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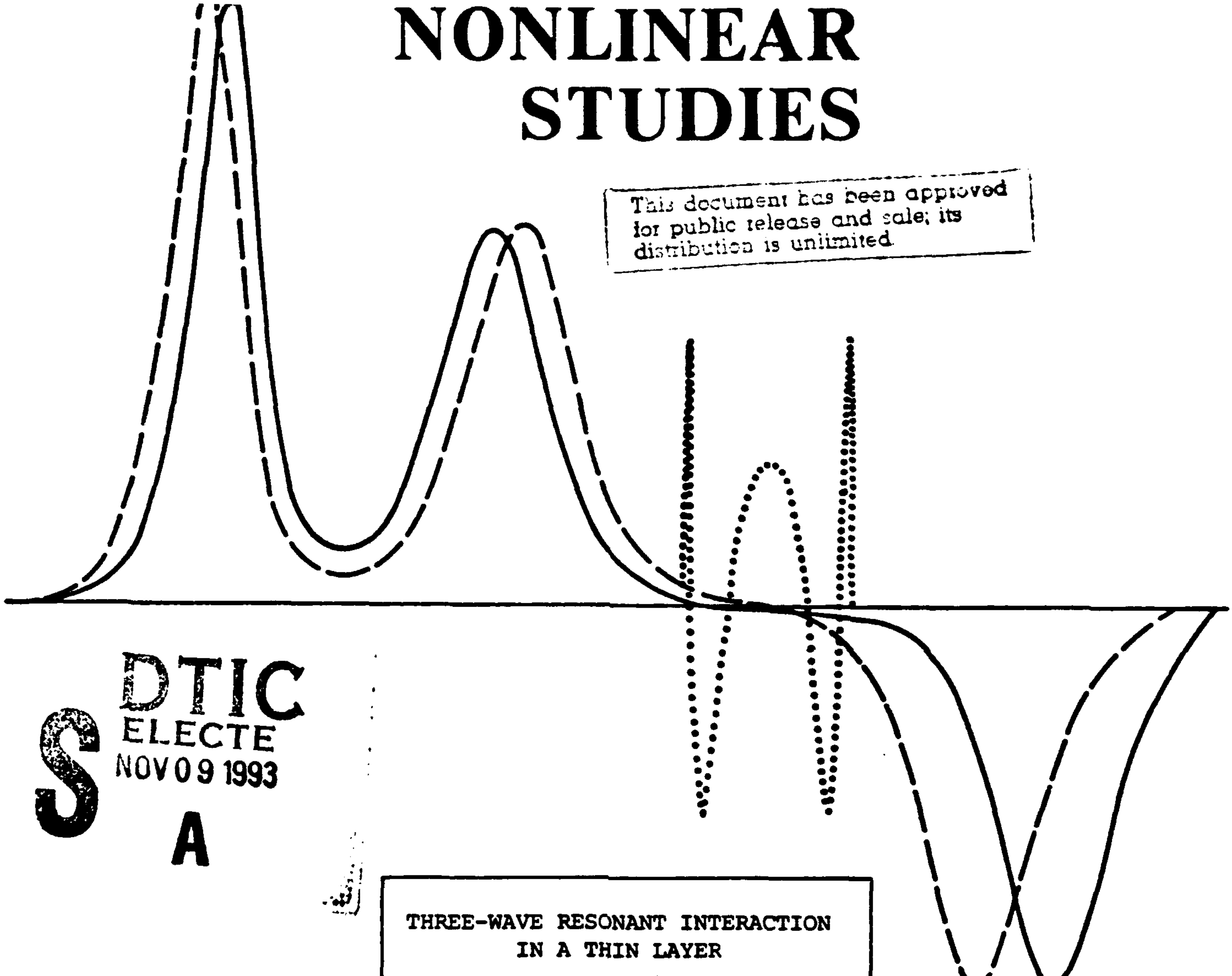

THREE-WAVE RESONANT INTERACTION IN A THIN LAYER

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Three-Wave Resonant Interaction In A Thin Layer

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Abstract

We analyze in detail conversion of a pump wave into two daughter waves (DW's) on an infinitely thin nonlinear layer, assuming that the conversion is initiated by weak seed pulses of the DW's. Three cases corresponding to different signs of the group velocities of the DW's are considered separately. In the case when both velocities are opposite to that of the pump wave, a threshold value of the "strength" of the nonlinear layer is found, above which oscillations between the conversion and the inverse process of recombination set in. In the case when the DW group velocities have different signs, the shapes of the generated DW pulses do not depend upon shapes of the corresponding "seed" pulses. In this case, multiple solutions are possible, provided the nonlinearity is strong enough.

1 Introduction

It is well known that three-wave resonant interactions (3WRI) play a fundamental role in a number of problems of plasma physics and nonlinear optics [1]. In many cases, it is necessary to analyze these interactions in inhomogeneous media [2,3]. An important physical realization of the 3WRI in an inhomogeneous medium is interaction of an intensive electromagnetic wave with a plasma. The linear interaction of electromagnetic waves with an inhomogeneous medium is a traditional problem of nonlinear optics [4] and plasma physics (see, e.g., refs. [5,6,7,8]). Recently, attention has been attracted to resonant nonlinear interactions of the waves with a strongly inhomoneous plasma layer (sheath), which can find a promising practical application to design means of communication with a space vehicle reentering the atmosphere [9].

In ref. [3], the analysis was developed for two situations: When the inhomogeneity was very smooth, and when, contrary to this, the nonlinearity was residing in a very narrow layer, so that the nonlinear terms in the governing equations were proportional to a delta-function. The objective of the present paper is to complete the analysis for the thin layer started in ref. [3].

The system of equations governing the 3WRI in an infinitely thin layer has the form [3]

$$(u_1)_t + v_1(u_1)_x = -i\epsilon\delta(x)u_p u_2^*, \tag{1.1a}$$

$$(u_2)_t + v_2(u_2)_x = -i\epsilon\delta(x)u_p u_1^*, \tag{1.1b}$$

$$(u_p)_t + v(u_p)_x = -i\epsilon\delta(x)u_1 u_2, \tag{1.1c}$$

where $u_1(x,t), u_2(x,t)$, and $u_p(x,t)$ are complex envelopes of the, respectively, DW's and pump, v_1, v_2, and v being their group velocities. In what follows, the pump velocity v will be assumed always positive by definition, while the DW velocities $v_{1,2}$ may have either sign.

In an infinitesimal vicinity of the point $x = 0$, one may neglect the temporal derivatives in eqs. (1), reducing them to the system of ODE's with singular coefficients. Even in the case of a finitely thin layer, this approximation is still valid, provided the envelope is slowly varying in time, such that $\Delta x|u_t| << |vu|$ where Δx is the width of the thin layer. We also remark that if we interchange x and t, we see that this problem is also solvable by the IST for the (3WRI) [1]. Upon this interchange, we now have the initial value problem where the profiles are specified at $t = 0^-$, they then interact for a short time with a very large coupling constant until $t = 0^+$, at which time we wish to know the final profiles. Although the IST can formally solve this problem the complexities of the inverse scattering equations can be bypassed with the following simpler analysis.

An important part of this problem concerns the relative phase of each envelope[4]. In general, each envelope has an amplitude and a phase,

$$u_1 = A_1 e^{i\alpha_1}, \tag{1.2a}$$

$$u_2 = A_2 e^{i\alpha_2}, \tag{1.2b}$$

$$u_p = A_p e^{i\alpha_p}. \tag{1.2c}$$

Then from (1.1), upon neglecting the time derivatives, we have

$$v_1 \frac{dA_1}{dx} = \epsilon\delta(x) A_p A_2 \sin\alpha \tag{1.3a}$$

$$v_2 \frac{dA_2}{dx} = \epsilon\delta(x) A_p A_1 \sin\alpha \tag{1.3b}$$

$$v \frac{dA_p}{dx} = -\epsilon\delta(x) A_1 A_2 \sin\alpha \tag{1.3c}$$

$$v_1 \frac{d\alpha_1}{dx} = -\epsilon\delta(x) \frac{A_p A_2}{A_1} \cos\alpha \tag{1.4a}$$

$$v_2 \frac{d\alpha_2}{dx} = -\epsilon\delta(x) \frac{A_p A_1}{A_2} \cos\alpha \tag{1.4b}$$

$$v \frac{d\alpha_p}{dx} = -\epsilon\delta(x) \frac{A_1 A_2}{A_p} \cos\alpha \tag{1.4c}$$

where

$$\alpha \equiv \alpha_p - \alpha_1 - \alpha_2 \tag{1.5}$$

Now from (1.4)

$$\frac{d\alpha}{dx} = \epsilon\delta(x) \left[A_p \left(\frac{A_2}{v_1 A_1} + \frac{A_1}{v_2 A_2} \right) - \frac{A_1 A_2}{v A_p} \right] \cos\alpha \tag{1.6}$$

Note that the smallest amplitude in (1.6) will determine to which fixed point ($\alpha = \pm\frac{\pi}{2}$) the system will be driven toward. If one amplitude is much smaller than the other two, then the sign of its group velocity determines which fixed point is stable and α will be driven very rapidly to that stable point. If the two smallest amplitudes are comparible, then if the signs of their group velocities are different, they can compete for determining the stable point. In any case, any value of α away from $\pm\frac{\pi}{2}$ is unstable and it is only near the values of $\pm\frac{\pi}{2}$ that each of the individual phases become stationary [see (1.4)]. We shall therefore assume that $\alpha = \pm\frac{\pi}{2}$ and which value we use shall be the same as the stable fixed point of (1.6). Of course, these two stable points only differ by a phase of π, which simply means a sign change of the amplitude. Sometimes we may find it convenient to simply let the amplitude carry the sign instead of worrying about the phase change, particularly when the amplitude passes through zero.

As it immediately follows from eqs. (1.3), one can eliminate the amplitudes A_1 and A_2 in favor of A_p [4]

$$A_n^2 = \frac{v}{v_n}[C_n(t) - A_p^2], n = 1, 2, \tag{1.7}$$

$C_n(t)$ being arbitrary functions of time. Actually, eq. (1.7) is nothing but the well-known Manly-Rowe relation [1]. At last, noting that A_1 and A_2 are each positive, inserting eq. (1.7) into eq. (1.3c), one obtains the first-order equation for $A_p(x)$

$$\frac{dA_p}{dx} = -\epsilon\delta(x)\sin\alpha\sqrt{(v_1 v_2)^{-1}(C_1 - A_p^2)(C_2 - A_p^2)} \tag{1.8}$$

upon noting that $v > 0$.

A general problem that we are going to discuss is to use eq. (1.8) to analyze the conversion of an arbitrary initial pump pulse $u^{(i)}(t - \frac{x}{v})$. To do this, it is necessary to specify the functions $C_n(t)(n = 1, 2)$ in eq. (1.8), which one can do considering boundary conditions at $x = \pm 0$. In turn, this analysis proves to be essentially different for different signs of the DW velocities v_n. Therefore, in what follows we will analyze separately three different cases.

2 Forward Conversion: $v_1, v_2 > 0$

For the finite slab case, this case of forward conversion could be either the decay case ($0 < v_1 < v < v_2$ or $0 < v_2 < v < v_1$) or the stimulated backscatter (SBS) case ($0 < v < v_1, v_2$ or $0 < v_1, v_2 < v$) depending on whether the magnitude of velocity of the pump lies between or outside of the velocities of the DWs. However, these distinctions and differences become irrelevant in the limit of a thin layer, with the decay case and SBS case having the same limit. Here, we expect that the initial pump pulse $u^{(i)}(t - \frac{x}{v})$ will be converted, in part, into DW pulses $u_n = u_n^{(f)}(t - \frac{x}{v_n})$, and will also give rise to a residual (final) pulse of the pump wave $u^{(f)}(t - \frac{x}{v})$. To initiate the conversion process, it is necessary to launch, together with $u^{(i)}(t - \frac{x}{v})$, small "seed" DW pulses $u_n^{(i)}(t - \frac{x}{v_n})$ (otherwise we will have the trivial solution $u_n \equiv 0$). Now for the value of α. In this case, since we initially only have small seed DW pulses, the last term in (1.6) can be neglected. Then a simple analysis easily shows that $\alpha = \frac{\pi}{2}$ is the stable value and (1.8) gives that the pump will initially deplete. As it depletes, the amplitudes can pass through zero. If that happens, we simply set $\alpha \equiv \frac{\pi}{2}$ and allow the amplitudes, if they pass through zero, to carry a sign. Next, the functions $C_n(t)$ can be found from eqs. (1.7) taken at $x = 0^-$ (i.e. before the conversion):

$$C_n(t) = [A^{(i)}(t)]^2 + \frac{v_n}{v}[A_n^{(i)}(t)]^2 \equiv (A^{(i)}(t))^2 + \frac{v_n}{v}U_n(t), n = 1,2, \tag{2.1}$$

where the initial amplitudes $A^{(i)}$ and $A_n^{(i)}$ are related to the initial pulses of the pump and daughter waves by eqs. (1.2).

According to what was said above, we assume that

$$U_n << (A^{(i)})^2. \tag{2.2}$$

Next one should insert eq. (2.1) into eq. (1.8) and integrate the latter equation, making use of the relation $\int_{-\infty}^{+\infty}\delta(x)dx = 1$. This must yield the profile of the residual (final) pulse $A^{(f)}(t)$ determined by $A^{(i)}(t)$ and $A_{1,2}^{(i)}(t)$. In a general case, this expression involves elliptic integrals and is rather complicated [6]. However, the integral may be simplified if one makes use of the inequality (2.2). Eventually, we find

$$A^{(f)}(t) = A^{(i)}(t)\frac{1-\kappa(t)}{1+\kappa(t)} \tag{2.3}$$

where

$$\kappa(t) \equiv \frac{v_1U_1(t) + v_2U_2(t)}{8v(A^{(i)}(t))^2}\exp\left[\frac{2\epsilon A^{(i)}(t)}{\sqrt{v_1v_2}}\right]. \tag{2.4}$$

Although the pre-exponential factor in the expression (2.4) is small according to eq. (2.2), its smallness can be readily compensated by the exponential if ϵ is sufficiently large, and the conversion rate, proportional to $\kappa(t)$, can thus be nonsmall. Note that $A^{(f)}(t)$ vanishes when $\kappa(t)$ attains the value unity, and with subsequent increase of $\kappa A^{(f)}$ the amplitude $A^{(f)}$ is again nonzero with the opposite sign, attaining the value $A^{(f)}(t) = -A^{(i)}(t)$ at $A^{(i)}(t) = \infty$. Although the amplitudes A have been defined in eqs. (1.2) as positive quantities, its negative values imply that, beyond the full conversion point ($A^{(f)} = 0$) the DW's produced by the decay of the pump start to recombine into it. As it was explained in ref. [3], one can actually expect oscillations between the pump and DW's, if the delta-function layer is replaced by a structured profile with a finite width. In the framework of the present analysis, the oscillations will appear if one lauches finite but small seed DW pulses $u_n^{(i)}(t)$ instead of the vanishingly small ones considered above. In this case, the approximation (2.4) is no longer valid and would have to be replaced by the appropriate elliptic integral which would contain this oscillatory behavior.

With the quantity $A^{(f)}(t)$ given by eqs. (2.3) and (2.4), one can find the DW pulses $u_n^{(f)}(t)$ generated by the pump conversion. ¿From eqs. (2.1) and (1.7), taken at $x = 0^+$ (after the conversion), it follows that

$$(A_n^{(f)}(t))^2 \approx \frac{v}{v_n}[(A^{(i)}(t))^2 - (A^{(f)}(t))^2] \tag{2.5}$$

The DW pulses will propagate in the region $x > 0$ in the form $u_n = u_n(t - \frac{x}{v_n})$. Thus, in the case when v_1 and v_2 are positive, the pump pulse passing through the nonlinear layer gives rise to a residual pump pulse and to DW pulses, all propagating in the same direction.

At last, it is relevant to analyze if the phase stability condition remains valid during the amplitude conversion. Inserting eqs. (1.7) and (2.1) into (1.6), in which U_n is neglected in comparison with $(A^{(i)})^2$, one can readily find that the fixed point $\alpha = \frac{\pi}{2}$ is no longer stable if

$$(A^{(f)})^2 < \frac{1}{3}(A^{(i)})^2 \tag{2.6}$$

i.e., when the conversion is too deep (this, in particular, means that the point of the full conversion, at which $A^{(f)}$ vanishes, lies in the unstable domain). Using eqs. (2.3) and (2.4), it is easy to find in an explicit form a boundary between stable and unstable domains in the parametric space. In this work we will not consider how the solutions should be modified to take account of the phase instability in the domain (2.6).

3 Backward Conversion: $v_1, v_2 < 0$

The backward conversion case corresponds only to SBS in the homogeneous case of a finite slab. There, the decay of the pump occurs through a backscattering process whereby the pump is partially reflected [1], which in a sense, is a mode conversion. In this case, the generated DW pulses will propagate backwards, i.e. they will look like reflected waves. To generate them, it is necessary to launch small "seed" (initial) pulses $u_n^{(i)}$ at $x = 0^+$. Actually, the conversion will proceed from $x = 0^+$ to $x = 0^-$. With regard to this, eq. (1.6) with negative v_1 and v_2 tell us that the stable value of the phase α is *always* $\frac{\pi}{2}$. Thus, in the present case the amplitude evolution equation (1.8) also gives rise to depletion of the inital pump pulse.

According to what was said above, the functions $C_n(t)$ are determined by eq. (1.7) taken at $x = 0^+$:

$$C_n(t) = (A^{(f)}(t))^2 + \frac{v_n}{v}(A_n^{(i)}(t))^2 \equiv (A^{(f)}(t))^2 + \frac{v_n}{v}U_n(t), \tag{3.1}$$

cf. eq. (2.1). This time, it is assumed

$$U_n << (A^{(f)})^2 \,, \tag{3.2}$$

cf. eq. (2.2). Note that with this approximation, we are assuming that complete conversion will not occur. The subsequent analysis can be developed essentially as it was done in the preceding section. We insert eq. (3.1) into eq. (1.8), and then, to calculate the resultant integral approximately, we make use of the inequality (3.2). Eventually, we arrive at the following transcendental equation relating $A^{(f)}$ to $A^{(i)}$:

$$\frac{8v}{(|v_1|U_1 + |v_2|U_2)} \frac{(A^{(i)} - A^{(f)})}{(A^{(i)} + A^{(f)})}(A^{(f)})^2 = \exp(\frac{2\epsilon A^{(f)}}{\sqrt{v_1 v_2}}). \tag{3.3}$$

Elementary analysis of this equation demonstrates that there is a threshold value ϵ_{thr} such that eq. (3.3) has two solutions at $\epsilon < \epsilon_{thr}$, and no solution at $\epsilon > \epsilon_{thr}$. One can find an approximate expression for ϵ_{thr} once again making use of the inequality (3.2),

$$\epsilon_{thr}^2 = \frac{8vv_1v_2}{|v_1|U_1 + |v_2|U_2}. \tag{3.4}$$

Note that the threshold value given by eq. (3.4) does not depend on $A^{(i)}$, and it is large in virtue of the smallness of U_1 and U_2. At $\epsilon > \epsilon_{thr}$, the above-mentioned oscillations between the pump's decay into the DW's and the reverse process of recombination of the DW's into the pump take place, so that ϵ_{thr} is, as a matter of fact, the threshold for the onset of oscillations. However, the oscillations cannot be considered in the framework of the approximation used. To do this, it is necessary to consider the exact elliptic integral corresponding to eq. (1.8).

The existance of two solutions below the threshold implies that one solution should be stable and one unstable. The stable branch corresponds to $\frac{dA^{(f)}}{dA^{(i)}} > 0$, and the unstable one to $\frac{dA^{(f)}}{dA^{(i)}} < 0$. In particular, in the region $A^{(i)} > \frac{\sqrt{v_1 v_2}}{\epsilon}$ the stable branch can be found in the following approximate form:

$$A^{(f)} = \frac{\sqrt{v_1 v_2}}{2\epsilon} \ln \frac{8v(A^{(i)})^2}{|v_1|U_1 + |v_2|U_2}. \tag{3.5}$$

In this region, this branch lies in a region of small $A^{(f)}$.

To conclude this section, let us emphasize once again that, in the case considered, the value $\alpha = \frac{\pi}{2}$ remains the stable fixed point of eq. (1.6) at all the values of the parameters.

4 Mixed Conversion: $v_1 > 0, v_2 < 0$

This case is another example where the interaction could be the soltion decay case ($v_1 < v$) or the stimulated backscatter case ($v < v_1$). In this case where v_1 and v_2 have different signs, the seed pulses of the two daughter waves should be taken, respectively, at $x = 0^+$ and at $x = 0^-$. Accordingly, the functions C_1 and C_2 are determined by eqs. (1.7) taken at $x = 0^+$ and at $x = 0^-$. To determine stable values of the phase α, one can consider the particular cases when either of the seed components $u_{1,2}^{(i)}$ is nonzero, while the other one is absent. If, for instance, only the first component ($n = 1$) is present, the conversion process proceeds from $x = 0^-$ to $x = 0^+$, and eq. (1.6) with positive v_1 defines $\alpha = \frac{\pi}{2}$ as the stable fixed value. It is easy to see that the same is true in the opposite particular case, when only the second seed component is different from zero. Thus, as well as in the two preceding cases, eq. (1.8) gives rise to depletion of the initial pump. However, the important difference from the previous cases is that this time the elliptic integral following from eq. (1.8) converges even when the initial seed values $A_n^{(i)}$ are exactly equal to zero. Actually, this means that the results which will be put forward below are valid provided that

$$(A^{(i)})^2 - (A^{(f)})^2 >> (A_n^{(i)})^2 \frac{v_n^2}{v^2}. \tag{4.1}$$

Neglecting the "seeds" according to what was said above, we obtain

$$C_1(t) = (u^{(i)}(t))^2; C_2(t) = (u^{(f)}(t))^2, \tag{4.2}$$

cf. eqs. (2.1) and (3.1). Then, it is straightforward to insert eqs. (4.2) into eq. (1.7) to obtain from it the relation between A^f and A^i in the following general form:

$$\epsilon(A^{(i)})^2 = (1+2N)\sqrt{|v_1 v_2|}K(\sqrt{1-[A^{(f)}/A^{(i)}]^2}), \tag{4.3}$$

$K(q)$ being the complete elliptic integral of the first order with the modulus $q \equiv \sqrt{1-[A^{(f)}/A^{(i)}]^2}$. The integer $N = 0, 1, 2, \ldots$ takes into account *explicitly* the above-mentioned circumstance that the pump and daughter waves may accomplish N cycles of the full conversion in the course of the interaction in the nonlinear layer. It will be demonstrated below that N cannot take indefinitely large values; nevertheless, we conclude that, in the case when N may take more than one value, the three-wave conversion problem has *multiple* solutions. At the same time, one should remember that each inversion of the pump decay into the opposite process of the pump recombination introduces a jump $\Delta\alpha = \pi$ into the phase α defined in eq. (1.5). So, stability of different branches of the multiple solution remains to be clarified.

To represent the final amplitude as a function of the initial one, we introduce the function $q = q(K)$ as an inverse to $K = K(q)$. Then, the representation following from eq. (4.3) is

$$(A^{(f)})^2 = (A^{(i)})^2 \left[1 - q^2\left(\frac{\epsilon(A^{(i)})^2}{(1+2N)\sqrt{|v_1 v_2|}}\right)\right]. \tag{4.4}$$

Note that eqs. (4.3) and (4.4) demonstrate correct asymptotic behavior in the limit $\epsilon \to \infty$ (at N fixed): $q^2(\infty) = 1$, so that $u^{(f)} = 0$, i.e. the full conversion takes place in this limit. On the other hand, eq. (4.3) has no solution in the region

$$\frac{\epsilon(A^{(i)})^2}{\sqrt{|v_1 v_2|}} < \frac{\pi}{2}(1+2N). \tag{4.5}$$

Thus, for any given ϵ we obtain a finite number of solutions of the mixed conversion problem. In particular, there is no solution of the form sought for if inequality (4.5) holds for $N = 0$. Actually, the basic assumption made above, i.e. omission of the "seeds" $A_n^{(i)}$, is not legitimate in the region (4.5), so that a solution in this region must take a more complicated form. One can also understand this from the viewpoint of the soliton decay case discussed in Ref. [1]. The solution for soliton decay is very sensitive to the "seeds", with the scalelength for the decay linearly proportionally to the natural logarithm of the seed amplitude. Also the relative phases of the decay products can be strongly affected by the relative phases and amplitudes of the seeds. Consequently it is not surprising that if we take the limit of vanishing seeds, we find a multitude of possible solutions. We also note that in the SBS case [1], that problem is a two-point boundary value problem as is here. Such problems can be remarkable sensitive to initial conditions, particularly near any homogeneous solutions. That may also provide a means of understanding these multiple soltuions.

At last, the final form of the DW's produced by the conversion of the pump can be obtained from eqs. (4.2). The waves u_1 and u_2 will propagate, respectively, as transmitted and reflected waves generated by the nonlinear layer. The important fact is that outside of the region (4.5), the shapes of the generated DW pulses do not depend on the form of the "seed" pulses.

Coming back to the phase stability problem, one can readily find that, apart from the phase jumps at the decay-recombination inversion points, the stability exponent determined by eq. (1.6) changes its sign at the point

$$A^2 = \frac{1}{3}[(A^{(i)})^2 + (A^{(f)})^2 + \sqrt{(A^{(i)})^4 + (A^{(f)})^4 - (A^{(i)})^2(A^{(f)})^2}] \,. \tag{4.6}$$

It is easy to check that the critical value of A^2 determined by eq. (4.6) always belongs to the interval $(A^{(f)})^2 < A^2 < (A^{(i)})^2$, so that this value is automatically attained in the course of the conversion. Thus, in any case the conversion index given by eq. (4.4) pertains, strictly speaking, to a solution which is not globally stable. Nevertheless, this result may be quite useful, as it should be relevant at least in the case when the initial rate of the phase disturbances is low enough.

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