemma, let  $x_{j-1} > m$  (mode of  $f$ ) and  $i \geq j$ . First assume  $d\delta_{j-1} / dx_1 \geq 0$ . From Lemma 5.3  $\partial \gamma_j / \partial x_j \geq 0$ , and from the first part of this lemma  $dx_j / dx_1 > 0$ . Applying this to (5.21) we have

$$\frac{d\gamma_j}{dx_1} = \frac{\partial \gamma_j}{\partial x_j} \frac{dx_j}{dx_1} + \frac{\partial \gamma_j}{\partial \delta_{j-1}} \frac{d\delta_{j-1}}{dx_1} \geq \frac{\partial \gamma_j}{\partial \delta_{j-1}} \frac{d\delta_{j-1}}{dx_1}.$$

From (5.19) in Lemma 5.3

$$\frac{\partial \gamma_j}{\partial \delta_{j-1}} = \frac{\partial u}{\partial \delta_{j-1}} \geq \frac{p_2(1+q_2)f(x_{j-1})}{f(x_j) - q_2^2 f(x_{j-1})}.$$

Then

$$\frac{d\gamma_j}{dx_1} \geq \frac{p_2(1+q_2)f(x_{j-1})}{f(x_j) - q_2^2 f(x_{j-1})} \frac{d\gamma_{j-1}}{dx_1}.$$

By (5.16)

$$\frac{d\gamma_j}{dx_1} \geq \frac{p_2(1+q_2)f(x_{j-1})}{f(x_j) - q_2^2 f(x_{j-1})} \left[ \frac{1}{p_2} \frac{d\gamma_{j-1}}{dx_1} - \frac{q_2}{p_2} \frac{d\gamma_j}{dx_1} \right].$$

Rearranging, we get

$$\frac{d\gamma_j}{dx_1} \geq \left[ \frac{(1+q_2)f(x_{j-1})}{f(x_j) + q_2f(x_{j-1})} \right] \frac{d\gamma_{j-1}}{dx_1}.$$

Hence it follows that

$$\frac{d\gamma_{j+k}}{dx_1} \geq \prod_{n=0}^k \left[ \frac{(1+q_2)f(x_{j+n-1})}{f(x_{j+n}) + q_2f(x_{j+n-1})} \right] \frac{d\gamma_{j-1}}{dx_1}.$$

$$\text{Let } \eta = \min_{0 \leq n < k} \left\{ \frac{(1+q_2)f(x_{j+n-1})}{f(x_{j+n}) + q_2f(x_{j+n-1})} \right\}.$$

By assumption  $x_{j-1} > m$  implies  $f(x_{j+n-1})/f(x_{j+n}) > 1$  for  $n \geq 0$ , which in turn implies  $\eta > 1$ . Then

$$\frac{d\gamma_{j+k}}{dx_1} \geq \eta^{k+1} \frac{d\gamma_{j-1}}{dx_1}.$$

Now let  $i-1 = j+k$  and we obtain the desired result assuming  $d\delta_j/dx_1 \geq 0$ .

Next assume  $d\delta_j/dx_1 < 0$ . Then from (5.16)

$$0 > \frac{d\delta_j}{dx_1} = \frac{1}{p_2} \frac{d\gamma_{j-1}}{dx_1} - \frac{q_2}{p_2} \frac{d\gamma_j}{dx_1}.$$

Rearranging we get  $\frac{d\gamma_j}{dx_1} > \frac{1}{q_2} \frac{d\gamma_{j-1}}{dx_1}$ .

Hence it follows that  $\frac{d\gamma_{j+k}}{dx_1} > \left( \frac{1}{q_2} \right)^{k+1} \frac{d\gamma_{j-1}}{dx_1}$ .

By Lemma 5.1, for  $n \geq 0$ ,  $\frac{1}{q_2} > \frac{(1+q_2)f(x_{j,n-1})}{f(x_{j,n}) + q_2f(x_{j,n-1})}$ , which

implies  $1/q_2 > \eta$ , thus  $\frac{d\gamma_{j+k}}{dx_1} > \eta^{k+1} \frac{d\gamma_{j-1}}{dx_1}$ .

Again let  $i-1 = j+k$  to obtain the desired result.  $\parallel$

Theorem 5.3. Let  $x_1^*$  be the optimal first inspection, and  $\{x_i\}$  be the schedule obtained by (5.2) from an arbitrary initial  $x_1$ . Then:

- a. if  $x_1 > x_1^*$  there is an  $i$  such that  $\delta_i > \delta_{i-1}$ ;
- b. if  $x_1 < x_1^*$  there is an  $i$  such that  $\delta_i < 0$ .

Proof. Let  $m$  be the mode of  $f$  and  $j$  be such that  $x_{j-1} > m$ .

From Lemma 5.5,  $d\gamma_{i-1}/dx_1 \geq 0$ ,  $i > 1$ , and

$$\frac{d\gamma_{i-1}}{dx_1} \geq \eta^{i-j} \frac{d\gamma_{j-1}}{dx_1}, \text{ where } \eta > 1,$$

for all  $i \geq j$ . Therefore, an increase in  $x_1$  from  $x_1^*$  will result in an increase in  $\gamma_{j-1}$  and  $\gamma_{i-1}$ . The latter, by Lemma 5.5, can be made arbitrarily large by appropriately choosing  $i$ . Therefore, for some  $i$ ,  $\gamma_i > \gamma_1$ , which, from (5.20) implies that  $\delta_{i+k} > \delta_{1+k}$  for some  $k$ . Likewise a decrease in  $x_1$  from  $x_1^*$  will eventually result in a negative  $\gamma_i$  entailing a negative  $\delta_{i+k}$ .  $\parallel$

As a result of Theorem 5.3, the optimal solution is unique. An optimum checking sequence can be obtained from (5.2) recursively once the first inspection time  $x_1$  is obtained, and the bisection algorithm to find the optimum  $x_1$  suggested by Barlow, Hunter, and Proschan (1963) for  $p_2=1$  can be extended to the case for  $0 < p_2 < 1$ . If the choice of  $x_1$  is too small, then eventually the checking intervals generated by (5.2) will be negative. If  $x_1$  is too large, then checking intervals will eventually increase, and thus are not optimal.

Alternatively an optimum sequence can be obtained from (5.12) and (5.15). Suppose a starting  $x_1$  is fixed. Then using (5.12) and (5.15) we obtain  $z_2$  from

$$z_2 = x_1 + \frac{F(x_1)}{f(x_1)} - \frac{c_1}{p_2 c_2}.$$

Using (5.12) again we obtain  $z_1$  from  $z_2$  and  $x_1$ . Then the following steps can be repeated:

1. When the values of  $z_i$ ,  $z_{i-1}$ , and  $x_{i-1}$  are known, use (5.12) to express the right hand side of (5.15) as a function of  $x_{i-1}$ ,  $z_i$ , and  $z_{i-1}$ , and then solve for  $z_{i+1}$ .
2. Compute  $x_i$  using (5.12).

Then to obtain the optimum inspection sequence  $\{x_i^*\}$  we apply the following algorithm.

Algorithm 5.1.

1. Compute  $\{x_k\}$  and  $\{z_k\}$  as above.

2. If for some  $k$ ,  $\gamma_k > \gamma_{k-1}$ , reduce  $x_1$  and repeat; if  $\gamma_k < 0$ , increase  $x_1$  and repeat.

3. Continue until  $\{x_k\}$  is determined to degree of accuracy required.

In this way both  $\{x_k\}$  and  $\{z_k\}$  can be determined. also at each iteration, (5.13) can be used to compute the terms of  $Cp_2$ , the expected cost.

Example 5.3. Let  $f$  be a Weibull density with  $\alpha=2$ ,  $\lambda=.01$  where  $f(t) = \lambda\alpha(\lambda t)^{\alpha-1}\exp[-(\lambda t)^\alpha]$ .

Choose an inspection cost of  $c_1=10$  and failure cost of  $c_2=1$ . Table 11 shows optimal first checking times,  $x_1^*$ , and expected costs,  $Cp_2$ , for various failure-detection probabilities,  $p_2$ .

Table 11. Optimal First Checking Times and Costs

$p_2$	$Cp_2$	$x_1^*$
1.0	42.227	68.15750
.9	46.237	68.87350
.8	50.789	70.02767
.7	56.104	71.69260

Table 12 below gives the optimal checking schedules  $\{x_i^*\}$  and inspection intervals  $\{\delta_i^*\}$  for  $p_2=1.0$ , .9, .8, and .7. As  $p_2$  decreases, the first checking time is delayed

Table 12. Optimal Checking Schedules for a Weibull Density with  $\lambda=.01$ ,  $\alpha=2$ ,  $c_1=10$ ,  $c_2=1$ ,  $p_2=1.0, .9, .8$ , and  $.7$

i	$p_2=1.0$		$p_2=.9$	
	$x_i^*$	$\delta_{i-1}^*$	$x_i^*$	$\delta_{i-1}^*$
1	68.157	68.157	68.874	68.874
2	101.534	33.377	99.093	30.219
3	129.052	27.518	124.013	24.920
4	153.384	24.332	146.029	22.016
5	175.597	22.213	166.106	20.077
6	196.254	20.657	184.757	18.651
7	215.698	19.445	202.295	17.538
8	234.160	18.462	218.929	16.634
9	251.801	17.641	234.809	15.879
10	268.742	16.942	250.044	15.235
11	285.077	16.334	264.721	14.677
12	300.877	15.800	278.907	14.186

i	$p_2=.8$		$p_2=.7$	
	$x_i^*$	$\delta_{i-1}^*$	$x_i^*$	$\delta_{i-1}^*$
1	70.028	70.028	71.693	71.693
2	97.240	27.212	95.970	24.277
3	119.766	22.526	116.207	20.238
4	139.684	19.918	134.144	17.937
5	157.849	18.165	150.519	16.375
6	174.720	16.871	165.734	15.215
7	190.580	15.860	180.038	14.304
8	205.617	15.037	193.601	13.562
9	219.968	14.351	206.541	12.940
10	233.733	13.766	218.949	12.408
11	246.994	13.261	230.894	11.945
12	259.816	12.822	242.432	11.538

longer, but subsequent intervals are shorter.

Example 5.4. For a skewed left Weibull density, as  $p_2$  increases the first checking time is delayed longer as seen in this example. Let  $f$  be a Weibull density with  $\alpha=5$ ,  $\lambda=.20$ ,  $c_1=1$ , and  $c_2=250$ . The optimal  $x_1$  is 1.106 at  $p_2=.25$  and 1.388 at  $p_2=.50$ .

#### 5.5.4. Majorization Results for Finite Inspection Sequences.

Next we will show that for a fixed  $k < \infty$  inspection sequences  $\{x_k\}$  generated by (5.2) are related by weak majorization. To simplify notation we will use  $F_k \equiv F(x_k)$  and  $f_k \equiv f(x_k)$ . Also from (5.2) an equivalent form for  $N_k$  is

$$N_k \equiv \sum_{i=1}^k \frac{(F_i - F_{i-1}) q_2^{k-i}}{f_k} = \frac{F_k - p_2 \sum_{i=1}^{k-1} F_{k-i} q_2^{i-1}}{f_k}.$$

Also we will show the long-run expected cost function,  $C_{p_1}$ , is an order preserving function. An interesting consequence is that a minimum cost finite checking sequence is not obtained by truncating an optimum infinite sequence. Some definitions and theorems on majorization from Marshall and Olkin (1979) are repeated here for convenience. For  $\underline{x} \in \mathbb{R}^n$ ,  $x_{(i)}$  denotes the increasing order of components of  $\underline{x}$ ,

i.e.  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$  and  $x_{[i]}$  denotes the decreasing order, i.e.,  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$ .

Definition 5.1. For  $\underline{x}, \underline{y} \in \mathbb{R}^n$ ,

$\underline{x}$  is weakly submajorized by  $\underline{y}$ , denoted

$$\underline{x} \prec_w \underline{y} \text{ if } \sum_1^k x_{[i]} \leq \sum_1^k y_{[i]}, \text{ for } k=1, \dots, n, \text{ and } \underline{x} \text{ is weakly}$$

supermajorized by  $\underline{y}$ , denoted

$$\underline{x} \prec^w \underline{y} \text{ if } \sum_1^k x_{(i)} \geq \sum_1^k y_{(i)}, \text{ for } k=1, \dots, n.$$

In either case  $\underline{x}$  is said to be weakly majorized by  $\underline{y}$  or  $\underline{y}$  weakly majorizes  $\underline{x}$ . One simple consequence of the definition, from p.11 of Marshall and Olkin (1979), is that

$$x_i \leq y_i \text{ for } i=1, \dots, n \Rightarrow \underline{x} \prec_w \underline{y} \text{ and } \underline{y} \prec^w \underline{x}.$$

We will show that this is the case for inspection sequences generated by (5.2). The proof is by induction, and requires the following lemma.

Lemma 5.6. Let  $f$  be a  $PF_2$  density and  $f'$  exist everywhere; then for  $k=1, 2, \dots$

$$\frac{\partial N_k}{\partial x_k} = 1 - \frac{f'_k}{f_k} N_k = 1 - \frac{f'_k}{f_k^2} \left[ F_k - p_2 \sum_{i=1}^{k-1} F_{k-i} q_2^{i-1} \right] > 0.$$

Proof. Case 1: If  $x_k < \text{mode of } f$ , then  $f'_k > 0$  and

$f_k/f'_k \geq F_k/f_k$ . Then

$$\begin{aligned} \frac{f'_k}{f_k} \left[ \frac{F_k}{f_k} - p_2 \sum_{i=1}^{k-1} \frac{F_{k-i}}{f_k} q_2^{i-1} \right] &\leq \frac{f'_k}{f_k} \left[ \frac{f_k}{f'_k} - p_2 \sum_{i=1}^{k-1} \frac{F_{k-i}}{f_k} q_2^{i-1} \right] \\ &= 1 - p_2 \sum_{i=1}^{k-1} \frac{f'_k F_{k-i}}{f_k^2} q_2^{i-1} < 1. \end{aligned}$$

Case 2: If  $x_k \geq \text{mode of } f$ , then  $f'_k \leq 0$  and it

suffices to show that  $F_k > p_2 \sum_{i=1}^{k-1} F_{k-i} q_2^{i-1}$ . Clearly

$$p_2 \sum_{i=1}^{k-1} F_{k-i} q_2^{i-1} < p_2 F_{k-1} \sum_{i=1}^{k-1} q_2^{i-1} = p_2 F_{k-1} \frac{(1-q_2^{k-1})}{(1-q_2)} = F_{k-1} (1-q_2^{k-1}) < F_{k-1}.$$

Therefore  $F_k > F_{k-1} > p_2 \sum_{i=1}^{k-1} F_{k-i} q_2^{i-1}$ .  $\parallel$

Theorem 5.4. Let  $\{x_n\}$  and  $\{y_n\}$  be two inspection sequences generated by (5.2). Let  $\underline{x}$  and  $\underline{y}$  in  $\mathbb{R}^{k+1}$  be the first  $k+1$  inspection times for these sequences. When  $f$  is a  $PF_2$  density and  $p_2=1$ , or when  $f$  is an exponential density and  $0 \leq p_2 \leq 1$ , then  $x_1 \leq y_1$  implies  $\underline{x} <_w \underline{y}$ .

Proof. The proof is by induction. For  $n=1$ ,

$$\delta_1 \equiv x_2 - x_1 = \frac{F_1}{f_1} - q_2 \left[ \frac{F_2 - p_2 F_1}{f_2} \right] - \frac{c_1}{c_2}.$$

By implicit differentiation we find

$$\frac{dx_2}{dx_1} = \frac{1 + \frac{\partial N_1}{\partial x_1} + p_2 q_2 \frac{f_1}{f_2}}{1 + q_2 \frac{\partial N_2}{\partial x_2}},$$

which is positive by Lemma 5.6. Thus  $x_2$  increases when  $x_1$  increases.

Furthermore  $dx_2/dx_1 \geq 1$  when  $p_2 q_2 \frac{f_1}{f_2} + \frac{\partial N_1}{\partial x_1} \geq q_2 \frac{\partial N_2}{\partial x_2}$ . This

is true by Lemma 5.6 when  $p_2=1$  ( $q_2=0$ ). Combining this condition with the definition of  $\delta_1$  we get the equivalent condition

$$p_2 q_2 \frac{f_1}{f_2} + p_2 \geq \frac{f_2'}{f_2} \left[ \delta_1 + \frac{c_1}{c_2} \right] + \frac{F_1}{f_1} \left[ \frac{f_1'}{f_1} - \frac{f_2'}{f_2} \right]$$

which is clearly satisfied for an exponential density since LHS  $> 0$  and RHS  $< 0$ . Now since  $\delta_1 = x_2 - x_1$ , we have

$$\frac{d\delta_1}{dx_1} = \frac{dx_2}{dx_1} - 1 \geq 0.$$

Next we will show  $d\delta_2/dx_1 \geq 0$  and  $dx_3/dx_1 \geq 1$ .

By definition

$$\begin{aligned} \delta_2 \equiv x_3 - x_2 &= \frac{F_2 - p_2 F_1}{f_2} - q_2 \left[ \frac{F_3 - p_2 F_2 - p_2 q_2 F_1}{f_3} \right] - \frac{c_1}{c_2} \\ &= \frac{F(x_2) - p_2 F(x_2 - \delta_1)}{f(x_2)} - q_2 \left[ \frac{F(x_2 + \delta_2) - p_2 F(x_2) - p_2 q_2 F(x_2 - \delta_1)}{f(x_2 + \delta_2)} \right] - \frac{c_1}{c_2}. \end{aligned}$$

Then we can find  $d\delta_2/dx_1$  by  $\frac{d\delta_2}{dx_1} = \frac{\partial \delta_2}{\partial \delta_1} \frac{d\delta_1}{dx_1} + \frac{\partial \delta_2}{\partial x_2} \frac{dx_2}{dx_1}$ .

First we find 
$$\frac{\partial \delta_2}{\partial \delta_1} = \frac{p_2 \left[ \frac{f_1}{f_2} - q_2^2 \frac{f_1}{f_3} \right]}{1 + q_2 \frac{\partial N_3}{\partial x_3}} > 0$$

since the numerator is positive by Lemma 5.1 and the denominator is positive by Lemma 5.6.

Next we find

$$\frac{\partial \delta_2}{\partial x_2} = \frac{\frac{\partial N_2}{\partial x_2} - q_2 \frac{\partial N_3}{\partial x_3} + p_2 q_2 \frac{f_2}{f_3} + p_2 q_2^2 \frac{f_1}{f_3}}{1 + q_2 \frac{\partial N_3}{\partial x_3}}.$$

The denominator is positive by Lemma 5.6. The numerator is positive when

$$p_2 q_2 \frac{f_2}{f_3} + p_2 q_2^2 \frac{f_1}{f_3} + \frac{\partial N_2}{\partial x_2} \geq q_2 \frac{\partial N_3}{\partial x_3},$$

which is true by Lemma 5.6 when  $p_2=1$  ( $q_2=0$ ).

An equivalent condition is

$$p_2 q_2 \frac{f_2}{f_3} + p_2 q_2^2 \frac{f_1}{f_3} + p_2 \geq \frac{f_3'}{f_3} \left[ \delta_2 + \frac{c_1}{c_2} \right] + \left[ \frac{F_2 - p_2 F_1}{f_2} \right] \left[ \frac{f_2'}{f_2} - \frac{f_3'}{f_3} \right],$$

which is clearly satisfied for an exponential density.

Therefore  $d\delta_2/dx_1 \geq 0$  and  $x_3 = x_1 + \delta_1 + \delta_2$  gives

$$\text{us } \frac{dx_3}{dx_1} = 1 + \frac{d\delta_1}{dx_1} + \frac{d\delta_2}{dx_1} \geq 1. \text{ So an increase in } x_1 \text{ results in}$$

increases in both  $x_2$  and  $x_3$ .

Now assume  $d\delta_{n-1}/dx_1 \geq 0$  and  $dx_n/dx_1 \geq 1$  for  $n=2, \dots, k$ .

To show  $d\delta_k/dx_1 \geq 0$  and  $dx_{k+1}/dx_1 \geq 1$ . By definition

$$\delta_k = \frac{F(x_k) - p_2 F(x_k - \delta_{k-1}) - p_2 q_2 F(x_k - \delta_{k-1} - \delta_{k-2}) - \dots - p_2 q_2^{k-2} F(x_k - \delta_{k-1} - \dots - \delta_1)}{f(x_k)} \\ - q_2 \frac{F(x_k + \delta_k) - p_2 F(x_k) - p_2 q_2 F(x_k - \delta_{k-1}) - \dots - p_2 q_2^{k-1} F(x_k - \delta_{k-1} - \dots - \delta_1)}{f(x_k + \delta_k)} - \frac{c_1}{c_2}.$$

Then we can find  $d\delta_k/dx_1$  from

$$\frac{d\delta_k}{dx_1} = \left[ \frac{\partial \delta_k}{\partial \delta_{k-1}} \right] \frac{d\delta_{k-1}}{dx_1} + \left[ \frac{\partial \delta_k}{\partial \delta_{k-2}} \right] \frac{d\delta_{k-2}}{dx_1} + \dots + \left[ \frac{\partial \delta_k}{\partial \delta_1} \right] \frac{d\delta_1}{dx_1} + \left[ \frac{\partial \delta_k}{\partial x_k} \right] \frac{dx_k}{dx_1}.$$

First we find for  $j=1, 2, \dots, (k-1)$

$$\frac{\partial \delta_k}{\partial \delta_{k-j}} = \frac{p_2 \left[ \frac{1}{f_k} - q_2^2 \frac{1}{f_{k+1}} \right] \sum_{i=j}^{k-1} f_{k-i} q_2^{i-1}}{1 + q_2 \frac{\partial N_{k+1}}{\partial x_{k+1}}} > 0$$

since the numerator is positive by Lemma 5.1 and the denominator is positive by Lemma 5.6. Next we find

$$\frac{\partial \delta_k}{\partial x_k} = \frac{\frac{\partial N_k}{\partial x_k} - q_2 \frac{\partial N_{k+1}}{\partial x_{k+1}} + p_2 q_2 \sum_{i=1}^k \frac{f_{k+1-i}}{f_{k+1}} q_2^{i-1}}{1 + q_2 \frac{\partial N_{k+1}}{\partial x_{k+1}}}.$$

The denominator is positive by Lemma 5.6. The numerator is positive when

$$p_2 q_2 \sum_{i=1}^k \frac{f_{k+1-i}}{f_{k+1}} q_2^{i-1} + \frac{\partial N_k}{\partial x_k} \geq q_2 \frac{\partial N_{k+1}}{\partial x_{k+1}},$$

which is true by Lemma 5.6 when  $p_2=1$  ( $q_2=0$ ). An equivalent

condition is

$$p_2 q_2 \sum_{i=1}^k \frac{f_{k+1-i}}{f_{k+1}} q_2^{i-1} + p_2 \geq \frac{f'_{k+1}}{f_{k+1}} \left[ \delta_k + \frac{c_1}{c_2} \right] + N_k \left[ \frac{f'_k}{f_k} - \frac{f'_{k+1}}{f_{k+1}} \right],$$

which is clearly satisfied by an exponential density.

Thus  $\partial \delta_k / \partial x_k \geq 0$ , and therefore  $d\delta_k / dx_k \geq 0$ .

Also  $x_{k+1} = x_1 + \delta_1 + \delta_2 + \dots + \delta_k$  implies

$$\frac{dx_{k+1}}{dx_1} = 1 + \frac{d\delta_1}{dx_1} + \frac{d\delta_2}{dx_1} + \dots + \frac{d\delta_k}{dx_1} \geq 1.$$

Thus an increase in  $x_1$  results in increases in  $x_2, \dots, x_{k+1}$ .

Therefore  $x_1 \leq y_1$  implies  $x_i \leq y_i$  for  $i=1, 2, \dots, k+1$  and we

have  $\underline{x} <_w \underline{y}$ .  $\parallel$

To show that  $C_{p_2}$ , the long-run expected cost function preserves the ordering of weak majorization, we need the following theorem of Ostrowski from Marshall and Olkin (1979) p. 59.

Theorem (Ostrowski, 1952). Let  $\phi$  be a real-valued function, defined and continuous on  $\mathcal{E}$ , and continuously differentiable on the interior of  $\mathcal{E}$ , where  $\mathcal{E} = \{ \underline{x} : x_1 \leq \dots \leq x_n \}$ .

Then

$\phi(\underline{x}) \leq \phi(\underline{y})$  whenever  $\underline{x} <_w \underline{y}$  on  $\mathcal{E}$  if and only if

$$\frac{\partial \phi}{\partial x_n}(\underline{z}) \geq \frac{\partial \phi}{\partial x_{n-1}}(\underline{z}) \geq \dots \geq \frac{\partial \phi}{\partial x_1}(\underline{z}) \geq 0,$$

i.e., the gradient  $\nabla\phi(z) \in \mathcal{E}_+$ , for all  $z$  in the interior of  $\mathcal{E}$ . Similarly,

$\phi(\underline{x}) \leq \phi(\underline{y})$  whenever  $\underline{x} \prec_w \underline{y}$  on  $\mathcal{E}$  if and only if

$$0 \geq \frac{\partial\phi}{\partial x_n}(z) \geq \frac{\partial\phi}{\partial x_{n-1}}(z) \geq \dots \geq \frac{\partial\phi}{\partial x_1}(z)$$

for all  $z$  in the interior of  $\mathcal{E}$ .

Theorem 5.5. Let  $\underline{x}$  and  $\underline{y}$  in  $\mathbb{R}^n$  be inspection schedules generated by (5.2). If  $x_1 < y_1$ , then  $C_{p_1}(\underline{x}) \leq C_{p_1}(\underline{y})$ , when the conditions of Theorem 5.4 are satisfied.

Proof. For any inspection sequence  $\underline{x}$  generated by (5.2) we have  $\partial C_{p_1}/\partial x_i = 0$  for  $i=1,2,\dots,n$ . By Theorem 5.4 we have  $\underline{x} \prec_w \underline{y}$ , and applying Ostrowski's Theorem,  $C_{p_1}(\underline{x}) \leq C_{p_1}(\underline{y})$ .  $\parallel$

A practical application of Theorems 5.4 and 5.5 involves choosing a finite checking sequence with minimum cost,  $C_{p_1}$ . Since  $C_{p_1}$  is an order preserving function, if we desire a minimum cost checking sequence of  $n$  inspection times, then we should choose an initial inspection time  $x_1$  that is less than  $x_1^*$ , the optimal initial inspection time for an infinite sequence of inspections. Then  $\underline{x} \prec_w \underline{x}^*$  and  $C_{p_1}(\underline{x}) \leq C_{p_1}(\underline{x}^*)$ .

Since  $x_1 < x_1^*$ , the checking sequence generated from  $x_1$

will eventually result in a negative inspection time, and thus a finite number of positive inspection times is obtained. The least cost initial value of  $x_1$  can be chosen so that a checking sequence of exactly  $n$  inspection times is obtained, or  $x_1$  can be chosen so that the checking sequence terminates at a specified inspection time.

As a consequence of choosing  $x_1 < x_1^*$ , we get smaller inspection intervals than those obtained by using  $x_1^*$ . Thus in the case  $f$  is an exponential density,  $0 < p_2 \leq 1$ , if we are interested in a least cost finite checking sequence, we will find that our inspection intervals will be decreasing as opposed to constant inspection intervals obtained for an optimal infinite sequence.

Example 5.5. Let  $f$  be an exponential density with  $\lambda=.01$ ,  $f(t)=\lambda\exp(-\lambda t)$ ,  $t \geq 0$ . Choose a cost per inspection,  $c_1=10$ , cost per unit of failure time,  $c_2=1$ , and probability of detecting a failure,  $p_2=.9$ . For an optimal infinite checking sequence,  $x_1^*=41.874$  and constant checking intervals are  $\delta_k^*=36.805$  resulting in a cost of  $C_{p_2} = 57.075$ .

To obtain an optimal finite schedule containing  $n=10$  inspection times, observe from Table 13 that an initial inspection time of  $x_1=39$  results in a checking sequence of only 9 inspection times; whereas, an initial inspection time of  $x_1=40$  results in a checking sequence of 10

inspection times. Also an initial inspection time of  $x_1=39.9$  also results in a checking sequence of 10 inspection times.

The cost of the 10-inspection sequence beginning at  $x_1=40$  is  $C_{p_2} = 18.898$ . The cost of the 10-inspection sequence beginning at  $x_1=39.9$  is  $C_{p_2} = 18.669$ . Thus the optimal sequence for  $n=10$  begins at  $x_1$ , where  $39 < x_1 < 39.9$ .

Table 13. Inspection Schedules and Intervals for  $p_2=.9$  for an Exponential Density  $\lambda=.01$ ,  $c_1=10$ ,  $c_2=1$

i	$C_{p_2}=5.535$		$C_{p_2}=18.669$		$C_{p_2}=18.898$	
	$x_i$	$\delta_{i-1}$	$x_i$	$\delta_{i-1}$	$x_i$	$\delta_{i-1}$
1	39.00	39.00	39.90	39.90	40.00	40.00
2	72.11	33.11	74.16	34.26	74.39	34.39
3	103.66	31.55	107.32	33.17	107.73	33.35
4	133.05	29.39	138.95	31.63	139.61	31.88
5	159.49	26.44	168.45	29.50	169.46	29.85
6	181.99	22.50	195.04	26.59	196.51	27.05
7	199.42	17.42	217.73	22.70	219.83	23.32
8	210.55	11.14	235.40	17.67	238.28	18.46
9	214.32	3.77	246.83	11.43	250.67	12.39
10	210.02	-4.30	250.93	4.11	255.87	5.20
11			246.96	-3.98	253.10	-2.78

## 6. IMPERFECT DETECTION OF SOFTWARE FAULTS

### 6.1. Introduction.

In this chapter we present the impact of Type II inspection error on two software fault detection models, where the probability of detecting a fault is  $0 < p_2 \leq 1$ . In Section 6.2 we introduce Type II error into the Koch and Kubat (1983) optimal release time model. In Section 6.3 we explore fault diversity in the Jelinski-Moranda (JM) model.

### 6.2. Imperfect Fault-Detection Model.

The Koch and Kubat (1983) software release time decision model is based on the following function for the average number of faults detected during testing:

$$E[N(T)] = M[1 - \exp(-\theta T)] \quad (6.1)$$

where  $T$  = software release time (total testing time)

$M$  = total number of faults in the software  
prior to testing

$\theta$  = proportionality constant.

Next the optimal time,  $T$ , to release software for operational use is obtained by maximizing the average gain

function,  $G(T)$ :

$$G(T) = cM[1 - \exp(-\theta T)] - c_1 T \quad (6.2)$$

where  $c_1$  = cost per rate of testing

$c_2$  = cost of correcting a fault during test phase

$c_3$  = cost of correcting a fault during operational  
phase ( $c_3 > c_2$ )

$$c = c_3 - c_2.$$

The above cost factors presented by Bai and Yun (1988) are a condensed version of the many different cost factors used by Koch and Kubat (1983), and are used here for simplicity and ease of understanding the model. The optimal release time,  $T$ , obtained by maximizing the average gain is the same as the optimal release time obtained by minimizing the expected loss function used by Okumoto and Goel (1980).

Equation (6.1) is identical in form to the mean value function of the Goel-Okumoto (1979) nonhomogeneous Poisson process (NHPP) software reliability growth model (SRGM) with intensity function  $dE[N(T)]/dT$ . Yamada and Osaki (1985) categorize SRGM's according to whether their fault detection rate per fault is increasing, constant, or decreasing, where fault detection rate per fault is defined as  $r(T) = \{dE[N(T)]\}/\{M - E[N(T)]\}$ . For the Goel-Okumoto NHPP SRGM,  $r(T) = \theta$ , which is a constant fault detection rate per fault. Note that in both the Koch and Kubat model and the Goel-Okumoto NHPP SRGM that fault "detection" is

operationally equivalent to fault "occurrence" because it is assumed that every fault that occurs is immediately detected and corrected.

In the present treatment we assume that a fault that occurs may not be detected, with probability  $q_2$ . Then  $p_2 = 1 - q_2$  is the probability that a fault which occurs is detected (and corrected immediately). This results in reducing the fault detection rate per fault,  $\theta$ , so that the imperfect fault detection rate per fault is  $p_2\theta$ . Thus we modify (6.1), the average number of faults detected during a test period of duration  $T$ , to

$$E[N(T)] = M[1 - \exp(-p_2\theta T)] \quad (6.3)$$

and the average gain (6.2) is modified as

$$G(T) = cM[1 - \exp(-p_2\theta T)] - c_1T. \quad (6.4)$$

We obtain the optimal release time  $T_{p_2}^*$  by solving

$dG(T)/dT = 0$  to get

$$T_{p_2}^* = \frac{1}{p_2\theta} \ln \left[ \frac{p_2\theta cM}{c_1} \right]. \quad (6.5)$$

Since  $d^2G(T)/dT^2 < 0$ ,  $T_{p_2}^*$  maximizes the gain. To compare

this optimal release time to the optimal release time  $T_1^*$  obtained by Koch and Kubat when  $p_2=1$ , we rewrite (6.5) to get

$$T_{p_2}^* = \frac{1}{p_2\theta} \ln p_2 + \frac{1}{p_2} T_1^*. \quad (6.6)$$

From equations (6.5) and (6.6) we see that if  $1/M\theta \geq c/c_1$ , then the optimal release time is  $T_{p_2}^* = 0$  for all  $p_2$ .

If  $1/M\theta < c/c_1$ , then the optimal release time is given by (6.6). For  $1/M\theta < c/c_1$ , Figure 6 compares  $T_{p_2}^*$  to  $T_1^*$ .  $T_{p_2}^*$  is a concave function of  $p_2$  with maximum value of  $T_{p_2}^* = cM/c_1e$  at  $p_2 = c_1e/cM\theta$ .

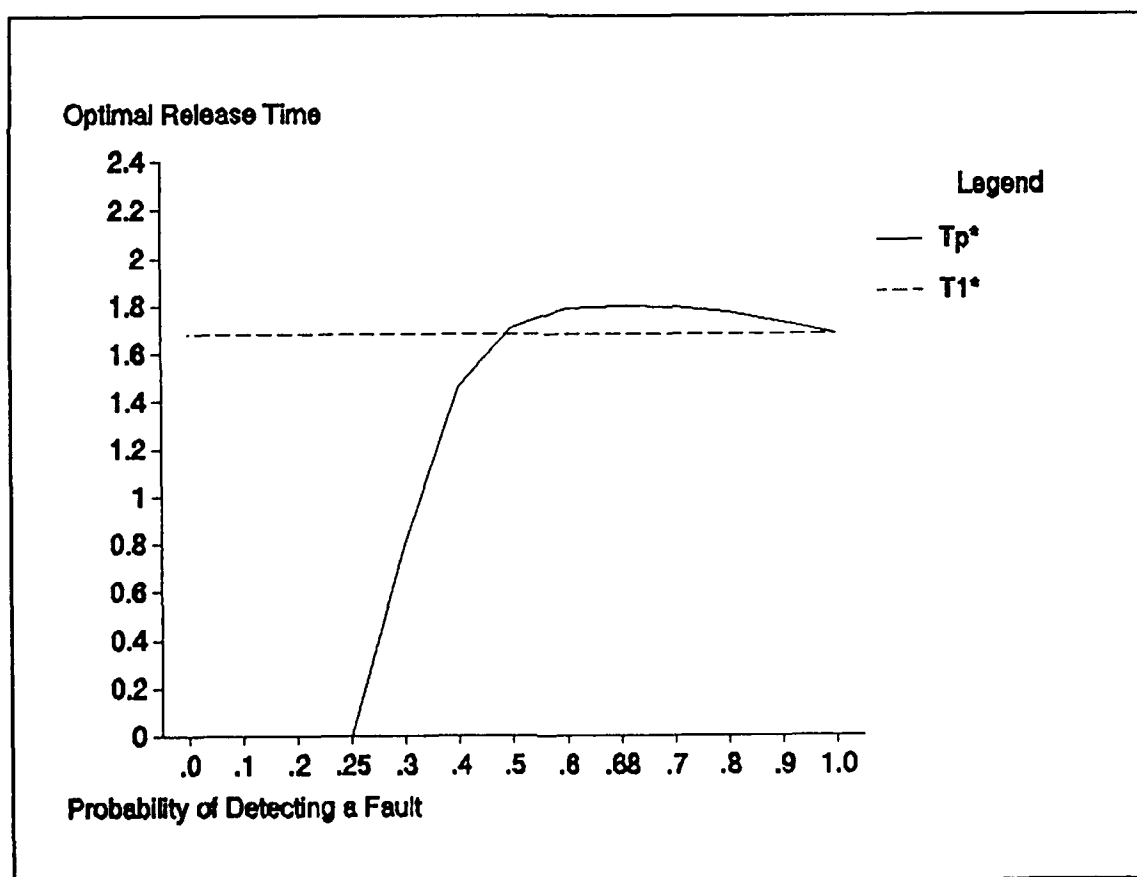


Figure 6. Optimal Release Time,  $T_p^*$ .

When  $0 < p_2 \leq 1$ , the optimum release time  $T_{p_2}^*$  differs from the Koch and Kubat optimum  $T_1^*$  at all but two values of  $p_2$ , when  $p_2=1$  and the value of  $p_2$  that solves the following equation.

$$\ln p_2 = (p_2 - 1) \ln \left[ \frac{cM\theta}{c_1} \right]. \quad (6.7)$$

From Figure 6 we see that when the probability of detecting a fault is smaller than the solution to (6.7), then the optimal release time  $T_{p_2}^*$  is smaller than the perfect-detection optimal release time  $T_1^*$ ; otherwise,  $T_{p_2}^* \geq T_1^*$ , when  $p_2$  is greater than the solution to (6.7).

Example 6.1. A numerical example is provided to compare the results of this imperfect fault-detection model to the results obtained by Koch and Kubat using the values in their example 1:  $c=3500$ ,  $c_1=21500$ ,  $M=30$ ,  $\theta=5/6$ . Then  $T_1^*=1.68$ ,  $cM/c_1e = 1.80$ ,  $c_1e/cM\theta = .68$ ,  $c_1/cM\theta = .25$ . The graph of this particular example is shown in Figure 6.

### 6.3. Fault Diversity in the Jelinski-Moranda Model.

Many software models make the erroneous assumption that each fault is equally likely to cause failure (and thus equally likely to be detected). Boland, Proschan, and

Tong (1987) p. 185 present the impact of fault diversity in models of the Jelinski-Moranda (JM) type. They assume that  $N$  unknown faults in a software system have detection times,  $Z_i$ , which are independent exponential random variables with failure rates  $\underline{\lambda} = (\lambda_1, \dots, \lambda_N)$ . Assuming that when a fault occurs, it is detected and eliminated, the vector  $\underline{\lambda}$  represents the diverse fault detection rates for the system. Define  $T_{n,\underline{\lambda}}$  to be the time until any  $n$  faults in the system are eliminated. Let  $\underline{\lambda} = (\lambda_1, \dots, \lambda_N)$  and  $\underline{\lambda}' = (\lambda_1', \dots, \lambda_N')$  be two vectors of fault detection rates for two software systems. Boland, Proschan, and Tong (1987) then prove that if  $\underline{\lambda} \succ^m \underline{\lambda}'$  ( $\underline{\lambda}$  majorizes  $\underline{\lambda}'$ ) then  $T_{n,\underline{\lambda}} \geq T_{n,\underline{\lambda}'}$ . Thus the more diverse the fault detection rates are, the longer it takes to eliminate any  $n$  faults.

This section presents similar results specifically for the Jelinski-Moranda model and shows how fault diversity in the JM model is related to imperfect fault detection.

Assume there are  $N$  unknown faults initially in the system and times between fault detections are independent, exponentially distributed random variables  $Z_i$ . Then we know that in the JM model for  $Z_i$ , the time between the detection of the  $(i-1)$ st and the  $i$ th fault, the fault detection rate is  $\lambda_i = (N-i+1)\lambda$ ,  $i=1, \dots, N$ , where  $\lambda$  is a proportionality constant. The model assumes that faults that occur are detected and corrected with certainty. We can characterize the JM model by the vector  $\underline{\lambda} =$

$(\lambda_1, \dots, \lambda_N)$ . For a given vector  $\underline{\lambda}$ , define the random variable  $T_{n,\underline{\lambda}}$  to be the time until the first  $n$  faults detected in the system are eliminated.

To show the impact of fault diversity, we need the following definition and lemma from pages 251 and 256 of Ross (1983).

Definition 6.1. The random variable  $X$  is stochastically larger than the random variable  $Y$ , written  $X \geq^s Y$  if  $P[X > a] \geq P[Y > a]$  for all  $a$ .

Lemma 6.1. Let  $X_1, X_2, \dots, X_n$  be independent and  $Y_1, Y_2, \dots, Y_n$  be independent. If  $X_i \geq^s Y_i$ , then for any increasing  $f$ ,  $f(X_1, \dots, X_n) \geq^s f(Y_1, \dots, Y_n)$ .

Theorem 6.1. Assuming the Jelinski-Moranda model, let  $\underline{\lambda} = (\lambda_1, \dots, \lambda_N)$  and  $\underline{\lambda}' = (\lambda_1', \dots, \lambda_N')$  be two vectors of fault detection rates for two software systems, where  $\lambda_i = (N-i+1)\lambda$  and  $\lambda_i' = (N-i+1)\lambda'$ . If  $\lambda \leq \lambda'$  then  $T_{n,\underline{\lambda}} \geq^s T_{n,\underline{\lambda}'}$  for any  $n \leq N$ .

Proof.  $\lambda_i = (N-i+1)\lambda \leq (N-i+1)\lambda' = \lambda_i'$  for all  $i=1, \dots, N$ . Then  $P[Z_i \geq a] = \exp(-\lambda_i a) \geq \exp(-\lambda_i' a) = P[Z_i' \geq a]$  so that  $Z_i \geq^s Z_i'$ . Define

$$f_n(Z_1, \dots, Z_N) = \sum_{i=1}^n Z_i = T_{n,\underline{\lambda}}.$$

Since  $f_n$  is an increasing function, by Lemma 6.1 we have  $T_{n,\lambda} \geq^{st} T_{n,\lambda'}$  for  $n \leq N$ .  $\parallel$

Clearly from the definition of weak supermajorization in Chapter 5, if  $\lambda \leq \lambda'$  then  $\underline{\lambda} >^w \underline{\lambda}'$ , thus for the JM model our result is similar to the Boland, Proschan, and Tong (1987) Theorem 5.2 p. 186 for models of the JM type which requires majorization instead of weak supermajorization to conclude  $T_{n,\lambda} \geq^{st} T_{n,\lambda'}$ .

From Theorem 6.1 we can conclude that if the fault detection rate  $\lambda_i$  for a software system is less than the fault detection rate  $\lambda_i'$  for another software system, then the time to correction of  $n$  faults for the first system,  $T_{n,\lambda}$  is longer than the  $T_{n,\lambda'}$  for the second system.

Now assume that the probability that a fault is detected upon occurrence is  $0 < p_2 < 1$ . Once the fault is detected it is immediately corrected. Suppose there are two teams working independently to correct faults in identical software systems. The first team detects faults with probability  $0 < p_2 < 1$ , while the second team detects faults with probability one. We can describe the fault correction rates for the two teams as  $\lambda_i = p_2 \lambda_i'$  and  $\lambda_i'$  respectively, then  $T_{n,\lambda} \geq^{st} T_{n,\lambda'}$ . Thus imperfect fault detection increases fault correction time, as expected.

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