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The Morphological Processing of Binary Images

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The morphological processing of binary images is treated in depth from the viewpoint of *ordered topology*. The basic ideas of Nachbin and Birkhoff concerning when a partial ordering of a set that has a topology is compatibile with that topology are used to obtain a definition of an *order resolvable topological ordered space*. This abstraction is then used to define semicontinuity in a sufficiently general way to unify its treatment in mathematical morphology. This is followed by a detailed review of the basic concepts of closed-set morphology. In this review, a more thorough treatment than is available elsewhere is given of the several limit concepts used in morphology; likewise, the continuity properties of the *erosion*, *dilation*, and *homothesis* operations are more thoroughly treated than is usual. The morphological transformation and transformation space theory of closed-set morphology is presented in a systematic way that incorporates the relevant contributions of Matheron and Maragos, as well as the recent important work of Banon and Barrera. Finally, a class of countable closed-set bases for the underlying *hit-miss topology* is used to derive a novel representation of morphological transformations.

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1 Introduction

As applied to the image-processing tasks required for automatic/aided target recognition (ATR), mathematical morphology (Giardina and Doughtery, 1988; Serra, 1982; Matheron, 1975) is a theory of certain transformations of a set of functions that represent images. These transformations, which are designed to extract geometrical shape information from the images they operate on, are called *morphological transformations.* The first task of the science or theory of *morphological image processing* (MIP) is therefore twofold: to choose a set of functions for the representational role and then define the morphological transformations of these functions.

This report treats this primary task in detail for the special case of *binary images.* For the more general case of *greyscale images,* MIP chooses the extended real valued (ERV) upper semicontinuous (USC) functions of two or more real variables to play the representational role. The reasons for this choice are more appropriately considered in treating the general case and will not be dealt with here. Suffice it to say that such functions are adequate for the greyscale images encountered in practice (which show occasional abrupt jumps against a background of general continuity) and have certain convenient technical properties. This report accordingly focuses on the USC functions whose only values are 0 and 1, and with good reason: Binary-valued USC functions are equivalent to the *closed subsets* of the independent variable space or image field. Indeed a binary-valued USC function f determines a unique closed subset F of the image field S through $F \equiv \{x \in S : f(x) = 1\}$; conversely, such a closed set determines a unique binary USC function f through $f(x) = 1$ for all $x \in F$ and $= 0$ otherwise. Because of this equivalence, the focus on binary f allows us to concentrate directly on geometrical shapes (the closed subsets of S) and develop a theory to extract information about them. At the level of binary f, then, MIP is *closed-set morphology.*

The transformations that can probe binary images for shape information have been thoroughly studied by Matheron (1975) and Serra (1982). The elementary forms of these transformations are known as *hit-miss transformations* and *erosions*; in fact, erosions are a species of hit-miss transformation.

A hit-miss transformation is a set-to-set mapping (set transformation) characterized by two fixed subsets, A and B, of the image field, where one of them (say B) includes the other; that is, $A \subset B$. It acts as follows. If the binary image that is operated on contains a set F that can be translated in the image field so as to include A and be included in B (i.e., if there is a translation vector x such that $A \subset F + x \subset B$), then the transformation will output a point at $-x$. If there is no such F, then the transformation will output nothing (i.e., the output set will be empty). Evidently, the more nearly identical A and B are in size and shape to the size and shape of a given set F, the more refined an F-recognizer the corresponding hit-miss transformation will be. Moreover, every *translationally invariant* (TI) set transformation is a union (i.e., a logical ORing) of hit-miss transformations (Banon and Barrera, 1991); hence every TI set transformation interacts in the way indicated with the shape/size content of a binary image to produce its output. When the including set B is the entire image field, the corresponding hit-miss transformation degenerates into one characterized by the single set A. In this case the transformation is called *erosion by A*. Erosions eliminate all sets from a binary image that are either too small or too misshaped relative to A to include A; the remaining sets are reduced in size (eroded) by an amount that depends roughly on the size of A and in more detail on the shape and scale of A relative to the shape and scale of the set eroded. Erosions are the elementary building blocks of the TI set transformations that preserve set inclusion relations. Such transformations are called *increasing* or *order preserving* and include the so-called *morphological filters*; they can always be expressed as a union of erosions (Matheron, 1975).

The TI set transformations of the image field accordingly form a class of (plainly nonlinear) transformations that act on the shape/size content of binary images. Closed-set morphology results when appropriate topological requirements are imposed on this class.

Closed-set morphology is both an *algebraic* and a *topological* theory. Its algebraic operations are of three kinds: (1) the set-theoretical operations of *union* ($\cup$) and *intersection* ($\cap$), (2) the usual *closure* and *boundary-extraction* operations that arise from the intrinsic metrical character of the image field, and (3) what I call the *Minkowski operations*. The latter, which depend on the vector structure of the image field, are *Minkowski addition* ($\oplus$), *Minkowski subtraction* ($\ominus$), and *multiplication by a real scalar* (*homothesis*). Minkowski addition, which is simply pointwise set addition, is essentially the same as *dilation* (the operation *dual* to erosion); likewise, erosion and Minkowski subtraction are essentially the same. The topological aspect of closed-set morphology operates at two distinct levels. The lower level is due to the obvious fact that the *closed* subsets of the image field, which are the elementary objects of interest at the higher level, have their own intrinsic metric-topological character. Besides this, however, it is necessary to define a notion of *convergence* of a sequence of closed sets. This notion gives *continuity* properties to the algebraic operations and is the source of the higher level topology known as *Matheron's hit-miss topology.* It is much less well-known than the algebraic aspect of the theory, because the *general topology* (Kelley, 1955) needed for its appreciation is unfamiliar to many investigators. As an aid for the reader, I have included an appendix that summarizes the required background in general topology. Further details can be found in Kelley (1955).

The need for a high-level topology arises from the way morphological transformations must behave to be of practical utility. Two images of the same scene formed with the same imaging apparatus will generally differ because of differing stochastic components. In applying an arbitrary transformation to such images, it may happen that a large difference in the output results when there is not a large stochastic difference in the inputs. Such a transformation should clearly not be admitted to the class of "morphological transformations." It is indeed often the case in applied mathematics that one has a transformation $\Lambda : u \longmapsto v$ and wants to ensure that a "small" perturbation in u will not produce a "large" or uncontrolled perturbation in the result $v = \Lambda(u)$. Such assurances are typically provided by imposing some degree of *continuity* on Λ, and this requires a transformation topology.

In classical applications of mathematics, continuity is usually treated as a *metric space* concept: that is, in terms of a *metric* such as the familiar *euclidean metric* or *Hilbert-space norm.* Topology is the most general theory of mathematical continuity, however; it is a generalization of metric-space theory. If the topology appropriate to a given application is *metrizable* and if the associated metric function is known explicitly, a direct confrontation with topology can be avoided. This is usually the case. In mathematical morphology, however, despite the metrizability of the relevant topologies, the associated metrics are in a practical sense unknown; here the confrontation cannot be avoided.

The most important use of Matheron's hit-miss topology in morphology is to make the solution of stochastic optimization problems possible (at least in principle). The relevance of this capability to ATR can be seen as follows. The goal of ATR is to optimally identify the presence or absence of certain important objects in imagery that includes all sorts of other things (background clutter and objects of no interest) in an essentially stochastic way. For probability theory to be effectively applied to the optimization problem thus posed, it would be helpful if morphology theory had a concept of a *random closed set* or *binary image*: that is, if we had a *theory of random binary images and noise* compatible with the morphological transformation theory. Matheron (1975) has in fact given us this; he introduced his topology into morphology mainly to arrive at a useful concept of a random closed set. (Despite its importance, however, the stochastic side of morphology is not further discussed herein. I mention it only for completeness.)

This report includes both original material and a review of the work of others. Section 2 outlines several mathematical concepts (*topological* and *compact ordered spaces*, *upper* and *lower topologies*, etc) that permit a unified treatment of the various semicontinuity concepts needed in mathematical morphology. This approach leads to valuable perspectives and economy of thought and to my knowledge has not been previously employed by morphology theorists. I use the basic ideas of Nachbin (1965) and Birkhoff (1948) concerning when a *partial ordering* of a set that has a topology is *compatibile* with that topology. By pursuing Nachbin's concept of a topological ordered space (TO-space), I arrive at a definition of what I call an *order resolvable* TO-space. This abstraction makes it possible to unify the treatment of semicontinuity.

Section 3 gives an in-depth review of the basic concepts of closed-set morphology. Here I give a more thorough treatment than is available elsewhere of the several limit concepts used in morphology. I also point out some interesting links with the basic work of Hausdorff (1927) and Frink (1942) on the limits of nets and sequences of sets. In discussing the continuity properties of the algebraic operations $\cup$, $\cap$, $\oplus$, $\ominus$, etc, I again provide a more thorough treatment than is available elsewhere, especially for the Minkowski operations, which are taken up in section 4. Section 5 discusses the morphological transformation and transformation space theory of closed-set morphology. Here I incorporate the relevant contributions of Matheron (1975) and Maragos (1989, 1985), as well as the recent important work of Banon and Barrera (1991); I also indicate the *poset/lattice* algebraic aspects of the morphological transformation space and somewhat generalize Matheron's kernel theory to apply to general rather than only *increasing* transformations. I conclude section 5 with a description of a novel representation of the morphological transformations that uses countable closed-set bases for Matheron's topology. Section 6 concludes the report with the general outlines of the next stage of morphology's image processing theory.

2 Ordered Topology

The spaces of interest in mathematical morphology are generally closed-set classes having the hit-miss topology introduced by Matheron (1975). Such spaces are both *partially ordered sets* (Def. 2.1)—*posets* (ordered by the subset relation)—and topological spaces (Def. A.1).

Nachbin (1965) has framed a concept of the compatibility of the topology of a poset and its ordering relation (Def. 2.2). (There are several different definitions of order-topological compatibility; Nachbin's is only one of them.) When this compatibility occurs, he calls the overall structure a *topological ordered space* (TO-space); when the topology in question is compact, he calls a TO-space a *compact ordered space.*

For a compact ordered space, one can give a general definition of the *upper and lower topologies* into which the space's topology resolves. Based on these notions one can then formulate a very general definition of the *upper* and *lower semicontinuity* of mappings into the space. This abstract framework is quite convenient for mathematical morphology because Matheron's hit-miss topology is both compact and compatible in Nachbin's sense with the ordering relation $\subset$, and furthermore because the abstract definition embraces the wide variety of semicontinuity types encountered in mathematical morphology. Hence, a more systematic account of the subject is made possible. I begin by outlining Nachbin's theory and develop it further where needed.

2.1 Topological and Compact Ordered Spaces

A *binary relation* $\mathbf{r}$ in a set X is a set of ordered pairs of elements from X. If $x, y \in X$ and $(x, y) \in \mathbf{r}$, we write $x\mathbf{r}y$. If $x\mathbf{r}x$ for all $x \in X$, then $\mathbf{r}$ is called *reflexive*. If $\forall\ x, y, z \in X$ we have $x\mathbf{r}y$ and $y\mathbf{r}z \implies x\mathbf{r}z$, then $\mathbf{r}$ is called *transitive*. Note that $\leq$ and $\subset$ are, respectively, binary relations in the set of real numbers ($\Re$) and in any class of subsets of a given set. These relations share other properties that may be abstracted to give the concept of a poset.

Definition 2.1 *A poset $(X, \preceq)$ is a set X of elements $x, y, z, \ldots$ in which a reflexive and transitive binary relation $\preceq$ is defined such that whenever $x \preceq y$ and $y \preceq x$, it follows that $x = y$. If $(X, \preceq)$ is a poset, then $\preceq$ is called a partial ordering of X (more simply an ordering) and the set $\{(x, y) \in X \times X : x \preceq y\}$ is called the graph of $\preceq$ on X.*

Thus $(\Re, \leq)$ and $(\mathcal{A}, \subset)$ (where $\mathcal{A}$ is a class of subsets of a given set) are posets. The poset $(\Re, \leq)$ is *totally ordered* because $\forall\ x, y \in \Re$, either

$x \leq y$ or $y \leq x$. For $(\mathcal{A}, \subset)$, this need not be so; hence the general term *partially ordered set.*

Definition 2.2 *Let $(X, \preceq)$ be a poset and let τ be a topology on X. If the graph of $\preceq$ on X is a closed subset of the product space $X \times X$, we say that $\preceq$ is a closed order in X and call $(X, \tau, \preceq)$ a topological ordered space or a TO-space. A TO-space $(X, \tau, \preceq)$ is called a (locally) compact ordered space if (X, τ) is (locally) compact.*

A topology is a collection of subsets designated as *open* (Def. A.1); the *closed* sets are the complements of the open sets. For the definitions of *product spaces* and *compactness*, see the appendix (A.12–A.13). *(Locally) compact ordered space* is abbreviated by *(L)CO-space.*

Definition 2.3 *Let $(X, \preceq)$ be a poset and let A be a subset of X. If $x \in A$ and $y \preceq x$ $(x \preceq y) \Longrightarrow y \in A$, then A is called a decreasing (increasing) set. If $y \in A$ whenever $x \preceq y \preceq z$, $x \in A$, and $z \in A$, then A is called a convex set.*

Remark 2.1 *X and $\emptyset$* (the empty set) *are increasing, decreasing, and convex. Unions and intersections of increasing (decreasing) sets are increasing (decreasing). Intersections of convex sets are convex.*

A *base* for a topology is a collection of open sets such that every open set is a union of sets from the collection. A *subbase* is a collection of open sets such that every open set is a union of finite intersections of sets from the collection. A *local base* at $x \in X$ is a collection of open sets containing x such that every open set containing x contains a member of the collection. A topological space (X, τ) is called *first countable* if it has a countable local base at every $x \in X$; it is called *second countable* if it has a countable base. A topological space is called *Hausdorff* if every pair of distinct points in it have disjoint open *neighborhoods* (Def. A.3).

Definition 2.4 *A TO-space $(X, \tau, \preceq)$ is called locally convex if it has a convex local base at each $x \in X$.*

Theorem 2.1 (Nachbin) *Every TO-space is Hausdorff and every CO-space is locally convex.*

Theorem 2.2 *Let $(X, \preceq)$ be a poset and let τ be a first countable Hausdorff topology on X. Then $(X, \tau, \preceq)$ is a TO-space if and only if $x_i \to x$ in X, $y_i \to y$ in X, and $x_i \preceq y_i$ for all i together imply that $x \preceq y$.*

Proof. It must be shown that the proposition $x_i \to x$ in X, $y_i \to y$ in X, and $x_i \preceq y_i \; \forall \; i \Longrightarrow x \preceq y$ holds if and only if the graph of $\preceq$ is closed in the product space $X \times X$. From general topology we know that the graph in question, say G, is closed if and only if every convergent sequence in G has its limit in G. By definition, a sequence $\{(x_i, y_i)\}$ in $X \times X$ is in the graph of $\preceq$ if and only if $x_i \preceq y_i$ for all i. Moreover $\{(x_i, y_i)\}$ converges to (x, y) in the product space $X \times X$ if and only if $x_i \to x$ in X and $y_i \to y$ in X. Since $(x, y) \in G$ if and only if $x \preceq y$, the proof is complete.

Remark 2.2 *Let $(X, \tau, \preceq)$ be a TO-space, let A be a subset of X, and let τ_A denote the relative topology of A in (X, τ). Then $(A, \preceq)$ is a poset and $(A, \tau_A, \preceq)$ is a TO-space; moreover, if A is a compact subset of X, then $(A, \tau_A, \preceq)$ is a CO-space.*

See section A.11 for *relative topologies.* If $(X, \tau, \preceq)$ is a TO-space, we indicate that a subset A of X is being considered as a TO-space relative to τ and $\preceq$ by calling A a *TO-subspace of* $(X, \tau, \preceq)$.

Remark 2.3 *If $(X, \tau, \preceq)$ is a CO-space and A is a subset of X, then the TO-subspace A is a CO-space if and only if A is closed in (X, τ).*

When $(X, \preceq)$ is also a *lattice* (Birkhoff, 1948) (i.e., when $x \wedge y \equiv \inf\{x, y\}$ and $x \vee y \equiv \sup\{x, y\}$ exist in X for all x and y in X), it is convenient to have some special terminology.

Definition 2.5 *Let $(X, \wedge, \vee)$ be a lattice and let $\preceq$ denote its induced ordering (i.e., the ordering defined by $x \preceq y \Longleftrightarrow x \wedge y = x$). If $(X, \tau, \preceq)$ is a TO-space, we call $(X, \tau, \wedge, \vee)$ a closed-order lattice (CO-lattice). A CO-lattice whose topology is (locally) compact will be called a (locally) compact closed-order lattice.*

A one-to-one mapping of a TO-space (CO-lattice) onto another is called a *TO-space* (*CO-lattice*) *isomorphism* if the mapping is both a poset (lattice) isomorphism (Birkhoff, 1948) and a *homeomorphism* (see Def. A.24). Similar definitions apply to (locally) compact ordered spaces and (locally) compact CO-lattices.

2.2 Upper and Lower Topologies and Semicontinuity

Remark 2.4 *If $(X, \tau, \preceq)$ is a TO-space, then the class τ_u of open decreasing subsets of X and the class τ_ℓ of open increasing subsets of X are topologies on X. We call them the decreasing and increasing topologies, respectively, of $(X, \tau, \preceq)$.*

If $(X, \tau, \preceq)$ is a TO-space, it is generally not true that τ is generated by $\tau_u \cup \tau_\ell$; i.e., τ is not the smallest topology on X that contains $\tau_u \cup \tau_\ell$. This deficiency is remedied when $(X, \tau, \preceq)$ is a CO-space.

Theorem 2.3 (Nachbin) *If $(X, \tau, \preceq)$ is a compact ordered space, then $\tau_u \cup \tau_\ell$ is a subbase for τ.*

For *generated topologies* and *subbases*, see sections A.1 and A.8.

Definition 2.6 *Let $(X, \tau, \preceq)$ be a TO-space and let A be a subset of X. Regarding A as a TO-subspace of $(X, \tau, \preceq)$, we can define the relative decreasing and increasing topologies $\tau_u(A)$ and $\tau_\ell(A)$ of A in $(X, \tau, \preceq)$ in the usual way by*

$$\tau_u(A) \equiv \{A \cap G : G \in \tau_u\} \quad \text{and} \quad \tau_\ell(A) \equiv \{A \cap G : G \in \tau_\ell\}.$$

Note that $\tau_u(A)$ and $\tau_\ell(A)$ are, respectively, the classes of open decreasing and open increasing subsets of the TO-subspace A; that is, they are the decreasing and increasing topologies of the TO-space $(A, \tau_A, \preceq)$ in accordance with Remark 2.4. We therefore have the following corollary to Theorem 2.3.

Corollary 2.1 *If $(X, \tau, \preceq)$ is a CO-space and A is a subset of X, then $\tau_u(A) \cup \tau_\ell(A)$ is a subbase for τ_A.*

Definition 2.7 *A TO-space $(X, \tau, \preceq)$ whose increasing and decreasing topologies together form a subbase for τ will be called order resolvable.*

Thus a CO-space and a TO-subspace of CO-space are order resolvable.

Definition 2.8 *If $(X, \tau, \preceq)$ is an order-resolvable TO-space, then two topologies μ and λ on X, such that $\mu \cup \lambda$ is a subbase for τ, $\mu \subset \tau_u$, and $\lambda \subset \tau_\ell$, will be called upper and lower topologies for $(X, \tau, \preceq)$. Thus τ_u and τ_ℓ are the maximal upper and lower topologies for $(X, \tau, \preceq)$.*

Remark 2.5 *If X and Y are TO-spaces, then a TO-space isomorphism of X onto Y maps the open increasing (decreasing) subsets of X onto the open increasing (decreasing) subsets of Y; this isomorphism is hence a homeomorphism relative to the maximal upper (lower) topologies of X and Y, and if X is order resolvable, then so is Y.*

We can now define what is generally meant in this setting by a USC (LSC) function or mapping; see Def. A.24.

Definition 2.9 *Let Ω be a topological space, let X be an order resolvable TO-space, let μ and λ be upper and lower topologies for X, let ω be a point in Ω, and let Λ map Ω to X. Then Λ is μ-USC (λ-LSC) [at ω] if Λ is continuous [at ω] with respect to μ (λ).*

Theorem 2.4 *Let Ω be a topological space, let X be an order resolvable TO-space, let μ and λ be upper and lower topologies for X, let ω be a point in Ω, and let Λ map Ω to X. Then*

1. *Λ is μ-USC (λ-LSC) $\Longleftrightarrow$ Λ is μ-USC (λ-LSC) at every ω.*
2. *Λ is continuous [at ω] $\Longleftrightarrow$ Λ is both μ-USC and λ-LSC [at ω].*

2.3 Familiar Examples

The following are commonplace but useful examples of the foregoing abstractions. Section 3 addresses a centrally important example in mathematical morphology proper.

First consider the posets formed by the real and extended real numbers, each with the ordering relation $\leq$. Let $\Re$ denote the set of real numbers, let $\emptyset$ denote the empty subset of $\Re$, and consider the collections $\mu \equiv \{\Re, \emptyset, (-\infty, t) : t \in \Re\}$ and $\lambda \equiv \{\Re, \emptyset, (t, \infty) : t \in \Re\}$.

Remark 2.6 *The collections μ and λ are topologies on $\Re$, $\mu \cup \lambda$ is a subbase for the usual topology τ of $\Re$, $(\Re, \tau, \leq)$ is an LCO-space, and μ and λ are its maximal upper and lower topologies.*

Let $\Re^{(e)}$ denote the extended real numbers, let $\emptyset$ denote the empty subset of $\Re^{(e)}$, and consider the collections

$$\mu_e \equiv \{\Re^{(e)}, \emptyset, [-\infty, t) : t \in \Re^{(e)}\} \quad \text{and} \quad \lambda_e \equiv \{\Re^{(e)}, \emptyset, (t, \infty] : t \in \Re^{(e)}\}.$$

Remark 2.7 *The collections μ_e and λ_e are topologies on $\Re^{(e)}$, $\mu_e \cup \lambda_e$ is a subbase for the usual topology τ_e of $\Re^{(e)}$, $(\Re^{(e)}, \tau_e, \leq)$ is a CO-space, and μ_e and λ_e are its maximal upper and lower topologies.*

Remark 2.8 $(\Re^{(e)}, \tau_e, \inf, \sup)$ *is a complete compact CO-lattice and* $(\Re, \tau, \inf, \sup)$ *is a conditionally complete locally compact CO-lattice.*

For poset/lattice properties such as *completeness* and *conditional completeness*, see Birkhoff (1948). Next, consider the semicontinuity of mappings from a topological space X to the CO-space $(\Re^{(e)}, \tau_e, \leq)$. By direct application of the general definition, we obtain the following:

Definition 2.10 *If f is ERV on a topological space X, then*

1. *f is USC $\Longleftrightarrow \{x \in X : f(x) < t\}$ is open in X for all $t \in \Re^{(e)}$.*
2. *f is LSC $\Longleftrightarrow \{x \in X : f(x) > t\}$ is open in X for all $t \in \Re^{(e)}$.*

This and the next definition yield (trivially) an alternative characterization in terms of f's *cross sections.*

Definition 2.11 *If $t \in \Re^{(e)}$ and f is ERV on X, then we call the set $X_t(f) \equiv \{x \in X : f(x) \geq t\}$ the horizontal cross section of f at t and also use the notation $X_t^-(f) \equiv \{x \in X : f(x) > t\}$.*

Theorem 2.5 *If f is ERV on a topological space X, then f is USC if and only if $X_t(f)$ is closed in X for all $t \in \Re^{(e)}$, and f is LSC if and only if $X_t^-(f)$ is open in X for all $t \in \Re^{(e)}$.*

Theorem 2.5 turns out to be quite useful for the more general morphology of ERV USC functions (greyscale morphology) and is particularly useful in treating the well-known *threshold decomposition method* of Serra (1982). When X is a first countable Hausdorff space, the theorem becomes the following:

Theorem 2.6 *If f is an ERV function on a first countable Hausdorff space X, then f is USC if and only if $f(x) \geq \limsup f(x_i)$ $\forall\, x \in X$ and $\forall\, \{x_i\}$ in X with limit x, and f is LSC if and only if $f(x) \leq \liminf f(x_i)$ $\forall\, x \in X$ and $\forall\, \{x_i\}$ in X with limit x.*

This result comes up again and again in different guises and settings as a criterion for semicontinuity. I refer to it and its relatives as the *usual semicontinuity criterion.* Finally, many authors limit the ERV USC (LSC) functions to those that do not take on the value ∞ $(-\infty)$. These may be considered special cases of the definition I am using.

3 Morphospace

For a less familiar example of the foregoing abstractions, I turn to mathematical morphology proper. The basic ingredient of Matheron's morphology theory is a structure that I call a *morphospace*; it is defined as follows. Let S be an LCS space (i.e., a locally compact, second countable Hausdorff space) and let $\mathbf{F}(S)$, $\mathbf{G}(S)$, and $\mathbf{K}(S)$ respectively denote the classes of closed, open, and compact subsets of S.

Remark 3.1 $(\mathbf{F}(S), \cap, \cup)$ *is a complete distributive lattice, and its induced ordering is* $\subset$.

The lattice $(\mathbf{F}(S), \cap, \cup)$ becomes a morphospace when it is given Matheron's hit-miss topology.

3.1 Hit-Miss Topology

If $K \in \mathbf{K}(S)$ and $G \in \mathbf{G}(S)$, we can define the collections $\mathbf{F}^K$ and $\mathbf{F}_G$ by $\mathbf{F}^K \equiv \{F \in \mathbf{F}(S) : F \cap K = \emptyset\}$ and $\mathbf{F}_G \equiv \{F \in \mathbf{F}(S) : F \cap G \neq \emptyset\}$. Indeed these notational conventions are used for arbitrary subsets K and G of S. Let $\boldsymbol{\kappa} \equiv \{\mathbf{F}^K : K \in \mathbf{K}(S)\}$ and $\boldsymbol{\eta} \equiv \{\mathbf{F}_G : G \in \mathbf{G}(S)\}$.

Definition 3.1 *The hit-miss topology* $\boldsymbol{\tau}$ *of* $\mathbf{F}(S)$ *is the topology generated by* $\boldsymbol{\kappa} \cup \boldsymbol{\eta}$*; hence,* $\boldsymbol{\kappa} \cup \boldsymbol{\eta}$ *is a subbase for* $\boldsymbol{\tau}$.

Note that the identities $\cup \mathbf{F}_{B_\alpha} = \mathbf{F}_{\cup B_\alpha}$ and $\cap \mathbf{F}^{B_\alpha} = \mathbf{F}^{\cup B_\alpha}$ hold for any collection $\{B_\alpha\}$ of subsets of S. On the other hand, $\mathbf{F}_{\{B_\alpha\}} \equiv \cap \mathbf{F}_{B_\alpha} \supset \mathbf{F}_{\cap B_\alpha}$ and $\cup \mathbf{F}^{B_\alpha} \subset \mathbf{F}^{\cap B_\alpha}$ are all that hold generally. With these notations and facts, the typical finite intersection of sets from the generating class $\boldsymbol{\kappa} \cup \boldsymbol{\eta}$ has the form

$$\mathbf{F}^{K_1} \cap \ldots \cap \mathbf{F}^{K_m} \cap \mathbf{F}_{G_1} \cap \ldots \cap \mathbf{F}_{G_k} = \mathbf{F}^{K_1 \cup \ldots \cup K_m} \cap \mathbf{F}_{G_1, \ldots, G_k}$$

where the nonnegative integers m and k may be zero but not simultaneously. Letting $K_1 \cup \ldots \cup K_m = K$, Matheron uses the notation $\mathbf{F}^K_{G_1, \ldots, G_k} \equiv \mathbf{F}^K \cap \mathbf{F}_{G_1, \ldots, G_k}$. Thus if K is an arbitrary compact subset of S and $\{G_1, \ldots, G_k\}$ is an arbitrary (possibly empty) finite set of open subsets of S, then the collection of sets of the form $\mathbf{F}^K_{G_1, \ldots, G_k}$ is a base for the hit-miss topology of $\mathbf{F}(S)$. Henceforth assume that $\mathbf{F}(S)$ is carrying its hit-miss topology.

Theorem 3.1 (Matheron) *$\boldsymbol{\tau}$ is compact, second countable, and Hausdorff; hence, $\boldsymbol{\tau}$ is normal, regular, and metrizable and* $\mathbf{F}(S)$ *has a countable dense subset.*

The second part of this theorem is a consequence of several results from general topology (Kelley, 1955): (1) compact Hausdorff spaces are normal, (2) normal Hausdorff spaces are regular, and (3) regular second countable spaces are equivalently metrizable and have a countable dense subset. See section A.9.

Theorem 3.2 (Matheron) *Let $\{E_i\}$ and $\{F_i\}$ be convergent sequences in $\mathbf{F}(S)$ with limits E and F, respectively, and suppose that $E_i \subset F_i$ for all i. Then $E \subset F$.*

Comparing this with Theorem 2.2 shows that $\subset$ is a closed order in $\mathbf{F}(S)$. Thus $(\mathbf{F}(S), \boldsymbol{\tau}, \subset)$ is a CO-space and $(\mathbf{F}(S), \boldsymbol{\tau}, \cap, \cup)$ is a compact CO-lattice. Note that $\boldsymbol{\kappa}$ $(\boldsymbol{\eta})$ consists entirely of decreasing (increasing) sets, contains the universal collection $\mathbf{F}(S)$ (the empty collection $\boldsymbol{\emptyset}$), and is closed under finite intersections (arbitrary unions). Even if we append $\boldsymbol{\emptyset}$ to $\boldsymbol{\kappa}$ and $\mathbf{F}(S)$ to $\boldsymbol{\eta}$, however, we will fail to convert either into a topology on $\mathbf{F}(S)$ because $\boldsymbol{\kappa} \cup \boldsymbol{\emptyset}$ is not closed under arbitrary unions and $\boldsymbol{\eta} \cup \mathbf{F}(S)$ is not closed under finite intersections. However, we do have the following.

Proposition 3.1 *Let $\boldsymbol{\mu}' = \boldsymbol{\kappa} \cup \boldsymbol{\emptyset}$ and $\boldsymbol{\lambda}' = \boldsymbol{\eta} \cup \mathbf{F}(S)$. Then $\boldsymbol{\mu}'$ is closed under finite intersections and is a base for the topology $\boldsymbol{\mu}$ that it generates on $\mathbf{F}(S)$, $\boldsymbol{\lambda}'$ is closed under arbitrary unions and is a subbase for the topology $\boldsymbol{\lambda}$ that it generates on $\mathbf{F}(S)$, and $\boldsymbol{\mu}$ $(\boldsymbol{\lambda})$ consists entirely of decreasing (increasing) $\boldsymbol{\tau}$-open sets.*

Proof. Note that $\boldsymbol{\mu}'$ is closed under finite intersections, so that for $\boldsymbol{\mu}'$ to be a subbase is the same as for it to be a base. For a collection $\boldsymbol{\mathcal{B}}$ of subsets of a set X to be a base for the topology that it generates on X, the following is sufficient: (a) for each $x \in X$ there is a member of $\boldsymbol{\mathcal{B}}$ that contains x and (b) $\boldsymbol{\mathcal{B}}$ is closed under finite intersections. Since $\boldsymbol{\mu}'$ is closed under finite intersections, the proposition *$\boldsymbol{\mu}'$ is a base for the topology that it generates* follows from $S = \mathbf{F}^{\boldsymbol{\emptyset}}$ and the empty subset of S is compact. Likewise, the proposition *$\boldsymbol{\lambda}'$ is a subbase for the topology that it generates* is equivalent to *the set of finite intersections of sets from $\boldsymbol{\lambda}'$ is a base for the topology that it generates*, and the truth of the latter is a direct consequence of $\mathbf{F}(S) \in \boldsymbol{\lambda}'$. The rest is trivial.

Clearly $\boldsymbol{\mu} \cup \boldsymbol{\lambda}$ is a subbase for the hit-miss topology of $\mathbf{F}(S)$, and it follows that $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are upper and lower topologies for $(\mathbf{F}(S), \boldsymbol{\tau}, \subset)$. The question raised by Proposition 3.1, however, is whether $\boldsymbol{\mu}$ ($\boldsymbol{\lambda}$) is *strictly weaker* than the decreasing (increasing) topology of $(\mathbf{F}(S), \boldsymbol{\tau}, \subset)$; i.e., whether $\boldsymbol{\mu}$ ($\boldsymbol{\lambda}$) contains all the decreasing (increasing) $\boldsymbol{\tau}$-open sets. I leave this question open.

I earlier referred to the morphospace $(\mathbf{F}(S), \boldsymbol{\tau}, \cap, \cup)$ as the basic ingredient of Matheron's morphology. Associated with $\mathbf{F}(S)$, however, are the two other classes $\mathbf{G}(S)$ and $\mathbf{K}(S)$ of topologically important subsets of S, and it would be more correct to say that this trio of spaces is the basic structure underlying Matheron's morphology. The topological and lattice-algebraic aspects of $\mathbf{G}(S)$ and $\mathbf{K}(S)$ therefore merit consideration (see next section).

3.2 The Dual and Myopic Topologies

Since the complementation operation in S (denoted $\cdot^c$) maps $\mathbf{F}(S)$ one-to-one onto $\mathbf{G}(S)$, we can obtain a natural topology $\boldsymbol{\tau}^*$ for $\mathbf{G}(S)$ (called the *dual topology*) by requiring this mapping to be a homeomorphism. If we let $\mathbf{G}_A \equiv \{G \in \mathbf{G}(S) : G \supset A\}$ and $\mathbf{G}^A \equiv \{G \in \mathbf{G}(S) : G \not\supset A\}$, then the dual topology is generated by the collection

$$\{\mathbf{G}_K : K \in \mathbf{K}(S)\} \cup \{\mathbf{G}^G : G \in \mathbf{G}(S)\}.$$

Proposition 3.2 $(\mathbf{G}(S), \boldsymbol{\tau}^*, \subset)$ *is a CO-space, and* $(\mathbf{G}(S), \boldsymbol{\tau}^*, \cap, \cup)$ *is a compact CO-lattice.*

Proof. By the definition of the dual topology, $(\mathbf{G}(S), \boldsymbol{\tau}^*)$ is second countable and Hausdorff. Theorem 2.2 therefore applies. Let $\{U_i\}$ and $\{V_i\}$ be convergent sequences in $\mathbf{G}(S)$ with the dual-topology limits U and V, respectively, and suppose that $U_i \subset V_i$ for all i. Then $U_i^c \to U^c$ and $V_i^c \to V^c$ in the hit-miss topology of $\mathbf{F}(S)$ and $V_i^c \subset U_i^c$ for all i. By Theorem 3.2 we see that $V^c \subset U^c$. Thus $U \subset V$ and the desired conclusion follows from Theorem 2.2 and the fact that $(\mathbf{G}(S), \boldsymbol{\tau}^*)$ is compact.

Remark 3.2 $(\mathbf{G}(S), \cap, \cup)$ *is a complete distributive lattice, and the complementation mapping of* $\mathbf{F}(S)$ *onto* $\mathbf{G}(S)$ *is a dual-lattice isomorphism between* $(\mathbf{F}(S), \boldsymbol{\tau}, \cap, \cup)$ *and* $(\mathbf{G}(S), \boldsymbol{\tau}^*, \cap, \cup)$.

$(\mathbf{G}(S), \boldsymbol{\tau}^*, \cap, \cup)$ is called the morphospace dual of $(\mathbf{F}(S), \boldsymbol{\tau}, \cap, \cup)$.

Remark 3.3 *If we let μ^* and λ^* denote the topologies generated subbasically on* $\mathbf{G}(S)$ *by the collections* $\{\mathbf{G}^G : G \in \mathbf{G}(S)\} \cup \mathbf{G}(S)$ *and* $\{\mathbf{G}_K : K \in \mathbf{K}(S)\} \cup \emptyset$, *respectively, it follows that μ^* and λ^* are upper and lower topologies for* $(\mathbf{G}(S), \tau^*, \subset)$.

Note that $\mu^* = \lambda^c$ and $\lambda^* = \mu^c$, where the complementation operation acts on the individual members of the collections respectively comprising λ and μ. For instance, if we let $\lambda = \{\mathcal{G} : \mathcal{G} \in \lambda\}$, then $\lambda^c = \{\{F^c : F \in \mathcal{G}\} : \mathcal{G} \in \lambda\}$. The lattice-algebraic and topological duality evident in the pair $(\mathbf{F}(S), \mathbf{G}(S))$ can be elaborated in the ***dual mapping*** concept, defined as follows.

Definition 3.2 *Let $\mathcal{P} = \mathcal{P}(S)$ denote the class of all subsets of S. If Ψ is a mapping of* $\mathbf{F}(S)$ $(\mathbf{G}(S))$ *into $\mathcal{P}$, then we define the corresponding dual mapping Ψ^* of* $\mathbf{G}(S)$ $(\mathbf{F}(S))$ *into $\mathcal{P}$ by $\Psi^*(A) \equiv [\Psi(A^c)]^c$; if Ψ is a mapping of* $\mathbf{F}(S) \times \mathbf{F}(S)$ $(\mathbf{G}(S) \times \mathbf{G}(S))$ *into $\mathcal{P}$, then we define the dual mapping Ψ^* of* $\mathbf{G}(S) \times \mathbf{G}(S)$ $(\mathbf{F}(S) \times \mathbf{F}(S))$ *into $\mathcal{P}$ by $\Psi^*(A, B) \equiv [\Psi(A^c, B^c)]^c$; and so on and likewise for the other products, mixed or otherwise, of* $\mathbf{F}(S)$ *and* $\mathbf{G}(S)$.

Since $[A^c \cup B^c]^c = A \cap B$ and $[A^c \cap B^c]^c = A \cup B$, we see, for instance, that $\cap$ $(\cup)$ on $\mathbf{X}^c \times \mathbf{Y}^c$ is the mapping dual to $\cup$ $(\cap)$ on $\mathbf{X} \times \mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ each stand for either $\mathbf{F}(S)$ or $\mathbf{G}(S)$. Finally, note that the dual mapping concept actually applies to any pair (L, L') of dual-lattice isomorphic lattices. For instance, if Ψ is a mapping of L to itself and $\mathcal{C}$ denotes a dual-lattice isomorphism of L onto L', then $\Psi'(a') \equiv \mathcal{C}[\Psi(\mathcal{C}^{-1}(a'))]$ $(a' \in L')$ defines the mapping of L' to itself, which is dual to Ψ relative to $\mathcal{C}$. This observation is used in section 4 in connection with the fact that the lattice $(\mathcal{P}(S), \cap, \cup)$ is self-dual under the complementation mapping.

Since $\mathbf{K}(S) \subset \mathbf{F}(S)$, the relative hit-miss topology of $\mathbf{K}(S)$ in $\mathbf{F}(S)$ seems a natural topology for $\mathbf{K}(S)$. When S is a compact space, there is no difference between $\mathbf{F}(S)$ and $\mathbf{K}(S)$, and using the relative hit-miss topology of $\mathbf{K}(S)$ would also result in the coincidence of their topologies. The hit-miss topology of $\mathbf{K}(S)$ is generally flawed, however, by the fact that $\mathbf{K}(S)$ is neither a closed nor an open subset of $\mathbf{F}(S)$ when the two are not identical. Moreover, a stronger topology on $\mathbf{K}(S)$ also coincides with the hit-miss topology when S is compact; this is the so-called *myopic topology*, whose origin may be seen as follows.

Suppose that S is not compact but merely locally compact (as when $S = \Re^n$, for example). Let us "compactify" S by means of the classical

Alexandroff one-point compactification procedure (Royden, 1968). In this procedure, we append a formal point ω (called the ***point at infinity***) to S and define the topology of the new space $S^* = S \cup \{\omega\}$ by taking a set in S^* to be open when it is either an open subset of S or a ***neighborhood of infinity***: i.e., the complement of a compact subset of S. With this definition, S^* becomes a compact second-countable Hausdorff space (and hence an LCS space), so that $\mathbf{F}(S^*)$ has a hit-miss topology. With this topology, $\mathbf{F}(S^*)$ can be expressed as a disjoint union of the two subspaces $\mathbf{F}^* \equiv \{F \cup \{\omega\} : F \in \mathbf{F}(S)\}$ and $\mathbf{K}(S)$, which are closed and open subspaces of $\mathbf{F}(S^*)$, respectively. Moreover, the relative topology of $\mathbf{F}^*$ in $\mathbf{F}(S^*)$ is topologically equivalent to the hit-miss topology of $\mathbf{F}(S)$ because the one-to-one mapping $F \cup \{\omega\} \longmapsto F$ of $\mathbf{F}^*$ onto $\mathbf{F}(S)$ is a homeomorphism. The relative topology $\boldsymbol{\upsilon}$ of $\mathbf{K}(S)$ in $\mathbf{F}(S^*)$ is called the *myopic topology*.

The myopic topology may be defined alternatively as the topology generated on $\mathbf{K}(S)$ by $\{\mathbf{K}^F : F \in \mathbf{F}(S)\} \cup \{\mathbf{K}_G : G \in \mathbf{G}(S)\}$, where

$$\mathbf{K}^F \equiv \{K \in \mathbf{K}(S) : K \cap F = \emptyset\}$$

and

$$\mathbf{K}_G \equiv \{K \in \mathbf{K}(S) : K \cap G \neq \emptyset\}.$$

The following results summarize basic information about $\boldsymbol{\upsilon}$ and its relation to the hit-miss topology of $\mathbf{F}(S)$; apart from Corollary 3.1, they are due to Matheron (1975).

Theorem 3.3 *$(\mathbf{K}(S),\boldsymbol{\upsilon})$ is an LCS space. If S is not a compact space, then neither is $(\mathbf{K}(S),\boldsymbol{\upsilon})$, and $\boldsymbol{\upsilon}$ is strictly stronger than the relative hit-miss topology of $\mathbf{K}(S)$. If $\mathcal{K}$ is a $\boldsymbol{\upsilon}$-compact subset of $\mathbf{K}(S)$, then the relative hit-miss topology of $\mathcal{K}$ and the relative myopic topology of $\mathcal{K}$ coincide. A subset $\mathcal{K}$ of $\mathbf{K}(S)$ is $\boldsymbol{\upsilon}$-compact if and only if $\mathcal{K}$ is closed in $\mathbf{F}(S)$ and there exists a $K_0 \in \mathbf{K}(S)$ such that $K_0 \supset K$ for all $K \in \mathcal{K}$.*

Corollary 3.1 *$(\mathbf{K}(S), \boldsymbol{\upsilon}, \subset)$ is an LCO-space, and $(\mathbf{K}(S), \boldsymbol{\upsilon}, \cap, \cup)$ is a locally compact CO-lattice.*

Proof. Theorem 2.2 applies because $(\mathbf{K}(S), \boldsymbol{\upsilon})$ is second countable and Hausdorff. Let $\{K_i\}$ and $\{K_i'\}$ be convergent sequences in $\mathbf{K}(S)$ with the myopic topology limits K and K', respectively, and suppose that $K_i \subset K_i'$ for all i. Then $K_i \to K$ and $K_i' \to K'$ in the hit-miss topology of $\mathbf{F}(S)$, and $K_i \subset K_i'$ for all i. By Theorem 3.2 we see that $K \subset K'$, so that the desired conclusion follows from Theorem 2.2 and the fact that $(\mathbf{K}(S), \boldsymbol{\upsilon})$ is locally compact.

Remark 3.4 ($\mathbf{K}(S)$, $\boldsymbol{\upsilon}$, $\cap, \cup$) *is a distributive lattice that has the universal lower bound $\emptyset \subset S$ but has no universal upper bound unless S is compact. If $\mathcal{K} \subset \mathbf{K}(S)$ is not empty, then* $\inf \mathcal{K}$ *exists in $\mathbf{K}(S)$ but* $\sup \mathcal{K}$ *need not, unless S is compact. Hence if S is not compact, then* ($\mathbf{K}(S)$, $\boldsymbol{\upsilon}$, $\cap, \cup$) *is neither complete nor conditionally complete.*

Theorem 3.4 *If S is metrizable, then the relative myopic topology of $\mathbf{K}'(S) \equiv \mathbf{K}(S) \setminus \emptyset$ coincides with the topology induced on $\mathbf{K}'(S)$ by the Hausdorff metric:*

$$d_H(K, K') \equiv \max \left\{ \sup_{x \in K} d(x, K'), \sup_{x' \in K'} d(x', K) \right\}.$$

Here K and K' are any nonempty compact subsets of S and d is a metric on S that induces its topology.

Finally, note that the collections

$$\{\mathbf{K}^F : F \in \mathbf{F}(S)\} \cup \emptyset \quad \text{and} \quad \{\mathbf{K}_G : G \in \mathbf{G}(S)\} \cup \mathbf{K}(S)$$

consist, respectively, of decreasing and increasing $\boldsymbol{\upsilon}$-open sets. Thus the LCO-space ($\mathbf{K}(S)$, $\boldsymbol{\upsilon}$, $\subset$) is order resolvable, and it follows that the topologies $\boldsymbol{\mu}_m$ and $\boldsymbol{\lambda}_m$ generated subbasically by these collections are upper and lower topologies for ($\mathbf{K}(S)$, $\boldsymbol{\upsilon}$, $\subset$).

3.3 Limits and Semicontinuity

Henceforth I drop the S in writing $\mathbf{F}(S)$, $\mathbf{K}(S)$, etc. If $\{F_i\}$ is a $\boldsymbol{\tau}$-convergent sequence in $\mathbf{F}$ with limit F, we write $F_i \to F$ and $\lim F_i = F$ equivalently. The two theorems that follow are due to Matheron (1975) and give technically useful convergence criteria for sequences in $\mathbf{F}$. I refer to (a) and (b) of the second one as *Matheron's convergence criteria.*

Theorem 3.5 *A sequence $\{F_i\}$ in $\mathbf{F}$ converges to $F \in \mathbf{F}$ if and only if* (1) *$G \subset S$ is open and $G \cap F \neq \emptyset \Longrightarrow G \cap F_i \neq \emptyset$ $\forall$ but at most finitely many F_i and* (2) *$K \subset S$ is compact and $K \cap F = \emptyset \Longrightarrow K \cap F_i = \emptyset$ $\forall$ but at most finitely many F_i.*

Theorem 3.6 *A sequence $\{F_i\}$ in $\mathbf{F}$ converges to $F \in \mathbf{F}$ if and only if (a) for each $x \in F$ there exist $x_i \in F_i$ for all but at most finitely many i such that $x_i \to x$ and (b) if $\{F_{i_k}\}$ is a subsequence of $\{F_i\}$, then every convergent sequence $x_{i_k} \in F_{i_k}$ has its limit in F. In addition, (a) and (b) are respectively equivalent to* (1) *and* (2) *of Theorem 3.5.*

For τ^*-convergence in **G** and υ-convergence in **K**, Matheron (1975) gives the following characterizations.

Theorem 3.7 *A sequence $\{G_i\}$ in* **G** *converges to $G \in$* **G** *if and only if (a) for each $x \notin G$ there exists a sequence $\{x_i\}$ in S such that $x_i \to x$ and $x_i \notin G_i$ for all but at most finitely many i and (b) if $\{G_{i_k}\}$ is a subsequence of $\{G_i\}$ and if $\{x_{i_k} \notin G_{i_k}\}$ converges to x, then $x \notin G$.*

Theorem 3.8 *A sequence $\{K_i\}$ in* **K** *converges in the myopic topology to $K \in$* **K** *if and only if $K_i \to K$ in the hit-miss topology of* **F** *and there exists a $K_0 \in$* **K** *such that $K_0 \supset K_i$ for all i.*

Corollary 3.2 *The one-point subset $\{\emptyset\}$ of* **K** *is open in the myopic topology; that is, the empty subset of S is an isolated point of the space* (**K**,υ). *Hence $K_i \to \emptyset$ in* **K** *implies that all but at most finitely many of the K_i are empty.*

Proposition 3.3 *The relative hit-miss and myopic topologies of the subspace of one-point subsets $\{x\}$ of S both coincide with the LCS topology of S under the identification $\{x\} \leftrightarrow x$.*

Proof. The proposition equivalently asserts that $x_i \to x$ in $S \iff \{x_i\} \to \{x\}$ in **F** $\iff \{x_i\} \to \{x\}$ in **K**. If $x_i \to x$ in S, then $\{x_i\} \to \{x\}$ in **F** by Matheron's convergence criteria and the fact that all subsequences of a convergent S-sequence converge to the limit of the original sequence. On the other hand, if $\{x_i\} \to \{x\}$ in **F**, then every subsequence of $\{x_i\}$ converges to x in S by the second of Matheron's convergence criteria. Thus $x_i \to x$ in S, and we have proved that $x_i \to x$ in $S \iff \{x_i\} \to \{x\}$ in **F**. For the **K**-convergence case, $x_i \to x$ in S implies that $\{x_i\}$ is contained in a compact set, and (by what was just proved) that $\{x_i\} \to \{x\}$ in **F**. Thus $\{x_i\} \to \{x\}$ in **K**. On the other hand, if $\{x_i\} \to \{x\}$ in **K**, then $\{x_i\} \to \{x\}$ in **F**, and by what we have already proved, this implies that $x_i \to x$ in S. This completes the proof.

The topological relationship indicated by this proposition is a feature that recommends the hit-miss topology over the more usual *Moore-Smith order* and *interval* topologies of lattices, which do not relativize in the desirable manner of Proposition 3.3 (Frink, 1942).

For the next definition, note that the limit of a convergent subsequence of a sequence $\{F_i\}$ ($\{G_i\}$, $\{K_i\}$) in **F** (**G**, **K**) is called a *limit point* of $\{F_i\}$ ($\{G_i\}$, $\{K_i\}$). Also, since **F** (**G**) is a compact space, it follows that every sequence in **F** (**G**) has at least one limit point.

Definition 3.3 *Let $\{F_i\}$ be a sequence in* $\mathbf{F}$ *and let $\mathcal{L}(\{F_i\})$ denote its set of limit points. Then we define $\underline{Lim}\ F_i \equiv \bigcap\{F : F \in \mathcal{L}(\{F_i\})\}$ and $\overline{Lim}\ F_i \equiv \bigcup\{F : F \in \mathcal{L}(\{F_i\})\}$ and call them the lower and upper limits of the sequence $\{F_i\}$.*

The next theorem (Matheron, 1975) gives alternative characterizations of lower and upper limits and tells us (as would be expected) that a sequence converges if and only if its upper and lower limits coincide.

Theorem 3.9 (Upper-Lower Limit Theorem) *If $\{F_i\}$ is a sequence in* $\mathbf{F}$*, then* (a) *$\underline{Lim}\ F_i$ is the largest $F \in \mathbf{F}$ that satisfies condition "a" of Theorem 3.6,* (b) *$\overline{Lim}\ F_i$ is the smallest $F \in \mathbf{F}$ that satisfies condition "b" of Theorem 3.6, and* (c) *$F_i \to F \iff \underline{Lim}\ F_i = \overline{Lim}\ F_i = F$.*

Hence both the upper and lower limits lie in $\mathbf{F}$.

For the next result I use the term *subsequence* in the unconventional manner of Kelley (1955). Generally, a subsequence $\{i_k\}$ is a strictly increasing function $k \longmapsto i_k$ whose domain and range are the positive integers (or natural numbers). For Kelley, a *subsequence* $\{i_k\}$ is a non-decreasing function $k \longmapsto i_k$ of the foregoing kind that eventually goes to infinity; i.e., given a positive integer N, there is a k such that $i_k \geq N$. The usual term for such a *subsequence* is *cofinal subset of* $\{1, 2, 3, ...\}$. I indicate the term *subsequence* in Kelley's sense by italics.

Proposition 3.4 *If $\{F_i\}$ is a sequence in* $\mathbf{F}$*, then*

$$\overline{Lim}\, F_i = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i \geq n} F_i}$$

where the last overbar denotes topological closure, and

$$\underline{Lim}\, F_i = \bigcap_{\{i_k\}} \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$$

where $\{i_k\}$ ranges over all cofinal subsets of $\{1, 2, 3, ...\}$.

Proof. For the first part, see Matheron (1975), Proposition 1–2–3. For the second part consider the following. If $\{F_{i_k}\}$ has a *subsequence* that converges to F, then $F \subset \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$ by Matheron's convergence criteria. Since every $\{F_{i_k}\}$ has a convergent subsequence, $\underline{Lim}\ F_i \subset \bigcap \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$ where the intersection extends over all *subsequences* $\{i_k\}$. It remains to prove the reverse inclusion. Let

$x \in \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$ for every *subsequence* $\{i_k\}$. We can show that this implies that x is in every *subsequential* limit of $\{F_i\}$. Let $F_{i_k} \to F$. It is sufficient to prove that $x \in F$. In particular, we know that $x \in \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$. Let $\{x_{1,j}\}$ in $\bigcup_{k=1}^{\infty} F_{i_k}$ have limit x. If $\{x_{1,j}\}$ has a *subsequence* $x_{1,j_\ell} \in F_{i_{k_\ell}}$, we are done. If not, then $\{x_{1,j}\}$ along with its limit x lies entirely in $\bigcup_{k=1}^{m_1} F_{i_k}$ for some positive integer m_1. Since x must lie as well in $\overline{\bigcup_{k=m_1+1}^{\infty} F_{i_k}}$, there is a sequence $\{x_{2,j}\}$ with limit x that lies in $\bigcup_{k=m_1+1}^{\infty} F_{i_k}$. If $\{x_{2,j}\}$ has a *subsequence* $x_{2,j_\ell} \in F_{i_{k_\ell}}$, we are again done. If not, then $\{x_{2,j}\}$ along with its limit x lies entirely in $\bigcup_{k=m_1+1}^{m_2} F_{i_k}$ for some positive integer $m_2 > m_1$. Since $x \in \overline{\bigcup_{k=m_2+1}^{\infty} F_{i_k}}$ $\exists$ $\{x_{3,j}\}$ with limit x that lies in $\bigcup_{k=m_2+1}^{\infty} F_{i_k}$. The proposition thus follows by induction.

Proposition 3.4 shows that the definitions of the lower and upper limits of a sequence in $\mathbf{F}$ (Def. 3.3) in fact do not coincide with the sequence version of the usual definitions of the *inferior* and *superior limits* (lim inf and lim sup) of a *net* in a complete lattice (see Birkhoff, 1948). If $\{x_\alpha : \alpha \in \mathcal{D}\}$ is such a net (where $(\mathcal{D}, \trianglerighteq)$ is a directed set), then the usual definitions are

$$\liminf x_\alpha \equiv \sup_\alpha \inf\{x_\beta : \beta \trianglerighteq \alpha\}$$

and

$$\limsup x_\alpha \equiv \inf_\alpha \sup\{x_\beta : \beta \trianglerighteq \alpha\}.$$

By the expression $\overline{Lim}\, F_i = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i\geq n} F_i}$ given in the first part of Proposition 3.4, it follows that $\overline{Lim}\, F_i$ is in fact equal to $\limsup F_i$. It is not, however, the case that $\underline{Lim}\, F_i = \liminf F_i$; that is, it is not true that $\underline{Lim}\, F_i = \overline{\bigcup_{n=1}^{\infty} \bigcap_{i\geq n} F_i}$. Frink (1942) has given the following net definitions of the lower and upper limits:

$$\underline{Lim}\, x_\alpha \equiv \inf_{\mathcal{D}'} \sup\{x_\beta : \beta \in \mathcal{D}'\}$$

where $\mathcal{D}'$ ranges over the *cofinal subsets* of $\mathcal{D}$ and

$$\overline{Lim}\, x_\alpha \equiv \limsup x_\alpha.$$

Since the directed set associated with a sequence is $(\{1,2,3,...\}, \geq)$, Proposition 3.4 shows that Frink's more general definitions reduce in the case of sequences to those of Definition 3.3. The origin of these concepts, as Frink (1942) points out, can be found in Hausdorff (1927): Frink's $\underline{Lim}$ and $\overline{Lim}$ are the generalizations to nets of Hausdorff's notions of the *lower and upper closed limits* of a sequence of sets.

The following are analogs of Definition 3.3 and Theorem 3.9 for $\mathbf{G}$.

Definition 3.4 *Let $\{G_i\}$ be a sequence in* $\mathbf{G}$ *and let $\mathcal{L}(\{G_i\})$ denote its set of limit points. Then define $\underline{Lim}\ G_i \equiv \bigcap\{G : G \in \mathcal{L}(\{G_i\})\}$ and $\overline{Lim}\ G_i \equiv \bigcup\{G : G \in \mathcal{L}(\{G_i\})\}$ and call them the lower and upper limits of the sequence $\{G_i\}$.*

Remark 3.5 *If $\{G_i\}$ is a sequence in* $\mathbf{G}$*, then $\underline{Lim}\ G_i = [\overline{Lim}\ G_i^c]^c$ and $\overline{Lim}\ G_i = [\underline{Lim}\ G_i^c]^c$.*

Corollary 3.3 *If $\{G_i\}$ is a sequence in* $\mathbf{G}$*, then $\underline{Lim}\ G_i$ is the largest $G \in \mathbf{G}$ that satisfies condition (b) of Theorem 3.7, $\overline{Lim}\ G_i$ is the smallest $G \in \mathbf{G}$ that satisfies condition (a) of Theorem 3.7, and $G_i \to G \iff \underline{Lim}\ G_i = \overline{Lim}\ G_i = G$.*

Matheron (1975) defines the semicontinuities of mappings into **F**, **G**, and **K** as follows.

Definition 3.5 *Let X be a topological space.*

1. *If $\Psi : X \longrightarrow \mathbf{F}$, then Ψ is USC if and only if $\Psi^{-1}(\mathbf{F}^K)$ is open in X for all $K \in \mathbf{K}$, and Ψ is LSC if and only if $\Psi^{-1}(\mathbf{F}_G)$ is open in X for all $G \in \mathbf{G}$.*
2. *If $\Psi : X \longrightarrow \mathbf{G}$, then Ψ is USC if and only if $\Psi^{-1}(\mathbf{G}^G)$ is open in X for all $G \in \mathbf{G}$, and Ψ is LSC if and only if $\Psi^{-1}(\mathbf{G}_K)$ is open in X for all $K \in \mathbf{K}$.*
3. *If $\Psi : X \longrightarrow \mathbf{K}$, then Ψ is USC if and only if $\Psi^{-1}(\mathbf{K}^F)$ is open in X for all $F \in \mathbf{F}$, and Ψ is LSC if and only if $\Psi^{-1}(\mathbf{K}_G)$ is open in X for all $G \in \mathbf{G}$.*

Matheron's definitions coincide with Definition 2.9 when (1) $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are the upper and lower topologies of $(\mathbf{F}, \boldsymbol{\tau}, \subset)$, (2) $\boldsymbol{\mu}^*$ and $\boldsymbol{\lambda}^*$ are the upper and lower topologies of $(\mathbf{G}, \boldsymbol{\tau}^*, \subset)$, and (3) $\boldsymbol{\mu}_m$ and $\boldsymbol{\lambda}_m$ are the upper and lower topologies of $(\mathbf{K}, \boldsymbol{\upsilon}, \subset)$. In the next theorem, Matheron (1975) particularizes statement (1) of the foregoing definition to first countable Hausdorff spaces X. This result is a form of the *usual semicontinuity criterion* first met in Theorem 2.6.

Theorem 3.10 *If X is first countable and Hausdorff and $\Psi : X \longrightarrow \mathbf{F}$, then Ψ is USC at $x \in X \iff \Psi(x) \supset \overline{Lim}\ \Psi(x_i)\ \forall\ \{x_i\}$ in X that converge to x, and Ψ is LSC at $x \in X \iff \Psi(x) \subset \underline{Lim}\ \Psi(x_i)\ \forall\ \{x_i\}$ in X that converge to x.*

For the analog of Theorem 3.10 for mappings into $\mathbf{G}$, simply replace $\mathbf{F}$ by $\mathbf{G}$ in the statement of that theorem. To get a similar result for mappings into $\mathbf{K}$ requires some preparation.

Remark 3.6 *A sequence $\{K_i\}$ in $\mathbf{K}$ has $\boldsymbol{v}$-limit points if and only if $\{K_i\}$ has a compact subsequence, i.e., if and only if there is a compact subset K of S and a subsequence $\{K_{i_k}\}$ such that $K_{i_k} \subset K$ for all k.*

Definition 3.6 *Let $\{K_i\}$ be a sequence in $\mathbf{K}$ with a compact subsequence, and let $\mathcal{L}(\{K_i\})$ denote its set of $\boldsymbol{v}$-limit points. Then we define the lower and upper $\boldsymbol{v}$-limits of $\{K_i\}$, respectively, as follows:*

1. $\boldsymbol{v}\text{-}\underline{Lim}\ K_i \equiv \bigcap\{K : K \in \mathcal{L}(\{K_i\})\}$.
2. $\boldsymbol{v}\text{-}\overline{Lim}\ K_i \equiv \bigcup\{K : K \in \mathcal{L}(\{K_i\})\}$.

Definition 3.7 *A sequence $\{K_i\}$ in $\mathbf{K}$ is called a c-sequence if there is a $K \in \mathbf{K}$ such that $K \supset K_i$ for all i. If X is a first countable Hausdorff space and $\Psi : X \longrightarrow \mathbf{K}$, then Ψ is called a c-mapping if $\{\Psi(x_i)\}$ is a c-sequence whenever $\{x_i\}$ converges in X.*

Remark 3.7 *If $\{K_i\}$ is a c-sequence, then $\boldsymbol{v}\text{-}\underline{Lim}\ K_i = \underline{Lim}\ K_i$ and $\boldsymbol{v}\text{-}\overline{Lim}\ K_i = \overline{Lim}\ K_i$.*

Lemma 3.1 *If X is a topological space and $\Psi : X \longrightarrow \mathbf{K}$, then*

1. *Ψ is USC at $x \in X \implies \Psi$ is continuous at x in the relative $\boldsymbol{\mu}$ topology of $\mathbf{K}$.*
2. *Ψ is LSC at $x \in X \iff \Psi$ is continuous at x in the relative $\boldsymbol{\lambda}$ topology of $\mathbf{K}$.*

Proof. If Ψ is USC (LSC) at x, then Ψ is continuous at x in the $\boldsymbol{\mu}_m$ ($\boldsymbol{\lambda}_m$) topology of $\mathbf{K}$. Now the relative $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ topologies of $\mathbf{K}$ are generated subbasically by the collections $\{\mathbf{K}^K : K \in \mathbf{K}\} \cup \emptyset$ and $\{\mathbf{K}_G : G \in \mathbf{G}\} \cup \mathbf{K}$. Thus the relative $\boldsymbol{\lambda}$ topology of $\mathbf{K}$ is precisely $\boldsymbol{\lambda}_m$ and the relative $\boldsymbol{\mu}$ topology of $\mathbf{K}$ is contained in $\boldsymbol{\mu}_m$. This completes the proof.

The analog of Theorem 3.10 for mappings into $\mathbf{K}$ can now be given:

Theorem 3.11 *If X is a first countable Hausdorff space and Ψ is a c-mapping on X, then*

1. *Ψ is USC at $x \in X \iff \Psi(x) \supset \overline{Lim}\ \Psi(x_i)\ \forall\ \{x_i\}$ in X that converge to x.*
2. *Ψ is LSC at $x \in X \iff \Psi(x) \subset \underline{Lim}\ \Psi(x_i)\ \forall\ \{x_i\}$ in X that converge to x.*

Proof. By Theorem 3.10, Remark 3.7, and Lemma 3.1, it is sufficient to prove that $\Psi(x) \supset \overline{Lim}\ \Psi(x_i)\ \forall\ \{x_i\}$ in X that converge to x $\Longrightarrow \Psi$ is USC at x. Since X is a first countable Hausdorff space, we may characterize the upper semicontinuity of Ψ as follows: Ψ is USC at $x \iff x_i \to x \Longrightarrow \Psi(x_i) \to \Psi(x)$ in the μ_m topology of $\mathbf{K}$. Thus Ψ is USC at $x \iff x_i \to x$, $F \in \mathbf{F}$, and $\Psi(x) \cap F = \emptyset$ imply that $\Psi(x_i) \cap F = \emptyset$ for all but at most finitely many i. It is hence sufficient to prove that $\Psi(x) \supset \overline{Lim}\ \Psi(x_i) \Longrightarrow F \in \mathbf{F}$ and $\Psi(x) \cap F = \emptyset$ together imply that $\Psi(x_i) \cap F = \emptyset$ for all but at most finitely many i. Suppose that F is a closed set that misses $\Psi(x)$ and that F hits infinitely many of the $\Psi(x_i)$. There is then a subsequence $\{i_k\}$ and a sequence $\{y_k\}$ in F such that $y_k \in \Psi(x_{i_k})$. Since $\{y_k\}$ is contained in a compact set (because $\{\Psi(x_i)\}$ is), it follows that $\{y_k\}$ has a convergent subsequence whose limit y is in F. But such a y must also lie in the upper limit of $\{\Psi(x_i)\}$ and hence in $\Psi(x)$. This contradiction completes the proof.

3.4 Lattice and Intrinsic Topological Operations

The following are three general and useful results that are used below to analyze the continuity properties of the lattice operations $\cap$ and $\cup$, and the intrinsic topological operations *closure*, *interior*, and *boundary*.

Theorem 3.12 *A continuous* (USC, LSC) *function of a continuous function is continuous* (USC, LSC).

Note, however, that a continuous function of a USC (LSC) function is not necessarily USC (LSC). The next result is due to Matheron (1975).

Proposition 3.5 *Let X be a topological space and let $\Psi : X \longrightarrow \mathbf{G}$. Then Ψ is* LSC (USC) *if and only if the complementary mapping of X to $\mathbf{F}$ given by $x \longmapsto [\Psi(x)]^c$ is* USC (LSC).

Proposition 3.6 *Let Ψ be a mapping from a product of* **F***'s and* **G***'s into either* **F** *or* **G** *and let Ψ^* be its dual mapping. Then Ψ is continuous* (USC, LSC) *if and only if Ψ^* is continuous* (LSC, USC).

Proof. Suppose that Ψ maps $\prod_{i=1}^{k} \mathbf{X}_i$ into $\mathbf{X}$ where the $\mathbf{X}_i$ and $\mathbf{X}$ are each either $\mathbf{F}$ or $\mathbf{G}$. Let X_i denote the generic element of $\mathbf{X}_i$. Then since $(X_1^c, ..., X_k^c) \longmapsto (X_1, ..., X_k)$ is a homeomorphism of $\prod_{i=1}^{k} \mathbf{X}_i^c$ onto $\prod_{i=1}^{k} \mathbf{X}_i$, it follows from Theorem 3.12 that the mapping $(X_1^c, .., X_k^c) \longmapsto \Psi(X_1, ..., X_k)$ is continuous (USC, LSC) as Ψ is continuous (USC, LSC). Thus it follows (from Prop. 3.5) that the dual mapping $\Psi^* : (X_1^c, ..., X_k^c) \longmapsto [\Psi(X_1, ..., X_k)]^c$ is continuous (LSC, USC) as Ψ is continuous (USC, LSC).

Matheron (1975) specifies the connection between the hit-miss topology of $\mathbf{F}$ (myopic topology of $\mathbf{K}$, dual topology of $\mathbf{G}$) and the lattice algebra $(\mathbf{F}, \cap, \cup)$ $((\mathbf{K}, \cap, \cup)$, $(\mathbf{G}, \cap, \cup))$ as follows.

Theorem 3.13 *$\cup$ is a continuous operation in* **F** *and* **K***, but $\cap$ is only USC in both spaces. On the other hand, $\cap$ is a continuous operation in* **G***, but $\cup$ is only LSC.*

The elementary intrinsic topological operations that can be performed on the closed, open, and compact subsets of S are the *closure*, *interior*, and *boundary* operations. The *interior* A° of a subset A of S is the largest open subset of A, and the *boundary* ∂A of A is defined as $\partial A \equiv \overline{A} \setminus A^\circ$. Concerning the continuity properties of these operations, Matheron (1975) gives us the following proposition.

Proposition 3.7 *The mapping $F \longmapsto \overline{F^c}$ of $\mathbf{F}(S)$ to itself is LSC, and if S is a locally connected space, then the mapping $F \longmapsto \partial F$ of $\mathbf{F}(S)$ to itself is LSC.*

Corollary 3.4 *We may thus deduce the following:*

1. *$G \longmapsto \overline{G}$ is an LSC mapping of* **G** *into* **F***.*
2. *$F \longmapsto F^\circ$ is a USC mapping of* **F** *onto* **G***.*
3. *$K \longmapsto K^\circ$ is a USC mapping of* **K** *into* **G***.*
4. *If S is locally connected, then*
   (a) *$K \longmapsto \partial K$ is an LSC mapping of $\mathbf{K}(S)$ to itself.*
   (b) *$G \longmapsto \partial G$ is an LSC mapping of $\mathbf{G}(S)$ into $\mathbf{F}(S)$.*

Proof. $G \longmapsto \overline{G}$ can be expressed as $G \longmapsto G^c \longmapsto \overline{[G^c]^c} = \overline{G}$: i.e., as an LSC function of a continuous function. This proves statement 1 above. If F is closed, then the complement of F° is $\overline{F^c}$. Thus statement 2 follows from Proposition 3.5. For statement 3, let $K_i \to K$ in $\mathbf{K}$. Then according to statement 2 we have $\underline{Lim}\ \overline{K_i^c} \supset \overline{K^c} = [K^\circ]^c$. Thus $K^\circ \supset [\underline{Lim}\ \overline{K_i^c}]^c$, and by Remark 3.5 we have $K^\circ \supset \overline{Lim}\ K_i^\circ$. Thus statement 3 follows. For statement 4a, let $K_i \to K$ in $\mathbf{K}$ once again. Then it follows (from Prop. 3.7) that $\underline{Lim}\ \partial K_i \supset \partial K$ if we understand the lower limit relative to convergence in $\mathbf{F}(S)$; but since the ∂K_i are all contained in a compact set (since the K_i are), it follows that the lower limits of $\{\partial K_i\}$ relative to convergence in $\mathbf{K}(S)$ and $\mathbf{F}(S)$ are the same. Thus statement 4a follows. For statement 4b, let $G_i \to G$ in $\mathbf{G}(S)$. Then $G_i^c \to G^c$ in $\mathbf{F}(S)$, and since $\partial G_i^c = \partial \overline{G_i} = \partial G_i$ and $\partial G^c = \partial \overline{G} = \partial G$, it follows that $\partial G \subset \underline{Lim}\ \partial G_i$. This completes the proof.

Note that $\Re^n$ with its usual topology is a connected, and hence locally connected, topological space. The useful concept of *monotonic sequential* (MS) *convergence* is defined as follows.

Definition 3.8 *Let X be a universal set and let $\{A_i\}$ be a sequence of subsets of X. $\{A_i\}$ is said to be decreasing (increasing) if $A_i \supset A_{i+1}$ ($A_i \subset A_{i+1}$) for all i. If $\{A_i\}$ is decreasing, we put $A = \cap_i A_i$, write $A_i \downarrow A$, and call A the MS-limit of $\{A_i\}$. If $\{A_i\}$ is increasing, we put $A = \cup_i A_i$, write $A_i \uparrow A$, and call A the MS-limit of $\{A_i\}$. If $\{A_i\}$ is either increasing or decreasing, we call it a monotone sequence.*

The relation between MS and τ-convergence is as follows:

Theorem 3.14 (Matheron) *If $\{F_i\}$ is a monotone sequence in $\mathbf{F}$, then $F_i \downarrow F \Longrightarrow F_i \to F$ and $F_i \uparrow A \Longrightarrow F_i \to \overline{A}$.*

Corollary 3.5 *We may thus deduce the following:*

1. *If $\{G_i\}$ is a monotone sequence in $\mathbf{G}$, then $G_i \uparrow G \Longrightarrow G_i \to G$ in $\mathbf{G}$ and $G_i \downarrow A \Longrightarrow G_i \to A^\circ$ in $\mathbf{G}$.*
2. *If $\{K_i\}$ is a monotone sequence in $\mathbf{K}$, then $K_i \downarrow K \Longrightarrow K_i \to K$ in $\mathbf{K}$ and $K_i \uparrow A \Longrightarrow K_i \to \overline{A}$ in $\mathbf{K}$ or $\mathbf{F}$, depending on whether $\overline{A}$ is compact or not.*

4 Minkowski Operations

When the underlying space S is $\Re^n$, the vector space operations become available to augment the lattice and intrinsic topological operations. We can, for instance, *translate* a subset A of $\Re^n$ by a vector $x \in \Re^n$ as follows: $A + x \equiv x + A \equiv \{z \in \Re^n : z = x + y, y \in A\}$; if A is empty, then $A + x$ is defined to be empty.

Remark 4.1 *If $A \subset \Re^n$ is closed (compact, open), then $A+x$ is closed (compact, open) for all $x \in \Re^n$.*

The *translation mapping* is accordingly defined on $\Re^n \times \mathbf{F}$, $\Re^n \times \mathbf{K}$, and $\Re^n \times \mathbf{G}$ into $\mathbf{F}$, $\mathbf{K}$, and $\mathbf{G}$, respectively. We can also *multiply* a subset A of $\Re^n$ by a real scalar α as follows: $\alpha A \equiv \{x \in \Re^n : x = \alpha y, y \in A\}$; if A is empty, then αA is defined to be empty.

Remark 4.2 *If $A \subset \Re^n$ is closed (compact) and $\alpha \in \Re$, then αA is closed (compact); if A is open and $\alpha \neq 0$, then αA is open.*

The *scalar multiplication mapping* is thus defined on $\Re \times \mathbf{F}$, $\Re \times \mathbf{K}$, and $(\Re \setminus \{0\}) \times \mathbf{G}$ into $\mathbf{F}$, $\mathbf{K}$, and $\mathbf{G}$, respectively. It is customary to distinguish the multiplication of a set A by -1 with the notation $(-1)A \equiv \check{A}$.

Remark 4.3 *If $F \in \mathbf{F}$ and $\alpha \neq 0$, then $[\alpha F]^c = \alpha F^c$. If $G \in \mathbf{G}$ and $\alpha \neq 0$, then $[\alpha G]^c = \alpha G^c$.*

If A and B are any subsets of $\Re^n$, then their *Minkowski sum* is defined by $A \oplus B \equiv \{x : x = y + z, y \in A, z \in B\}$; if A or B is empty, then $A \oplus B \equiv \emptyset$. Thus $A \oplus B = B \oplus A$ and if $x \in \Re^n$, then $A + x = A \oplus \{x\}$. We follow Matheron (1975) in defining the *dilation* of A by B as $A \oplus \check{B}$.

Remark 4.4 *If A and B are subsets of $\Re^n$, we have the following:*

1. *If A is open, then $A \oplus B$ is open.*
2. *If A and B are compact, then $A \oplus B$ is compact.*
3. *If A is closed and B is compact, then $A \oplus B$ is closed.*
4. *If A and B are closed, then $A \oplus B$ is not necessarily closed.*

Thus $\oplus$ (or equivalently dilation) gives us binary operations in **K** and **G**, and $(A, B) \longmapsto A \oplus B$ $(A \oplus \check{B})$ defines mappings of $\mathbf{F} \times \mathbf{K}$, $\mathbf{F} \times \mathbf{G}$, and $\mathbf{G} \times \mathbf{K}$ into **F**, **G**, and **G**, respectively.

Since the lattice $(\mathcal{P}(\Re^n), \cap, \cup)$ is self-dual under the complementation mapping, the mapping of $\mathcal{P}(\Re^n)$ thus dual to $A \longmapsto A \oplus B$ (namely, $A \longmapsto (A^c \oplus B)^c$) serves to define the *Minkowski difference* $A \ominus B$ of two arbitrary subsets of $\Re^n$; i.e., $A \ominus B \equiv (A^c \oplus B)^c$. Also, the mapping similarly dual to $A \longmapsto A \ominus B$ (namely, $A \longmapsto (A^c \ominus B)^c$) gives $A \oplus B$ in the same way: i.e., $A \oplus B = (A^c \ominus B)^c$. Both $\oplus$ and $\ominus$ are accordingly dual operations. Following Matheron (1975), the *erosion* of A by B is taken as $A \ominus \check{B} = \{x : x + y \in A \ \forall \ y \in B\}$. Dilation and erosion are thus also each other's dual.

Remark 4.5 *If A and B are subsets of $\Re^n$, we have the following:*

1. *A is closed (compact) $\Longrightarrow$ $A \ominus B$ is closed (compact).*
2. *A is open and B is compact (open) $\Longrightarrow$ $A \ominus B$ is open (closed).*
3. *A is open and B is closed $\Longrightarrow$ $A \ominus B$ is not necessarily open.*

Thus $\ominus$ (or equivalently erosion) gives us binary operations in **F** and **K**, and $(A, B) \longmapsto A \ominus B$ $(A \ominus \check{B})$ defines mappings of $\mathbf{F} \times \mathbf{K}$, $\mathbf{G} \times \mathbf{K}$, $\mathbf{F} \times \mathbf{G}$, and $\mathbf{G} \times \mathbf{G}$ into **F**, **G**, **F**, and **F**, respectively.

There is some confusion in the literature regarding the definitions of $\ominus$ and dilation. Everyone defines $A \oplus B$ in the same way, but some call it, rather than $A \oplus \check{B}$, the dilation of A by B (e.g., Maragos, 1989, and Haralick et al, 1987); also, everyone uses the term *erosion* in the same way, but some (Maragos, 1989, and Haralick et al, 1987) define "$\ominus$" so that the erosion of A by B is A"$\ominus$"B.

The operation $\oplus$ has some familiar algebraic properties; it is, for instance, *commutative* and *associative*. Consequently $(\mathbf{K}, \oplus)$ and $(\mathbf{G}, \oplus)$ are *commutative semigroups*. The set $\{0\}$ consisting of only the origin of $\Re^n$ qualifies as an identity element, since $A \oplus \{0\} = \{0\} \oplus A = A$ for all $A \subset \Re^n$. It is not unique, however (and $(\mathbf{K}, \oplus)$ and $(\mathbf{G}, \oplus)$ are therefore not *groups*), because (by definition) $\emptyset \oplus A = A \oplus \emptyset \equiv \emptyset \ \forall \ A \subset \Re^n$. Thus, $A \ominus \emptyset \equiv \Re^n$ for all A. On the other hand, $\emptyset \ominus A = \emptyset$ for all nonempty A, while $\emptyset \ominus \emptyset = \Re^n$. In general, $A \ominus \check{A} \supset \{0\}$, but if A is nonempty and bounded, then $A \ominus \check{A} = \{0\}$. Thus $K \in \mathbf{K}'$ implies that $K \ominus \check{K} = \{0\}$; thus, $(\mathbf{K}', \oplus)$ has properties reminiscent of those of a group: in $\mathbf{K}'$ the set $\{0\}$ is a unique identity element and every $K \in \mathbf{K}'$ has a unique "inverse" $\check{K}$ in the modified sense that $(K^c \oplus \check{K})^c = (\check{K} \oplus K^c)^c = \{0\}$.

I refer to $\oplus$, $\ominus$, and scalar multiplication as the ***Minkowski operations.*** Some of the important algebraic rules that hold among the Minkowski and lattice operations are summarized in the next three results; for the second and third, see Matheron (1975).

Remark 4.6 *For all subsets A and B of $\Re^n$ and all real α and β,*

1. $1A = A$ *and* $0A = \{0\}$.
2. $\alpha(\beta A) = (\alpha\beta)A$ *and* $(\alpha + \beta)A \subset \alpha A \oplus \beta A$.
3. $\alpha(A \cap B) = \alpha A \cap \alpha B$ *and* $\alpha(A \cup B) = \alpha A \cup \alpha B$.
4. $\alpha(A \oplus B) = \alpha A \oplus \alpha B$.

Proposition 4.1 *If A and B are subsets of $\Re^n$, then*

$$(A \ominus \check{B}) \oplus B \subset A \subset (A \oplus \check{B}) \ominus B.$$

Theorem 4.1 *For all subsets A, B, and C of $\Re^n$,*

1. $(A \ominus C) \oplus B \subset (A \oplus B) \ominus C$.
2. $(A \ominus B) \ominus C = (A \ominus C) \ominus B = A \ominus (B \oplus C)$.
3. $B \subset C \Longrightarrow A \oplus B \subset A \oplus C$, $A \ominus C \subset A \ominus B$ *and* $B \ominus A \subset C \ominus A$.
4. $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$ *and* $A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C)$.
5. $(B \cap C) \ominus A = (B \ominus A) \cap (C \ominus A)$ *and* $A \oplus (B \cap C) \subset (A \oplus B) \cap (A \oplus C)$.
6. $(A \ominus B) \cup (A \ominus C) \subset A \ominus (B \cap C)$ *and* $(B \ominus A) \cup (C \ominus A) \subset (B \cup C) \ominus A$.

The continuity and semicontinuity properties of the Minkowski operations relative to the hit-miss, dual, and myopic topologies are given in the following series of results.

Theorem 4.2 *Scalar multiplication is a continuous mapping, respectively, of $\Re \setminus \{0\} \times$ **F**, $\Re \times$ **K**, and $\Re \setminus \{0\} \times$ **G** onto **F**, **K**, and **G**.*

Proof. To prove that scalar multiplication is a continuous mapping of $\Re \setminus \{0\} \times \mathbf{F}$ onto $\mathbf{F}$, let $F_i \to F$ in $\mathbf{F}$ and let $\alpha_i \to \alpha$ in $\Re \setminus \{0\}$. We show that $\alpha_i F_i \to \alpha F$ in $\mathbf{F}$. If $F = \emptyset$, then every subsequence $\{x_{i_k} \in F_{i_k}\}$ fails to converge, and this implies that every subsequence $\{y_{i_k} \in \alpha_{i_k} F_{i_k}\}$ also fails to converge. So assume that $F \neq \emptyset$. If $x \in \alpha F$, then $x = \alpha y$ for some $y \in F$. Since $F_i \to F$ in $\mathbf{F}$, there are $y_i \in F_i$ such that $y_i \to y$. Thus $\alpha_i y_i \in \alpha_i F_i$, and it follows that $\alpha_i y_i \to \alpha y = x$. Thus Matheron's first convergence criterion is satisfied. For the second, suppose that $x_{i_k} \in \alpha_{i_k} F_{i_k}$ and that $x_{i_k} \to x \in \Re^n$. Since $x_{i_k} = \alpha_{i_k} y_{i_k}$ with $y_{i_k} \in F_{i_k}$, we have $y_{i_k} = x_{i_k}/\alpha_{i_k}$, so that $y_{i_k} \to x/\alpha \in F$. It therefore follows that $x \in \alpha F$, and this establishes Matheron's second convergence criterion. This proves the first part of the theorem.

To prove that scalar multiplication is a continuous mapping of $\Re \setminus \{0\} \times \mathbf{K}$ onto $\mathbf{K}$, let $K_i \to K$ in $\mathbf{K}$ and $\alpha_i \to \alpha$ in $\Re \setminus \{0\}$. By what we have proved, we see that $\alpha_i K_i \to \alpha K$ in $\mathbf{F}$. Since $\{\alpha_i\}$ is necessarily bounded and there is a compact set that contains all the K_i, we readily see that there is also a compact set that contains all the $\alpha_i K_i$. To prove continuity at $(0, K)$ for arbitrary compact K, let $\alpha_i \to 0$ and $K_i \to K$ in $\mathbf{K}$. We must show that (1) $\alpha_i K_i \to \{0\}$ when $K \neq \emptyset$ and (2) $\alpha_i K_i \to \emptyset$ when $K = \emptyset$. For (1) let $x \in K$ and let $x_i \in K_i$ be such that $x_i \to x$. Then $\alpha_i x_i \to \mathbf{0}$. To complete case (1) we let $y_{i_k} \in \alpha_{i_k} K_{i_k}$ converge to y and show that $y = \mathbf{0}$. There are clearly $x_{i_k} \in K_{i_k}$ such that $y_{i_k} = \alpha_{i_k} x_{i_k}$. Since $\{x_{i_k}\}$ is contained in a compact set, we assume without loss of generality that $x_{i_k} \to x$. Thus it is clear that $y_{i_k} \to \mathbf{0}$. Case (2) is an immediate consequence of the fact that all but finitely many of the K_i are empty.

Now note that the mapping $(\alpha, G) \longmapsto (\alpha, G^c)$ of $\Re \setminus \{0\} \times \mathbf{G}$ onto $\Re \setminus \{0\} \times \mathbf{F}$ is a homeomorphism. Thus this mapping followed by scalar multiplication is continuous, and the desired result for $\mathbf{G}$ follows from Proposition 3.5. This completes the proof.

Corollary 4.1 *If $\alpha \neq 0$, then $A \longmapsto \alpha A$ is a homeomorphism of* $\mathbf{F}$ ($\mathbf{K}$, $\mathbf{G}$) *onto itself.*

For the next result, see Matheron (1975), Proposition 1–5–1.

Proposition 4.2 *$\oplus$ is a continuous mapping of* $\mathbf{F} \times \mathbf{K}$ *and* $\mathbf{K} \times \mathbf{K}$ *into* $\mathbf{F}$ *and* $\mathbf{K}$, *respectively.*

Corollary 4.2 *The translation mapping is continuous on $\Re^n \times \mathbf{F}$ and $\Re^n \times \mathbf{K}$, and for each $x \in \Re^n$ it follows that $A \longmapsto A + x$ is a homeomorphism of* $\mathbf{F}$ ($\mathbf{K}$) *onto itself.*

Proposition 4.3 *$\oplus$ and $\ominus$ have the following continuity properties:*

1. *$\ominus$ is only a USC mapping of* $\mathbf{F} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{K}$, *and* $\mathbf{F} \times \mathbf{F}$ *into* $\mathbf{F}$, $\mathbf{K}$, *and* $\mathbf{F}$, *respectively.*
2. *$\oplus$ is only an LSC mapping of* $\mathbf{G} \times \mathbf{K}$ *into* $\mathbf{G}$.
3. *$\ominus$ is only a USC mapping of* $\mathbf{F} \times \mathbf{G}$ *into* $\mathbf{F}$.

Proof. Matheron (1975, Prop. 1-5-2) establishes that the mapping $(F, K) \longmapsto F \ominus K$ is USC on $\mathbf{F} \times \mathbf{K}'$. Since $F \ominus \emptyset = \Re^n \ \forall\, F \in \mathbf{F}$, we see that this mapping is USC on $\mathbf{F} \times \mathbf{K}$ as well. Matheron also shows that $(K', K) \longmapsto K' \ominus K$ is USC on $\mathbf{K} \times \mathbf{K}'$. Thus we find that $\ominus$ is also USC on $\mathbf{K} \times \mathbf{K}$.

To prove that $\ominus$ is USC on $\mathbf{F} \times \mathbf{F}$, let $E_i \to E$ and $F_i \to F$ in $\mathbf{F}$. It is sufficient to show that $x_{i_k} \in E_{i_k} \ominus F_{i_k}$ and $x_{i_k} \to x$ in $\Re^n$ together imply that $x \in E \ominus F$; recall here that $\overline{Lim}\, E_i \ominus F_i$ is the smallest closed set with this property. Since $\{x_{i_k}\} \subset E_{i_k} \ominus F_{i_k}$ for each k, it follows that $\{x_{i_k}\} \oplus \check{F}_{i_k} \subset (E_{i_k} \ominus F_{i_k}) \oplus \check{F}_{i_k} \subset E_{i_k}$; the first inclusion follows from the general property $A \oplus C \subset B \oplus C$ whenever $A \subset B$, and the second follows from $(A \ominus B) \oplus \check{B} \subset A$ for any sets A and B. We therefore have $\{x_{i_k}\} \oplus \check{F}_{i_k} \subset E_{i_k}$ for all k. Since $\oplus$ is a continuous mapping of $\mathbf{F} \times \mathbf{K}$, it follows from Theorem 3.2 and Proposition 3.3 that $\{x\} \oplus \check{F} \subset E$ and therefore that $(\{x\} \oplus \check{F}) \ominus F \subset E \ominus F$, because $A \ominus C \subset B \ominus C$ whenever $A \subset B$. Since $(A \oplus \check{B}) \ominus B \supset A$ for any sets A and B, we finally have $\{x\} \subset E \ominus F$ or $x \in E \ominus F$. Thus $\ominus$ is USC on $\mathbf{F} \times \mathbf{F}$.

To see that $(A, B) \longmapsto A \ominus B$ is not LSC on $\mathbf{F} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{K}$, or $\mathbf{F} \times \mathbf{F}$, consider the following example in $\Re^2$: Let $a > 0$, and for each $n = 1, 2, 3, \ldots$ let $K_n = \{(x_1, x_2) : x_1^2 + x_2^2 \le a - n^{-1}\}$ and let $K_n' = \{(x_1, x_2) : x_1^2 + x_2^2 \le a + n^{-1}\}$. Then $K_n \to K$ and $K_n' \to K$ in both $\mathbf{F}$ and $\mathbf{K}$, where $K = \{(x_1, x_2) : x_1^2 + x_2^2 \le a\}$. Since $K_n' \supset K_n$ for all n, we have that $K_n \ominus K_n' = K_n \ominus \check{K}_n' = \emptyset$ for all n because $\check{K}_n' = K_n'$ for all n. Therefore $K_n \ominus K_n' \to \emptyset$ and $\overline{Lim}\, K_n \ominus K_n' = \underline{Lim}\, K_n \ominus K_n' = \emptyset$. But $K \ominus K = \{0\}$ because K is compact and nonempty. Thus $K \ominus K \not\subset \underline{Lim}\, K_n \ominus K_n'$. This completes the proof of the first assertion.

The mapping $(G, K) \longmapsto (G \oplus K)^c = G^c \ominus K$ of $\mathbf{G} \times \mathbf{K}$ into $\mathbf{G}$ can be expressed as a USC function of a continuous function as follows: $(G, K) \longmapsto (G^c, K) \longmapsto G^c \ominus K$. Hence this mapping is USC, and the second assertion follows from Proposition 3.5.

To prove that $\ominus$ is USC on $\mathbf{F} \times \mathbf{G}$, let $E_i \to E$ in $\mathbf{F}$ and $G_i \to G$ in $\mathbf{G}$. By proving that $x_{i_k} \in E_{i_k} \ominus G_{i_k}$ and $x_{i_k} \to x$ in $\Re^n$ together imply that $x \in E \ominus G$, we again show that $E \ominus G \supset \overline{Lim}\ E_i \ominus G_i$. Since for each k we have $\{x_{i_k}\} \subset E_{i_k} \ominus G_{i_k}$, it follows as before that $\{x_{i_k}\} \oplus \check{G}_{i_k} \subset (E_{i_k} \ominus G_{i_k}) \oplus \check{G}_{i_k} \subset E_{i_k}$. We therefore have $\{x_{i_k}\} \oplus \check{G}_{i_k} \subset E_{i_k}$ for all k. Since $\oplus$ is LSC on $\mathbf{G} \times \mathbf{K}$, it again follows from Theorem 3.2 and Proposition 3.3 that $\{x\} \oplus \check{G} \subset E$ and therefore that $(\{x\} \oplus \check{G}) \ominus G \subset E \ominus G$. Hence $\{x\} \subset E \ominus G$ or $x \in E \ominus G$, and it follows that $\ominus$ is a USC mapping of $\mathbf{F} \times \mathbf{G}$ into $\mathbf{F}$. To see that $(A, B) \longmapsto A \ominus B$ is not generally LSC on $\mathbf{F} \times \mathbf{G}$, consider the example already used above and let $G_n = \{(x_1, x_2) : x_1^2 + x_2^2 < a + n^{-1}\}$. Then $K_n \to K$ and $G_n \to K^\circ$ in $\mathbf{F}$ and $\mathbf{G}$, respectively. Since $G_n \supset K_n$ for all n, we have $K_n \ominus G_n = \emptyset$, so that $K_n \ominus G_n \to \emptyset$ and

$$\overline{Lim}\ K_n \ominus G_n = \underline{Lim}\ K_n \ominus G_n = \emptyset.$$

Thus $K \ominus K^\circ \not\subset \underline{Lim}\ K_n \ominus G_n$. This completes the proof of the third assertion and the proposition.

Corollary 4.3 *The following are now readily deduced.*

1. $\ominus$ *is a continuous mapping of* $\mathbf{G} \times \mathbf{K}$ *into* $\mathbf{G}$.
2. $\oplus$ *is only an LSC mapping of* $\mathbf{G} \times \mathbf{F}$ *into* $\mathbf{G}$.
3. $\ominus$ *is only a USC mapping of* $\mathbf{G} \times \mathbf{G}$ *into* $\mathbf{F}$.
4. $\oplus$ *is only an LSC mapping of* $\mathbf{G} \times \mathbf{G}$ *into* $\mathbf{G}$.

Because $\ominus$ is continuous on $\mathbf{G} \times \mathbf{K}$ and since $A \ominus \{x\} = A - x$ for all $A \subset \Re^n$ and $x \in \Re^n$, we obtain the following:

Remark 4.7 $A \longmapsto A + x$ *is continuous on* $\Re^n \times \mathbf{G}$ *and is a homeomorphism of* $\mathbf{G}$ *for each* $x \in \Re^n$.

Since $A \longmapsto \check{A}$ is a homeomorphism of the domains $\mathbf{F}$, $\mathbf{K}$, and $\mathbf{G}$ onto themselves, it follows that the continuity properties established above for $\oplus$ and $\ominus$ hold as well for dilation and erosion.

Table 1 summarizes the continuity properties established in this and previous sections for the various algebraic operations. *Note that the entries for the boundary operation assume that the underlying space S is locally connected like* $\Re^n$.

Table 1.
Summary of Continuity Properties.

Operation	*Domain*	*Range*	*Continuity type*
scalar $\times$ set	$\Re \setminus \{0\} \times \mathbf{F}$	F	C
scalar $\times$ set	$\Re \times \mathbf{K}$	K	C
scalar $\times$ set	$\Re \setminus \{0\} \times \mathbf{G}$	G	C
$\oplus$ or dilation	$\mathbf{K} \times \mathbf{K}$	K	C
$\oplus$ or dilation	$\mathbf{F} \times \mathbf{K}$	F	C
$\oplus$ or dilation	$\mathbf{F} \times \mathbf{G}$	G	LSC
$\oplus$ or dilation	$\mathbf{G} \times \mathbf{K}$	G	LSC
$\oplus$ or dilation	$\mathbf{G} \times \mathbf{G}$	G	LSC
$\ominus$ or erosion	$\mathbf{F} \times \mathbf{F}$	F	USC
$\ominus$ or erosion	$\mathbf{K} \times \mathbf{K}$	K	USC
$\ominus$ or erosion	$\mathbf{F} \times \mathbf{K}$	F	USC
$\ominus$ or erosion	$\mathbf{G} \times \mathbf{K}$	G	C
$\ominus$ or erosion	$\mathbf{F} \times \mathbf{G}$	F	USC
$\ominus$ or erosion	$\mathbf{G} \times \mathbf{G}$	F	USC
union	$\mathbf{F} \times \mathbf{F}$	F	C
union	$\mathbf{K} \times \mathbf{K}$	K	C
union	$\mathbf{G} \times \mathbf{G}$	G	LSC
intersection	$\mathbf{F} \times \mathbf{F}$	F	USC
intersection	$\mathbf{K} \times \mathbf{K}$	K	USC
intersection	$\mathbf{G} \times \mathbf{G}$	G	C
A^c	F	F	LSC
closure	G	F	LSC
interior	F	G	USC
interior	K	G	USC
boundary	F	F	LSC
boundary	K	K	LSC
boundary	G	F	LSC

When the space $S = \Re^n$, therefore, we have a structure composed not only of the morphospace $(\mathbf{F}(\Re^n), \boldsymbol{\tau}, \cap, \cup)$, its morphospace dual $(\mathbf{G}(\Re^n), \boldsymbol{\tau}^*, \cap, \cup)$, and the locally compact CO-lattice $(\mathbf{K}(\Re^n), \boldsymbol{v}, \cap, \cup)$, but as well of the Minkowski operations $\oplus$, $\ominus$, and scalar multiplication in and among these spaces, along with their continuity properties. This, then, is the fundamental field of operations of the mathematical morphology of euclidean sets. The next section considers the "morphological transformation" theory appropriate to this structure.

5 $\mathcal{M}$-Transformations of $\mathbf{F}(\Re^n)$

A mapping Ψ of $\mathbf{F} = \mathbf{F}(\Re^n)$ to itself is called a *transformation* of (or on) $\mathbf{F}$. A transformation type that I call an *$\mathcal{M}$-transformation* is here defined and examined in detail. The *order-preserving* (see Def. 5.5) $\mathcal{M}$-transformations are more commonly known as *morphological filters*. For the following, let $\mathcal{P} = \mathcal{P}(\Re^n)$ denote the class of all subsets of $\Re^n$.

Definition 5.1 *$\Psi : \mathbf{F} \longrightarrow \mathcal{P}$ is said to be translationally invariant or TI if $\Psi(F + x) = \Psi(F) + x$ for all $F \in \mathbf{F}$ and $x \in \Re^n$. A TI mapping will be called an $\mathcal{M}$-transformation (of or on $\mathbf{F}$) if it is into $\mathbf{F}$ and USC. Similarly, $\Psi : \mathbf{G} \longrightarrow \mathcal{P}$ is said to be TI if $\Psi(G + x) = \Psi(G) + x$ for all $G \in \mathbf{G}$ and $x \in \Re^n$.*

Note that the mapping Ψ^* dual to an $\mathcal{M}$-transformation Ψ is a TI LSC mapping of $\mathbf{G} = \mathbf{G}(\Re^n)$ into $\mathbf{G}$. This, of course, is an $\mathcal{M}$-transformation of (or on) $\mathbf{G}$.

5.1 Matheron's Kernel Theory

Definition 5.2 *The kernel of a TI mapping $\Psi : \mathbf{F} \longrightarrow \mathcal{P}$ is the set* $\ker(\Psi) \equiv \{F \in \mathbf{F} : 0 \in \Psi(F)\}$. *Likewise, the kernel of a TI mapping $\Psi : \mathbf{G} \longrightarrow \mathcal{P}$ is the set* $\ker(\Psi) \equiv \{G \in \mathbf{G} : 0 \in \Psi(G)\}$.

Although I do not explicitly state them, it should be kept in mind that there are dual-space analogs of all the results that follow.

Theorem 5.1 *If Ψ is a TI mapping of $\mathbf{F}$ to $\mathcal{P}$ and $F \in \mathbf{F}$, then*

$$\Psi(F) = \{x \in \Re^n : F - x \in \ker(\Psi)\}.$$

If $\mathcal{K}$ is any subset of $\mathbf{F}$, then $F \longmapsto \{x \in \Re^n : F - x \in \mathcal{K}\}$ defines a TI mapping of $\mathbf{F}$ to $\mathcal{P}$ whose kernel is $\mathcal{K}$.

Theorem 5.2 (Closed Kernel Theorem) *A TI mapping Ψ of $\mathbf{F}$ to $\mathcal{P}$ is into $\mathbf{F}$ and USC if and only if* $\ker(\Psi)$ *is closed in $\mathbf{F}$.*

Matheron (1975, Prop. 8–2–1) establishes this result for TI closed-set mappings that are order-preserving (Def. 5.5). His proof in fact holds for all TI closed-set mappings and thus validates Theorem 5.2.

According to Theorem 5.1, a TI mapping is uniquely determined by its kernel, and there is a one-to-one correspondence between the TI mappings of $\mathbf{F}$ and $\mathcal{P}(\mathbf{F})$, the class of all subsets of $\mathbf{F}$. According to the closed kernel theorem, there is therefore a one-to-one correspondence between the class of $\mathcal{M}$-transformations of $\mathbf{F}$ and the class of closed subsets of $\mathbf{F}$, i.e., $\mathbf{F}(\mathbf{F})$. Moreover, since $\mathbf{F}$ itself is an LCS space, we can identify the class of $\mathcal{M}$-transformations of $\mathbf{F}$ with $\mathbf{F}(\mathbf{F})$ (understood to have its hit-miss topology) and thereby produce the *space* of $\mathcal{M}$-transformations. I denote this space by $\mathcal{M}(\mathbf{F})$.

A noteworthy feature of $\mathcal{M}(\mathbf{F}) \leftrightarrow \mathbf{F}(\mathbf{F})$ arises from the fact that its underlying space $\mathbf{F}$ is compact. Because of this, $\mathbf{F}(\mathbf{F})$ and $\mathbf{K}(\mathbf{F})$ are the same sets, and the myopic topology of $\mathbf{K}(\mathbf{F})$ coincides with the hit-miss topology of $\mathbf{F}(\mathbf{F})$. Because the hit-miss topology of $\mathbf{F}$ is metrizable, it additionally follows that the relative hit-miss topology of the subspace $\mathbf{F}'(\mathbf{F}) \equiv \mathbf{F}(\mathbf{F}) \setminus \emptyset$ coincides with the topology induced on $\mathbf{F}'(\mathbf{F})$ by the Hausdorff metric; the corresponding subspace $\mathcal{M}'(\mathbf{F})$ of $\mathcal{M}(\mathbf{F})$ is obtained from the latter by the deletion of the trivial transformation that maps all $F \in \mathbf{F}$ to the empty set.

For later convenience, the definitions of TI mappings and their kernels are here generalized as follows:

Definition 5.3 *If $\mathcal{A} \subset \mathcal{P}$ is nonempty and such that $A + x \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and $x \in \Re^n$, then $\mathcal{A}$ is said to be closed under translations. Let $\mathcal{A}$ be closed under translations.*

1. *If $\Psi : \mathcal{A} \longrightarrow \mathcal{P}$, then Ψ is called TI if $\Psi(A + x) = \Psi(A) + x$ for all $A \in \mathcal{A}$ and $x \in \Re^n$.*
2. *If $\Psi : \mathcal{A} \longrightarrow \mathcal{P}$ is TI, then the kernel of Ψ is the set*

$$\ker(\Psi) \equiv \{A \in \mathcal{A} : \mathbf{0} \in \Psi(A)\}.$$

It is also useful to expand our notion of dual mappings as follows:

Definition 5.4 *Let $\mathcal{A} \subset \mathcal{P}$ be nonempty and let*

$$\mathcal{A}^* \equiv \{A \in \mathcal{P} : A^c \in \mathcal{A}\}.$$

Then if $\Psi : \mathcal{A} \longrightarrow \mathcal{P}$, we call the mapping Ψ^* of $\mathcal{A}^*$ to $\mathcal{P}$ defined by $\Psi^*(A) = [\Psi(A^c)]^c$ the dual of Ψ.

If $(\mathcal{A}, \cap, \cup)$ is a lattice, then so is $(\mathcal{A}^*, \cap, \cup)$, and the two are then dual-lattice isomorphic under complementation. In this case, the definition of the dual mapping Ψ' that follows Definition 3.2 therefore coincides with the present one.

Remark 5.1 *Let $\mathcal{A}^* \equiv \{A \in \mathcal{P} : A^c \in \mathcal{A}\}$, where $\mathcal{A} \subset \mathcal{P}$ is closed under translations. Then $\mathcal{A}^*$ is closed under translations, and if Ψ is a TI mapping of $\mathcal{A}$ to $\mathcal{P}$, then so is Ψ^*.*

The next and final objective of this report is to outline and relate the representation theorems for $\mathcal{M}$-transformations established by Matheron (1975), Maragos (1985 and 1989), and Bannon and Barrera (1991). For this, however, the lattice/poset structure of $\mathcal{M}(\mathbf{F})$ and its associated CO-space aspects must first be considered.

5.2 Lattice Algebra and Partial Ordering in $\mathcal{M}(\mathbf{F})$

The space $\mathcal{M}(\mathbf{F})$ has a natural lattice and poset structure that it acquires from $\mathbf{F}(\mathbf{F})$ through the correspondence $\Psi \leftrightarrow \ker(\Psi)$. If Ψ and Ψ' are transformations in $\mathcal{M}(\mathbf{F})$, then the transformations $\Psi \cap \Psi'$ and $\Psi \cup \Psi'$ can be defined in terms of their kernels as follows:

$$\ker(\Psi \cap \Psi') \equiv \ker(\Psi) \cap \ker(\Psi')$$

and

$$\ker(\Psi \cup \Psi') \equiv \ker(\Psi) \cup \ker(\Psi').$$

Since these kernels are plainly closed, it follows that $\Psi \cap \Psi'$ and $\Psi \cup \Psi'$ are in $\mathcal{M}(\mathbf{F})$. For all $\Psi, \Psi' \in \mathcal{M}(\mathbf{F})$ and all $F \in \mathbf{F}$, we moreover have

$$(\Psi \cap \Psi')(F) = \Psi(F) \cap \Psi'(F)$$

and

$$(\Psi \cup \Psi')(F) = \Psi(F) \cup \Psi'(F).$$

Since $\cup$ and $\cap$ are clearly associative operations in $\mathcal{M}(\mathbf{F})$, we may define $\Psi_1 \cup ... \cup \Psi_k$ and $\Psi_1 \cap ... \cap \Psi_k$ inductively for any finite set $\{\Psi_1, ..., \Psi_k\}$ of transformations in $\mathcal{M}(\mathbf{F})$. Thus $\mathcal{M}(\mathbf{F})$ is closed under finite unions and intersections. That is,

Proposition 5.1 *If $\Psi_1, ..., \Psi_k \in \mathcal{M}(\mathbf{F})$, then*

1. *$\Psi_1 \cup ... \cup \Psi_k$ and $\Psi_1 \cap ... \cap \Psi_k$ are in $\mathcal{M}(\mathbf{F})$.*
2. *For all $F \in \mathbf{F}$,*
   (a) $(\Psi_1 \cup ... \cup \Psi_k)(F) = \Psi_1(F) \cup ... \cup \Psi_k(F)$.
   (b) $(\Psi_1 \cap ... \cap \Psi_k)(F) = \Psi_1(F) \cap ... \cap \Psi_k(F)$.
3. $\ker(\Psi_1 \cup ... \cup \Psi_k) = \ker(\Psi_1) \cup ... \cup \ker(\Psi_k)$.
4. $\ker(\Psi_1 \cap ... \cap \Psi_k) = \ker(\Psi_1) \cap ... \cap \ker(\Psi_k)$.

$\mathcal{M}(\mathbf{F})$ is in fact closed under arbitrary intersections.

Proposition 5.2 *If $\{\Psi_\alpha\}$ is a family of transformations in $\mathcal{M}(\mathbf{F})$, then the mapping $\cap_\alpha \Psi_\alpha$ defined on $\mathbf{F}$ by*

$$F \longmapsto \cap_\alpha \Psi_\alpha(F)$$

is in $\mathcal{M}(\mathbf{F})$ and $\ker(\cap_\alpha \Psi_\alpha) = \cap_\alpha \ker(\Psi_\alpha)$; in fact,

$$\inf\{\Psi_\alpha\} = \cap_\alpha \Psi_\alpha.$$

Remark 5.2 *$(\mathcal{M}(\mathbf{F}), \cap, \cup)$ is a complete distributive lattice isomorphic to $(\mathbf{F}(\mathbf{F}), \cap, \cup)$ under the correspondence $\leftrightarrow$ which topologically identifies $\mathcal{M}(\mathbf{F})$ with $\mathbf{F}(\mathbf{F})$.*

Proposition 5.3 *If $\{\Psi_\alpha\}$ is a family of transformations in $\mathcal{M}(\mathbf{F})$, then the mapping $\cup_\alpha \Psi_\alpha$ defined on $\mathbf{F}$ by*

$$F \longmapsto \overline{\cup_\alpha \Psi_\alpha(F)}$$

is in $\mathcal{M}(\mathbf{F})$ and $\ker(\cup_\alpha \Psi_\alpha) = \overline{\cup_\alpha \ker(\Psi_\alpha)}$; in fact,

$$\sup\{\Psi_\alpha\} = \cup_\alpha \Psi_\alpha.$$

Proposition 5.4 *The ordering $\subseteq$ induced in $\mathcal{M}(\mathbf{F})$ by its lattice operations may be characterized as follows: If $\Psi, \Psi' \in \mathcal{M}(\mathbf{F})$, then*

$$\Psi \subseteq \Psi' \iff \Psi(F) \subset \Psi'(F) \ \forall\ F \in \mathbf{F} \iff \ker(\Psi) \subset \ker(\Psi').$$

In the following, the hit-miss topologies of $\mathbf{F}(\mathbf{F})$ and $\mathcal{M}(\mathbf{F})$ are denoted by the (same) symbol $\boldsymbol{\nu}$.

Corollary 5.1 *Under the correspondence $\leftrightarrow$,*

1. *$(\mathcal{M}(\mathbf{F}), \subseteq)$ is poset isomorphic to $(\mathbf{F}(\mathbf{F}), \subset)$.*
2. *$(\mathcal{M}(\mathbf{F}), \boldsymbol{\nu}, \subseteq)$ is CO-space isomorphic to $(\mathbf{F}(\mathbf{F}), \boldsymbol{\nu}, \subset)$.*
3. *$(\mathcal{M}(\mathbf{F}), \boldsymbol{\nu}, \cap, \cup)$ is compact CO-lattice isomorphic to the compact CO-lattice $(\mathbf{F}(\mathbf{F}), \boldsymbol{\nu}, \cap, \cup)$.*

Therefore, $\cup$ is a continuous operation in $\mathcal{M}(\mathbf{F})$, but $\cap$ is only USC.

5.3 Matheron-Maragos Representations

The representation theory of Matheron (1975) and Maragos (1985 and 1989) is outlined in this section; it is essentially a representation theory for morphological filters, i.e., increasing $\mathcal{M}$-transformations.

Definition 5.5 *If Ψ maps $\mathbf{F}$ [or $\mathcal{A} \subset \mathcal{P}$] to $\mathcal{P}$, then Ψ is called increasing (decreasing) if $\Psi(E) \supset \Psi(F)$ ($\Psi(E) \subset \Psi(F)$) whenever E and F are in $\mathbf{F}$ [$\mathcal{A}$] and $E \supset F$. "Order preserving" and "order reversing" are synonyms for "increasing" and "decreasing."*

Remark 5.3 *If $\mathcal{A} \subset \mathcal{P}$ is nonempty and $\Psi : \mathcal{A} \longrightarrow \mathcal{P}$ is increasing (decreasing), then Ψ^* is increasing (decreasing).*

The utility of the concept of MS convergence (Def. 3.8) is revealed by the following important result (Matheron, 1975).

Proposition 5.5 *If $\Psi : \mathbf{F} \longrightarrow \mathbf{F}$ is increasing, then*

$$\Psi \text{ is USC} \iff F_i \downarrow F \text{ in } \mathbf{F} \Longrightarrow \Psi(F_i) \downarrow \Psi(F).$$

If we are interested only in morphological filters, this proposition tells us that the topological machinery outlined and developed so far can be almost entirely dispensed with. If we are interested in nonincreasing $\mathcal{M}$-transformations, however, we need the entire topological apparatus.

Remark 5.4 *The mappings $F \longmapsto \emptyset$ and $F \longmapsto \Re^n \; \forall F \in \mathbf{F}$ are $\mathcal{M}$-transformations that are at once increasing and decreasing. We denote them $\Psi_\emptyset$ and $\Psi_{\Re^n}$ and call them the trivial transformations.*

Theorem 5.3 *If Ψ is a TI mapping of $\mathbf{F}$ to $\mathcal{P}$, then $\Psi(\emptyset)$ is either $\emptyset$ or $\Re^n$, and likewise for $\Psi(\Re^n)$. Furthermore, if Ψ is nontrivial and increasing (decreasing), then $\Psi(\emptyset) = \emptyset$ ($\Re^n$) and $\Psi(\Re^n) = \Re^n$ ($\emptyset$).*

Theorem 5.4 *$\Psi \in \mathcal{M}(\mathbf{F})$ is increasing (decreasing) if and only if $\ker(\Psi)$ is an increasing (decreasing) set.*

I denote the subspace of increasing (decreasing) $\mathcal{M}$-transformations of $\mathbf{F}$ by $\mathcal{M}_\uparrow(\mathbf{F})$ ($\mathcal{M}_\downarrow(\mathbf{F})$).

Proposition 5.6 *$\mathcal{M}_\uparrow(\mathbf{F})$ and $\mathcal{M}_\downarrow(\mathbf{F})$ are closed subspaces of $\mathcal{M}(\mathbf{F})$.*

Proof. Let $\{\Psi_i\}$ be a sequence of increasing $\mathcal{M}$-transformations and suppose that $\Psi_i \to \Psi$ in $\mathcal{M}(\mathbf{F})$. To see that Ψ is increasing, let $\ker(\Psi_i) = \mathcal{F}_i$ and let $\ker(\Psi) = \mathcal{F}$. Then our hypothesis is equivalent to $\mathcal{F}_i \to \mathcal{F}$ in $\mathbf{F}(\mathbf{F})$ and each $\mathcal{F}_i$ is an increasing set. Let F be any set in $\mathcal{F}$ and let $E \supset F$. There are $F_i \in \mathcal{F}_i$ such that $F_i \to F$ in $\mathbf{F}$. Let $E_i = F_i \cup E$. Since each $\mathcal{F}_i$ is increasing and $E_i \supset F_i$, it follows that $E_i \in \mathcal{F}_i$ for all i and $E_i \to E$ in $\mathbf{F}$. Thus $E \in \mathcal{F}$, and it follows that Ψ is increasing. The proof for $\mathcal{M}_\downarrow(\mathbf{F})$ is similar.

Since $\mathcal{M}_\uparrow(\mathbf{F})$ and $\mathcal{M}_\downarrow(\mathbf{F})$ are also closed under the lattice operations, it follows that the posets $(\mathcal{M}_\uparrow(\mathbf{F}), \subseteq)$ and $(\mathcal{M}_\downarrow(\mathbf{F}), \subseteq)$ are compact ordered spaces, and that the lattices $(\mathcal{M}_\uparrow(\mathbf{F}), \cap, \cup)$ and $(\mathcal{M}_\downarrow(\mathbf{F}), \cap, \cup)$ are compact closed-order lattices (when $\mathcal{M}_\uparrow(\mathbf{F})$ and $\mathcal{M}_\downarrow(\mathbf{F})$ have their relative topologies in $\mathcal{M}(\mathbf{F})$).

The next result is Matheron's well-known *union of erosions formula.*

Theorem 5.5 *If* $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$, *then for all* $F \in \mathbf{F}$

$$\Psi(F) = \overline{\bigcup\{F \ominus \check{E} : E \in \ker(\Psi)\}}.$$

The corresponding result for $\Psi \in \mathcal{M}_\downarrow(\mathbf{F})$ is

Theorem 5.6 *If* $\Psi \in \mathcal{M}_\downarrow(\mathbf{F})$, *then for all* $F \in \mathbf{F}$

$$\Psi(F) = \overline{\bigcup\{\check{E} \ominus F : E \in \ker(\Psi)\}}.$$

It is well known that there is much redundancy in the above formulas. The *minimal basis kernel* (Maragos, 1985) $\mathcal{K}_{\min}(\Psi)$ of a $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$ is the collection of minimal elements under $\subset$ of $\ker(\Psi)$ (i.e., the set of $E \in \ker(\Psi)$ such that no $F \in \ker(\Psi)$ is a proper subset of E). Similarly, the *maximal basis kernel* $\mathcal{K}_{\max}(\Psi)$ of a $\Psi \in \mathcal{M}_\downarrow(\mathbf{F})$ is the collection of maximal elements under $\subset$ of $\ker(\Psi)$. These collections are empty when $\ker(\Psi)$ is empty (i.e., when $\Psi = \Psi_\emptyset$); otherwise, the existence of $\mathcal{K}_{\min}$ and $\mathcal{K}_{\max}$ as nonempty collections is guaranteed by *Zorn's lemma* and the fact that $\ker(\Psi)$ is closed in $\mathbf{F}$. The following lemma of Banon and Barrera (1991) helps to show this.

Lemma 5.1 *If $\mathcal{L}$ is a totally ordered subset of* $\mathbf{F}(S)$, *then* $\bigcap \mathcal{L}$ *and* $\overline{\bigcup \mathcal{L}}$ *lie in the closure of $\mathcal{L}$.*

Theorem 5.7 *If $\mathcal{F}$ is a nonempty increasing closed subset of $\mathbf{F}(S)$, then the set $\mathbf{M}_\wedge$ of minimal elements of $\mathcal{F}$ in $\mathcal{F}$ is nonempty and*

$$\mathcal{F} = \{F \in \mathbf{F}(S) : F \supset M, M \in \mathbf{M}_\wedge\}.$$

On the other hand, if $\mathcal{F}$ is a nonempty decreasing closed subset of $\mathbf{F}(S)$, then the set $\mathbf{M}_\vee$ of maximal elements of $\mathcal{F}$ in $\mathcal{F}$ is nonempty and

$$\mathcal{F} = \{F \in \mathbf{F}(S) : F \subset M, M \in \mathbf{M}_\vee\}.$$

Proof. Let E be an element of $\mathcal{F}$ and let $\mathcal{A}_E$ denote the class of elements in $\mathcal{F}$ that are subsets of E. Every totally ordered subset of $\mathcal{A}_E$ has an infimum in $\mathbf{F}(S)$ given by the intersection of all the elements of the subset. Because $\mathcal{F}$ is closed in $\mathbf{F}(S)$, it follows from Lemma 5.1 that this infimum lies in $\mathcal{F}$ and hence in $\mathcal{A}_E$. Thus, every totally ordered subset of $\mathcal{A}_E$ has a lower bound in $\mathcal{A}_E$; hence by Zorn's lemma, $\mathcal{A}_E$ has a minimal element that is also a minimal element of $\mathcal{F}$. Let $\mathbf{M}_\wedge$ denote the set of all such minimal elements as E ranges over $\mathcal{F}$. Since $\mathcal{F}$ is increasing and every $E \in \mathcal{F}$ contains an $M \in \mathbf{M}_\wedge$, it follows that

$$\mathcal{F} = \{F \in \mathbf{F}(S) : F \supset M, M \in \mathbf{M}_\wedge\}.$$

The proof for decreasing $\mathcal{F}$ is similar.

Since $B \supset C \Longrightarrow A \ominus \check{C} \supset A \ominus \check{B}$, the $E \in \ker(\Psi)$ of Theorem 5.5 can be limited to those in $\mathcal{K}_{\mathsf{min}}(\Psi)$; the $E \in \ker(\Psi)$ of Theorem 5.6 can likewise be limited to those in $\mathcal{K}_{\mathsf{max}}(\Psi)$.

Corollary 5.2 *We therefore have the following:*

1. *If $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$, then for all $F \in \mathbf{F}$*

$$\Psi(F) = \overline{\bigcup\{F \ominus \check{E} : E \in \mathcal{K}_{\mathsf{min}}(\Psi)\}}.$$

2. *If $\Psi \in \mathcal{M}_\downarrow(\mathbf{F})$, then for all $F \in \mathbf{F}$*

$$\Psi(F) = \overline{\bigcup\{\check{E} \ominus F : E \in \mathcal{K}_{\mathsf{max}}(\Psi)\}}.$$

I refer to the contents of Theorems 5.5 and 5.6 and Corollary 5.2 as the *Matheron-Maragos representations.* The first part of the next result (Matheron, 1975) is an algebraic form of Theorem 5.5.

Theorem 5.8 *If $\Psi : \mathcal{P} \longrightarrow \mathcal{P}$ is TI and increasing, then for all $A \in \mathcal{P}$*

1. $\Psi(A) = \bigcup\{A \ominus \check{B} : B \in \ker(\Psi)\}$.
2. $\Psi(A) = \bigcap\{A \oplus \check{B} : B \in \ker(\Psi^*)\}$.

The second part of this theorem is Matheron's ***intersection of dilations formula***. The topological version of this formula is an important addition to Theorem 5.5. To obtain it, consider the following definition.

Definition 5.6 *If Ψ maps $\mathbf{F}$ to $\mathcal{P}$ and is increasing, then we define the associated mapping $\Psi_\wedge$ for each $A \in \mathcal{P}$ by*

$$\Psi_\wedge(A) \equiv \bigcup\{\Psi(F) : F \in \mathbf{F}, F \subset A\}$$

where the union of the empty family is taken as empty.

$\Psi_\wedge$ is the *smallest increasing extension of Ψ to $\mathcal{P}$* and has the associated dual $(\Psi_\wedge)^*$ on $\mathcal{P}$. For convenience I denote the latter more briefly as ψ and write ψ_K for ψ restricted to $\mathbf{K}$.

Remark 5.5 *If Ψ is a TI mapping of $\mathbf{F}$ to $\mathcal{P}$, then so is $\Psi_\wedge$.*

If $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$, then for all $F \in \mathbf{F}$ (by Thm. 5.8),

$$\Psi(F) = \Psi_\wedge(F) = \bigcup_{B \in \ker(\Psi_\wedge)} F \ominus \check{B} = \bigcap_{B \in \ker(\psi)} F \oplus \check{B}.$$

Theorem 5.5 accordingly states that it is enough in the third member to take the union over the sets in $\ker(\Psi) \subset \ker(\Psi_\wedge)$ and then form the topological closure. Theorem 5.9 (below) is a topological rendering of the intersection of dilations formula (Matheron, 1975); it establishes (among other things) that it is sufficient in the last member to take the intersection over the sets in $\ker(\psi_K) \subset \ker(\psi)$.

Theorem 5.9 *Let Ψ be an increasing TI mapping of $\mathbf{F}$ to $\mathcal{P}$. Then,*

1. $\ker(\psi_K)$ *is closed in* $\mathbf{K}$.
2. Ψ *is into* $\mathbf{F}$ *and USC if and only if for all* $F \in \mathbf{F}$

$$\Psi(F) = \bigcap\{F \oplus \check{K} : K \in \ker(\psi_K)\}.$$

3. $\Psi \in \mathcal{M}_\uparrow(\mathbf{F}) \Longrightarrow \ker(\Psi) = \bigcap\{\mathbf{F}_K : K \in \ker(\psi_K)\}$.

Therefore, it is always possible to represent a morphological filter as an intersection of dilations by compact sets.

5.4 Banon-Barrera Representations

The representations of Banon and Barrera (1991) generalize those of Matheron-Maragos to arbitrary $\Psi \in \mathcal{M}(\mathbf{F})$.

If E and H are in $\mathbf{F}$ and $E \subset H$, then we can define the ***bracketing transformation*** $\cdot \Delta (E, H)$ for all $F \in \mathbf{F}$ by

$$F \Delta (E, H) \equiv \{x \in \Re^n : E + x \subset F \subset H + x\} = (F \ominus \check{E}) \cap (\check{H} \ominus F).$$

Bracketing transformations are essentially the same as Serra's ***hit-miss transformations*** $\cdot \otimes (E, H^c)$. Indeed, $F \otimes (E, H^c) = F \Delta (E, H)$ for all $F \in \mathbf{F}$. It is clear that $\cdot \Delta(E, H)$ is translationally invariant and that its kernel $\{F \in \mathbf{F} : E \subset F \subset H\} \equiv [E, H]$ (called a *closed interval* in $\mathbf{F}$) is a closed subset of $\mathbf{F}$. Thus, $\cdot \Delta (E, H) \in \mathcal{M}(\mathbf{F})$. The basic importance of bracketing transformations is made clear by Theorem 5.10, which represents the general $\Psi \in \mathcal{M}(\mathbf{F})$ as the union of those bracketing transformations whose kernels are contained in $\ker(\Psi)$.

Theorem 5.10 (Banon-Barrera) *If $\Psi \in \mathcal{M}(\mathbf{F})$, then for all $F \in \mathbf{F}$*

$$\Psi(F) = \overline{\bigcup\{F \Delta (E, H) : [E, H] \subset \ker(\Psi)\}}.$$

This result is perhaps the sharpest expression of the fact that *the members of the transformation space $\mathcal{M}(\mathbf{F})$ act directly on the shape/size content of binary images to produce their output.* To relate the representation of Theorem 5.10 to the Matheron-Maragos representations, note that if $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$, then

$$[E, H] \subset \ker(\Psi) \iff [E, \Re^n] \subset \ker(\Psi) \iff E \in \ker(\Psi).$$

Also, $F \Delta (E, \Re^n) = F \ominus \check{E}$. Thus, the expression

$$\Psi(F) = \overline{\bigcup\{F \Delta (E, H) : [E, H] \subset \ker(\Psi)\}}$$

becomes $\Psi(F) = \overline{\bigcup\{F \ominus \check{E} : E \in \ker(\Psi)\}}$. Also, if $\Psi \in \mathcal{M}_\downarrow(\mathbf{F})$, then

$$[E, H] \subset \ker(\Psi) \iff [\emptyset, H] \subset \ker(\Psi) \iff H \in \ker(\Psi).$$

Since in this case $F \Delta (\emptyset, H) = \check{H} \ominus F$, we can see that

$$\Psi(F) = \overline{\bigcup\{F \Delta (E, H) : [E, H] \subset \ker(\Psi)\}}$$

becomes $\Psi(F) = \overline{\bigcup\{\check{H} \ominus F : H \in \ker(\Psi)\}}$. Theorem 5.10 thus combines and generalizes Theorems 5.5 and 5.6 to arbitrary $\Psi \in \mathcal{M}(\mathbf{F})$.

Now consider the following definition (Banon and Barrera, 1991).

Definition 5.7 *If $\Psi \in \mathcal{M}(\mathbf{F})$, then a collection* $\mathbf{B}$ *of closed intervals contained in* $\ker(\Psi)$ *is said to satisfy the representation condition for* Ψ *if every closed interval contained in* $\ker(\Psi)$ *is contained in an interval of* $\mathbf{B}$. *The class of maximal closed intervals contained in* $\ker(\Psi)$ *is denoted* $\mathbf{B}(\Psi)$ *and is called the basis of* Ψ.

Remark 5.6 *If* $\Psi \in \mathcal{M}(\mathbf{F})$ *and* $\mathbf{B}$ *satisfies the representation condition for* Ψ, *then* $\mathbf{B}(\Psi) \subset \mathbf{B}$.

Theorem 5.11 (Banon-Barrera) *If* $\Psi \in \mathcal{M}(\mathbf{F})$ *and if* $\mathbf{B}$ *satisfies the representation condition for* Ψ, *then for all* $F \in \mathbf{F}$

$$\Psi(F) = \overline{\bigcup\{F \,\triangle\, (E,H) : [E,H] \in \mathbf{B}\}}.$$

In addition, $\mathbf{B}(\Psi)$ *itself satisfies the representation condition for* Ψ, *so that* $\Psi(\cdot) = \bigcup\{\cdot \,\triangle\, (E,H) : [E,H] \in \mathbf{B}(\Psi)\}$ *is a minimal representation of* Ψ *as a union of bracketing transformations.*

Note that if $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$, then $[E,H] \in \mathbf{B}_\Psi \iff E \in \mathcal{K}_{\min}(\Psi)$, and if $\Psi \in \mathcal{M}_\downarrow(\mathbf{F})$, then $[E,H] \in \mathbf{B}_\Psi \iff H \in \mathcal{K}_{\max}(\Psi)$; thus, the minimal representation of Theorem 5.11 generalizes Corollary 5.2 to $\mathcal{M}(\mathbf{F})$.

Let us now redefine $\cdot \,\triangle\, (E,H)$ on $\mathcal{P}$ for $E \subset H$ (both) also in $\mathcal{P}$. The dual of $\cdot \,\triangle\, (E,H)$ will be denoted $\cdot \,\nabla\, (E,H)$ and is given by

$$A \,\nabla\, (E,H) = \{x \in \Re^n : A \cap (E+x) \neq \emptyset \text{ or } A \cup (H+x) \neq \Re^n\} =$$

$$(A \oplus \check{E}) \cup (A^c \oplus \check{H}^c).$$

With this pair of mappings, Banon and Barrera (1991) have established an algebraic representation of *arbitrary* TI mappings that generalizes Theorem 5.8. Letting $[E,H]_{\mathcal{A}} \equiv \{A \in \mathcal{A} : E \subset A \subset H\}$, where E and H are assumed in $\mathcal{A}$ (and likewise for $\mathcal{A}^*$), they prove the following:

Theorem 5.12 *If* $\mathcal{A} \subset \mathcal{P}$ *is closed under translations and* $\Psi : \mathcal{A} \longrightarrow \mathcal{P}$ *is TI, then for all* $A \in \mathcal{A}$

1. $\Psi(A) = \bigcup\{A \,\triangle\, (E,H) : [E,H]_{\mathcal{A}} \subset \ker(\Psi)\}$.
2. $\Psi(A) = \bigcap\{A \,\nabla\, (E,H) : [E,H]_{\mathcal{A}^*} \subset \ker(\Psi^*)\}$.

This theorem reduces to Theorem 5.8 when $\mathcal{A} = \mathcal{P}$ and Ψ is increasing.

5.5 Representations from Countable Bases

In this final section, I describe a class of representations generated by certain countable bases for the hit-miss topology of $\mathbf{F}$. These bases are defined as follows. Let $\mathcal{B}$ denote a *relatively compact* countable base for the topology of $\Re^n$ such that each open $G = \bigcup\{B \in \mathcal{B} : \overline{B} \subset G\}$. Let $\boldsymbol{\beta}$ denote the class of subsets of $\mathbf{F}$ that have the form $\mathbf{F}^{\overline{S_1}\cup\overline{S_2}\cup\ldots\cup\overline{S_k}}_{B_1,B_2,\ldots,B_m}$, where m and k are arbitrary nonnegative integers and $B_1, \ldots, B_m, S_1, \ldots, S_k$ are arbitrary sets from $\mathcal{B}$. Then the countable collection $\boldsymbol{\beta}$ is a base for $\boldsymbol{\tau}$. With respect to such a $\boldsymbol{\beta}$, we can represent an arbitrary open subset of $\mathbf{F}$ as a countable union of sets of the form $\mathbf{F}^{\overline{S_1}\cup\overline{S_2}\cup\ldots\cup\overline{S_k}}_{B_1,B_2,\ldots,B_m}$. An arbitrary closed subset of $\mathbf{F}$ can therefore be represented as a countable intersection of sets of the complementary form $\mathbf{F}^{B_1}\cup\ldots\cup\mathbf{F}^{B_m}\cup\mathbf{F}_{\overline{S_1}\cup\ldots\cup\overline{S_k}}$. Such a representation of the kernel of a $\Psi \in \mathcal{M}(\mathbf{F})$ will be called an *elementary β-expansion of* $\ker(\Psi)$. Since $\mathbf{F}^{B_i} = \{F \in \mathbf{F} : F \subset B_i^c\}$ is a decreasing set, $\mathbf{F}^{B_i}$ is the kernel of a decreasing transformation in $\mathcal{M}(\mathbf{F})$. It therefore follows from Proposition 5.1 that

$$\ker(\varphi) = \mathbf{F}^{B_1} \cup \ldots \cup \mathbf{F}^{B_m} \tag{1}$$

is also the kernel of a decreasing $\varphi \in \mathcal{M}(\mathbf{F})$. Also,

$$\ker(\vartheta) = \mathbf{F}_{\overline{S_1}\cup\ldots\cup\overline{S_k}} \tag{2}$$

is the kernel of an increasing $\vartheta \in \mathcal{M}(\mathbf{F})$ because $\mathbf{F}_{\overline{S_1}\cup\ldots\cup\overline{S_k}}$ is plainly an increasing set. I accordingly adopt the following terminology (and denote $\mathcal{M}(\mathbf{F})$ more briefly as $\mathcal{M}$):

1. A $\varphi \in \mathcal{M}$ with a kernel of form (1) will be called an *elementary decreasing $\mathcal{M}$-transformation.*
2. A $\vartheta \in \mathcal{M}$ with a kernel of form (2) will be called an *elementary increasing $\mathcal{M}$-transformation.*

Again applying Proposition 5.1, we can now see that

$$\ker(\phi) = \mathbf{F}^{B_1} \cup \ldots \cup \mathbf{F}^{B_m} \cup \mathbf{F}_{\overline{S_1}\cup\ldots\cup\overline{S_k}} \tag{3}$$

is the kernel of a $\phi \in \mathcal{M}$ that has the form $\phi = \varphi\cup\vartheta$, where φ and ϑ are elementary decreasing and increasing $\mathcal{M}$-transformations, respectively.

Definition 5.8 *A $\phi \in \mathcal{M}$ with a kernel of form (3) will be called an elementary $\mathcal{M}$-transformation.*

In view of Proposition 5.2, we may summarize as follows:

Theorem 5.13 *Let $\Psi \in \mathcal{M}$ and let $\bigcap_i \mathbf{F}^{B_1^{(i)}} \cup \ldots \cup \mathbf{F}^{B_{m_i}^{(i)}} \cup \mathbf{F}_{\overline{S_1^{(i)}} \cup \ldots \cup \overline{S_{k_i}^{(i)}}}$ be an elementary β-expansion of* $\ker(\Psi)$. *Then $\Psi = \bigcap_i \phi_i$ where $\{\phi_i\}$ is an at most denumerable set of elementary $\mathcal{M}$-transformations such that* $\ker(\phi_i) = \mathbf{F}^{B_1^{(i)}} \cup \ldots \cup \mathbf{F}^{B_{m_i}^{(i)}} \cup \mathbf{F}_{\overline{S_1^{(i)}} \cup \ldots \cup \overline{S_{k_i}^{(i)}}}$. *Thus $\phi_i = \varphi_i \cup \vartheta_i$ where the φ_i and ϑ_i are, respectively, elementary decreasing and increasing members of $\mathcal{M}$ with kernels* $\ker(\varphi_i) = \mathbf{F}^{B_1^{(i)}} \cup \ldots \cup \mathbf{F}^{B_{m_i}^{(i)}}$ *and* $\ker(\vartheta_i) = \mathbf{F}_{\overline{S_1^{(i)}} \cup \ldots \cup \overline{S_{k_i}^{(i)}}}$.

I call the representation $\Psi = \bigcap_i \phi_i$ of this theorem an ***elementary β-expansion of Ψ***. For an elementary increasing ϑ, the minimal basis kernel is simply the class of all one-point subsets of

$$K = \overline{S_1} \cup \overline{S_2} \cup \ldots \cup \overline{S_k}.$$

According to Corollary 5.2, then, the action of the corresponding mapping ϑ on a set F is given by

$$\vartheta(F) = \bigcup_{x \in K} (F + x) = F \oplus K.$$

For an elementary decreasing φ, $\mathcal{K}_{\max}(\varphi)$ is contained in the finite collection $\{B_1^c, \ldots, B_m^c\}$, and the action of the corresponding φ is

$$\varphi(F) = \bigcup_{j=1}^{m} \check{B}_j^c \ominus F.$$

Theorem 5.14 follows from these observations.

Theorem 5.14 *If $\Psi \in \mathcal{M}$ and $F \in \mathbf{F}$, then $\Psi(F) = \bigcap_i \phi_i(F)$, where the elementary $\mathcal{M}$-transformations ϕ_i are given by*

$$\phi_i(F) = (F \oplus K_i) \cup \bigcup_{j=1}^{m_i} \left(E_j^{(i)} \ominus F\right)$$

where $K_i = \overline{S_1^{(i)}} \cup \ldots \cup \overline{S_{k_i}^{(i)}}$ and $\check{E}_j^{(i)} = \left[B_j^{(i)}\right]^c$.

This representation of the $\mathcal{M}$-transformations is useful in extending the present theory to ERV USC functions.

6 Conclusion

My concern in this report has been with closed-set morphology, with giving a detailed summary exposition of the algebraic and topological theory of the morphological transformations of euclidean sets. In the next stage of the theory, the more general ERV USC functions must be used to erect a morphology theory of greyscale images on the foundation of closed-set morphology.

This transition from closed-set morphology in real n-dimensional space to ERV USC function morphology in n independent real variables is often understood in terms of the *method of sections* or *threshold decomposition*. With the formal justification provided by *Serra's theorem* (Serra, 1982), this method applies an $\mathcal{M}$-transformation of $\mathbf{F}(\Re^n)$ to the horizontal cross sections of the function to obtain a "stack" of at most n-dimensional closed sets whose "top surface" defines the graph of the transformed function under suitable conditions. Serra's theorem provides the necessary and sufficient conditions for such a "stack of sets" to define an ERV USC function. Besides being USC and TI, the mapping used to transform the cross sections must be increasing.

A more general but indirect way to develop a function morphology is to define the *morphological function transforms* by means of the *umbrae* of the graphs of the functions (Matheron, 1969; Sternberg, 1979). This method takes advantage of the fact that the class of umbrae of the ERV USC functions defined on $\Re^n$ is a topological subspace of $\mathbf{F}(\Re^n \times [-\infty, \infty])$ when the latter has Matheron's hit-miss topology. Because of this, the $\mathcal{M}$-transformations of this umbra subspace can be used to indirectly define the morphological transforms of the ERV USC functions. Moreover, the class of order-preserving $\mathcal{M}$-transforms (the ones that preserve the $\leq$ relation and therefore correspond to increasing umbra transformations) turns out to be identical to the class of transforms defined by the threshold decomposition method. In other words, the conventional threshold decomposition method results in a theory of morphological filters of greyscale images. The general class of $\mathcal{M}$-transforms produced by the umbra method consists of a great deal more than morphological filters, however. In pursuing the umbra method, one can establish many function-morphological analogs of the set-morphological results summarized in this report. All this will be rigorously developed in a companion report.

Appendix. General Topology

This appendix gives a summary of background material in general topology. It mainly follows the treatise of Kelley (1955).

Definition A.1 *Let X be a set and let τ be a collection of subsets of X. We call the pair (X, τ) a topological space if $X \in \tau$, $\emptyset \in \tau$, and*

1. *If $\{U_1, \dots, U_k\}$ is a finite family of sets in τ, then $\bigcap_{i=1}^{k} U_i \in \tau$.*
2. *If $\{U_\alpha\}$ is any family of sets in τ, then $\bigcup_\alpha U_\alpha \in \tau$.*

Conditions 1 and 2 are often stated as τ *is closed under finite intersections* and τ *is closed under arbitrary unions.* τ itself is called a *topology* for (on, in, or of) X, and its members are called the *open subsets of* X. The elements of X are usually called *points.* When no confusion is likely, it is customary to say "the topological space X" rather than "the topological space (X, τ)" and "open set" rather than "open subset of X."

It is readily seen that the collections $\{X, \emptyset\}$ and $\mathcal{P}(X)$ (i.e., the collection of all subsets of X) are topologies for X. We call $\{X, \emptyset\}$ the *trivial* topology and $\mathcal{P}(X)$ the *discrete* topology (of X). These are the extreme cases for any given set.

Definition A.2 *If τ and τ' are topologies on X and $\tau \supset \tau'$, we say that τ is larger (stronger, finer) than τ' or that τ' is smaller (weaker, coarser) than τ.*

Thus the trivial topology is the smallest one that X can have, and the discrete topology is the largest.

A.1 Generated Topologies

Remark A.1 *If $\{\tau_\alpha\}$ is a collection of topologies on X, then $\cap_\alpha \tau_\alpha$ is a topology on X but $\cup_\alpha \tau_\alpha$ need not be. In fact,*

1. *$\cap_\alpha \tau_\alpha$ is the unique largest topology τ on X such that $\tau \subset \tau_\alpha \; \forall \, \alpha$.*
2. *There is a unique smallest topology τ' on X such that $\tau' \supset \bigcup_\alpha \tau_\alpha$.*

Remark A.2 *If X is a set and $\mathcal{A}$ is a collection of subsets of X, then the smallest topology $\tau(\mathcal{A})$ on X that contains $\mathcal{A}$ exists and equals the intersection of all the topologies on X that contain $\mathcal{A}$.*

We call $\tau(\mathcal{A})$ the topology *generated by $\mathcal{A}$ in X*. The topology τ' of Remark A.1 is accordingly called the topology generated by $\cup_\alpha \tau_\alpha$.

A.2 Neighborhoods

Definition A.3 *If x is a point in a topological space X, then a subset of X containing an open set that contains x is called a neighborhood of x. Thus an open neighborhood of x is an open set U with $x \in U$.*

The neighborhood concept turns out to be very useful. For instance, it leads to alternative characterizations of the open sets (Thm. A.1) and helps us analyze the structure of the so-called *closed sets*.

Theorem A.1 *If X is a topological space and $G \subset X$, then the following are equivalent assertions.*

1. *G is open.*
2. *G contains a neighborhood of each of its points.*
3. *G is a neighborhood of each of its points.*

A.3 Closed Sets

Definition A.4 *For the following, let X be a topological space.*

1. *A subset F of X is said to be "closed" if its complement is open.*
2. *A point $x \in X$ is called a "cluster point" of a subset A of X if every open neighborhood of x includes a point of $A \setminus \{x\}$.*
3. *A point $x \in X$ is called a "point of closure" of a subset A of X if every open neighborhood of x includes a point of A.*
4. *The set $\overline{A}$ of all points of closure of A is called the "closure of A."*

Remark A.3 *Let A be a subset of a topological space X.*

1. *A cluster point of A is a point of closure of A, but a point of closure of A need not be a cluster point of A.*
2. *If ΔA denotes the cluster points of A, then $\overline{A} = A \cup \Delta A$.*

Theorem A.2 *Let X be a topological space. Then,*

1. *X and $\emptyset$ are closed.*
2. *The intersection of any family of closed sets is closed.*
3. *The union of any finite family of closed sets is closed.*
4. *A subset F of X is closed if and only if $F = \overline{F}$.*

5. $\overline{A}$ *is closed for any subset* A *of* X.
6. *For each* $A \subset X$, $\overline{A}$ *is the intersection of all closed supsets of* A.

The closed subsets of a topological space are uniquely characterized by statements 1, 2, and 3 of Theorem A.2.

Theorem A.3 *Any collection* $\mathcal{A}$ *of subsets of a set* X *that includes* X *and* $\emptyset$ *and is closed under arbitrary intersections and finite unions defines the unique topology on* X *(the class of complements of the sets in* $\mathcal{A}$*) for which* $\mathcal{A}$ *is the class of closed sets.*

A.4 Closure Operators

Topologies can also be defined by means of *closure operators.*

Definition A.5 *A closure operator on a set* X *is a function that assigns to each* $A \subset X$ *a subset* $\tilde{A}$ *of* X *such that the following hold:*

1. $\tilde{\emptyset} = \emptyset$.
2. $A \subset \tilde{A}$ *for each* $A \subset X$.
3. $\tilde{\tilde{A}} = \tilde{A}$ *for each* $A \subset X$.
4. *For all subsets* A *and* B *of* X, $\widetilde{A \cup B} = \tilde{A} \cup \tilde{B}$.

Properties 1 to 4 are called the *Kuratowski closure axioms.* Closure operators are indeed sometimes called Kuratowski closure operators.

Theorem A.4 *Let* $\sim$ *be a closure operator on a set* X, *let* $\mathcal{A}$ *be the family of subsets* A *of* X *such that* $\tilde{A} = A$, *and let* τ *be the family of complements of the members of* $\mathcal{A}$. *Then* τ *is a topology on* X *and* $\tilde{A}$ *is the* τ*-closure of* A *for each subset* A *of* X.

A.5 Closed-Set Bases

Definition A.6 *Let* X *be a topological space.*

1. *A collection* $\mathcal{B}$ *of closed subsets of* X *is called a closed-set base for the topology of* X *if every closed subset of* X *is an intersection of sets in* $\mathcal{B}$.
2. *A collection* $\mathcal{B}$ *of closed subsets of* X *is called a closed-set subbase for the topology of* X *if the finite unions of sets in* $\mathcal{B}$ *form a closed-set base for the topology of* X.

A.6 Convergence

The concept of *convergence* in a topological space is one of the most important tools for the analysis of topological phenomena.

A.6.1 Sequences

Definition A.7 *Let X be a topological space and let* $\mathbf{N}$ *denote the set of natural numbers. A function $S : i \longmapsto x_i$ defined on* $\mathbf{N}$ *with values in X is called a sequence in X. We denote such sequences by $\{x_i : i \in \mathbf{N}\}$ or more simply by $\{x_i\}$.*

Definition A.8 *A sequence $\{x_i\}$ in X is said to converge to $x \in X$ if every open neighborhood of x contains all but at most a finite number of the x_i. We symbolize this situation by writing either $x_i \to x$ or*

$$\lim_{i \to \infty} x_i = x$$

(sometimes simply $\lim x_i = x$*). We call x a limit of the sequence $\{x_i\}$.*

Remark A.4 *If X is a topological space and $x \in X$, then the sequence $\{x_i\}$ in X with $x_i = x$ for all i converges to x.*

Definition A.9 *Let X be a topological space and let $\{x_i\}$ be a sequence in X. We say that $x \in X$ is a limit point of $\{x_i\}$ if every open neighborhood of x contains infinitely many of the x_i.*

Remark A.5 *Let X be a topological space and let $\{x_i\}$ be a sequence in X. Then $x \in X$ is a limit point of $\{x_i\}$ if and only if for each open neighborhood U of x and each natural number N there is a natural number $n \geq N$ such that $x_n \in U$.*

Remark A.6 *Let $\{x_i\}$ be a sequence in a topological space X.*

1. *If x is a cluster point of $\{x_i\}$, then x is a limit point of $\{x_i\}$.*
2. *If x is a limit point of $\{x_i\}$, then x is a point of closure of $\{x_i\}$.*
3. *The converses of statements* 1 *and* 2 *are generally false.*

A.6.2 Double Sequences

Definition A.10 *If X is a set, then an X-valued function $\mathcal{R}$ defined for all ordered pairs (i, k) of natural numbers is called a double sequence in X. We generally write $\mathcal{R}(i, k) = x_{i,k}$ and denote the double sequence by $\{x_{i,k} : i \in \mathrm{N}$ and $k \in \mathrm{N}\}$ or more simply by $\{x_{i,k}\}$.*

Theorem A.5 is called the *double sequence theorem.*

Theorem A.5 *Let $\{x_{i,k} : i \in \mathrm{N}$ and $k \in \mathrm{N}\}$ be a double sequence in X. If for each fixed i the sequence $\{x_{i,k} : k \in \mathrm{N}\}$ converges to $x_i \in X$, and if the sequence $\{x_i\}$ converges to $x \in X$, then $x_{i,f(i)} \to x$ for some N-valued function f defined on N.*

A.6.3 Subsequences

Definition A.11 *Let X be a topological space, let $\mathcal{S} : i \longmapsto x_i$ be a sequence $\{x_i\}$ in X, and let $\sigma : k \longmapsto i_k$ be an N-valued function defined on N such that $k > k' \Longrightarrow i_k > i_{k'}$ (i.e., σ is strictly increasing). Then the function $\mathcal{S} \circ \sigma$ defines the sequence $\{\mathcal{S} \circ \sigma(k) : k \in \mathrm{N}\} \equiv \{x_{i_k}\}$, which we call a subsequence of our original sequence $\{x_i\}$.*

Remark A.7 *Let $\{x_i\}$ be a sequence in a topological space X and let x be a point in X. If $\{x_i\}$ has a subsequence $\{x_{i_k}\}$ such that $x_{i_k} \to x$, then x is a limit point of $\{x_i\}$. The converse is generally false.*

Theorem A.6 *Let X be a topological space.*

1. *A sequence $\{x_i\}$ in X converges to $x \in X$ if and only if every subsequence of $\{x_i\}$ converges to x.*
2. *If $\{x_i\}$ is a sequence in X and $x_i \not\to x \in X$, then $\exists$ a subsequence $\{x_{i_k}\}$ of $\{x_i\}$ none of whose subsequences converge to x.*

A.6.4 Directed Sets and Nets

Sequence convergence is an inadequate tool in completely general topological spaces because of the following shortcomings:

1. A sequence may converge to more than one point.
2. Distinct topologies can have the same convergent sequences and limits; that is, sequence convergence can fail to fix the topology.

The more general *Moore-Smith* theory of *net convergence* overcomes the second shortcoming.

Definition A.12 *Let $\mathcal{D}$ be a nonempty set of elements $\alpha, \beta, \gamma, \ldots$ in which a reflexive and transitive binary relation $\trianglerighteq$ is defined. If $(\mathcal{D}, \trianglerighteq)$ also has the Moore-Smith property (namely, for each α and β in $\mathcal{D}$ there exists a $\gamma \in \mathcal{D}$ such that $\gamma \trianglerighteq \alpha$ and $\gamma \trianglerighteq \beta$), then $(\mathcal{D}, \trianglerighteq)$ is called a directed set and $\mathcal{D}$ is said to be directed by $\trianglerighteq$.*

Definition A.13 *A subset $\mathcal{D}'$ of a directed set $\mathcal{D}$ is called cofinal if for each $\alpha \in \mathcal{D}$ there exists a $\sigma \in \mathcal{D}'$ such that $\sigma \trianglerighteq \alpha$.*

Remark A.8 *Let $(\mathcal{D}, \trianglerighteq)$ be a directed set.*

1. *If $\mathcal{D}'$ is a subset of $\mathcal{D}$ that is not cofinal, then the complement of $\mathcal{D}'$ in $\mathcal{D}$ is cofinal.*
2. *If $\mathcal{D}'$ is a cofinal subset of $\mathcal{D}$, then $(\mathcal{D}', \trianglerighteq)$ is a directed set.*

Definition A.14 *If X is a set and $(\mathcal{D}, \trianglerighteq)$ is a directed set, then a function $\mathcal{N} : \alpha \longmapsto x_\alpha$ defined on $\mathcal{D}$ with values in X is called a net in X and is denoted $\{x_\alpha : \alpha \in \mathcal{D}\}$ or more simply $\{x_\alpha\}$. We also use the notation $(\mathcal{N}|\mathcal{D}, \trianglerighteq)$, or more simply $\mathcal{N}|\mathcal{D}$, or more simply still $\mathcal{N}$.*

Remark A.9 *The set of natural numbers together with the relation $\geq$ is a directed set; hence a sequence is an example of a net.*

A.6.5 Subnets

Definition A.15 *Let $(\mathcal{D}, \trianglerighteq)$ and $(\mathcal{D}', \trianglerighteq')$ be directed sets and let $\mathcal{N}|\mathcal{D}$ and $\mathcal{N}'|\mathcal{D}'$ be nets in X. We say that $\mathcal{N}'|\mathcal{D}'$ is a subnet of $\mathcal{N}|\mathcal{D}$ if there is a function $\Sigma : \mathcal{D}' \longrightarrow \mathcal{D}$ that satisfies the following:*

1. $\mathcal{N}' = \mathcal{N} \circ \Sigma$.
2. $\forall\, \alpha \in \mathcal{D}\ \exists\, \alpha' \in \mathcal{D}'$ *such that* $\beta' \trianglerighteq' \alpha' \Longrightarrow \Sigma(\beta') \trianglerighteq \alpha$.

Remark A.10 *Suppose that $\mathcal{N}$ is a net in X and that $\mathcal{N}'$ is a subnet of $\mathcal{N}$. If $\mathcal{N}''$ is a subnet of $\mathcal{N}'$, then $\mathcal{N}''$ is a subnet of $\mathcal{N}$.*

Remark A.11 *Let $(\mathcal{D}, \trianglerighteq)$ be a directed set and let $\mathcal{D}'$ be a cofinal subset of $\mathcal{D}$. If $\mathcal{N}|\mathcal{D}$ is a net in X and if $\mathcal{I}$ denotes the identity mapping of $\mathcal{D}'$ to $\mathcal{D}$, then $(\mathcal{N} \circ \mathcal{I})|\mathcal{D}'$ is a subnet of $\mathcal{N}|\mathcal{D}$.*

Thus, in particular, a subsequence of a sequence is a subnet of that sequence considered as a net. On the other hand, sequences may have subnets that are not subsequences.

A.6.6 Moore-Smith Convergence

Definition A.16 *Let X be a topological space, let $\{x_\alpha\}$ be a net in X, and let x be a point of X. We say that $\{x_\alpha\}$ converges to x if for each open neighborhood U of x there is a $\beta(U)$ such that*

$$\alpha \trianglerighteq \beta(U) \Longrightarrow x_\alpha \in U.$$

In this case we say equivalently that x is a limit of the net $\{x_\alpha\}$.

In general a net can converge to more than one point. This is the shortcoming of sequence convergence that net convergence retains.

Theorem A.7 *Let X be a topological space.*

1. *If $\mathcal{D}$ is a directed set and $x \in X$, then the net $\{x_\alpha : \alpha \in \mathcal{D}\}$ with $x_\alpha = x$ for all $\alpha \in \mathcal{D}$ converges to x.*
2. *A net $\mathcal{N}$ in X converges to $x \in X$ if and only if every subnet of $\mathcal{N}$ converges to x.*
3. *If $x \in X$ and $\mathcal{N}$ is a net in X that does not converge to x, then there is a subnet of $\mathcal{N}$ none of whose subnets converge to x.*

A.6.7 Product Directed Sets

Let $(\mathcal{D}, \trianglerighteq)$ and $(\mathcal{D}', \trianglerighteq')$ be any directed sets and let $\mathcal{D} \times \mathcal{D}'$ denote the set of ordered pairs (α, α') where $\alpha \in \mathcal{D}$ and $\alpha' \in \mathcal{D}'$. We define a reflexive and transitive binary relation $\gg$ in $\mathcal{D} \times \mathcal{D}'$ as follows:

$$(\alpha, \alpha') \gg (\beta, \beta') \Longleftrightarrow \alpha \trianglerighteq \beta \text{ and } \alpha' \trianglerighteq' \beta'.$$

$\gg$ has the Moore-Smith property and thus directs $\mathcal{D} \times \mathcal{D}'$. We call $(\mathcal{D} \times \mathcal{D}', \gg)$ the *product directed set.*

Suppose now that $\{(\mathcal{D}_\alpha, \trianglerighteq_\alpha) : \alpha \in \mathcal{A}\}$ is a family of directed sets indexed by the elements of an arbitrary set $\mathcal{A}$. The product set $\prod_{\alpha \in \mathcal{A}} \mathcal{D}_\alpha$ is then defined to be the class of all functions f defined on $\mathcal{A}$ such that $f(\alpha) \in \mathcal{D}_\alpha$. The *product directed set*

$$\left(\prod_{\alpha \in \mathcal{A}} \mathcal{D}_\alpha, \gg \right)$$

is defined as follows: If f and g are elements of $\prod_\alpha \mathcal{D}_\alpha$, then $f \gg g \Longleftrightarrow f(\alpha) \trianglerighteq_\alpha g(\alpha)$ for all $\alpha \in \mathcal{A}$.

A.6.8 Iterated Limits Theorem

Theorem A.8 is a generalization of the double sequence theorem to nets.

Theorem A.8 *Let $\mathcal{D}$ be a directed set and let $\{\mathcal{D}_\alpha : \alpha \in \mathcal{D}\}$ be a family of directed sets. Let $\widehat{\mathcal{D}}$ denote the product directed set*

$$\mathcal{D} \times \prod_{\alpha \in \mathcal{D}} \mathcal{D}_\alpha$$

and let $\mathcal{H}(\beta, f) = (\beta, f(\beta))$ for all $(\beta, f) \in \widehat{\mathcal{D}}$. Suppose that $\mathcal{N}(\beta, \lambda)$ lies in a topological space X for each $\beta \in \mathcal{D}$ and $\lambda \in \mathcal{D}_\beta$, so that $\{\mathcal{N}(\beta, \lambda) : \lambda \in \mathcal{D}_\beta\}$ is a net in X for each fixed β. Assume (for each fixed β) that this net converges to $x_\beta \in X$, and that $\{x_\beta : \beta \in \mathcal{D}\}$ converges to $x \in X$. Given all this, we conclude that the net $(\mathcal{N} \circ \mathcal{H})|\widehat{\mathcal{D}}$ converges to x.

A.6.9 Limit Points of Nets

Definition A.17 *If X is a topological space and $\mathcal{N} = \{x_\alpha : \alpha \in \mathcal{D}\}$ is a net in X, then a point $x \in X$ is called a limit point of $\mathcal{N}$ if for each open neighborhood U of x and each $\alpha \in \mathcal{D}$, there is a $\beta \in \mathcal{D}$ such that $\beta \trianglerighteq \alpha$ and $x_\beta \in U$.*

Theorem A.9 *A point x in a topological space X is a limit point of a net $\mathcal{N}$ in X if and only if there is a subnet of $\mathcal{N}$ that converges to x* (compare with Rmk. A.7).

A.6.10 Nets and Closed Sets

Theorem A.10 *Let X be a topological space. Then,*

1. *A point x is a cluster point of $A \subset X$ if and only if there is a net in $A \setminus \{x\}$ that converges to x.*
2. *A point x belongs to the closure of $A \subset X$ if and only if there is a net in A that converges to x.*
3. *A subset F of X is closed if and only if the limits of all convergent nets in F lie in F.*

Another useful result that characterizes the limit points of nets in terms of closure is as follows:

Theorem A.11 *Let $\mathcal{N}|\mathcal{D}$ be a net in a topological space X and for each $\beta \in \mathcal{D}$ let $A_\beta \equiv \{\mathcal{N}(\alpha) : \alpha \trianglerighteq \beta\}$. Then $x \in X$ is a limit point of $\mathcal{N}|\mathcal{D}$ if and only if x is in the closure of each A_β, i.e., if and only if*

$$x \in \bigcap_{\beta \in \mathcal{D}} \overline{A_\beta}.$$

The third assertion in Theorem A.10 shows that the class of convergent nets in a topological space, together with their corresponding limits, determines the class of closed sets and by complementation the topology; that is, no other topology has precisely the same convergent nets and limits. Net convergence is accordingly definitive of the topology of general topological spaces.

A.7 Convergence Classes

We can clarify the way in which net convergence is definitive of topology by considering the notion of a *convergence class.*

Definition A.18 *Let X be a set and let $\mathcal{C}$ be a class consisting of pairs $(\mathcal{N}, x)$, where $\mathcal{N}$ is a net in X and $x \in X$. We say that $\mathcal{C}$ is a convergence class for X if the following hold:*

1. *If $x \in X$, $\mathcal{D}$ is a directed set, and $\mathcal{N}$ is the net $\{x_\alpha : \alpha \in \mathcal{D}\}$ with $x_\alpha = x$ for all $\alpha \in \mathcal{D}$, then $(\mathcal{N}, x) \in \mathcal{C}$.*
2. *If $(\mathcal{N}, x) \in \mathcal{C}$, then $(\mathcal{N}', x) \in \mathcal{C}$ for all subnets $\mathcal{N}'$ of $\mathcal{N}$.*
3. *If $\mathcal{N}$ is a net in X, $x \in X$, and $(\mathcal{N}, x) \notin \mathcal{C}$, then $\mathcal{N}$ has a subnet $\mathcal{N}'$ such that $\mathcal{N}''$ is a subnet of $\mathcal{N}' \Longrightarrow (\mathcal{N}'', x) \notin \mathcal{C}$.*
4. *Let $\mathcal{D}$ be a directed set and let $\{\mathcal{D}_\alpha : \alpha \in \mathcal{D}\}$ be a family of directed sets. Let $\widehat{\mathcal{D}}$ denote the product directed set*

$$\mathcal{D} \times \prod_{\alpha \in \mathcal{D}} \mathcal{D}_\alpha$$

and let $\mathcal{H}(\beta, f) = (\beta, f(\beta))$ for all $(\beta, f) \in \widehat{\mathcal{D}}$. Suppose that $\mathcal{N}(\beta, \lambda)$ lies in X for each $\beta \in \mathcal{D}$ and $\lambda \in \mathcal{D}_\beta$. If

$$(\{\mathcal{N}(\beta, \lambda) : \lambda \in \mathcal{D}_\beta\}, x_\beta) \in \mathcal{C}$$

for each $\beta \in \mathcal{D}$ and if $(\{x_\beta\}, x) \in \mathcal{C}$, then $(\mathcal{N} \circ \mathcal{H}, x) \in \mathcal{C}$.

Theorem A.12 *Let C be a convergence class for X and for each subset A of X, let $\tilde{A}$ be the set*

$$\{x \in X : (\mathcal{N}, x) \in C \text{ for some net } \mathcal{N} \text{ in } A\}.$$

Then $\sim$ is a closure operator on X and $(\mathcal{N}, x) \in C \iff \mathcal{N}$ converges to x in the topology induced by $\sim$.

Let $\mathbf{T}(X)$ denote the set of all topologies that X admits and let $\mathbf{C}(X)$ denote the set of X's convergence classes. Then Theorem A.12 gives us a mapping $\mathcal{T} : \mathbf{C}(X) \longrightarrow \mathbf{T}(X)$, where $\mathcal{T}(C)$ is the topology induced on X by the closure operator $\sim$. We call $\mathcal{T}(C)$ the *topology induced by* (the convergence class) C *on* X. Now each $\tau \in \mathbf{T}(X)$ has a corresponding convergence class C_τ, namely, the class of τ-convergent nets together with their limits; moreover, $\mathcal{T}(C_\tau) = \tau$ for all $\tau \in \mathbf{T}(X)$. We may therefore conclude that $\mathcal{T}$ is one-to-one and onto $\mathbf{T}(X)$. This result is a precise statement of the equivalence of net convergence and topology.

Remark A.12 *Let C_1 and C_2 be convergence classes for X and let τ_1 and τ_2 denote the corresponding induced topologies. Then,*

1. *$C_1 \subset C_2$ if and only if $\tau_2 \subset \tau_1$.*
2. *$C_1 \cap C_2$ is a convergence class for X whose induced topology is generated by $\tau_1 \cup \tau_2$.*
3. *The convergence class inducing $\tau_1 \cap \tau_2$ is the smallest one containing $C_1 \cup C_2$.*

A.8 Open-Set Bases

Definition A.19 *Let X be a topological space.*

1. *By a base for the topology of X, we mean a collection $\mathcal{B}$ of open sets such that every open set is a union of sets in $\mathcal{B}$.*
2. *By a subbase for the topology of X, we mean a collection of open sets whose finite intersections form a base for the topology of X.*
3. *By a local base at $x \in X$, we mean a collection $\mathcal{B}_x$ of open neighborhoods of x such that every open neighborhood of x contains a member of $\mathcal{B}_x$.*

A.9 Special Topological Spaces

Two important species of topological space are related to the countability properties of local bases and bases.

Definition A.20 *Let X be a topological space.*

1. *If X has a countable local base at each $x \in X$, we call X a first countable space.*
2. *If X has a countable base, we call X a second countable space.*

There are also a number of important species of topological space that are related to the *separation properties* that a space may have.

Definition A.21 *Let X be a topological space.*

1. *X is a T_0-space if for every pair of distinct points in X there is a neighborhood of one of them in which the other does not lie.*
2. *X is a T_1-space if for every pair of distinct points x and y in X there is a neighborhood of x in which y does not lie and a neighborhood of y in which x does not lie.*
3. *If every pair of distinct points in X have disjoint neighborhoods, we say that X is a Hausdorff space or a T_2-space.*
4. *X is regular if $x \in X$, F is closed, and $x \notin F$ together imply that there are disjoint open sets U and V such that $x \in U$ and $F \subset V$. A regular T_1-space is called a T_3-space.*
5. *X is normal if for every pair of disjoint closed subsets E and F of X there are disjoint open sets U and V such that $U \supset E$ and $V \supset F$. A normal T_1-space is called a T_4-space.*

Theorem A.13 *We have the following equivalent characterizations.*

1. *A topological space X is Hausdorff if and only if each net in X has at most one limit.*
2. *A topological space X is a T_1-space if and only if every one-point (singleton) subset of X is closed.*
3. *A topological space X is a T_0-space if and only if $\overline{\{x\}} = \overline{\{y\}} \Longrightarrow x = y$ where x and y are points in X.*

Remark A.13 *Let X be a topological space.*

1. *X is second countable $\Longrightarrow$ X is first countable.*
2. *X is a T_4-space $\Longrightarrow$ X is a T_3-space $\Longrightarrow$ X is Hausdorff $\Longrightarrow$ X is a T_1-space $\Longrightarrow$ X is a T_0-space.*

By Theorem A.13 (1) we can write $\lim_\alpha x_\alpha$ for the unique limit of a convergent net $\{x_\alpha\}$ in a Hausdorff space. If we confine our attention to first countable spaces, we find the following:

Remark A.14 *At a point x in a first countable space, there is a countable local base $\mathcal{B}_x = \{B_i : i \in \mathrm{N}\}$ such that $B_{i+1} \subset B_i \ \forall\, i \in \mathrm{N}$.*

Proposition A.1 *Let X be a first countable topological space and let $x \in X$ be a limit of an arbitrary convergent net $\mathcal{N}|\mathcal{D}$ in X. Then there is a sequence $\{x_i : i \in \mathrm{N}\} \subset \{\mathcal{N}(\alpha) : \alpha \in \mathcal{D}\}$ that converges to x.*

In any topological space, F is closed $\Longleftrightarrow$ every convergent net in F has its limit in F $\Longrightarrow$ every convergent sequence in F has its limit in F. In first countable spaces, then, the last implication becomes an equivalence; hence *sequence convergence alone is definitive of the topology in first countable spaces.* Theorem A.14 summarizes the main properties of first countable spaces and should be compared with Theorem A.10.

Theorem A.14 *Let X be a first countable topological space.*

1. *X is Hausdorff $\Longleftrightarrow$ each sequence in X has at most one limit.*
2. *If $A \subset X$, then x is a cluster point of A $\Longleftrightarrow$ there is a sequence in $A \setminus \{x\}$ that converges to x.*
3. *$U \subset X$ is open $\Longleftrightarrow$ for each sequence $\{x_i\}$ that converges to a point of U all but at most finitely many of the x_i lie in U.*
4. *A point $x \in X$ is a limit point of a sequence $\{x_i\}$ in X $\Longleftrightarrow$ $\{x_i\}$ has a subsequence that converges to x.*
5. *If $A \subset X$, then $x \in \overline{A}$ if and only if there is a sequence in A that converges to x.*
6. *A subset F of X is closed if and only if every convergent sequence in F has its limit in F.*

Thus sequence convergence is a fully adequate tool (i.e., net convergence need not be considered) in first countable Hausdorff spaces.

A.9.1 Metric Spaces

Definition A.22 *Let X be a set of elements $x, y, z, \ldots$ and let $d(x,y)$ be a real valued function defined on $X \times X$ such that the following hold for all $x, y, z \in X$.*

1. $0 \leq d(x,y) < \infty$.
2. $d(x,y) = 0$ *if and only if* $x = y$.
3. $d(x,y) = d(y,x)$.
4. $d(x,z) \leq d(x,y) + d(y,z)$.

d is called a metric on X and the pair (X,d) is called a metric space.

Remark A.15 *Let d be a metric on X, for $x \in X$ and $\epsilon > 0$ let*

$$B_\epsilon(x) \equiv \{x' \in X : d(x,x') < \epsilon\},$$

and let $\mathcal{B} \equiv \{B_\epsilon(x) : x \in X \text{ and } 0 < \epsilon < \infty\}$. Then $\mathcal{B}$ is a base for a topology on X called the metric topology of (X,d).

A.9.2 Metrizable Topological Spaces

Definition A.23 *A topological space (X,τ) is said to be metrizable if there exists a metric d on X for which τ is the metric topology of (X,d). Such a metric is said to be compatible with the given topology.*

If X is a topological space, if $A \subset B \subset X$, and if $\overline{A} = B$, then we say that A is *dense in* B.

Theorem A.15 *If X is a T_1-space, then the following are equivalent.*

1. *X is regular and second countable.*
2. *X is metrizable and has a countable dense subset.*

A.10 Continuity

Definition A.24 *Let X and Y be topological spaces and let f be a function defined on X with values in Y.*

1. *We say that f is continuous if $f^{-1}(U) \equiv \{x \in X : f(x) \in U\}$ is an open subset of X for all open subsets U of Y. We call $f^{-1}(U)$ the inverse image of U under f.*
2. *If f maps X one-to-one and onto Y and if f and f^{-1} are continuous, we call f a homeomorphism and say that X and Y are homeomorphic or topologically equivalent.*

Remark A.16 *If f is a homeomorphism, then so is the inverse function f^{-1}. If X and Y are homeomorphic spaces, then as abstract topological spaces they are indistinguishable.*

Theorem A.16 *Let $f : X \longrightarrow Y$, where X and Y are topological spaces. Then the following are equivalent.*

1. *f is continuous.*
2. *The inverse image under f of each closed subset of Y is a closed subset of X.*
3. *The inverse image under f of each member of a subbase for the topology of Y is an open subset of X.*
4. *For each net $\mathcal{N}$ in X that converges to an $x \in X$, the composition net $f \circ \mathcal{N}$ in Y converges to $f(x)$.*
5. *$f(\overline{A}) \subset \overline{f(A)}$ for each $A \subset X$.*
6. *$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for each $B \subset Y$.*

Definition A.25 *Let $f : X \longrightarrow Y$, where X and Y are topological spaces. We say that f is continuous at $x \in X$ if the inverse image under f of every neighborhood of $f(x)$ is a neighborhood of x.*

Theorem A.17 *Let X and Y be topological spaces and let f be a function defined on X with values in Y.*

1. *f is continuous at $x \in X$ if and only if for each neighborhood U of $f(x)$ there is a neighborhood V of x such that $f(V) \subset U$.*
2. *f is continuous at $x \in X$ if and only if for each net $\mathcal{N}$ in X that converges to x the composition net $f \circ \mathcal{N}$ in Y converges to $f(x)$.*
3. *f is continuous if and only if f is continuous at all $x \in X$.*

In first countable spaces, the second characterization above of local continuity has the following more familiar form:

Corollary A.1 *Let X and Y be first countable spaces, let $f : X \longrightarrow Y$, and let x be a point in X. Then f is continuous at x if and only if $f(x_i) \to f(x)$ in Y for every sequence $\{x_i\}$ in X that converges to x.*

A.11 Relative Topologies

Definition A.26 *If X is a topological space and $A \subset X$, then the "relative" or "subspace" topology of A in X is the collection of $B \subset A$ such that $B = A \cap U$ for some open U in X. The relative topology of A in X is also called the topology that A "inherits" from X.*

Remark A.17 *Let X be a topological space, let $A \subset X$, and let A have the topology it inherits from X. If X is first (second) countable, then A is first (second) countable. If X is Hausdorff, then so is A.*

A.12 Compactness

Definition A.27 *If X is a topological space, then a family of open sets whose union contains a subset C of X is called an open cover of C. A subcollection of an open cover of C that is itself an open cover of C is naturally called a subcover. A cover (subcover) is called finite if the number of its sets is finite.*

1. *If K is a subset of X and if each open cover of K has a finite subcover, then we say that K is a compact subset of X.*
2. *X is called a compact space if X is a compact subset of itself.*
3. *We call X a locally compact space if every $x \in X$ has an open neighborhood with compact closure.*
4. *An open subset U of X is called relatively compact if $\overline{U}$ is compact.*
5. *A locally compact, second countable Hausdorff space is sometimes called an LCS space.*

Remark A.18 *If X is a topological space and K is a compact subset of X, then K with its relative topology is a compact space.*

Definition A.28 *A family $\mathcal{A}$ of subsets of a set has the finite intersection property if every finite subfamily of $\mathcal{A}$ has a nonempty intersection.*

Theorem A.18 *The following results are classical:*

1. *A subset K of a topological space X is compact if and only if every family of closed subsets of K with the finite intersection property has a nonempty intersection.*
2. *A closed subset of a compact set is compact, and a compact subset of a Hausdorff space is closed.*

3. *Thus if X is a compact space and F is a closed subset of X, then F with its relative topology in X is a compact space.*
4. *Every sequence in a compact space has at least one limit point.*
5. *If X is a compact Hausdorff space or a second countable regular space, then X is a normal space.*

A.13 Product Topologies

Definition A.29 *If X and Y are topological spaces, then the topology on $X \times Y$ that is generated basically by the sets $U \times V$ (where U is open in X and V is open in Y) is called the product topology; $X \times Y$ is called a product space when it carries its product topology.*

Remark A.19 *Let X and Y be topological spaces and let $\{(x_i, y_i)\}$ be a sequence in $X \times Y$. Then $\{x_i\}$ is a sequence in X, $\{y_i\}$ is a sequence in Y, and $\{(x_i, y_i)\}$ converges to (x, y) in the product space $X \times Y$ if and only if $x_i \to x$ in X and $y_i \to y$ in Y.*

Remark A.20 *Let X and Y be topological spaces and let $\{(x_\alpha, y_\alpha)\}$ be a net in $X \times Y$. Then $\{x_\alpha\}$ is a net in X, $\{y_\alpha\}$ is a net in Y, and $\{(x_\alpha, y_\alpha)\}$ converges to (x, y) in the product space $X \times Y$ if and only if $\{x_\alpha\}$ converges to x in X and $\{y_\alpha\}$ converges to y in Y.*

Remark A.21 *Let X and Y be topological spaces and let $X \times Y$ be the corresponding product space.*

1. *If X and Y are first (second) countable, then so is $X \times Y$.*
2. *If X and Y are Hausdorff, then $X \times Y$ is Hausdorff.*
3. *If X and Y are (locally) compact, then so is $X \times Y$.*

For details not covered in this summary, consult Kelley (1955).

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