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This paper describes a technique for the solution of systems of symmetric parabolic partial differential equations that are associated with the generation of biologic texture generating equations on the highly irregular regions associated with complex image geometries. This technique, when used with an automatic grid generation program allows for the direct and efficient generation of textures on complex surfaces. Since there is a one to one correspondence between the generated texture in the transformed domain and the original surface there is no need to map the generated texture pattern back to the surface of interest. This technique can be applied to large systems of reaction diffusion equations, thus expanding the set of patterns that can be produced using models based upon reaction-diffusion equations. In addition, textures based upon variable diffusion constants can be generated including a wide range of anisotropic textures.

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TEXTURE GENERATION ON IRREGULAR REGIONS

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ABSTRACT

This paper describes a technique for the solution of systems of symmetric parabolic partial differential equations that are associated with the generation of biologic textures in computer graphics. The technique presented allows for the solution of the texture generating equations on the highly irregular regions associated with complex image geometries. This technique, when used with an automatic grid generation program allows for the direct and efficient generation of textures on complex surfaces. Since there is a one to one correspondence between the generated texture in the transformed domain and the original surface there is no need to map the generated texture pattern back to the surface of interest. This technique can be applied to large systems of reaction diffusion equations, thus expanding the set of patterns that can be produced using models based upon reaction-diffusion equations. In addition, textures based upon variable diffusion constants can be generated including a wide range of anisotropic textures.

KEYWORDS: Texture ; Parabolic Equations ; Reaction-Diffusion ; Irregular Geometries ; Iterative methods ; Mathematical Modeling.

OVERVIEW

A considerable body of research exists describing the generation of biologic textures using a system of parabolic partial differential equations. These equations are called reaction-diffusion equations in applied mathematics [Turk91] [Witkin-Kass91]. Previous work in numerical analysis has presented techniques to solve such equations using finite difference techniques on problem domains that are square, circular or rectangular. For the solution of models that are defined upon irregular regions, previous work has produced techniques that superimpose square, circular or rectangular grids upon the irregular problem domain. Previous work has solved the equations on these superimposed grids by either extrapolating boundary values to grid points outside the problem domain or by modifying the finite difference stencils used in approximating the partial derivatives [Castillo et al.92].

This paper presents a technique for solving reaction-diffusion equations on an irregular region. In this technique, the vertices of the image are transformed from physical space to a new coordinate system which is called logical space. The system of reaction-diffusion equations are then transformed to logical space and differenced using a second order nearest neighbor scheme. Since the problem domain in logical space is a square or rectangle, almost any finite difference technique developed in the last

100 years can be used to produce a system of difference equations. The resulting system of difference equations is solved using an iterative scheme to produce the texture values.

The primary advantage of this method is that it eliminates the need to map the calculated texture values back onto the original untransformed surface of interest. Another advantage is that the method is very easy to parallelize for implementation on a massively parallel computer architecture such as a Single Instruction Multiple Data (SIMD) architecture.

METHOD

The technique described in this paper has three goals to meet for the generation of texture patterns: grid generation; equation transformation; and solution of the transformed system of equations.

The first goal of this method is to generate a grid. To accomplish this goal the coordinates of the vertices of the image to be textured are transformed from physical space to a rectangular or square grid in logical space. This transformation is accomplished using Castillo's variational grid generation method [Castillo91] [Castillo-Richardson93]. This grid generation approach controls three properties of the grid: grid spacing, grid cell areas, and grid orthogonality. The grid is generated by direct minimization of a discrete functional. By grid orthogonality is meant the angles of the lines between the lines comprising the computational grid. The variational grid generation method allows the user to vary the effects of above three properties upon the grid generation process. Grid generation provides a boundary fitted coordinate system which conforms to the geometry of the region where the PDE or system of PDE's are specified.

There are several advantages to the variational grid generation method. The weights can be chosen to prevent folding of the grid and to concentrate grid points in areas of the grid where there are large variations in the solution. In addition, the generated grid can be adapted to the solution.

The second goal of this method is to transform the system of reaction-diffusion equations to logical space. Consider the following system of reaction-diffusion equations:

$$(1)\qquad \begin{array}{c} L_1 * f_1 = g_1 \\ L_2 * f_2 = g_2 \\ | \\ | \\ | \\ L_n * f_n = g_n \end{array}$$

where L_i is the operator

$$(2)\qquad \frac{\partial}{\partial t} + \sum_{i,j} \left(\frac{\partial}{\partial x_i} \left(r_{i,j} \left(\frac{\partial}{\partial x_j} \right) \right) \right.$$

The r_{ij}'s are the coefficients of the reaction-diffusion equation and consist of the diffusion constants, etc. It will also be assumed that $r_{ij} = r_{ji}$ so that $L_1, \dots, L_n$ are symmetric. For the 3-D case the i's and j's are summed from 1 to 3. It should be noted that the r_{ij}'s can be constants or variable. The system of equations can produce anisotropic textures subject to the constraint that the operators must be symmetric. For the situation where the system of reaction-diffusion equations are linear or quasi linear this is not a problem since these types of systems can be modified to be symmetric. The coefficients of the cross derivatives can be averaged or similarly modified to produce symmetry in the system.

Given the transformation x = x(E) where E = (t, e_1, e_2 ,e_3) we have the following transformation of the equations, which is an extension of the transformation for elliptic partial differential equations given by [Steinberg91] to the case of parabolic partial differential equations of which reaction-diffusion equations are an example.

$$(3)\qquad \Lambda_1 * f_1 = \Gamma_1 ,$$

$$\Lambda_2 * f_2 = \Gamma_2 ,$$

ect. where Λ_i = Eq. (4)

(4) $$\frac{\partial}{\partial t}+\sum_i \frac{\partial e_i}{\partial t}\frac{\partial}{\partial e_i}+\sum_{i,j}(\frac{\partial}{\partial e_i}(s_{i,j}(\frac{\partial}{\partial e_j}))$$

(5) $$s_{i,j}=\frac{1}{J}\sum_m (c_{l,i}\, r_{l,m}\, c_{m,j})$$

(6) $$M=(m_{i,j})=\frac{\partial x_i}{\partial e_i}$$

(7) $$f_i=f_i(x(E))$$

(8) $$\Gamma_i=Jg_i(x(E))$$

In the above equations J is the Jacobian of the transformation and the c_{ij}'s are the entries in the cofactor matrix of J with J = determinant(M). Note that the s_{ij}'s are the transformed coefficients.

The transformed equations can then be differenced using a second order central difference scheme. Since the operators of the untransformed equation were symmetric then the transformed equations are symmetric and the difference scheme is a nearest neighbor scheme. It should be noted that the grid generation method provides the descretized values of the transformed coordinates. For example, in the two dimensional case, we have the following where the matrix of x and y values are provided by the grid generation process. Thus the partial derivatives in the transformed coordinate system are given by Eq. (9) and Eq. (10).

(9) $$x_{e_i}=\frac{(x(i+1,j)-x(i-1,j))}{2\delta x}$$

(10) $$y_{e_j}=\frac{(y(i,j+1)-y(i,j-1))}{2\delta y}$$

In order to preserve the original properties of the system of reaction-diffusion equations the elements of the cofactor matrix are constrained to satisfy the following equation called the metric identity. The metric identity, Eq. (11), guarantees the second order approximation, symmetry and the nearest neighbor property.

(11) $$\sum_j \frac{\partial c_{i,j}}{\partial e_j}=0$$

If the grid generation method transforms the coordinates of the corners of each of the vertices in rectangular patches comprising the image then the boundary conditions are restricted. If the grid generation method transforms the coordinates of the vertices and a harmonic or linear average of the corners is used to produce transformed coordinates at the transformed patch centers then the system of equations can be placed in conservative form and a much larger class of boundary conditions can be accommodated including neumann, dirichlet and robin boundary conditions. The Neumann or no-flux boundary boundary condition is represented by.

(12) $$\nabla u \bullet n=0$$

The Dirichlet boundary condition is

(13) $$u=b(x,(E))$$

The Robin boundary condition is

(14) $$\alpha(x(E))u+\beta(x(E)\nabla u \bullet n=c(x(E))$$

An arbitrary partial differential equation is said to be in conservative form if it can be written as the divergence of a vector field. Satisfaction of the metric identity implies the well mannered behavior of the conservative form and preservation of conservation laws throughout the transformation process. The vector field is called the flux F while the transformed flux is Φ and they are defined by the following.

(15) $$F_i = \sum_j r_{i,j} \frac{\partial f}{\partial x_j}$$

(16) $$\Phi_i = \sum_j s_{i,j} \frac{\partial f}{\partial e_j}$$

The system of transformed and differenced equations are then solved using iterative methods such as Successive Over Relaxation (SOR) or a conjugate gradient iterative scheme. Several standard textbooks exist on the subject of SOR and conjugate gradient methods [Smith85] [Strikwerda89]. It should be noted that the conjugate gradient method is more amenable to parallel computation than the SOR method. Preconditioning of the conjugate gradient scheme is recommended.

The metric identity guarantees that the symmetry of the original system of PDE's is preserved. Preserving the symmetry of the original system aids in producing a matrix of transformed coefficients for the resulting system of difference equations that is well banded and thus particularly suited for efficient implementation of the SOR and CG iterative methods.

The calculated texture values are the same in both logical and physical space. This is a consequence of the grid generation method and the metric identity. By indexing the transformed points in logical space with the same indices as the physical space the need to map the texture values back onto the original surface is eliminated. Thus, the warping and discontinuities produced by the mapping of textures to a surface by algebraic or procedural techniques are avoided.

TEXTURE MODELING

The purpose of texture generation in computer graphics is to provide texture values for the geometrical patches that comprise an image in 2-D or 3-D space. The basic assumption is that these graphical images are highly irregular in shape. Because of the irregular nature of realistic graphics images a large repertoire of techniques has been developed to map the generated texture values onto the image domain. Several examples are parametric texture mapping and projective techniques using translations, rotations and standard projections. The focus of this paper is to provide the theoretical framework for a direct production of texture values for a computer image free of the necessity to map the generated texture values onto the image geometry.

Over the years a large number of texture models have been produced using algebraic or procedural techniques to generate texture. Examples are marble generation, fractal textures and summation of sine waves. This paper focuses on models of biologic and chemical processes that produce patterns suitable for use in texture synthesis.

A class of models that describe pattern producing processes in nature consist of systems of reaction-diffusion equations. Reaction diffusion equations are parabolic partial differential equations of the form given by Eq. (1) and Eq. (2) above. If the g_i's are linear then the system of reaction-diffusion equations are linear. If the g_i's are nonlinear then the system of reaction-diffusion equations are quasi linear. Most of the interesting models in the literature, such as Meinhardt's stripe system are quasi linear.

Reaction-diffusion equations are examples of phennomena that obey conservation laws. If we are interested in a concentration or texture value at a point, the rate of change of this texture value is related to the inflow and outflow of particles (diffusion) through an area or volume element and the rate of particle creation or destruction (reaction).

There are many examples of quasi linear models in morphogenisis and population biology that embody systems of reaction-diffusion equations and that form patterns. One example of reaction-diffusion equations that form beautiful and interesting patterns is the model of reaction-diffusion wave propagation based upon the eikonal equation which has the following form [Grindrod91]. This model produces a series of spiral and toroidal patterns in nature. The most interesting patterns are produced in three dimensions. Eq. (17) and Eq. (18) describe these spiral and toroidal patterns.

$$(17) \quad \varepsilon \frac{\partial u}{\partial t} + \varepsilon^2 \sum_{i,j} \left(\frac{\partial u}{\partial x_i} \left(r_{i,j} \left(\frac{\partial u}{\partial x_j} \right) \right) \right) + F(u) = 0$$

The wave front propagates according to the following. N is the normal to the wave surface and K is the mean curvature of the wave.

$$(18) \quad \mathrm{N} + \varepsilon\, \mathrm{K} = \mathrm{c}$$

CURRENT AND FUTURE WORK

Previous work using this technique has been performed on models based upon elliptic partial differential equations. Results have been obtained for the transformation of elliptic partial differential equations using several test grids and a model of the otolith (eardrum) membrane [Castillo et al.92] [Tablewski90]. The elliptic partial differential model of the otolith membrane has also been implemented in parallel on a mesh type architecture [Castillo-Richardson93]. Implementation of the method in parallel for the case of elliptic PDE's resulted in a reduction of computation time by a factor of 50-250.

Current work involves the implementation of the grid generation method in parallel on mesh type architecture using the Applied Memory Technology DAP 610 computer. In addition, current work entails the implementation of the methods presented in this paper to the case of systems of parabolic partial differential equations.

Previous work on the otolith model resulted in a solution time of 40 minutes sequentially and 55 seconds in parallel for a 62 by 62 computational grid. It is anticipated that the timing in the parabolic case would be slightly longer for each time step.

The technique presented in this paper is a direct method for solving systems of symmetric parabolic partial differential equations on irregular regions. Application of this technique to systems of reaction-diffusion equations can be used to produce textures in a straight forward manner free from the problems of remapping back onto the surface.

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