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CLOSED LOOP VIBRATIONAL CONTROL: THEORY AND APPLICATIONS

FINAL REPORT

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In this project, a novel control technique, referred to as Closed Loop Vibrational Control, is developed and applied to the problem of fuselage vibrations suppression in helicopter dynamics. This technique is applicable to systems where the control input enters the open loop dynamics as an amplitude of a periodic, zero average function, and this amplitude can be chosen to depend on the system's outputs. An example of such a system is the helicopter with Higher Harmonic Control (HHC) where periodic feathering of rotor blades around a fixed pitch angle is introduced in order to suppress the fuselage vibrations.

For systems with this structure, a number of control-theoretic problems, including stabilizability, pole placement and robustness, have been solved, and the results are reported in this document.

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1 FOREWORD

In this project, a novel control technique, referred to as Closed Loop Vibrational Control, has been developed and applied to the problem of fuselage vibrations suppression in helicopter dynamics. From the theoretical standpoint, the technique developed is applicable to systems where the control input enters the open loop dynamics as an amplitude of a periodic, zero average function, and this amplitude can be chosen to depend on the system's outputs. An example of such a system is the helicopter with Higher Harmonic Control (HHC) where periodic feathering of rotor blades around a fixed pitch angle is introduced in order to suppress the fuselage vibrations. From the practical standpoint, the technique developed is useful for plants where conflicting control objectives must be achieved with an insufficient number of actuators. From this perspective, the technique developed is based on the frequency separation, i.e. the utilization of low and high frequency control signals so that, on the average, all control objectives are satisfied. In the HHC case, this frequency separation amounts to low frequency rotor blades pitch angle control to ensure the desired altitude of the hovercraft and the high frequency rotor blade pitch angle control to suppress the fuselage vibrations.

For systems with this structure, the following problems have been solved and are reported in this document:

1. Conditions for the state and dynamic output-feedback stabilizability by closed loop vibrational control have been derived.

2. Pole placement capabilities of vibrational controllers have been investigated.

3. Stability robustness of closed loop vibrational control has been analyzed.

4. Youla-type parametrization of closed loop vibrational controllers have been derived and

utilized for the design purposes.

5. A method for H_2–optimal zeros placement has been developed.

6. The results obtained have been applied to helicopter vibration suppression problem and a technique referred to as Very High Harmonic Control (VHHC) has been investigated.

In short, the main result can be formulated as follows: A novel control technique has been developed and its utility in helicopter vibrations suppression has been demonstrated.

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4 STATEMENT OF THE PROBLEM

The goal of this thesis is the development of a control theory for a class of dynamical systems described by the following equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)f(\frac{t}{\epsilon}) , \\ y(t) &= Cx(t) , \\ f(\tau) &= f(\tau + T) , \ T \neq 0 , \end{aligned} \tag{0.1}$$

$$\frac{1}{T}\int_0^T f(\tau)\,d\tau = 0 ,$$

$$0 < \epsilon \ll 1 ,$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the *control*, $f(t)$ is a *periodic*, average zero scalar function, and ϵ is a small positive parameter. Stabilizability properties of system (0.1) with state and output feedback are analyzed and the pole placement capabilities investigated. A characteristic feature of system (0.1) is that the control, u, enters the open loop dynamics as an amplitude of a periodic, zero average function. Such situations arise in a number of applications where two conflicting control goals have to be accomplished by a single actuator. For instance, in the helicopter control problem, a single actuator (the blades' pitch angle) is used to ensure both the desired altitude and the fuselage vibration suppression. These goals are conflicting in the sense that if the pitch angle is chosen to ensure the desired altitude, the fuselage vibrations are not suppressed; if the pitch angle is used to suppress the fuselage vibrations, the desired altitude is not attained. In order to accomplish the two goals simultaneously, a frequency separation approach may be employed. Specifically, a low frequency control may be used to stabilize the desired altitude and a high frequency, average zero control may be used to suppress the vibrations without compromising the first goal. When the control loop is closed with respect to the low frequency control, the equations have the form of system (0.1), and the goal is to choose the control, u, as a function of x or y so that the resulting system has the desired dynamical properties.

In particular, the above ideology has been successfully implemented in the Higher Harmonic Control (HHC) of helicopters, where periodic feathering of rotor blades around a fixed pitch angle is introduced in order to suppress the fuselage vibrations. Helicopter vibration is a long standing problem. Recent experiments [1]-[5] have shown that HHC may lead to an order of magnitude reduction in fuselage vibrations. The primary difficulty in implementation of the HHC systems is the complex interaction between inertia, structural and aerodynamical hub shears and moments for HHC-equipped helicopter rotors. These interactions, which are difficult to predict due to the highly complex dynamics of most helicopter systems, account for the extreme sensitivity of HHC efficacy to proper magnitude and phasing of the HHC inputs. Using the idea of closed loop vibrational control, we will explore alternative means of suppressing the vibratory airload without the phase dependencies in the input.

It is well known that unavoidable discrepancies between mathematical models and real-world systems can result in the degradation of control system performance. Thus, we also investigate the property of ***stability robustness*** for system (0.1). Both synthesis and analysis problems are addressed. In the synthesis problem, it is assumed that (0.1) is the nominal plant, whereas the true plant is defined by

$$\dot{x} = (A + \Delta A)x + Buf\left(\frac{t}{\epsilon}\right) , \tag{0.2}$$

where ΔA is the perturbation matrix. Given the perturbation matrix ΔA, we derive a condition which guarantees the existence of a controller

$$u = \frac{K}{\epsilon} x \ , \tag{0.3}$$

which robustly stabilizes the uncertain system (0.2). In the analysis problem, we determine a bound on ΔA which ensures that a controller (0.3) which stabilizes the nominal plant (0.1), also stabilizes the perturbed system (0.2).

We also investigate the parametrization of stabilizing controllers for the system (0.1) since such a characterization could be quite useful, e.g. to satisfy desired performance specifications [6]. We first consider the observer-based output controller, K_{nom}, defined by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Buf\left(\frac{t}{\epsilon}\right) + L(y - \hat{y}) \ , \\ u &= \frac{K}{\epsilon}\hat{x} \ , \\ \hat{y} &= C\hat{x} \ , \end{aligned} \tag{0.4}$$

and parametrize all stabilizing controllers for (0.1) in the class of rational transfer functions. Next, the parametrization of the averaged closed loop transfer function resulting from a stabilizing output feedback controller is derived.

Finally, since periodic controllers are known to relocate control loop zeros [7], we give a solution to the zero placement problem for open and closed loop system and characterize control-theoretic properties of the resulting system. In this thesis, we treat the above problem by considering a linear time-invariant SISO system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1u(t) + B_2w(t) \ , \\ y(t) &= Cx(t) \ , \quad x \in \mathbb{R}^n, u, w, y \in \mathbb{R} \ . \end{aligned} \tag{0.5}$$

In the design stage of the system (0.5), when no actuator positioning and disturbance protection measures are yet finalized, input vectors B_1 and B_2 and output vector C may be viewed as free parameters to be chosen so that appropriate performance specifications are satisfied. Among these, it seems reasonable to require that the transmission from the control u to the output y be maximized and the transmission from the disturbance w to y be minimized. Since a choice of B_1 and B_2 defines, for a given A and C, the system's zeros, the problem of choosing B_1 and B_2 also defines the zeros of the system (0.5). The purpose of this research is to give a solution to the zero placement problem for open and closed loop system and characterize control-theoretic properties of the resulting system. Note also that although the problem of optimal pole placement has received enormous attention during the last 30 years, that of optimal zero placement has been relatively neglected in control systems research. This latter problem could however be of substantial importance not only in the context of vibrational control but also in the placement of actuators and sensors since the latter determines the input and output vectors of the linear model (0.5) [8]-[10].

5 SUMMARY OF THE MOST IMPORTANT RESULTS

5.1 PART 1. STATE AND OUTPUT FEEDBACK STABILIZABILITY AND POLE PLACEMENT CAPABILITIES

5.1.1 INTRODUCTION

In this part of the report, we develop a control theory for a class of SISO dynamical systems described by the following equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)f\left(\frac{t}{\epsilon}\right) , \\ y(t) &= Cx(t) , \\ f(t) &= f(t+T) , T \neq 0 , \end{aligned} \tag{1.1}$$

$$\frac{1}{T}\int_0^T f(t)\, dt = 0 ,$$

$$0 < \epsilon \ll 1 ,$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}, C \in \mathbb{R}^{1\times n}, f(t)$ is a periodic, zero average scalar function, and ϵ is a small positive constant. A characteristic feature of this problem is that the control, u, enters the open loop dynamics as the amplitude of a periodic, zero average function. Such situations arise in a number of applications where two conflicting control goals have to be accomplished by a single actuator. For instance, in the helicopter control problem, a single actuator (the blades' pitch angle) is used to ensure both the desired altitude and the fuselage oscillations suppression. These goals are conflicting in the sense that if the pitch angle is chosen to ensure the desired altitude, the fuselage oscillations are not suppressed; if the pitch angle is used to suppress the fuselage oscillations, the desired altitude is not attained. In order to accomplish the two goals simultaneously, a frequency separation approach may be employed. Specifically, a low frequency control may be used to stabilize the desired altitude and a high frequency, average zero control may be used to suppress the vibrations. When the control loop is closed with respect to the low frequency control, the resulting equations have the form of system (1.1), and the goal is to choose the control, u, as a function of x or y so that the system has the desired dynamical properties. This ideology has been implemented in the Higher Harmonic Control (HHC) of helicopters, where periodic feathering of rotor blades around a fixed pitch angle is introduced in order to suppress the fuselage vibrations.

Another example is the periodic operation of chemical reactors [19]. Here again the problem is to choose the amplitude of input flow vibrations so that the closed loop system behaves as desired.

In order to simplify the analysis and obtain constructive results, following [11]-[17], we assume that the periodic function $f(t)$ is of high frequency as compared with the dynamics of $\dot{x} = Ax$. Formally, this means that function f has the asymptotic form $f(\frac{t}{\epsilon})$, where $\epsilon > 0$ is sufficiently small. Thus, more precisely the problem addressed in this part of the report is as follows:

Given system (1.1), determine under what conditions there exist $K \in \mathbb{R}^{1\times n}, L \in \mathbb{R}^{n\times 1}$, and $\epsilon_0 \ll 1$ such that for all $\epsilon \leq \epsilon_0$, the closed loop system composed of (1.1) with the time invariant state space controller

$$u = \frac{K}{\epsilon}x , \tag{1.2}$$

or a time invariant output controller

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Buf\left(\frac{t}{\epsilon}\right) + L(y - \hat{y}) , \\ \hat{y} &= C\hat{x} , \\ u &= \frac{K}{\epsilon}\hat{x} , \end{aligned} \tag{1.3}$$

is asymptotically stable. The state feedback gains of (1.2), (1.3) are restricted to be time invariant for reasons of practical implementation. Problem (1.1), (1.2) is considered in Section 5.1.2 and problem (1.1), (1.3) is discussed in Section 5.1.3. In addition, we characterize the pole placement capabilities ensured by closed loop vibrational control and present the corresponding results in Section 5.1.4. To illlustrate the results, in Section 5.1.5 we consider an example inspired by a helicopter with HHC.

5.1.2 STATE SPACE FEEDBACK

In this section, we present the result for stabilization of the system (1.1) with state space feedback (1.2).

Theorem 1.1: There exists a $K \in I\!R^{1\times n}$ and an $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ system (1.1), (1.2) is asymptotically stable if only if (A, B) is stabilizable and the sum of all the controllable eigenvalues of A is negative.

Proof: Necessity is proved by the following considerations. Represent the state space model (1.1) in Kalman canonical form,

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_{nc} \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{nc} \end{bmatrix} \begin{bmatrix} x_c \\ x_{nc} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} uf\left(\frac{t}{\epsilon}\right) , \tag{1.4}$$

where the pair (A_c, B_1) is controllable. Since A_{nc} is not affected by feedback, the stabilizability of (A, B) is necessary.

The stability of x_c, with $x_{nc}(0) = 0$, is governed by the following equation:

$$\dot{x}_c = A_c x_c + B_1 uf\left(\frac{t}{\epsilon}\right) , \quad x_c \in I\!R^m,\ u \in I\!R . \tag{1.5}$$

Introducing a state feedback $u = Kx_c/\epsilon$, we obtain

$$\dot{x}_c = \left[A_c + B_1 \frac{K}{\epsilon} f\left(\frac{t}{\epsilon}\right)\right] x_c . \tag{1.6}$$

Since (1.6) is periodic, there exists a Lyapunov transformation which reduces (1.6) to an equation with constant coefficients,

$$\dot{z} = \Lambda z ,$$

preserving the stability property. From the Jacobi-Liouville theorem [13],

$$\frac{1}{T}\int_0^T \operatorname{Tr}\left[A_c + B_1\frac{K}{\epsilon} f\left(\frac{t}{\epsilon}\right)\right] dt = \operatorname{Tr} \Lambda ,$$

where T is the period of $f(t/\epsilon)$. Thus,

$$\text{Tr } A_c = \text{Tr } \Lambda ,$$

where Tr A_c is equal to the sum of all the controllable eigenvalues. Therefore. Tr $A_c < 0$ is necessary and this completes the proof of necessity.

Suffiency is proved as follows: Consider (1.4) and assume that all the eigenvalues of A_{nc} have negative real parts. Without loss of generality, assume that (1.5) is in the controller canonical form, i.e.:

$$\dot{x}_c = A_c x_c + B_1 u f\left(\frac{t}{\epsilon}\right) , \quad x_c \in \mathbb{R}^m,\ u \in \mathbb{R} ,$$

where

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & 1 \\ -a_m & -a_{m-1} & \cdots & -a_1 \end{bmatrix} , \quad B_1 = \begin{bmatrix} 0 \\ \\ 0 \\ 1 \end{bmatrix} ,$$

and a_i are the coefficients of the characteristic polynomial of matrix A_c. Apply state feedback

$$u = \frac{K}{\epsilon} x_c = \left[\frac{k_m}{\epsilon} \cdots \frac{k_2}{\epsilon}\ 0\right] , \tag{1.7}$$

where $k_i \sim 1$, i=2, ..., m. The closed loop system is

$$\dot{x}_c = A_c x_c + \frac{1}{\epsilon} B_1 K f\left(\frac{t}{\epsilon}\right) x_c \tag{1.8}$$

$$= X_1(t, x_c) + \frac{1}{\epsilon} X_2\left(\frac{t}{\epsilon}, x_c\right) . \tag{1.9}$$

The generating equation, [17], for this system has the form

$$\frac{dx_c}{d\tau} = B_1 K f(\tau) x_c , \tag{1.10}$$

where $\tau = t/\epsilon$. The general solution of (1.10) is

$$x_c = \Phi(\tau) x_{c0} ,$$

where $\Phi(\tau)$ is a fundamental matrix for $BK_1 f(\tau)$ and x_{c0} is a constant. Consequently, introducing the substitution

$$x_c = \Phi(\tau)\xi = h(\tau, \xi) ,$$

we obtain the following equation in Bogoliuboff's standard form [21]:

$$\frac{d\xi}{d\tau} = \epsilon \left[\frac{\partial h}{\partial \xi}\right]^{-1} X_1(t, h(\tau, \xi)) \tag{1.11}$$

$$= \Phi^{-1}(\tau) A_c \Phi(\tau) \xi .$$

Applying the averaging principle [21], we obtain the following averaged equation

$$\dot{\bar{\xi}} = \overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right) A_c \Phi\left(\frac{t}{\epsilon}\right)}\bar{\xi} , \tag{1.12}$$

where

$$\overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right) A_c \Phi\left(\frac{t}{\epsilon}\right)} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & 1 \\ -a_m - k_2 k_m \overline{\phi^2} & -a_{m-1} - k_2 k_{m-1}\overline{\phi^2} & \cdots & -a_1 \end{bmatrix} , \tag{1.13}$$

$$\phi\left(\frac{t}{\epsilon}\right) = \int_0^{t/\epsilon} f(\tau) d\tau ,$$

and bar denotes the averaged value, i.e. $\overline{\beta(\tau)} = 1/T \int_0^T \beta(\tau) d\tau$. Let $\lambda_{o_1}, \ldots, \lambda_{o_m}$ be the open loop eigenvalues of (1.5) and choose the closed-loop eigenvalues $\lambda_{c_1}, \ldots, \lambda_{c_m}$ of (1.12) as follows:

$$\lambda_{c_i} = \frac{1}{m}\sum_{i=1}^{m} \lambda_{o_i} + j \text{ Im } \lambda_{o_i} . \tag{1.14}$$

Then the state feedback gains (1.7) can be found to be

$$k_i = \frac{(a_{c_i} - a_i)}{k_2 \overline{\phi^2}} , \quad i = 2, \ldots, m , \tag{1.15}$$

where $a_{c_1}, \ldots, a_{c_m}$ are the coefficients of the closed loop characteristic equation corresponding to $\lambda_{c_1}, \ldots, \lambda_{c_m}$.

The control gains (1.15) guarantee the asymptotic stability of the averaged system (1.12). As it follows [17], if (1.12) is asymptotically stable, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ equation (1.5) is also asymptotically stable. This proves the sufficiency. **Q.E.D.**

Corollary 1.1: Assume that the sum of the controllable eigenvalues of A in (1.1) is positive, then there exists an ϵ_0 such that no dynamic state feedback of the form

$$\begin{aligned} \dot{v} &= Fv + Gx , \\ u &= \frac{1}{\epsilon}[Hv + Jx] , \end{aligned} \tag{1.16}$$

will stabilize the system (1.1) whenever $0 < \epsilon \leq \epsilon_0$.

Proof: The resulting closed loop equations with the dynamic state feedback controller (1.16) are :

$$\begin{aligned} \dot{x} &= Ax + B\frac{H}{\epsilon} v f\left(\frac{t}{\epsilon}\right) + B\frac{J}{\epsilon} x f\left(\frac{t}{\epsilon}\right) , \\ \dot{v} &= Fv + Gx . \end{aligned}$$

In fast time $\tau = t/\epsilon$, the above equation becomes :

$$\begin{bmatrix} \frac{dx}{d\tau} \\ \frac{dv}{d\tau} \end{bmatrix} = \begin{bmatrix} \epsilon A + BJf(\tau) & BHf(\tau) \\ \epsilon G & \epsilon F \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} . \tag{1.17}$$

Let $\Phi(\tau)$ be a fundamental matrix for $BJf(\tau)$. Define

$$W(\tau) = \int_0^\tau \Phi(\tau - q)BHf(q)dq \ .$$

and the substitution

$$\begin{bmatrix} x(\tau) \\ v(\tau) \end{bmatrix} = \begin{bmatrix} \Phi(\tau) & W(\tau) \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ \varphi(\tau) \end{bmatrix} .$$

Using the above substitution, we obtain

$$\begin{bmatrix} \frac{d\xi}{d\tau} \\ \frac{d\varphi}{d\tau} \end{bmatrix} = \epsilon \begin{bmatrix} \Phi^{-1}A\Phi - \Phi^{-1}WG\Phi & \Phi^{-1}AW - \Phi^{-1}WGW - \Phi^{-1}WF \\ G\Phi & GW + F \end{bmatrix} \begin{bmatrix} \xi \\ \varphi \end{bmatrix} . \qquad (1.18)$$

which is an equation in the standard form [21]. Introduce the following Lyapunov transformation

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} I & \Phi^{-1}\left(\frac{t}{\epsilon}\right)W\left(\frac{t}{\epsilon}\right) \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi(t) \\ \varphi(t) \end{bmatrix} .$$

which will preserve the stability of (1.18), yields

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Phi^{-1}\left(\frac{t}{\epsilon}\right)A\Phi\left(\frac{t}{\epsilon}\right) & \Phi^{-1}\left(\frac{t}{\epsilon}\right)BHf\left(\frac{t}{\epsilon}\right) \\ G\Phi\left(\frac{t}{\epsilon}\right) & F \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} .$$

Applying the averaging principle [21], we obtain

$$\begin{bmatrix} \dot{\overline{z_1}} \\ \dot{\overline{z_2}} \end{bmatrix} = \begin{bmatrix} \overline{\Phi^{-1}A\Phi} & 0 \\ G\overline{\Phi} & F \end{bmatrix} \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \end{bmatrix} .$$

Thus the eignevalues of the averaged closed loop system are the union of those of $\overline{\Phi^{-1}A\Phi}$ and F. It follows from Theorem 1.1 and the averaging principle [21], that if Tr $A > 0$, there exists an ϵ_0 such that the closed loop system (1.1), (1.16) will be unstable whenever $0 < \epsilon \leq \epsilon_0$. **Q.E.D.**

5.1.3 OUTPUT FEEDBACK

This section deals with the problem of stabilizing system (1.1) with an output feedback (1.3).

Theorem 1.2: There exists a $K \in \mathbb{R}^{1\times n}, L \in \mathbb{R}^{n\times 1}$, and an $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ the system (1.1), (1.3) is asymptotically stable if and only if (A, B, C) is stabilizable and detectable and the sum of the controllable eigenvalues of A is negative. The separation principle holds, i.e. the choice of K and L can be carried out independently.

Proof: Necessity is proved by the following considerations. Consider the observer,

$$\dot{\hat{x}} = A\hat{x} + Buf\left(\frac{t}{\epsilon}\right) + L(y - C\hat{x}) ,$$

and the feedback law

$$u = \frac{K}{\epsilon}\hat{x} .$$

The dynamical equations for the closed loop system are:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & B\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right) \\ LC & (A-LC)+B\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right) \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} . \tag{1.19}$$

Defining the observation error $e = x - \hat{x}$, we obtain the following equivalent dynamical equations:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + B\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right) & -B\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right) \\ 0 & (A-LC) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} .$$

Thus, the stability of the closed loop system depends on those of $A+(1/\epsilon)BKf(t/\epsilon)$ and $A-LC$. Using the results of Theorem 1.1 completes the proof of necessity.

Sufficiency is proved as follows: Consider system (1.1) with (A, B, C) stabilizable and detectable. In fast time $\tau = t/\epsilon$, the resulting closed-loop equations with output feedback (1.3) are:

$$\begin{bmatrix} \frac{dx}{d\tau} \\ \frac{d\hat{x}}{d\tau} \end{bmatrix} = \begin{bmatrix} \epsilon A & BKf(\tau) \\ \epsilon LC & \epsilon(A-LC)+BKf(\tau) \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} . \tag{1.20}$$

Let $\Phi(\tau)$ be a fundamental matrix for $BKf(\tau)$. Define

$$\Phi_1(\tau) = \Phi^{-1}(\tau)A\Phi(\tau) ,$$

and the substitution

$$\begin{bmatrix} x(\tau) \\ \hat{x}(\tau) \end{bmatrix} = \begin{bmatrix} I & BK\int_0^\tau \Phi(q)f(q)dq \\ 0 & \Phi(\tau) \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ \varphi(\tau) \end{bmatrix} .$$

Using the above substitution and following the proof of Theorem 1.1, we reduce (1.20) to the standard form [21] and apply the averaging principle to obtain the following averaged equations:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} A+\overline{\Phi}LC-LC & \overline{\Phi}_1 - A + LC - \overline{\Phi}LC \\ \overline{\Phi}LC & \overline{\Phi}_1 - \overline{\Phi}LC \end{bmatrix} \begin{bmatrix} \xi \\ \varphi \end{bmatrix} . \tag{1.21}$$

To simplify (1.21), introduce the following transformation

$$\begin{bmatrix} \overline{z_1} \\ \overline{z_2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} \overline{\xi} \\ \overline{\varphi} \end{bmatrix} ,$$

which yields

$$\begin{bmatrix} \dot{\overline{z_1}} \\ \dot{\overline{z_2}} \end{bmatrix} = \begin{bmatrix} \overline{\Phi}_1 & -\overline{\Phi}_1 + A - LC + \overline{\Phi}LC \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \end{bmatrix} . \tag{1.22}$$

Using a construction similar to that of (1.10)-(1.12), one can compute the state feedback gain required to stabilize $\overline{\Phi}_1$ and assign all the eigenvalues of $A - LC$ through the choice of L so that the asymptotic stability of (1.21) is guaranteed. As it follows [17], if (1.21) is asymptotically stable, there exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ equation (1.19) is also asymptotically stable. **Q.E.D.**

5.1.4 POLE PLACEMENT CAPABILITIES

Consider again system (1.1) with feedback (1.2) and assume that

$$\frac{K}{\epsilon} = \left[\frac{k_n}{\epsilon} \cdots \frac{k_1}{\epsilon}\right] ,$$

where $k_i \sim 1$, $i = 1, \ldots, n$. Thus, the closed loop system is

$$\dot{x} = \left(A + B\frac{K_1}{\epsilon} f\left(\frac{t}{\epsilon}\right)\right) x . \qquad (1.23)$$

As it is shown in [17] and (1.12), this equation can be reduced to the averaged equation,

$$\dot{\bar{\xi}} = (A + \overline{B})\bar{\xi} , \qquad (1.24)$$

where

$$x(\tau) = \Phi(\tau)\xi(\tau) ,$$

$\Phi(\tau)$ is a fundamental matrix for $BKf(\tau)$ and $\tau = t/\epsilon$. As it follows from [17] and [21], (1.23) is asymptotically stable for sufficiently small ϵ if the averaged system (1.24) is asymptotically stable. Matrix $\overline{B}$ which, along with matrix A, defines the stability of (1.24) can be characterized as follows:

Theorem 1.3: Assume A and $\underline{B}$ are in the controller canonical form. Then there exists ϵ_0 such that for all $0 < \epsilon \leq \epsilon_0$, matrix $\overline{B}$ of (1.24) has the form:

$$\overline{B} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \\ 0 & \cdots & 0 & 0 \\ -\overline{b_n} & \cdots & -\overline{b_2} & 0 \end{bmatrix} \qquad (1.25)$$

and

$$\overline{b_i} = \sum_{s=1}^{\infty} \alpha_s^2 \frac{k_2\ k_i}{2s^2} ,$$

where α_s, $s = 1, 2, \ldots$, are the Fourier coefficients of $f(\tau)$, i.e.

$$f(\tau) = \sum_{s=1}^{\infty} \alpha_s \sin(s\tau + \varphi_s) .$$

Proof: Follows directly from Theorem 3 of [13].

Denote the characteristic polynomials of A and $(A + \overline{B})$ of (1.23) and (1.24), respectively, by

$$p_0(s) = s^n + a_{10}s^{n-1} + a_{20}s^{n-2} + \cdots + a_{n0} , \qquad (1.26)$$
$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n . \qquad (1.27)$$

It follows from Theorem 1.3 that

$$a_1 = a_{10} , \qquad (1.28)$$
$$a_2 \geq a_{20} , \qquad (1.29)$$

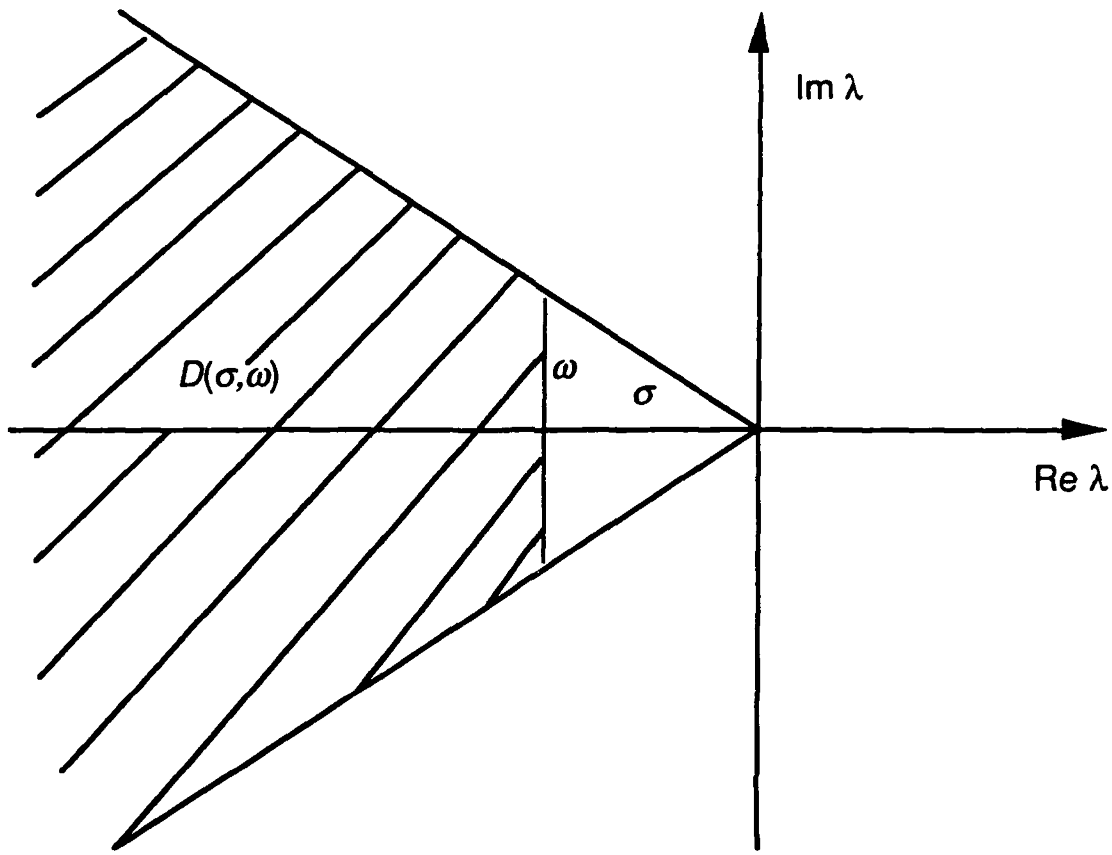

Figure 1.1: Sector region $D(\sigma,\omega)$

and a_j, $3 \leq j \leq n$, can be arbitrarily assigned. Below we analyze to what extent the constraints (1.28) and (1.29) prevent the control designer from assigning the closed loop eigenvalues of the averaged equation (1.24) to a desired region of the complex plane.

More specifically, considering the closed region $D(\sigma,\omega)$ of Figure 1.1, our purpose is to identify the conditions under which we can find n real or complex numbers $\lambda_1, \ldots, \lambda_n$, occurring as pairs of complex poles, such that they are the roots of the polynomial (1.27) which satisfy (1.28) and (1.29). When this is possible, we say that (real or complex) pole assignment in the region $D(\sigma,\omega)$ using closed loop vibrational control is possible. Note that if the closed loop poles are confined to this region, then the system modes of (1.24), and hence the averaged modes of (1.23), will have a time constant smaller than or equal to $-1/\sigma$ and decay exponentially at a rate greater than or equal to $-\sigma/\sqrt{\sigma^2+\omega^2}$.

Since, closed loop vibrational control can only modify the coefficients $a_{i0}, 2 \leq i \leq n$, of (1.26), we will assume throughout this section that $n \geq 2$. Lemmas A.1-A.5 are contained in Appendix A1.

Theorem 1.4: Pole assignment in the region $D(\sigma,\omega)$ using closed loop vibrational control is possible, only if

$$-a_{10} \leq n\sigma \ . \tag{1.30}$$

Proof: If $\lambda_1, \ldots, \lambda_n \in D(\sigma,\omega)$ are the roots of the polynomial (1.27), then

$$-a_1 = \sum_{i=1}^{n} \lambda_i \ , \tag{1.31}$$

and

$$\mathrm{Re}(\lambda_i) \leq \sigma \ . \tag{1.32}$$

Equation (1.30) is obtained by equating the real parts in (1.31) and using (1.32). **Q.E.D.**

Theorem 1.5: Real pole assignment in the region $D(\sigma,\omega)$ is possible using closed loop vibrational control feedback if and only if

$$-a_{10} \leq n\sigma \ , \tag{1.33}$$

$$a_{20} \leq \frac{n-1}{2n} a_{10}^2 \ . \tag{1.34}$$

Proof: The necessity of (1.33) follows from Theorem 1.4. The necessity of (1.34) follows from the fact that when $\lambda_1, \ldots, \lambda_n$ are the roots of (1.27), the coefficients a_1 and a_2 are

$$a_1 = -\sum_{i=1}^{n} \lambda_i \ , \tag{1.35}$$

$$a_2 = \sum_{\substack{i,j=1 \\ i>j}}^{n} \lambda_i \lambda_j \ . \tag{1.36}$$

The maximum value that (1.36) can achieve subject to the constraint (1.35), (1.28) is given by Lemma A.2 and is exactly the right hand side of (1.34). Therefore if (1.34) is violated, pole assignment to the region $D(\sigma,\omega)$ with real poles is not possible.

To prove the sufficiency of (1.33), (1.34), assume they both hold. Choose

$$\lambda_1^* = \lambda_2^* = \ldots = \lambda_n^* = \frac{a_{10}}{n} \ .$$

It is immediately checked that these real numbers solve the problem of pole assignment to the region $D(\sigma,\omega)$, which completes the proof. **Q.E.D.**

Theorem 1.5 gives a simple solution of the problem of pole assignment to the region $D(\sigma,\omega)$ with *real* poles. When *complex* poles are allowed, results similar to Theorem 1.5 are obtained:

Theorem 1.6: Complex pole assignment in the region $D(\sigma,\omega)$ is possible using closed loop vibrational control feedback if and only if

(i) for n even

$$-a_{10} \leq n\sigma \ , \tag{1.37}$$

and

$$a_{20} \leq \frac{a_{10}^2}{2n\sigma^2}\left((n-1)\sigma^2 + \omega^2\right) , \quad (\text{if } |\sigma| \geq \omega) \ , \tag{1.38}$$

or

$$a_{20} \leq (\omega^2 - \sigma^2)\left(\frac{n-2}{2}\right)\left(\frac{n}{2} + \frac{a_{10}}{\sigma}\right) + (\omega^2 + \sigma^2)\left(\frac{a_{10}}{2\sigma}\right)^2 , \quad (\text{if } |\sigma| < \omega) \ , \tag{1.39}$$

(ii) for n odd with one real pole placed at $p < \sigma$

$$-a_{10} \leq (n-1)\sigma + p \ , \tag{1.40}$$

and

$$a_{20} \leq -p(a_{10}+p) + \frac{(a_{10}+p)^2}{2(n-1)\sigma^2}\left((n-2)\sigma^2+\omega^2\right) , \qquad (\text{if } |\sigma| \geq \omega) \ , \tag{1.41}$$

or

$$\begin{aligned} a_{20} \leq & -p(a_{10}+p) + (\omega^2-\sigma^2)\left(\frac{n-3}{2}\right)\left(\frac{n-1}{2}+\frac{a_{10}+p}{\sigma}\right) \\ & +(\omega^2+\sigma^2)\left(\frac{a_{10}+p}{2\sigma}\right)^2 , \qquad (\text{if } |\sigma| < \omega) \ . \end{aligned} \tag{1.42}$$

Proof: We first consider the case when n is even and let $|\sigma| \geq \omega$. The necessity of (1.37) follows from Theorem 1.4. The necessity of (1.38) follows from the fact that when $\lambda_1, \ldots, \lambda_n$ are the roots of (1.27), the coefficients a_1 and a_2 are

$$a_1 = -\sum_{i=1}^{n} \lambda_i \ , \tag{1.43}$$

$$a_2 = \sum_{\substack{i,j=1 \\ i>j}}^{n} \lambda_i \lambda_j \ . \tag{1.44}$$

The maximum value that (1.44) can achieve subject to the constraint (1.43), (1.28) is given by Lemma A.2 and is exactly the right hand side of (1.38).

To prove the sufficiency of (1.37), (1.38), assume they both hold. Choose

$$\lambda_1^* = -\frac{a_{10}}{n\sigma}\left(\sigma + j(-1)^i\omega\right) \ , \qquad i = 1, \ldots, n.$$

It is immediately checked that these complex numbers solve the problem of pole assignment to the region $D(\sigma, \omega)$, which completes the proof.

When n is even and $|\sigma| < \omega$, the proof is similar and follows immediately from Theorem 1.4 and Lemma A.3.

The proof for the case when n is odd is completed by using Theorem 1.4, Lemma A.4 (if $|\sigma| \geq \omega$) and Lemma A.5 (if $|\sigma| < \omega$). **Q.E.D.**

Remark 1.1: If $\sigma^2 - \omega^2 = 0$, then conditions (1.38), (1.39) and (1.41), (1.42) in Theorem 2.6 are identical. Note also that the expressions in the right hand side of (1.39) and (1.42) have larger values than the corresponding expressions in the right hand side of (1.38) and (1.41) respectively. Consequently, it is easier to satisfy the condition for assigning complex poles into the region $D(\sigma, \omega)$ such that $\sigma^2 - \omega^2 \leq 0$ with closed loop vibrational control.

Remark 1.2: Some observations concerning the region $D(\sigma, \omega)$ are in order. Let $\lambda = -\zeta\omega_n \pm j\omega_d$ be a complex pole, where ζ is the damping ratio, $0 \leq \zeta \leq 1, \omega_n = |\lambda|$ is the natural (undamped) frequency, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. Then if $\lambda \in D(\sigma, \omega)$, it follows that $\zeta \geq -\sigma/\sqrt{\sigma^2+\omega^2}$ and $-\zeta\omega_n \leq \sigma$. To analyze the pole placement capabilities of the closed loop vibrational control, we note that in practice, design specifications

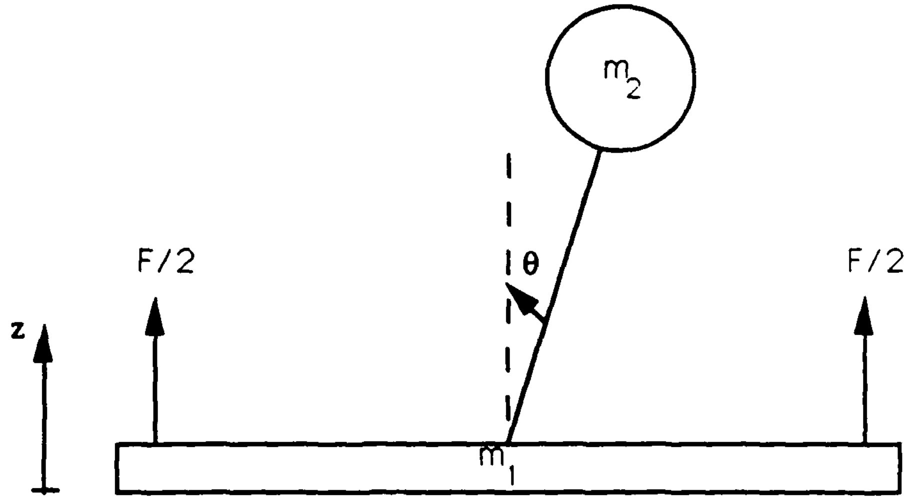

Figure 1.2: Inverted pendulum mounted on a platform

are often given in terms of $\zeta \geq \zeta_{min}$ and $-\zeta\omega_n \leq \alpha$. Such specifications will be satisfied by $D(\sigma,\omega)$ if $\sigma \leq \alpha$ and $-\sigma/\sqrt{\sigma^2+\omega^2} \geq \zeta_{min}$, or equivalently

$$\sigma \leq \alpha \,, \tag{1.45}$$

$$\omega \leq -\sigma\sqrt{1/\zeta_{min}^2 - 1} \,. \tag{1.46}$$

Hence, different values of σ and ω, subject to the constraints in Theorem 1.6, can be chosen to enforce different bounds on the damping ratio, natural frequency and damped natural frequency of the averaged closed loop system.

5.1.5 AN ILLUSTRATIVE EXAMPLE

Below we present an example inspired by a helicopter with HHC. The system is shown in Figure 1.2. Here, an inverted pendulum is mounted on a platform, and the goal is to maintain the platform altitude at the desired level, $z = 0$, and the pendulum at the upright position, $\theta = 0$, using a single force actuator, F, located on the platform. The two goals are clearly conflicting in the sense mentioned in Section 5.1.1. Specifically, if we use the actuator to maintain the platform at $z = 0$, the pendulum will topple down; if we use the actuator to force the platform to move with an acceleration lesser than $-g$, the pendulum will stay in the upright position but the platform will not be at $z = 0$. Therefore, closed loop vibrational control is the method of choice.

The Lagrange equations for the system at hand are :

$$(m_1+m_2)\ddot{z} - m_2 l\ddot{\theta}\sin\theta - m_2 l\dot{\theta}^2\cos\theta + (m_1+m_2)g = F - \zeta\dot{z} \,, \tag{1.47}$$

$$m_2 l^2\ddot{\theta} - m_2 l(\ddot{z}+g)\sin\theta = -\eta\dot{\theta} \,, \tag{1.48}$$

where m_1, z and ζ denote the mass, altitude and damping coefficient of the platform, m_2, θ, l, and η represent the mass, angle, length and damping coefficient of the pendulum. Following the ideology of closed loop vibrational control, we choose F with frequency-separated components :

$$F = (m_1+m_2)g - kz + \frac{\alpha}{\epsilon} f_1\left(\frac{t}{\epsilon}\right) \,. \tag{1.49}$$

Here the first term compensates in an open loop fashion for the system's weight, the second term is the low frequency feedback to maintain $z = 0$, and the third term is the high frequency control to maintain $\theta = 0$.

To reduce (1.47), (1.48) to the form (1.1), we linearize all the nonlinearities in (1.47) and (1.48) retaining, however, the interaction term $\ddot{z}\theta$ in (1.48). Assuming that $f_1(t/\epsilon) = \sin(t/\epsilon)$, solving (1.47) as $t \to \infty$ and substituting the resulting periodic function in (1.48), we obtain a system of the form (1.1) with

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & \frac{-\eta}{m_2 l^2} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, f\left(\frac{t}{\epsilon}\right) = \sin\left(\frac{t}{\epsilon} + \phi\right) ,$$

and

$$u = \frac{K}{\epsilon} x , \tag{1.50}$$

where

$$K = \begin{bmatrix} \frac{-\alpha}{l(m_1 + m_2)} & 0 \end{bmatrix} , \tag{1.51}$$

and ϕ is the phase shift introduced by the system (1.47) with input $f_1(t/\epsilon)$. Since Tr $A < 0$ and (A, B) is controllable, according to Theorem 1.1, there exists an ϵ_0 such that for all $0 < \epsilon \leq \epsilon_0$, the system (1.50) is stabilizable by a state space feedback. Since, in addition, (C, A) is observable, the system is stabilizable by output feedback as well (Theorem 1.2).

Further according to Theorem 1.5, real pole assignment in the region $D(\sigma, \omega)$ is possible if and only if

$$\begin{aligned} \sigma &\geq \frac{-\eta}{2m_2 l^2} , \\ \frac{-g}{l} &\leq \frac{\eta^2}{4m_2^2 l^4} . \end{aligned} \tag{1.52}$$

The second condition is always met. Therefore, for real pole assignment, σ has to be chosen so that the condition (1.52) is satisfied.

According to Theorem 1.6, complex pole assignment into the region $D(\sigma, \omega)$ is possible if and only if (1.52) is met and, in addition,

$$\frac{-g}{l} \leq \frac{\eta^2}{4m_2^2 l^4 \sigma^2}(\sigma^2 + \omega^2) ,$$

is satisfied. Again, since the last condition is always satisfied, (1.52) is necessary and sufficient for both real and complex pole assignment in $D(\sigma, \omega)$.

To design a specific controller for (1.47), (1.48), we average equation (1.50) to obtain

$$\dot{\bar{\xi}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{\alpha^2}{2l^2(m_1+m_2)^2} & \frac{-\eta}{m_2 l^2} \end{bmatrix} \bar{\xi} .$$

Then, if we choose the closed loop poles at

$$\lambda_{1c} = \lambda_{2c} = -\frac{\eta}{2m_2 l^2} ,$$

the resulting control gain K is given in (1.51) with

$$\alpha = \sqrt{2}l(m_1 + m_2)\sqrt{\frac{\eta^2}{4m_2^2l^4} + \frac{g}{l}} \ . \tag{1.53}$$

In general, if (1.52) is met, the gain K that assigns the closed loop poles at λ_{1c} and λ_{2c}, is expressed by (1.51) with:

$$\alpha = \sqrt{2}l(m_1 + m_2)\sqrt{\lambda_{1c}\lambda_{2c} + \frac{g}{l}} \ . \tag{1.54}$$

If we choose the observer gain $L = [l_1 \ \ l_2]$ to ensure that the observer poles are at

$$\lambda_{1o} = \lambda_{2o} = \gamma\lambda_{1c} = -\frac{\gamma\eta}{2m_2l^2} \ , \qquad \gamma > 0 \ ,$$

then

$$l_1 = \frac{(\gamma - 1)\eta}{m_2l^2} \ ,$$
$$l_2 = \frac{(\gamma - 2)^2\eta^2}{4m_2^2l^4} + \frac{g}{l} \ .$$

In general, the observer gain L that assigns the observer poles at $\lambda_{1o}, \lambda_{2o}$ is given by

$$l_1 = -\lambda_{1o} - \lambda_{2o} - \frac{\eta}{m_2l^2} \ ,$$
$$l_2 = \lambda_{1o}\lambda_{2o} - \frac{\eta l_1}{m_2l^2} + \frac{g}{l} \ .$$

To illustrate the behavior of the pendulum-platform system with closed loop vibrational control, we carried out numerical simulations of equations (1.47), (1.48) with the following parameters : $m_1 = 0.1$ kg, $m_2 = 0.01$ kg, $g = 9.8$ m/s^2, $l = 1$ m, $\eta = 0.1, \zeta = 0.6$. The control law has been chosen as in (1.49) with $k = 5, \alpha = 2$ (see (1.54)) and $\epsilon = 0.01$. The results are illustrated in Figures 1.3 and 1.4 for the initial conditions $z = 0$ m and $\theta = 0.1$ rad. In addition, we introduced impulsive perturbations at the platform (at $t = 2.5$ sec) and at the pendulum (at $t = 3.5$ sec). As it follows from these graphs, the closed loop vibrational control indeed ensures, on the average, simultaneous satisfaction of the two conflicting objectives using a single actuator. Note that the limit cycle in the dynamics of the platform is due to the fact that the controller (1.51), derived for the linear system (1.1), (1.50), is, in fact, applied to the original nonlinear system (1.47), (1.48). This is why closed loop vibrational control may lead to the satisfaction of the two conflicting goals not pointwise in time but only on the average.

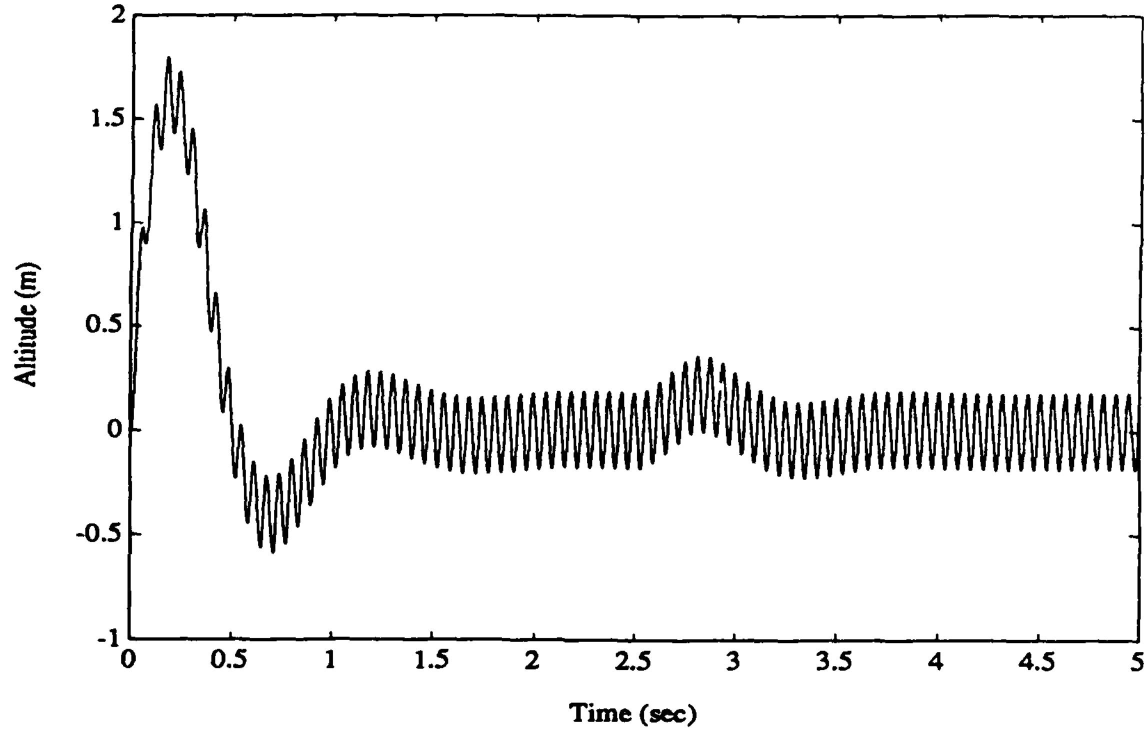

Figure 1.3: Platform Response with Closed Loop Vibrational Control

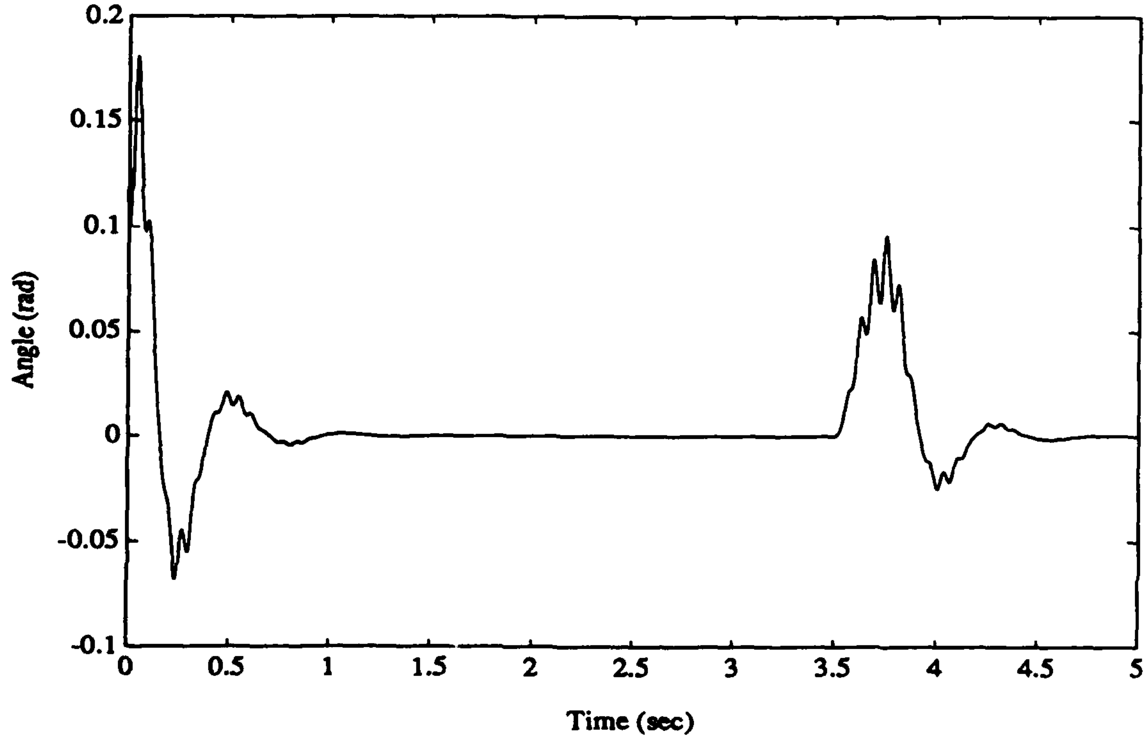

Figure 1.4: Pendulum Response with Closed Loop Vibrational Control

5.2 PART 2. STABILITY ROBUSTNESS IN CLOSED LOOP VIBRATIONAL CONTROL

5.2.1 INTRODUCTION

Consider the system

$$\begin{aligned} \dot{x} &= Ax + Buf\left(\frac{t}{\epsilon}\right) , \\ f(t) &= f(t+T) , \ T \neq 0 , \end{aligned} \tag{2.1}$$

$$\frac{1}{T}\int_0^T f(t)\,dt = 0 ,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and ϵ is a small positive parameter, $0 < \epsilon \ll 1$.

A control theory for system (2.1), referred to as closed loop vibrational control, has been developed in Part 1 of this report. In particular, the following has been proved :

Theorem 2.1: There exists K and $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ system (2.1) is stabilizable by a state space feedback

$$u = \frac{K}{\epsilon}x = \frac{1}{\epsilon}[k_n \ k_{n-1} \ \ldots \ k_2 \ k_1]x \tag{2.2}$$

if and only if (A, B) is stabilizable and the sum of all the controllable eigenvalues of A is negative.

This section is devoted to the property of *stability robustness* in closed loop vibrational control. Both synthesis and analysis problems are addressed. In the synthesis problem, it is assumed that (2.1) is the nominal plant, whereas the true plant is defined by

$$\dot{x} = (A + \Delta A)x + Buf\left(\frac{t}{\epsilon}\right) , \tag{2.3}$$

where ΔA is the perturbation matrix. Assuming that the characteristic polynomial of (2.3) belongs to a polytope, Section 5.2.2 below gives a condition which guarantees the existence of a controller (2.2) such that the closed loop system (2.3), (2.2) is asymptotically stable for all members of the polytope. In the analysis problem, Section 5.2.3 gives a bound in the spectral radius of ΔA so that for all ΔA's that meet this bound, a controller (2.2), which stabilizes the nominal plant (2.1), also stabilizes the perturbed system (2.3). In Section 5.2.4, we present a case where closed loop vibrational control modifies the structure of the perturbation matrix ΔA and, therefore, may lead to robustness properties stronger than those of conventional (time-invariant) control. In Section 5.2.5, we will consider the robustness of closed loop vibrational control in the presence of high frequency unmodeled dynamics.

5.2.2 SYNTHESIS

Assume that A and B in (2.1) are in the controllable canonical form

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{n0} & -a_{(n-1)0} & \cdots & -a_{10} \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} ,$$

where a_{i0} are the coefficients of the characteristic polynomial of the nominal system (2.1). Assume that the perturbation matrix ΔA affects the last row of the system matrix A, and that the characteristic polynomial of the perturbed system (2.3) belongs to the class $\mathcal{P}$ defined as follows:

$$\mathcal{P} = \left\{ p(s) = \sum_{j=1}^{m} \alpha_j p_j(s) : \alpha_j \geq 0, j = 1, \ldots, m; \sum_{j=1}^{m} \alpha_j = 1 \right\} .$$

Here

$$p_j(s) = s^n + a_1^j s^{n-1} + \ldots + a_n^j, j = 1, \ldots, m , \tag{2.4}$$

are the vertex polynomials and $a_i^j, i = 1, \ldots, n$ denote the ith coefficient of the jth vertex polynomial.

Theorem 2.2: There exists K and ϵ_0 such that for all $0 < \epsilon \leq \epsilon_0$, any perturbed system (2.3) with open loop characteristic polynomial in $\mathcal{P}$ can be stabilized by a single controller (2.2) if and only if all coefficients $a_1^j, j = 1, \ldots, m$, defined in (2.4), are positive.

The proof of Theorem 2.2, is based on the following lemmas:

Lemma 2.1: [61] Consider a polynomial

$$p(s) = s^n + \psi_1 s^{n-1} + \ldots + \psi_n , \tag{2.5}$$

where $n \geq 3, \psi_i > 0, i = 1, \ldots, n$. Let δ^* be the positive real solution of the equation $\delta(\delta+1)^2 = 1$. Then polynomial (2.5) is Hurwitz if the coefficients ψ_i satisfy the condition

$$\frac{\psi_{l-1}\psi_{l+2}}{\psi_l \psi_{l+1}} \leq \delta^*, \qquad l = 1, \ldots, n-2 . \tag{2.6}$$

Inequality (2.6) will henceforth be referred to as the Lipatov's conditon [61].

Lemma 2.2: Consider the following four polynomials:

$$\begin{aligned} p_{k1}(s) &= s^n + \gamma_1^+ s^{n-1} + \gamma_2^+ s^{n-2} + \gamma_3^- s^{n-3} + \gamma_4^- s^{n-4} + \ldots \\ p_{k2}(s) &= s^n + \gamma_1^- s^{n-1} + \gamma_2^- s^{n-2} + \gamma_3^+ s^{n-3} + \gamma_4^+ s^{n-4} + \ldots \\ p_{k3}(s) &= s^n + \gamma_1^- s^{n-1} + \gamma_2^+ s^{n-2} + \gamma_3^+ s^{n-3} + \gamma_4^- s^{n-4} + \ldots \\ p_{k4}(s) &= s^n + \gamma_1^+ s^{n-1} + \gamma_2^- s^{n-2} + \gamma_3^- s^{n-3} + \gamma_4^+ s^{n-4} + \ldots , \end{aligned} \tag{2.7}$$

where $n \geq 3, \gamma_i^- \leq \gamma_i^+, 1 \leq i \leq n$ and $\gamma_1^- > 0$. Let coefficients $\overline{a_i}, 1 \leq i \leq n$ be defined as

$$\overline{a_i}(k) = \begin{cases} \gamma_1^- \; k^{-\frac{(n-1)(i-1)}{4} + \frac{(i-1)^2}{4}} - \gamma_i^- & , \text{ if } i \text{ is odd} , \\ k^{-\frac{(n-1)(i+2)}{4} + \frac{i(i-2)}{4}} - \gamma_i^- & , \text{ if } i \text{ is even} ,. \end{cases} \tag{2.8}$$

where $k > 0$. Then there exists a k^* such that for all $0 < k < k^*$, the coefficient $\overline{a_2}$ is positive and the polynomials

$$\begin{aligned} p_{kcl1}(s) &= s^n + (\gamma_1^+ + \overline{a_1}(k))s^{n-1} + (\gamma_2^+ + \overline{a_2}(k))s^{n-2} + (\gamma_3^- + \overline{a_3}(k))s^{n-3} + \ldots \\ p_{kcl2}(s) &= s^n + (\gamma_1^- + \overline{a_1}(k))s^{n-1} + (\gamma_2^- + \overline{a_2}(k))s^{n-2} + (\gamma_3^+ + \overline{a_3}(k))s^{n-3} + \ldots \\ p_{kcl3}(s) &= s^n + (\gamma_1^- + \overline{a_1}(k))s^{n-1} + (\gamma_2^+ + \overline{a_2}(k))s^{n-2} + (\gamma_3^+ + \overline{a_3}(k))s^{n-3} + \ldots \\ p_{kcl4}(s) &= s^n + (\gamma_1^+ + \overline{a_1}(k))s^{n-1} + (\gamma_2^- + \overline{a_2}(k))s^{n-2} + (\gamma_3^- + \overline{a_3}(k))s^{n-3} + \ldots \end{aligned} \tag{2.9}$$

are Hurwitz.

Proof: Let γ_i^j denotes the ith coefficient, $1 \le i \le n$, of the jth polynomial, $1 \le j \le 4$, in (2.7). First, we show that there exists k^* such that for all $0 < k \le k^*$, polynomials (2.9) satisfy Lipatov's condition, i.e.

$$\max_{1 \le l \le n-2, 1 \le j \le 4} \frac{(\gamma_{l-1}^j + \overline{a_{l-1}}(k))(\gamma_{l+2}^j + \overline{a_{l+2}}(k))}{(\gamma_l^j + \overline{a_l}(k))(\gamma_{l+1}^j + \overline{a_{l+1}}(k))} < \delta^* , \tag{2.10}$$

where δ^* is defined in Lemma 3.1. Then, the statement of this lemma follows directly from Lemma 3.1. To show that Lipatov's condition holds, consider

$$f_l(k) \triangleq \max_{1 \le j \le 4} \frac{(\gamma_{l-1}^j + \overline{a_{l-1}}(k))(\gamma_{l+2}^j + \overline{a_{l+2}}(k))}{(\gamma_l^j + \overline{a_l}(k))(\gamma_{l+1}^j + \overline{a_{l+1}}(k))} , \quad 1 \le l \le n-2 . \tag{2.11}$$

It is easy to see that the function $f_l(k)$ is monotonically increasing with respect to k. Indeed, define $\overline{a_0}(k) = 0$ and $\gamma_0^- = \gamma_0^+ = 1$. Let $c_q^+(k) = \gamma_q^+ + \overline{a_q}(k), c_q^-(k) = \gamma_q^- + \overline{a_q}(k)$, and $\Delta c_q = c_q^+(k) - c_q^-(k) = \gamma_q^+ - \gamma_q^-$ for $0 \le q \le n$. Then

$$\begin{aligned} f_l(k) &= \frac{c_{l-1}^+(k)c_{l+2}^+(k)}{c_l^-(k)c_{l+1}^-(k)} , \quad 1 \le l \le n-2 , & (2.12)\\ &= \frac{c_{l-1}^-(k)c_{l+2}^-(k)}{c_l^-(k)c_{l+1}^-(k)} \left(1 + \frac{\Delta c_{l-1}}{c_{l-1}^-(k)}\right)\left(1 + \frac{\Delta c_{l+2}}{c_{l+2}^-(k)}\right) \\ &= \psi(k)\left(1 + \frac{\Delta c_{l-1}}{c_{l-1}^-(k)}\right)\left(1 + \frac{\Delta c_{l+2}}{c_{l+2}^-(k)}\right) , & (2.13) \end{aligned}$$

where

$$\psi(k) = \begin{cases} k^{\frac{n+1}{2}} & , \text{ if } l = 1 , \\ k & , \text{ if } 2 \le l \le n-2 . \end{cases} \tag{2.14}$$

We will show that each of the three factors in the right hand side of (2.13) is a non-decreasing function of k. Indeed, $c_0^- = 1, c_1^- = \gamma_1^- > 0$, and since $c_i^-(k) = \gamma_i^- + \overline{a_i}(k)$ for $2 \le i \le n$, we observe from (2.8) that the exponent of k in $c_i^-(k)$ is negative or zero. Also $\Delta c_q \ge 0$ for $0 \le q \le n$ and from (2.14), $\psi(k_1) < \psi(k_2)$ for $0 < k_1 < k_2$. Thus, $f_l(k)$ is a continuous and monotonically increasing function of k. Also, as it follows from (2.13), $f_l(0) = 0$, and $\lim_{k \to \infty} f_l(k) = \infty$. Therefore, by the intermediate value theorem, there exists k_l^* such that

$$f_l(k_l^*) = \delta^* , \quad 1 \le l \le n-2 . \tag{2.15}$$

Let $k^* = \min_l k_l^*$. Since $f_l(k)$ is monotonically increasing with k, it follows that $f_l(k^*) \le f_l(k_l^*), 1 \le l \le n-2$. Therefore,

$$\max_l f_l(k) = \max_{l,j} \frac{(\gamma_{l-1}^j + \overline{a_{l-1}}(k))(\gamma_{l+2}^j + \overline{a_{l+2}}(k))}{(\gamma_l^j + \overline{a_l}(k))(\gamma_{l+1}^j + \overline{a_{l+1}}(k))} < \max_l f_l(k^*) \le \delta^* ,$$

is satisfied for all $0 < k < k^*$. Finally, to ensure that $\overline{a_2} > 0$, i.e.

$$k^{1-n} - \gamma_2^- > 0 , \tag{2.16}$$

choose k in (2.8) as follows:

(i) if $\gamma_2^- \leq 0$, let $0 < k < k^*$,

(ii) if $\gamma_2^- > 0$, let $0 < k < \min[(\gamma_2^-)^{\frac{1}{1-n}}, k^*]$. **Q.E.D.**

Proof: (of Theorem 2.2) Consider the closed loop system (2.3), (2.2) :

$$\dot{x} = (A + \Delta A + BKf\left(\frac{t}{\epsilon}\right))x \ . \tag{2.17}$$

In the fast time $\tau = t/\epsilon$, the closed loop system (2.17) is

$$\frac{dx}{d\tau} = (\epsilon(A + \Delta A) + BKf(\tau))x \ . \tag{2.18}$$

Let $\Phi(\tau)$ be a fundamental matrix for $BKf(\tau)$. Reducing (2.18) into the standard form [21] and then applying the averaging principle, we have the following averaged system

$$\dot{\overline{x}} = \overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right)(A + \Delta A)\Phi\left(\frac{t}{\epsilon}\right)} \ \overline{x} \ , \tag{2.19}$$

where the bar denotes time averaging operation. According to Theorem 2 of [17], there exists ϵ_0 such that for all $0 < \epsilon \leq \epsilon_0$, system (2.17) is asymptotically stable if the averaged system (2.19) is also asymptotically stable. Moreover, it was shown in [52] that the averaged closed loop system (2.19) has the following characteristic polynomial:

$$p_{cl}(s) = s^n + (a_1 + \overline{a_1})s^{n-1} + (a_2 + \overline{a_2})s^{n-2} + \ldots + (a_n + \overline{a_n}) \ , \tag{2.20}$$

where

$$\begin{aligned} \overline{a_i} &= k_2 k_i \overline{\phi^2} \ , \qquad i = 1, \ldots, n \ , \\ k_1 &= 0 \ , \\ \phi(t) &= \int_0^t f(\tau) \ d\tau \ , \end{aligned} \tag{2.21}$$

and the coefficients a_i are the coefficients of the characteristic polynomial of the perturbed system (2.3).

The necessity part is obvious: Since the a_1 coefficient in (2.20) cannot be adjusted, it is necessary that $a_1^j > 0, j = 1, \ldots, m$ for (2.20) to be Hurwitz.

Sufficiency is based on Kharitonov's theorem [58] and Lemma 3.2. For $0 < n \leq 2$, it is easy to construct $\overline{a_i}, 1 \leq i \leq n$ to stabilize (2.20). Hence, we will consider the case when $n \geq 3$. Let $\gamma_i^- = \min_j a_i^j, \gamma_i^+ = \max_j a_i^j$, where a_i^j denotes the ith coefficient, $1 \leq i \leq n$, of the jth vertex polynomial $1 \leq j \leq m$. We first construct four interval polynomials which contain the polynomial (2.20):

$$\begin{aligned} p_{kcl1}(s) &= s^n + c_1^+ s^{n-1} + c_2^+ s^{n-2} + c_3^- s^{n-3} + c_4^- s^{n-4} + \ldots \\ p_{kcl2}(s) &= s^n + c_1^- s^{n-1} + c_2^- s^{n-2} + c_3^+ s^{n-3} + c_4^+ s^{n-4} + \ldots \\ p_{kcl3}(s) &= s^n + c_1^- s^{n-1} + c_2^+ s^{n-2} + c_3^+ s^{n-3} + c_4^- s^{n-4} + \ldots \\ p_{kcl4}(s) &= s^n + c_1^+ s^{n-1} + c_2^- s^{n-2} + c_3^- s^{n-3} + c_4^+ s^{n-4} + \ldots \ , \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} c_i^- &= \gamma_i^- + \overline{a_i} \ , \qquad 1 \leq i \leq n \ , \\ c_i^+ &= \gamma_i^+ + \overline{a_i} \ . \end{aligned} \tag{2.23}$$

Next, we need to determine $\overline{a_i}$ in (2.23) satisfying $\overline{a_1} = 0, \overline{a_2} > 0$ so that the four interval polynomials (2.22) are Hurwitz. It follows from Kharitonov's theorem that if the polynomials (2.22) are Hurwitz, then (2.20) is also Hurwitz. Choose $\overline{a_i}, 1 \leq i \leq n$ as defined in (2.8), then from Lemma 3.2, there exists a k^* such that for all $0 < k < k^*$, polynomials (2.22) are Hurwitz. The corresponding stabilizing state feedback gain K can be computed from (2.21). As it follows [17], for each Hurwitz averaged closed loop polynomial $p_{cl}(s)$ in (2.20) there exists $\epsilon_{\mathcal{P}} > 0$ such that for all $0 < \epsilon \leq \epsilon_{\mathcal{P}}$ the corresponding closed loop system of (2.3), (2.2) is also asymptotically stable. In [17], a lower bound of $\epsilon_{\mathcal{P}}$ was derived. This bound for $\epsilon_{\mathcal{P}}$ is a continuous function of the coefficients of the open loop characteristic polynomial $p(s)$. Since the set of open loop characteristic polynomial $p(s) \in \mathcal{P}$ is closed and bounded, it follows from the property of continous functions that a uniform lower bound of $\epsilon_{\mathcal{P}}$ exists. The proof is completed by setting $\epsilon_0 = \min_{\mathcal{P}} \epsilon_{\mathcal{P}}$. **Q.E.D.**

Remark 2.1 : The condition that the coefficient $a_1^j > 0, j = 1, \ldots, m$, in Theorem 2.2 is equivalent to the requirement that the trace of the perturbed matrix $A + \Delta A$ be negative.

Remark 2.2 : Theorem 2.2 is an extension of the result obtained in [62] for interval polynomials. The assumption that $\mathcal{P}$ is polytopic is weaker than Kharitonov's interval polynomial assumption because it allows for linearly dependent coefficient perturbations.

Example 2.1 : Consider a 6th order system (2.3) with open loop characteristic polynomial $p(s) \in \mathcal{P}$ where $\mathcal{P}$ is a polytope of polynomials (convex hull) with the following four vertex polynomials

$$
\begin{aligned}
p_1(s) &= s^6 + 0.5s^5 - 0.6s^4 + 1.5s^3 + 2.5s^2 + 3.4s - 3 , \\
p_2(s) &= s^6 + 0.7s^5 - s^4 + 2s^3 + 2s^2 + 4s - 1 , \\
p_3(s) &= s^6 + s^5 + s^4 + s^3 + 3s^2 + 3s + 1 , \\
p_4(s) &= s^6 + 0.6s^5 + 0.2s^4 + 1.1s^3 + 2.9s^2 + 3.8s + 2 .
\end{aligned}
$$

This uncertain open loop system is unstable since the vertex polynomials $p_1(s)$ and $p_2(s)$ are obviously unstable. With vibrational state feedback control (2.2), the characteristic polynomial (2.20) of the resulting averaged closed loop system is bounded by the interval polynomials (2.22). Next, we construct $\overline{a_i}, 1 \leq i \leq 6$, as defined in (2.8) and determine $k > 0$ so that the interval polynomials (2.22) are Hurwitz. Following (2.12), we define the function $f_l(k)$ as:

$$
f_l(k) = \frac{c_{i-1}^+(k) c_{i+2}^+(k)}{c_i^-(k) c_{i+1}^-(k)} , \quad 1 \leq l \leq 4 ,
$$

and hence

$$
\begin{aligned}
f_1(k) &= k^{3.5}\left(1 + \frac{\Delta c_3 k^{1.5}}{\gamma_1^-}\right) , \\
f_2(k) &= k\left(1 + \frac{\Delta c_1}{\gamma_1^-}\right)\left(1 + \Delta c_4 k^{5.5}\right) , \\
f_3(k) &= k\left(1 + \Delta c_2 k^5\right)\left(1 + \frac{\Delta c_5 k}{\gamma_1^-}\right) , \\
f_4(k) &= k\left(1 + \frac{\Delta c_3 k^{1.5}}{\gamma_1^-}\right)\left(1 + \Delta c_6 k^4\right) ,
\end{aligned}
$$

where $\Delta c_i = c_i^+ - c_i^- = \max_j a_i^j - \min_j a_i^j$. Consequently, with $\gamma_1^- = \min_j a_1^j = 0.5, \delta^* = 0.4656$, we compute $k_l^*, 1 \leq l \leq 4$, such that

$$
f_1(k_1^*) = (k_1^*)^{3.5}(1 + 2(k_1^*)^{1.5}) = \delta^* , \tag{2.24}
$$

$$f_2(k_2^*) = k_2^*(1+1)(1+(k_2^*)^{5.5}) = \delta^* , \quad (2.25)$$
$$f_3(k_3^*) = k_3^*(1+2(k_3^*)^5)(1+2k_3^*) = \delta^* , \quad (2.26)$$
$$f_4(k_4^*) = k_4^*(1+2(k_4^*)^{1.5})(1+5(k_4^*)^4) = \delta^* . \quad (2.27)$$

Solving (2.24)-(2.27), we obtain $k_1^* = 0.6536, k_2^* = 0.2327, k_3^* = 0.2925, k_4^* = 0.3232$. Thus $k^* = \min_l k_l^* = k_2^* = 0.2327$. Since $\min_j a_2^j < 0$, for $0 < k \leq 0.2327$, the interval polynomials (2.22) and consequently, the characteristic polynomial (2.20) will be stabilized. Arbitrarily choose $k = 0.23$, then $\overline{a_2} = 1554.7, \overline{a_3} = 3.5, \overline{a_4} = 3237.6, \overline{a_5} = -0.8$, and $\overline{a_6} = 360.3$. With $f(\tau) = \sin\tau$, the corresponding feedback gains in (2.2) was $k_1 = 0, k_2 = 55.8, k_3 = 0.1255, k_4 = 116.1, k_5 = -0.0287$, and $k_6 = 12.9$. The stability of the characteristic polynomial (2.20) of the averaged closed loop system with the feedback K, can be verified by performing eigenvalue tests on appropriate Hurwitz matrices corresponding to each of the vertices of the polytope of polynomials $\mathcal{P}$ [63]. From Theorem 2 of [17], the asymptotic stability of the averaged closed loop system will ensure the asymptotic stability of the original system (2.3).

5.2.3 ANALYSIS

Consider system (2.3) with feedback (2.2):

$$\dot{x} = (A + \Delta A + B\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right))\ x . \quad (2.28)$$

Assume that K and ϵ are chosen according to Theorem 2.1 so that (2.28) is asymptotically stable when $\Delta A = 0$. Under this condition, we obtain the following results:

Theorem 2.3: There exists ϵ_0 such that for all $0 < \epsilon \leq \epsilon_0$, the system (2.28) is aymptotically stable if

$$\sigma_{max}\left(\overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right)\Delta A\Phi\left(\frac{t}{\epsilon}\right)}\right) \leq \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} , \quad (2.29)$$

where P is the unique positive definite solution of the Lyapunov equation

$$P\overline{A} + \overline{A}^T P + 2Q = 0 , \quad (2.30)$$

Q is some positive definite matrix and

$$\overline{A} = \overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right)A\Phi\left(\frac{t}{\epsilon}\right)} . \quad (2.31)$$

Proof: From (2.19), the averaged closed loop system's equation of (2.28) is:

$$\begin{aligned}\dot{\overline{x}} &= \overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right)(A+\Delta A)\Phi\left(\frac{t}{\epsilon}\right)}\ \overline{x} , \\ &= (\overline{A} + \overline{\Delta A})\ \overline{x} , \end{aligned} \quad (2.32)$$

where

$$\overline{\Delta A} = \overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right)\Delta A\Phi\left(\frac{t}{\epsilon}\right)} .$$

and the bar denotes time averaging operation.

Note that (2.32) is obtained by the linearity of the averaging operation. From [67], we know that the averaged linear time-invariant system (2.32) is stable if

$$\sigma_{max}(\overline{\Delta A}) \leq \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} . \tag{2.33}$$

where P is the unique matrix that satisfies the Lyapunov equation (2.30), Q is any positive definite matrix and $\sigma_{max}(M)$ represents the largest singular value of M.

Hence, for each ΔA satisfying (2.29), the averaged system (2.32) is asymptotically stable, and there exists $\epsilon_{\Delta A}$ such that for all $0 < \epsilon \leq \epsilon_{\Delta A}$, system (2.28) is also asymptotically stable. In [17], it was shown that $\epsilon_{\Delta A}$ is a continuous function of the elements of the matrix ΔA. Since the set of ΔA satisfying (2.29) is closed and bounded, it follows that there exists a uniform lower bound for $\epsilon_{\Delta A}$. The proof is completed by setting $\epsilon_0 = \min_{\Delta A} \epsilon_{\Delta A}$. **Q.E.D.**

Note that the perturbation bounds given by (2.29) are indirectly imposed on ΔA through the averaged perturbed matrix $\overline{\Delta A}$. Assume that A and B are in the controller canonical form, then

$$\overline{\Delta A} = \Delta A + \Delta A_1 , \tag{2.34}$$

where

$$\Delta A_1 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \\ k_n \sum_{i=1}^{n-1} \Delta A_{in} k_{n-i+1} \overline{\phi^2} & \cdots & k_2 \sum_{i=1}^{n-1} \Delta A_{in} k_{n-i+1} \overline{\phi^2} & 0 \end{bmatrix} , \tag{2.35}$$

and ΔA_{ij} denotes the (i,j)th element of the perturbation matrix ΔA.

Corollary 2.1: Assume that A and B are in controller canonical form. Then there exists an ϵ_0 such that for all $0 < \epsilon \leq \epsilon_0$, the system (2.28) is stable if

$$\sigma_{max}(\Delta A) + \sigma_{max}(\Delta A_1) \leq \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} . \tag{2.36}$$

Proof: The proof follows directly from Theorem 2.1, (2.34) and the fact that $\sigma_{max}(\Delta A + \Delta A_1) \leq \sigma_{max}(\Delta A) + \sigma_{max}(\Delta A_1)$. **Q.E.D.**

Example 2.2 : Consider the system (2.28) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} , B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , f(\tau) = \sin(\tau) ,$$

and $K = [2 \;\; 0]$. Applying the averaging principle, we obtain the averaged system (2.32) with

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} .$$

Choose $Q = I$. From (2.36), the allowable range of the perturbation matrix which ensures the asymptotic stability of the original system (2.28) is given by

$$\sigma_{max}(\Delta A) + 2|\Delta A_{12}| \leq 0.382 .$$

5.2.4 A SPECIAL CASE

In a special case, it is possible to show that the structure of $\overline{\Delta A}$ may be different from that of ΔA. This difference can be exploited in order to derive stronger robustness results for closed loop vibrational control than linear time-invariant control. Below we give a necessary condition that describes this situation.

Theorem 2.4: Assume that A and B in (2.1) are in controller canonical form. Then vibrational feedback (2.2) modifies the structure of the perturbation matrix ΔA affecting the averaged system (2.32) only if the first $n-1$ elements in the last column of ΔA are not all zero.

Proof: From equation (2.35), we see that the first $n-1$ elements in the last row of the averaged perturbation matrix $\overline{\Delta A}$ can be modified by the feedback gains $k_i, i = 2, \ldots, n$, only when $\Delta A_{jn} \neq 0, j = 1, \ldots, n-1$. **Q.E.D.**

The above result shows that vibrational control is capable of changing some of the elements of the perturbation matrix ΔA which affects the averaged system. Since the asymptotic stability of the averaged system (2.32) ensures the asymptotic stability of the original system (2.28), the modification in the perturbation matrix affecting the averaged system may change the stability robustness of the original system.

Indeed, given a Hurwitz matrix A_0 and a matrix perturbation direction A_1, it is shown in [69] that the largest interval (r_{min}, r_{max}) such that

$$A_r = A_0 + rA_1$$

is strictly stable for all $r \in (r_{min}, r_{max})$ is given by

$$r_{min} = \frac{1}{\lambda^-_{min}(-T(A_0)^{-1}T(A_1))}, \quad r_{max} = \frac{1}{\lambda^+_{max}(-T(A_0)^{-1}T(A_1))}, \tag{2.37}$$

where $T(\cdot)$ is any linear mapping which transforms the stability problem into a nonsingularity problem, $\lambda^+_{max}(M)$ denotes the maximum positive (real) eigenvalue of a square matrix M, and $\lambda^-_{min}(M)$ denotes the minimum negative (real) eigenvalue of M.

Since the perturbation bounds in (2.37) depend on the perturbation direction A_1, the modification of ΔA into $\overline{\Delta A}$ in the averaged system by closed loop vibrational control may lead to improved stability robustness. This is illustrated in the following example :

Example 2.3 : Consider the system (2.3) with linear time-invariant state feedback

$$\begin{aligned} A_0 &= A + BK \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix}, \end{aligned} \tag{2.38}$$

where A_0 represents the closed loop system matrix. Assume a unidirectional perturbation matrix ΔA of the form

$$\Delta A = rA_1 = r \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix},$$

then the maximal perturbation bounds (2.37) under which stability is preserved is given by $r \in (-1, 4)$.

Next, we consider the same open loop system (2.3) with $f(\tau) = \sin\tau$ and introduce closed loop vibrational feedback of the form (2.2). Choose $K = [2\sqrt{2}\ \ \sqrt{2}\ \ 0]$ so that the averaged closed loop matrix $\overline{A}$ in (2.32) is identical to A_0. This ensures that the dynamics of the system (2.3) with closed loop vibrational control is similar to the linear time-invariant state feedback (2.38). Then $\overline{\Delta A}$ in (2.32) will have the form

$$\overline{\Delta A} = r\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

From (2.37), we determine system (2.32) is stable if and only if $r \in (-1, 8)$, which compares favorably with the interval $(-1, 4)$ obtained for linear time-invariant state feedback. Unfortunately, this situation does not always take place:

Example 2.4 : Consider the same nominal closed loop system as in Example 2.3. However, assume that ΔA now has the form :

$$\Delta A = r\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then with linear time-invariant state feedback (2.38), the maximal range which ensures the stability of the closed loop system is $r \in (-1, 4)$.

With closed loop vibrational feedback (2.2), the averaged perturbation matrix $\overline{\Delta A}$ becomes

$$\overline{\Delta A} = r\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$

and closed loop stability of the averaged system is ensured if $r \in (-1, 3.6298)$.

5.2.5 UNMODELED DYNAMICS

Generally, a model of the system to be controlled may become less accurate at high frequencies because of unknown or unmodeled parasitic dynamics. Moreover, these parasitic dynamics may change with time or other physical parameters, and so cannot be confidently modeled.

In state space plant descriptions, the addition of high frequency parasitic dynamics is called a singular perturbation, because the perturbed plant has more states than the plant. Consider the following singularly perturbed form of (2.1)

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1 u f\left(\frac{t}{\epsilon}\right), \\ \mu\dot{z} &= A_{21}x + A_{22}z + B_2 u f\left(\frac{t}{\epsilon}\right), \end{aligned} \qquad (2.39)$$

where $x \in I\!R^n$ is the modeled state, $z \in I\!R^m$ is the unmodeled high frequency state, $u \in I\!R$ is the control, $f(t)$ is a periodic, average zero scalar function, μ is a small positive constant and $0 < \epsilon \ll 1$.

Following (2.2), we synthesize a state feedback of the form

$$u = \frac{K}{\epsilon}x, \qquad (2.40)$$

where only the modeled state x is available for feedback.

The resulting closed loop system (2.39), (2.40) is given by

$$\begin{aligned} \dot{x} &= \tilde{A}_{11}(t)x + \tilde{A}_{12}(t)z \ , \\ \mu\dot{z} &= \tilde{A}_{21}(t)x + \tilde{A}_{22}(t)z \ , \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} \tilde{A}_{11}(t) &= A_{11} + B_1\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right) \ , \\ \tilde{A}_{12}(t) &= A_{12} \ , \\ \tilde{A}_{21}(t) &= A_{21} + B_2\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right) \ , \\ \tilde{A}_{22}(t) &= A_{22} \ . \end{aligned} \tag{2.42}$$

In this section, we determine conditions on $A_{11}, A_{12}, A_{21}, A_{22}, K$, and μ which ensure the stability of the system (2.41). To answer this question, we depend strongly on the following result:

Theorem 2.5: [70] Consider the linear time-varying system

$$\begin{aligned} \dot{x} &= \tilde{A}_{11}(t)x + \tilde{A}_{12}(t)z \ , \\ \mu\dot{z} &= \tilde{A}_{21}(t)x + \tilde{A}_{22}(t)z \ . \end{aligned} \tag{2.43}$$

Suppose $\tilde{A}_{ij}(t)$ are continuously differentiable, $\dot{\tilde{A}}_{ij}(t)$ are bounded, for $i,j = 1,2$, Re $\lambda(\tilde{A}_{22}(t)) < 0$ and the reduced system

$$\dot{x} = \left[\tilde{A}_{11}(t) - \tilde{A}_{12}(t)\tilde{A}_{22}^{-1}(t)\tilde{A}_{21}(t)\right]x \ , \tag{2.44}$$

is uniformly asymptotically stable. Then there exists a positive number μ_0 such that whenever μ belongs to $(0,\mu_0)$, the system (2.43) is uniformly asymptotically stable.

Inspecting (2.41), it is clear that $\tilde{A}_{ij}(t)$ are continuously differentiable, $\dot{\tilde{A}}_{ij}(t)$ are bounded for $i,j = 1,2$. Furthermore, the reduced system of (2.41) corresponding to the form of equation (2.44) is

$$\dot{x} = \left[A_{11} - A_{12}A_{22}^{-1}A_{21} + (B_1 - A_{12}A_{22}^{-1}B_2)\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right)\right]x \ , \tag{2.45}$$

Hence the stability of the singularly perturbed system with closed loop vibrational control is answered by the following theorem :

Theorem 2.6: Consider the system (2.39). Suppose that $(A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1 - A_{12}A_{22}^{-1}B_2)$ is stabilizable and the sum of all the controllable eigenvalues of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is negative. Then there exists a K, and an $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, system (2.45) is uniformly asymptotically stable. In addition, if A_{22} is Hurwitz, there exists a positive number μ_0 such that whenever μ belongs to $(0,\mu_0)$, the closed loop system (2.39), (2.40) is uniformly asymptotically stable.

Proof: The stabilizability condition for (2.45) follows directly from the results of Theorem 2.1. From Theorem 2.5, the uniform stability of (2.45) and the asymptotic stability of A_{22} will in turn ensure the existence of a $\mu_0 > 0$ such that the closed loop system (2.39), (2.40) is uniformly asymptotically stable for all $0 < \mu < \mu_0$. **Q.E.D.**

A conservative estimate of the value μ_0 can be computed as in [71] or [72]. In particular, the estimate obtained by Kokotovic, Khalil and O'Reilly [72], based on the solutions of two Lyapunov equations, is shown in most cases to be less conservative than that obtained by Javid [71].

5.3 PART 3. DESIGN OF VIBRATIONAL CONTROLLER FOR PERFORMANCE AND DISTURBANCE REJECTION

5.3.1 INTRODUCTION

Consider the class of dynamical systems described by the following equations:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t)f\left(\frac{t}{\epsilon}\right) + B_w w(t) , \qquad (3.1)\\
z(t) &= C_z x(t) + D_{zu}u(t) + D_{zw}w(t) ,\\
y(t) &= Cx(t) + D_{yw}w(t) ,\\
f(t) &= f(t+T) , \; T \neq 0 ,
\end{aligned}$$

$$\frac{1}{T}\int_0^T f(t)\,dt = 0 ,$$

$$0 < \epsilon \ll 1 ,$$

where $x \in I\!R^n$ is the state, $y \in I\!R$ is the measured output, $z \in I\!R$ is the regulated output, $u \in I\!R$ is the control input, $w \in I\!R$ is the exogenous input, $f(t)$ is a known periodic, average zero scalar function, and ϵ is a small positive constant. A characteristic feature of this system is that the control enters the open loop system dynamics as an amplitude of a periodic, zero average function, and this amplitude can be chosen to depend on the system's states or, more generally, output. An example of such a system is the helicopter with Higher Harmonic Control (HHC), where periodic feathering of rotor blades around a fixed pitch angle is introduced in order to suppress the fuselage vibrations [3]-[5]. Another example is the periodic operation of chemical reactors [19] where the input flow vibrations are introduced so that the closed loop system behaves as desired.

The theory for the control of system (3.1), referred to as closed loop vibrational control, has been developed in Section 5.1 of this report. In particular, necessary and sufficient conditions for the output feedback stabilizability of the system (3.1) with the observer-based output controller, K_{nom}, defined by

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Buf\left(\frac{t}{\epsilon}\right) + L(y - \hat{y}) , \qquad (3.2)\\
u &= \frac{K}{\epsilon}\hat{x} ,\\
\hat{y} &= C\hat{x} ,
\end{aligned}$$

have been established. However, the class of all stabilizing output feedback controllers has not been characterized, although such a characterization could be quite useful, e.g. to satisfy desired performance specification [6].

The purpose of this section is to present such a characterization using the parametrization approach. The results obtained are quite similar to the Youla parametrization [73]. Specifically, we show that the averaged closed loop transfer function resulting from a stabilizing output feedback controller is an affine function of an arbitrary stable transfer function.

This section is organized as follows : Section 5.3.2 determines the set of all stabilizing controllers for system (3.1). The parametrization of the corresponding averaged closed loop transfer function is described in Section 5.3.3. To illustrate the results, Section 5.3.4 presents an example where the parametrization of the averaged closed loop transfer function is used to design a controller that achieves certain step response specifications for the original, non-averaged system. The disturbance decoupling capability of closed loop vibrational control is discussed in Section 5.3.5.

5.3.2 PARAMETRIZATION OF STABILIZING OUTPUT CONTROLLERS

The following result for stabilization of the system (3.1) with output feedback (3.2) has been derived in Section 5.1:

Theorem 3.1: There exists K, L and an $0 < \epsilon_0 \ll 1$ such that for all $0 < \epsilon \leq \epsilon_0$ system (3.1) is stabilizable by the output feedback (3.2) if and only if (A, B, C) is stabilizable and detectable and the sum of the controllable eigenvalues of A is negative. Moreover, the separation principle holds, i.e. the choice of K and L can be carried out independently.

Although (3.2) contains a time varying function, $f(t/\epsilon)$, it nevertheless can be viewed as a time invariant controller since, firstly, $f(t/\epsilon)$ is a part of the control signal and, secondly, K and L are constant gains. Therefore, it is natural to parametrize all stabilizing controllers for (3.1) in the class of rational transfer functions.

To accomplish this, augment the closed loop system (3.1), (3.2) by an auxilliary controller K_Q as shown in Figure 3.1, where K_Q is defined by :

$$\begin{aligned} \dot{x}_Q &= A_Q x_Q + B_Q e \ , \\ v &= \frac{C_Q}{\epsilon} x_Q + \frac{D_Q}{\epsilon} e \ , \\ e &= y - C\hat{x} \ , \end{aligned} \tag{3.3}$$

and y and $\hat{x}$ are defined in (3.1) and (3.2) respectively. Note that K_Q is a compensator characterized by the high gain $\frac{1}{\epsilon}$:

$$Q(s) = \frac{1}{\epsilon}\left[C_Q(sI - A_Q)^{-1}B_Q + D_Q\right] \ .$$

The state space realization of the augmented controller (K_{nom}, K_Q) is :

$$\dot{x}_e = A_e x_e + B_e u f\left(\frac{t}{\epsilon}\right) + L_e(y - C_e x_e) \ , \tag{3.4}$$

$$u = \frac{K_e}{\epsilon} x_e + \frac{D_Q}{\epsilon}(y - C_e x_e) \ , \tag{3.5}$$

where

$$\begin{aligned} x_e &= \begin{bmatrix} \hat{x} \\ x_Q \end{bmatrix}, A_e = \begin{bmatrix} A & 0 \\ 0 & A_Q \end{bmatrix}, B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, \\ C_e &= \begin{bmatrix} C & 0 \end{bmatrix}, K_e = \begin{bmatrix} K & C_Q \end{bmatrix}, L_e = \begin{bmatrix} L \\ B_Q \end{bmatrix}. \end{aligned}$$

Theorem 3.2: Assume (3.1) is internally stabilized by the nominal controller (3.2). Then it is also internally stabilized by the augmented controller (3.4), (3.5) if and only if A_Q is stable.

Proof: The necessity is proved by the following consideration. The internal dynamical equations for the closed loop system with the augmented controller (3.2), (3.3) are

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \\ \dot{x}_Q \end{bmatrix} = \begin{bmatrix} A + B\frac{D_Q}{\epsilon}Cf\left(\frac{t}{\epsilon}\right) & (B\frac{K}{\epsilon} - B\frac{D_Q}{\epsilon}C)f\left(\frac{t}{\epsilon}\right) & B\frac{C_Q}{\epsilon}f\left(\frac{t}{\epsilon}\right) \\ (L + B\frac{D_Q}{\epsilon}f\left(\frac{t}{\epsilon}\right))C & A - LC + (B\frac{K}{\epsilon} - B\frac{D_Q}{\epsilon}C)f\left(\frac{t}{\epsilon}\right) & B\frac{C_Q}{\epsilon}f\left(\frac{t}{\epsilon}\right) \\ B_Q C & -B_Q C & A_Q \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ x_Q \end{bmatrix}.$$

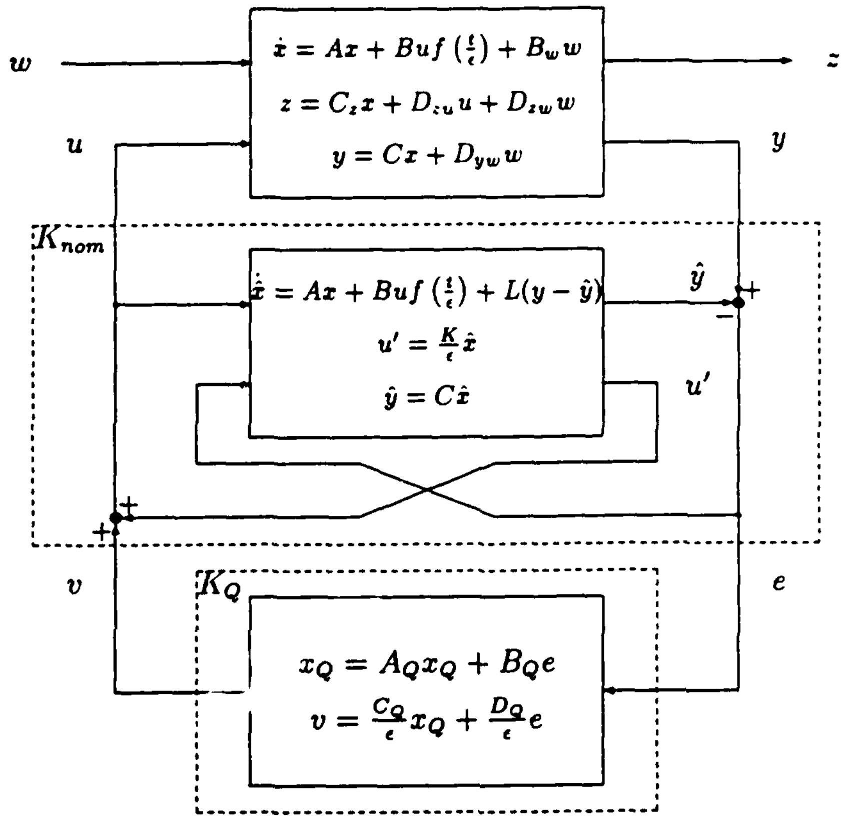

Figure 3.1: Plant (4.1) with the augmented controller (K_{nom}, K_Q).

Defining the observation error $e_1 = x - \hat{x}$, we obtain the following equivalent dynamical equations :

$$\begin{bmatrix}\dot{x}\\ \dot{e_1}\\ \dot{x_Q}\end{bmatrix} = \begin{bmatrix} A+B\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right) & -B\frac{K}{\epsilon}f\left(\frac{t}{\epsilon}\right)+B\frac{D_Q}{\epsilon}Cf\left(\frac{t}{\epsilon}\right) & B\frac{C_Q}{\epsilon}f\left(\frac{t}{\epsilon}\right) \\ 0 & A-LC & 0 \\ 0 & B_QC & A_Q\end{bmatrix}\begin{bmatrix}x\\ e_1\\ x_Q\end{bmatrix}.$$

Thus the stability of the closed loop system depends on the stability of $A + BK/\epsilon f(t/\epsilon)$, A_Q and $A - LC$.

Sufficiency is proved as follows : Since the stability of the closed loop system does not depend on D_Q, we let $D_Q = 0$. The equations for the controller (3.4), (3.5) can be rewritten as

$$\dot{x_e} = (A_e + B_eK_ef\left(\frac{t}{\epsilon}\right) - L_eC_e)x_e + L_ey, \tag{3.6}$$

$$u = \frac{K_e}{\epsilon}x_e . \tag{3.7}$$

In fast time $\tau = t/\epsilon$, the resulting closed loop equations with the controller (3.4), (3.5) are :

$$\begin{bmatrix}\frac{dx}{d\tau}\\ \frac{dx_e}{d\tau}\end{bmatrix} = \begin{bmatrix}\epsilon A & BK_ef(\tau)\\ \epsilon L_eC & \epsilon(A_e - L_eC_e) + B_eK_ef(\tau)\end{bmatrix}\begin{bmatrix}x\\ x_e\end{bmatrix}. \tag{3.8}$$

Let $\Phi_1(\tau)$ be a fundamental matrix for $B_eK_ef(\tau)$. Define

$$\Phi(\tau) = \Phi_1^{-1}(\tau)A_e\Phi_1(\tau) ,$$

and

$$\begin{bmatrix} x(\tau) \\ x_e(\tau) \end{bmatrix} = \begin{bmatrix} I & [I\ 0](\Phi_1(\tau) - I) \\ 0 & \Phi_1(\tau) \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ \varphi(\tau) \end{bmatrix} . \tag{3.9}$$

Using the transformation (3.9), we reduce (3.8) into the standard form [21], apply the averaging principle and obtain the following averaged equations:

$$\begin{bmatrix} \dot{\overline{\xi}} \\ \dot{\overline{\varphi}} \end{bmatrix} = \begin{bmatrix} A + [I\ 0](\overline{\Phi_1^{-1}}L_eC - L_eC) & [I\ 0](\overline{\Phi} - A_e - \overline{\Phi_1^{-1}}L_eC_e + L_eC_e) \\ \overline{\Phi_1^{-1}}L_eC & \overline{\Phi} - \overline{\Phi_1^{-1}}L_eC_e \end{bmatrix} \begin{bmatrix} \overline{\xi} \\ \overline{\varphi} \end{bmatrix} , \tag{3.10}$$

where the bar denotes time averaging operation. To simplify (3.10), introduce the following substitution

$$\begin{bmatrix} \overline{z_1} \\ \overline{z_2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & -I \end{bmatrix} \begin{bmatrix} \overline{\xi} \\ \overline{\varphi} \end{bmatrix} ,$$

which yields

$$\begin{bmatrix} \dot{\overline{z_1}} \\ \dot{\overline{z_2}} \end{bmatrix} = \begin{bmatrix} \overline{\Phi_2^{-1}A\Phi_2} & [I\ 0](-\overline{\Phi} + A_e + \overline{\Phi_1^{-1}}L_eC_e - L_eC_e) \\ 0 & \begin{bmatrix} A - LC & 0 \\ -B_QC & A_Q \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \end{bmatrix} , \tag{3.11}$$

where $\Phi_2(\tau)$ is a fundamental matrix for $BKf(\tau)$. Hence, if A_Q is stable, the stability of the original closed loop system is ensured since K and L are chosen to stabilize $\overline{\Phi_2^{-1}A\Phi_2}$ and $A - LC$ for all $0 < \epsilon \le \epsilon_0$ [17]. **Q.E.D.**

Remark 3.1 : Note that the augmented controller equations (3.4), (3.5) are actually the equations of an observer-based controller for the system (3.1), with its dynamics augmented in such a way that the the signal e is uncontrollable from v. This structure is the same as the state space version of the Youla parametrization of all stabilizing controllers for linear time-invariant systems [74].

5.3.3 PARAMETRIZATION OF THE AVERAGED CLOSED LOOP TRANSFER FUNCTIONS

As it has been shown in the proof of Theorem 3.2, the dynamics of (3.1) with the augmented controller (3.2), (3.3), and $D_Q = 0$ are characterized in the average by the equation (3.10). In this section, we derive an explicit parametrization of the averaged closed loop transfer function from input w to the averaged output, $\overline{z}$, defined below. We also show that this averaged output approximates asymptotically the actual non-averaged output z as ϵ approaches zero.

Assume that A and B in (3.1) are in the controller canonical form, (if (A, B) is only stabilizable, we assume that the controllable part of the Kalman decomposition is in the controller form) and let

$$K = [\ k_n\ \cdots\ k_2\ k_1\ ] ,$$

where $k_i \sim 1, i = 1, \ldots, n$. The averaged closed loop equations (3.10) with $D_Q = 0$, reduces to

$$\begin{aligned} \dot{\overline{\xi}} &= A\xi + [-B\overline{K} \quad k_2\overline{\phi^2}BC_Q]\overline{\varphi} , \\ \dot{\overline{\varphi}} &= \begin{bmatrix} LC \\ B_QC \end{bmatrix} \overline{\xi} + \begin{bmatrix} A - B\overline{K} - LC & k_2\overline{\phi^2}BC_Q \\ -B_QC & A_Q \end{bmatrix} \overline{\varphi} , \end{aligned}$$

where

$$\overline{K} = [\ k_2k_n\overline{\phi^2} \quad k_2k_{n-1}\overline{\phi^2} \quad \ldots \quad k_2^2\overline{\phi^2} \quad 0\ ] , \tag{3.12}$$

$$\phi(\frac{t}{\epsilon}) = \int_0^{t/\epsilon} f(\tau)d\tau . \tag{3.13}$$

Define the averaged output $\overline{z}$ as follows :

$$\overline{z} = C_z\overline{\xi} + D_{zu}\overline{u} + D_{zw}w, \tag{3.14}$$

where $\overline{u} = [-\overline{K} \quad k_2\overline{\phi^2}C_Q]\overline{\varphi}$. Hence, from (3.10) and (3.14), the averaged closed loop transfer function from w to $\overline{z}$ resulting from the augmented controller (K_{nom}, K_Q) can be represented by

$$\overline{z} = (T_{11} + T_{12}\overline{Q}T_{21})w , \tag{3.15}$$

where

$$\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & 0 \end{bmatrix} = C_T(sI - A_T)^{-1}B_T + D_T , \tag{3.16}$$

$$\begin{aligned} A_T &= \begin{bmatrix} A & -B\overline{K} \\ LC & A - B\overline{K} - LC \end{bmatrix} , \\ B_T &= \begin{bmatrix} B_w & B \\ LD_{yw} & B \end{bmatrix} , \\ C_T &= \begin{bmatrix} C_z & -D_{zu}\overline{K} \\ C & -C \end{bmatrix} , \\ D_T &= \begin{bmatrix} D_{zw} & D_{zu} \\ D_{yw} & 0 \end{bmatrix} , \end{aligned}$$

and $\overline{K}$ and L are the state feedback and observer gains of the nominal estimated-state feedback controller, $\overline{K_{nom}}$, of the averaged closed loop system. The averaged closed loop transfer function is depicted in Figure 3.2 where the system (3.1) and the nominal controller $\overline{K_{nom}}$ are combined into the block T.

The stable transfer function $\overline{Q(s)}$ has the following state space realization

$$\begin{aligned} \dot{\overline{x_Q}} &= A_Q\overline{x_Q} + B_Q\overline{e} , \\ \overline{v} &= k_2\overline{\phi^2}C_Q\overline{x_Q} . \end{aligned} \tag{3.17}$$

Expression (3.15) shows that the averaged closed loop transfer function with $D_Q = 0$ is affine in $\overline{Q(s)}$, where $\overline{Q(s)}$ is any asymptotically stable strictly proper transfer function with the gain

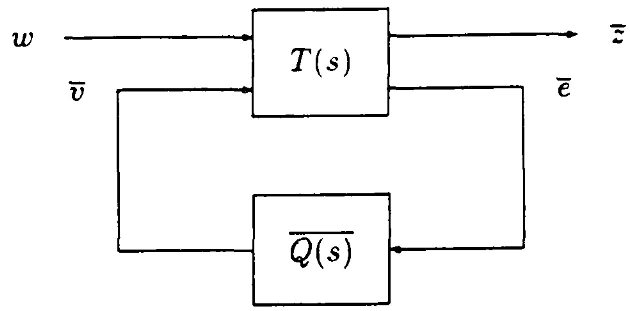

Figure 3.2: Parametrized form of averaged closed loop transfer function

of order 1 (compare with (3.3)). This flexibility in choosing $\overline{Q(s)}$ can be used to yield a closed loop system which satisfies certain design criteria.

Remark 3.2 : For $D_Q \neq 0$, the algebra is considerably more involved, but a similar result can be obtained which parametrizes the averaged transfer function in terms of an asymptotically stable transfer function.

Next we establish the correspondence between the averaged output $\overline{z}$ and the actual non-averaged output z.

Theorem 3.3: Assume that $D_{zu} = 0$ and the transfer function corresponding to (A, B, C_z) in (3.1) has relative degree greater than or equal to 2. Then, if the averaged system (3.10) is asymptotically stable, for any $\delta > 0$ there exists $\epsilon_0(\delta)$ such that for all $0 < \epsilon \leq \epsilon_0$, system (3.1) is also asymptotically stable and the following inequality

$$\|z(t) - \overline{z(t)}\| \leq \delta \,, \qquad t \in [0, \infty), \tag{3.18}$$

holds.

Proof: Let A and B be in the controller canonical form, (if (A, B) is only stabilizable, we assume that the controllable part of the Kalman decomposition is in the controller form). From (3.1) and (3.14),

$$\|z(t) - \overline{z(t)}\| \;=\; \|C_z x(t) + D_{zu} w(t) - C_z \overline{\xi} - D_{zu} w(t)\| \,,$$

where $\overline{\xi}$ is the averaged steady state of system (3.10). Substituting relation (3.9) into the above equation, we have

$$\|z(t) - \overline{z(t)}\| \;=\; \|C(\xi - \overline{\xi}) + C[\Phi_2\left(\frac{t}{\epsilon}\right) - I \,]\, \frac{1}{k_1}(e^{k_1\phi(t/\epsilon)} - 1) B C_Q]\overline{\varphi}\| \,, \tag{3.19}$$

where

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T, C = \begin{bmatrix} c_n & c_{n-1} & \cdots & c_2 & 0 \end{bmatrix}, \Phi_2\left(\frac{t}{\epsilon}\right) = \exp(BK\phi(t/\epsilon)).$$

Thus (3.19) simplifies to

$$\begin{aligned} \|z(t) - \overline{z(t)}\| &= \|C(\xi - \overline{\xi})\| \,, \\ &\leq \max_{0<i\leq n} |c_i| \|\xi - \overline{\xi}\| \,. \end{aligned}$$

Consequently, the result (3.18) follows directly from Theorem 1.3 of [75]. **Q.E.D.**

5.3.4 DESIGN EXAMPLE

In this section, we will use the parametrization of the averaged closed loop transfer function to design a controller of the form (3.4), (3.5) so that the original system meets certain step response specifications.

Consider now an example of system (3.1) with

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -10 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B_w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\
C_z &= \begin{bmatrix} 10 & -1 & 0 \end{bmatrix}, D_{zu} = \begin{bmatrix} 0 \end{bmatrix}, D_{zw} = \begin{bmatrix} 0 \end{bmatrix}, \\
C &= \begin{bmatrix} -10 & 1 & 0 \end{bmatrix}, D_{yw} = \begin{bmatrix} 1 \end{bmatrix}, f(\frac{t}{\epsilon}) = sin(\frac{t}{\epsilon}),
\end{aligned}
$$

where z represents the step response output of the system and y is the measured tracking error to a step input.

With $\epsilon = 0.002$, a controller (3.2) that stabilizes (3.1) can be defined by

$$K = [\ 3.7032 \quad 3.2404 \quad 0\ ], \quad L = [\ -12.5 \quad -75 \quad -50\ ]^T .$$

The corresponding transfer functions $T_{11}(s), T_{12}(s)$, and $T_{21}(s)$ can be computed from (3.16) with

$$\overline{K} = [\ 6 \quad 5.25 \quad 0\ ], \quad L = [\ -12.5 \quad -75 \quad -50\ ]^T .$$

Assume that the design specifications for the averaged closed loop system are defined in terms of the overshoot, undershoot, and the rise time:

$$\overline{z_{os}} = \sup_{t \geq 0} \overline{z(t)} - 1 \leq 0.25 , \tag{3.20}$$

$$\overline{z_{us}} = \sup_{t \geq 0} -\overline{z(t)} \leq 0.7 , \tag{3.21}$$

$$t_{rise} = \inf \{T \mid \overline{z(t)} > 0.8 \text{ for } t \geq T\} \leq 1 . \tag{3.22}$$

The design specifications (3.20)-(3.22) are closed loop quasiconvex [76], since the set of closed loop transfer functions satisying the design specificatic ns is quasiconvex. Thus the controller design problem can be solved via quasiconvex optimization. Following the approach of [76], we use the Ritz approximation of the augmented controller $\overline{K_N(x)}$ that consists of the nominal controller $\overline{K_{nom}}$, and a stable transfer function $\overline{Q(s)}$ defined as a linear combination of the fixed transfer functions $Q_1(s), \ldots, Q_N(s)$:

$$\overline{Q(s)} = \sum_{i=0}^{N} \alpha_i Q_i(s) , \quad N = 5 .$$

Vector $\alpha = [\alpha_1 \ldots \alpha_N] \in I\!R^N$, has to be determined so that the the controller $\overline{K_N(x)}$ ensures the desired closed loop specification. With $T = 1$ in (3.22), the overshoot, undershoot, and the rise time specifications are

$$\sum_{1}^{5} \alpha_i s_i(t) \leq 1.25 , \qquad 1.0 \leq t \leq 10.0 , \tag{3.23}$$

$$\sum_{1}^{5} \alpha_i s_i(t) \geq -0.7 , \qquad 0 \leq t \leq 1.0 , \tag{3.24}$$

$$\sum_{1}^{5} \alpha_i s_i(t) \geq 0.8 , \qquad 1.0 \leq t \leq 10.0 , \tag{3.25}$$

where s_i is the step response of the averaged closed loop system with the controller $\overline{K_N(x)}$ when $\alpha_i = 1$ for $i = 1, \ldots, 5$. As in [76], we finely discretize t in (3.23)-(3.25) to obtain a set of L linear inequality constraints on α of the form :

$$c_k^T \alpha \ \leq \ h_k \ , \qquad k = 1, \ldots, L \ ,$$

where c_k and h_k are constants. The following solution

$$\alpha = [\ 495.9832 \quad -197.1771 \quad -400.5335 \quad -317.782 \quad -179.7225\ ] \ ,$$

was found by minimizing $\|\alpha\|_2$ subject to the constraints (3.23)-(3.25) with

$$Q_i = \left(\frac{1}{s+1}\right)^i \ , \quad i = 1, \ldots, 5 \ .$$

The corresponding stable transfer function $Q(s)$ in (3.3) for the original non-averaged system (3.1) was computed to be :

$$A_Q = \begin{bmatrix}
-15 & -105 & -455 & -1365 & -3003 & -5005 & -6435 & -6435 & -5005 & -3003 & -1365 & -455 & -105 & -15 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} ,$$

$$Bq = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T ,$$

$$Cq = \frac{61721}{\epsilon} [\ 0.0050 \quad 0.0675 \quad 0.4217 \quad 1.6003 \quad 4.0998 \quad 7.4458 \quad 9.7688 \quad 9.2020 \quad 5.9643 \quad 2.2986 \quad 0.1462 \quad -0.3797 \quad -0.2225 \quad -0.0572 \quad -0.0060\ ] ,$$

$$Dq = 0 \ .$$

The resulting closed loop step response of the averaged system with the nominal controller $\overline{K_{nom}}$ and the augmented controller $\overline{K_N(x)}$ are shown in Figure 3.3.

The closed loop response of the original non-averaged system (3.1) with the nominal controller K_{nom} and the augmented controller (K_{nom}, Q) are shown in Figure 3.4. It can be seen that the actual and averaged closed loop step responses are almost identical which is in agreement with the result of Theorem 3.3. Also, both the actual and averaged closed loop step responses with the respective augmented controllers satisfy the design criteria.

5.3.5 DISTURBANCE REJECTION

The question of when the disturbance w can be completely decoupled by feedback control from the regulated output z in the system (3.1) led to the development of geometric control theory. The so-called disturbance decoupling problems for linear time-invariant systems have been investigated extensively in the last two decades. The problem is to find a compensator such that the closed loop transfer function from disturbance w to desired output z is equal to zero.

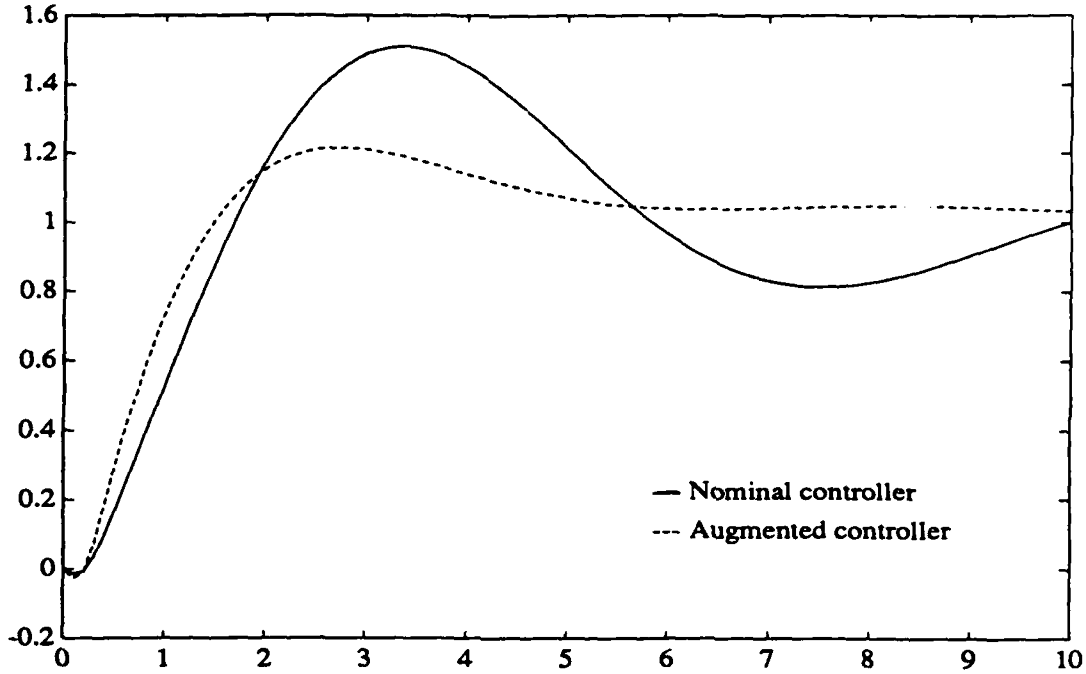

Figure 3.3: Step Response of Averaged System

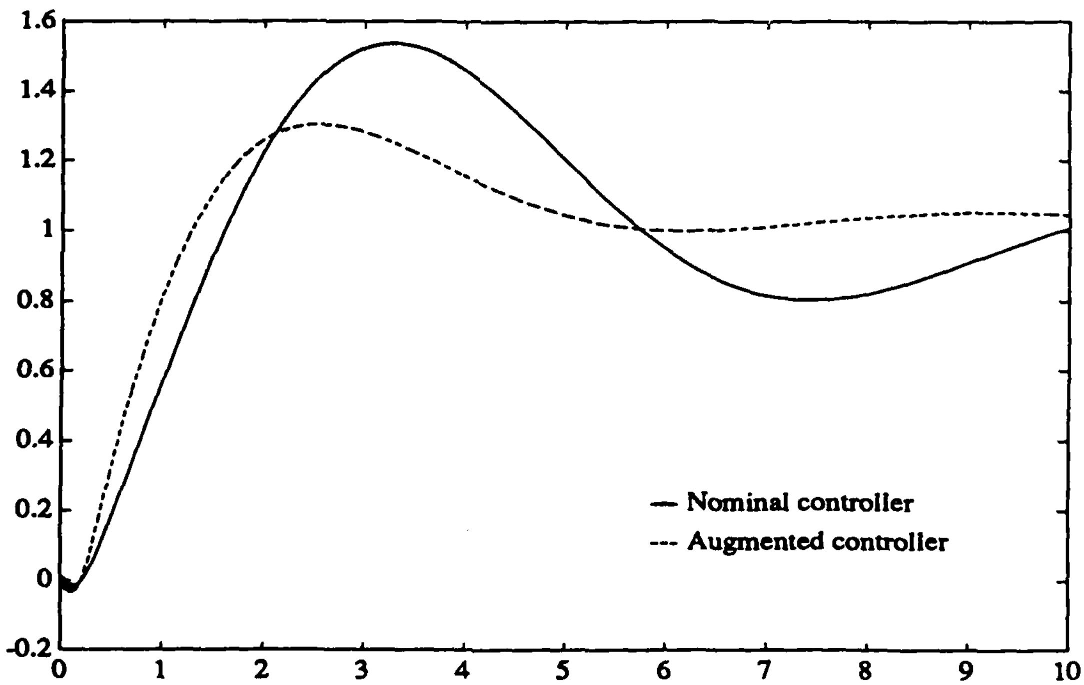

Figure 3.4: Step Response of Actual System

Using the concept of (A, B)-invariance, the disturbance decoupling problem with state feedback (DPP) was solved in [77]. The problem of disturbance decoupling with state feedback and the extra requirement of internal stability (DPPS) was solved in [78], [79]. A detailed reference for the above mentioned problems can be found in [80].

In this section, we study the disturbance decoupling problem with closed loop vibrational control. In particular, for the system (3.1) with $D_{zw} = 0$, i.e.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)f\left(\frac{t}{\epsilon}\right) + B_w w(t) , \\ z(t) &= C_z x(t) , \end{aligned} \tag{3.26}$$

we will establish conditions under which a state feedback of the form

$$u = \frac{K}{\epsilon} x , \tag{3.27}$$

can be found which decouples z from w. Similar to the approach in Section 5.3.3, we will first derive the averaged equation for the closed loop system (3.26), (3.27). Let $\Phi(\tau)$ be a fundamental matrix for $BKf(\tau)$ and introduce the substitution

$$\xi(\tau) = \Phi(\tau)x(\tau) .$$

Then the corresponding averaged equation of the closed loop system (3.26), (3.27) is

$$\begin{aligned} \dot{\overline{\xi}} &= \overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right) A\Phi\left(\frac{t}{\epsilon}\right)}\overline{\xi} + \overline{\Phi^{-1}\left(\frac{t}{\epsilon}\right)} B_w w , \\ \overline{z} &= \overline{C\Phi\left(\frac{t}{\epsilon}\right)}\overline{\xi} . \end{aligned} \tag{3.28}$$

Assume that (A, B) in (3.26) is controllable, then without loss of generality, let (A, B) be in the controller canonical form:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ \\ 0 \\ 1 \end{bmatrix} , \\ C_z &= \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{bmatrix} . \end{aligned}$$

With state feedback

$$u = \frac{K}{\epsilon} x = \frac{1}{\epsilon} \begin{bmatrix} k_n & \cdots & k_2 & k_1 \end{bmatrix} ,$$

the averaged closed loop system (3.28) reduces to

$$\begin{aligned} \dot{\overline{\xi}} &= (A - B\overline{K})\overline{\xi} + B_w w , \\ \overline{z} &= C\overline{\xi} , \end{aligned} \tag{3.29}$$

where $\overline{K}$ is defined in (3.12).

It follows from Theorem 3.3, that if the transfer function corresponding to (A, B, C_z) has relative degree ≥ 2, the actual regulated output z will be arbitrarily close to

$\bar{z}$. Hence, our goal is to determine $\overline{K}$ for the averaged linear time-invariant system (3.29) so that $\bar{z}$ is decoupled from w. This will in turn ensures that for the original non-averaged system (3.26), the regulated output z will also be decoupled from the disturbance w.

To solve the disturbance decoupling problem for the averaged system (3.29), we first introduce the following definition :

Definition 3.1: A subspace $V \subset X = \mathbb{R}^n$ is said to be (A, B)-invariant (or controlled-invariant) if there exists $K : X \rightarrow U$ such that $(A - BK)V \subset V$.

Theorem 3.4: [80] A subspace $V \subset \mathbb{R}^n$ is controlled-invariant if and only if $AV \subset V + \mathrm{Im}B$.

Corollary 3.1: Consider the system (3.29) with $C \neq 0$. Then $V \subset \ker C$ does not contain the vector $e_n = [\ 0 \ \ \cdots \ \ 0 \ \ 1\ ]^T$.

Proof: We shall prove by contradiction. Let e_i denotes the $i-th$ standard basis vector. Assume that $e_n \in V$. From Theorem 3.4, for $v_i = e_n$, there exists a $w_i \in V$ and $u_i \in U = \mathbb{R}$ such that

$$Ae_n = w_i + Bu_i \ .$$

Hence $w_i = e_{n-1} \in V$. Applying Theorem 3.3 again with $v_i = e_{n-1}$, we see that $e_{n-2} \in V$. Repeating this procedure, we see that $e_1, e_2, \ldots, e_n \in V$. Hence $\dim(V) = n$ and we obtain a contradiction. **Q.E.D.**

Definition 3.2: The system (3.29) is said to be disturbance decoupled by closed loop vibrational control if there exists $\overline{K}$ of the form (3.12) such that

$$< A - B\overline{K}|\mathrm{Im}B_w > \ \subset \ \ker C \ ,$$

where

$$< A|\mathrm{Im}B > \triangleq \mathrm{Im}B + A\mathrm{Im}B + \ldots + A^{n-1}\mathrm{Im}B.$$

Theorem 3.5: Assume that the transfer function corresponding to (A, B, C_z) in system (3.29) has relative degree 2. Then the system (3.29) can be disturbance decoupled by closed loop vibrational control if and only if there exists a controlled-invariant subspace V satisfying $\mathrm{Im}B_w \subset V \subset \ker C$.

Proof: The necessity is well known and established in [80]. To prove sufficiency, we need to show that there exists a $\overline{K} \in \overline{K}(V) = \{\overline{K} : X \rightarrow U | (A - B\overline{K})V \subset V \text{ and } V \subset \ker C\}$ satisfying (3.12). Let $v_1, \ldots, v_\mu$ be a basis for V. From Corollary 4.1, we know that $e_n \notin V$. Now $e_{n-1} \notin V$ since $c_{n-2} \neq 0$. Applying Theorem 3.4, for each $v_i, i = 1, \ldots, \mu$, there exists a $w_i \in V$ and $u_i \in U = \mathbb{R}$ such that

$$Av_i = w_i + Bu_i \ ,$$

where $B : U \rightarrow X$. Let $v_{\mu+1}, \ldots, v_{n-2}, e_{n-1}, e_n$ be a set of vectors such that $v_1, \ldots, v_\mu, v_{\mu+1}, \ldots, v_{n-2}, e_{n-1}, e_n$ is a basis for $\mathbb{R}^n$. Define $\overline{K}$ by the equation

$$\overline{K}\,[v_1 \ldots v_\mu \ v_{\mu+1} \ldots v_{n-2} \ e_{n-1} \ e_n] = [u_1 \ldots u_\mu \ u_{\mu+1} \ldots u_{n-2} \ u_{n-1} \ u_n] \ ,$$

where the scalars $u_{\mu+1}, \ldots, u_{n-2}$ are arbitrary and $u_{n-1} > 0$ and $u_n = 0$. Clearly, $\overline{K}$ satisfying (3.12) exists and is computable since the matrix $[v_1 \ldots v_\mu v_{\mu+1} \ldots v_{n-2} e_{n-1} e_n]$ is non-singular. **Q.E.D.**

It follows from Theorem 3.3, that if the averaged output $\bar{z}$ is decoupled from w, then the effects of w will also be decoupled from the actual output z. The corresponding state feedback gain K for the original non-averaged system (3.26) can be easily computed from (3.12).

5.4 PART 4. VERY HIGH HARMONIC CONTROL IN HELICOPTERS

5.4.1 INTRODUCTION

Helicopter vibration is a long standing problem. From the earliest days of rotorcraft development, oscillatory motions of the non-rotating portion of the airframe have been a matter of serious concern. Vibration affects adversely pilot performance and passenger comfort and increases operation costs and maintenance requirements, limiting the use of helicopters in commercial aviation and military operations.

Traditionally, rotor induced vibration reduction efforts have focused on altering blade and fuselage dynamic characteristics or supplementing the rotorcraft with vibration isolators and attenuators. These passive methods have met with limited success [3]. Considerable research has been done on "active" techniques, which make direct use of vehicle control. One of the most widely used method of active control is the *Higher Harmonic Control* (*HHC*). HHC involves driving the blade pitch angle at the frequency N/rev (N is the number of blades) in order to cancel the effect of vibratory airload transmitted to the fuselage. Recent experiments, [3]-[5], have shown that HHC may lead to an order of magnitude reduction in fuselage vibrations. For this method to be effective, not only the amplitude but also the phase of the oscillations introduced should be chosen appropriately, depending on the flight conditions. The purpose of *this research is to show that if the frequency of the oscillations introduced is much greater than* N/rev, the phase dependency disappears. To distinguish this approach from HHC, we call it VHHC, the Very High Harmonic Control. This part of the report presents a simple model and a hypothetical explanation of the efficacy of VHHC. The development presented here is based on the ideas of vibrational control. Vibrational control is a method for changing dynamic properties of systems by introducing a control which enters the open loop dynamics as an amplitude of a periodic, zero average function. The theory of this control technique has been developed in Section 5.1. As it has been pointed out in Section 5.1, vibrational control is a useful tool for achieving two conflicting control goals when only one actuator is available. This is exactly the case in helicopters where the rotor blade pitch is used to accomplish both the desired aircraft altitude and the fuselage oscillations suppression.

The application of closed loop vibrational control requires the knowledge of a dynamic model of a system under consideration. Unfortunately, due to uncertainties associated with dynamics, measurement data and model structure, the mathematical modeling of helicopters for handling qualities, performance, and flight control is a very difficult problem. The modeling of helicopters is further complicated by numerous energy sources which cause fuselage vibrations. These energy sources are (i) alternating aerodynamics forces acting on the rotors, (ii) engine vibrations, and (iii) aerodynamics forces acting on the fuselage and nonrotating parts of the machine. The rotor system, which transmits the vibratory airloads to the fuselage through the rotor shaft, is one of the most significant contributors to helicopter vibrations [23]. The rotating blades create vibratory airloads containing all harmonics of the rotor rotational frequency which are passed from the blades to the pylon and then to the cabin through complicated load paths. These loads are felt as vibratory forces and moments whose frequencies are integer multiples of the blade passage frequency (number of blades times rotational frequency). No simple models describing these phenomena are available. Therefore, for the purposes of this research, we develop a simplified *qualitative model* of a helicopter and consider only the fixed airframe vibrations excited by the rotor. Aerodynamic effects are faked as follows: Vibratory air load generated by the rotating blades is modeled as a periodic function of the rotational frequency. The aerodynamic perturbations induced by the rotor on the fuselage are modeled also as a periodic function with frequency equal to the blade passage frequency. Finally, incremental aerodynamic forces generated by VHHC are modeled by another periodic function with high frequency. For simplicity, all these functions are chosen as sinusoids, however this choice does not affect the qualitative results of this thesis.

Using this model, this thesis gives a phenomenological explanation of the efficiacy of VHHC in helicopter oscillations suppression. The outline of this section is as follows: In Section 5.4.2, a simple model of a helicopter is developed. Sections 5.4.3 and 5.4.4 are devoted to analysis of

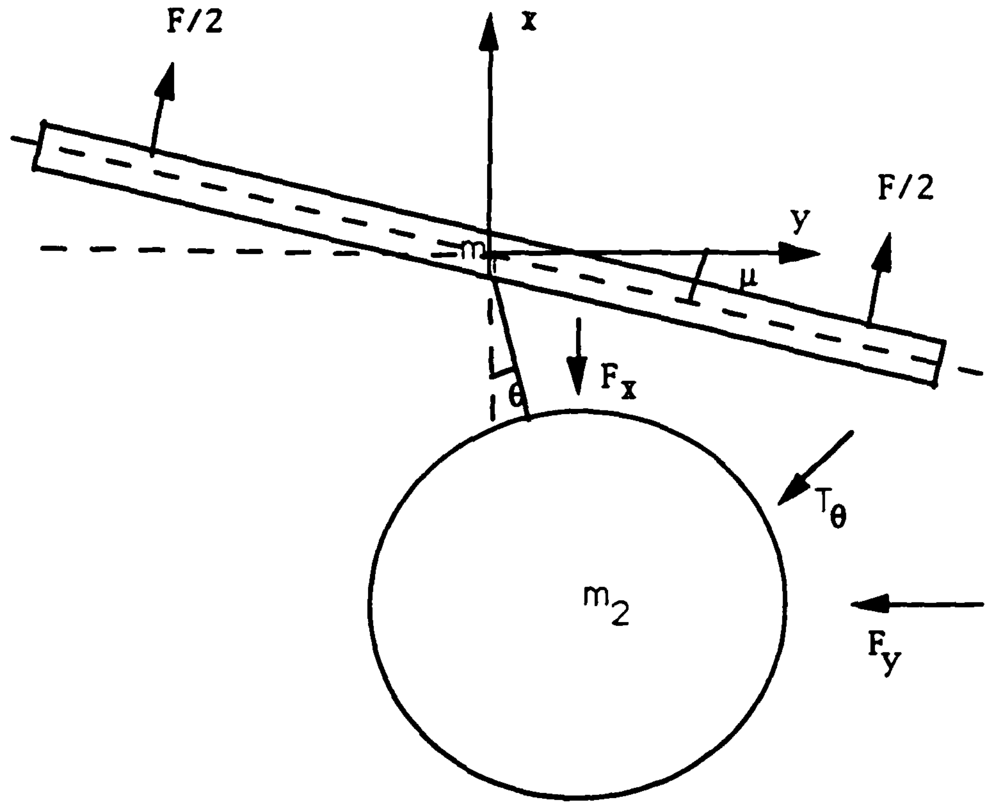

Figure 4.1: Simple model of a helicopter

this model with and without VHHC in hover and forward flight, respectively. In Section 5.4.5, the effects of wind gusts are examined.

5.4.2 QUALITATIVE MODEL

Consider a simple, qualitative model of a helicopter shown in Figure 4.1. The platform with mass m_1 and the pendulum with mass m_2 are intended to model the rotor and the fuselage, respectively. The rotor generates the lift force, F, and induces aerodynamic perturbations, T_θ, on the rotational degree of freedom of the fuselage. Force F and torque T_θ are modelled as

$$\begin{aligned} F &= F_0 + \alpha_0 \sin \omega t \ , \\ T_\theta &= \beta \sin N\omega t \ , \end{aligned} \tag{4.1}$$

where F_0 is the steady lift due to the conventional rotor blade pitch control, $\alpha_0 \sin \omega t$ is the periodic part of the lift due to vibratory airload, ω is the rotor rotational frequency, N is the number of rotor blades, α_0 and β are parameters that model the strength of the periodic components.

When VHHC is introduced, F is defined as follows:

$$F = F_0 + \alpha_0 \sin \omega t + \tfrac{\alpha}{\epsilon} \sin \tfrac{t}{\epsilon} \ , \quad 0 < \epsilon \ll 1 \ . \tag{4.2}$$

The small parameter ϵ is intended to model the high frequency of VHHC. Although in general the amplitude of blade feathering is small (i.e. of the order ϵ), the aerodynamic forces generated by oscillating airfoils are proportional not only to the law of oscillations but also to the first and second derivative of this law [83]. Therefore, the amplitude of the VHHC generated force is (4.2) is assumed to be $\frac{\alpha}{\epsilon}$. Note that the effect of differentiation, mentioned above, has been observed experimentally in [29] where higher HHC frequencies required smaller HHC amplitudes to suppress the fuselage vibrations.

The constant part of the lift, F_0, is chosen to compensate for the system weight and to produce a control force to ensure the desired helicopter altitude. Therefore, F_0 is defined as

$$F_0 = \frac{(m_1 + m_2)g - kx}{\cos\mu} , \tag{4.3}$$

where g is the acceleration of gravity, μ is the rotor tilt angle and k is the gain of the altitude controller.

In addition to force F and torque T_θ, two more forces act on the system, F_x and F_y. These are intended to model wind gusts and are assumed to be of an impulsive nature. A specific form of F_x and F_y is described in Section 5.4.5.

Given the system of Figure 4.1 and forces (4.1), (4.2), we write the Lagrange equation for the platform m_1 (rotor) and for the pendulum m_2 (fuselage) as follows:

$$(m_1 + m_2)\ddot{x} + m_2 l\ddot{\theta}\sin\theta + m_2\ell\dot{\theta}^2\cos\theta + (m_1 + m_2)g = F\cos\mu - F_x - \zeta\dot{x} , \tag{4.4}$$

$$(m_1 + m_2)\ddot{y} + m_2\ell\ddot{\theta}\cos\theta - m_2\ell\dot{\theta}^2\sin\theta = F\sin\mu - F_y - \zeta\dot{y} , \tag{4.5}$$

$$m_2\ell^2\ddot{\theta} + m_2\ell(\ddot{x}\sin\theta + \ddot{y}\cos\theta) + m_2\ell g\sin\theta = T_\theta + F_x\ell\sin\theta + F_y\ell\cos\theta - \eta\dot{\theta} , \tag{4.6}$$

where x, y and ζ denote the altitude, horizontal distance and dampling coefficient of the platform and θ, ℓ and η the angle, length and damping coefficient of the pendulum.

The dynamics of this model are analyzed in Sections 5.4.3-5.4.5 using both a theoretical study and numerical simulations. For simulation purposes, the parameters involved in (4.4)-(4.6) have been chosen as follows: $m_1 = 0.01$kg, $m_2 = 0.1$kg, $\ell = 1/\sqrt{10}$m, $\zeta = 0.6$, $\eta = 0.1$, k=5, g=9.8 m/sec^2, $\alpha_0 = \beta = 0.1$, N=3, $\omega = 1$ and $\epsilon = 0.01$.

5.4.3 ANALYSIS: HOVER

To model a helicopter in hover, we set $\mu = 0$. Assuming that $F_x = F_y = 0$ and considering only the vertical and angular motions of the pendulum-platform system, we reduce equations (4.4)-(4.6) to the following form:

$$(m_1 + m_2)\ddot{x} + m_2\ell\ddot{\theta}\sin\theta + m_2\ell\dot{\theta}^2\cos\theta + (m_1 + m_2)g = F - \zeta\dot{x} , \tag{4.7}$$

$$m_2\ell^2\ddot{\theta} + m_2\ell(\ddot{x} + g)\sin\theta = T_\theta - \eta\dot{\theta} . \tag{4.8}$$

The effect of VHHC in suppressing the vertical acceleration $\ddot{x}$ and the angular acceleration $\ddot{\theta}$ of the system (4.7), (4.8) is analyzed by considering three different cases. In the first case, the system is constrained to move only in the vertical direction (one degree of freedom), i.e. $\theta \equiv 0$. In the second case, the system is analyzed with no constraint on its angular motion (two degrees of freedom) but without VHHC, i.e. $\alpha = 0$. Finally, in the third case we investigate the system performance with VHHC, i.e. $\alpha \neq 0$.

5.4.3.1 One Degree of Freedom without VHHC

If we assume that the pendulum is fixed rigidly to the platform and therefore cannot execute angular motion, equations (4.7) and (4.8) reduce to the following single dynamical equation:

$$(m_1 + m_2)\ddot{x} + (m_1 + m_2)g = F_0 + \alpha_0\sin\omega t - \zeta\dot{x} . \tag{4.9}$$

where

$$A_1 = \frac{1}{\sqrt{(m_2\ell g - m_2\ell^2N^2\omega^2)^2 + \eta^2N^2\omega^2}} \cong 2.69 \ , \tag{4.16}$$

and

$$\sigma_1 = -\tan^{-1}\left(\frac{\eta N\omega}{m_2\ell g - m_2\ell^2N^2\omega^2}\right) .$$

Thus, the steady state angular acceleration is given by:

$$\ddot{\theta}_{ss}(t) = -N^2\omega^2A_1\beta\sin(N\omega t + \sigma_1) . \tag{4.17}$$

Substituting (4.15) into (4.7), we obtain the steady state solution for the altitude of the platform:

$$x_{ss}(t) = M_1\alpha_0\sin(\omega t + \varphi_1) - (N\omega A_1\beta)^2M_2m_2\ell\cos(2(N\omega t + \sigma_1) + \Omega_1) \ , \tag{4.18}$$

where

$$M_2 = \frac{1}{\sqrt{(k - (m_1 + m_2)4N^2\omega^2)^2 + 4N^2\omega^2\xi^2}} , \cong 0.267 \ , \tag{4.19}$$

and

$$\Omega_1 = -\tan^{-1}\left(\frac{2N\omega\xi}{k - (m_1 + m_2)4N^2\omega^2}\right) .$$

Thus, the steady state vertical acceleration of the system is:

$$\ddot{x}_{ss}(t) = -\omega^2M_1\alpha_0\sin(\omega t + \varphi_1) + 4N^2\omega^2M_2m_2\ell(A_1\beta N\omega)^2\cos(2(N\omega t + \sigma_1) + \Omega_1) . \tag{4.20}$$

A comparison of (4.20) with (4.12) shows that the two degrees of freedom system exhibits vertical oscillations with both the rotational and the blade passage frequencies. The latter component contributes significantly to the fuselage vibrations when the helicopter is in hover [85]. This leads to a conclusion that, in the two degrees of freedom system, the aerodynamic perturbations, T_θ, generated by the rotor, excite the fuselage vibrations which then are passed on back to the rotor. This positive feedback is responsible for the large vertical acceleration of the system at hand.

This conclusion is supported by numerical simulations of (4.13), (4.14). Results are illustrated in Figures 4.3 and 4.4 ($\alpha = 0$), where equations (4.13), (4.14) are solved under zero initial conditions. These figures, which are in a good agreement with estimates (4.15)-(4.20), show that aerodynamic perturbations, T_θ, excite oscillations of the pendulum with a frequency equal to the blade passage frequency of $1.5/\pi$ Hz. As a result, the pendulum has an angular acceleration of the same frequency and an amplitude of about 2.7 rad/s^2 as can be seen in Figure 4.4(b). The dynamic excitation of the pendulum is subsequently passed on to the platform and generates additional vertical motion. Compared with Figure 4.2(a), Figure 4.3(a) shows that the altitude of the platform is excited by both the $0.5/\pi$ Hz vibratory airload generated by the rotor and the $3/\pi$ Hz vibration due to the oscillations of the pendulum. Thus, as shown in Figure 4.3(b), the resulting vertical acceleration of the platform also exhibits the effects of both the rotor vibratory airload and the oscillations of the pendulum. This latter component accounts for the large vertical acceleration of about 0.2 m/s^2, i.e., an order of magnitude increase in comparison with the one degree of freedom system.

5.4.3.3 Two Degrees of Freedom with VHHC

With VHHC, the vertical and angular accelerations of the system can be obtained by substituting (4.2) into equations (4.7) and (4.8):

$$
\begin{aligned}
\ddot{x} &= \frac{1}{m_1+m_2-m_2\sin^2\theta}\left(\frac{\eta\dot{\theta}}{\ell}\sin\theta + m_2 g\sin^2\theta - m_2\ell\cos\theta - kx - \zeta\dot{x}\right. \\
&\left. -\frac{\beta}{\ell}\sin N\omega t\sin\theta\alpha_0\sin\omega t + \frac{\alpha}{\epsilon}\sin\left(\frac{t}{\epsilon}\right)\right) , \qquad (4.21) \\
\ddot{\theta} &= \frac{m_1+m_2}{m_1+m_2-m_2\sin^2\theta}\left(\frac{m_2}{2(m_1+m_2)}\dot{\theta}^2\sin 2\theta + \frac{kx}{\ell(m_1+m_2)}\sin\theta\right. \\
&+\frac{\zeta\dot{x}}{\ell(m_1+m_2)}\sin\theta - \frac{g}{\ell}\sin\theta - \frac{\eta}{m_2\ell^2}\dot{\theta} - \frac{1}{\ell(m_1+m_2)}\alpha_0\sin\omega t\sin\theta \\
&\left. -\frac{1}{\ell(m_1+m_2)}\frac{\alpha}{\epsilon}\sin\left(\frac{t}{\epsilon}\right)\sin\theta + \frac{1}{m_2\ell^2}\beta\sin N\omega t\right) . \qquad (4.22)
\end{aligned}
$$

The terms in the right hand side of (4.21) and (4.22) contain the aerodynamic loads associated with the rotor vibratory airload, aerodynamic perturbations, and the VHHC. The VHHC term is of much higher frequency than the other two components. In practice, high frequency component of the vertical acceleration is attenuated by passive damping in the cabin. It is therefore justifiable to consider only the filtered, or averaged with respect to high frequency, vertical and angular accelerations of the system. This is the standard methodology of vibrational control [11]- [13], where high frequency parametric oscillations are introduced in order to improve the averaged (or filtered) behavior of the system. To obtain the averaged equations we first use the generating equations technique of [20] that reduces (4.21), (4.22) to the standard Bogoliubov's form and then use the averaging principle of [21], to derive the averaged equation. For the system at hand the resulting averaged equations are (see Appendix A2 for details) :

$$
\begin{aligned}
\ddot{z} &= \frac{1}{m_1+m_2-m_2\sin^2\phi}\left(\frac{\eta\dot{\phi}}{\ell}\sin\phi + m_2 g\sin^2\phi -\right. \\
&-m_2\ell\left(\frac{-\alpha^2\sin^2\phi}{2\ell^2(m_1+m_2-m_2\sin^2\phi)^2}\dot{\phi}^2\right) \\
&\left.\cos\phi - kz - \zeta\dot{z} - \frac{\beta}{\ell}\sin N\omega t\sin\phi + \alpha_0\sin\omega t\right) , \qquad (4.23) \\
\ddot{\phi} &= \frac{m_1+m_2}{m_1+m_2-m_2\sin^2\phi}\left(\frac{m_2}{2(m_1+m_2)}\left(\frac{-\alpha^2\sin^2\phi}{2\ell^2(m_1+m_2-m_2\sin^2\phi)^2}\right.\right. \\
&\left. -\frac{\alpha^2}{2m_2\ell^2(m_1+m_2-m_2\sin^2\phi)} + \dot{\phi}^2\right)\sin 2\phi \\
&+\frac{kz}{\ell(m_1+m_2)}\sin\phi + \frac{\zeta\dot{z}}{\ell(m_1+m_2)}\sin\phi - \frac{g}{\ell}\sin\phi - \frac{\eta}{m_2\ell^2}\dot{\phi} \\
&\left. -\frac{1}{\ell(m_1+m_2)}\alpha_0\sin\omega t\sin\phi + \frac{1}{m_2\ell^2}\beta\sin N\omega t\right) , \qquad (4.24)
\end{aligned}
$$

where z and ϕ denote the averaged altitude and angle, respectively.

Comparing (4.21), (4.22) and (4.23), (4.24), we observe that the introduction of VHHC results in additional terms containing amplitude α. To analyze the effects of these terms, we first reduce

(4.23), (4.24) to the form (4.7), (4.8). Then, since ϕ is small, we neglect the first term containing α^2 in (4.24) and approximate the denominator of the remaining term containing α^2 by $m_1 + m_2$. Hence we write (4.23), (4.24) as:

$$(m_1 + m_2)\ddot{z} + m_2\ell\ddot{\phi}\sin\phi + m_2\ell\dot{\phi}^2 cos\phi = \alpha_0 \sin\omega t - kz - \xi\dot{z} \ , \qquad (4.25)$$

$$m_2\ell^2\ddot{\phi} + m_2\ell\left(\ddot{z} + g + \frac{\alpha^2\cos\phi}{2\ell(m_1+m_2)^2}\right)\sin\phi = \beta\sin N\omega t - \eta\dot{\phi} \ . \qquad (4.26)$$

Equation (4.25) is identical to (4.7) but unlike (4.8), equation (4.26) has an additional term containing α. Since the vertical acceleration $\ddot{z}$ of the fuselage is generally of the order of 0.1g [84], for small ϕ we can neglect the term $m_2\ell\ddot{z}\sin\phi$ and replace $\sin\phi$ by ϕ and $\cos\phi$ by 1. Then the steady state solution of (4.8) can be represented as

$$\theta_{ss}(t) = A_2\beta\sin(N\omega t + \sigma_2) \ ,$$

where

$$\begin{aligned} A_2 &= \frac{1}{\sqrt{(m_2\ell g + \frac{m_2\alpha^2}{2(m_1+m_2)^2} - m_2\ell^2N^2\omega^2)^2 + \eta^2N^2\omega^2}} \ , \qquad (4.27) \\ &\cong 0.23, \text{ for } \alpha = 1 \ , \end{aligned}$$

and

$$\sigma_2 = -\tan^{-1}\left(\frac{\eta N\omega}{m_2\ell g + \frac{m_2\alpha^2}{2(m_1+m_2)^2} - m_2\ell^2N^2\omega^2}\right) \ .$$

A comparison of (4.27) with (4.16) shows that the VHHC term in (4.26) leads to an order of magnitude reduction of the angular vibrations of the fuselage, i.e. to the stiffening of the system. As it follows from (4.20) (with A_1, replaced by A_2 and σ_1 by σ_2), this results in an order of magnitude reduction in the vertical acceleration of the system. Therefore, the effect of VHHC can be interpreted as the decoupling of the positive feedback from the rotor through the aerodynamic perturbations to the fuselage and then back to the rotor.

This conclusion is supported by numerical investigation of (4.23), (4.24) (see Figures 4.3 and 4.4 for $\alpha \neq 0$): Figure 4.4(a) shows that VHHC significantly reduces the oscillations in the pendulum induced by the rotor vibratory airload. The resulting vibratory load suppression in the angular acceleration is illustrated in Figure 4.4(b). The attenuation of the oscillations in the pendulum produces a reduction in the dynamic load transferred back to the platform. As can be seen in Figure 4.3(a), the $3/\pi$ Hz vibrations generated by the oscillations in the pendulum on the vertical motion are greatly attenuated. Note that this attenuation improves as the amplitude of VHHC is increased (see Figure 4.5). For $\alpha = 1$, the effect of these oscillations is virtually eliminated. In particular, the steady state vertical acceleration is about 0.02 m/s^2. The $3/\pi$ Hz vibration due to the oscillations in the pendulum has been eliminated and the remaining vibration in the vertical acceleration is at a frequency of $0.5/\pi$ due to the rotor vibratory airload $\alpha_0 \sin\omega t$. Thus, the introduction of VHHC forced the model to behave almost as a one degree of freedom system.

5.4.4 ANALYSIS: FORWARD FLIGHT

To study the performance of a VHHC-equipped helicopter in forward flight, we consider equations (4.4)-(4.6) and assume that there is no wind gusts, i.e. $F_x = F_y = 0$. Similar to Section 5.4.3, we analyze the efficacy of VHHC in suppressing the vertical acceleration $\ddot{x}$, horizontal acceleration $\ddot{y}$ and angular acceleration $\ddot{\theta}$ by considering three different cases: (i) the system is constrained to move only in the vertical and horizontal direction (two degrees of freedom), (ii) the system has no constraints on its angular motion (three degrees of freedom) but without VHHC, i.e. $\alpha = 0$, and (iii) the system with three degrees of freedom and with VHHC, i.e. $\alpha \neq 0$. We assume that the system's parameters are the same as those defined in Section 5.4.2 and choose $\mu = 5$ deg.

5.4.4.1 Two Degrees of Freedom without VHHC

If the system (4.4)-(4.6), with $F_x = F_y = 0$, cannot execute angular motion, then the system's dynamics are governed by the following two equations:

$$(m_1 + m_2)\ddot{x} + (m_1 + m_2)g = (F_0 + \alpha_0 \sin \omega t) \cos \mu - \xi \dot{x} \ , \tag{4.28}$$

$$(m_1 + m_2)\ddot{y} = (F_0 + \alpha_0 \sin \omega t) \sin \mu - \xi \dot{y} \ . \tag{4.29}$$

Here again we observe the absence of the coupling terms due to angular motion. The response of the vertical and horizontal acceleration is affected only by F_0 and $\alpha_0 \sin \omega t$ and no blade passage frequency is involved. Numerical solutions of (4.28), (4.29) are illustrated in Figures 4.6 and 5.7 for 0.155 m/sec initial horizontal velocity and all other initial conditions equal to zero. From these figures we conclude that steady state vertical and horizontal accelerations are small (about 0.02 m/s^2 and 0.002 m/s^2 respectively) and are of $0.5/\pi$ Hz frequency due to the rotor vibratory airload $\alpha_0 \sin \omega t$.

5.4.4.2 Three Degrees of Freedom without VHHC

Consider equations (4.4)-(4.6) with angular motion but without VHHC (i.e. $\alpha = 0$). The vertical, horizontal and angular accelerations can be expressed as:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & 0 & m_2 \ell \sin\theta \\ 0 & m_1 + m_2 & m_2 \ell \cos\theta \\ m_2 \ell \sin\theta & m_2 \ell \cos\theta & m_2 \ell^2 \end{bmatrix}^{-1} \begin{bmatrix} -m_2 \ell \dot{\theta}^2 \cos\theta - (m_1 + m_2)g + F \cos\mu - \xi \dot{x} \\ m_2 \ell \dot{\theta}^2 \sin\theta + F \sin\mu - \xi \dot{y} \\ -m_2 \ell g \sin\theta + T_\theta - \eta \dot{\theta} \end{bmatrix} , \tag{4.30}$$

where F is given in (4.1). An approximate analysis of these equations can be carried out by using the method of Section 5.4.3.2. Here, however, we investigate the response of system (4.30) by performing simulation of these nonlinear equations with an initial horizontal velocity of 0.155 m/sec and all other initial conditions equal to zero. Similar to Figure 4.4(a), in Figure 4.10(a) with $\alpha = 0$ we observe that the aerodynamic perturbation T_θ generates oscillations in the pendulum with a frequency of $1.5/\pi$ Hz. These oscillation are, in turn, transferred back to the platform and produce additional motion in the vertical and horizontal directions. Thus, as shown in Figures 4.8(a) and 4.9(a), the resulting steady state altitude and horizontal velocity display the effects of both vibratory airload $\alpha_0 \sin \omega t$ and the oscillations of the pendulum. Again, this latter component accounts for most of the vertical and horizontal accelerations.

5.4.4.3 Three Degrees of Freedom with VHHC

With VHHC, the system's dynamic equations are governed by equations (4.30) with F defined in (4.2). As mentioned in Section 5.4.3.3, the high frequency component due to VHHC is filtered out by passive damping of the cabin, and hence it is of interest to consider the filtered vertical, horizontal and angular accelerations of the system. Using the generating equation method proposed in [20], we reduce (4.30) to the standard Bogoliubov form and apply averaging analysis [21]. The averaged equations obtained are as follows (see Appendix A2 for details) :

$$\begin{bmatrix} \ddot{z} \\ \ddot{\nu} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{\alpha m_2(\cos\mu \sin 2\phi + \sin\mu \cos 2\phi)}{m_1(m_1+m_2)} & \frac{m_1+m_2-m_2\cos^2\phi}{m_1(m_1+m_2)} & \frac{m_2 \sin 2\phi}{2m_1(m_1+m_2)} & -\frac{\sin\phi}{m_1 \ell} \\ \frac{\alpha m_2(\cos\mu \sin 2\phi - \sin\mu \cos 2\phi)}{m_1(m_1+m_2)} & \frac{m_2 \sin 2\phi}{2m_1(m_1+m_2)} & \frac{m_1+m_2-m_2\cos^2\phi}{m_1(m_1+m_2)} & -\frac{\cos\phi}{m_1 \ell} \\ \frac{\alpha(\sin\mu \sin\phi - \cos\mu \cos\phi)}{m_1 \ell} & -\frac{\sin\phi}{m_1 \ell} & -\frac{\cos\phi}{m_1 \ell} & \frac{m_1+m_2}{m_1 m_2 \ell^2} \end{bmatrix} \times$$

$$\begin{bmatrix} \frac{\alpha}{2}\left(\frac{\sin\phi\cos\mu+\cos\phi\sin\mu}{m_1}\right) \\ -m_2\ell\left(\phi^2+\frac{\alpha^2(\sin\phi\cos\mu+\cos\phi\sin\mu)^2}{2(m_1\ell)^2}\right)\cos\phi - kz + \alpha_0\cos\mu\sin\omega t - \xi\dot{z} \\ m_2\ell\left(\phi^2+\frac{\alpha^2(\sin\phi\cos\mu+\cos\phi\sin\mu)^2}{2(m_1\ell)^2}\right)\sin\phi + (m_1+m_2)g\tan\mu - kz\tan\mu + \alpha_0\sin\mu\sin\omega t - \xi\dot{\mu} \\ -m_2\ell g\sin\phi + T_\theta - \eta\dot{\phi} \end{bmatrix} \tag{4.31}$$

where z, ν and ϕ represent the averaged altitude, horizontal distance and angle respectively.

The effects of VHHC when a helicopter is in forward flight and when a helicopter is in hover are very similar: VHHC suppresses the oscillations of the pendulum induced by aerodynamic perturbations T_θ; this, in turn, leads to a reduction of the dynamic load passed on back to the vertical and horizontal motions. This can be verified by performing the same analysis as in Section 5.4.3.3. The effectiveness of VHHC in suppressing the vertical and horizontal accelerations when the helicopter is in forward flights is also illustrated by simulating nonlinear equation (4.31) with the initial horizontal velocity of 0.155 m/sec and all other initial conditions equal to zero. For clarity of illustrations, we present here the simulations only for $\alpha = 0.25$ (other values of α are characterized in Figures 4.11 (a) and (b)). As it can be seen from Figures 4.8 and 4.9, the $3/\pi$ Hz vibrations in the vertical and horizontal motion are greatly attenuated. By comparing with Figures 4.6 and 4.7, we notice that VHHC stiffens the pendulum and forces the model to behave as a 2 degrees of freedom system. We observe in Figure 4.10(a) that with VHHC the steady state angular position of the pendulum is perpendicular to the platform. Since the angular position of the pendulum is initially at zero, this accounts for the large transient overshoots in Figures 4.8 and 4.9.

5.4.5 EFFECTS OF WIND GUSTS

Next we analyze the effects of wind gusts on the performance of VHHC-equipped helicopters during forward flight. We introduce impulsive perturbations F_x (at $t = 1$ sec and $t = 4$ sec) and F_y (at $t = 2.5$ sec and $t = 4$ sec) into the system; the pulse duration and amplitude, respectively, are: 0.5 sec and 0.1 N. As it follows from Figures 4.12 and 4.13, VHHC still cancels the vibrations. However, since the gusts F_x and F_y enter the system's equations directly in (4.4) and (4.5), VHHC is not very effective in suppressing their effects on the vertical and horizontal accelerations of the system.

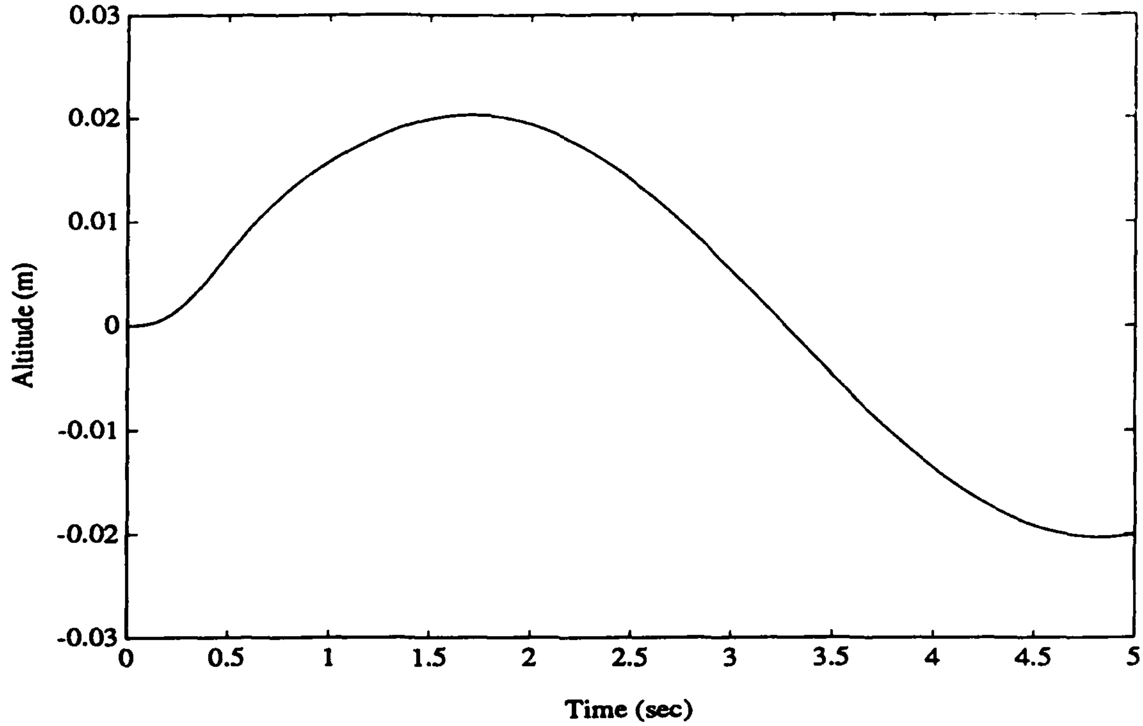

Figure 4.2: (a) Altitude of helicopter in hover (1 degree of freedom)

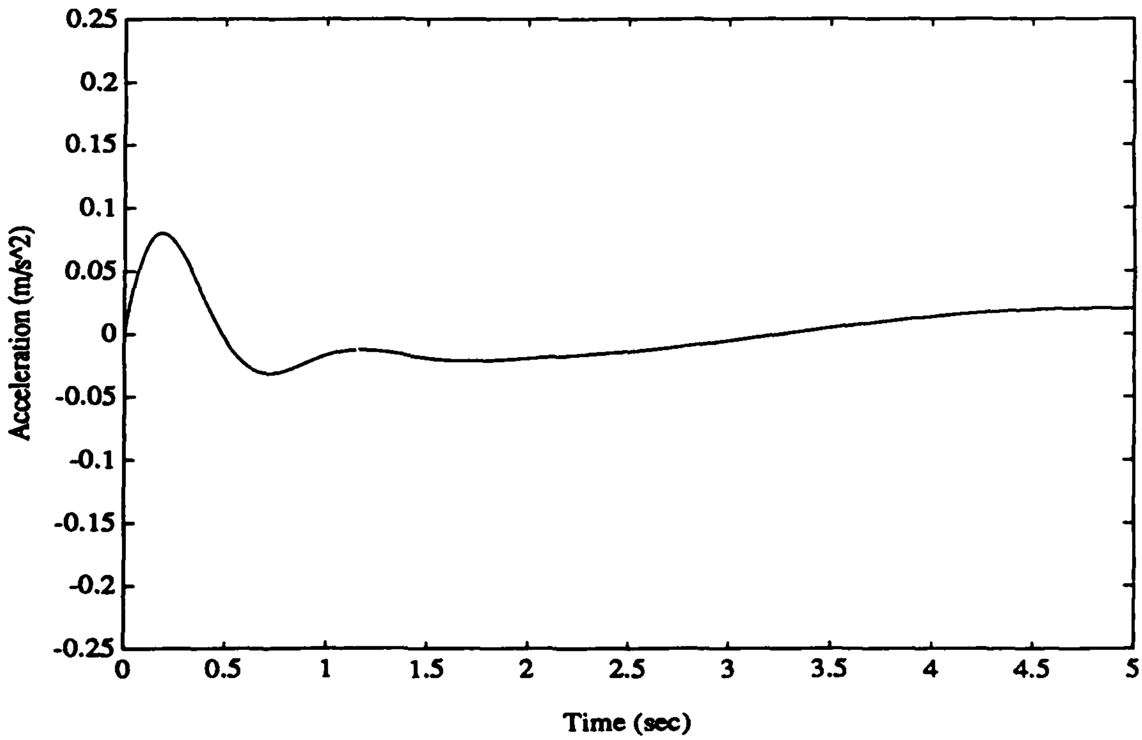

Figure 4.2: (b) Vertical acceleration of helicopter in hover (1 degree of freedom)

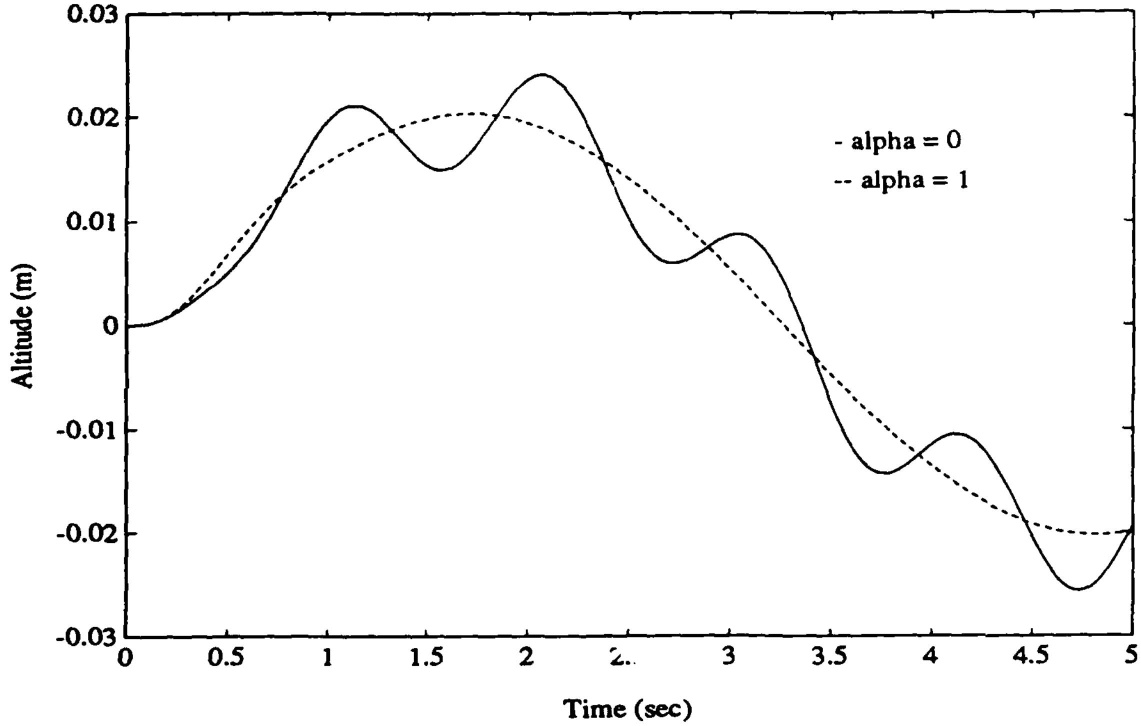

Figure 4.3: (a) Altitude of helicopter in hover (2 degrees of freedom)

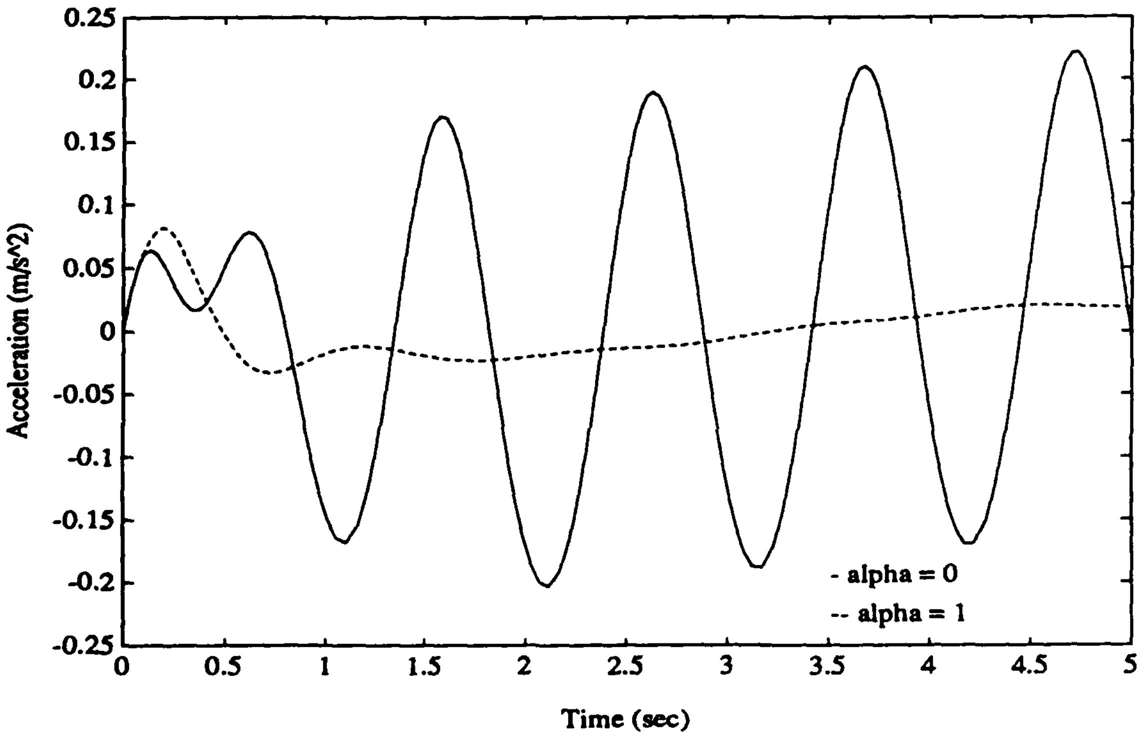

Figure 4.3: (b) Vertical acceleration of helicopter in hover (2 degrees of freedom)

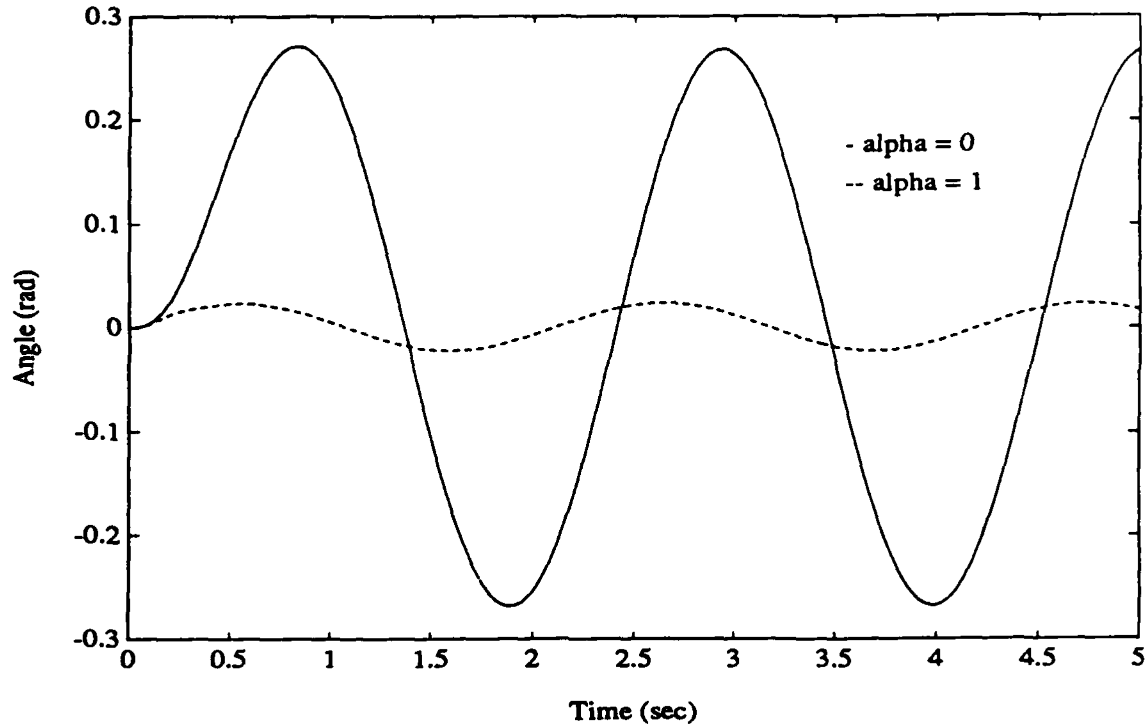

Figure 4.4: (a) Angle of the fuselage in hover (2 degrees of freedom)

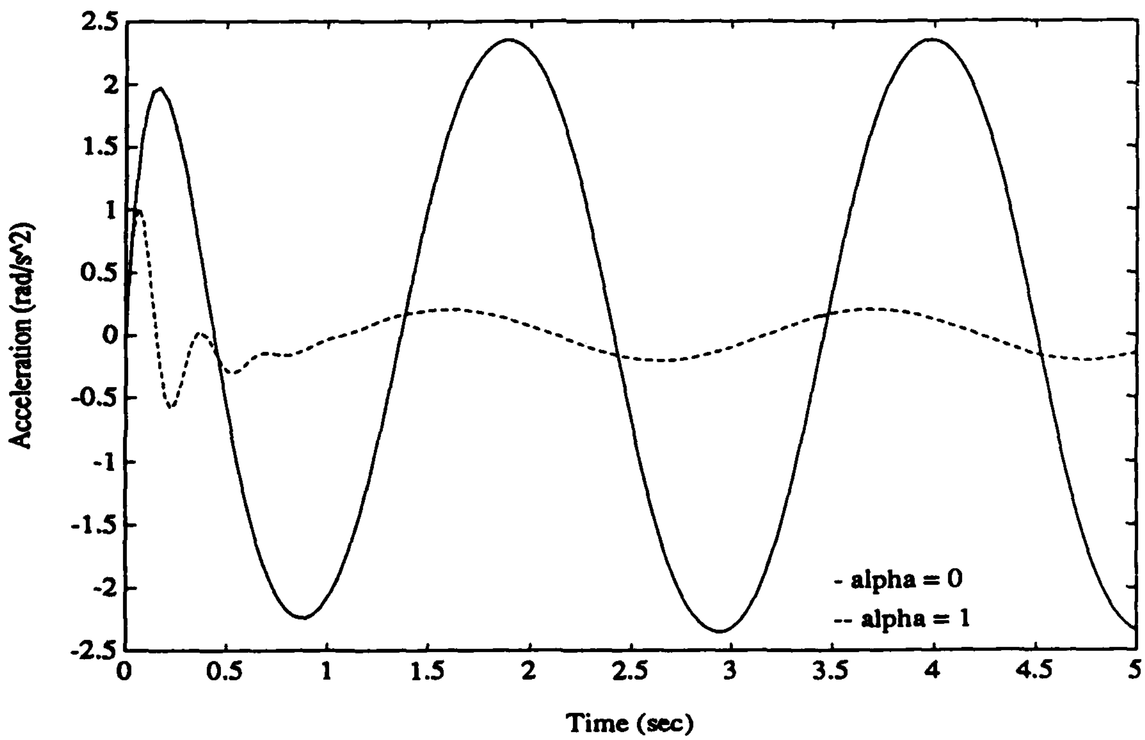

Figure 4.4: (b) Anglular acceleration of the fuselage in hover (2 degrees of freedom)

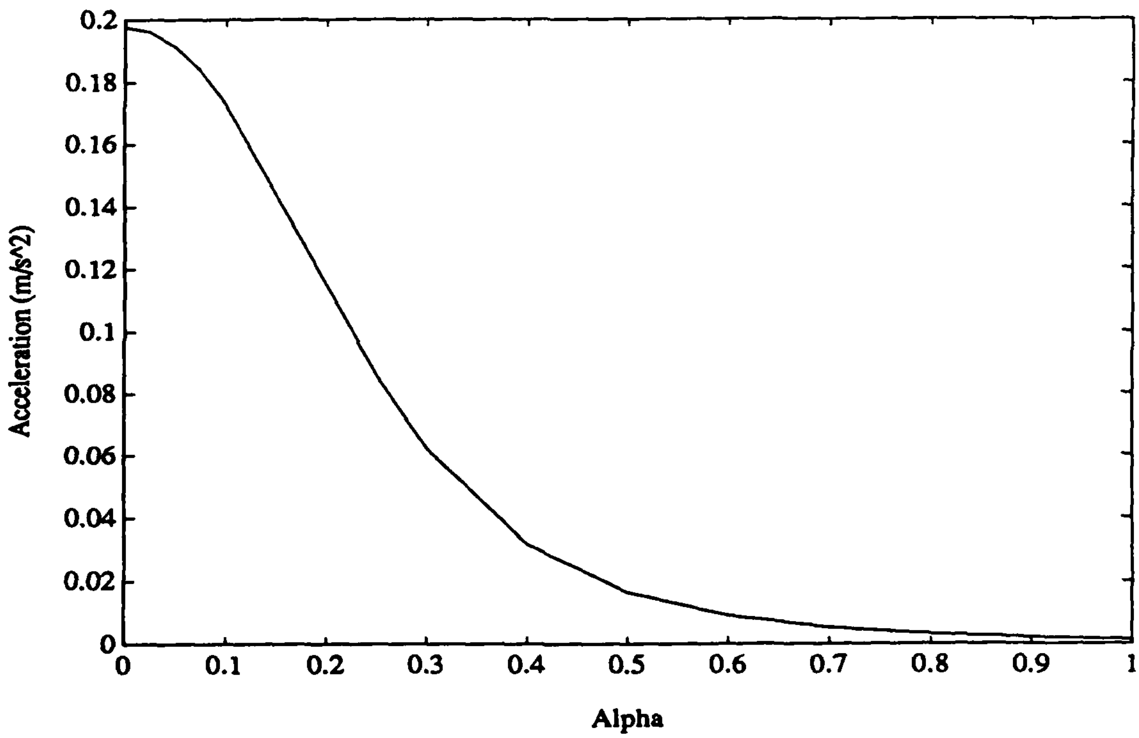

Figure 4.5: Vertical acceleration of helicopter in hover for different α

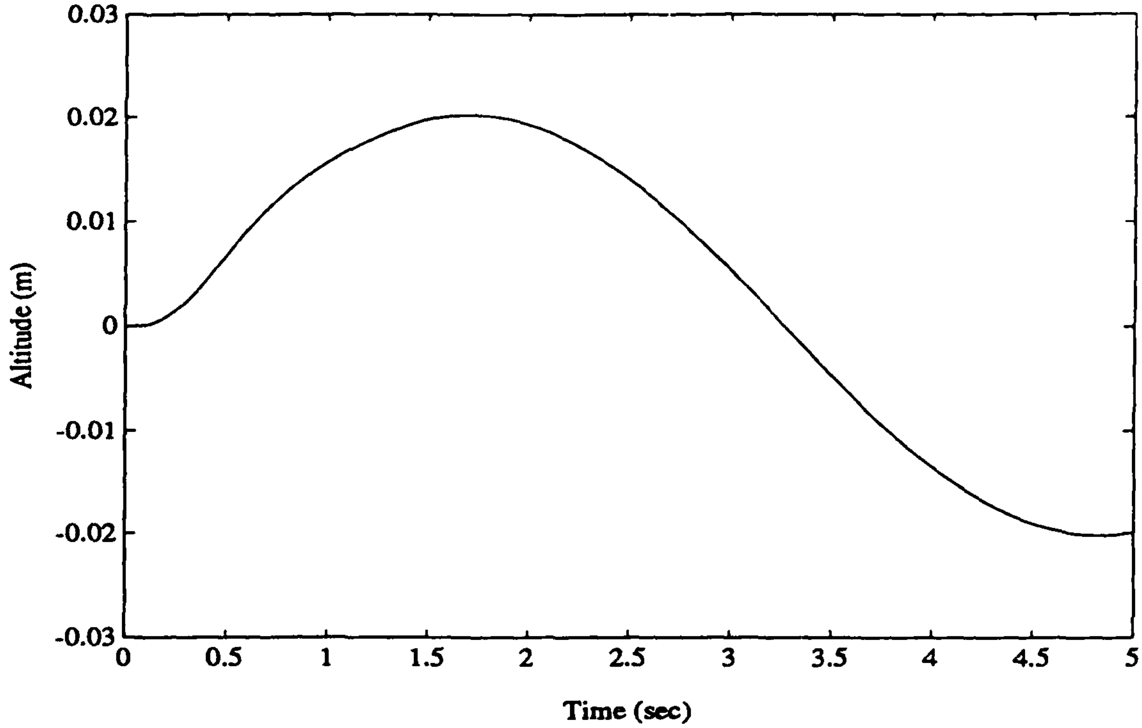

Figure 4.6: (a) Altitude of helicopter in flight (2 degrees of freedom)

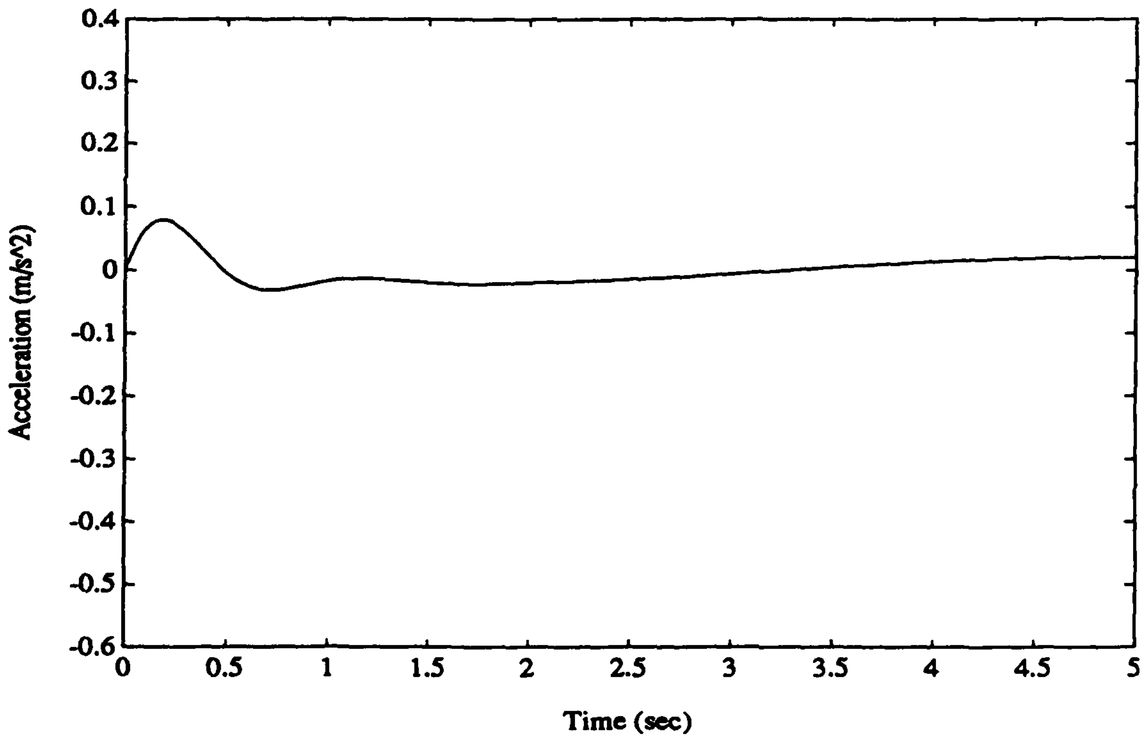

Figure 4.6: (b) Vertical acceleration of helicopter in flight (2 degrees of freedom)

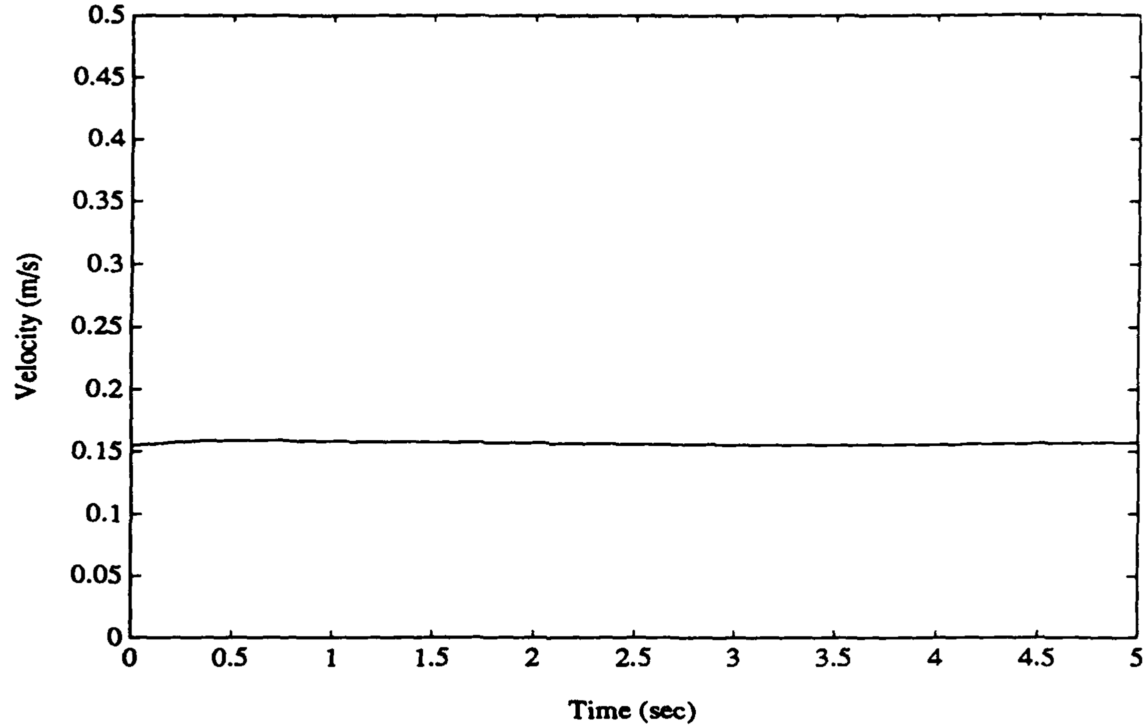

Figure 4.7: (a) Horizontal velocity of helicopter in flight (2 degrees of freedom)

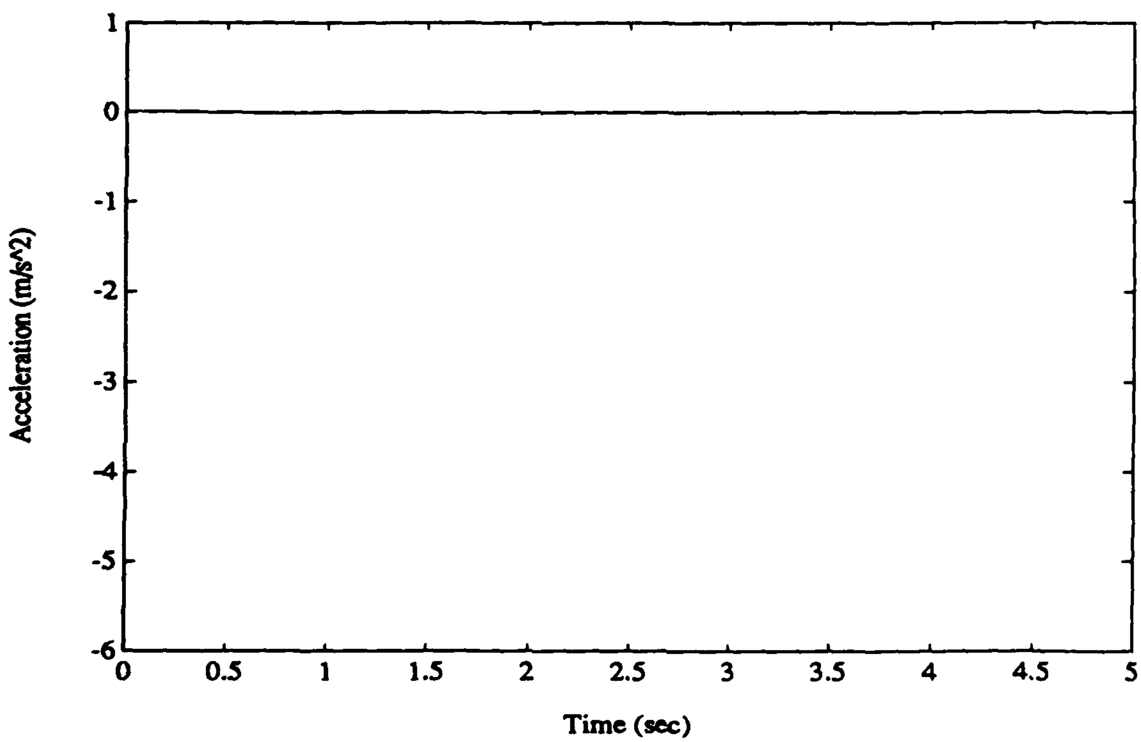

Figure 4.7: (b) Horizontal acceleration of helicopter in flight (2 degrees of freedom)

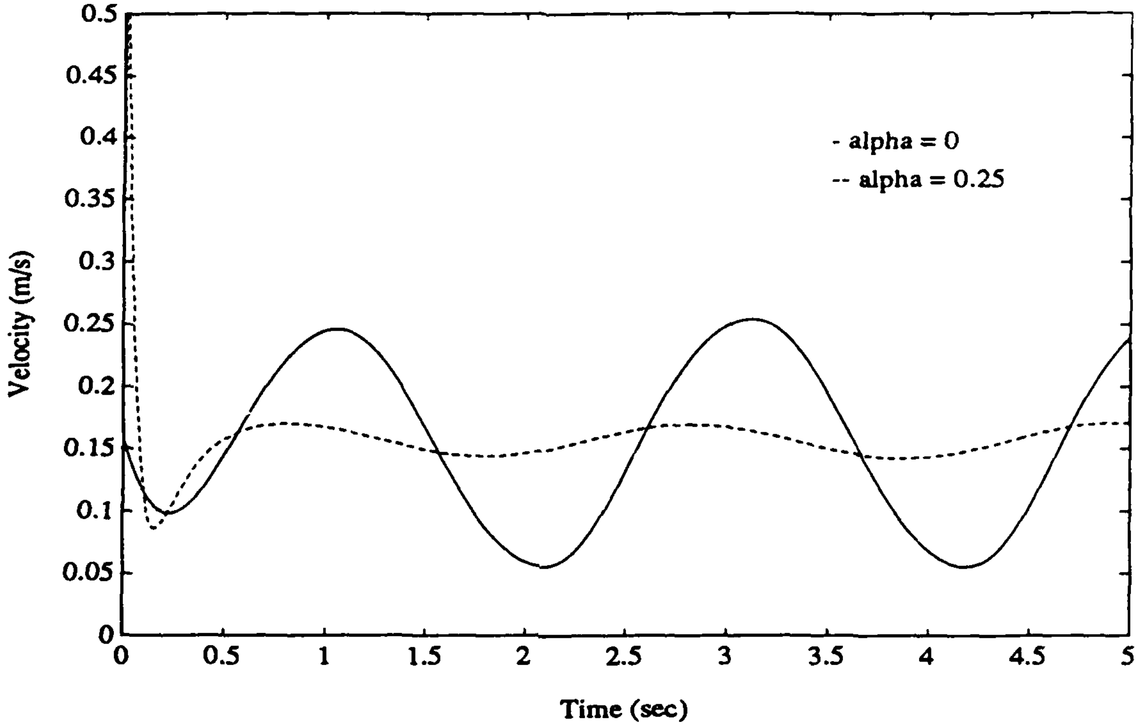

Figure 4.9: (a) Horizontal velocity of helicopter in flight (3 degrees of freedom)

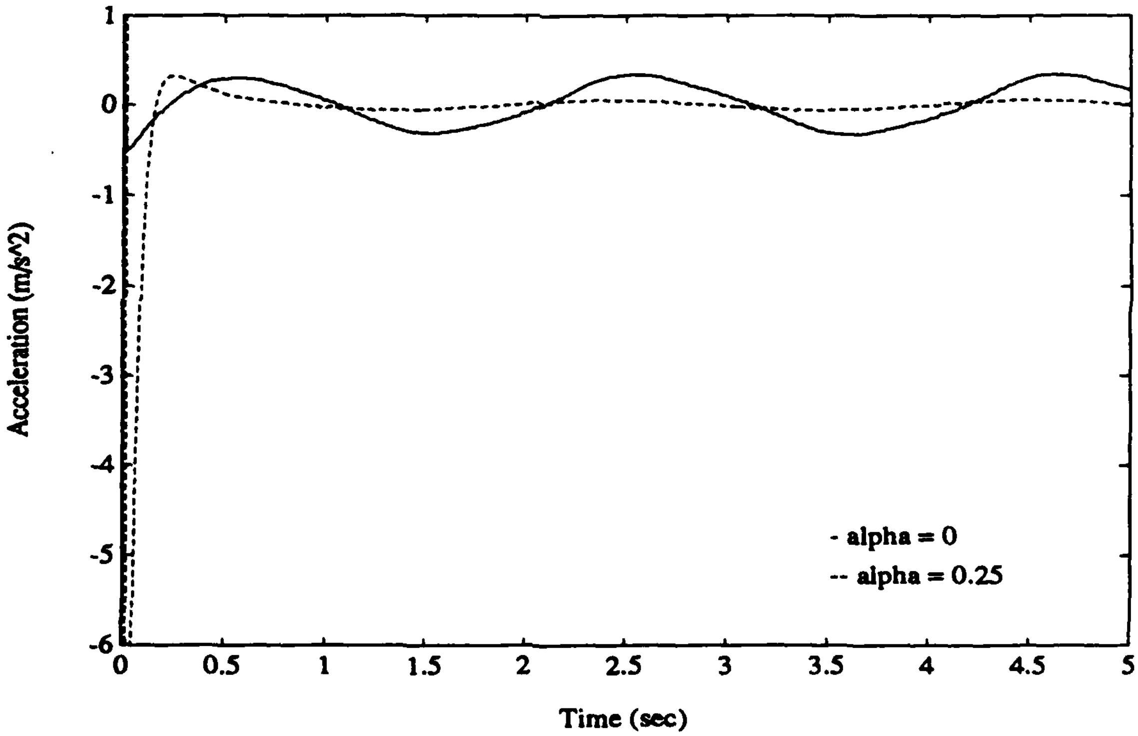

Figure 4.9: (b) Horizontal acceleration of helicopter in flight (3 degrees of freedom)

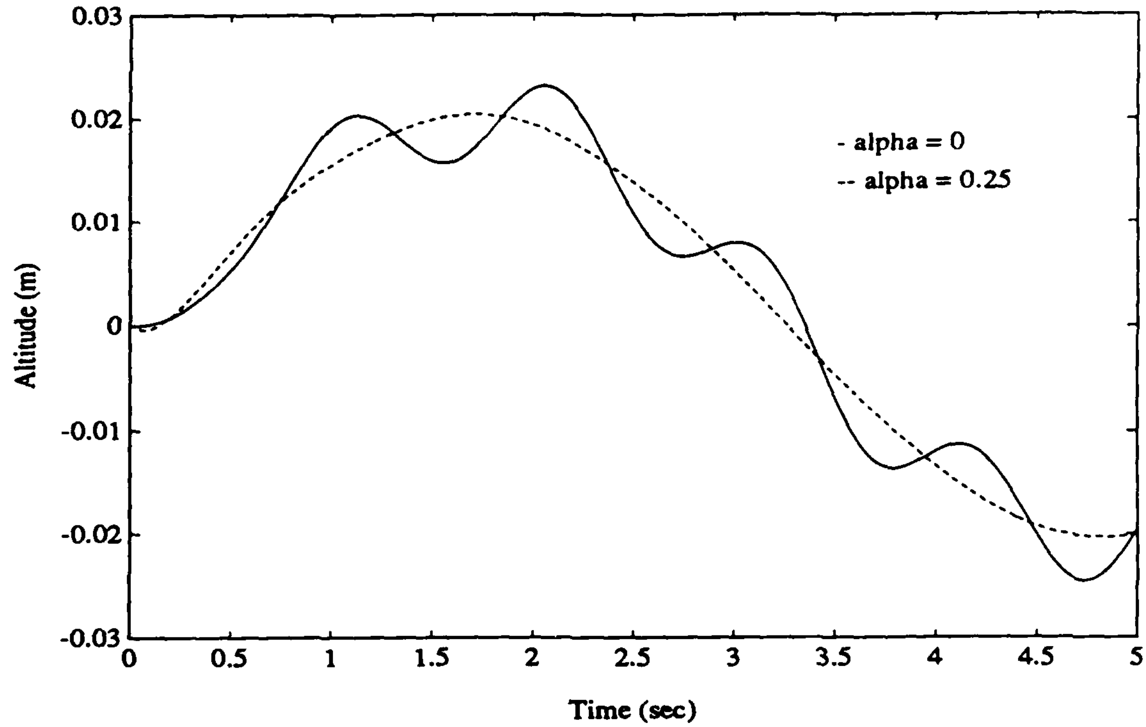

Figure 4.8: (a) Altitude of helicopter in flight (3 degrees of freedom)

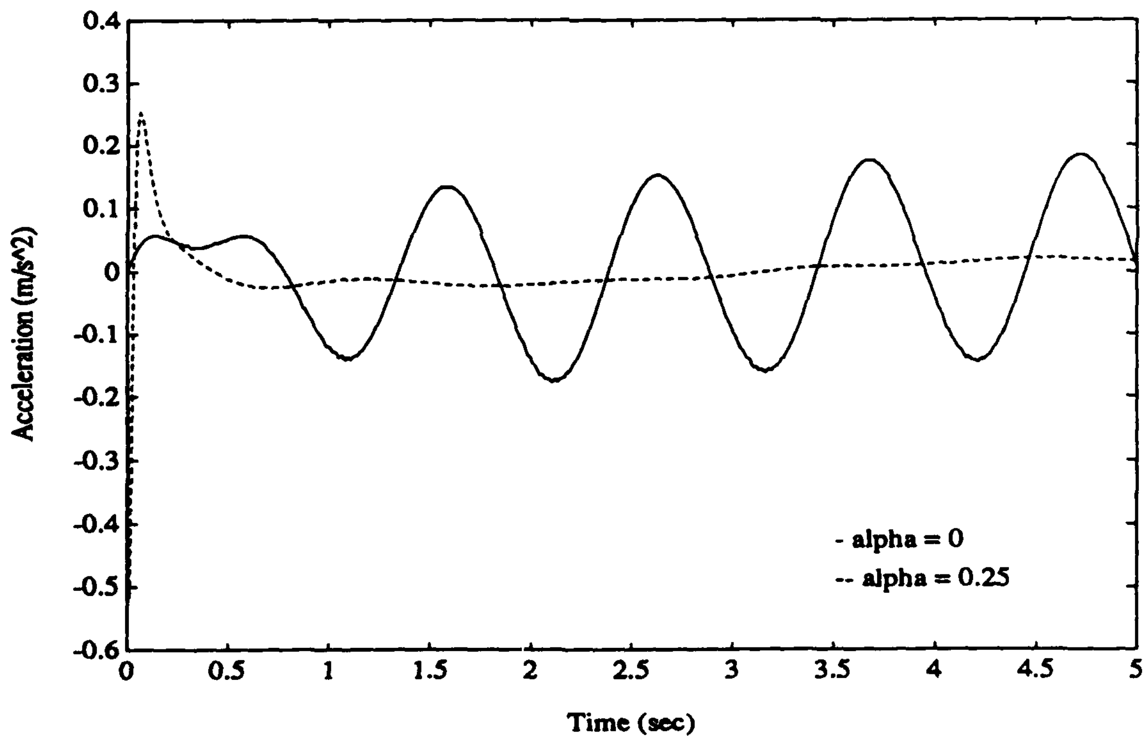

Figure 4.8: (b) Vertical acceleration of helicopter in flight (3 degrees of freedom)

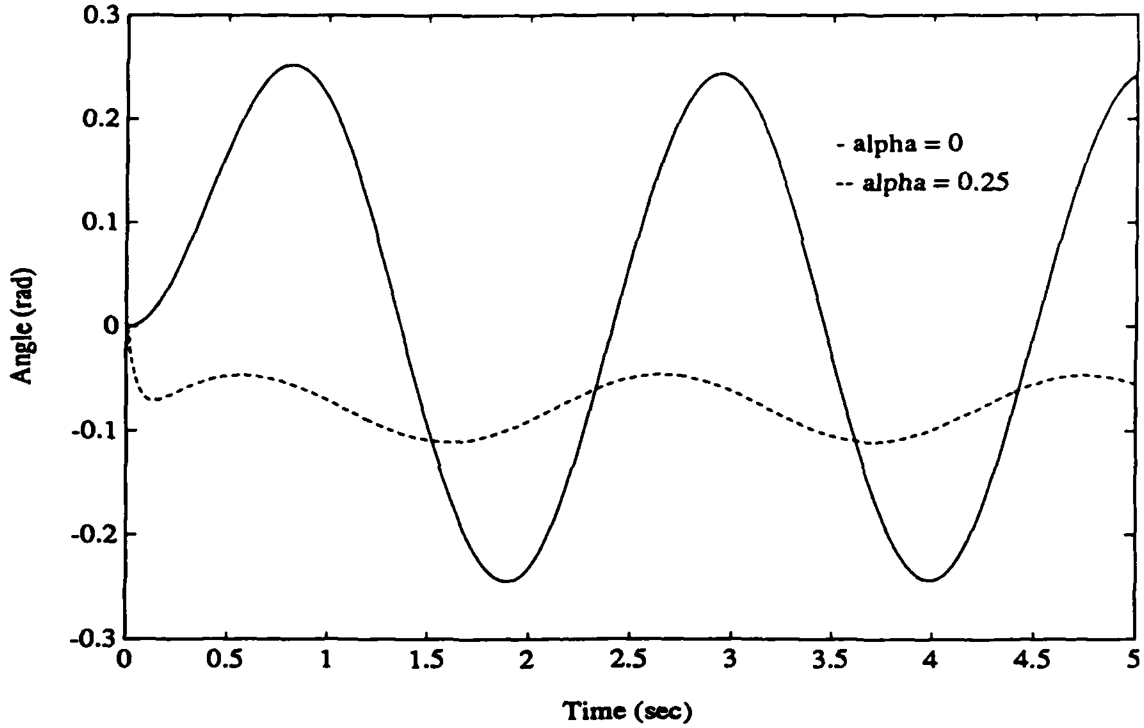

Figure 4.10: (a) Angle of the fuselage in flight (3 degrees of freedom)

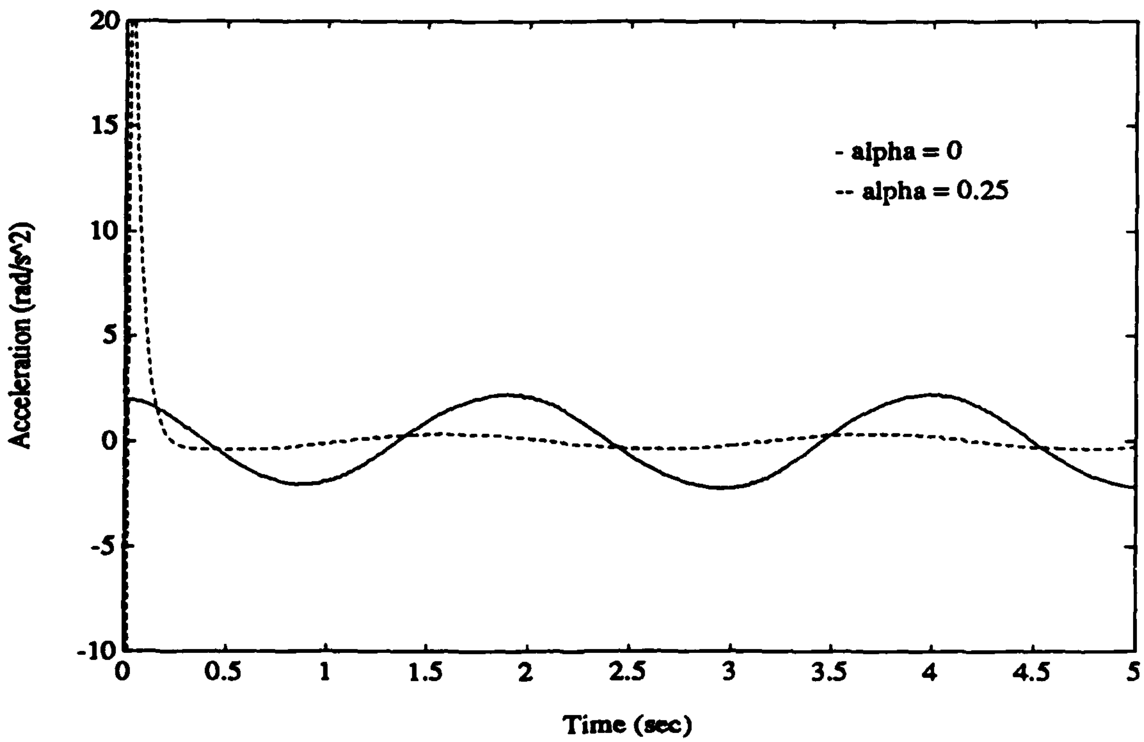

Figure 4.10: (b) Angular acceleration of the fuselage in flight (3 degrees of freedom)

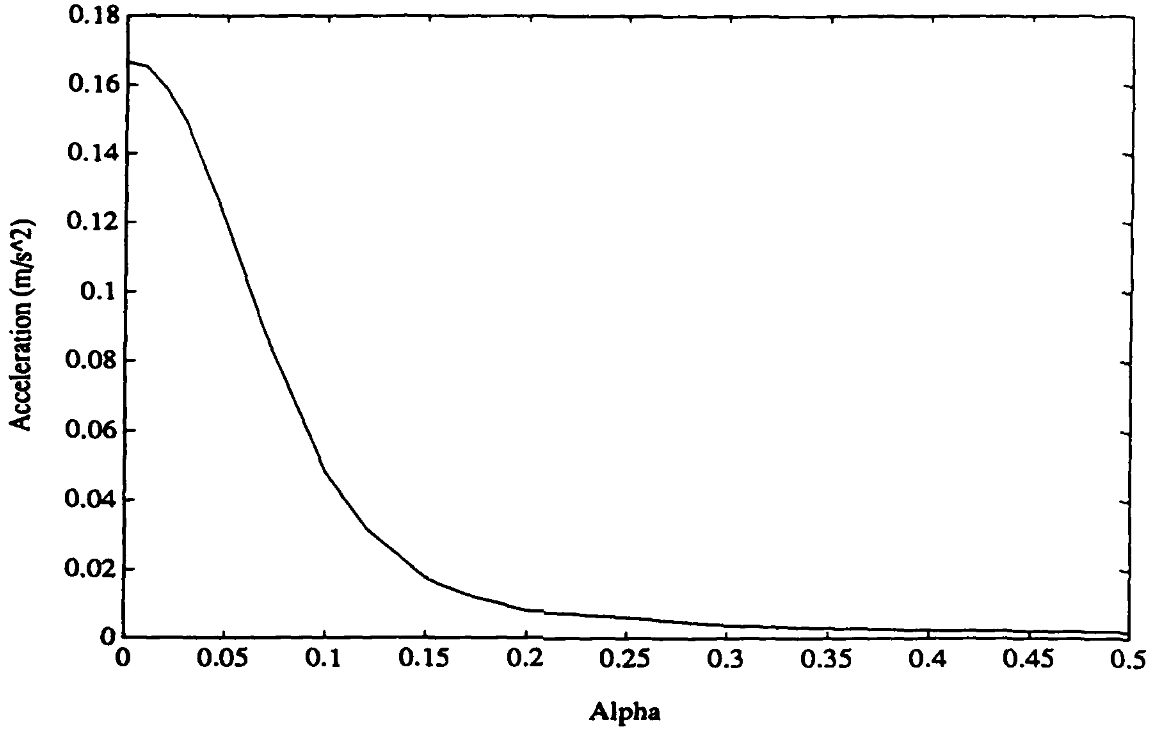

Figure 4.11: (a) Vertical acceleration of helicopter in flight for different α

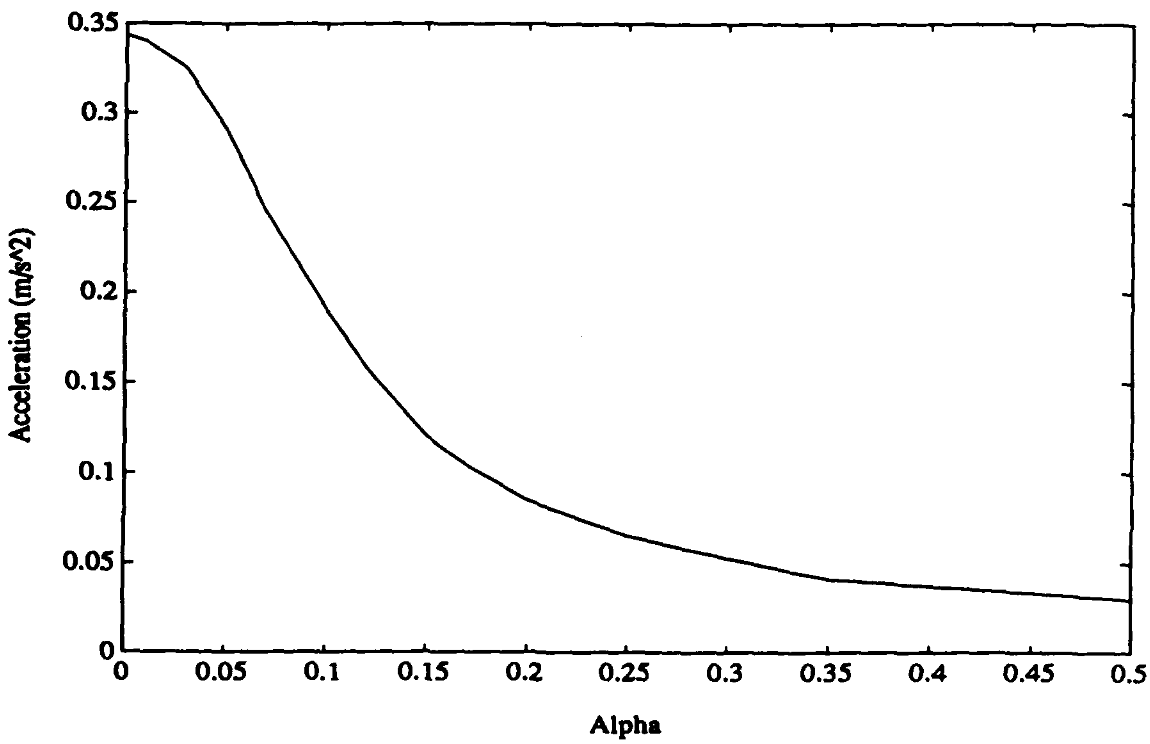

Figure 4.11: (b) Horizontal acceleration of helicopter in flight for different α

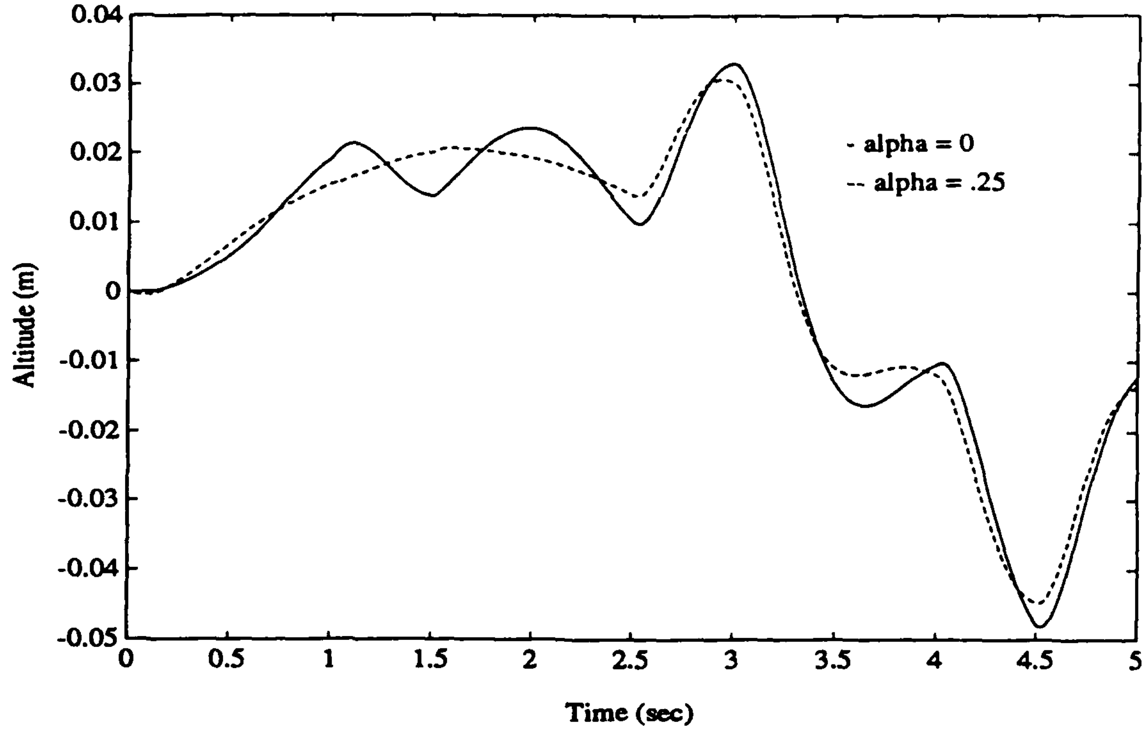

Figure 4.12: (a) Altitude of helicopter in forward flight with wind gust

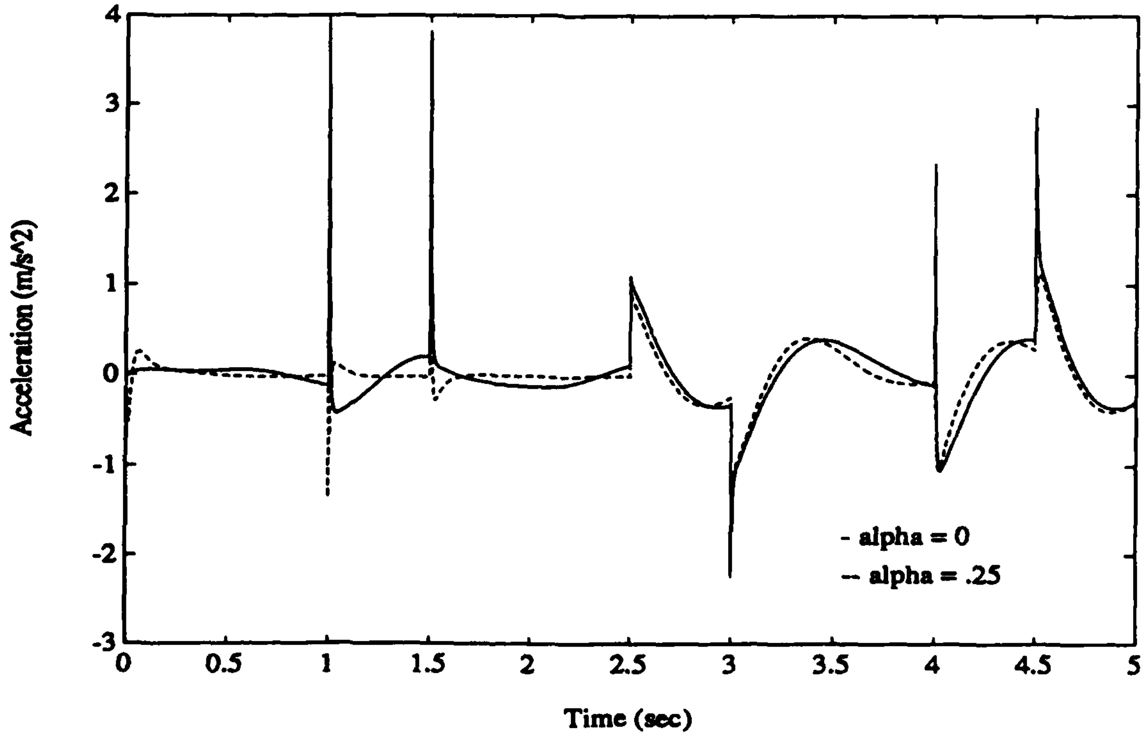

Figure 4.12: (b) Vertical acceleration of helicopter in flight with wind gust

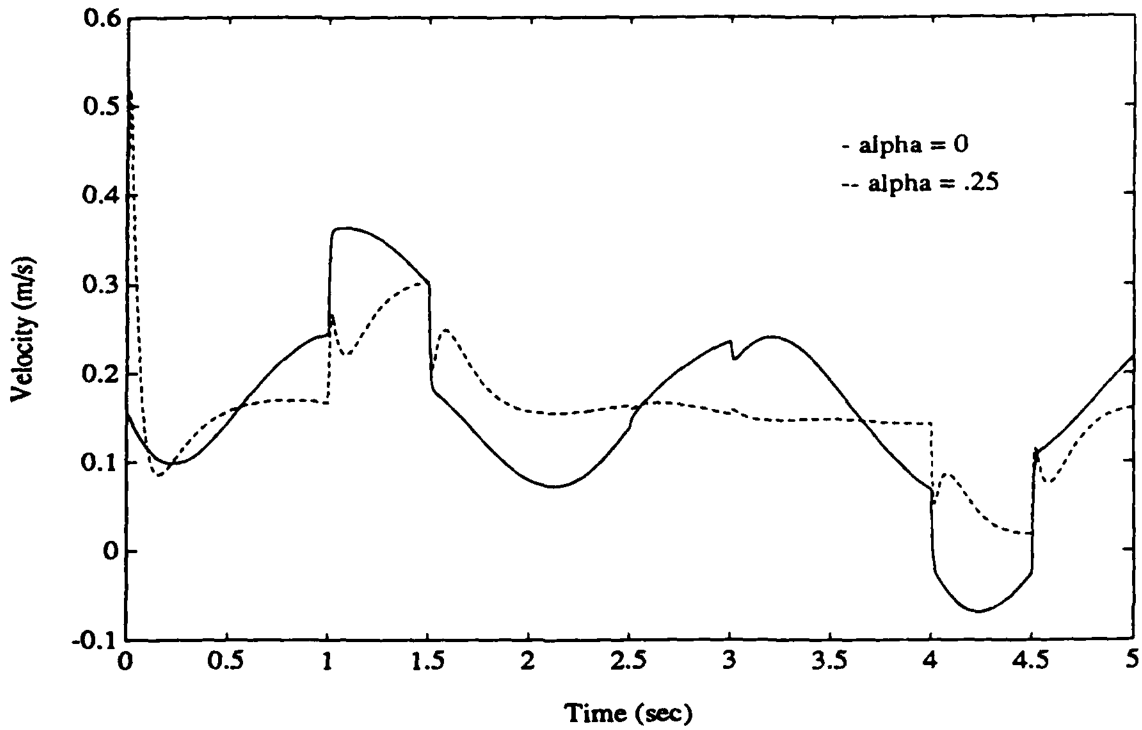

Figure 4.13: (a) Horizontal velocity of helicopter in flight with wind gust

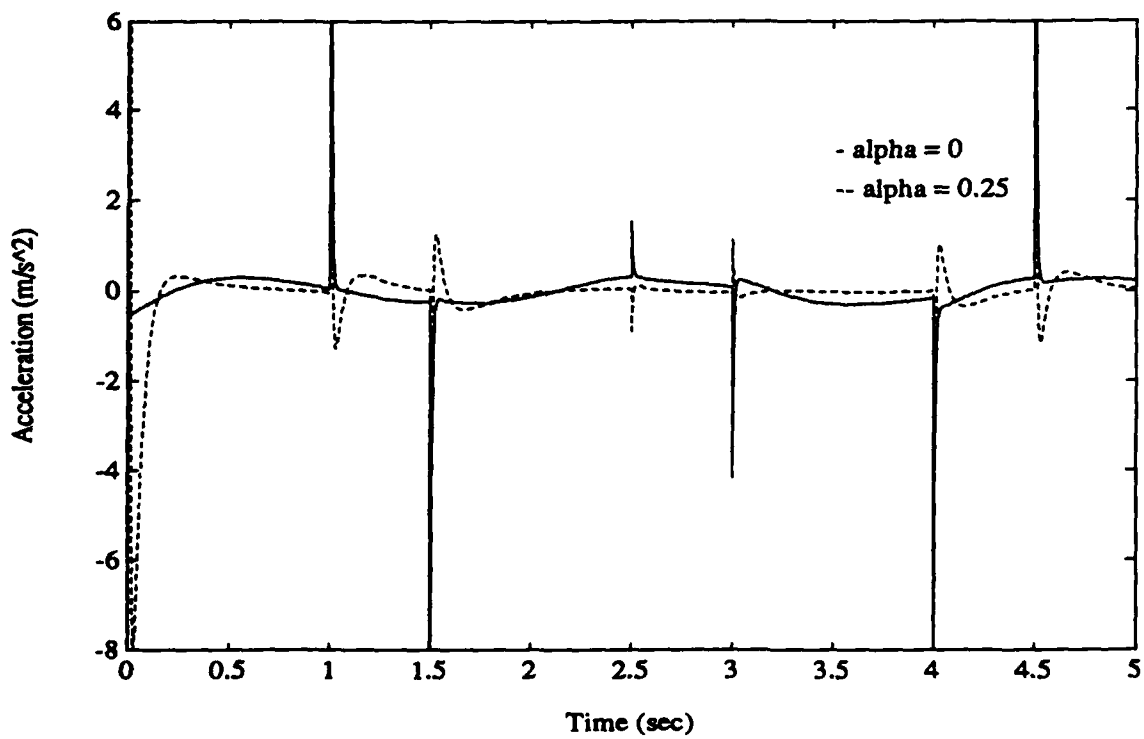

Figure 4.13: (b) Horizontal acceleration of helicopter in flight with wind gust

5.5 PART 5. H_2-OPTIMAL ZEROS

5.5.1 INTRODUCTION

Another important aspect in the control design of systems, besides ensuring sufficient number of actuators as discussed in Section 5.1, is to determine the location of the control system components, namely actuators and sensors. In this part of the report, we treat this problem by considering a linear time-invariant SISO system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1u(t) + B_2w(t) , \\ y(t) &= Cx(t) , \quad x \in \mathbb{R}^n, u, w, y \in \mathbb{R}. \end{aligned} \tag{5.1}$$

In the design stage of the system (5.1), when no actuator positioning and disturbance protection measures are yet finalized, input vectors B_1 and B_2 may be viewed as free parameters to be chosen so that appropriate performance specifications are satisfied. Among these, it seems reasonable to require that the transmission from the control u to the output y be maximized and the transmission from the disturbance w to y be minimized. The term "transmission" could be formalized in a number of ways. We choose here a formalization in terms of transfer functions. Specifically, we say that the transmission from u to y is maximized if the H_2-norm of $G_u(s) := C(sI - A)^{-1}B_1$ is maximized with respect to all $B_1 \in \mathbb{R}^n$ such that $||B_1|| = 1$ ($|| \cdot ||$ denotes the Euclidean norm of a vector). Similarly, the transmission from w to y is minimized (i.e. the disturbance rejection is optimized) if the H_2-norm of $G_w(s) := C(sI - A)^{-1}B_2$ is minimized with respect to all $B_2 \in \mathbb{R}^n$ such that $||B_2|| = 1$. Since a choice of B_1 and B_2 defines, for a given A and C, the system's zeros, optimal B_1 and B_2 define the zeros, optimal with regard to the above stated criteria. The purpose of this research is to give a solution to the H_2-optimal zero placement problem for open and closed loop system (Sections 5.5.2 and 5.5.4) and characterize control-theoretic properties of the resulting system (Section 5.5.3). In addition, we analyze the relationship between the open and closed loop H_2-optimal zeros (Section 5.5.5).

Since the L_∞ norm of y is often used for control system performance specifications, the H_2-norm as a measure of optimality seems quite reasonable because it is induced by the L_2 norm on the input and the L_∞ norm on the output [86]. Other measures of optimality are, however, possible. For instance, Hughes and Skelton [87] optimize the controllability and observability "norms" to ensure good actuators and sensors configuration. Optimization of controllability, observability or the information matrix has been addressed in [39], [90], and [91]. An LQ-type approach is developed in [48]. In [38], actuator location has been addressed under a control effort saturation assumption. The H_2-optimization of this paper is related to the above cited literature because sensor and actuator locations typically determine the matrices C and B_1 respectively. In the present work, however, we assume that the directions of B_1 and B_2 can be chosen arbitrarily whereas in practical applications components of vectors B_1 and B_2 might be functionally related. Accounting for these functional relationships will be a topic of future work.

5.5.2 H_2-OPTIMAL ZEROS IN OPEN LOOP ENVIRONMENT

5.5.2.1 Problem Formulation

Consider an asymptotically stable SISO plant defined by a transfer function $G(s) = C(sI - A)^{-1}B$. Assume that vector B is subject to the constraint

$$B^TMB = 1 , \tag{5.2}$$

where M is a symmetric, positive definite matrix.

The problem of H_2-optimal zero placement in open loop environment is formulated as follows: Find vector B, satisfying (5.2), such that $||G||_2$ is maximized (minimized).

Remark 5.1: A natural choice for M in (5.2) seems to be $M = I$, resulting in $||B|| = 1$. It turns out, however, that this normalization may lead to a coordinate dependent optimal B and consequently coordinate dependent zeros (see Section 5.5.3.3). If (C, A) is observable, a better choice is

$$M = C^T C + A^T C^T C A + \ldots + (A^{n-1})^T C^T C A^{n-1} \; .$$

resulting in the normalization

$$(CB)^2 + (CAB)^2 + \ldots + (CA^{n-1}B)^2 = 1 \; . \tag{5.3}$$

Since $CB, CAB, \ldots, CA^{n-1}B$ are the Markov parameters of $G(s)$ and since

$$CA^i B = \left. \frac{d^i g(t)}{dt^i} \right|_{t=0^+} , \quad i = 0, 1, \ldots, n-1 \; ,$$

where

$$g(t) = \mathcal{L}^{-1}\{G(s)\},$$

the normalization (5.3) constrains B to those that satisfy

$$\sum_{i=0}^{n-1} \left[\left. \frac{d^i g(t)}{dt^i} \right|_{t=0^+} \right]^2 = 1 \; . \tag{5.4}$$

Thus, the optimal B is selected from all those that result in an impulse response which at $t = 0^+$ satisfy the normalization (5.4). This in fact defines a unit sphere in the space of impulse responses of SISO, LTI systems of dynamic order at most n. It turns out that this normalization gives a coordinate independent optimal B and as a consequence, coordinate independent zeros (see Section 5.5.3.3).

5.5.2.2 Main Result

Theorem 5.1: Assume that A in (5.1) is Hurwitz and M in (5.2) is positive definite and symmetric. Let L_o be the observability Gramian, defined by $A^T L_o + L_o A + C^T C = 0$. Then $\max_{B_1^T M B_1 = 1} \|G_u\|_2 \left(\min_{B_2^T M B_2 = 1} \|G_w\|_2 \right)$ is attained at $B_1{}^*$ $(B_2{}^*)$ collinear with the eigenvector of $M^{-1} L_o$ corresponding to its largest (smallest) eigenvalue $\lambda_{\max}(\lambda_{\min})$. Under this choice $\max_{B_1^T M B_1 = 1} \|G_u\|_2^2 = \lambda_{\max} \left(\min_{B_2^T M B_2 = 1} \|G_w\|_2^2 = \lambda_{\min} \right)$.

Proof: First, note that $M^{-1} L_o$ is a product of two symmetric, positive definite matrices and so its eigenvalues are real and positive. Second, we observe that since $\|G\|_2^2 = B^T L_o B$ (see [92]) and since $(\cdot)^2$ is a monotonic operation, the Lagrangian for the problem at hand is

$$L(B, \gamma, \lambda) = \gamma(B^T L_0 B) - \lambda(B^T M B - 1) \; ,$$

where the multipliers $\gamma, \lambda \in \mathbb{R}$ are not simultaneously zero. The first order necessary condition is [93]

$$\nabla_B L = \gamma L_0 B - \lambda M B = 0 \; ,$$

which yields

$$\gamma M^{-1} L_0 B = \lambda B \; .$$

Since $B^T M B = 1, \gamma = 0$ implies $\lambda = 0$. Hence, it can be assumed without loss of generality that $\gamma = 1$. Furthermore, $\lambda \geq 0$. Thus, any optimal B^* should lie along the direction of an eigenvector of $M^{-1} L_o$ and

$$\|G^*\|_2^2 = B^{*T} L_o B^* = \lambda B^{*T} M B^* = \lambda \; .$$

Hence,

$$\max_{B_1^T M B_1=1} \|G_u\|_2^2 = B_1^{*T} L_0 B_1^* = \lambda_{\max} B_1^{*T} M B_1^* = \lambda_{\max} ,$$
$$\min_{B_2^T M B_2=1} \|G_w\|_2^2 = B_2^{*T} L_o B_2^* = \lambda_{\min} B_2^{*T} M B_2^* = \lambda_{\min} .$$

Q.E.D.

Remark 5.2: Theorem 5.1 can be reformulated for choosing the measurement vector C to maximize (minimize) the H_2-norm of a transfer function $G(s) = C(sI - A)^{-1}B$ subject to $CMC^T = 1$. The optimal C^* lies in the direction of the left eigenvector of $L_c M^{-1}$ corresponding to the largest (smallest) eigenvalue of $L_c M^{-1}$ for maximum (minimum) $\|G\|_2$, where L_c is the controllability Gramian, defined by $AL_c + L_c A^T + BB^T = 0$.

Remark 5.3: For optimal disturbance rejection, when the pair (C, A) in (5.1) is not observable, Theorem 5.1 yields a choice of B_2 corresponding to a null eigenvector of $M^{-1}L_o$. This choice leads to the invariance of the output with respect to the disturbance, i.e., $[CB_2\, CAB_2 \ldots CA^{n-1}B_2] = 0$.

Remark 5.4: For unstable SISO systems, one can introduce the notion of a α-shifted H_2-norm defined as follows:

$$\begin{aligned} \|G\|_{2\alpha}^2 &= \int_0^\infty \left(Ce^{(A-\alpha I)t}B\right)^2 dt , \\ &= CL_{c\alpha}C^T = B^T L_{o\alpha} B , \quad \alpha > 0, \end{aligned}$$

where α is chosen so that $(A - \alpha I)$ is Hurwitz, and $L_{c\alpha}$ and $L_{o\alpha}$ are the controllability and observability Gramians of the pair $(A - \alpha I, B)$ and $(C, A - \alpha I)$ respectively. Here, the optimal B^* would then lie along the direction of the eigenvector of $M^{-1}L_{o\alpha}$ corresponding to the largest (smallest) eigenvalue of $M^{-1}L_{o\alpha}$ for maximum (minimum) $\|G\|_{2\alpha}$.

5.5.3 QUALITATIVE PROPERTIES OF OPEN LOOP SYSTEMS WITH THE H_2-OPTIMAL ZEROS

This section examines several control-theoretic properties of systems with the input vector chosen according to Theorem 5.1. The dual results for the optimal measurement vector C can be obtained similarly.

5.5.3.1 Optimal Control Effort

The optimal transmission from u to y is formalized in Section 5.5.2 as the H_2-optimization of the corresponding transfer function. A question arises : What does H_2-optimization of $G(s)$ means in terms of the control effort necessary to accomplish a given task. An answer is given below.

Consider the system,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) , \\ y(t) &= Cx(t) . \quad x \in I\!R^n, u, y \in I\!R \end{aligned} \tag{5.5}$$

Let U be the set of all bounded measurabl controls $u(t)$ that transfer (5.5) from the initial condition $x(0) = x_o$ to a final condition satis ng $y(t_1) = y_1$ during $[0, t_1]$. Define

$$J(x_0, y_1, t_1) = \min_{u(t)\in U} \int_0^{t_1} u^2(\tau)d\tau .$$

Theorem 5.2: Assume that A in (5.5) is Hurwitz and M in (5.2) is positive definite and symmetric. Then $\min_{B^T MB=1} \lim_{t_1\to\infty} J(x_0, y_1, t_1) \left(\max_{B^T MB=1} \lim_{t_1\to\infty} J(x_0, y_1, t_1)\right)$ is achieved at B which is collinear with the eigenvector of $M^{-1}L_o$ corresponding to its largest (smallest) eigenvalue.

Proof: First we prove that

$$J(x_0, y_1, t_1) = \frac{(y_1 - Ce^{At_1}x_0)^2}{\int_0^{t_1} Ce^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}C^T d\tau} . \tag{5.6}$$

Indeed, a control $u^0(t)$ that transfers (5.5) from the initial condition $x(0) = x_o$ to a final condition satisfying $y(t_1) = y_1$ over $[0, t_1]$ can be calculated as follows:

$$u^0(t) = B^T e^{A^T(t_1-t)}C^T W^{-1}(0,t_1)[y_1 - Ce^{At_1}x_0], \quad t \in [0, t_1] , \tag{5.7}$$

where

$$W(0,t_1) = \int_0^{t_1} Ce^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}C^T d\tau .$$

This control is optimal in the sense that if $u^1(t)$ is any other control in U, then

$$\int_0^{t_1} \left(u^1(t)\right)^2 dt \geq \int_0^{t_1} \left(u^0(t)\right)^2 dt . \tag{5.8}$$

To prove this we note that

$$y_1 - Ce^{At_1}x_0 = \int_0^{t_1} Ce^{A(t_1-\tau)}Bu^0(\tau)d\tau = \int_0^{t_1} Ce^{A(t_1-\tau)}Bu^1(\tau)d\tau .$$

Therefore,

$$\int_0^{t_1} Ce^{A(t_1-\tau)}B(u^1(\tau) - u^0(\tau))d\tau = 0 .$$

Then

$$W^{-1}(0,t_1)\left[y_1 - Ce^{At_1}x_0\right] \int_0^{t_1} Ce^{A(t_1-\tau)}B(u^1(\tau) - u^0(\tau))d\tau = 0 ,$$

which is equivalent to

$$\int_0^{t_1} \left(u^1(\tau) - u^0(\tau)\right) \left(Ce^{A(t_1-\tau)}B\right)^T W^{-1}(0,t_1)\left[y_1 - Ce^{At_1}x_0\right] d\tau = 0 .$$

From the last expression and (5.7), we obtain

$$\int_0^{t_1} \left(u^1(\tau) - u^0(\tau)\right) u^0(\tau)d\tau = 0 .$$

Therefore,

$$\begin{aligned}
\int_0^{t_1} \left(u^1(\tau)\right)^2 d\tau &= \int_0^{t_1} \left(u^1(\tau) - u^0(\tau) + u^0(\tau)\right)^2 d\tau \\
&= \int_0^{t_1} \left(u^1(\tau) - u^0(\tau)\right)^2 d\tau + \int_0^{t_1} \left(u^0(\tau)\right)^2 d\tau \\
&\quad +2\int_0^{t_1} \left(u^1(\tau) - u^0(\tau)\right) u^0(\tau)d\tau \\
&= \int_0^{t_1} \left(u^1(\tau) - u^0(\tau)\right)^2 d\tau + \int_0^{t_1} \left(u^0(\tau)\right)^2 d\tau
\end{aligned}$$

which results in (5.8).

Thus, it follows from (5.8) that the minimum energy required to transfer the system (5.5) from $x(0) = x_o$ to a final condition satisfying $y(t_1) = y_1$ is indeed given by (5.6). Hence if A is Hurwitz,

$$\lim_{t_1 \to \infty} J(x_0, y_1, t_1) = \frac{y_1^2}{B^T L_o B} . \tag{5.9}$$

Since $B^T L_o B$, subject to the constraint $B^T M B = 1$, achieves its maximum (minimum) at B collinear with the maximum (minimum) eigenvalue of $M^{-1} L_o$, Theorem 5.2 is proved. **Q.E.D.**

Theorem 5.2 offers an alternative characterization of H_2-optimal transmission: The optimal B_1^* (B_2^*) ensures minimum control (maximum disturbance) energy in transferring the system state from an arbitrary initial state to a final state, consistent with the specified output, over an asymptotically infinite time interval.

5.5.3.2 Controllability Properties

As stated in Theorem 5.1, system (5.1) with $B_1 = B_1{}^*$ has H_2-optimal (maximal) transmission properties from u to y. A question arises: Is there always at least some transmission from u to every component of x in such a system? An answer is given below.

Theorem 5.3: Assume A in (5.1) is Hurwitz, M in (5.2) is positive definite and symmetric and the maximum eigenvalue of the matrix $M^{-1} L_o$, where L_o is the observability Gramian of (5.1), is simple. Then system (5.1) with $B_1 = B_1{}^*$ and $B_2 = 0$ is controllable if and only if there exists no left eigenvector of A orthogonal to the eigenvector of $M^{-1} L_o$ corresponding to its largest eigenvalue.

Proof: Follows directly from Theorem 5.1 and the PBH controllability test [94]. **Q.E.D.**

Theorem 5.4: Assume A in (5.1) is Hurwitz and $B_1 = B_1{}^*$, $B_2 = 0$. Assume further that $M = I$, the observability Gramian L_o has distinct eigenvalues and rank $\begin{bmatrix} \lambda_{max} I - L_o \\ L_o A^T - A^T L_o \end{bmatrix} = n$. Then system (5.1) is controllable only if $(L_o, L_o A^T - A^T L_o)$ is observable.

Proof: *The theorem is proved by contradiction.* Suppose $B_1 = B_1^*$ and $B_2 = 0$. If $(L_o, L_o A^T - A^T L_o)$ is unobservable, then for some $\bar{\lambda}$, there exists $x \neq 0$ such that

$$\begin{bmatrix} \bar{\lambda} I - L_o \\ L_o A^T - A^T L_0 \end{bmatrix} x = 0 ,$$

i.e.

$$L_o A^T x = \bar{\lambda} A^T x , \quad \bar{\lambda} \neq \lambda_{\max} .$$

Since $\bar{\lambda}$ is a simple eigenvalue of L_o with a unique eigenvector x, this implies that $A^T x$ must be collinear with x, i.e. there must exist ρ such that $A^T x = \rho x$. This shows that x^T is a left eigenvector of A. Since both, x and $B_1{}^*$, are eigenvectors of L_o,

$$\begin{aligned} x^T B_1^* &= 0 , \\ x^T A &= \rho x^T . \end{aligned}$$

Therefore, according to the PBH test, system (5.1) with $B_1 = B_1{}^*$ and $B_2 = 0$ is uncontrollable [95]. **Q.E.D.**

5.5.3.3 Coordinate Independence

Theorem 5.1 may yield optimal B_1 and B_2 which result in coordinate dependent zeros. The following statement gives condition on M which ensures coordinate independence.

Theorem 5.5: Assume that A in (5.1) is Hurwitz and M in (5.2) is positive definite and symmetric. Let S be a similarity transformation and $\hat{M}$ be the representation of M in the new basis. Then system (5.1) with $B_1 = B_1^*$ and $B_2 = B_2^*$ yields zeros which are independent of the coordinate transformation S if $\hat{M} = S^T M S$.

Proof: From Theorem 5.1, B_1^* and B_2^* satisfy

$$\begin{aligned} M^{-1} L_o B_1^* &= \lambda_{\max} B_1^*, \\ M^{-1} L_o B_2^* &= \lambda_{\min} B_2^*. \end{aligned}$$

Under the similarity transformation S, system (5.1) has the following representation:

$$\hat{A} = S^{-1}AS, \hat{B}_1 = S^{-1}B_1, \hat{B}_2 = S^{-1}B_2, \hat{C} = CS \text{ and } \hat{L}_o = S^T L_o S \ .$$

The optimal $\hat{B}_1^*$ and $\hat{B}_2^*$ are given by $\hat{M}^{-1}\hat{L}_o\hat{B}_1^* = \hat{\lambda}_{\max}\hat{B}_1^*$ and $\hat{M}^{-1}\hat{L}_o\hat{B}_2^* = \hat{\lambda}_{\min}\hat{B}_2^*$ respectively, i.e.,

$$\hat{M}^{-1}S^T L_o S\hat{B}_1^* = \hat{\lambda}_{\max}\hat{B}_1^* \text{ and } \hat{M}^{-1}S^T L_o S\hat{B}_2^* = \hat{\lambda}_{\min}\hat{B}_2^*.$$

Hence, if $\hat{M} = S^T MS$,

$$\hat{B}_1^* = S^{-1}B_1^* \text{ and } \hat{B}_2^* = S^{-1}B_2^* \ .$$

Since optimal $\hat{B}_1^*$ and $\hat{B}_2^*$ are related to B_1^* and B_2^*, respectively, by the transformation S, the zeros defined by B_1^* and B_2^* are independent of the coordinate transformation S. **Q.E.D.**

If $M = C^TC + A^TC^TCA + \ldots + (A^{n-1})^TC^TCA^{n-1}$, then $\hat{M} = S^T(C^TC + A^TC^TCA + \ldots + (A^{n-1})^TC^TCA^{n-1})S = S^TMS$. Therefore this normalization indeed results in coordinate independent zeros. If, however, $M = I$, the resulting H_2-optimal zeros may be coordinate dependent. Indeed, consider the system (5.1) with $M = I$ and A, C as follows:

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, \ C = [1 \ 0] \ .$$

In this case, B_1^* is given by

$$B_1^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that the zero of the system is at 0. Introducing the similarity transformation

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix},$$

results in

$$\hat{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \ \hat{C} = [0 \ 1] \ .$$

This gives

$$\hat{B}_1^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and the zero lies at -3.277. Hence, the normalization $M = I$ may lead to coordinate dependent zeros.

5.5.3.4 Non-minimum Phase Properties

It should be pointed out that the transfer functions $G_u(s)$ and $G_w(s)$ associated with the optimal vectors B_1^* and B_2^* may not be minimum phase.

Theorem 5.6: Assume $M = I$ and the maximum and minimum eigenvalues of the observability Gramian L_o are different. Suppose (5.1) is in the observer canonical form with $B_1 = B_1^*$ and $B_2 = B_2^*$. Then at least one of the transfer functions, $G_u(s)$ or $G_w(s)$, is non-minimum phase.

Proof: For system (5.1) in the observer canonical form, the elements of vectors B_1^* and B_2^* correspond to the numerator coefficients of the transfer functions $G_u(s)$ and $G_w(s)$ respectively. Thus, $G_u(s)$ and $G_w(s)$ are minimum phase only if all the elements of B_1^* and B_2^* are of the same sign. We show below that this is, in fact, impossible. Indeed, B_1^* and B_2^* are collinear with two eigenvectors of the observability Gramian L_o. As a symmetric matrix, L_o cannot have two eigenvectors with all components of the same sign. To prove this, assume that x and y are two eigenvectors of L_o, associated with two eigenvalues, λ and μ ($\mu \neq \lambda$) respectively. Assume that x has all components of the same sign. Then, since x and y are orthogonal, i.e. $\sum_{i=1}^n x_i y_i = 0$, not all components of y are of the same sign. Thus, B_1^* and B_2^* cannot have all components of the same sign. **Q.E.D.**

5.5.4 H_2-OPTIMAL ZEROS IN CLOSED LOOP ENVIRONMENT

5.5.4.1 Problem Formulation

In the closed loop environment, a number of problems concerning H_2-optimal zeros can be formulated. The following three are of importance in various control-theoretic situations:

5.5.4.1.1 H_2-optimal zeros in the disturbance rejection problem

Here the system (5.1) is considered in a closed loop form, for instance, with $u = Kx$. Define $G(s) = C(sI - A - B_1K)B_2$ and we want to choose B_2 such that $||G||_2$ is minimized subject to $||B_2|| = 1$ (or $B_2^T M B_2 = 1$, $M = M^T, M > 0$). Obviously, if $(A + B_1K)$ is Hurwitz, the answer to this question follows from Section 5.5.2 above: The optimal B_2 is collinear with the eigenvector of the closed loop observability Gramian, L_o^{cl}, corresponding to its smallest eigenvalue. Here L_o^{cl} is defined by

$$(A + B_1K)^T L_o^{cl} + L_o^{cl}(A + B_1K) + C^T C = 0 .$$

Due to the separation property, the same result holds not only for state feedback, but also for observer-based output feedback.

5.5.4.1.2 H_2-optimal zeros in the regulator problem

Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu , \\ y &= Cx , \quad x \in I\!R^n, u, y \in I\!R \\ E\{x_0\} &= 0, \ E\{x_0 x_0^T\} = Q , \end{aligned} \tag{5.10}$$

where $x_o = x(0)$. Assume that

$$u = Kx , \tag{5.11}$$

where K is chosen so that

$$J = E\left\{\int_0^\infty \left(x^T C_1^T C_1 x + u^2\right) dt\right\} , \tag{5.12}$$

is minimized. If (A, B) is stabilizable, (C_1, A) is detectable, it is well known that the optimal K is given by

$$K = -B^T P , \tag{5.13}$$

where P is the positive semi-definite solution of

$$A^T P + PA - PBB^T P + C_1^T C_1 = 0 . \tag{5.14}$$

The optimal value attained by functional (5.12) with feedback (5.13) is :

$$J^* = Tr(PQ) . \tag{5.15}$$

Obviously, the value J^* may be further improved by minimizing J^* over all admissible vectors B. To cast this problem in H_2-norm minimization form, we define $Q = B_1 B_1^T$ and rewrite (5.14) as

$$(A - BB^T P/2)^T P + P(A - BB^T P/2) + C_1^T C_1 = 0 . \tag{5.16}$$

Hence

$$J^* = Tr(PB_1B_1^T) = \mathrm{Tr}(B_1 P B_1^T) , \tag{5.17}$$

and from (5.16) and (5.17), the functional J^* is equivalent to the square of the H_2-norm of the following transfer function

$$G_J(s) = C_1(sI - A - BK/2)^{-1} B_1 . \tag{5.18}$$

It should be noted that the matrix $(A + BK/2)$ is Hurwitz due to the fact that LQ designs have a guaranteed gain margin from 1/2 to $+\infty$ [96]. Also, for a given A, B_1 and C_1, the choice of B, which determines K, will generally affect both the zeros and poles of the transfer function $G_J(s)$ (see Example 5.2).

Thus, the problem of H_2-optimal zero placement regulator problem can be stated as follows: Given (5.10), (5.11), (5.13) and (5.14), find vector B satisfying the constraint $||B|| = 1$ (or $B^TMB = 1$, $M = M^T, M > 0$) such that $||G_J||_2$ *is minimized.*

A necessary condition for the solution of this problem is given in Section 5.5.4.2.

5.5.4.1.3 **H_2-optimal zeros in the servomechanism problem**

Consider again system (5.10) and assume that

$$u = Kx + v \ , \tag{5.19}$$

where v is an exogenous signal. The control gain K is assumed to be chosen according to the designer's favorite methodology, such as pole placement, LQG, H_∞ design, or any other. The problem of H_2-optimal zero placement in the servomechanism problem is formulated as follows:

Given (5.10) and a design methodology for K, find B satisfying $||B|| = 1$ (or $B^TMB = 1$, $M = M^T, M > 0$) which maximizes $||G_v||_2$ where

$$G_v(s) = C(sI - A - BK)^{-1}B \ . \tag{5.20}$$

A solution to this problem is given below under the assumption that K is designed using the LQR approach, i.e.,

$$K = -B^TP \ ,$$

$$A^TP + PA - PBB^TP + C_1^TC_1 = 0 \ .$$

Remark 5.5: As it follows from (5.18) and (5.20), the main difference between the open and closed loop H_2-optimal zero placement is in the fact that in the latter case, vector B affects not only the zeros but also the poles of the transfer function.

Remark 5.6: The problems formulated above can be extended to dynamic output feedback as well. Indeed, assume that we use a controller

$$u = K\hat{x} + v \ , \tag{5.21}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \ . \tag{5.22}$$

Then, it is well known that the closed loop transfer function from v to y is also given by (5.20).

5.5.4.2 Main Result

5.5.4.2.1 H_2-optimal zeros in the regulator problem

Theorem 5.7: Consider the closed loop system (5.10), (5.11), (5.13) and (5.14). Assume that (A, B, C_1) is stabilizable and detectable. Then $\min_{||B||=1} ||G_J||_2$ is attained at vector B only if it is collinear with an eigenvector of matrix $S = PGP$, i.e.,

$$(PGP - \lambda I)B = 0 \ , \tag{5.23}$$

where $P, G \in \mathbb{R}^{n\times n}$, are defined by

$$(A - BB^TP)^TP + P(A - BB^TP) + PBB^TP + C_1^TC_1 = 0 \ , \tag{5.24}$$

$$(A - BB^TP)G + G(A - BB^TP)^T + B_1B_1^T = 0 \ . \tag{5.25}$$

In addition,

$$||G_J||_2^2 = \lambda + C_1GC_1^T \ .$$

Proof: The Lagrangian for the problem is

$$L(B,P,G,\gamma,\lambda) = \gamma B_1^T P B_1 + Tr(G[A^TP + PA - PBB^TP + C_1^TC_1]) + \lambda(B^TB - 1),$$

where γ, $\lambda \in I\!R$ and $G \in I\!R^{n\times n}, G = G^T$, are the Lagrange multipliers which are not simultaneously zero. The first order necessary conditions for an extremum are as follows:

$$\begin{aligned}\frac{\partial L}{\partial P} &= \gamma B_1 B_1^T + AG + GA^T - GPBB^T - BB^TPG = 0 \ ,\\ \frac{\partial L}{\partial B} &= -2PGPB + 2\lambda B = 0 \ .\end{aligned}$$

Since $A - BB^TP$ is stable with (C_1, A) detectable, $\gamma = 0$ implies $\lambda = 0$, $G = 0$ in the above equations. Hence, it can be assumed without loss of generality that $\gamma = 1$. Furthermore, G is nonnegative definite.

Equations (5.23) and (5.25) are thus obtained and (5.24) is equivalent to (5.14). From (5.25), we have

$$\begin{aligned}Tr(PB_1B_1^T) &= -Tr(P[A - BB^TP]G) - Tr(PG[A - BB^TP]^T)\\ &= -2Tr(P[A - BB^TP]G)\\ &= -2Tr(PAG) + 2Tr(PBB^TPG)\\ &= -2Tr(PAG) + 2B^TPGPB \ .\end{aligned}$$

From (5.23), (5.24) and knowing $B^TB = 1$, we have

$$\begin{aligned}||G_J||_2^2 &= Tr(PB_1B_1^T)\\ &= -2Tr(PAG) + 2\lambda\\ &= -\lambda + C_1GC_1^T + 2\lambda\\ &= \lambda + C_1GC_1^T \ .\end{aligned}$$

Q.E.D.

5.5.4.2.2 H_2-optimal zeros in the servomechanism problem

Theorem 5.8: Consider the closed loop system (5.10), (5.19), (5.13) and (5.14). Assume that (A, B, C_1) is stabilizable and detectable. Then $\max_{||B||=1} ||G_v||_2$ is attained at vector B only if it is collinear with an eigenvector of matrix $S = L_o^{cl} - PGP - 2PL_c^{cl}L_o^{cl}$, i.e.,

$$(L_o^{cl} - PGP - 2PL_c^{cl}L_o^{cl} - \lambda I)B = 0 \ , \tag{5.26}$$

where P, L_o^{cl}, G and $L_c^{cl} \in I\!R^{n\times n}$ are defined by the following equations:

$$\begin{aligned}(A - BB^TP)^TP + P(A - BB^TP) + PBB^TP + C_1^TC_1 &= 0, && (5.27)\\ (A - BB^TP)^TL_o^{cl} + L_o^{cl}(A - BB^TP) + C^TC &= 0, && (5.28)\\ (A - BB^TP)G + G(A - BB^TP)^T - L_c^{cl}L_o^{cl}BB^T - BB^TL_o^{cl}L_c^{cl} &= 0, && (5.29)\\ (A - BB^TP)L_c^{cl} + L_c^{cl}(A - BB^TP)^T + BB^T &= 0. && (5.30)\end{aligned}$$

In addition,

$$\max_{||B||=1} ||G_v||_2^2 = CL_c^{cl}C^T \ .$$

Proof: By forming the Lagrangian.

$$
\begin{aligned}
L(B, L_o^{cl}, P, G, L_c^{cl}, \gamma, \lambda) &= \gamma B^T L_o^{cl} B + Tr(G[A^T P + PA - PBB^T P + C_1^T C_1] + \\
&\quad L_c^{cl}[(A - BB^T P)^T L_o^{cl} + L_o^{cl}(A - BB^T P) + C^T C]) - \\
&\quad \lambda(B^T B^{-1}),
\end{aligned}
$$

where $\gamma, \lambda \in I\!R$ and $G = G^T, L_c^{cl} = (L_c^{cl})^T$ are the Lagrange multipliers which are not simultaneously zero. The closed loop observability Gramian L_o^{cl} is defined by (5.28). The following conditions are necessary for an extremum:

$$
\frac{\partial L}{\partial P} = AG + GA^T - GPBB^T - BB^T PG - L_c^{cl} L_o^{cl} BB^T - BB^T L_o^{cl} L_c^{cl} = 0 \,, \tag{5.31}
$$

$$
\frac{\partial L}{\partial L_o^{cl}} = (A - BB^T P) L_c^{cl} + L_c^{cl}(A - BB^T P)^T + \gamma BB^T = 0 \,, \tag{5.32}
$$

$$
\frac{\partial L}{\partial B} = 2\gamma L_o^{cl} B - 2PGPB - 4PL_c^{cl} L_o^{cl} B - 2\lambda B = 0 \,. \tag{5.33}
$$

Since $A - BB^T P$ is stable with (C_1, A) detectable, $\gamma = 0$ implies $L_c^{cl} = G = 0$ and $\lambda = 0$ in (5.32), (5.31) and (5.33) respectively. Hence, it can be assumed without loss of generality that $\gamma = 1$. Furthermore, L_c^{cl} is nonnegative definite.

Equations (5.26), (5.29)-(5.30) are thus obtained and (5.27) is equivalent to (5.14). From (5.28) and (5.30) , we have

$$
\begin{aligned}
Tr(BB^T L_o^{cl}) &= -2Tr(AL_c^{cl} L_o^{cl}) + 2Tr(L_c^{cl} PBB^T L_o^{cl}) \\
&= Tr(L_c^{cl} C^T C) = CL_c^{cl} C^T.
\end{aligned}
$$

Q.E.D.

Remark 5.7: Since equations (5.23)-(5.25) and (5.26)-(5.30) are all coupled, they are formidable, indeed. Their relative utility may be justified by the following two arguments. First, these equations can be used to verify whether vector B optimal for an open loop system remains optimal in the closed loop environment. Section 5.5.5 below examines this relationship. Secondly, iterative procedures can be suggested for solving these equations (see below). Although the convergence conditions for these procedures are presently unknown, it is easy to show that they always generate sequences which contain convergent subsequences. Indeed, since the optimization is performed over the compact set $\{B : B^T B = 1\}$, any sequence contains at least one convergent subsequence [97].

The following iterative procedure was tested on several examples for solving equations (5.26)-(5.30):

1) Choose a vector $B(n)$, for $n = 0$ such that $B^T(n)B(n) = 1$ (e.g. $B(n) = [1\ 0\ 0\ \dots\ 0]^T$).

2) Solve the algebraic Ricatti equation (5.27) for $n = 0$ giving $P(n)$.

3) Solve the Lyapunov equations (5.28) and (5.30) for $n = 0$ giving $L_o^{cl}(n)$ and $L_c^{cl}(n)$.

4) Solve the Lyapunov equation (5.29) for $n = 0$ giving $G(n)$.

5) Solve the eigenvalue equation (5.26) for $n = 1$ such that $\lambda(n)$ is the largest real eigenvalue of $L_o^{cl} - PGP - 2PL_c^{cl}L_o^{cl}$ and $B(n)$ is the corresponding eigenvector.

6) Iterate steps (2), (3), (4), and (5) for $n = 1, 2, 3, \ldots$, giving sequences $\{P(n)\}$, $\{L_o^{cl}(n)\}$, $\{L_c^{cl}(n)\}, \{G(n)\}, \{\lambda(n)\}$, and $\{B(n)\}$.

A similar iterative procedure for solving equations (5.23)-(5.25) is obtained by making the appropriate modifications on the above algorithm.

5.5.5 RELATIONSHIP BETWEEN THE OPEN LOOP AND CLOSED LOOP H_2-OPTIMAL ZEROS

The H_2-optimal input vector B calculated according to Theorem 5 .1 for the open loop environment may or may not be optimal in the closed loop case pertaining to Theorem 5.7. Below we illustrate this assertion by two examples and formulate a conjecture as to when one or another situation may take place.

Example 5.1: Consider the system (5.10) with A and C as follows:

$$A = \begin{bmatrix} 0 & -10000 \\ 1 & -200 \end{bmatrix}, \quad C = [0\ 1] . \tag{5.34}$$

In this case, the open loop observability Gramian is

$$L_o = \begin{bmatrix} 2.5E-07 & 0 \\ 0 & 2.5E-03 \end{bmatrix},$$

and, therefore, vector B_{ol}^* that achieves $\max_{||B||=1} ||G||_2$, where $G(s) = C(sI - A)^{-1}B$, is given by

$$B_{ol}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Next, substituting (5.34) in equations (5.23)-(5.25), we calculate the vector B optimal in the regulator problem. Let $C_1 = C, B_1 = [1\ 1]^T$, and $M = I$. Invoking the iterative procedure described in Section 5.5.4.2, we obtain

$$G = \begin{bmatrix} 25.0122 & 0 \\ 0 & 0.0025 \end{bmatrix}, P = \begin{bmatrix} 2.5E-07 & 0 \\ 0 & 2.5E-03 \end{bmatrix},$$

and, finally,

$$B_{cl}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which was verified by performing an exhaustive search over the set $\{B : ||B|| = 1\}$. Thus, in this particular situation, the optimal open loop B_{ol}^* and the optimal closed loop B_{cl}^* are the same.

Example 5.2: Consider again (5.10) with A and C given by

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad C = [0\ 1], \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and $C_1 = C, M = I$. The open loop optimal B^*_{ol} for this case turns out to be

$$B^*_{ol} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

whereas the regulator problem results in

$$\begin{aligned} B^*_{cl} &= \begin{bmatrix} 0.4384 \\ 0.8988 \end{bmatrix} , \\ K &= [0.0347 \; 0.1465] . \end{aligned}$$

Obviously, the two vectors B^*_{ol} and B^*_{cl} differ significantly. Note also that the zero of the triple (A, B_1, C_1) is at -1 while that of the triple $(A + B^*_{cl}K/2, B_1, C_1)$ lies at -0.984.

Based on our experience in computing the optimal input vector B in open loop (Theorem 5.1) and closed loop (Theorem 5.7), we are led to the following conjecture.

Conjecture 5.1: When the LQR methodology is used to compute the feedback gain K in the closed loop case, optimal vectors B^*_{ol} and B^*_{cl} are close to each other for system with "fast" open-loop poles.

The reason behind such a conjecture is that for systems possessing open loop poles in the far left half complex plane, the location of the closed loop poles will not be significantly different from that of the open loop poles with a LQR controller. Hence optimal vector B^*_{ol} obtained from the open loop system dynamics will be close to that of B^*_{cl}.

However, in general, in a closed loop system with a specified feedback controller, for maximum H_2-norm transmission from an exogeneous input to the output y, the choice of the optimal input vector must take into consideration the dynamics of the closed-loop system and not the open-loop dynamics. To be more precise, the choice of the input vector B and the controller design should be performed simultaneously as presented in Theorems 5.7 and 5.8.

6 LIST OF PUBLICATIONS AND TECHNICAL REPORTS

[1] P.T. Kabamba, S.M. Meerkov, and E.-K. Poh, "H_2−Optimal Zeros," *Proceedings of the 30th Conf. on Decision and Control*, Brighton, U.K., pp. 2289-2290, Dec. 1991. An extended version of this paper has been accepted for publication in *IEEE Transactions on Automatic Control*. The full paper is in *Univ. of Michigan Contr. Grp Report*, CGR-92-16, Aug. 1992.

[2] P.T. Kabamba, S.M. Meerkov, and E.-K. Poh, "Closed Loop Vibrational Control:State and Output Feedback Stabilizability," *Proceedings of the 30th Conf. on Decision and Control*, Brighton, U.K., pp. 2375-2376, Dec. 1991.

[3] P.T. Kabamba, S.M. Meerkov, and E.-K. Poh, "Closed Loop Vibrational Control:State and Output Feedback Stabilizability and Pole Placement Capabilities," *Univ. of Michigan Contr. Grp Report*, CGR-92-17, Aug. 1992. An extended version of this paper was submitted for publication in *IEEE Transactions on Automatic Control*.

[4] P.T. Kabamba, S.M. Meerkov, and E.-K. Poh, "VHHC in Helicopters," *Proc. American Control Conference*, San Francisco, CA, June 1993. An extended version of this paper was submitted for publication in *Journal of Guidance, Control and Dynamics*. The full paper is in *Univ. of Michigan Contr. Grp Report*, CGR-93-11, July 1993.

[5] P.T. Kabamba, S.M. Meerkov, and E.-K. Poh, "Stability Robustness in Closed Loop Vibrational Control," *Univ. of Michigan Contr. Grp Report*, CGR-93-4, Feb. 1993. An extended abstract of this report was submitted for publication in *IEEE Transactions on Automatic Control* and the 32nd Conf. on Decision and Control, San Antonio, TX, Dec. 1993.

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7 LIST OF PARTICIPATING SCIENTIFIC PERSONNEL

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9 APPENDICES

APPENDIX A1: LEMMAS A.1-A.5

Lemma A.1: Let

$$P(\lambda) = \sum_{\substack{i,j=1 \\ i<j}}^{m} \lambda_i \lambda_j \ ,$$

where $\lambda = \{\lambda_1, \ldots, \lambda_m\}$ is a set of self conjugate points in $D(\sigma, \omega)$. The function P achieves its global maximum subject to the constraint

$$\sum_{i=1}^{m} \lambda_i = a \ ,$$

at $\lambda_i^* \in \partial D, i = 1, \ldots, m$, where ∂D denotes the boundary of $D(\sigma, \omega)$.

Proof: Consider a set of self conjugate points $\lambda = \{\lambda_1, \ldots, \lambda_m\}$. Then

$$\sum_{\substack{i,j=1 \\ i<j}}^{m} \lambda_i \lambda_j = \sum_{i=1}^{m/2} (\text{Re } \lambda_i)^2 + (\text{Imag } \lambda_i)^2 + 4 \sum_{\substack{i,j=1 \\ i<j}}^{m/2} (\text{Re } \lambda_i)(\text{Re } \lambda_j) \ , \tag{A.1}$$

where

$$|\text{Imag } \lambda_i| \leq \left| \frac{\omega}{\sigma} \text{Re } \lambda_i \right| \ . \tag{A.2}$$

From (A.1) and (A.2), it follows that $\max_{\lambda \in D(\sigma,\omega)} P(\lambda)$ is achieved at

$$|\text{Imag } \lambda_i| = \left| \frac{\omega}{\sigma} \text{Re } \lambda_i \right| \ , \qquad i = 1, \ldots, m \ ,$$

i.e. when $\lambda_i^* \in \partial D, i = 1, \ldots, m$. **Q.E.D.**

Consider the optimization problem: For m even, and $\sigma^2 - \omega^2 > 0$, find a self conjugate set of eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_m\} \in D(\sigma, \omega)$ such that

$$P(\lambda) = \sum_{\substack{i,j=1 \\ i<j}}^{m} \lambda_i \lambda_j \ , \tag{A.3}$$

is maximized subject to

$$\sum_{i=1}^{m} \lambda_i = a < 0 \ . \tag{A.4}$$

Lemma A.2: The global solution $\lambda^* = \{\lambda_1^*, \ldots, \lambda_m^*\}$ of problem (A.3), (A.4) is given by

$$\lambda_i^* = \frac{a}{m\sigma}(\sigma + j\omega) \ , \qquad i = 1, \ldots, m/2 \ , \tag{A.5}$$

$$\lambda_{m/2+i}^* = \frac{a}{m\sigma}(\sigma - j\omega) \ , \qquad i = 1, \ldots, m/2 \ , \tag{A.6}$$

and

$$P(\lambda^*) = \frac{a^2}{2m\sigma^2}\left((m-1)\sigma^2 + \omega^2\right) \ . \tag{A.7}$$

Proof: As a consequence of Lemma A.1, the global solution λ of (A.3), subject to the constraint (A.4), must have the form

$$\begin{aligned} \lambda_i &= r_i(\sigma + j\omega) \ , \qquad r_i \geq 1 \ , \qquad i = 1, \ldots, m/2 \ , \\ \lambda_{m/2+i} &= r_i(\sigma - j\omega) \ , \qquad\qquad\qquad i = 1, \ldots, m/2 \ . \end{aligned}$$

Therefore,

$$\begin{aligned} P(\lambda) &= \sum_{\substack{i,j=1 \\ i<j}}^{m} \lambda_i \lambda_j \ , \\ &= 2\left(\sigma^2 - \omega^2\right) \sum_{\substack{i,j=1 \\ i<j}}^{m/2} r_i r_j + \left(\sigma^2 + \omega^2\right)\left(\frac{a}{2\sigma}\right)^2 \ , \end{aligned} \tag{A.8}$$

and the constraint (A.4) reduces to

$$\sum_{i=1}^{m/2} r_i = \frac{a}{2\sigma} \ . \tag{A.9}$$

Since $\sigma^2 - \omega^2 > 0, \max P(\lambda)$ is achieved at r which maximizes $\sum_{\substack{i,j=1 \\ i<j}}^{m/2} r_i r_j$.

Define the set $r = \{r_1, \ldots, r_{m/2}\}$. Since the constraint (A.9) is always regular, we apply the Lagrange multiplier rule. The Lagrangrian is

$$L(r,\beta) = \sum_{\substack{i,j=1 \\ i<j}}^{m/2} r_i r_j - \beta\left(\sum_{i=1}^{m/2} r_i - \frac{a}{2\sigma}\right)$$

The first order necessary conditions

$$\begin{aligned} \frac{\partial L(r,\beta)}{\partial r} &= 0 \ , \\ \frac{\partial L(r,\beta)}{\partial \beta} &= 0 \ , \end{aligned}$$

yield a linear system of equations whose solution is

$$r_1^* = \ldots = r_{m/2}^* = \frac{a}{m\sigma} \ ,$$

together with

$$\beta = \frac{2-m}{2\sigma m} a \ .$$

The second order conditions ensure that (A.5)-(A.6) is a strict maximum. Moreover, since under the constraint (A.9) the cost function (A.8) is quadratic, this maximum is the global maximum, and the proof is complete. **Q.E.D.**

Consider the optimization problem : For m even, and $\sigma^2 - \omega^2 < 0$, find a self conjugate set of eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_m\} \in D(\sigma, \omega)$ such that

$$P(\lambda) = \sum_{\substack{i,j=1 \\ i<j}}^{m} \lambda_i \lambda_j \ , \tag{A.10}$$

is maximized subject to

$$\sum_{i=1}^{m} \lambda_i = a \ . \tag{A.11}$$

Lemma A.3: The global solution $\lambda^* = \{\lambda_1^*, \ldots, \lambda_m^*\}$ of problem (A.10), (A.11) is given by

$$\lambda_i^* = r_i^*(\sigma + j\omega) \ , \qquad i = 1, \ldots, m/2 \ , \tag{A.12}$$

$$\lambda_{m/2+i}^* = r_i^*(\sigma - j\omega) \ , \qquad i = 1, \ldots, m/2 \ , \tag{A.13}$$

where

$$r_1^* = \frac{a}{2\sigma} - \left(\frac{m}{2} - 1\right) \quad \text{and} \quad r_i^* = 1 \ , \qquad i = 2, \ldots, m/2 \ . \tag{A.14}$$

In addition,

$$P(\lambda^*) = (\omega^2 - \sigma^2)\left(\frac{m}{2} - 1\right)\left(\frac{m}{2} - \frac{a}{\sigma}\right) + (\sigma^2 + \omega^2)\left(\frac{a}{2\sigma}\right)^2 \ . \tag{A.15}$$

Proof: As a consequence of Lemma A.1, the global solution λ of (A.10), subject to the constraint (A.11), must have the form

$$\lambda_i = r_i(\sigma + j\omega) \ , \qquad r_i \geq 1 \ , \qquad i = 1, \ldots, m/2 \ ,$$

$$\lambda_{m/2+i} = r_i(\sigma - j\omega) \ , \qquad i = 1, \ldots, m/2 \ .$$

Since $\sigma^2 - \omega^2 < 0$, and from (A.8), $\max P(\lambda)$ is achieved at r which minimizes $\sum_{\substack{i,j=1 \\ i<j}}^{m/2} r_i r_j$. Hence, we need to determine the set $r = \{r_1 \cdots r_{m/2}\}$ which minimizes

$$P(r) = \sum_{\substack{i,j=1 \\ i<j}}^{m/2} r_i r_j \ ,$$

subject to the constraints

$$r_i \geq 1 \ , \qquad i = 1, \ldots, m/2 \ , \tag{A.16}$$

$$\sum_{i=1}^{m/2} r_i = \frac{a}{2\sigma} \geq \frac{m}{2} \ . \tag{A.17}$$

Since the constraints (A.16), (A.17) are regular, we apply the Karush-Kuhn-Tucker rule. The Lagrangian is

$$L(r, \mu, \beta) = \sum_{\substack{i,j=1 \\ i<j}}^{m/2} r_i r_j - \sum_{i=1}^{m/2} \mu_i (r_i - 1) - \beta \left(\sum_{i=1}^{m/2} r_i - \frac{a}{2\sigma}\right) \ .$$

where the non-negative Lagrange multiplier $\mu = [\mu_1 \cdots \mu_{m/2}]$ satisfies $\mu_i(r_i - 1) = 0$, for all $i = 1, \ldots, m/2$. The first order necessary conditions yield a linear system of equations whose solution generates a set of candidate extremal points given by

$$\begin{aligned} r_i &= \frac{\frac{a}{2\sigma} - k}{\frac{m}{2} - k}, \qquad \mu_i = 1, \qquad i \notin A(r), \\ r_i &= 1, \qquad \mu_i = 0, \qquad i \in A(r), \\ \beta &= \frac{a}{2\sigma} - \frac{\frac{a}{2\sigma} - k}{\frac{m}{2} - k}, \end{aligned} \tag{A.18}$$

where $A(r) = \{r : r = 1\}$ is the set of active constraints and $k < m$ denotes the number of active constraints. It can be shown that

$$P(k) - P(k-1) = -\frac{(\frac{a}{2\sigma} - \frac{m}{2})^2}{2(\frac{m}{2} - k)(\frac{m}{2} + 1 - k)} < 0 .$$

Hence, among the candidate extremal points given by (A.18), the solution $k = m/2 - 1$ yields the global minimum

$$\begin{aligned} r_1^* &= \frac{a}{2\sigma} - (\frac{m}{2} - 1) \geq 1, \\ r_i^* &= 1, \qquad i = 2, \ldots, m/2, \end{aligned}$$

together with

$$\begin{aligned} \beta &= \frac{m}{2} - 1, \\ \mu_1 &= 0, \quad \mu_i = \frac{a}{2\sigma} - \frac{m}{2}, \qquad i = 2, \ldots, m/2 . \end{aligned}$$

The results (A.12)-(A.15) then follow immediately. **Q.E.D.**

Consider the optimization problem: For m even, and $\sigma^2 - \omega^2 > 0$, find a set of self conjugate eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_m\} \in D(\sigma, \omega)$ such that

$$P(\lambda) = \sum_{\substack{i,j=0 \\ i<j}}^{m} \lambda_i \lambda_j , \tag{A.19}$$

is maximized subject to

$$\lambda_0 = r_0 \sigma, \qquad r_0 \geq 1, \tag{A.20}$$

$$\sum_{i=0}^{m} \lambda_i = a . \tag{A.21}$$

Lemma A.4: The global solution $\lambda^* = \{\lambda_1^*, \ldots, \lambda_m^*\}$ of problem (A.19)-(A.21) is given by

$$\begin{aligned} \lambda_i^* &= \frac{a - r_0\sigma}{m\sigma}(\sigma + j\omega), \qquad i = 1, \ldots, m/2, \\ \lambda_{m/2+i}^* &= \frac{a - r_0\sigma}{m\sigma}(\sigma - j\omega), \qquad i = 1, \ldots, m/2, \end{aligned}$$

and

$$P(\lambda^*) = r_0\sigma(a - r_0\sigma) + \frac{(a - r_0\sigma)^2}{2m\sigma^2}\left((m-1)\sigma^2 + \omega^2\right) .$$

Proof: From (A.20), the expression (A.19) can be expressed as:

$$\begin{aligned} P(\lambda) &= \sum_{\substack{i,j=0 \\ i<j}}^{m} \lambda_i\lambda_j , \\ &= r_0\sigma(a - r_0\sigma) + \sum_{\substack{i,j=1 \\ i<j}}^{m} \lambda_i\lambda_j . \end{aligned} \tag{A.22}$$

Also the constraints (A.20) and (A.21) are equivalent to the following single constraint:

$$\sum_{i=1}^{m} \lambda_i = a - r_0\sigma . \tag{A.23}$$

Since the first term in the right hand side of (A.22) is constant, the solution of the optimization problem (A.22), (A.23) is obtained from Lemma A.2. **Q.E.D.**

Consider the optimization problem : For m even, and $\sigma^2 - \omega^2 < 0$, find a set of self conjugate eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_m\} \in D(\sigma, \omega)$ such that

$$P(\lambda) = \sum_{\substack{i,j=0 \\ i<j}}^{m} \lambda_i\lambda_j , \tag{A.24}$$

is maximized subject to

$$\lambda_0 = r_0\sigma , \qquad r_0 \geq 1 , \tag{A.25}$$

$$\sum_{i=0}^{m} \lambda_i = a . \tag{A.26}$$

Lemma A.5: The global solution $\lambda^* = \{\lambda_1^*, \ldots, \lambda_m^*\}$ is given by

$$\lambda_i^* = r_i^*(\sigma + j\omega) , \qquad i = 1, \ldots, m/2 , \tag{A.27}$$

$$\lambda_{m/2+i}^* = r_i^*(\sigma - j\omega) , \qquad i = 1, \ldots, m/2 , \tag{A.28}$$

where

$$r_1^* = \frac{a - r_0\sigma}{2\sigma} - \left(\frac{m}{2} - 1\right) \quad \text{and} \quad r_i^* = 1 , \quad i = 2, \ldots, m/2 . \tag{A.29}$$

In addition,

$$\begin{aligned} P(\lambda^*) &= r_0\sigma(a - r_0\sigma) + (\omega^2 - \sigma^2)\left(\frac{m-2}{2}\right)\left(\frac{m}{2} - \frac{a - r_0\sigma}{\sigma}\right) \\ &+ (\omega^2 + \sigma^2)\left(\frac{a - r_0\sigma}{2\sigma}\right)^2 . \end{aligned} \tag{A.30}$$

Proof: The results (A.27)-(A.30) follow immediately from (A.22), (A.23), and Lemma A.3. **Q.E.D.**

APPENDIX A2: DERIVATION OF AVERAGED EQUATIONS FOR HELICOPTER WITH VHHC

A2.1 AVERAGED EQUATIONS FOR HELICOPTER IN HC'.ER

Let $x_1 = x, x_2 = \theta, x_3 = \dot{x}, x_4 = \dot{\theta}$. Equations (4.21) and (4.22) are equivalent to :

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1+m_2-m_2\sin^2 x_2} & \frac{-\sin x_2}{\ell(m_1+m_2-m_2\sin^2 x_2)} \\ 0 & 0 & \frac{-\sin x_2}{\ell(m_1+m_2-m_2\sin^2 x_2)} & \frac{m_1+m_2}{m_2\ell^2(m_1+m_2-m_2\sin^2 x_2)} \end{bmatrix} \\
&\times \begin{bmatrix} x_3 \\ x_4 \\ -m_2\ell x_4^2\cos x_2 - kx_1 + \alpha_0\sin\omega t - \zeta x_3 \\ -m_2\ell g\sin x_2 + \beta\sin N\omega t - \eta x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha}{\epsilon(m_1+m_2-m_2\sin^2 x_2)}\sin\left(\frac{t}{\epsilon}\right) \\ \frac{-\alpha\sin x_2}{\epsilon\ell(m_1+m_2-m_2\sin^2 x_2)}\sin\left(\frac{t}{\epsilon}\right) \end{bmatrix} \\
&= X_1(t,x) + \frac{\alpha}{\epsilon}X_2\left(\frac{t}{\epsilon},x\right) .
\end{aligned}
$$

The generating equation [20] for this case is :

$$
\begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \\ \frac{dx_3}{d\tau} \\ \frac{dx_4}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha\sin\tau}{(m_1+m_2-m_2\sin^2 x_2)} \\ \frac{-\alpha\sin x_2\sin\tau}{\ell(m_1+m_2-m_2\sin^2 x_2)} \end{bmatrix} ,
$$

where $\tau = t/\epsilon$.

We obtain the general solution of the above equation in the fast time τ as

$$
\begin{aligned}
x_1 &= c_1 , \\
x_2 &= c_2 , \\
x_3 &= \frac{-\alpha\cos\tau}{m_1+m_2-m_2\sin^2 c_2} + c_3 , \\
x_4 &= \frac{\alpha\sin x_2\cos\tau}{m_1+m_2-m_2\sin^2 c_2} + c_4 ,
\end{aligned}
$$

where $c_i, i = 1, \ldots, 4$, are constants.

Conser :ntly, the substitution for this case is

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \frac{-\alpha\cos\tau}{m_1+m_2-m_2\sin^2 z_2} + z_3 \\ \frac{\alpha\sin z_2\cos\tau}{\ell(m_1+m_2-m_2\sin^2 z_2)} + z_4 \end{bmatrix} = h(\tau, z) ,
$$

and the equation in Bogoliubov's standard form can be written as [20] :

$$
\begin{aligned}
\frac{dz}{d\tau} &= \epsilon\left[\frac{\partial h}{\partial z}\right]^{-1} X_1(t, h(\tau, z)) \\
&= \epsilon\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2\alpha m_2\sin z_2\cos z_2\cos\tau}{(m_1+m_2-m_2\sin^2 z_2)^2} & 1 & 0 \\ & \frac{-2\alpha m_2\sin^2 z_2\cos z_2\cos\tau}{(m_1+m_2-m_2\sin^2 z_2)^2} - \frac{\alpha\cos z_2\cos\tau}{\ell(m_1+m_2-m_2\sin^2 z_2)} & 0 & 1 \end{bmatrix} \times
\end{aligned}
$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1+m_2-m_2\sin^2 z_2} & \frac{-\sin z_2}{\ell(m_1+m_2-m_2\sin^2 z_2)} \\ 0 & 0 & \frac{-\sin z_2}{\ell(m_1+m_2-m_2\sin^2 z_2)} & \frac{m_1+m_2}{m_2\ell^2(m_1+m_2-m_2\sin^2 z_2)} \end{bmatrix} \times$$

$$\begin{bmatrix} \frac{-\alpha\cos\tau}{m_1+m_2-m_2\sin^2 z_2} + z_3 \\ \frac{\alpha\sin z_2\cos\tau}{\ell(m_1+m_2-m_2\sin^2 z_2)} + z_4 \\ -m_2\ell\left(\frac{\alpha\sin z_2\cos\tau}{\ell(m_1+m_2-m_2\sin^2 z_2)} + z_4\right)^2\cos z_2 - kz_1 + \alpha_0\sin\omega t - \zeta\left(\frac{-\alpha\cos\tau}{m_1+m_2-m_2\sin^2 z_2} + z_3\right) \\ -m_2\ell g\sin z_2 + \beta\sin N\omega t - \eta\left(\frac{\alpha\sin z_2\cos\tau}{\ell(m_1+m_2-m_2\sin^2 z_2)} + z_4\right) \end{bmatrix}.$$

Therefore, applying the averaging principle , we obtain equations (4.23) and (4.24) where $\overline{z_1} = z$ and $\overline{z_2} = \phi$.

A2.2 AVERAGED EQUATIONS FOR HELICOPTER IN FORWARD FLIGHT

Let $x_1 = x, x_2 = y, x_3 = \theta, x_4 = \dot{x}, x_5 = \dot{y}, x_6 = \dot{\theta}$. Equations (4.4)-(4.6) are equivalent to :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{m_1+m_2-m_2\cos^2 x_3}{m_1(m_1+m_2)} & \frac{m_2\sin x_3\cos x_3}{m_1(m_1+m_2)} & \frac{-\sin x_3}{m_1\ell} \\ 0 & 0 & 0 & \frac{m_2\sin x_3\cos x_3}{m_1(m_1+m_2)} & \frac{m_1+m_2-m_2\sin^2 x_3}{m_1(m_1+m_2)} & \frac{-\cos x_3}{m_1\ell} \\ 0 & 0 & 0 & \frac{-\sin x_3}{m_1\ell} & \frac{-\cos x_3}{m_1\ell} & \frac{m_1+m_2}{m_1 m_2\ell^2} \end{bmatrix} \times$$

$$\begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ -m_2\ell x_6^2\cos x_3 - kx_1 + \alpha_0\cos\mu\sin\omega t - \zeta x_4 \\ m_2\ell x_6^2\sin x_3 + (m_1+m_2)g\tan\mu - kx_1\tan\mu + \alpha_0\sin\mu\sin\omega t - \eta x_5 \\ -m_2\ell g\sin x_3 + \beta\sin N\omega t - \eta x_6 \end{bmatrix} + \quad -$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{alpha}{\epsilon}\left(\frac{(m_1+m_2-m_2\cos^2 x_3)\cos\mu + m_2\sin x_3\cos x_3\sin\mu}{m_1(m_1+m_2)}\right)\sin\left(\frac{t}{\epsilon}\right) \\ \frac{alpha}{\epsilon}\left(\frac{(m_1+m_2-m_2\sin^2 x_3)\sin\mu + m_2\sin x_3\cos x_3\cos\mu}{m_1(m_1+m_2)}\right)\sin\left(\frac{t}{\epsilon}\right) \\ \frac{\alpha}{\epsilon}\left(\frac{-\sin x_3\cos\mu - \cos x_3\sin\mu}{m_1\ell}\right)\sin\left(\frac{t}{\epsilon}\right) \end{bmatrix}.$$

The generating equation for this case is :

$$\begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \\ \frac{dx_3}{d\tau} \\ \frac{dx_4}{d\tau} \\ \frac{dx_5}{d\tau} \\ \frac{dx_6}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha\left(\frac{(m_1+m_2-m_2\cos^2 x_3)\cos\mu + m_2\sin x_3\cos x_3\sin\mu}{m_1(m_1+m_2)}\right)\sin\tau \\ \alpha\left(\frac{(m_1+m_2-m_2\sin^2 x_3)\sin\mu + m_2\sin x_3\cos x_3\cos\mu}{m_1(m_1+m_2)}\right)\sin\tau \\ \alpha\frac{-\sin x_3\cos\mu - \cos x_3\sin\mu}{m_1\ell}\sin\tau \end{bmatrix},$$

where $\tau = t/\epsilon$.

The general solution of the above equation is :

$$\begin{aligned} x_1 &= c_1 \quad , \\ x_2 &= c_2 \quad , \\ x_3 &= c_3 \quad , \end{aligned}$$

$$x_4 = -\alpha\left(\frac{(m_1+m_2-m_2\cos^2 c_3)\cos\mu + m_2\sin c_3\cos c_3\sin\mu}{m_1(m_1+m_2)}\right)\cos\tau + c_4 .$$

$$x_5 = -\alpha\left(\frac{(m_1+m_2-m_2\sin^2 c_3)\sin\mu + m_2\sin c_3\cos c_3\cos\mu}{m_1(m_1+m_2)}\right)\cos\tau + c_5 .$$

$$x_6 = \alpha\left(\frac{\sin c_3\cos\mu + \cos c_3\sin\mu}{m_1\ell}\right)\cos\tau + c_6 .$$

where $c_i, i = 1,\ldots,6$, are constants.

The resulting substitution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ -\alpha\left(\frac{(m_1+m_2-m_2\cos^2 z_3)\cos\mu+m_2\sin z_3\cos z_3\sin\mu}{m_1(m_1+m_2)}\right)\cos\tau + z_4 \\ -\alpha\left(\frac{(m_1+m_2-m_2\sin^2 z_3)\sin\mu+m_2\sin z_3\cos z_3\cos\mu}{m_1(m_1+m_2)}\right)\cos\tau + z_5 \\ \alpha\left(\frac{\sin z_3\cos\mu+\cos z_3\sin\mu}{m_1\ell}\right)\cos\tau + z_6 \end{bmatrix},$$

and the equation in Bogoliubov's standard form is [20]

$$\frac{dz}{d\tau} = \epsilon\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha m_2\frac{(-\sin 2z_3\cos\mu-\cos 2z_3\sin\mu)}{m_1(m_1+m_2)}\cos\tau & 1 & 0 & 0 \\ 0 & 0 & \alpha m_2\frac{(\sin 2z_3\sin\mu-\cos 2z_3\cos\mu)}{m_1(m_1+m_2)}\cos\tau & 0 & 1 & 0 \\ 0 & 0 & \alpha\frac{(\cos z_3\cos\mu-\sin z_3\sin\mu)}{m_1\ell}\cos\tau & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{m_1+m_2-m_2\cos^2 z_3}{m_1(m_1+m_2)} & \frac{m_2\sin z_3\cos z_3}{m_1(m_1+m_2)} & \frac{-\sin z_3}{m_1\ell} \\ 0 & 0 & 0 & \frac{m_2\sin z_3\cos z_3}{m_1(m_1+m_2)} & \frac{m_1+m_2-m_2\sin^2 z_3}{m_1(m_1+m_2)} & \frac{-\cos z_3}{m_1\ell} \\ 0 & 0 & 0 & \frac{-\sin z_3}{m_1\ell} & \frac{-\cos z_3}{m_1\ell} & \frac{m_1+m_2}{m_1 m_2\ell^2} \end{bmatrix} \times$$

$$\begin{bmatrix} -\alpha\left(\frac{(m_1+m_2-m_2\cos^2 z_3)\cos\mu+m_2\sin z_3\cos z_3\sin\mu}{m_1(m_1+m_2)}\right)\cos\tau + z_4 \\ -\alpha\left(\frac{(m_1+m_2-m_2\sin^2 z_3)\sin\mu+m_2\sin z_3\cos z_3\cos\mu}{m_1(m_1+m_2)}\right)\cos\tau + z_5 \\ \alpha\left(\frac{\sin z_3\cos\mu+\cos z_3\sin\mu}{m_1\ell}\right)\cos\tau + z_6 \\ -m_2\ell\alpha^2\left(\frac{(\sin z_3\cos\mu+\cos z_3\sin\mu)\cos\tau}{m_1\ell} + z_6\right)^2\cos z_3 - kz_1 + \alpha_0\cos\mu\sin\omega t - \zeta t_1 \\ m_2\ell\alpha^2\left(\frac{(\sin z_3\cos\mu+\cos z_3\sin\mu)\cos\tau}{m_1\ell} + z_6\right)^2\sin z_3 + (m_1+m_2)g\tan\mu - kz_1\tan\mu + \alpha_0\sin\mu\sin\omega t - \zeta t_2 \\ -m_2\ell g\sin z_3 + \beta\sin N\omega t - \eta\left(\frac{(\sin z_3\cos\mu+\cos z_3\sin\mu)\alpha\cos\tau}{m_1\ell} + z_6\right) \end{bmatrix},$$

where

$$t_1 = z_4 - \frac{((m_1+m_2-m_2\cos^2 z_3)\cos\mu + m_2\sin z_3\cos z_3\sin\mu)\alpha\cos\tau}{m_1(m_1+m_2)},$$

$$t_2 = z_5 - \frac{((m_1+m_2-m_2\sin^2 z_3)\sin\mu + m_2\sin z_3\cos z_3\cos\mu)\alpha\cos\tau}{m_1(m_1+m_2)}.$$

Upon applying the averaging principle, we obtain equations (4.31) where $\overline{z_1} = z, \overline{z_2} = \nu$ and $\overline{z_3} = \phi$.