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Abstract. Consider the fluid dynamic limit problem for the Broadwell system of the kinetic theory of gases, for Riemannn, Maxwellian initial data. The formal limit is the Riemann problem for a pair of conservation laws and is invariant under dilations of coordinates. The approach of self-similar fluid dynamic limits consists in replacing the mean free path in the Broadwell model so that the resulting problem preserves the invariance under dilations. The limiting procedure was justified in [ST]. Here, we study the structure of the emerging solutions. We show that they consist of two wave fans separated by a constant state. Each wave fan is associated with one of the characteristic fields and is either a rarefaction wave or a shock wave. The shocks satisfy the Lax shock conditions and have the internal structure of a Broadwell shock profile.

AMS (MOS) Subject Classifications: 35L60, 35Q20, 76P05

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WAVE STRUCTURE INDUCED BY FLUID DYNAMIC LIMITS IN THE BROADWELL MODEL

§1. Introduction

Being among the simplest models in the kinetic theory of gases, the Broadwell model has served as a paradigm to understand the phenomenon of relaxation and the transition from a microscopic to a macroscopic description of gases. It consists of the system of semilinear hyperbolic equations

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\varepsilon}\left(f_3^2 - f_1 f_2\right) \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= \frac{1}{\varepsilon}\left(f_3^2 - f_1 f_2\right) \\ \frac{\partial f_3}{\partial t} &= -\frac{1}{2\varepsilon}\left(f_3^2 - f_1 f_2\right), \end{aligned} \tag{1.1}$$

and is associated with a six-velocity model describing a system of particles with identical masses that move along three mutually orthogonal directions with speeds $\pm c$ and interact according to a mechanism of equiprobable binary collisions (Broadwell [B]). The system (1.1) derives from the six-velocity model when specializing to one-dimensional flows, for which the densities of particles moving in the directions orthogonal to the flow are all equal. (We refer to Platkowski and Illner [PI] for the derivation and references).

In this context the function $f = (f_1, f_2, f_3)$, defined for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, describes the densities of particles: f_1 for particles moving in the positive x-direction, f_2 in the negative x-direction and f_3 in each of the positive or negative y- or z-directions. The speed c is taken here equal to one and ε stands for the mean free path, a measure of the average distance between successive collisions. The right hand side is called the collision operator and measures the rate of gain (or loss) in densities of particles effected through collisions. It is characterized by the quantity

$$Q(f) = f_3^2 - f_1 f_2 \,. \tag{1.2}$$

The zeroes of $Q(f)$ are the states of equilibrium for the system, $f_3^2 = f_1 f_2$, and are called Maxwellians. Finally, associated with each f are the quantities

$$\rho_f = f_1 + f_2 + 4f_3 \,, \qquad m_f = f_1 - f_2 \tag{1.3}$$

measuring the local density and momentum flux in the $x-$direction, respectively.

The limit when the mean free path approaches zero is known as the fluid dynamic limit. For small mean free path the strong interactions of particles entail a macroscopic description of the flow to become meaningful. In the case of the Broadwell model the induced macroscopic "Euler equations" are easy to identify. First rewrite (1.1) in the form

$$\begin{aligned} \frac{\partial}{\partial t}(f_1 + f_2 + 4f_3) + \frac{\partial}{\partial x}(f_1 - f_2) &= 0 \,, \\ \frac{\partial}{\partial t}(f_1 - f_2) + \frac{\partial}{\partial x}(f_1 + f_2) &= 0 \,, \\ \frac{\partial f_3}{\partial t} &= -\frac{1}{2\varepsilon}(f_3^2 - f_1 f_2) \,. \end{aligned} \tag{1.4}$$

Formally, as $\varepsilon \rightarrow 0$, it is expected that the first two equations pass in the limit, while the third causes the limiting f to be a local Maxwellian. Thus the limit fluid equations become

$$\begin{aligned} \frac{\partial}{\partial t}\Big(f_1 + f_2 + 4(f_1 f_2)^{1/2}\Big) + \frac{\partial}{\partial x}(f_1 - f_2) &= 0 \,, \\ \frac{\partial}{\partial t}(f_1 - f_2) + \frac{\partial}{\partial x}(f_1 + f_2) &= 0 \,. \end{aligned} \tag{1.5}$$

The corresponding macroscopic density and momentum of the fluid are given by

$$\rho = f_1 + 4(f_1 f_2)^{1/2} + f_2 \,, \qquad m = \rho u = f_1 - f_2 \,. \tag{1.6}$$

The algebraic system can be easily inverted and leads to an alternative form of the limit "Euler equations", in terms of the macroscopic variables $(\rho\,,\,u)$,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\,, \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho g(u)) = 0\,, \end{aligned} \tag{1.7}$$

where $g(u) := \frac{1}{3}\left[2(1+3u^2)^{1/2} - 1\right]$. They form a strictly hyperbolic, genuinely nonlinear system of conservation laws (Caflisch [C]).

The justification of the fluid dynamic limit has been the object of several investigations. We refer to Cercignani [Ce] for a survey of the literature on the Boltzmann equation and to Platkowski and Illner [PI] for results on discrete velocity models of kinetic theory. For the Broadwell model, the fluid dynamic limit is understood for smooth solutions of the limit "Euler equations" (Inoue and Nishida [NI], Caflisch and Papanicolaou [CP]). Regarding the case of solutions with shocks, we mention the studies on stability (in time) of traveling wave solutions (Kawashima and Matsumura [KM]) or rarefaction wave solutions (Matsumura [M]) for the Broadwell model, and a recent study by Xin [X], showing that a given piecewise smooth solution with noninteracting shocks of the limit fluid equations can be approximated by solutions of the Broadwell system as $\varepsilon \to 0$, that gives a definitive answer to one direction of the problem. The converse problem, to show that a given family of solutions to the Broadwell system converges globally in time to a fluid-dynamical solution, remains at present open.

Insight in the latter direction is provided by the approach of self-similar fluid dynamic limits (Slemrod and Tzavaras [ST]). For Riemann, Maxwellian data

$$f(x,0) = \begin{cases} f^+, & x > 0 \\ f^-, & x < 0 \end{cases} \qquad \text{with } f^\pm = (f_1^\pm, f_2^\pm, f_3^\pm) \tag{1.8}$$

$$Q(f^-) = Q(f^+) = 0, \quad f_1^\pm,\, f_2^\pm,\, f_3^\pm > 0, \tag{M}$$

the solutions of the limit fluid equations (1.5) are expected to be self-similar functions of $\xi = x/t$. On the other hand, the Broadwell system does not possess space-time dilational invariance and does not admit self-similar solutions of that type. Motivated by an analogous idea for systems of conservation laws (Dafermos [D_1]), one considers a modified Broadwell system

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\varepsilon t}(f_3^2 - f_1 f_2), \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= \frac{1}{\varepsilon t}(f_3^2 - f_1 f_2), \\ \frac{\partial f_3}{\partial t} &= -\frac{1}{2\varepsilon t}(f_3^2 - f_1 f_2), \end{aligned} \tag{1.9}$$

which does preserve the dilational invariance $(x,t) \to (ax, at)$, $a > 0$. The resulting Riemann problem $(1.8 - 1.9)$ admits self-similar solutions of the form $f = f(x/t)$. These are constructed by solving the singular boundary value problem

$$\begin{aligned} -(\xi - 1) f_1'(\xi) &= (f_3^2 - f_1 f_2)/\varepsilon \\ -(\xi + 1) f_2'(\xi) &= (f_3^2 - f_1 f_2)/\varepsilon \\ -\xi f_3' &= -(f_3^2 - f_1 f_2)/2\varepsilon \end{aligned} \tag{1.10}$$

$$f(-1) = f^-,\ f(+1) = f^+,$$

for $\xi \in [-1,1]$ and $f^\pm$ subject to (M), referred to as problem $(\mathcal{P}_\varepsilon)$. It has the following property [ST]: Any family of solutions $\{f^\varepsilon\}_{\varepsilon>0}$ to $(\mathcal{P}_\varepsilon)$ corresponding to fixed data $f^\pm$ is of uniformly bounded variation and, as a consequence, there exists a subsequence $\{f^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ and a function of bounded variation f such that $f^{\varepsilon_n} \to f$ pointwise. The limiting f is a local Maxwellian, satisfying $f_3 = (f_1 f_2)^{1/2}$ a.e in $[-1,1]$. If f is extended to the entire real line, by setting $f = f^-$ on $(-\infty,-1)$ and $f = f^+$ on $(1,\infty)$, the function $f(x/t)$ becomes a weak solution of the Riemann problem (1.5, 1.8).

The goal of this article is to investigate the structure of the limiting function f constructed via self-similar fluid dynamic limits. Of special interest is to clarify the behavior of $f(\xi)$ at points of discontinuity. The Broadwell model, being among the simplest in kinetic theory, is a case study to understand the mechanism of relaxation and the associated admissibility restrictions imposed on shocks.

The presence of relaxation mechanisms is natural in many physical contexts and has been investigated extensively both for specific models, such as Broadwell, but also for systems of two conservation laws with relaxation (*e.g.* Liu [L]). For (1.1), the study of shock profiles by Broadwell [B] captures the regularizing effect induced on shocks by relaxation. These shock profiles have been compared, and are in close agreement for weak shocks, with traveling wave solutions of an associated system of viscous conservation laws arising from (1.1) via the Chapman-Enskog expansion (Caflish [C]). An important difference of the self-similar relaxation investigated here is that it penalizes the whole wave fan simultaneously. Comparisons between the structure entailed by self-similar limits and the Broadwell shock profiles are carried out in the text.

The structure of the limiting solution is completely characterized via this approach. It turns out that f consists of two wave fans separated by a constant state. Each wave fan is associated with one of the characteristic fields of the limit fluid equations and consists of either a single rarefaction or a single shock. The shocks satisfy the Lax shock conditions and have the internal structure of a Broadwell shock profile.

The approach of self-similar viscous limits to study the structure of the Riemann problem was initiated by Dafermos [D$_2$], in the context of systems of two conservation laws. Several of the ideas developed in [D$_2$] are used here, but the basic strategy is different. One starts with some representation formulas expressing the rate of collisions $Q(f^\varepsilon)/\varepsilon$ (and thus also the derivatives $f_j^{\varepsilon\prime}$) as an averaging process depending on functions of the solution f^ε. Using the estimates on the total variation of f^ε from [ST], it can be shown that there is a finite Borel-Stieltjes measure so that along a subsequence

$$\frac{Q(f^\varepsilon)}{\varepsilon} \rightharpoonup \nu \qquad \text{weak-}\star \text{ in measures}.$$

The measure ν incorporates the form of the limiting f. Although ν can not be explicitly computed via this approach, it is possible to establish several properties of ν. Its support coincides with the set where f is nonconstant. Most important, using the representation formulas, it turns out that at points of $\operatorname{supp}\nu$ a function g related to the antiderivatives of the eigenvalues is maximized (Proposition 4.4). The maximization property captures (a large part of) the restrictions imposed on solutions by the relaxation process. It provides the behavior of the eigenvalues to the limit fluid equations on $\operatorname{supp}\nu$ (Corollary 4.5). The first part of the proof, carried out in Sections 3 and 4, is completed by characterizing the behavior of right and left derivatives at points in the wave fans (Proposition 4.6).

The emerging structure is that typical (at least for small data) of strictly hyperbolic but not necessarily genuinely nonlinear systems. The reason is that although the analysis in Sections 3 and 4 makes essential use of the strict hyperbolicity property of the limit fluid equations, it never uses its genuine nonlinearity properties. Complementing this picture, the geometry of the shock curves for (1.5) is employed in Section 6 to rule out the possibility of contact discontinuities and arrive at the stated result. Finally in Section 5, an idea from [D$_2$] is adapted to show that at points of discontinuity the family f^ε has the internal structure of a Broadwell shock profile (Proposition 5.1).

§2. Preliminaries. The boundary value problem $(\mathcal{P}_\varepsilon)$.

First we review some preliminary information on solutions of the boundary value problem $(\mathcal{P}_\varepsilon)$. The material presented in this section is taken out of [ST].

As the underlying differential equations are singular, the meaning of solutions needs clarification. A function $f = (f_1, f_2, f_3)$ defined on $[-1,1]$ solves $(\mathcal{P}_\varepsilon)$ if $f_j \in C[-1,1]$ satisfy the boundary conditions $f_j(\pm 1) = f_j^{\pm}$, $j = 1,2,3$, and the weak form of the equations

$$-(\xi - 1)f_1(\xi)\Big|_a^b + \int_a^b f_1(\tau)d\tau = \frac{1}{\varepsilon}\int_a^b Q(f(\tau))d\tau\,, \tag{2.1}$$

$$-(\xi + 1)f_2(\xi)\Big|_a^b + \int_a^b f_2(\tau)d\tau = \frac{1}{\varepsilon}\int_a^b Q(f(\tau))d\tau\,, \tag{2.2}$$

$$-\xi f_3(\xi)\Big|_a^b + \int_a^b f_3(\tau)d\tau = -\frac{1}{2\varepsilon}\int_a^b Q(f(\tau))d\tau\,, \tag{2.3}$$

for all a, $b \in [-1,1]$. It follows immediately that a solution in the above sense is continuously differentiable (except possibly at the singular points ± 1 and 0) and satisfies the strong form of equations (1.10) on $(-1,0)$ and $(0,1)$. A more detailed analysis shows that for positive, Maxwellian data:

(i) $f_j(\xi) > 0$ for $\xi \in [-1,1]$

(ii) $Q(f(-1)) = Q(f(0)) = Q(f(+1)) = 0$

(iii) $f_j \in C^{\alpha_\varepsilon}[-1,1]$ for some $0 < \alpha_\varepsilon < 1$, with α_ε increasing as ε decreases; moreover, there is $\varepsilon_0(f^{\pm}) > 0$ such that for $\varepsilon < \varepsilon_0$ the functions f_j are Lipshitz continuous on $[-1,1]$.

(iv) $Q(f)$ does not change sign on the intervals $(-1,0)$ or $(0,1)$. As a consequence the shapes of the functions f_j belong to one of the following categories:

C$_1$: $Q(f) > 0$ on $(-1,0)$ and $(0,1)$; f_1 is increasing on $(-1,1)$, f_2 is decreasing on $(-1,1)$, f_3 is decreasing on $(-1,0)$ and increasing on $(0,1)$.

C$_2$: $Q(f) < 0$ on $(-1,0)$ while $Q(f) > 0$ on $(0,1)$; f_1 is decreasing on $(-1,0)$ and increasing on $(0,1)$, f_2 is increasing on $(-1,0)$ and decreasing on $(0,1)$, f_3 is increasing on $(-1,1)$.

C$_3$: $Q(f) < 0$ on $(-1,0)$ and $(0,1)$; f_1 is decreasing on $(-1,1)$, f_2 is increasing on $(-1,1)$, f_3 is increasing on $(-1,0)$ and decreasing on $(0,1)$.

C$_4$: $Q(f) > 0$ on $(-1,0)$ while $Q(f) < 0$ on $(0,1)$; f_1 is increasing on $(-1,0)$ and decreasing on $(0,1)$, f_2 is decreasing on $(-1,0)$ and increasing on $(0,1)$, f_3 is decreasing on $(-1,1)$.

C$_5$: $Q(f) = 0$ on $(-1,0)$ and/or $Q(f) = 0$ on $(0,1)$; In this case f_1, f_2, f_3 are constant on the region where $Q(f) = 0$, and f_1, f_2, f_3 have the behavior indicated in Cases 1 - 4 where $Q(f) > 0$ or $Q(f) < 0$.

Henceforth, the data $f^{\pm}$ are fixed subject to (M) and it is assumed that for each $\varepsilon > 0$ the problem $(\mathcal{P}_\varepsilon)$ admits a solution f^ε. The question of existence is considered in [ST]. The difficulty of this problem is that the underlying equations are singular and that one needs to guarantee existence of solutions for every value of the parameter $\varepsilon > 0$. This has only been accomplished for a certain class of boundary data $f^{\pm}$ (*c.f.* [ST]). However, such restrictions do not enter in any other way in the present analysis.

The functions f^ε are extended to the whole real line by setting $f^\varepsilon = f^-$ on $(-\infty, -1)$ and $f^\varepsilon = f^+$ on $(1, \infty)$. The restrictions on the shapes of solutions are the key ingredient of the following theorem [ST, Lemma 1.2, Theorem 2.1].

Theorem 2.1. *Let $\{f^\varepsilon\}_{\varepsilon>0}$ be a family of extended solutions of $(\mathcal{P}_\varepsilon)$ corresponding to data $f^{\pm}$ satisfying (M). Then:*

(a) *There are positive constants m_j, M_j and K_j, $j = 1, 2, 3$, depending on the boundary data $f^{\pm}$ but independent of ε such that*

$$0 < m_j \leq f_j^\varepsilon(\xi) \leq M_j\,, \qquad \xi \in [-1, 1] \tag{2.4}$$

$$TV_{[-1,1]}\, f_j^\varepsilon \leq K_j \tag{2.5}$$

(b) *There exists a subsequence $\{f^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ and a positive, bounded function f of bounded variation such that $f^{\varepsilon_n} \to f$ pointwise on the reals. The function f satisfies*

$$f = \begin{cases} f^- & \text{on } (-\infty, -1] \\ f^+ & \text{on } [1, \infty) \end{cases}\,, \qquad f_3 = \sqrt{f_1\, f_2} \quad \text{for a.e. } \xi \in [-1, 1] \tag{2.6}$$

and the balance of mass and momentum equations

$$
\begin{aligned}
-\xi \frac{d}{d\xi}(f_1 + f_2 + 4(f_1 f_2)^{1/2}) + \frac{d}{d\xi}(f_1 - f_2) = 0\,, \\
-\xi \frac{d}{d\xi}(f_1 - f_2) + \frac{d}{d\xi}(f_1 + f_2) = 0\,,
\end{aligned}
\tag{2.7}
$$

in the sense of distributions and in the sense of measures.

The function

$$
f(\frac{x}{t}) = \left(f_1(\frac{x}{t}),\ f_2(\frac{x}{t}) \right), \qquad (x,t) \in (-\infty, \infty) \times (0, \infty), \tag{2.8}
$$

is a weak solution of the Riemann problem (1.5, 1.8). Indeed, (1.8) is certainly satisfied. That the weak form of (1.5) is satisfied follows from the weak form of (2.7) by using a change of test functions.

§3. Structure of solutions I

Let $\{f^\varepsilon\}_{\varepsilon>0}$ be a family of solutions to $(\mathcal{P}_\varepsilon)$ corresponding to fixed boundary data $f^\pm$ subject to (M). The functions f^ε satisfy

$$
f_1^{\varepsilon\prime} = \frac{1}{1-\xi} \frac{Q(f^\varepsilon)}{\varepsilon} \tag{3.1}
$$

$$
f_2^{\varepsilon\prime} = -\frac{1}{\xi+1} \frac{Q(f^\varepsilon)}{\varepsilon} \tag{3.2}
$$

$$
f_3^{\varepsilon\prime} = \frac{1}{2\xi} \frac{Q(f^\varepsilon)}{\varepsilon} \tag{3.3}
$$

with

$$
Q(f^\varepsilon) = Q \circ f^\varepsilon = {f_3^\varepsilon}^2 - f_1^\varepsilon f_2^\varepsilon\,. \tag{3.4}
$$

The form of the above equations suggests to monitor the quantity $Q(f^\varepsilon)/\varepsilon$ and study its limiting behavior as $\varepsilon \to 0$.

Representation formulas

First certain representation formulas for $Q(f^\varepsilon)/\varepsilon$ and the derivatives $f_j^{\varepsilon\prime}$ are derived. We introduce the notation

$$\begin{aligned} c^\varepsilon &= \frac{f_1^\varepsilon}{\xi+1} - \frac{f_2^\varepsilon}{1-\xi} + \frac{f_3^\varepsilon}{\xi} \\ &= -\frac{1}{(1-\xi^2)\,\xi}\left[(f_1^\varepsilon + f_2^\varepsilon + f_3^\varepsilon)\,\xi^2 - (f_1^\varepsilon - f_2^\varepsilon)\,\xi - f_3^\varepsilon\right]. \end{aligned} \tag{3.5}$$

and remark that the function c^ε depends implicitly on ε through the dependence on the solution f^ε. A simple computation, using $(3.1-3.3)$, shows that

$$\frac{dQ(f^\varepsilon)}{d\xi} = \frac{1}{\varepsilon}\left(\frac{f_1^\varepsilon}{\xi+1} - \frac{f_2^\varepsilon}{1-\xi} + \frac{f_3^\varepsilon}{\xi}\right) Q(f^\varepsilon) = \frac{1}{\varepsilon}\, c^\varepsilon\, Q(f^\varepsilon). \tag{3.6}$$

After an integration (3.6) leads to

$$\frac{Q(f^\varepsilon(\xi))}{\varepsilon} = \begin{cases} \frac{1}{\varepsilon} Q(f^\varepsilon(\alpha_-))\ \exp\left\{\frac{1}{\varepsilon}\int_{\alpha_-}^{\xi} c^\varepsilon(s)\,ds\right\} & \text{for } \xi \in (-1,0) \\ \frac{1}{\varepsilon} Q(f^\varepsilon(\alpha_+))\ \exp\left\{\frac{1}{\varepsilon}\int_{\alpha_+}^{\xi} c^\varepsilon(s)\,ds\right\} & \text{for } \xi \in (0,1) \end{cases}, \tag{3.7}$$

where α_-, α_+ are any two points with $-1 < \alpha_- < 0 < \alpha_+ < 1$. It is convenient to introduce a notation

$$g^\varepsilon(\xi) = \begin{cases} \int_{\alpha_-}^{\xi} c^\varepsilon(s)\,ds & \xi \in (-1,0) \\ \int_{\alpha_+}^{\xi} c^\varepsilon(s)\,ds & \xi \in (0,1) \end{cases} \tag{3.8}$$

for the integrals appearing in (3.7).

Integrating (3.1) and using (3.7), we obtain

$$
\begin{aligned}
f_1^\varepsilon(0) - f_1^- &= \frac{Q(f^\varepsilon(\alpha_-))}{\varepsilon} \int_{-1}^{0} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta \\
f_1^+ - f_1^\varepsilon(0) &= \frac{Q(f^\varepsilon(\alpha_+))}{\varepsilon} \int_{0}^{1} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta
\end{aligned}
\tag{3.9}
$$

In turn, substitution of (3.7) and (3.9) in (3.1) gives

$$
f_1^{\varepsilon\prime}(\xi) = \left(f_1^\varepsilon(0) - f_1^-\right) \frac{\frac{1}{1-\xi}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\xi)}}{\int_{-1}^{0} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta}, \qquad \xi \in (-1, 0), \tag{3.10}
$$

$$
f_1^{\varepsilon\prime}(\xi) = \left(f_1^+ - f_1^\varepsilon(0)\right) \frac{\frac{1}{1-\xi}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\xi)}}{\int_{0}^{1} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta}, \qquad \xi \in (0, 1). \tag{3.11}
$$

Relations (3.10 – 3.11) are conceptually interesting as they express $f_1^{\varepsilon\prime}$ as an averaging process. Note that the weighting factors g^ε depend on the solution f^ε. Analogous relations hold for the derivatives $f_2^{\varepsilon\prime}$ and $f_3^{\varepsilon\prime}$.

In a similar manner using (3.1 – 3.3) and (3.7), we obtain

$$
\frac{Q(f^\varepsilon(\alpha_-))}{\varepsilon} = \frac{f_1^\varepsilon(0) - f_1^-}{\int_{-1}^{0} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} = \frac{f_2^\varepsilon(0) - f_2^-}{-\int_{-1}^{0} \frac{1}{\zeta+1}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} = \frac{f_3^\varepsilon(0) - f_3^-}{\int_{-1}^{0} \frac{1}{2\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} \tag{3.12}
$$

$$
\frac{Q(f^\varepsilon(\alpha_+))}{\varepsilon} = \frac{f_1^+ - f_1^\varepsilon(0)}{\int_{0}^{1} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} = \frac{f_2^+ - f_2^\varepsilon(0)}{-\int_{0}^{1} \frac{1}{\zeta+1}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} = \frac{f_3^+ - f_3^\varepsilon(0)}{\int_{0}^{1} \frac{1}{2\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} \tag{3.13}
$$

Equations (3.12 – 3.13), in conjunction with (3.7) and (3.1 – 3.3), show that $Q(f^\varepsilon)/\varepsilon$ and the derivatives $f_j^{\varepsilon\prime}$ may be expressed in a variety of ways. In the sequel we work with

$$
\frac{Q(f^\varepsilon)}{\varepsilon} = \begin{cases} (f_1^\varepsilon(0) - f_1^-)\, \mu_-^\varepsilon & \text{on } (-1, 0) \\ (f_2^\varepsilon(0) - f_2^+)\, \mu_+^\varepsilon & \text{on } (0, 1) \end{cases} \tag{3.14}
$$

where μ_-^ε is defined on the interval $(-1, 0)$ by

$$
\mu_-^\varepsilon(\xi) = \frac{e^{\frac{1}{\varepsilon} g^\varepsilon(\xi)}}{\int_{-1}^{0} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} = \frac{e^{\frac{1}{\varepsilon} \int_{\alpha_-}^{\xi} c^\varepsilon(s)\, ds}}{\int_{-1}^{0} \frac{1}{1-\zeta}\, e^{\frac{1}{\varepsilon} \int_{\alpha_-}^{\zeta} c^\varepsilon(s)\, ds}\, d\zeta}, \tag{3.15}
$$

μ_+^ε is defined on the interval $(0,1)$ by

$$\mu_+^\varepsilon(\xi) = \frac{e^{\frac{1}{\varepsilon} g^\varepsilon(\xi)}}{\int_0^1 \frac{1}{\zeta+1} e^{\frac{1}{\varepsilon} g^\varepsilon(\zeta)}\, d\zeta} = \frac{e^{\frac{1}{\varepsilon}\int_{\alpha_+}^{\xi} c^\varepsilon(s)\, ds}}{\int_0^1 \frac{1}{\zeta+1} e^{\frac{1}{\varepsilon}\int_{\alpha_+}^{\zeta} c^\varepsilon(s)\, ds}\, d\zeta} \quad , \tag{3.16}$$

and $\alpha_- \in (-1,0)$, $\alpha_+ \in (0,1)$ are arbitrary constants. The functions $\mu_\pm^\varepsilon$ are normalized versions of $Q(f^\varepsilon)/\varepsilon$ with a different normalization used in each of the intervals $(-1,0)$ and $(0,1)$. It is easy to check that they are independent of the specific choice of $\alpha_\pm$.

Our next task is to rewrite c^ε in a more suggestive form. Given $f = (f_1, f_2, f_3) \in \mathbb{R}^3$ with $f_j > 0$, the roots of the quadratic

$$(f_1 + f_2 + f_3)\,\xi^2 - (f_1 - f_2)\,\xi - f_3 = 0$$

are real and given by

$$\begin{aligned}\lambda_{1,2}(f) &= \frac{1}{2\,(f_1 + f_2 + f_3)} \left[f_1 - f_2 \pm \sqrt{(f_1 - f_2)^2 + 4 f_3\,(f_1 + f_2 + f_3)} \right] \\ &= \frac{1}{2\,(f_1 + f_2 + f_3)} \left[f_1 - f_2 \pm \sqrt{(f_1 + f_2 + 2f_3)^2 - 4 f_1 f_2} \right],\end{aligned} \tag{3.17}$$

where λ_1 corresponds to the minus sign and λ_2 to the plus. They enjoy the properties

$$-1 < -\frac{f_2 + f_3}{f_1 + f_2 + f_3} < \lambda_1(f) < 0 < \lambda_2(f) < \frac{f_1 + f_3}{f_1 + f_2 + f_3} < 1\,. \tag{3.18}$$

Using (3.17), c^ε may be expressed in the form

$$\begin{aligned}c^\varepsilon &= \frac{f_1^\varepsilon}{\xi + 1} - \frac{f_2^\varepsilon}{1 - \xi} + \frac{f_3^\varepsilon}{\xi} \\ &= -\frac{1}{(1 - \xi^2)\,\xi} \left(f_1^\varepsilon + f_2^\varepsilon + f_3^\varepsilon \right) \left(\xi - \lambda_1(f^\varepsilon) \right) \left(\xi - \lambda_2(f^\varepsilon) \right).\end{aligned} \tag{3.19}$$

Remark. It is instructive to compare the functions $\lambda_{1,2}(f)$ with the eigenvalues $\bar{\lambda}_{1,2}(f)$ of *the limit fluid equations* (1.5). The latter is a pair of conservation laws for $f = (f_1, f_2)$ of the type

$$\frac{\partial}{\partial t}A(f) + \frac{\partial}{\partial x}B(f) = 0\,. \tag{3.20}$$

Such a system is strictly hyperbolic provided that $\nabla A(f)$ is nonsingular and the roots of the characteristic polynomial $\det\left(\nabla B(f) - \lambda\,\nabla A(f)\right) = 0$ are *real and distinct*. For the case of (1.5), in the range of interest $f_1 > 0$ and $f_2 > 0$, the functions $A(f)$ and $B(f)$ are smooth. An easy computation shows that

$$\det \nabla A(f) = -\frac{2}{\sqrt{f_1 f_2}}\,(f_1 + f_2 + \sqrt{f_1 f_2}) \neq 0 \tag{3.21}$$

while the eigenvalues are real and given by

$$\bar{\lambda}_{1,2}(f) = \frac{1}{2(f_1 + f_2 + \sqrt{f_1 f_2})}\left[f_1 - f_2 \pm \sqrt{\left(f_1 + f_2 + 2\sqrt{f_1 f_2}\right)^2 - 4 f_1 f_2}\,\right]. \tag{3.22}$$

Note that $\bar{\lambda}_{1,2}(f)$ are determined from a two-vector $(f_1,\, f_2)$ while $\lambda_{1,2}(f)$ depend on a three-vector $(f_1,\, f_2,\, f_3)$ and thus are not directly related. However, the two functions coincide along Maxwellians, what suggests that $\lambda_{1,2}(f^\varepsilon)$ trace the eigenvalues of (1.5) along the fluid dynamic limit $\varepsilon \to 0$.

Behavior near the singular points

Next we analyze the limiting behavior of f^ε in the neighborhood of the singular points. In the sequel C will stand for a generic constant independent of ε.

Lemma 3.1. There are constants $\lambda_{1\pm}$, $\lambda_{2\pm}$ with $-1 < \lambda_{1-} < \lambda_{1+} < 0 < \lambda_{2-} < \lambda_{2+} < 1$ depending on m_j, M_j but not ε such that

$$\left|\frac{Q(f^\varepsilon(\xi))}{\varepsilon}\right| \le \begin{cases} \frac{C}{\varepsilon}\left(\frac{\xi+1}{\lambda_{1-}+1}\right)^{\frac{m_1}{2\varepsilon}} & \text{for } \xi \in (-1, \lambda_{1-}) \\ \frac{C}{\varepsilon}\left|\frac{\xi}{\lambda_{1+}}\right|^{\frac{m_3}{2\varepsilon}} & \text{for } \xi \in (\lambda_{1+}, 0) \\ \frac{C}{\varepsilon}\left|\frac{\xi}{\lambda_{2-}}\right|^{\frac{m_3}{2\varepsilon}} & \text{for } \xi \in (0, \lambda_{2-}) \\ \frac{C}{\varepsilon}\left(\frac{1-\xi}{1-\lambda_{2+}}\right)^{\frac{m_2}{2\varepsilon}} & \text{for } \xi \in (\lambda_{2+}, 1) \end{cases} \tag{3.23}$$

As a consequence

$$\frac{1}{\varepsilon} Q(f^\varepsilon) \to 0 \qquad \text{pointwise for } \xi \in \mathcal{L} := [-1, \lambda_{1-}) \cup (\lambda_{1+}, \lambda_{2-}) \cup (\lambda_{2+}, 1] \tag{3.24}$$

and uniformly on compact subsets of $\mathcal{L}$.

Proof. First we use (2.4) to uniformly estimate the behavior of c^ε in the neighborhood of the singular points.

Claim: There exist constants $-1 < \lambda_{1-} < \lambda_{1+} < 0 < \lambda_{2-} < \lambda_{2+} < 1$ such that

$$\begin{aligned} \frac{m_1}{2(s+1)} &< c^\varepsilon(s) && \text{for} \quad -1 < s < \lambda_{1-} \\ c^\varepsilon(s) &< \frac{m_3}{2s} && \text{for} \quad \lambda_{1+} < s < 0 \\ \frac{m_3}{2s} &< c^\varepsilon(s) && \text{for} \quad 0 < s < \lambda_{2-} \\ c^\varepsilon(s) &< -\frac{m_2}{2(1-s)} && \text{for} \quad \lambda_{2+} < s < 1 \end{aligned}$$

We show the first two inequalities. It follows from (3.19) and (2.4) that for $s \in (-1, -\frac{1}{2}]$

$$\begin{aligned} c^\varepsilon(s) &= \frac{f_1^\varepsilon(s)}{s+1} - \frac{f_2^\varepsilon(s)}{3/2} - \frac{f_3^\varepsilon(s)}{1/2} \\ &\ge \frac{m_1}{s+1} - \frac{M_2}{3/2} - \frac{M_3}{1/2} =: \frac{m_1}{s+1} - A \end{aligned}$$

If we set $\lambda_{1-} = \min\{-1/2\,,\,-1 + (m_1/2A)\}$, we see that

$$c^\varepsilon(s) \geq \frac{m_1}{s+1} - A > \frac{m_1}{2\,(s+1)} \qquad \text{on} \;\; (-1, \lambda_{1-})\,.$$

which completes the first inequality. Similarly, for $s \in [-\frac{1}{2}, 0)$

$$c^\varepsilon(s) \leq \frac{M_1}{1/2} - \frac{m_2}{3/2} + \frac{m_3}{s} =: \frac{m_3}{s} + B$$

Choose now $\lambda_{1+} = \max\{-1/2\,,\,-m_3/(2B)\}$ and observe that

$$c^\varepsilon(s) \leq \frac{m_3}{s} + B < \frac{m_3}{2s} \qquad \text{on} \;\; (\lambda_{1+}, 0)\,.$$

By construction the thresholds $\lambda_{1-}\,,\,\lambda_{1+}$ have the property $-1 < \lambda_{1-} < \lambda_{1+} < 0$. This proves the first part of the claim; the second part is shown in a similar fashion.

Next we set $\alpha_- = \lambda_{1-}$ in (3.7) and use (3.4), (2.4) and the first inequality in the claim to obtain for $-1 < \xi < \lambda_{1-}$

$$\begin{aligned}
\Big|\frac{Q(f^\varepsilon(\xi))}{\varepsilon}\Big| &\leq \frac{C}{\varepsilon}\,\exp\Big\{-\frac{1}{\varepsilon}\int_\xi^{\lambda_{1-}} c^\varepsilon(s)\,ds\Big\} \\
&\leq \frac{C}{\varepsilon}\,\exp\Big\{-\frac{1}{\varepsilon}\int_\xi^{\lambda_{1-}} \frac{m_1}{2\,(s+1)}\,ds\Big\} \\
&= \frac{C}{\varepsilon}\,\Big(\frac{\xi+1}{\lambda_{1-}+1}\Big)^{\frac{m_1}{2\varepsilon}}
\end{aligned}$$

Similarly, using (3.7) with $\alpha_- = \lambda_{1+}$ and the second inequality in the claim, we deduce for $\lambda_{1+} < \xi < 0$

$$\begin{aligned}
\Big|\frac{Q(f^\varepsilon(\xi))}{\varepsilon}\Big| &\leq \frac{C}{\varepsilon}\,\exp\Big\{\frac{1}{\varepsilon}\int_{\lambda_{1+}}^{\xi} c^\varepsilon(s)\,ds\Big\} \\
&\leq \frac{C}{\varepsilon}\,\exp\Big\{\frac{1}{\varepsilon}\int_{\lambda_{1+}}^{\xi} \frac{m_3}{2\,s}\,ds\Big\} \\
&= \frac{C}{\varepsilon}\,\Big|\frac{\xi}{\lambda_{1+}}\Big|^{\frac{m_3}{2\varepsilon}}
\end{aligned}$$

Hence, the first two assertions in (3.23) are proved. The proof of the other two is similar and is omitted. ■

§4. Structure of solutions II

Consider the family $\{f^\varepsilon\}_{\varepsilon>0}$ of extended solutions to the problem $(\mathcal{P}_\varepsilon)$ corresponding to positive, Maxwellian data $f^\pm$. The functions f^ε take the values f^- on $(-\infty,-1]$ and f^+ on $[1,\infty)$, they satisfy the equations $(2.1-2.3)$ for $a, b \in \mathbb{R}$, and enjoy the properties described in the preceding sections. Our objective is to study the structure of the limit points of the family $\{f^\varepsilon\}_{\varepsilon>0}$.

Helly's theorem and the uniform estimates $(2.4-2.5)$ imply that there exists a subsequence (denoted by) $\{f^\varepsilon\}$ and a function f such that $f^\varepsilon \to f$ pointwise on the reals. The limit $f = (f_1, f_2, f_3)$ is a function of bounded variation, its components f_j are strictly positive, and it satisfies $f = f^-$ on $(-\infty,-1]$, $f = f^+$ on $[1,\infty)$. Henceforth, attention is restricted to the subsequence $\{f^\varepsilon\}$, and the notation $f^\varepsilon \to f$ will mean that a subsequence of the original family converges to the function f; the choice of subsequences and the sense of convergence is specified in context.

The first lemma establishes that the sequence $\{\frac{Q(f^\varepsilon)}{\varepsilon}\}$ is uniformly bounded in L^1. This estimate is a consequence of the variation bounds and stronger than the corresponding estimate provided by the balance of entropy identity (*c.f.* [ST]). It measures how fast $\{f^\varepsilon\}$ relaxes to a local Maxwellian.

Lemma 4.1. For any a, $b \in [-1,1]$, $a < b$,

$$\int_a^b \left|\frac{Q(f^\varepsilon)}{\varepsilon}\right| d\tau \le C \tag{4.1}$$

where C is a constant independent of ε.

Proof. By (iv) in Section 2, $Q(f^\varepsilon)$ has a sign on each of the subintervals $(-1,0)$ and $(0,1)$. The proof will follow by a case analysis. Consider the case $0 \le a < b \le 1$ and $Q(f^\varepsilon) > 0$ on $(0,1)$. Using (2.3) and (2.4), we obtain

$$\int_a^b \frac{Q(f^\varepsilon(\tau))}{\varepsilon}\, d\tau = 2\left(b\, f_3^\varepsilon(b) - a\, f_3^\varepsilon(a)\right) - 2\int_a^b f_3^\varepsilon(\tau)\, d\tau \le C\,,$$

which shows (4.1) in this case.

All other cases are treated similarly. When a, b take values on different subintervals, (4.1) follows from a combination of the estimates on each subinterval. ∎

Since $f^\varepsilon \to f$, it follows from (4.1) and (2.4) that

$$\int_a^b |Q(f)|\, d\tau = 0 \qquad \text{for any } a,\, b \in [-1,1]\,.$$

Hence,

$$f_3(\xi) = \left(f_1(\xi)\, f_2(\xi)\right)^{1/2} \qquad \text{for } \xi \in [-1,1] - \mathcal{E}\,,$$

where the set $\mathcal{E}$ of exceptional points has Lebesgue measure $m(\mathcal{E}) = 0$.

Since f is of bounded variation, its domain can be decomposed into two disjoint subsets $\mathcal{C} \cup \mathcal{S}$, such that on $\mathcal{C}$ all three components of f are continuous while on $\mathcal{S}$ at least one of the f_j is discontinuous. $\mathcal{S}$ is at most countable and the right and left limits $f(\xi+)$ and

$f(\xi-)$ exist at any $\xi \in \mathcal{S}$. The functions f_j inherit the monotonicity properties of f_j^ε; as a result $\xi = 0$ is the only possible point where $f(\xi-) = f(\xi+) \neq f(\xi)$ (but this is excluded as well later). So $\mathcal{S}$ includes no points of removable discontinuity. It follows

$$
\begin{aligned}
f_3(\xi) &= \left(f_1(\xi)\, f_2(\xi) \right)^{1/2} \qquad \text{for } \xi \in \mathcal{C}\,, \\
f_3(\xi+) = \left(f_1(\xi+)\, f_2(\xi+) \right)^{1/2}, \quad f_3(\xi-) &= \left(f_1(\xi-)\, f_2(\xi-) \right)^{1/2} \qquad \text{for } \xi \in \mathcal{S}\,,
\end{aligned}
\tag{4.2}
$$

and $\mathcal{E} \subset \mathcal{S}$ is at most countable. It is in this precise sense that f is a local Maxwellian. We show next that at points of discontinuity the Rankine-Hugoniot conditions are satisfied.

Lemma 4.2. For any point $\xi \in \mathcal{S}$

$$
\begin{aligned}
-\xi \left(\left[f_1 + f_2 + 4\sqrt{f_1\, f_2} \right] \Big|_{\xi-}^{\xi+} \right) + \left[f_1 - f_2 \right] \Big|_{\xi-}^{\xi+} &= 0 \\
-\xi \left(\left[f_1 - f_2 \right] \Big|_{\xi-}^{\xi+} \right) + \left[f_1 + f_2 \right] \Big|_{\xi-}^{\xi+} &= 0
\end{aligned}
\tag{4.3}
$$

Proof. Fix $\xi \in \mathcal{S}$ and pick any $a,\, b \in [-1,1] - \mathcal{S}$ with $a < \xi < b$. In view of $(2.1 - 2.3)$ the sequence $\{f^\varepsilon\}$ satisfies the identities

$$
\begin{aligned}
&-b \left[f_1^\varepsilon(b) + f_2^\varepsilon(b) + 4\, f_3^\varepsilon(b) \right] + a \left[f_1^\varepsilon(a) + f_2^\varepsilon(a) + 4\, f_3^\varepsilon(a) \right] \\
&\qquad + \left[f_1^\varepsilon(b) - f_2^\varepsilon(b) \right] - \left[f_1^\varepsilon(a) - f_2^\varepsilon(a) \right] + \int_a^b \left(f_1^\varepsilon(\tau) + f_2^\varepsilon(\tau) + 4\, f_3^\varepsilon(\tau) \right) d\tau = 0
\end{aligned}
$$

$$
\begin{aligned}
&-b \left[f_1^\varepsilon(b) - f_2^\varepsilon(b) \right] + a \left[f_1^\varepsilon(a) - f_2^\varepsilon(a) \right] + \left[f_1^\varepsilon(b) + f_2^\varepsilon(b) \right] \\
&\qquad - \left[f_1^\varepsilon(a) + f_2^\varepsilon(a) \right] + \int_a^b \left(f_1^\varepsilon(\tau) - f_2^\varepsilon(\tau) \right) d\tau = 0\,.
\end{aligned}
$$

First we pass to the limit $\varepsilon \to 0$ using the property $f_3 = \sqrt{f_1 f_2}$ on $\mathcal{C}$, and then let the points $a \to \xi-$ and $b \to \xi+$. In the combined limit we obtain the jump conditions (4.3). ∎

Since f may have discontinuities, the appropriate framework for passing to the $\varepsilon \to 0$ limit in $(3.1 - 3.3)$ is that of measures. Consider the functions

$$\Phi^\varepsilon(\xi) = \int_{-\infty}^{\xi} \frac{Q(f^\varepsilon(\tau))}{\varepsilon}\, d\tau \tag{4.4}$$

and note that Φ^ε takes constant values outside $[-1,1]$. By Lemma 4.1, the sequence $\{\Phi^\varepsilon\}$ is uniformly bounded and of uniformly bounded variation on the reals. Helly's theorem implies the existence of a subsequence, denoted again by $\{\Phi^\varepsilon\}$, and a function of bounded variation Φ such that $\Phi^\varepsilon \to \Phi$ pointwise on $\mathbb{R}$. Henceforth attention is restricted to this convergent subsequence.

Let $\varphi \in C_c(\mathbb{R})$ be any continuous function with compact support in the reals and consider the Riemann-Stieltjes integrals

$$\begin{aligned} < \nu^\varepsilon, \varphi > &:= \int \varphi(\xi)\, d\Phi^\varepsilon(\xi) = \int \varphi(\xi)\, \frac{Q(f^\varepsilon(\xi))}{\varepsilon}\, d\xi \\ < \nu, \varphi > &:= \int \varphi(\xi)\, d\Phi(\xi)\,. \end{aligned} \tag{4.5}$$

Helly's convergence theorem (Natanson [N, VII.7]) implies that

$$< \nu^\varepsilon, \varphi >= \int \varphi\, d\Phi^\varepsilon \to \int \varphi\, d\Phi =< \nu, \varphi > \qquad \text{for any } \varphi \in C_c(\mathbb{R})\,. \tag{4.6}$$

By the Riesz representation theorem ν^ε, ν may be viewed as finite (signed) Borel-Stieltjes measures. They are both supported on $[-1,1]$, ν^ε is generated by Φ^ε while ν is generated by $\bar{\Phi}$, the right continuous version of Φ defined by $\bar{\Phi}(x) = \Phi(x+)$ (*c.f.* Folland [F, Ch. 3.5, Ch. 7]). Equation (4.6) states that $\nu^\varepsilon \rightharpoonup \nu$ in the weak-$\star$ topology of measures. It allows to pass to the limit $\varepsilon \to 0$ in $(3.1 - 3.3)$ and obtain

$$\begin{aligned}
\int f_1\,((\xi-1)\varphi)'\,d\xi &= <\nu,\varphi> \\
\int f_2\,((\xi+1)\varphi)'\,d\xi &= <\nu,\varphi> \\
\int f_3\,(\xi\,\varphi)'\,d\xi \qquad &= -\frac{1}{2} <\nu,\varphi>
\end{aligned} \tag{4.7}$$

for any $\varphi \in C_c^1(\mathbb{R})$.

The next lemma characterizes the support of ν, $\operatorname{supp}\nu$, as precisely the set of points where f is not a constant state.

Lemma 4.3. Let ν^ε and ν be as in (4.5). Then

(i) There are constants $-1 < \lambda_{1-} < \lambda_{1+} < 0 < \lambda_{2-} < \lambda_{2+} < 1$ such that

$$f = \begin{cases} f_- & \text{on } (-\infty, \lambda_{1-}) \\ f(0) & \text{on } (\lambda_{1+}, \lambda_{2-}) \\ f_+ & \text{on } (\lambda_{2+}, \infty) \end{cases} \tag{4.8}$$

and $\operatorname{supp}\nu \subset [\lambda_{1-}, \lambda_{1+}] \cup [\lambda_{2-}, \lambda_{2+}]$.

(ii) $\xi \notin \operatorname{supp}\nu$ if and only if there exists an open interval I containing ξ such that f is constant on I.

(iii) $\mathcal{S} \subset \operatorname{supp}\nu$.

Proof. Let $\lambda_{1\pm}, \lambda_{2\pm}$ be as in Lemma 3.1. Integration of (3.3) over $(0,\xi)$, $0 < \xi < \lambda_{2-}$, and use of (3.23) give

$$\begin{aligned}
|f_3^\varepsilon(\xi) - f_3^\varepsilon(0)| &= \Big|\int_0^\xi \frac{Q(f^\varepsilon(\zeta))}{2\,\varepsilon\,\zeta}\,d\zeta\,\Big| \\
&\le \frac{C}{\varepsilon}\int_0^\xi \Big(\frac{\zeta}{\lambda_{2-}}\Big)^{\frac{m_3}{2\varepsilon}}\,\frac{1}{\zeta}\,d\zeta = \frac{2\,C}{m_3}\Big(\frac{\xi}{\lambda_{2-}}\Big)^{\frac{m_3}{2\varepsilon}}.
\end{aligned}$$

Passing to the limit $\varepsilon \to 0$, we see that $f_3 = f_3(0)$ for $\xi \in [0, \lambda_{2-})$. Similar arguments establish all other statements in (4.8).

Next we turn to the properties related to the support of ν. Recall that $\xi \notin \operatorname{supp} \nu$ if there exists an open interval $I \ni \xi$ such that $< \nu, \varphi >= 0$ for any $\varphi \in C_c(I)$. Let φ be continuous with $\operatorname{supp} \varphi \subset (-\infty, \lambda_{1-}) \cup (\lambda_{1+}, \lambda_{2-}) \cup (\lambda_{2+}, \infty)$. Then (4.6) and (3.23) imply

$$< \nu^{\varepsilon}, \varphi >= \int \varphi \frac{Q(f^{\varepsilon})}{\varepsilon} d\xi \to 0 =< \nu, \varphi > \qquad \text{as } \varepsilon \to 0.$$

Hence, $\operatorname{supp} \nu \subset [\lambda_{1-}, \lambda_{1+}] \cup [\lambda_{2-}, \lambda_{2+}]$.

It suffices to show (ii) for points $\xi \in [\lambda_{1-}, \lambda_{1+}] \cup [\lambda_{2-}, \lambda_{2+}]$. It follows as a direct implication of the statement:

Claim. Let I be an open interval such that $\bar{I} \subset (0,1)$ or $\bar{I} \subset (-1, 0)$. Then f is constant on I if and only if $< \nu, \varphi >= 0$ for any $\varphi \in C_c^1(I)$.

One direction in the claim is clear from (4.7). To show the converse, suppose that $\bar{I} \subset (0.1)$ (for concreteness) and that $< \nu, \varphi >= 0$ for any $\varphi \in C_c^1(I)$. The third identity in (4.7) then implies

$$\int_I f_3 \, \psi' \, d\xi = 0 \qquad \text{for any } \psi \in C_c^1(I).$$

Fix $\chi \in C_c(I)$ with $\int_I \chi \, ds = 1$. Given $\phi \in C_c(I)$ define

$$\psi = \int_{-\infty}^{\xi} \left(\phi - \chi \left(\int_I \phi \, d\tau\right)\right) ds$$

Then $\psi \in C_c^1(I)$ and using that as a test function we obtain

$$\int_I f_3 \left(\phi - \chi \left(\int_I \phi \, d\tau\right)\right) d\xi = \int_I \left(f_3 - \int_I (f_3 \, \chi) \, ds\right) \phi \, d\xi = 0 \qquad \text{for any } \phi \in C_c(I).$$

It follows that $f_3 = \int_I (f_3\,\chi)\,d\xi$ a.e. on I. Since f_3 is monotone in $(0,1)$, f_3 is constant on I. In a similar fashion it follows that f_1 and f_2 are also constant on I.

This completes the proof of part (ii). Part (iii) is a consequence of part (ii). ∎

Along the convergent sequence $f^\varepsilon \to f$ the functions c^ε converge pointwise (for $\xi \neq \pm 1, 0$) to a function c. In view of (3.19) and (3.8), we write

$$\begin{aligned} c^\varepsilon &= -\frac{1}{(1-\xi^2)\,\xi}\left(f_1^\varepsilon + f_2^\varepsilon + f_3^\varepsilon\right)\left(\xi - \lambda_1(f^\varepsilon)\right)\left(\xi - \lambda_2(f^\varepsilon)\right) \\ \to c &:= -\frac{1}{(1-\xi^2)\,\xi}\left(f_1 + f_2 + f_3\right)\left(\xi - \lambda_1(f)\right)\left(\xi - \lambda_2(f)\right) \\ &= \frac{f_1}{\xi+1} - \frac{f_2}{1-\xi} + \frac{f_3}{\xi} \qquad \text{pointwise on } (-1,0)\cup(0,1), \end{aligned} \tag{4.9}$$

and introduce the function

$$g(\xi) = \begin{cases} \int_{\alpha_-}^{\xi} c(s)\,ds & \xi \in (-1,0) \\ \int_{\alpha_+}^{\xi} c(s)\,ds & \xi \in (0,1) \end{cases} \tag{4.10}$$

with $\alpha_- \in (-1,0)$, $\alpha_+ \in (0,1)$ fixed constants.

The following proposition is the main ingredient of the analysis. It characterizes points in $\operatorname{supp}\nu$ as being points of global maxima (in appropriate intervals) for the function g. This property incorporates admissibility restrictions induced by the relaxation process.

Proposition 4.4. Let $\xi \in \operatorname{supp}\nu$.

(i) If $\xi \in \operatorname{supp}\nu \cap (0,1)$ then $g(\zeta) \le g(\xi)$ for any $\zeta \in (0,1)$

(ii) If $\xi \in \operatorname{supp}\nu \cap (-1,0)$ then $g(\zeta) \le g(\xi)$ for any $\zeta \in (-1,0)$.

Proof. We show (i). The proof proceeds in two steps.

Step 1. Fix some $\alpha_+ \in (0,1)$ and consider the functions g^ε defined in (3.8). The dominated convergence theorem, (4.9) and (2.4) imply that

$$g^\varepsilon = \int_{\alpha_+}^{\xi} c^\varepsilon(s)\,ds \to g = \int_{\alpha_+}^{\xi} c(s)\,ds \quad \text{pointwise for } \xi \in (0,1)\,.$$

By virtue of (2.4), (3.19) and (3.17), on any compact $[a,b] \subset (0,1)$ the sequence $\{g^\varepsilon\}$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $\{g^{\varepsilon'}\}$ converges uniformly. Since the whole sequence converges pointwise to g,

$$g^\varepsilon \to g = \int_{\alpha_+}^{\xi} c(s)\,ds \quad \text{uniformly on any } [a,b] \subset (0,1)\,. \tag{4.11}$$

The limit g is uniformly continuous on $[a,b] \subset (0,1)$.

Step 2. The second step is to show:

If $\xi \in (0,1)$ is such that for some $\alpha > 0$ the set

$$\mathcal{A} = \{\, \zeta \in (0,1) : \quad g(\zeta) - g(\xi) = \int_{\xi}^{\zeta} c(s)\,ds > \alpha \,\} \tag{4.12}$$

has positive Lebesgue measure $m(\mathcal{A}) > 0$, then $\xi \notin \operatorname{supp} \nu$.

To show (4.12), let $\xi \in (0,1)$ be fixed and let $\alpha > 0$ and $\mathcal{A}$ be as above with $m(\mathcal{A}) > 0$. Using (3.16) and (3.8), we write

$$\mu_+^\varepsilon(\xi) = \frac{1}{\int_0^1 \frac{1}{\zeta+1}\, e^{\frac{1}{\varepsilon}(g^\varepsilon(\zeta) - g^\varepsilon(\xi))}\, d\zeta} = \frac{1}{\int_0^1 \frac{1}{\zeta+1}\, e^{\frac{1}{\varepsilon}\int_\xi^\zeta c^\varepsilon(s)\,ds}\, d\zeta}\,. \tag{4.13}$$

Our goal is to estimate the integral in the denominator.

First, note that the estimates for the function c^ε in Lemma 3.1 can be carried to the limit $\varepsilon \to 0$ to show: There are constants $0 < \lambda_{2-} < \lambda_{2+} < 1$ such that

$$\begin{aligned} \frac{m_3}{2s} &< c(s) && \text{for} \quad 0 < s < \lambda_{2-} \\ c(s) &< -\frac{m_2}{2(1-s)} && \text{for} \quad \lambda_{2+} < s < 1 \end{aligned}$$

Hence, for $0 < \zeta < \lambda_{2-}$,

$$\int_{\lambda_{2-}}^{\zeta} c(s)\,ds < \frac{m_3}{2} \ln \frac{\zeta}{\lambda_{2-}} \to -\infty, \qquad \text{as } \zeta \to 0+,$$

while, for $\lambda_{2+} < \zeta < 0$,

$$\int_{\lambda_{2+}}^{\zeta} c(s)\,ds < \frac{m_2}{2} \ln \frac{1-\zeta}{1-\lambda_{2+}} \to -\infty, \qquad \text{as } \zeta \to 1-.$$

It follows that points ζ near the endpoints of $(0,1)$ violate the inequality in (4.12), and that $\mathcal{A}$ is contained in some compact interval $[c,d] \subset (0,1)$.

Fix now $\delta > 0$ such that

$$\xi' \in (\xi - \delta,\, \xi + \delta) \quad \text{implies} \quad |\, g(\xi) - g(\xi') \,| < \frac{\alpha}{6}$$

Because of the uniform convergence on compact subsets (4.11), we can choose $\varepsilon_0 > 0$ such that

$$\xi' \in (\xi - \delta,\, \xi + \delta), \quad \varepsilon < \varepsilon_0, \quad \zeta \in \mathcal{A} \quad \text{imply} \quad g^\varepsilon(\zeta) - g^\varepsilon(\xi') > \frac{\alpha}{2}.$$

In turn, (4.13) yields for $\xi' \in (\xi - \delta, \xi + \delta)$

$$0 < \mu_+^\varepsilon(\xi') \le \frac{1}{\int_{\mathcal{A}} \frac{1}{\zeta+1} \exp\{\frac{1}{\varepsilon} \int_{\xi'}^{\zeta} c^\varepsilon(s)\,ds\}\,d\zeta} < \frac{2}{m(\mathcal{A})} e^{-(\alpha/2\varepsilon)}. \tag{4.14}$$

Let φ be a continuous function with $\operatorname{supp}\varphi \subset (\xi - \delta, \xi + \delta)$. Then (4.5), (4.6), (3.14) and (4.14) together imply

$$
\begin{aligned}
< \nu^\varepsilon, \varphi > &= \int \frac{Q(f^\varepsilon(\xi'))}{\varepsilon} \varphi(\xi')\, d\xi' \\
&= (f_2^\varepsilon(0) - f_2^+) \int_{(\xi-\delta,\xi+\delta)} \mu_+^\varepsilon(\xi') \varphi(\xi')\, d\xi' \to 0
\end{aligned}
$$

Thus $\xi \notin \operatorname{supp}\nu$ and the proof of the statement (4.12) is complete.

It follows that for $\xi \in \operatorname{supp}\nu$ it is $m(\mathcal{A}) = 0$ for any positive α. Since g is continuous, that implies

$$
g(\zeta) - g(\xi) = \int_\xi^\zeta c(s)\, ds \leq \alpha \qquad \text{for all } \zeta \in (0,1)\,,\ \alpha > 0\,.
$$

Hence, $g(\zeta) \leq g(\xi)$ for all $\zeta \in (0,1)$. ∎

The maximization properties of g at points in the support of ν provide information on the structure of the limit function f. In particular, a weak form of the Lax shock conditions is satisfied at any point of jump discontinuity.

Corollary 4.5. Let ξ, $\xi' \in [-1,1]$, $\xi < \xi'$.

(a) If $\xi \in \mathcal{C} \cap \operatorname{supp}\nu$ then

$$
\begin{aligned}
\xi &= \lambda_1(f(\xi)) \qquad \text{for } \xi < 0\,, \\
\xi &= \lambda_2(f(\xi)) \qquad \text{for } \xi > 0\,.
\end{aligned}
\tag{4.15}
$$

(b) If $\xi \in \mathcal{S}$ then f satisfies at ξ the jump conditions (4.3) and the inequalities

$$
\begin{aligned}
\lambda_1(f(\xi+)) \leq \xi \leq \lambda_1(f(\xi-)) \qquad \text{for } \xi < 0\,, \\
\lambda_2(f(\xi+)) \leq \xi \leq \lambda_2(f(\xi-)) \qquad \text{for } \xi > 0\,.
\end{aligned}
\tag{4.16}
$$

(c) If ξ, $\xi' \in \operatorname{supp}\nu \cap (-1,0)$ then $\lambda_1(f(\xi+)) = \xi$, $\lambda_1(f(\xi'-)) = \xi'$ and for $\theta \in (\xi, \xi')$:

$$\theta = \lambda_1(f(\theta)) \qquad \text{if } \theta \in \mathcal{C} \tag{4.17}$$

$$\lambda_1(f(\theta+)) = \theta = \lambda_1(f(\theta-)) \qquad \text{if } \theta \in \mathcal{S} \tag{4.18}$$

If ξ, $\xi' \in \operatorname{supp}\nu \cap (0,1)$ then $\lambda_2(f(\xi+)) = \xi$, $\lambda_2(f(\xi'-)) = \xi'$ and for $\theta \in (\xi, \xi')$:

$$\theta = \lambda_2(f(\theta)) \qquad \text{if } \theta \in \mathcal{C} \tag{4.19}$$

$$\lambda_2(f(\theta+)) = \theta = \lambda_2(f(\theta-)) \qquad \text{if } \theta \in \mathcal{S} \quad . \tag{4.20}$$

Proof. The proof is presented for $\xi \in (0,1)$; the case of $(-1,0)$ is similar.

Consider the function g in (4.10) restricted to the interval $(0,1)$. Note that g is continuous and $g(\zeta) \to -\infty$ as $\zeta \to 0+$ or $\zeta \to 1-$. Since c is of bounded variation

$$\lim_{\zeta \to \xi\pm} \frac{g(\zeta) - g(\xi)}{\zeta - \xi} = \lim_{\zeta \to \xi\pm} \frac{1}{\zeta - \xi} \int_\xi^\zeta c(s)\, ds = c(\xi\pm)\,. \tag{4.21}$$

Thus $\frac{dg}{d\xi}$ exists and is continuous at points of $\mathcal{C}$, while only the right and left derivatives exist at points of $\mathcal{S}$. Let $\xi \in \operatorname{supp}\nu \cap (0,1)$. By Proposition 4.4 the function g achieves its global maximum on $(0,1)$ at ξ

$$g(\zeta) \le g(\xi) \qquad \text{for any } \ \zeta \in (0,1)\,.$$

First we show (a) and (b): At ξ

$$c(\xi+) \le 0\,, \qquad c(\xi-) \ge 0\,,$$

from where, using (4.9), (3.18) and noting that $\xi > 0$, we conclude

$$\xi - \lambda_2(f(\xi+)) \geq 0\,, \qquad \xi - \lambda_2(f(\xi-)) \leq 0\,. \tag{4.22}$$

If $\xi \in \mathcal{C}$ then (4.22) leads to (4.15). If $\xi \in \mathcal{S}$ then the jump conditions (4.3) are satisfied at ξ, by Lemma 4.2, and (4.22) gives (4.16).

Next we show (c). The points ξ, $\xi' \in \operatorname{supp}\nu \cap (0,1)$, $\xi < \xi'$, are both global maxima for g. Therefore $g(\xi) = g(\xi')$. We claim:

$$g(\theta) = g(\xi) \quad \text{for any } \theta \in (\xi, \xi')\,. \tag{4.23}$$

If (4.23) is violated at some point, there exist a, b with $\xi \leq a < b \leq \xi'$ such that

$$g(a) = g(b) = g(\xi)\,, \qquad g(\theta) < g(\xi) \quad \text{for } a < \theta < b\,.$$

At the points a, b we have

$$\lambda_2(f(a+)) \leq a \leq \lambda_2(f(a-))$$

$$\lambda_2(f(b+)) \leq b \leq \lambda_2(f(b-))$$

On the other hand, it follows from Lemma 4.3 and Proposition 4.4 that f is constant in the interval (a,b) and thus $\lambda_2(f(a+)) = \lambda_2(f(b-))$. The inequalities then imply $b \leq a$, which contradicts $a < b$; hence (4.23) follows.

This establishes that g stays constant at its maximum value on $[\xi,\ \xi']$. Therefore $c(\xi+) = c(\xi'-) = 0$ and $c(\theta-) = c(\theta+) = 0$ for $\xi < \theta < \xi'$. In turn, these yield

$$\theta = \lambda_2(f(\theta)) \qquad \text{for } \theta \in \mathcal{C}$$

$$\lambda_2(f(\theta+)) = \theta = \lambda_2(f(\theta-)) \qquad \text{for } \theta \in \mathcal{S}$$

$$\xi = \lambda_2(f(\xi+))$$

$$\xi' = \lambda_2(f(\xi'-))$$

and the proof is complete. ∎

Behavior of solutions in the wave fans

It follows that the region where f is nonconstant consists of two disjoint closed intervals: $I_{\lambda_1} = [a_1, b_1] \subset (-1, 0)$ associated with the first characteristic speed $\lambda_1(f)$ of (1.5), and $I_{\lambda_2} = [a_2, b_2] \subset (0, 1)$ associated with the second characteristic speed $\lambda_2(f)$. Each of I_{λ_1} or I_{λ_2} could be empty or consist of just a single point. The function f takes constant values on the complement of $I_{\lambda_1} \cup I_{\lambda_2}$, while it has the behavior indicated in Corollary 4.5 at points of I_{λ_1} or I_{λ_2}.

Our next objective is to obtain a fuller description of the behavior of f on the two wave fans. The following proposition does not use the specific form of (2.7), but only its structural properties and relations (4.17 – 4.20). It is thus convenient to work with the abstract form of (2.7)

$$-\xi \frac{d}{d\xi} A(f) + \frac{d}{d\xi} B(f) = 0$$

and the associated weak form

$$-b\,A(f(b)) + a\,A(f(a)) + B(f(b)) - B(f(a)) + \int_a^b A(f(s))\,ds = 0 \qquad a,\, b \in \mathbb{R}\,. \quad (4.24)$$

Consider the strictly hyperbolic system (3.20) with $A(f)$, $B(f)$ at least twice continuously differentiable and $\nabla A(f)$ nonsingular. Let $\lambda_i(f)$ be its eigenvalues, $l_i(f)$ the left eigenvectors and $r_i(f)$ the right eigenvectors. They are connected through the relations

$$\begin{aligned} \nabla B(f)\, r_i(f) &= \lambda_i(f)\, \nabla A(f)\, r_i(f) \\ l_i(f)\cdot \nabla B(f) &= \lambda_i(f)\, l_i(f)\cdot \nabla A(f) \end{aligned} \qquad (4.25)$$

and satisfy the normalization conditions

$$l_i(f)\cdot \nabla A(f)\, r_j(f) = \delta_{ij}\,. \qquad (4.26)$$

Proposition 4.6 Suppose that $I_{\lambda_k} = [a_k, b_k]$ is a full interval, $a_k < b_k$.

(i) For each $\xi \in [a_k, b_k)$ such that $\nabla\lambda_k(f(\xi+))\cdot r_k(f(\xi+)) \neq 0$ it is

$$\lim_{h\to 0\,,\,h>0} \frac{1}{h}\big(f(\xi + h-) - f(\xi+)\big) = \frac{1}{\nabla\lambda_k(f(\xi+))\cdot r_k(f(\xi+))}\, r_k(f(\xi+)) \qquad (4.27)$$

(ii) For each $\xi \in (a_k, b_k]$ such that $\nabla\lambda_k(f(\xi-))\cdot r_k(f(\xi-)) \neq 0$ it is

$$\lim_{h\to 0\,,\,h<0} \frac{1}{h}\big(f(\xi + h+) - f(\xi-)\big) = \frac{1}{\nabla\lambda_k(f(\xi-))\cdot r_k(f(\xi-))}\, r_k(f(\xi-)) \qquad (4.28)$$

Proof. We show (i). Fix $\xi \in [a_k, b_k)$ and let $h > 0$ such that $\xi + h \in I_{\lambda_k}$. The weak form (4.24) taken between the points $\xi+$ and $\xi + h-$ gives

$$\begin{aligned}
&\left[-\xi \ \ \nabla A(f(\xi+)) + \nabla B(f(\xi+))\right] \left(f(\xi + h-) - f(\xi+)\right) \qquad (4.29)\\
&\quad = (\xi + h) \left[A(f(\xi + h-) - A(f(\xi+) - \nabla A(f(\xi+)) \left(f(\xi + h-) - f(\xi+)\right) \right]\\
&\quad - \left[B(f(\xi + h-) - B(f(\xi+) - \nabla B(f(\xi+)) \left(f(\xi + h-) - f(\xi+)\right) \right]\\
&\quad - \int_{\xi}^{\xi+h} \left[A(f(s)) - A(f(\xi+) \right] ds + h \nabla A(f(\xi+)) \left(f(\xi + h-) - f(\xi+)\right)
\end{aligned}$$

The increment $(f(\xi + h-) - f(\xi+))$ is expanded in the basis of right eigenvectors

$$\omega(h) := f(\xi + h-) - f(\xi+) = \sum_i \omega_i(h)\, r_i(f(\xi+) \qquad (4.30)$$

Note that for a function of bounded variation $\omega(h) \to 0$ as $h \to 0+$, and that by (4.26)

$$\omega_i(h) = l_i(f(\xi+) \cdot \nabla A(f(\xi+))\, \omega(h)\,. \qquad (4.31)$$

Taking the inner product of (4.29) with $l_i(f(\xi+)$ and using (4.25), (4.31) and the Taylor expansion, we obtain

$$\left[-\xi + \lambda_i(f(\xi+))\right] \omega_i(h) = O\left(|\omega(h)|^2\right) + O\Big(\int_0^h |\omega(s)|\, ds \Big) + O\left(h\, |\omega(h)|\right), \qquad (4.32)$$

On account of Corollary 4.5 and the strict hyperbolicity property of (1.5), the coefficient $(-\xi + \lambda_i(f(\xi+))$ is nonzero for $i \neq k$ but vanishes for $i = k$.

Next, using (4.17 – 4.20) and the Taylor expansion of λ_k, we see that

$$\begin{aligned}
&\lambda_k(f(\xi + h-)) - \lambda_k(f(\xi+)) = h \qquad (4.33)\\
&\quad = \nabla \lambda_k(f(\xi+)) \cdot \left(f(\xi + h-) - f(\xi+)\right) + O\left(|\omega(h)|^2\right).
\end{aligned}$$

If we set $g_k = \nabla\lambda_k(f(\xi+)) \cdot r_k(f(\xi+))$, $g_k \neq 0$ by hypothesis, and use (4.33), (4.30) and relations (4.32) for $i \neq k$ we arrive at the estimate

$$\begin{aligned} g_k\,\omega_k(h) - h &= O\big(\sum_{i\neq k} |\omega_i(h)|\,\big) + O\big(|\omega(h)|^2\big) \\ &= O\big(|\omega(h)|^2\big) + O\big(\int_0^h |\omega(s)|\,ds\,\big) + O\big(h\,|\omega(h)|\big)\,. \end{aligned} \tag{4.34}$$

Adding (4.32) for $i \neq k$ with (4.34) gives

$$\begin{aligned} \varphi(h) :&= |g_k\,\omega_k(h) - h| + \sum_{i\neq k} |\omega_i(h)| \\ &= O\big(\,(|\omega(h)| + h)\,|\omega(h)|\,\big) + O\big(\int_0^h |\omega(s)|\,ds\,\big) \\ &= O\big(\,(|\omega(h)| + h)\,\varphi(h)\,\big) + O\big(\int_0^h \varphi(s)\,ds\,\big) + O(h^2)\,. \end{aligned} \tag{4.35}$$

Since $\omega(h) \to 0$ as $h \to 0+$, we can choose δ sufficiently small so that for $0 < h \leq \delta$

$$\varphi(h) \leq C\,h^2 + C\int_0^h \varphi(s)\,ds$$

The integral inequality, in turn, yields

$$0 \leq \varphi(h) \leq C'\,h^2 \qquad \text{for } 0 < h \leq \delta$$

and thus

$$\lim_{h\to 0+} \frac{\omega_i(h)}{h} = 0 \quad \text{for } i \neq k\,, \qquad \lim_{h\to 0+} \frac{\omega_k(h)}{h} = \frac{1}{g_k}\,.$$

This shows (4.27). Part (ii) is proved similarly. ∎

Proposition 4.6 implies that f has right and left derivatives at any point ξ which is not an accumulation point of $\mathcal{S}$. If such a point ξ belongs to $\mathcal{C}$ then f is Lipshitz there, and if, in addition, it is an interior point of I_{λ_k} then f is differentiable there. It also completes the picture regarding the structure of the wave fans. There are the following cases:

(i) If I_{λ_k} consists of a single point then the solution is a shock wave satisfying the weak form of the Lax shock conditions (4.16).

(ii) If I_{λ_k} is a full interval of points in $\mathcal{C}$ the solution is a k-rarefaction wave (provided that $\nabla\lambda_k \cdot r_k \neq 0$ on I_{λ_k} which is anyway necessary for rarefactions).

(iii) In general I_{λ_k} consists of an alternating sequence of shock waves and k-rarefaction waves such that each shock adjacent to a rarefaction from one side is a contact discontinuity on that side.

The emerging picture is that typical for small data of strictly hyperbolic but not genuinely nonlinear systems. It will be further simplified in Section 6, by using the geometry of the shock curves for (1.5).

§5. Self-similar limits and shock profiles

Our next task is to discuss the relation between self-similar fluid dynamic limits and shock profiles for the Broadwell model. To this end fix ξ a point of discontinuity for f, and note that $f(\xi-) \neq f(\xi+)$ are both Maxwellian states and satisfy (4.3).

Consider a sequence of points $\{\xi_\varepsilon\}$ with the property $\xi_\varepsilon \to \xi$, to be specified later. Define functions $v_j^\varepsilon(\zeta) = f_j^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta)$ in the new variable $-\infty < \zeta < \infty$. This accounts for a shift of the original solution and the introduction of the stretched variable ζ. The uniform estimates $(2.4 - 2.5)$ imply

$$0 < m_j \leq f_j^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) \leq M_j\,,$$

$$TV_\zeta\, f_j^\varepsilon(\xi_\varepsilon + \varepsilon\,\cdot) = TV_\xi\, f_j^\varepsilon(\cdot) \leq K_j\,. \tag{5.1}$$

Using Helly's theorem and a diagonal argument we establish the existence of a subsequence and a function $v = (v_1, v_2, v_3)$ such that

$$f_j^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) \to v_j(\zeta) \qquad \text{pointwise for } -\infty < \zeta < \infty\,. \tag{5.2}$$

The sequence $\{\Phi^\varepsilon\}$ defined in (4.4) is of uniformly bounded variation. Upon restricting to further subsequences, if necessary, we also have

$$\Phi^\varepsilon(\xi) = \int_{-\infty}^{\xi} \frac{Q(f^\varepsilon(\tau))}{\varepsilon}\, d\tau \;\to\; \Phi(\xi) \qquad \text{pointwise for } -\infty < \xi < \infty\,, \tag{5.3}$$

$$\Phi^\varepsilon(\xi_\varepsilon) \;\to\; \Phi^0\,.$$

The following lemma shows that the phase shift $\{\xi_\varepsilon\}$ can be arranged so that v is a shock profile for the Broadwell system connecting the states $f(\xi-)$ with $f(\xi+)$.

Proposition 5.1 There exists a choice of the sequence $\{\xi_\varepsilon\}$ such that $v(\zeta)$, defined by (5.2), is continuously differentiable and satisfies the differential equations

$$\begin{aligned} -\xi\frac{dv_1}{d\zeta} + \frac{dv_1}{d\zeta} &= Q(v) \\ -\xi\frac{dv_2}{d\zeta} - \frac{dv_2}{d\zeta} &= Q(v) \\ -\xi\frac{dv_3}{d\zeta} &= -\frac{1}{2}\,Q(v) \end{aligned} \tag{5.4}$$

and the boundary conditions

$$v(-\infty) = f(\xi-)\,, \quad v(+\infty) = f(\xi+)\,. \tag{5.5}$$

Proof. The proof has two steps. The first step, to show (5.4), is independent of the particular choice of the sequence $\xi_\varepsilon \to \xi$.

We evaluate (2.1) between the points $\xi_\varepsilon + \varepsilon\,\zeta$ and θ and obtain

$$-(\xi_\varepsilon + \varepsilon\,\zeta)\, f_1^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) + \theta\, f_1^\varepsilon(\theta) + f_1^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) - f_1^\varepsilon(\theta) \\ + \int_\theta^{\xi_\varepsilon + \varepsilon\,\zeta} f_1^\varepsilon(\tau)\, d\tau = \frac{1}{\varepsilon} \int_\theta^{\xi_\varepsilon + \varepsilon\,\zeta} Q(f^\varepsilon(\tau))\, d\tau\,. \tag{5.6}$$

Observe that, along a subsequence and as $\varepsilon \to 0$,

$$\int_\theta^{\xi_\varepsilon + \varepsilon\,\zeta} f_1^\varepsilon(\tau)\, d\tau \;\to\; \int_\theta^{\xi} f_1(\tau)\, d\tau$$

while

$$\begin{aligned}\frac{1}{\varepsilon} \int_\theta^{\xi_\varepsilon + \varepsilon\,\zeta} Q(f^\varepsilon(\tau))\, d\tau &= \frac{1}{\varepsilon} \int_\theta^{\xi_\varepsilon} Q(f^\varepsilon(\tau))\, d\tau + \frac{1}{\varepsilon} \int_{\xi_\varepsilon}^{\xi_\varepsilon + \varepsilon\,\zeta} Q(f^\varepsilon(\tau))\, d\tau \\ &= \big(\Phi^\varepsilon(\xi_\varepsilon) - \Phi^\varepsilon(\theta)\big) + \int_0^\zeta Q(f^\varepsilon(\xi_\varepsilon + \varepsilon\, s))\, ds \\ &\to \big(\Phi^0 - \Phi(\theta)\big) + \int_0^\zeta Q(v(s))\, ds\,.\end{aligned}$$

Passing to the limit in (5.6), we deduce that for θ, $\zeta \in \mathbb{R}$

$$-\xi v_1(\zeta) + v_1(\zeta) - \int_0^\zeta Q(v(s))\, ds = -\theta\, f_1(\theta) + f_1(\theta) + \int_\xi^\theta f_1(\tau)\, d\tau + \big(\Phi^0 - \Phi(\theta)\big)\,. \tag{5.7}$$

In a similar fashion from $(2.2 - 2.3)$ we obtain the relations

$$-\xi v_2(\zeta) - v_2(\zeta) - \int_0^\zeta Q(v(s))\, ds = -\theta\, f_2(\theta) - f_2(\theta) + \int_\xi^\theta f_2(\tau)\, d\tau + \big(\Phi^0 - \Phi(\theta)\big)\,, \tag{5.8}$$

$$-\xi v_3(\zeta) + \frac{1}{2} \int_0^\zeta Q(v(s))\, ds = -\theta\, f_3(\theta) + \int_\xi^\theta f_3(\tau)\, d\tau - \frac{1}{2}\big(\Phi^0 - \Phi(\theta)\big)\,. \tag{5.9}$$

Since the above equations hold for any θ, $\zeta \in \mathbb{R}$, both the left and the right hand sides are constant. That is, there are six equations hidden in $(5.7-5.9)$: three for f, and three for v describing the internal structure of the shock profile. The functions v_j are continuously differentiable and satisfy the system of ordinary differential equations (5.4).

We proceed to show (5.5). In preparation, note that $(5.7-5.9)$ may be combined to obtain

$$\begin{aligned} -\xi\,(\,v_1(\zeta)+2\,v_3(\zeta)\,)+v_1(\zeta) &= A \\ -\xi\,(\,v_2(\zeta)+2\,v_3(\zeta)\,)-v_2(\zeta) &= B \\ -\xi\,\frac{dv_3}{d\zeta}(\zeta) &= -\frac{1}{2}\,(\,{v_3}^2(\zeta)-v_1(\zeta)\,v_2(\zeta)\big)\,, \end{aligned} \tag{5.10}$$

where the constants A, B are calculated from the right hand sides of $(5.7-5.9)$, as $\theta \to \xi\pm$, and are given by the formulas

$$A = -\xi\,\big(\,f_1(\xi+)+2\,f_3(\xi+)\big)+f_1(\xi+) = -\xi\,\big(\,f_1(\xi-)+2\,f_3(\xi-)\big)+f_1(\xi-)$$

$$B = -\xi\,\big(\,f_2(\xi+)+2\,f_3(\xi+)\big)-f_2(\xi+) = -\xi\,\big(\,f_2(\xi-)+2\,f_3(\xi-)\big)-f_2(\xi-)$$

Also recall from (4.2)

$${f_3}^2(\xi+)-f_1(\xi+)\,f_2(\xi+)=0\,, \qquad {f_3}^2(\xi-)-f_1(\xi-)\,f_2(\xi-)=0\,.$$

The equilibria (u_1, u_2, u_3) of (5.10) are determined by solving the algebraic system

$$\begin{aligned} -\xi\,(\,u_1+2\,u_3\,)+u_1 &= A \\ -\xi\,(\,u_2+2\,u_3\,)-u_2 &= B \\ {u_3}^2-u_1\,u_2 &= 0 \end{aligned} \tag{5.11}$$

Substituting the first two expressions in the third, we see that u_3 is determined from the roots of the quadratic

$$(1-\xi^2)\,{u_3}^2 + (A + 2\,\xi\,u_3)(B + 2\,\xi\,u_3)$$

and then u_1, u_2 are obtained by solving the linear equations. Therefore (5.11) has at most two real solutions, and as a result the equilibria of (5.10) are precisely $f(\xi-)$ and $f(\xi+)$.

We turn now to (5.5). To fix ideas, suppose that $\xi \in (0,1)$ and that f_j^ε is monotone increasing (for some j) in $(0,1)$ but may be increasing or decreasing in $(-1,0)$. Let $\delta > 0$ be fixed. Given any bounded interval $J \ni 0$ and provided that ε is chosen sufficiently small, the function $f_j^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta)$ is monotone increasing on J and

$$f_j^\varepsilon(\xi - \delta) \le f_j^\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) \le f_j^\varepsilon(\xi + \delta)\,, \qquad \text{for } \ \zeta \in J\,.$$

Passing to the limit $\varepsilon \to 0$, we deduce that v_j is monotone increasing on any J, and thus also on $(-\infty,\infty)$, and that

$$f_j(\xi-) \le v_j(-\infty) \le v_j(0) \le v_j(+\infty) \le f_j(\xi+)\,. \tag{5.12}$$

Both the inequalities and the monotonicity properties are reversed when we start with a component f_j^ε that is monotone decreasing in $(0,1)$. In either case the limits $v(-\infty)$ and $v(+\infty)$ exist and are finite.

For $\xi \in \mathcal{S}$ it is $f_3(\xi-) \ne f_3(\xi+)$; otherwise $f(\xi-) = f(\xi+)$ by the analysis of (5.11). Suppose for concreteness that $f_3(\xi-) < f_3(\xi+)$ and fix a state $v_3(0)$ such that

$$f_3(\xi-) < v_3(0) < f_3(\xi+)\,.$$

Since f_3^ε is monotone near ξ and $f_3^\varepsilon \to f_3$ pointwise, for each sufficiently small ε we can choose ξ_ε such that $f_3^\varepsilon(\xi_\varepsilon) = v_3(0)$. The sequence $\{\xi_\varepsilon\}$ has the property $\xi_\varepsilon \to \xi$. With this choice of $\{\xi_\varepsilon\}$ define the associated v by (5.2). Then v satisfies (5.10) and the limits $v(-\infty)$ and $v(+\infty)$ exist and satisfy (5.12). Since $f(\xi-)$ and $f(\xi+)$ are the only equilibria of (5.10) and $v(0)$, by virtue of selection, is not an equilibrium state, we conclude that

$$f(\xi-) = v(-\infty) \neq v(+\infty) = f(\xi+)$$

and the proof is complete. ∎

§6. The limit fluid equations

In this section we use the special properties of the limit fluid equations (1.5) to complete the description of the wave fans. The properties used are the genuine nonlinearity of the characteristic fields of (1.5) and the geometry of the shock curves. Although they are both discussed in Caflisch [C], we give an independent presentation for completeness and to account for extended differences in notation.

The eigenvalues $\lambda_1(f)$, $\lambda_2(f)$ satisfy the characteristic polynomial

$$(f_1 + f_2 + \sqrt{f_1 f_2})\,\lambda_i^{\,2} - (f_1 - f_2)\,\lambda_i - \sqrt{f_1 f_2} = 0 \tag{6.1}$$

and are given by (3.22) or (3.17) for $f_3 = \sqrt{f_1 f_2}$. The corresponding right eigenvectors are easily computed

$$r_i = \begin{bmatrix} \lambda_i + 1 \\ \lambda_i - 1 \end{bmatrix}, \qquad i = 1, 2\,.$$

We differentiate (6.1) in order to express $\frac{\partial \lambda_i}{\partial f_1}$, $\frac{\partial \lambda_i}{\partial f_2}$ in terms of λ_i, and use the resulting expressions to compute

$$\nabla \lambda_i \cdot r_i = \frac{(1-{\lambda_i}^2)}{2\sqrt{f_1 f_2}} \left(\frac{(f_1 + f_2 + 4\sqrt{f_1 f_2})\,\lambda_i - (f_1 - f_2)}{(f_1 + f_2 + \sqrt{f_1 f_2})\,2\,\lambda_i - (f_1 - f_2)} \right). \tag{6.2}$$

The denominator in (6.2) is positive for the positive eigenvalue λ_2 and negative for λ_1. On the other hand, (3.17) yields

$$\begin{aligned}
\lambda_1(f) &< \frac{(f_1 - f_2) - |f_1 - f_2|}{2\,(f_1 + f_2 + \sqrt{f_1 f_2})} \le \frac{f_1 - f_2}{f_1 + f_2 + 4\sqrt{f_1 f_2}} = u \\
\lambda_2(f) &> \frac{(f_1 - f_2) + |f_1 - f_2|}{2\,(f_1 + f_2 + \sqrt{f_1 f_2})} \ge \frac{f_1 - f_2}{f_1 + f_2 + 4\sqrt{f_1 f_2}} = u
\end{aligned}$$

It follows from (6.2) and (3.18) that $\nabla \lambda_i \cdot r_i > 0$ for $i = 1, 2$.

Since (1.5) is genuinely nonlinear, the possibility of contact discontinuities is ruled out at least for weak shocks. For the case of strong shocks we need to study the geometry of the shock curves.

The shock curves are defined by solving the Rankine-Hugoniot conditions

$$\begin{aligned}
-s\,(\,u_1 + 2\,u_3\,) + u_1 &= -s\,(\,v_1 + 2\,v_3\,) + v_1 \\
-s\,(\,u_2 + 2\,u_3\,) - u_2 &= -s\,(\,v_2 + 2\,v_3\,) - v_2 \\
{u_3}^2 - u_1\,u_2 &= {v_3}^2 - v_1\,v_2 = 0
\end{aligned} \tag{6.3}$$

for the states $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ and the shock speed s. To this end fix one state, say v, with $v_3 = \sqrt{v_1 v_2}$ and consider the increments $[u_i] = u_i - v_i$. Then

$$2\,s\,[u_3] = (1-s)\,[u_1] = -(1+s)\,[u_2] \tag{6.4}$$

$$[u_3]\,([u_3] + 2\,v_3) = [u_1]\,[u_2] + v_2\,[u_1] + v_1\,[u_2] \tag{6.5}$$

Substituting (6.4) into (6.5) yields

$$[u_3]\left[(1+3\,s^2)\,[u_3]-2\left[\,(v_1+v_2+v_3)\,s^2-(v_1-v_2)\,s-v_3\,\right]\right]=0$$

If $[u_3]=0$ there is no shock. Thus, using (6.1), we obtain a representation of the shock curve parametrized by the speed s in the form of

$$[u_3]=\frac{2\,(v_1+v_2+\sqrt{v_1 v_2})}{1+3\,s^2}\,\big(s-\lambda_1(v)\big)\,\big(s-\lambda_2(v)\big)\,. \tag{6.6}$$

together with (6.4). As a consequence, if the shock is a contact discontinuity (at either side) then the strength of the shock $[u]=0$.

This observation excludes the possibility of contact discontinuities and simplifies the structure of the solutions for (1.5) considerably. We conclude then that each wave fan is either a single rarefaction or a single shock which satisfies the Lax shock conditions.

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