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The renormalization group (RNG) theory has been used to derive the cutoff-dependent subgrid eddy viscosity (Yakhot and Orszag, 1986). Here, the possibility of deriving the cusp for wavenumbers near the cutoff is investigated. It is explained why the cusp does not appear at lowest order in the RNG analysis of the Navier Stokes equations, and why previously proposed derivations of the cusp using RNG lead to mathematically or physically unreasonable results. However, the cusp may be derived from the lowest-order RNG equation for the energy spectrum using the RNG eddy damping. This energy equation is equivalent to the EDQNM model energy equation with a theoretically derived Kolmogorov constant (Dannevik, Yakhot and Orszag, 1987).

2

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Exploration of the Cusp Behavior of the RNG Eddy Viscosity Near the Cutoff Wavenumber

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by

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1. Introduction

The renormalization group (RNG) procedure for nonlinear, dissipative systems, introduced by Wilson (1975), has since been applied to the Navier Stokes equations (Forster, Nelson and Stephen, 1978, De Dominicis and Martin, 1979, Fournier and Frisch, 1983, and Yakhot and Orszag, 1986). Dynamical equations for the large scales and long times are derived by averaging over an infinitesimal band of small scales to remove the small scales from explicit consideration. This procedure yields infinitesimal modifications in the equations for the large scales. The removal process is iterated and these corrections accumulate to give finite changes. In the case of the Navier Stokes equations, the resultant model equations for the large scales are characterized by an eddy viscosity that replaces the molecular viscosity.

The present formulation of the RNG theory assumes that inertial-range turbulence can be modeled by isotropic turbulence driven by a random force. The force represents the net effect of the energy cascade from the very large scales on eddies in the inertial range, and is chosen to reproduce the Kolmogorov energy spectrum when opposed by the modified viscosity. Homogeneity in space and time and isotropy in space allow transformation to Fourier space, where the equations are algebraic.

For the purpose of large-eddy simulations, one would like to derive an eddy viscosity for all wavenumbers $0 < k < \Lambda$, where Λ is the cutoff wavenumber defining the resolution of the simulation. *Assuming a purely eddy viscosity model*, other studies of isotropic turbulence (Kraichnan, 1976, Chollet and Lesieur, 1981, and Domaradzki, Metcalfe, Rogallo and Riley, 1987) indicate that a realistic eddy viscosity exhibits a plateau value far from the cutoff ($k << \Lambda$) and increases to several times its plateau value very near the cutoff ($k \approx \Lambda$). This "cusp-up" behavior of the eddy viscosity is predicted by the Test-Field model of Kraichnan (1976) and EDQNM (Chollet *et al.* 1981), and is consistent with direct numerical simulations (Domaradzki *et al.* 1987).

At each iteration of the RNG fine-scale removal, the correction to the viscosity is found in the low-wavenumber limit $k \to 0$. Thus the RNG subgrid-scale model (Yakhot and Orszag, 1986) includes the traditional eddy-viscosity type triads. These triads have $k << \Lambda$, and $p, q > \Lambda$, where $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$. As discussed by Kraichnan (1976), the triads leading to the cusp are of a different nature, with k just below the cutoff Λ, $p << \Lambda$ and q just above Λ. This work explores the possibility of relaxing the limit $k \to 0$ in the derivation of the RNG eddy viscosity.

In fact, the RNG procedure involves the two limits $k \to 0$ and $\delta\Lambda \to 0$, where

$\delta\Lambda$ is the width of the removed high-wavenumber shell, and we have determined that these two limiting processes do not commute. The eddy viscosity derived by Yakhot and Orszag (1986) is found for the ratio $k/\delta\Lambda \to 0$, and we show that $k/\delta\Lambda \to 0$ is necessary to recover physically meaningful results.

Since $\delta\Lambda$ is an unphysical parameter, we believe that the restriction $k/\delta\Lambda \to 0$ must be re-interpreted as a condition on a Reynolds number, and may be equivalent to the so-called ϵ-expansion. It is possible that the cusp-up can be derived from extrapolation of an expansion of the effective viscosity near $k \to 0$ (for example a Padé scheme). However, it seems likely that the interactions that lead to the cusp behavior in a purely eddy viscosity model are represented by different terms in the RNG model, for example Galilean invariant higher-order nonlinearities.

Also while at Université Libre de Bruxelles, the principle investigator undertook to derive the vorticity equation for long times and large scales using the RNG method. The results are described in the enclosed preprint which will be submitted as a brief communication to Physics of Fluids A. The restriction $k/\delta\Lambda \to 0$ is implicitly enforced, and the long-time, large-scale vorticity equation is compared to the long-time, large-scale velocity equation. Information about the large-scale, long-time vorticity equation should be helpful for large-eddy simulations.

The second section of this report reviews the RNG procedure to remove small scales. The third section shows that iterative removal of scales from the Navier-Stokes quadratic nonlinearity gives the Yakhot-Orzsag eddy viscosity for $k/\delta\Lambda \to 0$, and zero for $\delta\Lambda/k \to 0$. In the fourth section, we attempt to clarify the work of Zhou, Vahala and Hossain (1988, 1989). In an attempt to derive the cusp, they considered iterative removal of scales from the cubic nonlinearity that arises after removal of one shell from the original quadratic nonlinearity. We show that the results for $k/\delta\Lambda \to 0$ are not mathematically meaningful. Carati (1991) previously showed that the results are unphysical for $\delta\Lambda/k \to 0$, and we give a review of Carati's (1991) work. In Section 5, it is suggested that the source of problems with the cubic term is non-Galilean invariance in the limit of long times.

2. The RNG scale-removal procedure

The RNG model is for a Newtonian fluid in an infinite domain, stirred by a Gaussian random force **f**,

$$\frac{\partial v_i}{\partial t} + v_j \nabla_j v_i = -\nabla_i p + \nu_o \nabla^2 v_i + f_i$$

$$\nabla_i v_i = 0 \tag{2.1}$$

where ν_o is the molecular viscosity and the density ρ has been absorbed into the pressure p. The force is defined by its two-point correlation function in wavevector and frequency space,

$$\begin{aligned} < \hat{f}_i(\mathbf{k},\omega)\hat{f}_j(\mathbf{k}',\omega') >= & \ 2D_o(2\pi)^{d+1}k^{-y}P_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'), \quad \Lambda_L < k \leq \Lambda_o \\ = & \ 0, \qquad 0 < k \leq \Lambda_L \end{aligned} \tag{2.2}$$

where $\mathbf{k}$ is the wavevector, ω is the frequency, $k = |\mathbf{k}|$ and d is the number of space dimensions. The parameter y is chosen to give the Kolmogorov form of the energy spectrum and in three dimensions $y = d = 3$ (Yakhot and Orszag, 1986). The wavenumber Λ_o is an ultraviolet cutoff, beyond which the energy spectrum is taken to be zero and the viscosity is the molecular viscosity ν_o. Statistical homogeneity in space and time is guaranteed through $\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')$. The projection operator $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ makes the force statistically isotropic and divergence-free. In the limit of an infinite Reynolds number, the integral scale $\Lambda_L \to 0$.

Small-scale fluctutations are eliminated from explicit appearance in (2.1) by averaging over the force field at small scales. The force given by (2.2) is postulated to represent the end result of the energy cascade from large to small scales, and therefore models the production due to the quadratic nonlinearity.

The Fourier transform of (2.1) leads to

$$\hat{v}_i(\hat{\mathbf{k}}) = G(\hat{\mathbf{k}})\hat{f}_i(\hat{\mathbf{k}}) - \frac{i\lambda_o}{2}G(\hat{\mathbf{k}})P_{imn}(\mathbf{k})\int \hat{v}_m(\hat{\mathbf{q}})\hat{v}_n(\hat{\mathbf{k}}-\hat{\mathbf{q}})\frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \tag{2.3}$$

where $\Lambda_L < k < \Lambda_o$, $\hat{\mathbf{k}} = (\mathbf{k}, \omega)$, $G(\hat{\mathbf{k}}) = (-i\omega + \nu_o k^2)^{-1}$, $P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$ and $d\hat{\mathbf{q}}$ denotes the $(d+1)$-dimensional integral over the wavevector and frequency components of $\hat{\mathbf{q}}$. The parameter $\lambda_o = 1$ is simply an ordering parameter for the dimensional equations, used to identify where the Reynolds number would appear in the nondimensional form of (2.3).

- To begin the RNG scale-removal, one first defines the high-wavenumber band below Λ_o to be removed, $\Lambda < k < \Lambda_o$, where the shell width is denoted $\delta\Lambda$,

$$\delta\Lambda = [\Lambda, \Lambda_o]. \tag{2.4}$$

Then, one separates the velocity modes into $\hat{\mathbf{v}}^<$ and $\hat{\mathbf{v}}^>$ according to scale,

$$\begin{aligned} \hat{\mathbf{v}}^<(\hat{\mathbf{k}}) &= \hat{\mathbf{v}}(\hat{\mathbf{k}}), && k < \Lambda \\ &= 0, && \Lambda \le k \le \Lambda_o \\ \hat{\mathbf{v}}^>(\hat{\mathbf{k}}) &= 0, && k \le \Lambda \\ &= \hat{\mathbf{v}}(\hat{\mathbf{k}}), && \Lambda \le k < \Lambda_o. \end{aligned} \tag{2.5}$$

Then equation (2.3) is written

$$\hat{v}_i(\hat{\mathbf{k}}) = G(\hat{\mathbf{k}})\hat{f}_i(\hat{\mathbf{k}}) - \frac{i\lambda_o}{2} G(\hat{\mathbf{k}}) P_{imn}(\mathbf{k}) \int (\hat{v}_m^<(\hat{\mathbf{q}})\hat{v}_n^<(\hat{\mathbf{k}}-\hat{\mathbf{q}}) +$$

$$2\hat{v}_m^>(\hat{\mathbf{q}})\hat{v}_n^<(\hat{\mathbf{k}}-\hat{\mathbf{q}}) + \hat{v}_m^>(\hat{\mathbf{q}})\hat{v}_n^>(\hat{\mathbf{k}}-\hat{\mathbf{q}}))\frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}}. \tag{2.6}$$

- Since the local Reynolds number of the modes in the shell is small, one expands $\hat{\mathbf{v}}^>$ in a perturbation series in powers of λ_o

$$\hat{\mathbf{v}}^> = \hat{\mathbf{v}}^{(0)} + \lambda_o \hat{\mathbf{v}}^{(1)} + \lambda_o^2 \hat{\mathbf{v}}^{(2)} + O(\lambda_o^3)$$

One finds

$$\hat{v}_i^>(\hat{\mathbf{k}}) = G(\hat{\mathbf{k}})\hat{f}_i^>(\hat{\mathbf{k}}) + \frac{-i\lambda_o}{2} G(\hat{\mathbf{k}}) P_{imn}(\mathbf{k}) \int (\hat{v}_m^<(\hat{\mathbf{q}})\hat{v}_n^<(\hat{\mathbf{k}}-\hat{\mathbf{q}}) +$$

$$2G(\hat{\mathbf{q}})\hat{f}_m^>(\hat{\mathbf{q}})\hat{v}_n^<(\hat{\mathbf{k}}-\hat{\mathbf{q}}) + G(\hat{\mathbf{q}})\hat{f}_m^>(\hat{\mathbf{q}})G(\hat{\mathbf{k}}-\hat{\mathbf{q}})\hat{f}_n^>(\hat{\mathbf{k}}-\hat{\mathbf{q}}))\frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \tag{2.7}$$

keeping in mind that λ_o identifies where the local Reynolds number appears in the nondimensional equations. The velocity scale for the local Reynolds number is the square root of the energy beyond the cutoff Λ_o, and the time scale is based on the viscosity at the cutoff.

- One substitutes the perturbation expansion for $\hat{\mathbf{v}}^{>}(\hat{\mathbf{q}})$ into the equations for the velocity modes $\hat{\mathbf{v}}^{<}(\hat{\mathbf{k}})$ at large scales $k \to \Lambda_L \to 0$ and long times $\omega \to 0$. An ensemble average over the force field in the high-wavenumber band using (2.2) eliminates the small scales from the equations for the large scales. The large-scale variables are assumed independent of the ensemble average over the force at high wavenumbers, and this condition is nearly satisfied for $\hat{\mathbf{k}} \to 0$. The resulting equation for $\hat{\mathbf{v}}^{<}$ are written symbolically, without subscripts, arguments, or coefficients, as

$$(-i\omega+h(k,\Lambda))\hat{v}^{<} = \hat{f}^{<}+\hat{F}^{<}+\lambda_o\int(\hat{v}^{<})^2+\lambda_o^2\int(\hat{v}^{<})^3+O(\lambda_o^3\int(\hat{v}^{<})^4). \quad (2.8)$$

where $h(k,\Lambda_o) = \nu_o k^2$,

$$h(k,\Lambda)-h(k,\Lambda_o) = \frac{\lambda_o^2 D_o}{(2\pi)^d}\frac{P_{lb}(\mathbf{k})}{(d-1)}P_{lmn}(\mathbf{k})\int\frac{P_{nab}(\mathbf{k}-\mathbf{q})P_{ma}(\mathbf{q})q^{-y}d\mathbf{q}}{\nu_o q^2(\nu_o q^2+\nu_o|\mathbf{k}-\mathbf{q}|^2)}+O(\lambda_o^4) \quad (2.9)$$

and $\hat{\mathbf{F}}^{<}$ is an induced force acting at low wavenumbers (Yakhot and Orszag, 1986). The frequency integration over Ω has already been carried out using the Residue theorem, where $\hat{\mathbf{q}} = (\mathbf{q},\Omega)$, and we have taken the limit $\omega \to 0$, where $\hat{\mathbf{k}} = (\mathbf{k},\omega)$. The wavevector integration in (2.9) is restricted by $\Lambda \le q \le \Lambda_o$ and $\Lambda \le |\mathbf{k}-\mathbf{q}| \le \Lambda_o$. The origin of the $O(\lambda_o^2)$ term in (2.9) is the $\hat{v}_m^{>}(\hat{\mathbf{q}})\hat{v}_n^{>}(\hat{\mathbf{k}}-\hat{\mathbf{q}})$ term in (2.6), and the implications of this are discussed below. Yakhot and Orszag (1986) took the limit $k/q \to 0$ (also implicitly assuming $k/\delta\Lambda \to 0$, see Section 4) and found

$$h(k,\Lambda)-h(k,\Lambda_o) = (\nu_T(\Lambda)-\nu_T(\Lambda_o))k^2$$

$$\nu_T(\Lambda)-\nu_T(\Lambda_o) = \frac{A_d\lambda_o^2}{\nu_o^2}\frac{D_o S_d}{(2\pi)^d}\frac{(\Lambda^{d-y-4}-\Lambda_o^{d-y-4})}{(4+y-d)}+O(\lambda_o^4) \quad (2.10)$$

$$A_d = \frac{d^2-d-\epsilon}{2d(d+2)}, \quad \epsilon \equiv 4+y-d \quad (2.11)$$

where S_d is the area of a sphere in d dimensions. The reason for the introduction of the new parameter ϵ will become clear below.

Iteration of the scale removal procedure can be performed by integration of the differential equation for $\nu_T(\Lambda)$,

$$\frac{d\nu_T(\Lambda)}{d\Lambda} = \lim_{\delta\Lambda\to 0} \frac{\nu_T(\Lambda) - \nu_T(\Lambda_o)}{\delta\Lambda} = -\frac{A_d}{2}\frac{\nu_T(\Lambda)\bar{\lambda}^2(\Lambda)}{\Lambda} + O(\bar{\lambda}^4) \tag{2.12}$$

$$\bar{\lambda}^2(\Lambda) = \frac{2D_o S_d}{(2\pi)^d}\frac{\lambda_o^2}{\nu_T^3(\Lambda)\Lambda^\epsilon}. \tag{2.13}$$

The quantity $\bar{\lambda}(\Lambda)$ is the effective Reynolds number at Λ, based on the modified viscosity $\nu_T(\Lambda)$.

Notice that in writing the differential equation (2.13) one has assumed that the value of ν_T derived in the limit $k \to 0$ is valid for all k up to the cutoff Λ (Smith and Reynolds, 1991, Section 3). Analysis of direct numerical simulation data (She, 1991) shows that the effective eddy viscosity due to triad interactions between wavevectors $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{q}$, where $\mathbf{k}+\mathbf{q}+\mathbf{p} = 0$, $\mathbf{k} < \Lambda$ and $\mathbf{p}, \mathbf{q} > \Lambda$, is indeed flat for $0 < k < \Lambda$ (She, 1991). Thus this assumption is consistent with the origin of the RNG eddy viscosity being the $\hat{v}_m^>(\hat{\mathbf{q}})\hat{v}_n^>(\hat{\mathbf{k}} - \hat{\mathbf{q}})$ term in (2.6).

With initial condition $\nu_T(\Lambda_o) = \nu_o$, integration of (2.12) leads to

$$\nu_T(\Lambda) = \nu_o\left[1 + \frac{3A_d}{\nu_o^3}\frac{D_o S_d}{(2\pi)^d}\frac{(\Lambda^{-\epsilon} - \Lambda_o^{-\epsilon})}{\epsilon}\right]^{1/3} \tag{2.14}$$

with (2.14) strictly valid for $\Lambda << \Lambda_o$ where it is independent of ν_o and Λ_o in keeping with the assumed similarity range. For $\Lambda << \Lambda_o$,

$$\nu_T(\Lambda) \sim \left(\frac{3A_d}{2\epsilon})\right)^{1/3}\left(\frac{2D_o S_d}{(2\pi)^d}\right)^{1/3}\Lambda^{-\epsilon/3} \tag{2.15}$$

and by (2.13)

$$\bar{\lambda}^2 \sim \frac{\epsilon}{3A_d}. \tag{2.16}$$

For $\epsilon \to 0$ and using (2.11), one sees that

$$\bar{\lambda}^2 \sim \frac{\epsilon}{3A_d^0} + O(\epsilon^2) \tag{2.17}$$

where $A_d^0 = (d^2 - d)/(2d(d+2))$. Therefore, for $\epsilon \to 0$, (2.12) becomes

$$\frac{d\nu_T(\Lambda)}{d\Lambda} = -\frac{(A_d^0 + O(\epsilon))}{2}\frac{\nu_T(\Lambda)\bar{\lambda}^2(\Lambda)}{\Lambda} + O(\epsilon\lambda^2). \tag{2.18}$$

For a consistent asymptotic expansion, *all* $O(\epsilon\bar{\lambda}^2)$ terms must be dropped, and the lowest-order solution for $\nu_T(\Lambda)$ is

$$\nu_T(\Lambda) = \nu_o\Big[1 + \frac{3A_d^0}{\nu_o^3}\frac{D_o S_d}{(2\pi)^d}\frac{(\Lambda^{-\epsilon} - \Lambda_o^{-\epsilon})}{\epsilon}\Big]^{1/3} \tag{2.19}$$

strictly valid in the limit $\Lambda << \Lambda_o$. Thus for $\epsilon \to 0$ and $\Lambda << \Lambda_o$,

$$\nu_T(\Lambda) \sim \left(\frac{3A_d^0}{2\epsilon}\right)^{1/3}\left(\frac{2D_o S_d}{(2\pi)^d}\right)^{1/3}\Lambda^{-\epsilon/3}. \tag{2.20}$$

Since $\bar{\lambda}$ is the effective Reynolds number of the flow at Λ, all perturbation expansions used in the scale-removal procedure are exact for $\epsilon \to 0$. Furthermore, the modified equation for $\hat{\mathbf{v}}^<(\hat{\mathbf{k}})$ is given at lowest-order in an expansion in powers of ϵ. The Yakhot-Orszag theory is an extrapolation of the RNG results at lowest order in ϵ to the case $\epsilon = 4$, which is the value for Kolmogorov scaling (Yakhot and Orszag, 1986).

Notice that the cubic nonlinearity in the nondimensional form of (2.8) is $O(\bar{\lambda}^2) = O(\epsilon)$ and can be neglected in comparison to the quadratic nonlinearity, which is $O(\bar{\lambda}) = O(\epsilon^{1/2})$. However, Zhou *et al.* (1988, 1989) considered the contribution to $h(k, \Lambda')$ after removal of the next band $\Lambda' \leq k \leq \Lambda$, and furthermore the accumulated contribution from the cubic after iterative removal of scales. Their goal was to derive the cusp in the eddy viscosity for $k \approx \Lambda$, where Λ is then the final cutoff, since the cubic term arises from the $\hat{v}_m^>(\hat{\mathbf{q}})\hat{v}_n^<(\hat{\mathbf{k}} - \hat{\mathbf{q}})$ term in (2.6). Kraichnan (1976) showed that the cusp is a result of triad interactions with k just below the cutoff, q just above the cutoff, and p much smaller than the cutoff. This has been verified by analysis of direct numerical simulation data (She, 1991).

In Section 4, we show that the contribution to $h(k, \Lambda')$ does not make mathematical sense for $k/\delta\Lambda \to 0$, which condition is necessary to derive the eddy viscosity from the quadratic nonlinearity, and is probably related to the ϵ-expansion. Carati (1991) showed that the contribution to $h(k, \Lambda')$ from the cubic is unphysical in the case $\delta\Lambda/k \to 0$. A fundamental problem with the cubic is that it is not Galilean invariant in the limit of long times ($\omega \to 0$), and this is probably the reason why the equations including the cubic do not give meaningful results when $\delta\Lambda/k \to 0$.

Section 3 shows that the condition $k/\delta\Lambda \to 0$ is necessary to derive the RNG eddy viscosity from the Navier Stokes quadratic nonlinearity. If this condition is relaxed, there is no contribution to $h(k, \Lambda)$ from the quadratic nonlinearity. Section 4 shows that the cubic nonlinearity yields nonsensical results for $k/\delta\Lambda \to 0$. It yields unphysical results for $\delta\Lambda/k \to 0$, probably because it is not Galilean invariant in the limit of long times (Section 5).

3. The renormalized viscosity

The equation for $v^<$ is, from (2.6)

$$G^{-1}\hat{v}_i^<(\hat{\mathbf{k}}) = \hat{f}_i^<(\hat{\mathbf{k}}) + \frac{-i\lambda_o}{2} P_{imn}(\mathbf{k}) \int [\,\hat{v}_m^<(\hat{\mathbf{q}})\,\hat{v}_n^<(\hat{\mathbf{k}}-\hat{\mathbf{q}}) +$$

$$2\,\hat{v}_m^>(\hat{\mathbf{q}})\,\hat{v}_n^<(\hat{\mathbf{k}}-\hat{\mathbf{q}}) + \hat{v}_m^>(\hat{\mathbf{q}})\,\hat{v}_n^>(\hat{\mathbf{k}}-\hat{\mathbf{q}})\,]\,\frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \tag{3.1}$$

where $G^{-1} = (-i\omega + h(k,\Lambda))$. Substituting the λ-expansion (2.7) for $v^>$, up to second order in λ_0^2, generates two terms linear in $v^<$ and quadratic in $f^>$. One comes from the $v^> v^<$ term but it vanishes after ensemble averaging over $f^>$ and integration over $\hat{\mathbf{q}}$. The other linear term is generated from the $v^> v^>$ term in (3.1), symbolically this contribution is $2(G^> f^>)(\lambda_0\, 2G^> f^> v^<)$. Ensemble averaging over $f^>$ using (2.2), the explicit form of this second linear term is, from (3.1) and (2.7),

$$2D_0\,\lambda_0^2 \int P_{imn}(\mathbf{k}) P_{mrs}(\mathbf{q}) P_{nr}(\mathbf{k}-\mathbf{q}) \times$$

$$G(\hat{\mathbf{k}}-\hat{\mathbf{q}}) G(\hat{\mathbf{q}}-\hat{\mathbf{k}}) G(\hat{\mathbf{q}})\,|\mathbf{k}-\mathbf{q}|^{-y} \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}}\;\; \hat{v}_s(\hat{\mathbf{k}}) \tag{3.2}$$

where the integration domain is such that $\Lambda' \le |\mathbf{k}-\mathbf{q}|,\, q \le \Lambda$. This is the expression (2.9). This expression has the form $T_{is}(\hat{\mathbf{k}})\hat{v}_s(\hat{\mathbf{k}})$, but from isotropy the tensor $T_{is}(\hat{\mathbf{k}})$ must be proportional to P_{is}: $T_{is}(\hat{\mathbf{k}}) = \delta G^{-1}\, P_{is}(\mathbf{k})$ where the scalar $\delta G^{-1} = P_{is}(\mathbf{k}) T_{is}(\hat{\mathbf{k}})/(d-1)$. Hence this contribution can be combined with the propagator $G(\hat{\mathbf{k}},\Lambda)$ to define a new propagator:

$$(G')^{-1}(\hat{\mathbf{k}},\Lambda') = G^{-1}(\hat{\mathbf{k}},\Lambda) + \delta G^{-1}$$

where

$$\delta G^{-1} = \frac{2M}{(2\pi)^2} \int P_{smn}(\mathbf{k}) P_{mrs}(\mathbf{q}) P_{nr}(\mathbf{p})\; G(\hat{\mathbf{p}}) G(-\hat{\mathbf{p}}) G(\hat{\mathbf{q}})\; p^{-y} d\hat{\mathbf{q}} \tag{3.3}$$

with $\hat{\mathbf{p}} = \hat{\mathbf{k}} - \hat{\mathbf{q}}$ and $\Lambda' \le p,\, q \le \Lambda$. The factor M stands for $M = D_0/[(d-1)(2\pi)^{d-1}]$.

Taking the long-time limit, integrating over large frequencies Ω (where $\hat{\mathbf{q}} = (\mathbf{q}, \Omega)$) using the Residue theorem, and making the usual change of variable

$$\int d\mathbf{q} = \int_0^{2\pi} d\phi \int\int \frac{pq}{k}\, dp dq$$

where ϕ determines the orientation of the triad around $\mathbf{k}$, the equation for $h(k,\Lambda)$ becomes, from (3.3)

$$h(k,\Lambda') - h(k,\Lambda) = M\int_{D2} \frac{A(k,p,q)\,p^{-y}}{h(p,\Lambda)\,[h(p,\Lambda)+h(q,\Lambda)]}\,\frac{pq}{k}dpdq \tag{3.4}$$

The integration domain $D2$ is such that $\Lambda' \le p,\ q \le \Lambda$, and

$$A(k,p,q) = P_{smn}(\mathrm{k})P_{nrs}(\mathrm{q})P_{mr}(\mathrm{p}) = [(k^2-p^2)(q^2-p^2)+k^2q^2]\frac{Q^2}{4k^2p^2q^2}$$

with $Q^2 = 2k^2p^2 + 2k^2q^2 + 2p^2q^2 - k^4 - p^4 - q^4$.

3.1. $k < \delta\Lambda$

When $k < \delta\Lambda$, the domain of integration for the contribution from the quadratic term (darkest area in Fig. 1), is composed of three distinct parts in the p,q-space,

$$\int_{D2} = \int_{\Lambda-k}^{\Lambda} dp \int_{\Lambda-k}^{\Lambda} dq + \int_{\Lambda'}^{\Lambda-k} dp \int_{p}^{p+k} dq + \int_{\Lambda'}^{\Lambda-k} dq \int_{q}^{q+k} dp$$

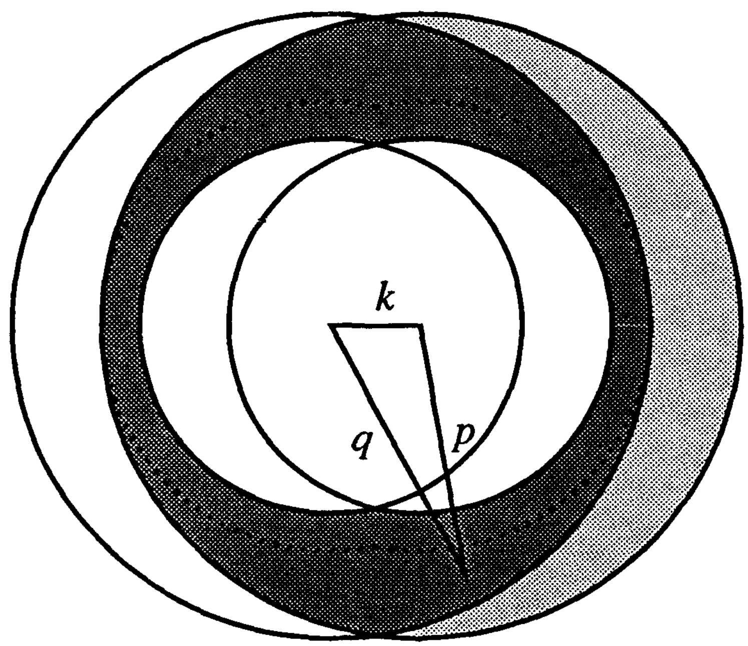

Fig. 1. Domain of integration when $k < \delta\Lambda$.

When $\Lambda = \Lambda_0$, $h(k, \Lambda) = \nu_0 k^2$ where ν_0 is the molecular viscosity. *Assume* that $h(k, \Lambda) = \nu(\Lambda) k^2$, for all k. Expanding in powers of k (with $q = p + kx, -1 \leq x \leq 1$) and $\delta\Lambda = \Lambda - \Lambda'$, one finds after some lengthy but straightforward manipulations,

$$h(k, \Lambda') - h(k, \Lambda) = -k^2 \, \delta\nu = \frac{M}{\nu^2(\Lambda)} \times$$

$$[\frac{10 - 2y}{15} k^2 \Lambda^{-2-y} \delta\Lambda + \frac{y - 3}{12} k^3 \Lambda^{-2-y} + \frac{y^2 - y - 6}{24} k^3 \Lambda^{-3-y} \delta\Lambda + O(k^4, \delta\Lambda^2)]$$

In the limit $k/\delta\Lambda \to 0$, one obtains a differential equation for $\nu(\Lambda) = h(k, \Lambda)/k^2$ which is readily integrated to find

$$\frac{\nu^3(\Lambda) - \nu^3(\Lambda_0)}{3} = M \, \frac{10 - 2y}{15} \, \frac{\Lambda^{-1-y} - \Lambda_0^{-1-y}}{(1 + y)} \tag{3.5}$$

This expression is identical to that (2.14) obtained by Yakhot and Orszag (1986) when $d = 3$ and $y = \epsilon - 1$ (only $d = 3$ is considered in this report).

3.2. $k > \delta\Lambda$

Figure 2. shows clearly that when $k > \delta\Lambda$ the contribution from the quadratic term is second order in $\delta\Lambda$. The domain of integration $D2$ (darkest area in Fig.2) in (3.4) is simply

$$\int_{D2} = \int_{\Lambda'}^{\Lambda} dp \int_{\Lambda'}^{\Lambda} dq$$

then (3.4) with $h(k, \Lambda) = \nu(\Lambda) k^2$ gives

$$\begin{aligned} h(k, \Lambda - \delta\Lambda) - h(k, \Lambda) &= \frac{M}{\nu^2} \, \frac{A(k, \Lambda, \Lambda) \, \Lambda^{-y}}{4\Lambda^4} \, \frac{\Lambda^2}{k} (\delta\Lambda)^2 + O(\delta\Lambda)^3 \\ &= \frac{M}{\nu^2} \, \frac{\Lambda^{-4-y}(4\Lambda k - k^3)}{8} (\delta\Lambda)^2 + O(\delta\Lambda)^3 \end{aligned}$$

One obtains

$$k^2 \, \delta\nu \propto k \, (\delta\Lambda)^2$$

hence the quadratic term does not contribute in the limit $\delta\Lambda \to 0$ for fixed k. Note that the form of the increment ($\propto k$) is not consistent with the assumed k^2 dependency for $h(k, \Lambda)$.

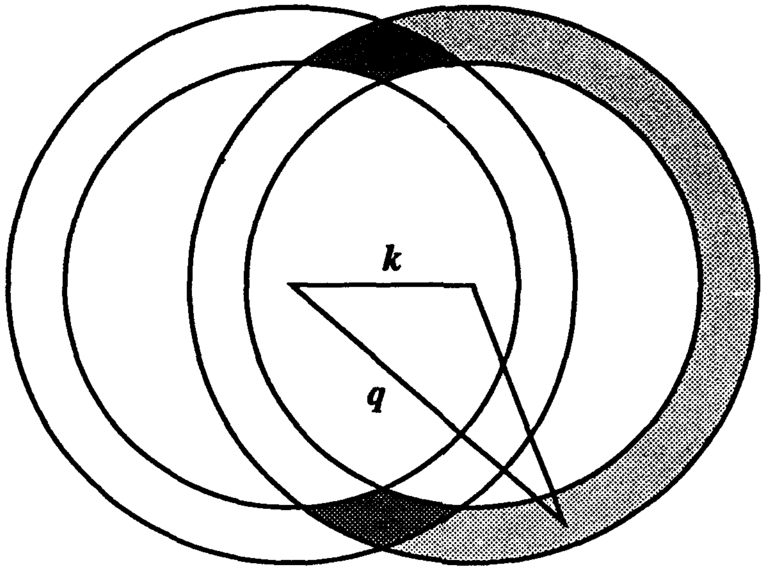

Fig. 2. Domain of integration when $k > \delta\Lambda$.

4. The cusp and the triple non-linear terms

The *subgrid scale* eddy-viscosity, $\nu(k/\Lambda)$, as defined by Kraichnan (1976), shows a very steep rise (a "cusp") as k approaches the numerical cut-off Λ. As explained by Kraichnan, this cusp behavior is due to the effect on a wavenumber just below the cut-off of triad interactions with a very small wavenumber and a wavenumber just above the cut-off. In the language of the previous sections, this would originate from the $2v^{<}v^{>}$-term in the $v^{<}$-equation. The sharp cusp in the eddy-viscosity shows that this effect is not well-modeled by an eddy-viscosity. Note also that the cusp is not universal but depends on the large scales (Kraichnan 1976).

After *one step* of the shell-elimination procedure, the $2\,v^{<}v^{>}$-term generates a *triple non-linear term* which is not present in the original Navier-Stokes equations. (It is shown below that the triple term is *not* Galilean invariant). However, at the *next* shell-elimination this triple non-linear term generates a new linear term. Zhou *et al.* (1988,1989) proposed that this new contribution to the propagator could be the one responsible for the cusp behavior of the subgrid scale eddy-viscosity. They went on to implement a numerical version of the iterative shell-elimination procedure. Unfortunately, their results are strongly dependent on the shell-width ($\delta\Lambda$) selected for the numerics. There is no evidence that the results converge when the shell-width tends to zero. To clarify this issue, Carati (1991) studied the problem in the limit of infinitesimal shells. He showed that the contribution from the triple term to the propagator is proportional to k for small k and not to k^2 as for a viscous term. This k contribution does not seem to make physical sense.

The shell-elimination procedure (Sect.2) starts by introducing a cutoff $\Lambda_1 < \Lambda_0$, and splitting the velocity field v into $v^<$ and $v^>$. Substituting the λ-expansion (2.7) for $\hat{v}^>_m(\hat{\mathbf{q}})$ in the $v^>v^<$-term of equation (3.1), generates a triple non-linearity on the right-hand side and a term linear in $v^<$ (Sect.3). The triple non-linearity has the form:

$$\frac{-i\lambda_o}{2}P_{imn}(\mathbf{k})\int\int\frac{d\hat{\mathbf{q}}\,d\hat{\mathbf{Q}}}{(2\pi)^{2d+2}}\,2\,\frac{-i\lambda_o}{2}G_0(\hat{\mathbf{q}})P_{mrs}(\mathbf{q})\,[\hat{v}^<_r(\hat{\mathbf{q}})\,\hat{v}^<_s(\hat{\mathbf{q}}-\hat{\mathbf{Q}})\,\hat{v}^<_n(\hat{\mathbf{k}}-\hat{\mathbf{q}})] \tag{4.1}$$

At the next step, a new cutoff $\Lambda_2 < \Lambda_1$ is introduced and the velocity field (formerly $v^<$) is separated once more into $<$ and $>$ components. One uses the λ-expansion for the new $v^>$ terms. The triple non-linearity generates various terms, in particular a linear term of the form

$$\begin{aligned}\frac{-i\lambda_o}{2}P_{imn}(\mathbf{k})\int\int\frac{d\hat{\mathbf{q}}\,d\hat{\mathbf{Q}}}{(2\pi)^{2d+2}}\,2\,\frac{-i\lambda_o}{2}\,G_0(\hat{\mathbf{q}})P_{mrs}(\mathbf{q})\,\times&\\
[\,G_1(\hat{\mathbf{Q}})G_1(\hat{\mathbf{q}}-\hat{\mathbf{Q}})\;\;\hat{f}^>_r(\hat{\mathbf{Q}})\,\hat{f}^>_s(\hat{\mathbf{q}}-\hat{\mathbf{Q}})\,\hat{v}^<_n(\hat{\mathbf{k}}-\hat{\mathbf{q}})\,+&\\
G_1(\hat{\mathbf{q}}-\hat{\mathbf{Q}})G_1(\hat{\mathbf{k}}-\hat{\mathbf{Q}})\;\;\hat{v}^<_r(\hat{\mathbf{Q}})\,\hat{f}^>_s(\hat{\mathbf{q}}-\hat{\mathbf{Q}})\,\hat{f}^>_n(\hat{\mathbf{k}}-\hat{\mathbf{q}})\,+&\\
G_1(\hat{\mathbf{Q}})G_1(\hat{\mathbf{k}}-\hat{\mathbf{q}})\;\;\hat{f}^>_r(\hat{\mathbf{Q}})\,\hat{v}^<_s(\hat{\mathbf{q}}-\hat{\mathbf{Q}})\,\hat{f}^>_n(\hat{\mathbf{k}}-\hat{\mathbf{q}})\,]&\end{aligned}$$

Averaging over realizations of the force with (2.2)

$$< f_i(\hat{\mathbf{k}})f_j(\hat{\mathbf{k}}') >= 2D_0(2\pi)^{d+1}k^{-y}\delta(\hat{\mathbf{k}}+\hat{\mathbf{k}}')P_{ij}(\mathbf{k})$$

the term proportional to $P_{mrs}(\mathbf{q}) < \hat{f}^>_r(\hat{\mathbf{Q}})\,\hat{f}^>_s(\hat{\mathbf{q}}-\hat{\mathbf{Q}}) >$ vanishes upon integration over $\hat{\mathbf{Q}}$, $\hat{\mathbf{q}}$. One is left with

$$2D_o\,\lambda_o^2\int P_{imn}(\mathbf{k})P_{mrs}(\mathbf{q})P_{nr}(\mathbf{p})\;G_1(\hat{\mathbf{p}})G_1(-\hat{\mathbf{p}})G_0(\hat{\mathbf{q}})\;p^{-y}\frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}}\,\hat{v}_s(\hat{\mathbf{k}}) \tag{4.2}$$

where $\hat{\mathbf{p}} = \hat{\mathbf{k}}-\hat{\mathbf{q}}$. This expression is quite similar to that obtained from the original quadratic non-linearity (3.2) and one proceeds as before. Taking the long-time limit ($\omega \to 0$) and integrating over large frequencies, one defines a new propagator at Λ_2:

$$h(k,\Lambda_2) - h(k,\Lambda_1) = \frac{D_o\lambda_o^2}{(d-1)(2\pi)^d} \int \frac{P_{smn}(\mathbf{k})P_{mrs}(\mathbf{q})P_{nr}(\mathbf{p})\, p^{-y}}{h(p,\Lambda_1)\,[h(p,\Lambda_1) + h(q,\Lambda_0)]} d\hat{\mathbf{q}}$$

The differences with expression (3.4) reside in the domain of integration, which is not symmetric in p, q, and the Λ-dependency of the functions $h(k,\Lambda)$ in the integrand.

Of course a *new* triple non-linearity has been generated after this second shell-elimination and it will also contribute to the propagator at the next shell-elimination. After $N+1$ eliminations, the sum of all contributions from the triple non-linearities to the propagator can be written:

$$h(k,\Lambda_{N+1}) - h(k,\Lambda_N) = M \sum_{j=0}^{N-1} \int_{B_j^N} \frac{A(k,p,q)\, p^{-y}}{h(p,\Lambda_N)\,[h(p,\Lambda_N) + h(q,\Lambda_j)]} \frac{pq}{k} dp dq \quad (4.3)$$

The integration domain B_j^N is such that $\Lambda_{N+1} \leq p \leq \Lambda_N$ and $\Lambda_{j+1} \leq q \leq \Lambda_j$.

4.1. $k < \delta\Lambda$

Suppose that all wavenumbers down to Λ have already been eliminated and consider now the next step of the procedure where another shell of wavenumbers will be eliminated to reach $\Lambda' < \Lambda$. When $k < \delta\Lambda$ there is only one term in the sum (4.3), that where $\Lambda_{N+1} \leq p \leq \Lambda_N$ and $\Lambda_N \leq q \leq \Lambda_{N-1}$. The other domains B_j^N are empty (Fig.1). The domain of integration $D3 = B_{N-1}^N$ (lightly shaded area in Fig.1) for the cubic term is independent of Λ' when $k < \delta\Lambda$, it is given by

$$\int_{D3} = \int_{\Lambda}^{\Lambda+k} dq \int_{q-k}^{\Lambda} dp$$

Expanding in powers of k and $\delta\Lambda$, as before, the contribution of the cubic to $h(k,\Lambda') - h(k,\Lambda) = -k^2\delta_{(c)}\nu$ is equal to

$$-k^2\,\delta_{(c)}\nu = \frac{M}{\nu^2(\Lambda)}\,[\frac{-2}{15}\,\Lambda^{-1-y}\,k^2 + \frac{3-y}{24}\,\Lambda^{-2-y}\,k^3 + O(k^4, \delta\Lambda^2)] \quad (4.4)$$

This contribution does not make mathematical sense as the right-hand side should be of order $k^2\delta\Lambda$ but is of order k^2 only.

4.2. $k > \delta\Lambda$

This case has been studied in detail by Carati (1991). The analysis is sketched below. In the limit of infinitesimal shells, $\Lambda_{j+1} - \Lambda_j = \delta\Lambda_j \to 0$, $N \to \infty$, letting $\Lambda_N = \Lambda$, the equation (3.8) becomes

$$\frac{\partial h(k,\Lambda)}{\partial \Lambda} = -\frac{M\,\Lambda^{1-y}}{k\,h(\Lambda,\Lambda)} \int_{\Lambda}^{\Lambda+k} \frac{\Lambda_j\, A(k,\Lambda,\Lambda_j)}{[h(\Lambda,\Lambda) + h(\Lambda_j,\Lambda_j)]} d\Lambda_j \tag{4.5}$$

This is an integro-differential equation for $h(k,\Lambda)$. Carati (1991) looked for a self-similar solution of the form

$$h(k,\Lambda) = \Lambda^c \varphi(k/\Lambda) \tag{4.6}$$

Substituting in (4), one finds $c = (5-y)/3$ for self-similarity and

$$\varphi(l) = \frac{M}{\varphi^2(1)} \int_0^1 x^{-c-1}\, F(lx)dx \tag{4.7}$$

with

$$F(v) = v^{-1} \int_1^{1+v} dw \frac{w}{1+w^c} A(v,1,w) \tag{4.8}$$

To determine the form of the propagator (4.6) in the limit of very large scales, one has to expand $\varphi(l)$ around $l = 0$. Let

$$F(v) = \sum_{i=0}^{\infty} F_i\, v^i$$

then integrating (4.7) term by term with $v = lx$ yields

$$\varphi(l) = \frac{M}{\varphi^2(1)} \sum_{i=0}^{\infty} l^i \frac{F_i}{i-c} \tag{4.9}$$

From (4.8), one finds

$$F(v) = -\frac{v}{4} + \frac{4+c}{15} v^2 + \frac{26-3c}{48} v^3 + O(v^4) \tag{4.10}$$

Writing $c = (5-y)/3 = 2 - \epsilon/3$, the final result reads

$$h(k,\Lambda) = \frac{M}{\varphi^2(1)} \Lambda^{2-\epsilon/3} \times$$
$$[\frac{-3}{\epsilon-3}\frac{k}{4\Lambda} + \frac{3}{\epsilon}\frac{18-\epsilon}{45}\frac{k^2}{\Lambda^2} + \frac{3}{\epsilon+3}\frac{20+\epsilon}{48}\frac{k^3}{\Lambda^3} + O(\frac{k}{\Lambda})^4] \tag{4.11}$$

The lowest order term is proportional to k and this term does not have any clear physical interpretation. It does no' . :rer'ond to a renormalized viscous damping.

5. Non-Galilean invariance of triple non-linearities

In this section it is shown that the triple non-linear term generated by the shell-elimination procedure is not Galilean invariant. It is believed that this non-invariance of the resulting equations is the cause for the non-sensical results presented in Sect.4. The reason is that *at the second cut* one performs the shell-elimination on an equation which has lost an essential symmetry of the original Navier-Stokes equations, the Galilean invariance. Thus keeping the triple non-linearity is in conflict with the spirit of the Renormalization Group analysis of the Navier-Stokes. At each step of the RNG procedure, the equations are renormalized to obtain new equations formerly identical to the original equations, and in particular preserving all the symmetries of those original equations.

Consider a Galilean transformation :

$$\begin{aligned} \mathbf{x} &= \mathbf{x}' + \mathbf{V}\,t \\ t &= t' \\ \mathbf{v}(\mathbf{x},t) &= \mathbf{V} + \mathbf{v}'(\mathbf{x}',t') \end{aligned} \tag{5.1}$$

In Fourier space:

$$\begin{aligned} \mathbf{k} &= \mathbf{k}' \\ \omega &= \omega' + \mathbf{k}'\cdot\mathbf{V} \\ \mathbf{v}(\mathbf{k},\omega) &= \mathbf{V}\delta(\mathbf{k})\delta(\omega) + \mathbf{v}'(\mathbf{k},\omega') \end{aligned} \tag{5.2}$$

As is well-known, a Galilean transformation of the Navier-Stokes equations (2.3) leaves them unchanged. Omitting viscosity and the force, which are irrelevant for our purpose here (the white-noise nature of the force assures Galilean invariance on average), (2.3) reads

$$-i\omega\, \hat{v}_i(\hat{\mathbf{k}}) = \frac{-i\lambda_o}{2} P_{imn}(\mathbf{k}) \int \hat{v}_m(\hat{\mathbf{q}})\hat{v}_n(\hat{\mathbf{k}}-\hat{\mathbf{q}}) \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \tag{5.3}$$

The Galilean transformation (5.2) of (5.3) leads to

$$\begin{aligned} (-i\,\omega' - i\,\mathbf{k}'\cdot\mathbf{V})\,\hat{v}'_i(\hat{\mathbf{k}}') = {} & -iP_{imn}(\mathbf{k}')V_m\hat{v}'_n(\hat{\mathbf{k}}') + \\ & \frac{-i}{2}P_{imn}(\mathbf{k}') \int \hat{v}_m(\hat{\mathbf{q}}')\hat{v}'_n(\hat{\mathbf{k}}'-\hat{\mathbf{q}}') \frac{d\hat{\mathbf{q}}'}{(2\pi)^{d+1}} \end{aligned} \tag{5.4}$$

This has the same form as (5.3) because $P_{imn}(\mathbf{k}')V_m\hat{v}'_n(\hat{\mathbf{k}}') = (\mathbf{k}\cdot\mathbf{V})\,\hat{v}'_i$.

After a first elimination, a triple non-linearity of the form (cf. (4.1)):

$$P_{imn}(\mathbf{k}) \int\!\!\int d\hat{\mathbf{q}}\, d\hat{\mathbf{Q}}\; P_{mrs}(\mathbf{q})\, [\hat{v}^<_r(\hat{\mathbf{q}})\, \hat{v}^<_s(\hat{\mathbf{q}}-\hat{\mathbf{Q}})\, \hat{v}^<_n(\hat{\mathbf{k}}-\hat{\mathbf{q}})] \tag{5.5}$$

is added to (5.3). A Galilean transformation (5.1) of that triple non-linearity (5.5) transforms it into, after some reductions using the definition of the function P_{imn} and continuity:

$$
\begin{aligned}
P_{imn}(\mathbf{k}') \int\!\!\int d\hat{\mathbf{q}}' d\hat{\mathbf{Q}}' P_{mrs}(\mathbf{q}')\, \hat{v}'_n(\hat{\mathbf{k}}' - \hat{\mathbf{q}}')\, \hat{v}'_s(\hat{\mathbf{q}}' - \hat{\mathbf{Q}}')\, \hat{v}'_r(\hat{\mathbf{Q}}') &+ \\
P_{imn}(\mathbf{k}') \int d\hat{\mathbf{Q}}'\, (\mathbf{k}' \cdot \mathbf{V})\, \hat{v}'_n(\hat{\mathbf{k}} - \hat{\mathbf{Q}})\, \hat{v}'_m(\hat{\mathbf{Q}}) &+ \\
2\, P_{imn}(\mathbf{k}) \int d\hat{\mathbf{q}}\, (\mathbf{q} \cdot \mathbf{V})\, \hat{v}'_n(\hat{\mathbf{k}} - \hat{\mathbf{q}})\, \hat{v}'_m(\hat{\mathbf{q}}) &+ \\
2(\mathbf{k} \cdot \mathbf{V})^2 \hat{v}'_i(\hat{\mathbf{k}}')&
\end{aligned}
$$

The appearance of the last three terms makes the equations non-invariant in the presence of the triple non-linearity. The third term is proportional to k.

Note finally from (5.2) that the limit of large times $\omega \to 0$ seems to require that the large scale limit $k \to 0$ be taken simultaneously, in order for the result to be Galilean invariant, i.e. $\omega' = \omega - \mathbf{k} \cdot \mathbf{V} \to 0$, $\mathbf{k}' \to 0$, when $\omega \to 0$ *and* $k \to 0$. This might suggest why the constraint $k/\delta\Lambda \to 0$ is required to obtain sensible results using the renormalization procedure.

6. Conclusions

The analysis presented above seems to indicate that the RNG procedure is unable to derive the cusp in the eddy-viscosity, at least at the lowest order of the technique. However, it should be pointed out that this might be more a problem of interpretation than a shortcoming of the procedure. The cusp arises when one *insist* on representing the effect of the subgrid scales on the resolved scales as an eddy-viscosity. Two remarks are in order.

First, it is very plausible that the RNG procedure correctly determines that part of the interactions with the subgrid scales which can actually be modeled by an eddy-viscosity. This part would be the only one of importance when k is less than about $\Lambda/2$. When eliminating all wavenumbers down to $\Lambda = k$ while still preserving the form $\nu(\Lambda)k^2$ (thus ignoring any cusp), the final result should rather be interpreted as the *eddy-damping* in the inertial range. The RNG procedure can then be used to obtain the EDQNM model as in Dannevik *et al.* (1987) from which one can derive the cusp behavior in an appropriately defined eddy-viscosity, exactly as in Kraichnan (1976). In effect, the RNG procedure is used in this case to determine the free parameter in the EDQNM model.

Second, it should be pointed out that the cusp in the eddy-viscosity is a *spectral space* feature. One must include the cusp when performing Large Eddy Simulations (LES) using *spectral methods* with a sharp cut-off. It is not clear at all that one needs to worry about the cusp when using finite-difference type methods. The reason for this is that the cusp represent a statistical "squishing out of resolution" of small scales. The nature of the conservative triad interactions in Fourier space makes it impossible for a spectral method to let energy leak out at the cutoff and one observes an artificial pile-up of energy near the high-wavenumber end if it is not removed by an increased eddy-viscosity in that region. In a finite-difference method, it is possible for small eddies to be squished beyond resolution, and to disappear without having to be removed by an extra amount of viscosity.

References

Carati, D. (1991) Phys. Rev. A. **44**, 6932.

Chollet, J. P. & Lesieur, M. (1981) J. Atmos. Sci. **38**, 2747.

Dannevik, W. P., Yakhot, V. & Orszag, S. A. (1987) Phys. Fluids **30**, 2021.

De Dominicis, C. & Martin, P. C. (1979) Phys. Rev. A **19**, 419.

Domaradzki, J. A., Metcalfe, R. W., Rogallo, R. S. & Riley, J. J. (1987) Phys. Rev. Lett. **58**, 547.

Forster, D., Nelson, D. R. & Stephen, M. J. (1977) Phys. Rev. A **16**, 732.

Fournier, D. & Frisch, U. (1983) Phys. Rev. A **28**, 1000.

Kraichnan, R. H. (1976) J. Atmos. Sci. **33**, 1521.

She, Z. S. (1991) private communication.

Smith, L. M. & Reynolds, W. C. (1992) Phys. Fluids A **2**, 364.

Wilson, K. G. (1975) Rev. Mod. Phys. **47**, 773.

Yakhot, V. & Orszag, S. A. (1986) J. Sci. Comput. **1**, 3.

Zhou, Y., Vahala, G. & Hossain, M. (1988) Phys. Rev. A **37**, 2590.

Zhou, Y., Vahala, G. & Hossain, M. (1989) Phys. Rev. A **40**, 5865.