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Minimization of Computational Requirements in the Hybrid Stress Finite Element Method

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The hybrid stress method has demonstrated many improvements over conventional displacement-based elements. A main detraction from the method, however, has been the higher computational cost in forming element stiffness coefficients due to matrix inversions and manipulations as required by the technique. By utilizing special transformations of initially assumed stress fields, a spanning set of orthonormalized stress modes can be generated which simplify the matrix equations and allow explicit expressions for element stiffness coefficients to be derived. The developed methodology is demonstrated using several selected 2-D quadrilateral and 3-D hexahedral elements.

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1 Introduction

Alternative approaches to displacement-based element formulations began with the pioneering work of Pian [1] who developed the stress-based approach for computing element stiffness matrices based on the principle of minimum complementary energy. The development of the hybrid stress technique followed using the Hellinger-Reissner functional [2] wherein both stresses and displacements are assumed as independent quantities and from which the purely stress-based approach can be formally derived. Current variational bases for finite element formulations range from single field functionals to the most general Hu-Washizu functional in which displacements, strains and stresses are all assumed as independent quantities [3,4]. The use of multiple independent field variables in element formulations has created a rich arena of theoretical approaches with which to maximize finite element performance and has yielded elements with improved convergence behavior and stress prediction, avoidance of locking in constrained media problems, and the inherent capability to represent traction-free edge conditions and singular stress fields. However, this has resulted in additional theoretical requirements in the formulation of robust elements which have been addressed by various researchers. These include the requirement of element invariance under coordinate transformation [5,6], suppression of spurious zero energy modes [7], minimum expansions for the independent field variables[7,8], and optimal sampling points for stress recovery [5]. A major detraction of element formulations based on multi-field functionals over displacement-based elements has been the computational cost associated with matrix inversions and additional numerical manipulations required in generating element stiffness matrices. As used herein, computational cost or efficiency is meant to refer to the minimum number of sequential operations formally required in mathematical statements and not to the efficiency of specific algorithmic implementations. The present work focuses on the hybrid stress method in which the structure of the element matrices is defined by the Hellinger-Reissner functional. A novel procedure is detailed herein for minimizing the numerical cost by making full use of the freedom in selecting and manipulating assumed stress fields which enable explicit forms of element stiffness matrices to be derived. The developed procedure is based on a detailed examination of the complementary energy matrix inherent to the hybrid technique and leads to stress field transformations which include an orthonormalization approach hitherto not attempted. The application of the technique is demonstrated with 2-D quadrilateral and 3-D hexahedral elements incorporating incompatible displacement fields. By simplifying the constituent matrices involved in forming element stiffness coefficients such that an explicit evaluation can be accomplished, the hybrid stress method is shown to offer a computational advantage over similar displacement-based element formulations.

2 Variational Basis for the Hybrid Stress Model

The extended Hellinger-Reissner functional may be stated as

$$\Pi_R = \int_v [(-1/2)\sigma^T S\sigma + \sigma^T(Lu_q) + \sigma^T(Lu_\lambda)]dv - \int_s u_q^T t ds \tag{1}$$

where σ is the assumed stress field, S is the material compliance matrix, u_q and u_λ are the assumed compatible and incompatible displacement fields, L is the differential operator relating strains to displacements, and t are applied surface tractions over a portion of the element boundary, s.

The assumed stresses may be represented by

$$\sigma = P\beta \tag{2}$$

where P is a matrix of polynomial terms and β is a vector of undetermined expansion coefficients. Each independent stress mode is, therefore, represented by a column in P. The displacement field is assumed over the element domain as

$$u = u_q + u_\lambda = Nq + M\lambda \tag{3}$$

where N and M are compatible and incompatible displacement shape functions, respectively, q are nodal displacements, and λ are Lagrange multipliers which enforce internal constraints. In the form of (1), the incompatible displacements act to variationally enforce the orthogonality of the stresses to the incompatible strain modes in a weak sense. Neglecting applied tractions and substituting (2) and (3) into (1) yields

$$\Pi_R = \int_v [(-1/2)\beta^T P^T S P\beta + \beta^T P^T(LN)q + \beta^T P^T(LM)\lambda]dv \tag{4}$$

or

$$\Pi_R \cong (-1/2)\beta^T H\beta + \beta^T Gq + \beta^T R\lambda \tag{5}$$

where

$$H = \int_v P^T SPdv \tag{6}$$

$$G = \int_v P^T(LN)dv = \int_v P^T Bdv \tag{7}$$

$$R = \int_v P^T(LM)dv = \int_v P^T \bar{B}dv \tag{8}$$

Seeking a stationary value of the functional by taking the first variation with respect to β and λ yields

$$\beta = H^{-1}(Gq - R\lambda) \tag{9}$$

and

$$R^T\beta = 0 \tag{10}$$

By eliminating λ and substituting the resulting expression for β into (5), the variation with respect to q yields the element stiffness matrix as

$$K = \bar{G}^T H^{-1}\bar{G} \tag{11}$$

where

$$\bar{G} = [I - R(R^T H^{-1}R)^{-1}R^T H^{-1}]G \tag{12}$$

3 Computational Minimization in the Hybrid Technique

Since the introduction of the hybrid stress method, minimizing the computational cost associated with equation (11) has been an ongoing concern. Counterbalanced with the ostensibly greater computational cost and demonstrated improvement in element behavior afforded by the two-field hybrid stress technique has been the computational simplicity of displacement-based elements. Thus, the selection of element formulations has been a form of Occam's Razor in what minimum degree of computational cost is required to implement a useful, convergent element to obtain accurate solutions to practical problems. To make hybrid element formulations competitive, various approaches have been applied to minimize the cost in evaluating equation (11). In formulations involving only compatible displacements, it has been noted that H^{-1} is not required separately but the product $H^{-1}G$ can be obtained via equation solver techniques using equation (9) by treating the columns of G as multiple right hand sides which leads to a significant reduction in computation [5]. Other simplifications have been achieved by the use of functionals such as (1) in which incompatible displacement modes are used to variationally enforce equilibrium or orthogonality constraints [2,8,9]. This permits the use of unconstrained and, therefore, uncoupled stress expansions which lead to block diagonal representations of the H-matrix in which the calculation of H^{-1} can be limited to inversions of the submatrix blocks. Recently, a theoretical basis has been developed for making admissible variations of terms in the complementary energy matrix which permit simplifying approximations to be made based on stability and convergence considerations to minimize the cost of computing H^{-1} [10]. Nevertheless, the various treatments of equation (11) still represent a significant computational cost in terms of numerical integrations and manipulations. The aim of the present effort is to strictly adhere to the variational constraint expressed by equation (11) by simplifying the fundamental mathematical statements through formal procedures without introducing additional assumptions or approximations.

4 Assumed Stress Expansions

The conventional procedure in the hybrid stress method is to define stress expansions in the natural or mapped coordinate system. This definition has been used to develop rational procedures for eliminating spurious kinematic modes and maintaining element invariance while keeping the number of independent stress modes to a minimum. However, one drawback of this approach has been the contravariant transformation required to express stresses in physical coordinates. This transformation will, in general, cause coupling between the constant and higher-order stress terms and the element will subsequently fail the patch test.

A solution to this problem has involved the use of an approximation to the Jacobian by using its constant value at the element centroid. This approximation - or 'variational crime' - is reasonably accurate for constant strain elements with linear interpolation functions but should be expected to demonstrate increasing inaccuracy with general element distortion and element order and be a limiting factor in coarse mesh accuracy. In the present development stresses will be assumed in both physical and natural coordinates to demonstrate a procedure for minimizing computational cost in all basic formulations using the hybrid technique. In using stresses defined in physical coordinates all *ad hoc* simplifications will be avoided and 'exact' expressions will be derived within the formal approximation framework based on the order of element interpolants and variational constraints. This is done in anticipation of future study in developing explicit higher-order element matrices where the degradation in accuracy of simplified transformations between natural and physical coordinate systems may become unacceptable. With stresses assumed in physical coordinates, the present study does not consider minimum expansions of assumed stress modes and the expansions are assumed complete to the highest order present in the strain field. Although this will, in cases, result in significantly greater number of stress terms than the minimum required to suppress zero energy modes, the completeness property of the assumed stresses preserves invariance and avoids spurious kinematic modes as the vector space formed by the assumed stresses is guaranteed to span the strain space. In the use of the Hellinger-Reissner functional, stress constraints need not be enforced *a priori*, however, they can be applied to satisfy field equilibrium and compatibility conditions pointwise in order to reduce the number of independent stress modes. In physical coordinates, element invariance will be preserved under field operations of elasticity if complete expansions for the stresses are used. Stress expansions assumed in natural coordinates permit greater freedom in the selection of expansions for specific stress components. This flexibility allows a degree of tailoring of element strain energy mode representation while maintaining invariance and a reduced sensitivity to mesh distortion.

5 Determination of Explicit Forms

A procedure for simplifying the expressions involved in (11) by utilizing permissible transformations of assumed stress fields is now detailed. A two-step transformation of the assumed stresses is suggested by an examination of the flexibility matrix given in equation (6) written out fully as

$$H = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[|J|P^T SP]d\xi d\eta d\zeta \tag{13}$$

An initial observation is that the structure of the integrand in (13) can be simplified through an apportioning transformation of the material compliance matrix, S, to the stress modes in $\mathbf{P}$ thereby allowing further simplifications in the flexibility matrix to be achieved. Towards this aim, the assumed stresses are first transformed through the introduction of a symmetric 'distributing' matrix, D, which acts to subsume the S matrix into P via an identity operator as

$$P = IP = (DD^{-1})P = D(D^{-1}P) = D\bar{P} \tag{14}$$

where the distributing matrix is defined as

$$D = S^{-1/2} \tag{15}$$

Although formally permissible, this operation transforms the initially assumed stress modes into a set of vector polynomials, $\bar{P}$, which do not have a direct physical interpretation. Instead of introducing new terminology, however, the transformed stresses in $\bar{P}$ will simply be referred to as stress modes. The inverse square root of the compliance matrix is obtained via a standard spectral decomposition as

$$S = Q\Lambda Q^T \tag{16}$$

in which Q is a column matrix of the normalized eigenvectors, Φ_i, of S given by

$$Q = [\Phi_1|\Phi_2|\dots|\Phi_n]$$

which is a unitary matrix with the property that

$$Q^T = Q^{-1}$$

and Λ is a diagonal matrix formed by the eigenvalues, φ_i, of S given by

$$\Lambda = diag[\varphi_1, \varphi_2, \dots, \varphi_n]$$

where $n = dim(S)$. With the above definitions, the D matrix is given by

$$D = S^{-1/2} = Q\Lambda^{-1/2}Q^T \tag{17}$$

where

$$\Lambda^{-1/2} = diag[\varphi_1^{-1/2}, \varphi_2^{-1/2}, \dots, \varphi_n^{-1/2}]$$

The symmetry and positive definiteness of the material property matrix guarantees the existence of the decomposition and explicit expressions for Q and Λ for both 2-D and 3-D orthotropic compliance matrices are presented in Appendix I.

Substitution of (14) into (13) yields

$$H = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[|J|\bar{P}^T D^T S D \bar{P}]d\xi d\eta d\zeta \tag{18}$$

where, from the definition of D and the symmetry of both S and D, we obtain

$$D^T S D = S^{-1/2} S S^{-1/2} = S S^{-1} = I \tag{19}$$

and the flexibility matrix reduces to

$$H = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[|J|\bar{P}^T \bar{P}]d\xi d\eta d\zeta \tag{20}$$

A second field transformation is motivated by the form of equation (20) which suggests its use to define an inner product space where a Gram-Schmidt procedure can be employed to generate an orthonormal spanning set of stress modes, P^*, which are a special linear combination of the modes present in $\bar{P}$. The weighted inner product is therefore defined as

$$< \bar{P}_i, \bar{P}_j >= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[|J|\bar{P}_i^T \bar{P}_j]d\xi d\eta d\zeta = \delta_{ij} \tag{21}$$

where δ_{ij} is the Kronecker delta function. The linear combination yielding a sequence of orthogonal stress modes is defined by

$$V_i = \bar{P}_i - \sum_{j=1}^{i-1} < P_j^*, \bar{P}_i > P_i^* \tag{22}$$

which are normalized to form basis vectors, P_i^*, as

$$P_i^* =< V_i, V_i >^{-1/2} V_i \tag{23}$$

Substitution of P^* into equation (20) yields by definition

$$H = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[|J|P^{*T} P^*]d\xi d\eta d\zeta \equiv I \tag{24}$$

Hence, by fully exploiting permissible operations on the assumed stress modes, the transformations due to the application of a distributing matrix and the generation of a weighted orthonormalized basis yield the element stress field as

$$P = DP^* \tag{25}$$

and the flexibility or complementary energy matrix $'H'$ is eliminated by formally reducing it to a matrix identity. The expression for the element stiffness matrix is now given by

$$K = \bar{G}^T \bar{G} \tag{26}$$

where

$$\bar{G} = [I - R(R^T R)^{-1} R^T] G \quad (27)$$

Separating out the Jacobian determinant from the compatible and incompatible strains as

$$B = (LN) = \frac{1}{|J|} B^*(\xi, \eta, \zeta) \quad (28)$$

$$\bar{B} = (LM) = \frac{1}{|J|} \bar{B}^*(\xi, \eta, \zeta) \quad (29)$$

and substituting into (7) and (8), the constituent matrices become

$$G = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} [P^{*T} D B^*] d\xi d\eta d\zeta \quad (30)$$

$$R = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} [P^{*T} D \bar{B}^*] d\xi d\eta d\zeta \quad (31)$$

With the removal of the flexibility matrix, the element stiffness matrix is fully determined by the two constituent matrices in (30) and (31) which represent the elastic strain energy contributions of the assumed stresses and the compatible and incompatible strain modes. The absence of the Jacobian determinant in the denominator allows a direct computation of linear algebraic forms for the G and R matrices based on the regular structure of the strain modes and the transformed stress fields. Explicit expressions for various element matrices will be developed in subsequent sections. In the computation and manipulation of element matrices, however, not all operations are most efficiently obtained explicitly and numerical procedures will be prescribed for certain procedures when computationally advantageous. For example, equation (27) requires the inversion of an inner matrix product in the definition of $\bar{G}$ given by

$$A = Z^{-1} = (R^T R)^{-1} \quad (32)$$

However, because the columns in R are equal to the number of incompatible modes, the dimension of A is given by

$$dim(A) = dim(\lambda) \times dim(\lambda) \quad (33)$$

which is usually small. In addition, because A is symmetric, the inversion effectively reduces to the computational effort of inverting a matrix of triangular form which can be computed explicitly in general but, with increasing matrix order, a numerical scheme is preferred. A second example is the Gram-Schmidt procedure which quickly leads to cumbersome expressions for the coefficients of the orthogonal stress modes when performed symbolically. However, with an initial evaluation of the basic scalar integrals required in the weighted inner product involving the Jacobian determinant and powers of the assumed stress polynomials, a simple numerical procedure may be used for efficiently computing the linear combination of modes present in $\bar{P}$ to generate the orthonormalized basis set P^*. Symbolic representations will, however, be generated for selected elements used in the present study.

The explicit form of the element stiffness matrix, decomposed into contributions due to compatible and incompatible displacements, is given by

$$K_{ij} = K_{q_{ij}} + K_{\lambda_{ij}} = g_{ni} g_{nj} - g_{ni} r_{ns} a_{sm} r_{km} g_{kj} \;\; ; \;\; K_{ji} = K_{ij} \quad (34)$$

where the components of G and R, g_{ij} and r_{ij}, are obtained by integrating (30) and (31) and a_{ij} are the components of the A matrix given in equation (32). The various indices range from

$$i = 1, 2, 3..., N_q; \quad j = 1, 2, 3, ..., i; \quad n, k = 1, 2, 3..., N_\beta; \quad s, m = 1, 2, 3..., N_\lambda$$

where N_q denotes the number of element degrees of freedom, N_β number of independent assumed stress modes, and N_λ the number of incompatible displacement modes.

The above method for determining explicit forms of element stiffness matrices will be demonstrated with 4-node quadrilateral plane and 8-node hexahedral solid element formulations. The explicit integration of (26) offers a significant decrease in computational cost over a purely numerical evaluation of (11).

6 4-Node Plane Quadrilateral Elements

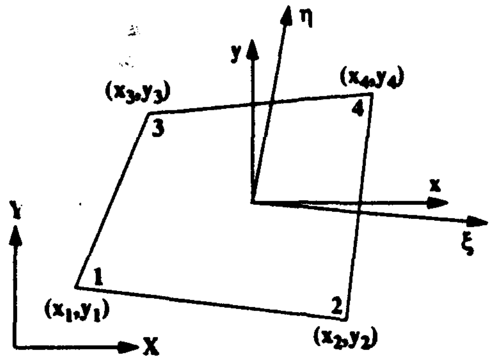


Figure 1. Quadrilateral element.

The stiffness matrices of several different 4-node plane elements will be explicitly derived in this section. Element configuration and node numbering is shown in Figure 1. Two complatable elements are presented to highlight features of both the developed procedure and the hybrid stress method followed by two element formulations incorporating incompatible displacement modes to optimize element performance. Stress expansions are assumed in both physical and natural coordinates to demonstrate the generality of the developed methodology. The correspondence to existing element formulations will be identified.
The displacement functions $\mathbf{u}_q$ are given by

$$\mathbf{u}_q = \begin{pmatrix} u_q \\ v_q \end{pmatrix} = \sum_{i=1}^{4} \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)\begin{pmatrix} u_i \\ v_i \end{pmatrix} = N_i q \tag{35}$$

The mapping from local physical coordinates to natural coordinates is given by

$$\begin{aligned} x &= a_1\xi + a_2\eta + a_3\xi\eta \\ y &= b_1\xi + b_2\eta + b_3\xi\eta \end{aligned} \tag{36}$$

where

$$\begin{bmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{37}$$

The compatible Element E4PQ

A 4-node plane element, designated E4PQ, is explicitly derived using stresses assumed in physical coordinates. The stress field is selected as complete linear expansions with the constraint $L^T\sigma = 0$ applied resulting in the following seven equilibrating stress modes

$$P = \begin{bmatrix} 1 & 0 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & x & y \\ 0 & 0 & 1 & -y & 0 & 0 & -x \end{bmatrix} \tag{38}$$

The derivation of weighted orthonormalized stress modes are obtained in the form

$$P_i^* = \begin{Bmatrix} p_{i1} + p_{i2}x + p_{i3}y \\ p_{i4} + p_{i5}x + p_{i6}y \\ p_{i7} + p_{i8}x + p_{i9}y \end{Bmatrix} \tag{39}$$

and are given by

$$P^* = \begin{bmatrix} p_{11} & 0 & 0 & p_{41}+p_{43}y & p_{51}+p_{52}x+p_{53}y & p_{61}+p_{62}x+p_{63}y & p_{71}+p_{72}x+p_{73}y \\ 0 & p_{24} & 0 & 0 & 0 & p_{64}+p_{66}y & p_{74}+p_{75}x+p_{76}y \\ 0 & 0 & p_{37} & p_{47}+p_{48}x & p_{57}+p_{58}x+p_{59}y & p_{67}+p_{68}x+p_{69}y & p_{77}+p_{78}x+p_{79}y \end{bmatrix} \tag{40}$$

The coefficients of the stress modes, p_{ij}, are presented in Appendix II. Because the stress modes in (38) are self equilibrating, equilibrium is unaffected through the linear combination of modes leading to (40).
The integration of equation (26) is obtained in a straightforward fashion. Using the following constants

$$\begin{aligned} e_{1k} &= b_2 z_1^k - b_1 z_2^k & e_{4k} &= a_1 z_2^k - a_2 z_1^k \\ e_{2k} &= b_3 z_1^k - b_1 z_3^k & e_{5k} &= a_1 z_3^k - a_3 z_1^k \\ e_{3k} &= b_2 z_3^k - b_3 z_2^k & e_{6k} &= a_3 z_2^k - a_2 z_3^k \end{aligned} \quad \text{where} \quad \begin{array}{|c|c|c|c|} \hline k & z_1^k & z_2^k & z_3^k \\ \hline 1 & -1 & -1 & 1 \\ 2 & 1 & -1 & -1 \\ 3 & -1 & 1 & -1 \\ 4 & 1 & 1 & 1 \\ \hline \end{array} \tag{41}$$

the components g_{nk} of (30) are given by

$$\begin{aligned} g_{n(2k-1)} &= e_{1k}(d_{11}p_{n1} + d_{12}p_{n4}) + d_{33}p_{n7}e_{4k} + \tfrac{1}{3}\{(p_{n2}d_{11} + p_{n5}d_{12})(a_1e_{2k} + a_2e_{3k}) + \\ &\quad (p_{n3}d_{11} + p_{n6}d_{12})(b_1e_{2k} + b_2e_{3k}) + d_{33}p_{n8}(a_1e_{5k} + a_2e_{6k}) + \\ &\quad d_{33}p_{n9}(b_1e_{5k} + b_2e_{6k})\} \\ g_{n(2k)} &= e_{4k}(d_{12}p_{n1} + d_{22}p_{n4}) + d_{33}p_{n7}e_{1k} + \tfrac{1}{3}\{(p_{n2}d_{12} + p_{n5}d_{22})(a_1e_{5k} + a_2e_{6k}) + \\ &\quad (p_{n3}d_{12} + p_{n6}d_{22})(b_1e_{5k} + b_2e_{6k}) + d_{33}p_{n8}(a_1e_{2k} + a_2e_{3k}) + \\ &\quad d_{33}p_{n9}(b_1e_{2k} + b_2e_{3k})\} \end{aligned} \tag{42}$$

where d_{ij} are elements of the distributing matrix. The stiffness matrix is then given explicitly as

$$K_{ij} = g_{ni}g_{nj} \quad ; \quad i = 1,2,3,...,8; \quad j = 1,2,3,...,i; \quad n = 1,2,3,...,7 \tag{43}$$

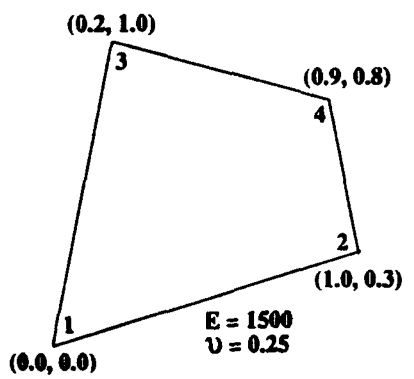
and

$$K_{ji} = K_{ij}$$

The stiffness matrix given above is equivalent to the 7-β hybrid element using the assumed field in (38) and is closely related to the standard minimum 5-β hybrid element of Pian which incorporates equilibrating stress expansions given by

$$P = \begin{bmatrix} 1 & 0 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{44}$$

A validation of the procedure is presented in Table 1 by showing the equivalence of eigenvalues obtained from the explicit element stiffness matrix given by (43) and the numerical computation using equation (11). For generating the results in Table 1, a general quadrilateral configuration was arbitrarily selected as shown in Figure 2. A 2 × 2 Gaussian integration rule was used for the numerical stiffness matrix.



Fiqure 2. General quadrilateral element.

Table 1. Equivalence of eigenvalues obtained from the explicit and numerically integrated form of the element stiffness matrix.

λ	Numerical K matrix	Explicit K matrix
1	0.0	0.0
2	0.0	0.0
3	0.0	0.0
4	500.69450	500.69450
5	585.41852	585.41852
6	1190.1325	1190.1325
7	1297.3502	1297.3502
8	2172.8470	2172.8470

The compatible Element E4PR

The E4PR element is formulated using unconstrained stress expansions resulting in the following nine independent stress modes

$$P = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y \end{bmatrix} \tag{45}$$

In performing the initial apportioning transformation of the assumed stress field as given by

$$\bar{P} = D^{-1}P \tag{46}$$

the stress modes become coupled due to the action of the distributing matrix which complicates the subsequent orthonormalization procedure. However, a simplification can be made through a linear combination of the modes in (46) with constant terms being absorbed into the vector of unknown expansion coefficients, β. With stresses expressed by

$$\sigma = P\beta = D\bar{P}\beta \tag{47}$$

the modes may be rearranged and the P matrix decomposed to give

$$D^{-1}\sigma = D^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\beta_1 + D^{-1}\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}\beta_2 + D^{-1}\begin{bmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix}\beta_3 \tag{48}$$

where β_i are subvectors of β. Rewriting the above as

$$D^{-1}\sigma = D^{-1}I\beta_1 + xD^{-1}I\beta_2 + yD^{-1}I\beta_3 \tag{49}$$

suggests a linear combination of the stress modes for each partition defined by

$$\bar{\beta}_i = D^{-1}\beta_i \tag{50}$$

which leads to the simplification

$$D^{-1}\sigma \quad = \quad D^{-1}ID\bar{\beta}_1 + xD^{-1}ID\bar{\beta}_2 + yD^{-1}ID\bar{\beta}_3 \quad = \quad I\bar{\beta}_1 + xI\bar{\beta}_2 + yI\bar{\beta}_3 \tag{51}$$

and equation (47) becomes

$$\sigma = DP\bar{\beta} \tag{52}$$

Although for arbitrary stress expansions equation (14) is strictly valid, because the stress expansions defined in (45) are balanced for all components, the inverse distributing matrix, D^{-1}, can be completely removed and the form of $\bar{P}$ is made identical to P. With β defined by equations (9) and (10), the linear operations on $\bar{P}$ are automatically accounted for and the distinction between $\bar{\beta}$ and β may be neglected. Orthogonalizing the assumed stress is thus reduced to determining an orthonormal sequence of scalar functions using the weighted inner product defined in (21) and the basis functions given by $[1, x, y]$.
The weighted orthonormalized stress modes are obtained as

$$P^* = \begin{bmatrix} p_1^* & p_2^* & p_3^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1^* & p_2^* & p_3^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_1^* & p_2^* & p_3^* \end{bmatrix} \tag{53}$$

where

$$\begin{aligned} p_1^* &= p_{11} \\ p_2^* &= p_{21} + p_{22}x \\ p_3^* &= p_{31} + p_{32}x + p_{33}y \end{aligned} \tag{54}$$

The coefficients, p_{ij}, are presented in Appendix II.

The integration of (30) yields the components g_{ij} with $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ as

$$\begin{aligned} g_{(i)(2j-1)} &= d_{11}\phi_{ij} & g_{(i+3)(2j-1)} &= d_{12}\phi_{ij} & g_{(i+6)(2j-1)} &= d_{33}\psi_{ij} \\ g_{(i)(2j)} &= d_{12}\psi_{ij} & g_{(i+3)(2j)} &= d_{22}\psi_{ij} & g_{(i+6)(2j)} &= d_{33}\phi_{ij} \end{aligned} \tag{55}$$

where

$$\begin{aligned}\phi_{ij} &= p_{i1}e_{1j} + [p_{i2}(a_1e_{2j} + a_2e_{3j}) + p_{i3}(b_1e_{2j} + b_2e_{3j})]/3 \\ \psi_{ij} &= p_{i1}e_{4j} + [p_{i2}(a_1e_{5j} + a_2e_{6j}) + p_{i3}(b_1e_{5j} + b_2e_{6j})]/3\end{aligned} \tag{56}$$

and where d_{ij} are elements of the distributing matrix and e_{ij} are defined in (41).
The stiffness matrix is explicitly given by

$$K_{ij} = g_{mi}g_{mj} \quad ; \quad i = 1,2,3,...,8; \quad j = 1,2,3,...,i; \quad m = 1,2,3,...,9$$

The interest in this hybrid formulation lies in the limitation principle of [11] which states that, in the limit of the order of unconstrained stress expansions, the hybrid stress method converges to the stiffness matrix obtained from a purely displacement-based formulation. For a parallelogram, the hybrid E4PR element exactly duplicates the displacement-based stiffness matrix. It is of interest to quantify how well the explicit hybrid element formulation can be substituted for the displacement-based element which does not permit a simple integrated representation under general element distortion. Figure 3 shows an initial unit element geometry and its change as a funtion of a distortion parameter, e. Table 2 presents ratios of traces of the element stiffness matrices of increasingly distorted element configurations as a function of the distortion parameter comparing the hybrid element with the corresponding displacement-based element. A 2×2 gaussian quadrature rule was used in computing the displacement - based matrix. It is shown that the explicit hybrid stress formulation yields a consistently more flexible element and offers a clear computational advantage.

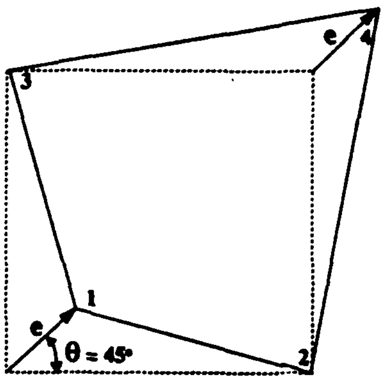


Figure 3. Definition of distortion parameter e.

Table 2. Ratios of stiffness matrix traces comparing explicit hybrid method, K_h, with numerical displacement-based method, K_d under increasing element distortion.

e	$Trace(K_h)/Trace(K_d)$
0.0	1.000000
0.1	0.998322
0.2	0.993153
0.3	0.984084
0.4	0.970401
0.5	0.951032
0.6	0.924311

The Incompatible Element E4PL

The E4PL element incorporates the unconstrained stress modes given by equation (45) with the addition of incompatible displacement modes. The incompatible displacement functions are selected as $[\xi^2, \eta^2]$ which are then modified according to Reference 12 to identically satisfy the strong form of the convergence requirement on incompatible modes given by

$$\int_v L u_\lambda dv = 0 \tag{57}$$

which yields

$$\left\{ \begin{array}{c} u \\ v \end{array} \right\}_\lambda = \left[\begin{array}{cccc} M_1 & 0 & M_2 & 0 \\ 0 & M_1 & 0 & M_2 \end{array} \right] \left\{ \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{array} \right\} \tag{58}$$

where

$$\begin{aligned} M_1 &= \xi^2 - \tfrac{4}{3}(f_1\xi + f_2\eta) \\ M_2 &= \eta^2 + \tfrac{4}{3}(f_1\xi + f_2\eta) \end{aligned} \tag{59}$$

By integrating (31), the elements of the R matrix with $n = 1,2,3$ are given by

$$\begin{array}{llll} r_{n,1} = d_{11}\Theta_n^1 & r_{n,2} = d_{12}\Theta_n^2 & r_{n,3} = d_{11}\Theta_n^4 & r_{n,4} = d_{12}\Theta_n^5 \\ r_{n+3,1} = d_{12}\Theta_n^1 & r_{n+3,2} = d_{22}\Theta_n^2 & r_{n+3,3} = d_{12}\Theta_n^4 & r_{n+3,4} = d_{22}\Theta_n^5 \\ r_{n+6,1} = d_{33}\Theta_n^2 & r_{n+6,2} = d_{33}\Theta_n^3 & r_{n+6,3} = d_{33}\Theta_n^5 & r_{n+6,4} = d_{33}\Theta_n^6 \end{array} \tag{60}$$

where

$$\begin{aligned}
\Theta_n^1 &= 4(p_{n1}h_1 + p_{n2}h_2 + p_{n3}h_3)/3 & \Theta_n^4 &= 4(p_{n1}h_7 + p_{n2}h_8 - p_{n3}h_6)/3 \\
\Theta_n^2 &= 4(p_{n1}h_4 + p_{n2}h_5 + p_{n3}h_6)/3 & \Theta_n^5 &= 4(p_{n1}h_9 + p_{n2}h_{10} - p_{n3}h_6)/3 \\
\Theta_n^3 &= 4(p_{n1}h_7 + p_{n2}h_2 + p_{n3}h_3)/3 & \Theta_n^6 &= 4(p_{n1}h_7 + p_{n2}h_8 - p_{n3}h_3)/3
\end{aligned} \tag{61}$$

where the constant terms f_i and h_i are presented in Appendix III. The elements of the Z matrix defined in (32) are computed as

$$z_{ij} = r_{ni}r_{nj} \quad ; \quad z_{ji} = z_{ij} \tag{62}$$

where

$$i = 1,2,3,4; \quad j = 1,2,...,i; \quad n = 1,2,3,...,9$$

The components a_{ij} of the inner product defined in (32) are obtained through a closed-form inversion of the symmetric 4×4 Z matrix. The element stiffness matrix, K, is given by the sum of contributions due to compatible and incompatible displacements

$$K_{ij} = K_{q_{ij}} + K_{\lambda_{ij}} = g_{ni}g_{nj} - g_{ni}r_{ns}a_{sm}r_{km}g_{kj} \quad ; \quad K_{ji} = K_{ij}$$

where

$$i = 1,2,3,...,8 \; ; \quad j = 1,2,3,...,i \; ; \quad n,k = 1,2,3,...,9 \; ; \quad s,m = 1,2,3,4$$

The Incompatible Element E4NL

The E4NL element is formulated using unconstrained stress expansions assumed in natural coordinates resulting in the following nine independent stress modes

$$\begin{Bmatrix} \tau^{11} \\ \tau^{22} \\ \tau^{12} \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \xi & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \xi & \eta \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_9 \end{Bmatrix} = \Gamma\beta \tag{63}$$

In order to preserve the constant stress terms, the natural stresses are mapped to physical space through a contravariant transformation using Jacobians computed at the element centroid

$$\sigma^{kl} = (J_o)_i^k (J_o)_j^l \tau^{ij} \tag{64}$$

or

$$P = T_\sigma \Gamma \tag{65}$$

The initial stresses are transformed as

$$\bar{P} = D^{-1}T_\sigma\Gamma = \bar{T}\Gamma \tag{66}$$

where

$$\bar{T} = \begin{bmatrix} \alpha_1 & \alpha_4 & \alpha_7 \\ \alpha_2 & \alpha_5 & \alpha_8 \\ \alpha_3 & \alpha_6 & \alpha_9 \end{bmatrix} \tag{67}$$

and the constants, a_i, are given by

$$\begin{aligned}
\alpha_1 &= \bar{d}_{11}a_1^2 + \bar{d}_{12}b_1^2 & \alpha_4 &= \bar{d}_{11}a_2^2 + \bar{d}_{12}b_2^2 & \alpha_7 &= 2(\bar{d}_{11}a_1a_2 + \bar{d}_{12}b_1b_2) \\
\alpha_2 &= \bar{d}_{12}a_1^2 + \bar{d}_{22}b_1^2 & \alpha_5 &= \bar{d}_{12}a_2^2 + \bar{d}_{22}b_2^2 & \alpha_8 &= 2(\bar{d}_{12}a_1a_2 + \bar{d}_{22}b_1b_2) \\
\alpha_3 &= \bar{d}_{33}a_1a_2 & \alpha_6 &= \bar{d}_{33}b_1b_2 & \alpha_9 &= \bar{d}_{33}(a_1b_2 + a_2b_1)
\end{aligned} \tag{68}$$

in which $\bar{d}_{ij}$ are elements of the inverse distributing matrix. By introducing a linear combination of stress modes defined by

$$\bar{\beta} = \begin{bmatrix} \bar{T} & & \\ & \bar{T} & \\ & & \bar{T} \end{bmatrix} \beta \tag{69}$$

the coupling of the transformed stresses can be eliminated resulting in

$$\bar{P} = \begin{bmatrix} 1 & \xi & \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \xi & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \xi & \eta \end{bmatrix} \tag{70}$$

and the orthonormalization is effectively reduced to determining an orthonormal sequence of scalar functions using the basis set $[1, \xi, \eta]$.
The weighted orthonormalized stress modes are obtained as

$$P^* = \begin{bmatrix} p_1^* & p_2^* & p_3^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1^* & p_2^* & p_3^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_1^* & p_2^* & p_3^* \end{bmatrix} \tag{71}$$

where

$$\begin{aligned} p_1^* &= p_{11} \\ p_2^* &= p_{21} + p_{22}\xi \\ p_3^* &= p_{31} + p_{32}\xi + p_{33}\eta \end{aligned} \tag{72}$$

and the coefficients, p_{ij}, are presented in Appendix III.

The integration of (30) yields the components g_{mn} with $m = 1,2,3$ and $n = 1,2,3,4$ as

$$\begin{aligned} g_{(m)(2n-1)} &= d_{11}\phi_{mn} & g_{(m+3)(2n-1)} &= d_{12}\phi_{mn} & g_{(m+6)(2n-1)} &= d_{33}\psi_{mn} \\ g_{(m)(2n)} &= d_{12}\psi_{mn} & g_{(m+3)(2n)} &= d_{22}\psi_{mn} & g_{(m+6)(2n)} &= d_{33}\phi_{mn} \end{aligned} \tag{73}$$

where

$$\begin{aligned} \phi_{mn} &= p_{m1}e_{1n} + [p_{m2}e_{2n} + p_{m3}e_{3n}]/3 \\ \psi_{mn} &= p_{m1}e_{4n} + [p_{m2}e_{5n} + p_{m3}e_{6n}]/3 \end{aligned} \tag{74}$$

and where d_{ij} are elements of the distributing matrix and e_{ik} are defined in (41).
The incompatible displacement modes are the same as those used in E4PL and yield the components r_{mn} of (31) with $m = 1,2,3$ as

$$\begin{aligned} r_{m,1} &= d_{11}\Theta_m^1 & r_{m,2} &= d_{12}\Theta_m^2 & r_{m,3} &= d_{11}\Theta_m^4 & r_{m,4} &= d_{12}\Theta_m^5 \\ r_{m+3,1} &= d_{12}\Theta_m^1 & r_{m+3,2} &= d_{22}\Theta_m^2 & r_{m+3,3} &= d_{12}\Theta_m^4 & r_{m+3,4} &= d_{22}\Theta_m^5 \\ r_{m+6,1} &= d_{33}\Theta_m^2 & r_{m+6,2} &= d_{33}\Theta_m^3 & r_{m+6,3} &= d_{33}\Theta_m^5 & r_{m+6,4} &= d_{33}\Theta_m^6 \end{aligned} \tag{75}$$

where

$$\begin{aligned} \Theta_m^1 &= p_{m1}h_1 + p_{m2}h_2 + p_{m3}h_3 & \Theta_m^4 &= p_{m1}h_7 + p_{m2}h_8 - p_{m3}h_3 \\ \Theta_m^2 &= p_{m1}h_4 + p_{m2}h_5 + p_{m3}h_6 & \Theta_m^5 &= p_{m1}h_9 + p_{m2}h_{10} - p_{m3}h_4 \\ \Theta_m^3 &= p_{m1}h_7 + p_{m2}h_2 + p_{m3}h_3 & \Theta_m^6 &= p_{m1}h_7 + p_{m2}h_8 - p_{m3}h_1 \end{aligned} \tag{76}$$

The constant terms f_i and h_i are contained in Appendix III. The elements of the symmetric Z matrix are computed as

$$z_{ij} = r_{ni}r_{nj} \ ; \ z_{ji} = z_{ij}$$

where

$$i = 1,2,3,4; \quad j = 1,2,...,i; \quad n = 1,2,3,...,9$$

The components a_{ij} are then obtained through the inversion of the symmetric 4×4 Z matrix. The element stiffness matrix, K, is given by

$$K_{ij} = K_{q_{ij}} + K_{\lambda_{ij}} = g_{ni}g_{nj} - g_{ni}r_{ns}a_{sm}r_{km}g_{kj} \quad ; \quad K_{ji} = K_{ij}$$

where

$$i = 1,2,3,...,8 \ ; \quad j = 1,2,3,...,i \ ; \quad n,k = 1,2,3,...,9 \ ; \quad s,m = 1,2,3,4$$

The E4PL and E4NL elements are related to the Pian-Sumihara element [13] which has demonstrated exceptional performance in plane stress/plane strain problems. An elegant derivation of an explicit form for the Pian-Sumihara element is presented in Reference [14] utilizing a scaling procedure and stabilization matrices to obtain stiffness coefficients. The present method avoids the need for numerical stabilization and provides a generic approach for obtaining explicit element stiffness matrices in hybrid element formulations.

7 8-Node Hexahedral Continuum Elements

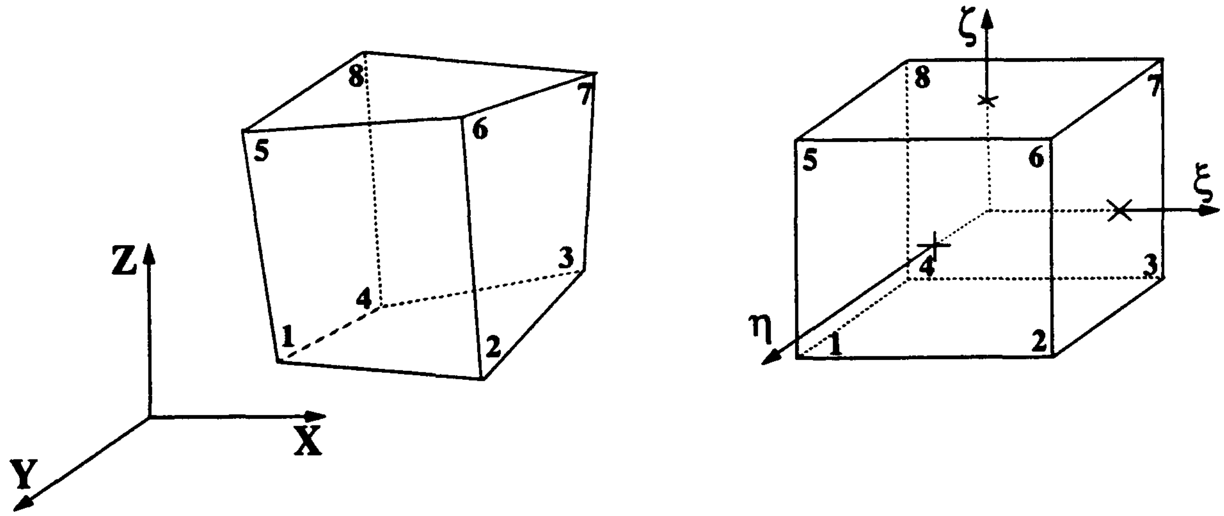


Figure 4. Hexahedral element configuration.

Two 8-node hexahedral continuum elements incorporating incompatible displacements will be considered in this section. Element geometry and node numbering convention is shown in Figure 4. The first element, designated E8PL, is based on unconstrained stress expansions assumed in physical coordinates. The second element, designated E8NL, is based on stresses assumed in natural coordinates. The selection of incompatible modes are identical to those used in [12].

The displacement functions $\mathbf{u}_q$ are given by

$$\mathbf{u}_q = \left\{ \begin{array}{c} u_q \\ v_q \\ w_q \end{array} \right\} = \sum_{i=1}^{8} \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta) \left\{ \begin{array}{c} u_i \\ v_i \\ w_i \end{array} \right\} = N_i q \tag{77}$$

The isoparametric mapping between local physical and natural coordinates is given by

$$\begin{aligned} x &= a_1\xi + a_2\eta + a_3\zeta + a_4\xi\eta + a_5\xi\zeta + a_6\eta\zeta + a_7\xi\eta\zeta \\ y &= b_1\xi + b_2\eta + b_3\zeta + b_4\xi\eta + b_5\xi\zeta + b_6\eta\zeta + b_7\xi\eta\zeta \\ z &= c_1\xi + c_2\eta + c_3\zeta + c_4\xi\eta + c_5\xi\zeta + c_6\eta\zeta + c_7\xi\eta\zeta \end{aligned} \tag{78}$$

where

$$\begin{bmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \\ a_7 & b_7 & c_7 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \\ x_7 & y_7 & z_7 \\ x_8 & y_8 & z_8 \end{bmatrix} \tag{79}$$

The E8PL Element

For an 8-node isoparametric continuum element with trilinear displacement interpolants, the number of independent strain modes is 18. A detailed study of strain modes in Reference [9] has determined that a linear expansion field is insufficient and will give rise to spurious zero energy modes unless selected quadratic terms are added. In physical coordinates, using the completeness property to guarantee element invariance, the number of assumed stress modes is 60. Equilibrium and compatibility constraints can be applied to reduce the independent modes to 42 but is not performed in the present analysis. Several versions of the E8PL element are formulated using different assumed stress fields to assess the effect on element performance. In $E8PL_1$, stresses are assumed as complete linear expansions for all stress components. This selection yields an element containing 2 zero energy modes which would have to be removed through stabilization to yield a useful element. Stress expansions including quadratic cross terms are incorporated into the $E8PL_2$ element. This element possesses the requisite number of rigid body modes but is not invariant under coordinate transformation. A third element, $E8PL_3$, incorporates complete quadratic expansions which precludes spurious kinematic modes and ensures invariance. This element, however, is shown to suffer from significant sensitivity to distortion. The three versions of the E8PL element are compared to a 3-D element formulated in natural coordinates which demonstrates optimal behavior.
To encompass the various stress fields assumed in the E8PL element, the initial stress field is selected as

$$P = \begin{bmatrix} [P_1] & & & & & \\ & [P_1] & & & & \\ & & [P_1] & & & \\ & & & [P_1] & & \\ & & & & [P_1] & \\ & & & & & [P_1] \end{bmatrix} \tag{80}$$

which are complete quadratic expansions given by

$$[P_1] = [1, x, y, z, xy, yz, zx, x^2, y^2, z^2] \tag{81}$$

Orthonormalizing yields

$$[P_1] = [p_1^*, p_2^*, p_3^*, p_4^*, p_5^*, p_6^*, p_7^*, p_8^*, p_9^*, p_{10}^*] \tag{82}$$

where, for $i = 1, 2, 3, ..., 10$, the general form of each mode is given by

$$p_i^* \;=\; p_{i1}^* + p_{i2}^* x + p_{i3}^* y + p_{i4}^* z + p_{i5}^* xy + p_{i6}^* yz + p_{i7}^* zx + p_{i8}^* x^2 + p_{i9}^* y^2 + p_{i10}^* z^2 \tag{83}$$

and

$$p_{ik}^* = 0 \quad for \quad k > i$$

A procedure for generating the constant coefficients in the expressions for p_i^* is presented in Appendix II.

The stress fields are selected in the following expressions by setting N_e equal to 4, 7 and 10 for the $E8PL_1$, $E8PL_2$ and $E8PL_3$ elements respectively. Integrating equation (30), the components g_{mn} where $m = 1, 2, 3, ..., N_e$ and $n = 1, 2, 3, ..., 8$ result in lengthy linear algebraic expressions which are, however, highly structured and allow a compact notation to be used. The expressions for g_{mn} are given by

$$\begin{aligned}
g_{(m)(3n-2)} &= d_{11}\Theta^1_{mn} & g_{(m+2N_e)(3n-2)} &= d_{31}\Theta^1_{mn} & g_{(m+4N_e)(3n-2)} &= d_{55}\Theta^3_{mn} \\
g_{(m)(3n-1)} &= d_{12}\Theta^2_{mn} & g_{(m+2N_e)(3n-1)} &= d_{32}\Theta^2_{mn} & g_{(m+4N_e)(3n-1)} &= 0 \\
g_{(m)(3n)} &= d_{13}\Theta^3_{mn} & g_{(m+2N_e)(3n)} &= d_{33}\Theta^3_{mn} & g_{(m+4N_e)(3n)} &= d_{55}\Theta^1_{mn} \\
g_{(m+N_e)(3n-2)} &= d_{21}\Theta^1_{mn} & g_{(m+3N_e)(3n-2)} &= 0 & g_{(m+5N_e)(3n-2)} &= d_{66}\Theta^2_{mn} \\
g_{(m+N_e)(3n-1)} &= d_{22}\Theta^2_{mn} & g_{(m+3N_e)(3n-1)} &= d_{44}\Theta^3_{mn} & g_{(m+5N_e)(3n-1)} &= d_{66}\Theta^1_{mn} \\
g_{(m+N_e)(3n)} &= d_{23}\Theta^3_{mn} & g_{(m+3N_e)(3n)} &= d_{44}\Theta^2_{mn} & g_{(m+5N_e)(3n)} &= 0
\end{aligned} \tag{84}$$

where

$$\Theta^k_{mn} = \sum_{s=1}^{m} p^*_{ms} \phi^k_{sn} \tag{85}$$

For $m = 1$

$$\phi^k_{1i} = -w^k_{1i}/3 \tag{86}$$

For $m = 2, 3, 4$

$$\phi^k_{ni} = -(3\alpha^n_1 w^k_{2i} + 3\alpha^n_2 w^k_{3i} + 3\alpha^n_3 w^k_{4i} + \alpha^n_4 w^k_{5i} + \alpha^n_5 w^k_{6i} + \alpha^n_6 w^k_{7i} + 2\alpha^n_7 w^k_{8i})/27 \tag{87}$$

For $m = 5, 6, 7, 8, 9, 10$

$$\begin{aligned}\phi^k_{ni} = -[&\alpha^n_1(10\beta^n_6 w^k_{8i} + \beta^n_5 w^k_{9i} + \beta^n_7 w^k_{12i} + \beta^n_1 w^k_{13i} + \beta^n_4 w^k_{20i} + 5\beta^n_3 w^k_{6i} + 5\beta^n_2 w^k_{5i} + \\ &\alpha^n_2(10\beta^n_5 w^k_{8i} + \beta^n_6 w^k_{10i} + 5\beta^n_3 w^k_{7i} + \beta^n_2 w^k_{14i} + \beta^n_7 w^k_{23i} + 5\beta^n_1 w^k_{5i} + \beta^n_4 w^k_{25i} + \\ &\alpha^n_3(10\beta^n_4 w^k_{8i} + 5\beta^n_2 w^k_{7i} + \beta^n_3 w^k_{15i} + \beta^n_6 w^k_{21i} + 5\beta^n_1 w^k_{6i} + \beta^n_7 w^k_{24i} + \beta^n_5 w^k_{26i} + \\ &\alpha^n_4(10\beta^n_3 w^k_{8i} + \beta^n_7 w^k_{11i} + \beta^n_5 w^k_{12i} + \beta^n_4 w^k_{16i} + \beta^n_1 w^k_{20i} + \beta^n_6 w^k_{23i} + \beta^n_2 w^k_{25i} + \\ &\alpha^n_5(10\beta^n_2 w^k_{8i} + \beta^n_1 w^k_{9i} + \beta^n_4 w^k_{12i} + \beta^n_5 w^k_{17i} + \beta^n_7 w^k_{22i} + \beta^n_6 w^k_{24i} + \beta^n_3 w^k_{26i} + \\ &\alpha^n_6(10\beta^n_1 w^k_{8i} + \beta^n_2 w^k_{10i} + \beta^n_6 w^k_{18i} + \beta^n_3 w^k_{21i} + \beta^n_4 w^k_{23i} + \beta^n_5 w^k_{24i} + \beta^n_7 w^k_{27i} + \\ &\alpha^n_7(\beta^n_4 w^k_{11i} + \beta^n_1 w^k_{12i} + \beta^n_7 w^k_{19i} + \beta^n_5 w^k_{22i} + \beta^n_2 w^k_{23i} + \beta^n_3 w^k_{24i} + \beta^n_6 w^k_{27i}]/135\end{aligned} \tag{88}$$

The constants w^k_{ji} are given by

$$\begin{aligned}
w^k_{1i} &= (\delta^k_{54} + 3\delta^k_{32})z^i_1 + (\delta^k_{62} + \delta^k_{15})z^i_2 + (\delta^k_{36} + \delta^k_{41})z^i_3 + (\delta^k_{46} + 3\delta^k_{13})z^i_5 + (\delta^k_{53} + \delta^k_{24})z^i_6 + (\delta^k_{65} + 3\delta^k_{21})z^i_7 \\
w^k_{2i} &= 3(\delta^k_{34} + \delta^k_{52})z^i_1 + (\delta^k_{72} + 3\delta^k_{13})z^i_2 + (\delta^k_{37} + 3\delta^k_{21})z^i_3 + (\delta^k_{53} + \delta^k_{24})z^i_4 + (\delta^k_{47} + 3\delta^k_{15})z^i_5 + (\delta^k_{75} + 3\delta^k_{41})z^i_7 \\
w^k_{3i} &= (\delta^k_{74} + 3\delta^k_{62})z^i_1 + (3\delta^k_{32} + \delta^k_{17})z^i_2 + (\delta^k_{36} + \delta^k_{41})z^i_4 + 3(\delta^k_{43} + \delta^k_{16})z^i_5 + (\delta^k_{73} + 3\delta^k_{21})z^i_6 + (\delta^k_{67} + 3\delta^k_{24})z^i_7 \\
w^k_{4i} &= (\delta^k_{57} + 3\delta^k_{36})z^i_1 + (3\delta^k_{32} + \delta^k_{71})z^i_3 + (\delta^k_{62} + \delta^k_{15})z^i_4 + (\delta^k_{76} + 3\delta^k_{53})z^i_5 + (3\delta^k_{13} + \delta^k_{27})z^i_6 + 3(\delta^k_{61} + \delta^k_{25})z^i_7 \\
w^k_{5i} &= 3(\delta^k_{72} + \delta^k_{64})z^i_1 + 3(\delta^k_{52} + \delta^k_{16})z^i_2 + (\delta^k_{67} + 3\delta^k_{24})z^i_3 + (\delta^k_{56} + 3\delta^k_{21})z^i_4 + 3(\delta^k_{45} + \delta^k_{17})z^i_5 + (\delta^k_{75} + 3\delta^k_{41})z^i_6 \\
w^k_{6i} &= 3(\delta^k_{37} + \delta^k_{56})z^i_1 + (\delta^k_{76} + 3\delta^k_{53})z^i_2 + 3(\delta^k_{34} + \delta^k_{61})z^i_3 + (3\delta^k_{13} + \delta^k_{64})z^i_4 + (\delta^k_{47} + 3\delta^k_{15})z^i_6 + 3(\delta^k_{45} + \delta^k_{71})z^i_7 \\
w^k_{7i} &= (\delta^k_{57} + 3\delta^k_{36})z^i_2 + (\delta^k_{74} + 3\delta^k_{62})z^i_3 + (3\delta^k_{32} + \delta^k_{45})z^i_4 + 3(\delta^k_{56} + \delta^k_{73})z^i_5 + 3(\delta^k_{43} + \delta^k_{25})z^i_6 + 3(\delta^k_{64} + \delta^k_{27})z^i_7 \\
w^k_{8i} &= \delta^k_{56}z^i_2 + \delta^k_{64}z^i_3 + \delta^k_{45}z^i_6 \\
w^k_{9i} &= (9\delta^k_{57} + 15\delta^k_{36})z^i_1 + (9\delta^k_{71} + 15\delta^k_{32})z^i_3 + (5\delta^k_{62} + 9\delta^k_{15})z^i_4 + 5(\delta^k_{76} + 3\delta^k_{53})z^i_5 + 5(\delta^k_{27} + 3\delta^k_{13})z^i_6 + 15(\delta^k_{61} + \delta^k_{25})z^i_7 \\
w^k_{10i} &= 5(\delta^k_{57} + 3\delta^k_{36})z^i_1 + 5(\delta^k_{71} + 3\delta^k_{32})z^i_3 + (9\delta^k_{62} + 5\delta^k_{15})z^i_4 + (9\delta^k_{76} + 15\delta^k_{53})z^i_5 + (9\delta^k_{27} + 15\delta^k_{13})z^i_6 + 15(\delta^k_{61} + \delta^k_{25})z^i_7 \\
w^k_{11i} &= (3\delta^k_{57} + 5\delta^k_{36})z^i_1 + (3\delta^k_{71} + 5\delta^k_{32})z^i_3 + 3(\delta^k_{62} + \delta^k_{15})z^i_4 + (3\delta^k_{76} + 5\delta^k_{53})z^i_5 + (3\delta^k_{27} + 5\delta^k_{13})z^i_6 + 5(\delta^k_{61} + \delta^k_{25})z^i_7 \\
w^k_{12i} &= (3\delta^k_{57} + 5\delta^k_{36})z^i_2 + (3\delta^k_{74} + 5\delta^k_{62})z^i_3 + (3\delta^k_{45} + 5\delta^k_{32})z^i_4 + 5(\delta^k_{56} + \delta^k_{73})z^i_5 + 5(\delta^k_{43} + \delta^k_{25})z^i_6 + 5(\delta^k_{64} + \delta^k_{27})z^i_7 \\
w^k_{13i} &= 3[(9\delta^k_{54} + 15\delta^k_{32})z^i_1 + (5\delta^k_{62} + 9\delta^k_{15})z^i_2 + (5\delta^k_{36} + 9\delta^k_{41})z^i_3 + 5(\delta^k_{46} + 3\delta^k_{13})z^i_5 + 5(\delta^k_{53} + \delta^k_{24})z^i_6 + 5(\delta^k_{65} + 3\delta^k_{21})z^i_7] \\
w^k_{14i} &= 3[5(\delta^k_{54} + 3\delta^k_{32})z^i_1 + (9\delta^k_{62} + 5\delta^k_{15})z^i_2 + 5(\delta^k_{36} + \delta^k_{41})z^i_3 + (9\delta^k_{46} + 15\delta^k_{13})z^i_5 + (5\delta^k_{53} + 9\delta^k_{24})z^i_6 + 5(\delta^k_{65} + 3\delta^k_{21})z^i_7] \\
w^k_{15i} &= 3[5(\delta^k_{54} + 3\delta^k_{32})z^i_1 + 5(\delta^k_{62} + \delta^k_{15})z^i_2 + (9\delta^k_{36} + 5\delta^k_{41})z^i_3 + 5(\delta^k_{46} + 3\delta^k_{13})z^i_5 + (9\delta^k_{53} + 5\delta^k_{24})z^i_6 + (9\delta^k_{65} + 15\delta^k_{21})z^i_7] \\
w^k_{16i} &= (9\delta^k_{54} + 15\delta^k_{32})z^i_1 + 9(\delta^k_{62} + \delta^k_{15})z^i_2 + (5\delta^k_{36} + 9\delta^k_{41})z^i_3 + (9\delta^k_{46} + 15\delta^k_{13})z^i_5 + (5\delta^k_{53} + 9\delta^k_{24})z^i_6 + 5(\delta^k_{65} + 3\delta^k_{21})z^i_7 \\
w^k_{17i} &= (9\delta^k_{54} + 15\delta^k_{32})z^i_1 + (5\delta^k_{62} + 9\delta^k_{15})z^i_2 + 9(\delta^k_{36} + \delta^k_{41})z^i_3 + 5(\delta^k_{46} + 3\delta^k_{13})z^i_5 + (9\delta^k_{53} + 5\delta^k_{24})z^i_6 + (9\delta^k_{65} + 15\delta^k_{21})z^i_7 \\
w^k_{18i} &= 5(\delta^k_{54} + 3\delta^k_{32})z^i_1 + (5\delta^k_{15} + 9\delta^k_{62})z^i_2 + (9\delta^k_{36} + 5\delta^k_{41})z^i_3 + (9\delta^k_{46} + 15\delta^k_{13})z^i_5 + 9(\delta^k_{53} + \delta^k_{24})z^i_6 + (9\delta^k_{65} + 15\delta^k_{21})z^i_7 \\
w^k_{19i} &= (3\delta^k_{54} + 5\delta^k_{32})z^i_1 + 3(\delta^k_{62} + \delta^k_{15})z^i_2 + 3(\delta^k_{36} + \delta^k_{41})z^i_3 + (3\delta^k_{46} + 5\delta^k_{13})z^i_5 + 3(\delta^k_{53} + \delta^k_{24})z^i_6 + (3\delta^k_{65} + 5\delta^k_{21})z^i_7 \\
w^k_{20i} &= (9\delta^k_{74} + 15\delta^k_{62})z^i_1 + (9\delta^k_{17} + 15\delta^k_{32})z^i_2 + (5\delta^k_{36} + 9\delta^k_{41})z^i_4 + 15(\delta^k_{43} + \delta^k_{16})z^i_5 + 5(\delta^k_{73} + 3\delta^k_{21})z^i_6 + 5(\delta^k_{67} + 3\delta^k_{24})z^i_7 \\
w^k_{21i} &= 5(\delta^k_{74} + 3\delta^k_{62})z^i_1 + 5(\delta^k_{17} + 3\delta^k_{32})z^i_2 + (9\delta^k_{36} + 5\delta^k_{41})z^i_4 + 15(\delta^k_{43} + \delta^k_{16})z^i_5 + (9\delta^k_{73} + 15\delta^k_{21})z^i_6 + (9\delta^k_{67} + 15\delta^k_{24})z^i_7 \\
w^k_{22i} &= (3\delta^k_{74} + 5\delta^k_{62})z^i_1 + (3\delta^k_{17} + 5\delta^k_{32})z^i_2 + 3(\delta^k_{36} + \delta^k_{41})z^i_4 + 5(\delta^k_{43} + \delta^k_{16})z^i_5 + (3\delta^k_{73} + 5\delta^k_{21})z^i_6 + (3\delta^k_{67} + 5\delta^k_{24})z^i_7 \\
w^k_{23i} &= 5(\delta^k_{37} + \delta^k_{56})z^i_1 + (3\delta^k_{76} + 5\delta^k_{53})z^i_2 + 5(\delta^k_{34} + \delta^k_{61})z^i_3 + (3\delta^k_{64} + 5\delta^k_{13})z^i_4 + (3\delta^k_{47} + 5\delta^k_{15})z^i_6 + 5(\delta^k_{45} + \delta^k_{71})z^i_7 \\
w^k_{24i} &= 5(\delta^k_{64} + \delta^k_{72})z^i_1 + 5(\delta^k_{16} + \delta^k_{52})z^i_2 + (3\delta^k_{67} + 5\delta^k_{24})z^i_3 + (3\delta^k_{56} + 5\delta^k_{21})z^i_4 + 5(\delta^k_{17} + \delta^k_{45})z^i_5 + (3\delta^k_{7} + 5\delta^k_{41})z^i_6 \\
w^k_{25i} &= 15(\delta^k_{34} + \delta^k_{52})z^i_1 + (9\delta^k_{72} + 15\delta^k_{13})z^i_2 + 5(\delta^k_{37} + 3\delta^k_{21})z^i_3 + (5\delta^k_{53} + 9\delta^k_{24})z^i_4 + (9\delta^k_{47} + 15\delta^k_{15})z^i_5 + 5(\delta^k_{75} + 3\delta^k_{41})z^i_7 \\
w^k_{26i} &= 15(\delta^k_{34} + \delta^k_{52})z^i_1 + 5(\delta^k_{72} + 3\delta^k_{13})z^i_2 + (9\delta^k_{37} + 15\delta^k_{21})z^i_3 + (9\delta^k_{53} + 5\delta^k_{24})z^i_4 + 5(\delta^k_{47} + 3\delta^k_{15})z^i_5 + (9\delta^k_{75} + 15\delta^k_{41})z^i_7 \\
w^k_{27i} &= 5(\delta^k_{34} + \delta^k_{52})z^i_1 + (3\delta^k_{72} + 5\delta^k_{13})z^i_2 + (3\delta^k_{37} + 5\delta^k_{21})z^i_3 + 3(\delta^k_{53} + \delta^k_{24})z^i_4 + (3\delta^k_{47} + 5\delta^k_{15})z^i_5 + (3\delta^k + 5\delta^k_{41})z^i_7
\end{aligned} \tag{89}$$

and the geometric constants α_m^k, β_m^k, and δ_{mn}^k are given by

$$\begin{aligned} \alpha_m^2 &= a_m \\ \alpha_m^3 &= b_m \\ \alpha_m^4 &= c_m \\ \alpha_m^5 &= a_m \\ \alpha_m^6 &= b_m \\ \alpha_m^7 &= c_m \\ \alpha_m^8 &= a_m \\ \alpha_m^9 &= b_m \\ \alpha_m^{10} &= c_m \end{aligned} \qquad \begin{aligned} \beta_m^5 &= a_m \\ \beta_m^6 &= b_m \\ \beta_m^7 &= c_m \\ \beta_m^8 &= a_m \\ \beta_m^9 &= b_m \\ \beta_m^{10} &= c_m \end{aligned} \qquad \begin{aligned} \delta_{mn}^1 &= b_n c_m - b_m c_n \\ \delta_{mn}^2 &= c_n a_m - c_m a_n \\ \delta_{mn}^3 &= a_n b_m - a_m b_n \end{aligned} \tag{90}$$

The values for z_j^i for i = 1,2,3,...,8 are given by

i	z_1^i	z_2^i	z_3^i	z_4^i	z_5^i	z_6^i	z_7^i
1	-1	1	1	-1	-1	1	-1
2	1	-1	-1	1	-1	1	-1
3	1	1	-1	-1	1	-1	-1
4	-1	-1	1	1	1	-1	-1
5	-1	1	-1	1	-1	-1	1
6	1	-1	1	-1	-1	-1	1
7	1	1	1	1	1	1	1
8	-1	-1	-1	-1	1	1	1

(91)

The stiffness coefficients due to compatible displacements are then given explicitly as

$$K_{q_{ij}} = g_{ki} g_{kj} \quad ; \quad i = 1,2,3,...,24; \quad j = 1,2,3,...,i; \quad k = 1,2,3,...,6N_e \tag{92}$$

In computing the stiffness contributions of the incompatible displacements, the selected nonconforming displacement modes are identical to those presented in [12] and are given by.

$$\left\{ \begin{matrix} u \\ v \\ w \end{matrix} \right\}_\lambda = \begin{bmatrix} M_1 & M_2 & M_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_1 & M_2 & M_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_1 & M_2 & M_3 \end{bmatrix} \left\{ \begin{matrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_9 \end{matrix} \right\} \tag{93}$$

where

$$\begin{aligned} M_1 &= \xi^2 - f_1\xi - f_2\eta - f_3\zeta \\ M_2 &= \eta^2 - f_4\xi - f_5\eta - f_6\zeta \\ M_3 &= \zeta^2 - f_7\xi - f_8\eta - f_9\zeta \end{aligned} \tag{94}$$

The derivation of the incompatible modes and all constants are contained in Appendix III.

The components of equation (31), r_{ij}, where $i = 1,2,3,...,N_e$ and $j = 1,2,3$ are given by

$$\begin{aligned} r_{i,j} &= d_{11}\Theta_{ij}^1 & r_{i+2N_e,j} &= d_{31}\Theta_{ij}^1 & r_{i+4N_e,j} &= d_{55}\Theta_{ij}^3 \\ r_{i,j+3} &= d_{12}\Theta_{ij}^2 & r_{i+2N_e,j+3} &= d_{32}\Theta_{ij}^2 & r_{i+4N_e,j+3} &= 0 \\ r_{i,j+6} &= d_{13}\Theta_{ij}^3 & r_{i+2N_e,j+6} &= d_{33}\Theta_{ij}^3 & r_{i+4N_e,j+6} &= d_{55}\Theta_{ij}^1 \\ r_{i+N_e,j} &= d_{21}\Theta_{ij}^1 & r_{i+3N_e,j} &= 0 & r_{i+5N_e,j} &= d_{66}\Theta_{ij}^2 \\ r_{i+N_e,j+3} &= d_{22}\Theta_{ij}^2 & r_{i+3N_e,j+3} &= d_{44}\Theta_{ij}^3 & r_{i+5N_e,j+3} &= d_{66}\Theta_{ij}^1 \\ r_{i+N_e,j+6} &= d_{23}\Theta_{ij}^3 & r_{i+3N_e,j+6} &= d_{44}\Theta_{ij}^2 & r_{i+5N_e,j+6} &= 0 \end{aligned} \tag{95}$$

where

$$\Theta_{ij}^k = \sum_{s=1}^{i} p_{is}^* \phi_{sj}^k \tag{96}$$

For $i = 1$

$$\begin{aligned}
\phi^k_{i1} &= 8[f_1(\delta^k_{54} + 3\delta^k_{32}) + f_2(\delta^k_{46} + 3\delta^k_{13}) + f_3(\delta^k_{65} + 3\delta^k_{21}) + 2(\delta^k_{25} + \delta^k_{43})]/3 \\
\phi^k_{i2} &= 8[f_4(\delta^k_{54} + 3\delta^k_{32}) + f_5(\delta^k_{46} + 3\delta^k_{13}) + f_6(\delta^k_{65} + 3\delta^k_{21}) + 2(\delta^k_{61} + \delta^k_{34})]/3 \\
\phi^k_{i3} &= 8[f_7(\delta^k_{54} + 3\delta^k_{32}) + f_8(\delta^k_{46} + 3\delta^k_{13}) + f_9(\delta^k_{65} + 3\delta^k_{21}) + 2(\delta^k_{52} + \delta^k_{16})]/3
\end{aligned} \tag{97}$$

For $i = 2, 3, 4$

$$\begin{aligned}
\phi^k_{i1} &= [16w^k_{i1} + 40(w^k_{i4}f_1 + w^k_{i5}f_2 + w^k_{i6}f_3)]/45 \\
\phi^k_{i2} &= [16w^k_{i2} + 40(w^k_{i4}f_4 + w^k_{i5}f_5 + w^k_{i6}f_6)]/45 \\
\phi^k_{i3} &= [16w^k_{i3} + 40(w^k_{i4}f_7 + w^k_{i5}f_8 + w^k_{i6}f_9)]/45
\end{aligned} \tag{98}$$

For $i = 5, 6, 7, 8, 9, 10$

$$\begin{aligned}
\phi^k_{i1} &= -[\psi^k_{i1} + f_1\psi^k_{i4} + f_2\psi^k_{i5} - f_3\psi^k_{i6}]/135 \\
\phi^k_{i2} &= -[\psi^k_{i2} + f_4\psi^k_{i4} + f_5\psi^k_{i6} - f_6\psi^k_{i6}]/135 \\
\phi^k_{i3} &= -[\psi^k_{i3} + f_4\psi^k_{i4} + f_5\psi^k_{i5} - f_6\psi^k_{i6}]/135
\end{aligned} \tag{99}$$

The constants ψ^k_{ij} and w^k_{ij} are given by

$$\begin{aligned}
\psi^k_{i1} &= 48\alpha^i_1 w^k_{i7} + 16(\alpha^i_2 w^k_{i10} - \alpha^i_3 w^k_{i11} + \alpha^i_4 w^k_{i12} - \alpha^i_5 w^k_{i14} - \alpha^i_6 w^k_{i16} - \alpha^i_7 w^k_{i17}) \\
\psi^k_{i2} &= 48\alpha^i_2 w^k_{i8} + 16(\alpha^i_7 w^k_{i24} - \alpha^i_6 w^k_{i21} - \alpha^i_5 w^k_{i27} + \alpha^i_4 w^k_{i19} - \alpha^i_3 w^k_{i26} + \alpha^i_1 w^k_{i23}) \\
\psi^k_{i3} &= 48\alpha^i_3 w^k_{i9} + 16(\alpha^i_5 w^k_{i28} - \alpha^i_6 w^k_{i30} + \alpha^i_7 w^k_{i32} + \alpha^i_1 w^k_{i34} - \alpha^i_2 w^k_{i35} - \alpha^i_4 w^k_{i36}) \\
\psi^k_{i4} &= 24\alpha^i_1 w^k_{i1} + \ 8(\alpha^i_7 w^k_{i16} - \alpha^i_4 w^k_{i10} + \alpha^i_5 w^k_{i11}) + 40(\alpha^i_6 w^k_{i18} + \alpha^i_3 w^k_{i15} - \alpha^i_2 w^k_{i13}) \\
\psi^k_{i5} &= 24\alpha^i_2 w^k_{i2} - \ 8(\alpha^i_7 w^k_{i27} - \alpha^i_4 w^k_{i23} + \alpha^i_6 w^k_{i26}) - 40(\alpha^i_3 w^k_{i22} + \alpha^i_5 w^k_{i25} - \alpha^i_1 w^k_{i20}) \\
\psi^k_{i6} &= 24\alpha^i_3 w^k_{i3} + \ 8(\alpha^i_5 w^k_{i34} - \alpha^i_6 w^k_{i35} - \alpha^i_7 w^k_{i36}) + 40(\alpha^i_1 w^k_{i29} - \alpha^i_2 w^k_{i31} - \alpha^i_4 w^k_{i33})
\end{aligned} \tag{100}$$

$$\begin{aligned}
w^k_{i1} &= \beta^i_2(9\delta^k_{45} + 15\delta^k_{23}) + 5\beta^i_3(\delta^k_{27} + \delta^k_{46}) + 5\beta^i_4(\delta^k_{73} + \delta^k_{65}) + \beta^i_5(3\delta^k_{47} + 5\delta^k_{26}) + \beta^i_6(3\delta^k_{75} + 5\delta^k_{63}) \\
w^k_{i2} &= 5\beta^i_2(\delta^k_{71} + \delta^k_{54}) + \beta^i_3(9\delta^k_{64} + 15\delta^k_{31}) + 5\beta^i_4(\delta^k_{37} + \delta^k_{65}) + \beta^i_5(3\delta^k_{74} + 5\delta^k_{51}) + \beta^i_7(3\delta^k_{67} + 5\delta^k_{35}) \\
w^k_{i3} &= 5\beta^i_2(\delta^k_{54} + \delta^k_{17}) + 5\beta^i_3(\delta^k_{72} + \delta^k_{46}) + \beta^i_4(9\delta^k_{56} + 15\delta^k_{12}) + \beta^i_6(3\delta^k_{57} + 5\delta^k_{14}) + \beta^i_7(3\delta^k_{76} + 5\delta^k_{42}) \\
w^k_{i4} &= 3\beta^i_2(\delta^k_{34} + \delta^k_{52}) + \beta^i_3(\delta^k_{74} + 3\delta^k_{62}) + \beta^i_4(\delta^k_{57} + 3\delta^k_{36}) + \beta^i_5(\delta^k_{72} + \delta^k_{64}) + \beta^i_6(\delta^k_{37} + \delta^k_{56}) \\
w^k_{i5} &= \beta^i_2(\delta^k_{47} + 3\delta^k_{15}) + 3\beta^i_3(\delta^k_{43} + \delta^k_{16}) + \beta^i_4(\delta^k_{76} + 3\delta^k_{53}) + \beta^i_5(\delta^k_{45} + \delta^k_{17}) + \beta^i_7(\delta^k_{73} + \delta^k_{56}) \\
w^k_{i6} &= \beta^i_2(\delta^k_{75} + 3\delta^k_{41}) + \beta^i_3(\delta^k_{67} + 3\delta^k_{24}) + 3\beta^i_4(\delta^k_{61} + \delta^k_{25}) + \beta^i_6(\delta^k_{45} + \delta^k_{71}) + \beta^i_7(\delta^k_{27} + \delta^k_{64}) \\
w^k_{i7} &= 3\beta^i_6(\delta^k_{37} + \delta^k_{56}) + 3\beta^i_5(\delta^k_{72} + \delta^k_{64}) + \beta^i_4(3\delta^k_{57} + 5\delta^k_{36}) + \beta^i_3(3\delta^k_{74} + 5\delta^k_{62}) + 9\beta^i_2(\delta^k_{34} + \delta^k_{52}) \\
w^k_{i8} &= 3\beta^i_7(\delta^k_{37} + \delta^k_{65}) + 3\beta^i_5(\delta^k_{54} + \delta^k_{71}) + \beta^i_4(3\delta^k_{67} + 5\delta^k_{35}) + 9\beta^i_3(\delta^k_{34} + \delta^k_{61}) + \beta^i_2(5\delta^k_{51} + 3\delta^k_{74}) \\
w^k_{i9} &= 3\beta^i_7(\delta^k_{27} + \delta^k_{64}) + 3\beta^i_6(\delta^k_{45} + \delta^k_{71}) + 9\beta^i_4(\delta^k_{25} + \delta^k_{61}) + \beta^i_3(3\delta^k_{67} + 5\delta^k_{24}) + \beta^i_2(3\delta^k_{75} + 5\delta^k_{41}) \\
w^k_{i10} &= \beta^i_8(3\delta^k_{57} + 5\delta^k_{36}) + 3\beta^i_2(3\delta^k_{74} + 5\delta^k_{62}) + 5\beta^i_7(\delta^k_{37} + \delta^k_{56}) + 3\beta^i_5(3\delta^k_{54} + 5\delta^k_{32}) + 15\beta^i_3(\delta^k_{52} + \delta^k_{34}) \\
w^k_{i11} &= \beta^i_8(3\delta^k_{47} + 5\delta^k_{26}) + 3\beta^i_2(3\delta^k_{75} + 5\delta^k_{63}) + 5\beta^i_7(\delta^k_{27} + \delta^k_{46}) + 3\beta^i_6(3\delta^k_{45} + 5\delta^k_{23}) + 15\beta^i_4(\delta^k_{25} + \delta^k_{43}) \\
w^k_{i12} &= 3\beta^i_8(\delta^k_{37} + \delta^k_{56}) + \beta^i_7(3\delta^k_{57} + 5\delta^k_{36}) + 9\beta^i_2(\delta^k_{72} + \delta^k_{64}) + 3\beta^i_3(3\delta^k_{54} + 5\delta^k_{32}) + 9\beta^i_5(\delta^k_{52} + \delta^k_{34}) \\
w^k_{i13} &= \beta^i_8(\delta^k_{37} + \delta^k_{56}) + \beta^i_7(\delta^k_{57} + 3\delta^k_{36}) + 3\beta^i_2(\delta^k_{72} + \delta^k_{64}) + 3\beta^i_3(\delta^k_{54} + 3\delta^k_{32}) + 3\beta^i_5(\delta^k_{34} + \delta^k_{52}) \\
w^k_{i14} &= 3\beta^i_8(\delta^k_{27} + \delta^k_{46}) + \beta^i_7(3\delta^k_{47} + 5\delta^k_{26}) + 9\beta^i_2(\delta^k_{73} + \delta^k_{65}) + 9\beta^i_6(\delta^k_{25} + \delta^k_{43}) + 3\beta^i_4(3\delta^k_{45} + 5\delta^k_{23}) \\
w^k_{i15} &= \beta^i_8(\delta^k_{27} + \delta^k_{46}) + \beta^i_7(\delta^k_{47} + 3\delta^k_{26}) + 3\beta^i_2(\delta^k_{73} + \delta^k_{65}) + 3\beta^i_6(\delta^k_{25} + \delta^k_{43}) + 3\beta^i_4(\delta^k_{45} + 3\delta^k_{23}) \\
w^k_{i16} &= \beta^i_8(3\delta^k_{45} + 5\delta^k_{23}) + \beta^i_5(3\delta^k_{75} + 5\delta^k_{63}) + \beta^i_6(3\delta^k_{47} + 5\delta^k_{26}) + 5\beta^i_3(\delta^k_{65} + \delta^k_{73}) + 5\beta^i_4(\delta^k_{27} + \delta^k_{46}) + 5\beta^i_7(\delta^k_{25} + \delta^k_{43}) \\
w^k_{i17} &= 3\beta^i_8(\delta^k_{25} + \delta^k_{43}) + \beta^i_3(3\delta^k_{75} + 5\delta^k_{63}) + \beta^i_4(3\delta^k_{47} + 5\delta^k_{26}) + 3\beta^i_5(\delta^k_{73} + \delta^k_{65}) + 3\beta^i_6(\delta^k_{27} + \delta^k_{46}) + \beta^i_7(3\delta^k_{45} + 5\delta^k_{23}) \\
w^k_{i18} &= \beta^i_8(\delta^k_{25} + \delta^k_{43}) + \beta^i_3(\delta^k_{75} + 3\delta^k_{63}) + \beta^i_4(\delta^k_{47} + 3\delta^k_{26}) + \beta^i_5(\delta^k_{65} + \delta^k_{73}) + \beta^i_6(\delta^k_{27} + \delta^k_{46}) + \beta^i_7(\delta^k_{45} + 3\delta^k_{23}) \\
w^k_{i19} &= 3\beta^i_8(\delta^k_{37} + \delta^k_{65}) + \beta^i_6(3\delta^k_{67} + 5\delta^k_{35}) + 9\beta^i_3(\delta^k_{71} + \delta^k_{54}) + 3\beta^i_2(3\delta^k_{64} + 5\delta^k_{31}) + 9\beta^i_5(\delta^k_{34} + \delta^k_{61}) \\
w^k_{i20} &= \beta^i_8(\delta^k_{37} + \delta^k_{65}) + \beta^i_6(\delta^k_{67} + 3\delta^k_{35}) + 3\beta^i_3(\delta^k_{54} + \delta^k_{71}) + 3\beta^i_2(\delta^k_{64} + 3\delta^k_{31}) + 3\beta^i_5(\delta^k_{34} + \delta^k_{61}) \\
w^k_{i21} &= 3\beta^i_8(\delta^k_{17} + \delta^k_{45}) + \beta^i_6(3\delta^k_{47} + 5\delta^k_{15}) + 9\beta^i_3(\delta^k_{56} + \delta^k_{73}) + 9\beta^i_7(\delta^k_{16} + \delta^k_{43}) + 3\beta^i_4(3\delta^k_{46} + 5\delta^k_{13}) \\
w^k_{i22} &= \beta^i_8(\delta^k_{17} + \delta^k_{45}) + \beta^i_6(\delta^k_{47} + 3\delta^k_{15}) + 3\beta^i_3(\delta^k_{73} + \delta^k_{56}) + 3\beta^i_7(\delta^k_{16} + \delta^k_{43}) + 3\beta^i_4(\delta^k_{46} + 3\delta^k_{13}) \\
w^k_{i23} &= \beta^i_8(3\delta^k_{67} + 5\delta^k_{35}) + 3\beta^i_3(3\delta^k_{74} + 5\delta^k_{51}) + 5\beta^i_6(\delta^k_{37} + \delta^k_{65}) + 3\beta^i_5(5\delta^k_{31} + 3\delta^k_{64}) + 15\beta^i_2(\delta^k_{34} + \delta^k_{61}) \\
w^k_{i24} &= 3\beta^i_8(\delta^k_{61} + \delta^k_{34}) + \beta^i_2(3\delta^k_{67} + 5\delta^k_{35}) + \beta^i_4(5\delta^k_{51} + 3\delta^k_{74}) + 3\beta^i_5(\delta^k_{37} + \delta^k_{65}) + 3\beta^i_7(\delta^k_{71} + \delta^k_{54}) + \beta^i_6(3\delta^k_{64} + 5\delta^k_{31}) \\
w^k_{i25} &= \beta^i_8(\delta^k_{16} + \delta^k_{43}) + \beta^i_2(\delta^k_{76} + 3\delta^k_{53}) + \beta^i_4(\delta^k_{47} + 3\delta^k_{15}) + \beta^i_5(\delta^k_{73} + \delta^k_{56}) + \beta^i_7(\delta^k_{17} + \delta^k_{45}) + \beta^i_6(\delta^k_{46} + 3\delta^k_{13})
\end{aligned} \tag{101}$$

$$
\begin{aligned}
w_{i26}^{k} &= \beta_8^i(3\delta_{47}^k + 5\delta_{15}^k) + 3\beta_3^i(3\delta_{76}^k + 5\delta_{53}^k) + 5\beta_6^i(\delta_{17}^k + \delta_{45}^k) + 3\beta_7^i(3\delta_{46}^k + 5\delta_{13}^k) + 15\beta_4^i(\delta_{16}^k + \delta_{43}^k) \\
w_{i27}^{k} &= \beta_8^i(3\delta_{46}^k + 5\delta_{13}^k) + \beta_5^i(3\delta_{76}^k + 5\delta_{53}^k) + \beta_7^i(3\delta_{47}^k + 5\delta_{15}^k) + 5\beta_2^i(\delta_{56}^k + \delta_{73}^k) + 5\beta_4^i(\delta_{17}^k + \delta_{45}^k) + 5\beta_6^i(\delta_{16}^k + \delta_{43}^k) \\
w_{i28}^{k} &= 3\beta_8^i(\delta_{27}^k + \delta_{64}^k) + \beta_5^i(3\delta_{67}^k + 5\delta_{24}^k) + 9\beta_4^i(\delta_{45}^k + \delta_{71}^k) + 3\beta_2^i(3\delta_{65}^k + 5\delta_{21}^k) + 9\beta_6^i(\delta_{25}^k + \delta_{61}^k) \\
w_{i29}^{k} &= \beta_8^i(\delta_{27}^k + \delta_{64}^k) + \beta_5^i(\delta_{67}^k + 3\delta_{24}^k) + 3\beta_4^i(\delta_{45}^k + \delta_{71}^k) + 3\beta_2^i(\delta_{65}^k + 3\delta_{21}^k) + 3\beta_6^i(\delta_{25}^k + \delta_{61}^k) \\
w_{i30}^{k} &= 3\beta_8^i(\delta_{17}^k + \delta_{54}^k) + \beta_5^i(3\delta_{57}^k + 5\delta_{14}^k) + 9\beta_4^i(\delta_{46}^k + \delta_{72}^k) + 9\beta_7^i(\delta_{16}^k + \delta_{52}^k) + 3\beta_3^i(3\delta_{56}^k + 5\delta_{12}^k) \\
w_{i31}^{k} &= \beta_8^i(\delta_{17}^k + \delta_{54}^k) + \beta_5^i(\delta_{57}^k + 3\delta_{14}^k) + 3\beta_4^i(\delta_{46}^k + \delta_{72}^k) + 3\beta_7^i(\delta_{16}^k + \delta_{52}^k) + 3\beta_3^i(\delta_{56}^k + 3\delta_{12}^k) \\
w_{i32}^{k} &= 3\beta_8^i(\delta_{16}^k + \delta_{52}^k) + \beta_2^i(5\delta_{42}^k + 3\delta_{76}^k) + \beta_3^i(3\delta_{57}^k + 5\delta_{14}^k) + 3\beta_6^i(\delta_{46}^k + \delta_{72}^k) + 3\beta_7^i(\delta_{17}^k + \delta_{54}^k) + \beta_5^i(3\delta_{56}^k + 5\delta_{12}^k) \\
w_{i33}^{k} &= \beta_8^i(\delta_{16}^k + \delta_{52}^k) + \beta_2^i(\delta_{76}^k + 3\delta_{42}^k) + \beta_3^i(\delta_{57}^k + 3\delta_{14}^k) + \beta_6^i(\delta_{72}^k + \delta_{46}^k) + \beta_7^i(\delta_{17}^k + \delta_{54}^k) + \beta_5^i(\delta_{56}^k + 3\delta_{12}^k) \\
w_{i34}^{k} &= \beta_8^i(3\delta_{67}^k + 5\delta_{24}^k) + 3\beta_4^i(5\delta_{41}^k + 3\delta_{75}^k) + 5\beta_5^i(\delta_{27}^k + \delta_{64}^k) + 3\beta_6^i(3\delta_{65}^k + 5\delta_{21}^k) + 15\beta_2^i(\delta_{61}^k + \delta_{25}^k) \\
w_{i35}^{k} &= \beta_8^i(3\delta_{57}^k + 5\delta_{14}^k) + 3\beta_4^i(3\delta_{76}^k + 5\delta_{42}^k) + 5\beta_5^i(\delta_{17}^k + \delta_{54}^k) + 3\beta_7^i(3\delta_{56}^k + 5\delta_{12}^k) + 15\beta_3^i(\delta_{16}^k + \delta_{52}^k) \\
w_{i36}^{k} &= \beta_8^i(3\delta_{56}^k + 5\delta_{12}^k) + \beta_6^i(3\delta_{76}^k + 5\delta_{42}^k) + \beta_7^i(3\delta_{57}^k + 5\delta_{14}^k) + 5\beta_2^i(\delta_{46}^k + \delta_{72}^k) + 5\beta_3^i(\delta_{17}^k + \delta_{54}^k) + 5\beta_5^i(\delta_{16}^k + \delta_{52}^k)
\end{aligned}
$$

and the geometric constants α_m^k, β_m^k, and δ_{mn}^k are given in (90).

The computation of the inner product given in (32) results in a 9×9 Z matrix which must be inverted to give the coefficients a_{ij}. Although Z is symmetric and specific explicit inversion schemes can be applied to minimize computations, this matrix order is perhaps at the limit in which closed-form expressions for the inverse may be succintly expressed and a numerical scheme may be preferred. The stiffness contributions due to the incompatible contributions, K_λ, are given by

$$K_{\lambda_{ij}} = -g_{ni} r_{ns} a_{sm} r_{km} g_{kj}$$

where

$$i = 1,2,3,...,6N_e \ ; \quad j = 1,2,3,...,i \ ; \quad n,k = 1,2,3,...,24 \ ; \quad s,m = 1,2,3,...9$$

The complete element stiffness matrix is therefore given by the sum

$$K = K_q + K_\lambda$$

The E8NL Element

The E8NL element is formulated using stress expansions defined in natural coordinates. Incompatible modes are introduced to complete the quadratic bending terms and enforce orthogonality between the stress and incompatible strain fields variationally. Two versions are presented to demonstrate the effect of different stress expansions on element behavior. E8NL$_1$ is based on incomplete quadratic expansions for the inplane stress components with complete linear expansions for the shear stresses which is identical to the FE1 element presented in Reference [10]. The selected incompatible displacement modes used in E8NL are identical to those used in the E8PL element.

The assumed stress field is given by

$$\Gamma = \begin{bmatrix} [\Gamma_1] & & & & & \\ & [\Gamma_1] & & & & \\ & & [\Gamma_1] & & & \\ & & & [\Gamma_2] & & \\ & & & & [\Gamma_2] & \\ & & & & & [\Gamma_2] \end{bmatrix} \tag{102}$$

where

$$\begin{aligned} \Gamma_1 &= [1, \xi, \eta, \zeta, \xi\eta, \eta\zeta, \zeta\xi] \\ \Gamma_2 &= [1, \xi, \eta, \zeta] \end{aligned} \tag{103}$$

The natural stresses are transformed to physical coordinates using a contravariant transformation based on centroidal Jacobians and premultiplied by the distributing matrix to yield the transformed stress field as

$$\bar{P} = D^{-1} T_\sigma \Gamma \tag{104}$$

Without attempting a simplification of the modes in (104) through linear combinations of the unknown expansion coefficients, the orthonormalizing process yields stress modes of the general form

$$P_i^* = \left\{ \begin{array}{l} p_{i1}^* + p_{i2}^*\xi + p_{i3}^*\eta + p_{i4}^*\zeta + p_{i5}^*\xi\eta + p_{i6}^*\eta\zeta + p_{i7}^*\zeta\xi \\ p_{i8}^* + p_{i9}^*\xi + p_{i10}^*\eta + p_{i11}^*\zeta + p_{i12}^*\xi\eta + p_{i13}^*\eta\zeta + p_{i14}^*\zeta\xi \\ p_{i15}^* + p_{i16}^*\xi + p_{i17}^*\eta + p_{i18}^*\zeta + p_{i19}^*\xi\eta + p_{i20}^*\eta\zeta + p_{i21}^*\zeta\xi \\ p_{i22}^* + p_{i23}^*\xi + p_{i24}^*\eta + p_{i25}^*\zeta \\ p_{i26}^* + p_{i27}^*\xi + p_{i28}^*\eta + p_{i29}^*\zeta \\ p_{i30}^* + p_{i31}^*\xi + p_{i32}^*\eta + p_{i33}^*\zeta \end{array} \right\} \tag{105}$$

where the coefficients are obtained numerically. The components g_{mn} with $m = 1,2,3,...,33$ and $n = 1,2,3,...,8$ result in the following linear algebraic expressions

$$\begin{aligned} g_{(m)(3n-2)} &= \sum_{k=1}^{7}(d_{11}p^*_{(m,k)} + d_{21}p^*_{(m,k+7)} + d_{31}p^*_{(m,k+14)})\Theta^1_{nk} + \\ &\quad \sum_{k=1}^{4}(d_{55}p^*_{(m,k+25)}\Theta^3_{nk} + d_{66}p^*_{(m,k+29)}\Theta^2_{nk}) \\ g_{(m)(3n-1)} &= \sum_{k=1}^{7}(d_{12}p^*_{(m,k)} + d_{22}p^*_{(m,k+7)} + d_{32}p^*_{(m,k+14)})\Theta^2_{nk} + \\ &\quad \sum_{k=1}^{4}(d_{44}p^*_{(m,k+21)}\Theta^3_{nk} + d_{66}p^*_{(m,k+29)}\Theta^1_{nk}) \\ g_{(m)(3n)} &= \sum_{k=1}^{7}(d_{13}p^*_{(m,k)} + d_{23}p^*_{(m,k+7)} + d_{33}p^*_{(m,k+14)})\Theta^3_{nk} + \\ &\quad \sum_{k=1}^{4}(d_{44}p^*_{(m,k+21)}\Theta^2_{nk} + d_{55}p^*_{(m,k+25)}\Theta^1_{nk}) \end{aligned} \tag{106}$$

where

$$\begin{array}{lll} \Theta^i_{n1} = -(3\phi^i_{n1} + \phi^i_{n8} + \phi^i_{n9} + \phi^i_{n10})/3 & \quad & \Theta^i_{n5} = -(3\phi^i_{n5} + \phi^i_{n20})/27 \\ \Theta^i_{n2} = -(3\phi^i_{n2} + \phi^i_{n13} + \phi^i_{n15})/9 & & \Theta^i_{n6} = -(3\phi^i_{n6} + \phi^i_{n18})/27 \\ \Theta^i_{n3} = -(3\phi^i_{n3} + \phi^i_{n11} + \phi^i_{n16})/9 & & \Theta^i_{n7} = -(3\phi^i_{n7} + \phi^i_{n19})/27 \\ \Theta^i_{n4} = -(3\phi^i_{n4} + \phi^i_{n12} + \phi^i_{n14})/9 & & \end{array} \tag{107}$$

and

$$\begin{array}{ll} \phi^i_{n1} = z^n_1\delta^i_{32} - z^n_5\delta^i_{13} - \delta^i_{12}z^n_7 & \phi^i_{n10} = z^n_3\delta^i_{36} + z^n_6\delta^i_{53} - z^n_7\delta^i_{56} \\ \phi^i_{n2} = z^n_1(\delta^i_{34} - \delta^i_{52}) + z^n_2\delta^i_{13} - z^n_3\delta^i_{12} + z^n_5\delta^i_{15} - z^n_7\delta^i_{14} & \phi^i_{n11} = z^n_1\delta^i_{74} + z^n_2\delta^i_{17} - z^n_4\delta^i_{14} \\ \phi^i_{n3} = z^n_1\delta^i_{62} + z^n_2\delta^i_{32} + z^n_5(\delta^i_{43} + \delta^i_{16}) - z^n_6\delta^i_{12} - z^n_7\delta^i_{42} & \phi^i_{n12} = z^n_1\delta^i_{57} - z^n_3\delta^i_{17} + z^n_4\delta^i_{15} \\ \phi^i_{n4} = z^n_1\delta^i_{36} + z^n_3\delta^i_{32} + z^n_5\delta^i_{53} + z^n_6\delta^i_{13} - z^n_7(\delta^i_{16} + \delta^i_{52}) & \phi^i_{n13} = z^n_2\delta^i_{72} - z^n_4\delta^i_{42} + z^n_5\delta^i_{47} \\ \phi^i_{n5} = z^n_1(\delta^i_{64} + \delta^i_{72}) + z^n_2(\delta^i_{16} + \delta^i_{52}) + z^n_5(\delta^i_{45} + \delta^i_{17}) - & \phi^i_{n14} = z^n_4\delta^i_{62} + z^n_5\delta^i_{76} - z^n_6\delta^i_{72} \\ \qquad z^n_3\delta^i_{42} - z^n_4\delta^i_{12} - z^n_6\delta^i_{14} & \phi^i_{n15} = z^n_3\delta^i_{37} - z^n_7\delta^i_{57} + z^n_4\delta^i_{53} \\ \phi^i_{n6} = z^n_6(\delta^i_{43} - \delta^i_{52}) - z^n_7(\delta^i_{46} + \delta^i_{72}) + z^n_5(\delta^i_{73} + \delta^i_{56}) + & \phi^i_{n16} = z^n_4\delta^i_{36} + z^n_6\delta^i_{73} - z^n_7\delta^i_{76} \\ \qquad z^n_2\delta^i_{36} + z^n_3\delta^i_{62} + z^n_4\delta^i_{32} & \phi^i_{n17} = 2(z^n_2\delta^i_{56} + z^n_3\delta^i_{64} + z^n_6\delta^i_{45}) \\ \phi^i_{n7} = z^n_1(\delta^i_{37} + \delta^i_{56}) + z^n_3(\delta^i_{34} - \delta^i_{16}) - z^n_7(\delta^i_{17} + \delta^i_{54}) + & \phi^i_{n18} = z^n_2\delta^i_{57} + z^n_3\delta^i_{74} + z^n_4\delta^i_{45} + \\ \qquad z^n_2\delta^i_{53}z^n_4\delta^i_{13} + z^n_6\delta^i_{15} & \phi^i_{n19} = z^n_2\delta^i_{76} + z^n_4\delta^i_{64} + z^n_6\delta^i_{47} \\ \phi^i_{n8} = z^n_1\delta^i_{54} + z^n_2\delta^i_{15} - z^n_3\delta^i_{14} & \phi^i_{n20} = -z^n_3\delta^i_{76} + z^n_4\delta^i_{56} - z^n_6\delta^i_{57} \\ \phi^i_{n9} = z^n_2\delta^i_{62} - z^n_5\delta^i_{46} - z^n_6\delta^i_{42} & \end{array} \tag{108}$$

The values for z_j^n are given in (91).

The components r_{mn}, with $m = 1,2,3,...,33$ and $n = 1,2,3$, are given by

$$\begin{aligned} r_{m,n} &= d_{11}\Theta^1_{mn1} + d_{21}\Theta^1_{mn2} + d_{31}\Theta^1_{mn3} + d_{55}\Phi^3_{mn2} + d_{66}\Phi^2_{mn3} \\ r_{m,n+3} &= d_{12}\Theta^2_{mn1} + d_{22}\Theta^2_{mn2} + d_{32}\Theta^2_{mn3} + d_{44}\Phi^3_{mn1} + d_{66}\Phi^1_{mn3} \\ r_{m,n+6} &= d_{13}\Theta^3_{mn1} + d_{23}\Theta^3_{mn2} + d_{33}\Theta^3_{mn3} + d_{44}\Phi^2_{mn1} + d_{55}\Phi^1_{mn2} \end{aligned} \tag{109}$$

where the following constants are defined

$$\begin{aligned} \Theta^i_{m1k} &= -\psi^i_{m1k} + f_1\psi^i_{m4k} + f_2\psi^i_{m5k} + f_3\psi^i_{m6k} & \Phi^i_{m1k} &= -\psi^i_{m7k} + f_1\psi^i_{m10k} + f_2\psi^i_{m11k} + f_3\psi^i_{m12k} \\ \Theta^i_{m2k} &= -\psi^i_{m2k} + f_4\psi^i_{m4k} + f_5\psi^i_{m5k} + f_6\psi^i_{m6k} & \Phi^i_{m2k} &= -\psi^i_{m8k} + f_4\psi^i_{m10k} + f_5\psi^i_{m11k} + f_6\psi^i_{m12k} \\ \Theta^i_{m3k} &= \psi^i_{m3k} + f_7\psi^i_{m4k} + f_8\psi^i_{m5k} + f_9\psi^i_{m6k} & \Phi^i_{m3k} &= \psi^i_{m9k} + f_7\psi^i_{m10k} + f_8\psi^i_{m11k} + f_9\psi^i_{m12k} \end{aligned} \tag{110}$$

$$\begin{aligned} \psi^i_{m1k} &= 3p_{(m,7k-6)}\lambda^i_4 + p_{(m,7k-2)}\lambda^i_{22} + p_{(m,7k-4)}\lambda^i_{14} + p_{(m,7k)}\lambda^i_{26} + p_{(m,7k-3)}\lambda^i_{19} + p_{(m,7k-5)}\lambda^i_9 \\ \psi^i_{m2k} &= 3p_{(m,7k-6)}\lambda^i_5 + p_{(m,7k-2)}\lambda^i_{23} + p_{(m,7k-4)}\lambda^i_{15} + p_{(m,7k-1)}\lambda^i_{24} + p_{(m,7k-3)}\lambda^i_{20} + p_{(m,7k-5)}\lambda^i_{10} \\ \psi^i_{m3k} &= 3p_{(m,7k-6)}\lambda^i_6 + p_{(m,7k-4)}\lambda^i_{16} + p_{(m,7k)}\lambda^i_{27} + p_{(m,7k-1)}\lambda^i_{25} + p_{(m,7k-3)}\lambda^i_{21} + p_{(m,7k-5)}\lambda^i_{11} \\ \psi^i_{m4k} &= 3p_{(m,7k-5)}\lambda^i_4 + p_{(m,7k-3)}\lambda^i_{17} + p_{(m,7k-4)}\lambda^i_{13} + 2p_{(m,7k-2)}\lambda^i_{14} + 2p_{(m,7k)}\lambda^i_{19} + 3p_{(m,7k-6)}\lambda^i_3 \\ \psi^i_{m5k} &= 3p_{(m,7k-6)}\lambda^i_2 + p_{(m,7k-5)}\lambda^i_8 + p_{(m,7k-3)}\lambda^i_{18} + 2p_{(m,7k-1)}\lambda^i_{20} + 2p_{(m,7k-2)}\lambda^i_{10} + 3p_{(m,7k-4)}\lambda^i_5 \\ \psi^i_{m6k} &= 3p_{(m,7k-6)}\lambda^i_1 + p_{(m,7k-5)}\lambda^i_7 + p_{(m,7k-4)}\lambda^i_{12} + 2p_{(m,7k)}\lambda^i_{11} + 3p_{(m,7k-3)}\lambda^i_6 + 2p_{(m,7k-1)}\lambda^i_{16} \\ \psi^i_{m7k} &= 3p_{(m,4k+18)}\lambda^i_4 + p_{(m,4k+20)}\lambda^i_{14} + p_{(m,4k+21)}\lambda^i_{19} + p_{(m,4k+19)}\lambda^i_9 \\ \psi^i_{m8k} &= 3p_{(m,4k+18)}\lambda^i_5 + p_{(m,4k+20)}\lambda^i_{17} + p_{(m,4k+21)}\lambda^i_{23} + p_{(m,4k+19)}\lambda^i_{11} \\ \psi^i_{m9k} &= 3p_{(m,4k+18)}\lambda^i_6 + p_{(m,4k+19)}\lambda^i_{12} + p_{(m,4k+21)}\lambda^i_{24} + p_{(m,4k+20)}\lambda^i_{18} \\ \psi^i_{m10k} &= 3p_{(m,4k+19)}\lambda^i_4 + p_{(m,4k+21)}\lambda^i_{17} + p_{(m,4k+20)}\lambda^i_{13} + 3p_{(m,4k+18)}\lambda^i_3 \\ \psi^i_{m11k} &= 3p_{(m,4k+20)}\lambda^i_5 + p_{(m,4k+19)}\lambda^i_8 + p_{(m,4k+21)}\lambda^i_{18} + 3p_{(m,4k+18)}\lambda^i_2 \\ \psi^i_{m12k} &= 3p_{(m,4k+18)}\lambda^i_1 + p_{(m,4k+19)}\lambda^i_7 + p_{(m,4k+20)}\lambda^i_{12} + 3p_{(m,4k+21)}\lambda^i_6 \end{aligned} \tag{111}$$

$$\begin{aligned} \lambda^i_1 &= 8(\delta^i_{56} + 3\delta^i_{12})/9 & \lambda^i_{10} &= 16(\delta^i_{17} + \delta^i_{45}) & \lambda^i_{19} &= 16(\delta^i_{37} + \delta^i_{56})/9 \\ \lambda^i_2 &= 8(\delta^i_{46} + 3\delta^i_{13})/9 & \lambda^i_{11} &= 16(\delta^i_{17} + \delta^i_{54}) & \lambda^i_{20} &= 16(\delta^i_{56} + \delta^i_{73})/9 \\ \lambda^i_3 &= 8(\delta^i_{54} + 3\delta^i_{32})/9 & \lambda^i_{12} &= 8(\delta^i_{76} + 3\delta^i_{42}) & \lambda^i_{21} &= (144\delta^i_{56} + 225\delta^i_{12})/45 \\ \lambda^i_4 &= 8(\delta^i_{34} + \delta^i_{52})/9 & \lambda^i_{13} &= 8(\delta^i_{74} + 3\delta^i_{62}) & \lambda^i_{22} &= (48\delta^i_{74} + 80\delta^i_{62})/45 \\ \lambda^i_5 &= 8(\delta^i_{43} + \delta^i_{16})/9 & \lambda^i_{14} &= 16(\delta^i_{72} + \delta^i_{64}) & \lambda^i_{23} &= (48\delta^i_{47} + 80\delta^i_{15})/45 \\ \lambda^i_6 &= 8(\delta^i_{16} + \delta^i_{52})/9 & \lambda^i_{15} &= (144\delta^i_{46} + 225\delta^i_{13})/5 & \lambda^i_{24} &= (48\delta^i_{76} + 80\delta^i_{53})/45 \\ \lambda^i_7 &= 8(\delta^i_{57} + 3\delta^i_{14})/9 & \lambda^i_{16} &= 16(\delta^i_{46} + \delta^i_{72}) & \lambda^i_{25} &= (48\delta^i_{76} + 80\delta^i_{42})/45 \\ \lambda^i_8 &= 8(\delta^i_{47} + 3\delta^i_{15})/9 & \lambda^i_{17} &= 8(\delta^i_{57} + 3\delta^i_{36}) & \lambda^i_{26} &= (48\delta^i_{57} + 80\delta^i_{36})/45 \\ \lambda^i_9 &= (144\delta^i_{54} + 225\delta^i_{32})/45 & \lambda^i_{18} &= 8(\delta^i_{76} + 3\delta^i_{53}) & \lambda^i_{27} &= (48\delta^i_{57} + 80\delta^i_{14})/45 \end{aligned} \tag{112}$$

and

$$\delta^1_{mn} = b_n c_m - b_m c_n \;;\quad \delta^2_{mn} = c_n a_m - c_m a_n \;;\quad \delta^3_{mn} = a_n b_m - a_m b_n \tag{113}$$

The components a_{ij} are then obtained through the inversion of the symmetric 9×9 Z matrix. The element stiffness matrix is given explicitly by

$$K_{ij} = K_{q_{ij}} + K_{\lambda_{ij}} = g_{ni}g_{nj} - g_{ni}r_{ns}a_{sm}r_{km}g_{kj} \;;\quad K_{ji} = K_{ij}$$

where

$$i = 1,2,3,...,24 \;;\quad j = 1,2,3,...,i \;;\quad n,k = 1,2,3,...,33 \;;\quad s,m = 1,2,3,...,9$$

A variation of the above element formulation, designated E8NL$_2$, is presented to show the simplification possible if balanced expansions are used for the stress components. The assumed stress field is given by

$$\Gamma = \begin{bmatrix} [\Gamma_1] & & & & & \\ & [\Gamma_1] & & & & \\ & & [\Gamma_1] & & & \\ & & & [\Gamma_1] & & \\ & & & & [\Gamma_1] & \\ & & & & & [\Gamma_1] \end{bmatrix} \tag{114}$$

where

$$\Gamma_1 = [1, \xi, \eta, \zeta, \xi\eta, \eta\zeta, \zeta\xi] \tag{115}$$

The natural stresses are transformed to physical coordinates using a contravariant transformation based on centroidal Jacobians and premultiplied by the distributing matrix to yield the transformed stress field as

$$\bar{P} = D^{-1} T_\sigma \Gamma = \bar{T}\Gamma \tag{116}$$

Through a linear combination as defined by

$$\bar{\beta} = diag[\bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}]\beta \tag{117}$$

the $\bar{P}$ matrix can be reduced to the uncoupled form of stress modes given in (114). Performing the orthonormalization of the fundamental modes yields

$$P = [p_1^*, p_2^*, p_3^*, p_4^*, p_5^*, p_6^*, p_7^*] \tag{118}$$

where, for $i = 1, 2, 3, ..., 7$, the general form of each mode is given by

$$p_i^* = p_{i1}^* + p_{i2}^*\xi + p_{i3}^*\eta + p_{i4}^*\zeta + p_{i5}^*\xi\eta + p_{i6}^*\eta\zeta + p_{i7}^*\zeta\xi \tag{119}$$

and

$$p_{ik}^* = 0 \quad for \quad k > i$$

Explicit expressions for the coefficients p_{ik}^* are presented in Appendix III.
The components g_{mn} with $m = 1, 2, 3, ..., 7$ and $n = 1, 2, 3, ..., 8$ result in the following linear algebraic expressions

$$\begin{aligned}
g_{(m)(3n-2)} &= \sum_{k=1}^{7} d_{11} p_{mk}^* \Theta_{nk}^1 & g_{(m)(3n-1)} &= \sum_{k=1}^{7} d_{12} p_{mk}^* \Theta_{nk}^2 & g_{(m)(3n)} &= \sum_{k=1}^{7} d_{13} p_{mk}^* \Theta_{nk}^3 \\
g_{(m+7)(3n-2)} &= \sum_{k=1}^{7} d_{12} p_{mk}^* \Theta_{nk}^1 & g_{(m+7)(3n-1)} &= \sum_{k=1}^{7} d_{22} p_{mk}^* \Theta_{nk}^2 & g_{(m+7)(3n)} &= \sum_{k=1}^{7} d_{23} p_{mk}^* \Theta_{nk}^3 \\
g_{(m+14)(3n-2)} &= \sum_{k=1}^{7} d_{13} p_{mk}^* \Theta_{nk}^1 & g_{(m+14)(3n-1)} &= \sum_{k=1}^{7} d_{23} p_{mk}^* \Theta_{nk}^2 & g_{(m+14)(3n)} &= \sum_{k=1}^{7} d_{33} p_{mk}^* \Theta_{nk}^3 \quad (120) \\
g_{(m+21)(3n-2)} &= 0.0 & g_{(m+21)(3n-1)} &= \sum_{k=1}^{7} d_{44} p_{mk}^* \Theta_{nk}^3 & g_{(m+21)(3n)} &= \sum_{k=1}^{7} d_{44} p_{mk}^* \Theta_{nk}^2 \\
g_{(m+28)(3n-2)} &= \sum_{k=1}^{7} d_{55} p_{m,k}^* \Theta_{nk}^3 & g_{(m+28)(3n-1)} &= 0.0 & g_{(m+28)(3n)} &= \sum_{k=1}^{7} d_{55} p_{m,k}^* \Theta_{nk}^1 \\
g_{(m+35)(3n-2)} &= \sum_{k=1}^{7} d_{66} p_{m,k}^* \Theta_{nk}^2 & g_{(m+35)(3n-1)} &= \sum_{k=1}^{7} d_{66} p_{m,k}^* \Theta_{nk}^1 & g_{(m+35)(3n)} &= 0.0
\end{aligned}$$

where $\Theta^j_{n\kappa}$, ϕ^i_{nk}, and z^n_j are defined above in equations (107), (108), and (91), respectively.

The components r_{mn} with $m = 1,2,3,\dots,7$ and $n = 1,2,3$, are given by

$$
\begin{array}{lcl lcl lcl}
r_{m,n} & = & d_{11}\Theta^1_{mn} & r_{m,n+3} & = & d_{12}\Theta^2_{mn} & r_{m,n+6} & = & d_{13}\Theta^3_{mn} \\
r_{m+7,n} & = & d_{12}\Theta^1_{mn} & r_{m+7,n+3} & = & d_{22}\Theta^2_{mn} & r_{m+7,n+6} & = & d_{23}\Theta^3_{mn} \\
r_{m+14,n} & = & d_{13}\Theta^1_{mn} & r_{m+14,n+3} & = & d_{23}\Theta^2_{mn} & r_{m+14,n+6} & = & d_{33}\Theta^3_{mn} \\
r_{m+21,n} & = & 0.0 & r_{m+21,n+3} & = & d_{44}\Theta^3_{mn} & r_{m+21,n+6} & = & d_{44}\Theta^2_{mn} \\
r_{m+28,n} & = & d_{55}\Theta^3_{mn} & r_{m+28,n+3} & = & 0.0 & r_{m+28,n+6} & = & d_{55}\Theta^3_{mn} \\
r_{m+35,n} & = & d_{66}\Theta^2_{mn} & r_{m+35,n+3} & = & d_{66}\Theta^1_{mn} & r_{m+35,n+6} & = & 0.0
\end{array}
\tag{121}
$$

where the following constants are defined

$$
\begin{aligned}
\Theta^i_{m1} &= -\psi^i_{m1} + f_1\psi^i_{m4} + f_2\psi^i_{m5} + f_3\psi^i_{m6} \\
\Theta^i_{m2} &= -\psi^i_{m2} + f_4\psi^i_{m4} + f_5\psi^i_{m5} + f_6\psi^i_{m6} \\
\Theta^i_{m3} &= \psi^i_{m3} + f_7\psi^i_{m4} + f_8\psi^i_{m5} + f_9\psi^i_{m6}
\end{aligned}
\tag{122}
$$

$$
\begin{aligned}
\psi^i_{m1} &= 3p^*_{m1}\lambda^i_4 + p^*_{m5}\lambda^i_{22} + p^*_{m3}\lambda^i_{14} + p^*_{m7}\lambda^i_{26} + p^*_{m4}\lambda^i_{19} + p^*_{m2}\lambda^i_9 \\
\psi^i_{m2} &= 3p^*_{m1}\lambda^i_5 + p^*_{m5}\lambda^i_{23} + p^*_{m3}\lambda^i_{15} + p^*_{m6}\lambda^i_{24} + p^*_{m4}\lambda^i_{20} + p^*_{m2}\lambda^i_{10} \\
\psi^i_{m3} &= 3p^*_{m1}\lambda^i_6 + p^*_{m3}\lambda^i_{16} + p^*_{m7}\lambda^i_{27} + p^*_{m6}\lambda^i_{25} + p^*_{m4}\lambda^i_{21} + p^*_{m2}\lambda^i_{11} \\
\psi^i_{m4} &= 3p^*_{m2}\lambda^i_4 + p^*_{m4}\lambda^i_{17} + p^*_{m3}\lambda^i_{13} + 2p^*_{m5}\lambda^i_{14} + 2p^*_{m7}\lambda^i_{19} + 3p^*_{m1}\lambda^i_3 \\
\psi^i_{m5} &= 3p^*_{m1}\lambda^i_2 + p^*_{m2}\lambda^i_8 + p^*_{m4}\lambda^i_{18} + 2p^*_{m6}\lambda^i_{20} + 2p^*_{m5}\lambda^i_{10} + 3p^*_{m3}\lambda^i_5 \\
\psi^i_{m6} &= 3p^*_{m1}\lambda^i_1 + p^*_{m2}\lambda^i_7 + p^*_{m3}\lambda^i_{12} + 2p^*_{m7}\lambda^i_{1} + 3p^*_{m4}\lambda^i_6 + 2p^*_{m6}\lambda^i_{16}
\end{aligned}
\tag{123}
$$

and the constants λ^i_j and δ^1_{mn} are defined above in equations (112) and (113), respectively.

The element stiffness matrix, K, is given by

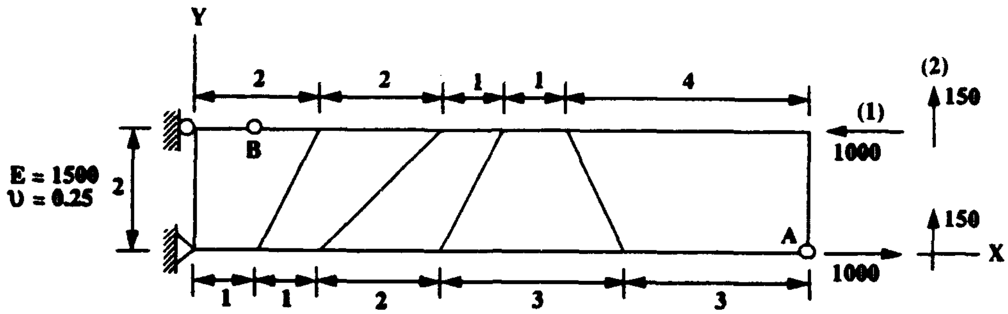
$$
K_{ij} = K_{q_{ij}} + K_{\lambda_{ij}} = g_{ni}g_{nj} - g_{ni}r_{ns}a_{sm}r_{km}g_{kj} \quad ; \quad K_{ji} = K_{ij}
$$

where

$$
i = 1,2,3,\dots,24 \; ; \quad j = 1,2,3,\dots,i \; ; \quad n,k = 1,2,3,\dots,42 \; ; \quad s,m = 1,2,3,\dots,9
$$

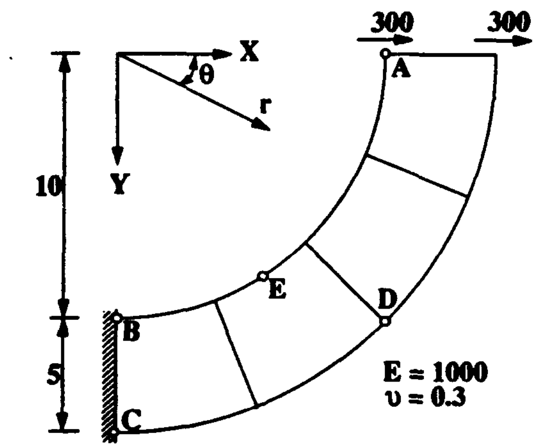
8 Numerical Demonstrations

In order to validate the performance of the explicitly derived 4 and 8-node elements, several standard benchmark problems are analyzed. Solutions to plane stress problems are presented in Figures 5 and 6 while Figure 7 depicts solutions to a solid cantilevered beam problem.



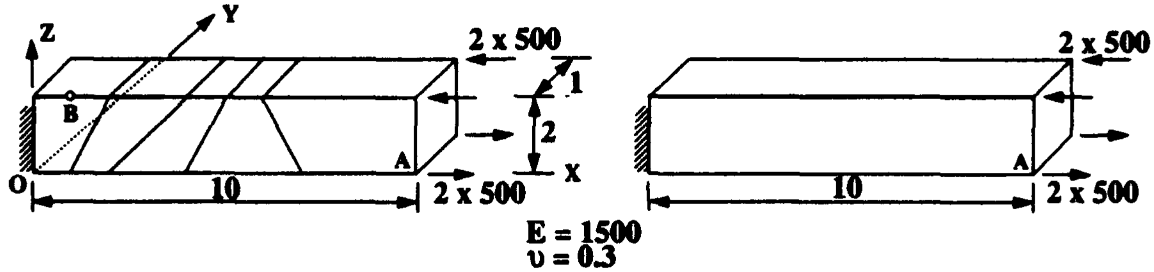
Elements	$v_A^{(1)}$	$\sigma_{xB}^{(1)}$	$v_A^{(2)}$	$\sigma_{xB}^{(2)}$
Q4 Bilinear isoparametric	45.7	-1761	50.7	-2448
Q6 (Wilson *et al* [15])	98.4	-2427.5	100.4	-3354.6
QM6 (Taylor *et al* [16])	96	-2511	97.9	-3388
NQ10 (Wu *et al* [12])	96	-2986	97.9	-4021
Pian, Sumihara [13]	96.18	-3014	98.2	-4137
E4PL	96.50	-3013	98.3	-4073
E4NL	96.18	-3014	98.1	-4074
Exact	100	-3000	102.6	-4050

Figure 5. 5-element cantilevered beam subjected to (1) pure bending and (2) end shear.



Elements	v_A	σ_{xB}	σ_{xC}	$\sigma_{\theta D}$	$\sigma_{\theta E}$
Q4	58.32	1773	-840.7	-629	1336
Q6	62.75	1545	-941.8	-694	1400
QM6	82.67	1783	-1320	-991	1643
NQ10	84.66	1781	-1509	-1097	1566
E4PL	85.25	1752	-1484	-1088	1311
E4NL	85.24	1753	-1485	-1089	1311
Exact	90.41	2214	-1476	-1044	1230

Figure 6. Clamped circular beam under end shear loading.



Cases	Five elements		Single element	
Elements	u_A	σ_{xB}	u_A	σ_{xB}
Q8 Trilinear isoparametric	45.7	-1761	9	3000
Wilson *et al* [15]	98.2	-2473	100	3000
NQ10 (Wu *et al* [12])	93.7	-2484	100	3000
$E8PL_1$	96.4	-2956	100	3000
$E8PL_2$	92.7	-2805	100	3000
$E8PL_3$	86.3	-1964	100	3000
$E8NL_1$	94.9	-2955	100	3000
$E8NL_2$	94.9	-2955	100	3000
Exact	100	3000	100	3000

Figure 7. Solid cantilevered beam under end moment loading.

9 Conclusion

The aim of introducing special stress field transformations has been to maximize the efficiency of hybrid element formulations by allowing explicit forms of element stiffness matrices to be derived. By fully exploiting the freedom in selecting and transforming independent stress fields in the hybrid stress technique, the computational cost associated with numerical matrix integration can be eliminated and inversions can be reduced substantially. In the extention to higher-order element formulations such as the 8-node plane and 20-node solid elements, without introducing incompatible displacement modes, matrix inversions are eliminated entirely. The approach has been demonstrated by deriving explicit linear algebraic forms for the stiffness matrices of selected 4-node quadrilateral and 8-node hexahedral elements. The computational advantage over purely displacement-based element formulations is clearly evident and the method outlined in this study should be expected to find general application in various hybrid/mixed methods to minimize computations in determining element stiffness coefficents.

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APPENDIX I

DETERMINATION OF DISTRIBUTING MATRICES

In computing the 2-D distributing matrix, D, the inversion of the material compliance matrix, S, is not required as the D matrix may be directly related to the material stiffness matrix C as

$$D = S^{-1/2} = C^{1/2} \quad ; \quad C^{1/2} = Q\Lambda^{1/2}Q^T$$

where the C and D matrix are given for an orthotropic material as

$$C = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} \qquad D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

The eigenvalues are computed as

$$\begin{aligned} \varphi_1 &= (-\sqrt{c_{22}^2 - 2c_{11}c_{22} + 4c_{12}^2 + c_{11}^2} + c_{22} + c_{11})/2 \\ \varphi_2 &= (\ \sqrt{c_{22}^2 - 2c_{11}c_{22} + 4c_{12}^2 + c_{11}^2} + c_{22} + c_{11})/2 \\ \varphi_3 &= c_{33} \end{aligned}$$

yielding the $\Lambda^{1/2}$ matrix as

$$\Lambda^{1/2} = diag[\varphi_1^{1/2}, \varphi_2^{1/2}, \varphi_3^{1/2}]$$

The Q matrix is defined as

$$Q = [\Phi_1|\Phi_2|\Phi_3]$$

where the eigenvectors are given by

$$\bar{\Phi}_1 = \begin{Bmatrix} c_{12} \\ \varphi_1 - c_{11} \\ 0 \end{Bmatrix}, \quad \bar{\Phi}_2 = \begin{Bmatrix} \varphi_2 - c_{22} \\ c_{12} \\ 0 \end{Bmatrix}, \quad \bar{\Phi}_3 = \begin{Bmatrix} 0 \\ 0 \\ c_{33} \end{Bmatrix}$$

and normalized as

$$\begin{aligned} N_i &= (\bar{\Phi}_i^T \bar{\Phi}_i)^{-1/2} \\ \Phi_i &= N_i \bar{\Phi}_i \end{aligned}$$

The computation of the 3-D distributing matrix using the symmetric C matrix for an orthotropic material is performed as follows.

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \qquad D = \begin{bmatrix} d_{11} & d_{12} & d_{13} & 0 & 0 & 0 \\ d_{12} & d_{22} & d_{23} & 0 & 0 & 0 \\ d_{13} & d_{23} & d_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{66} \end{bmatrix}$$

The eigenvalues are obtained as

$$\begin{aligned} \varphi_1 &= t_1 + t_2 - a/3 & \varphi_4 &= c_{44} \\ \varphi_2 &= (-(t_1 + t_2) - \tfrac{2a}{3} + \sqrt{-3}(t_1 - t_2))/2 & \varphi_5 &= c_{55} \\ \varphi_3 &= (-(t_1 + t_2) - \tfrac{2a}{3} - \sqrt{-3}(t_1 - t_2))/2 & \varphi_6 &= c_{66} \end{aligned}$$

where

$$
\begin{aligned}
t_1 &= (r + \sqrt{q^3 + r^2})^{1/3} \\
t_2 &= (r - \sqrt{q^3 + r^2})^{1/3} \\
q &= b/3 - a^2/9 \;\; ; \;\; r = (ab - 3c)/6 - a^3/27 \\
a &= -(c_{33} + c_{22} + c_{11}) \\
b &= ((c_{22} + c_{11})c_{33} - c_{23}^2 - c_{13}^2 + c_{11}c_{22} - c_{12}^2) \\
c &= ((c_{11}c_{22} + c_{12}^2)c_{33} + c_{11}c_{23}^2 - 2c_{12}c_{13}c_{23} + c_{13}^2 c_{22})
\end{aligned}
$$

yielding the $\Lambda^{1/2}$ matrix as

$$\Lambda^{1/2} = diag[\varphi_1^{1/2}, \varphi_2^{1/2}, \varphi_3^{1/2}, \varphi_4^{1/2}, \varphi_5^{1/2}, \varphi_6^{1/2}]$$

The Q matrix is given by

$$Q = [\Phi_1 | \Phi_2 | \Phi_3 | \Phi_4 | \Phi_5 | \Phi_6]$$

where the eigenvectors are determined by

$$\bar{\Phi}_1 = \left\{ \begin{array}{c} (c_{22} - \varphi_1)(c_{33} - \varphi_1) - c_{23}^2 \\ c_{13}c_{23} - c_{12}(c_{33} - \varphi_1) \\ c_{12}c_{23} - c_{13}(c_{22} - \varphi_1) \\ 0 \\ 0 \\ 0 \end{array} \right\}, \quad \bar{\Phi}_2 = \left\{ \begin{array}{c} c_{13}c_{23} - c_{12}(c_{33} - \varphi_2) \\ (c_{11} - \varphi_2)(c_{33} - \varphi_2) - c_{13}^2 \\ c_{12}c_{13} - c_{23}(c_{11} - \varphi_2) \\ 0 \\ 0 \\ 0 \end{array} \right\}$$

$$\bar{\Phi}_3 = \left\{ \begin{array}{c} c_{12}c_{23} - c_{13}(c_{22} - \varphi_3) \\ c_{12}c_{13} - c_{23}(c_{11} - \varphi_3) \\ (c_{11} - \varphi_3)(c_{22} - \varphi_3) - c_{12}^2 \\ 0 \\ 0 \\ 0 \end{array} \right\}, \bar{\Phi}_4 = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ c_{44} \\ 0 \\ 0 \end{array} \right\}, \quad \bar{\Phi}_5 = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ c_{55} \\ 0 \end{array} \right\}, \quad \bar{\Phi}_6 = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_{66} \end{array} \right\}$$

and are normalized to yield

$$N_i = (\bar{\Phi}_i^T \bar{\Phi}_i)^{-1/2}$$

$$\Phi_i = N_i \bar{\Phi}_i$$

In the case of repeated eigenvalues corresponding to a coupled submatrix partition, the eigenvectors associated with the degenerate eigenvalues are discarded and are replaced by vectors orthonormalized to the independent eigenvectors using the Gram-Schmidt procedure.

APPENDIX II

DETERMINATION OF STRESS MODE COMPONENTS USING THE GRAM-SCHMIDT PROCEDURE

An elaboration of (22) yields an automated procedure for the Gram-Schmidt process and is presented to indicate the numerical simplicity of developing orthonormal stress polynomials. A generic algorithm may be developed by first defining variables representing the initial stresses, intermediate combinations and the final orthonormalized stresses given by p_{ijk}, p'_{ijk} and p^*_{ijk}, respectively, where i indicates the mode number, j denotes the stress component and k represents the coefficient in the polynomial expansion for the j^{th} component. Next, the basic scalar integrals involving the Jacobian determinant and the polynomial powers arising from the inner product defined in (21) are computed and assigned to a variable, ϕ_{ij}, accounting for the order of expansion and the arrangement of terms. For example, in the 2-D case, if the expansion for each stress component is given by

$$P_i = \left\{ \begin{array}{c} p_{i1} + p_{i2}\bar{x} + p_{i3}\bar{y} \\ p_{i4} + p_{i5}\bar{x} + p_{i6}\bar{y} \\ p_{i7} + p_{i8}\bar{x} + p_{i9}\bar{y} \end{array} \right\}$$

where $\bar{x}$ and $\bar{y}$ are expressed in physical or natural coordinates, the scalar integrals required are given by

$$\phi = \int_{-1}^{1}\int_{-1}^{1}[|J|\left\{ \begin{array}{c} 1 \\ \bar{x} \\ \bar{y} \end{array} \right\}[1, \bar{x}, \bar{y}]d\bar{x}d\bar{y} = \left[\begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_4 & \lambda_6 \\ \lambda_3 & \lambda_6 & \lambda_5 \end{array} \right]$$

The computation of scalar integrals is detailed in Appendix IV.
With ϕ defined, the orthonormalizing procedure is given by the following sequence of operations which may be performed symbolically or numerically

$$i)\ \ p'_{smn} = p_{smn} - \sum_{r=1}^{s-1}(p_{rki}p^{*}_{skj}\phi_{ij})p^{*}_{rmn}$$

$$ii)\ \ p^{*}_{smn} = p'_{smn}(p'_{ski}p'_{skj}\phi_{ij})^{-1/2}$$

As shown, the nature of the Gram-Schmidt procedure is such that in developing an orthonormal set, each mode is computed in an 'evolutionary' fashion through the process of orthogonalizing the i^{th} mode to the preceding $i-1$ modes. When performed symbolically, this leads to increasingly cumbersome expressions for coefficients as a function of the number of modes and degree of stress coupling and is most efficiently implemented as a numerical procedure. Explicit expressions are presented below as examples for select 2-D elements to indicate the unwieldly nature of symbolic representation.

For the E4PQ element, the coefficients in the orthonormalized stress modes are given by

$$\begin{array}{lllll}
p_{11} = N_1 & p_{49} = N_4L_2 & p_{59} = N_5P_4 & p_{67} = N_6Q_4 & p_{74} = N_7\alpha_1 \\
p_{25} = N_1 & p_{51} = N_5\alpha_3 & p_{61} = N_6Q_1 & p_{68} = N_6Q_5 & p_{75} = N_7R_4 \\
p_{39} = N_1 & p_{52} = N_5P_1 & p_{62} = N_6Q_2 & p_{69} = N_6Q_6 & p_{76} = N_7R_5 \\
p_{42} = N_4\alpha_1 & p_{53} = N_5P_2 & p_{63} = N_6Q_3 & p_{71} = N_7R_1 & p_{77} = N_7R_6 \\
p_{43} = N_4L_1 & p_{57} = N_5P_3 & p_{65} = N_6\alpha_4 & p_{72} = N_7R_2 & p_{78} = N_7R_7 \\
p_{47} = -N_4\alpha_2 & p_{58} = -N_5\alpha_2 & p_{66} = N_6L_3 & p_{73} = N_7R_3 & p_{79} = N_7R_8
\end{array}$$

where

$$\begin{array}{ll}
D_1 = N_4^2(\alpha_1\alpha_3 + \alpha_2^2)(\lambda_6 - \lambda_2\lambda_3/\lambda_1) & D_4 = N_4^2\alpha_2^2(\lambda_6 - \lambda_2\lambda_3/\lambda_1) \\
P_1 = -\alpha_1 D_1 & D_5 = N_5^2\alpha_2(\alpha_2\lambda_5 - P_3\lambda_6 - P_4\lambda_3) \\
P_2 = (\alpha_1\lambda_3 D_1 - \alpha_3\lambda_2)/\lambda_1 & D_6 = -N_6^2\alpha_2(Q_5\lambda_5 + Q_6\lambda_3) \\
P_3 = \alpha_2 D_1 & D_7 = N_6^2[(\alpha_1\alpha_4 - \alpha_2 Q_4)\lambda_6 + L_3\alpha_1\lambda_2 - D_6] \\
P_4 = \alpha_2(\lambda_3 - \lambda_2 D_1)/\lambda_1 & R_1 = -\alpha_3 D_5 - Q_1 D_7 \\
D_2 = N_5^2(\alpha_2\lambda_6 - P_3\lambda_4 - P_4\lambda_2) & R_2 = -\alpha_1 D_4 - P_1 D_5 - Q_2 D_7 \\
D_3 = N_4^2(\lambda_4 - \lambda_2^2/\lambda_1) & R_3 = -L_1 D_4 - P_2 D_5 - Q_3 D_7 \\
Q_1 = -\alpha_2\alpha_3 D_2 & R_4 = -\alpha_4 D_7 \\
Q_2 = -\alpha_2(\alpha_1\alpha_2 D_3 + P_1 D_2) & R_5 = -L_3 D_7 + L_4 \\
Q_3 = \alpha_2(\alpha_1 L_5 D_3 - P_2 D_2) & R_6 = \alpha_2 D_4 - P_3 D_5 - Q_4 D_7 \\
Q_4 = \alpha_2(\alpha_2^2 D_3 - P_3 D_2 - 1) & R_7 = \alpha_2(D_5 - 1) - Q_5 D_7 \\
Q_5 = \alpha_2^2 D_2 & R_8 = L_5 - L_2 D_4 - P_4 D_5 - Q_6 D_7 \\
Q_6 = L_2(1 - \alpha_2^2 D_3) - \alpha_2 P_4 D_2 &
\end{array}$$

Normalizing constants are given by

$$
\begin{aligned}
N_1 &= \lambda_1^{-1/2} \\
N_4 &= [\alpha_1^2(\lambda_5 - \lambda_3^2/\lambda_1) + \alpha_2^2(\lambda_4 - \lambda_2^2/\lambda_1)]^{-1/2} \\
N_5 &= [(\alpha_3^2 + P_3^2)\lambda_4 + (2\alpha_3 P_1 - 2\alpha_2 P_3)\lambda_6 + 2(\alpha_3 P_2 + P_3 P_4)\lambda_2 + \\
&\quad (P_1^2 + \alpha_2^2)\lambda_5 + 2(P_1 P_2 - \alpha_2 P_4)\lambda_3 + (P_2^2 + P_4^2)\lambda_1]^{-1/2} \\
N_6 &= [(Q_1^2 + Q_4^2)\lambda_4 + 2(Q_1 Q_2 + Q_4 Q_5)\lambda_6 + 2(Q_1 Q_3 + Q_4 Q_6)\lambda_2 + \\
&\quad (Q_2^2 + \alpha_4^2 + Q_5^2)\lambda_5 + 2(Q_2 Q_3 + \alpha_4 L_3 + Q_5 Q_6)\lambda_3 + \\
&\quad (Q_3^2 + L_3^2 + Q_6^2)\lambda_1]^{-1/2} \\
N_7 &= [(R_1^2 + \alpha_1^2 + R_6^2)\lambda_4 + 2(R_1 R_2 + \alpha_1 R_4 + R_6 R_7)\lambda_6 + \\
&\quad 2(R_1 R_3 + \alpha_1 R_5 + R_6 R_8)\lambda_2 + (R_2^2 + R_4^2 + R_7^2)\lambda_5 + \\
&\quad 2(R_2 R_3 + R_4 R_5 + R_7 R_8)\lambda_3 + (R_3^2 + R_5^2 + R_8^2)\lambda_1]^{-1/2}
\end{aligned}
$$

and material constants are given by

$$
\begin{aligned}
L_1 &= -\alpha_1\lambda_3/\lambda_1 & \alpha_1 &= 1/d_{12} \\
L_2 &= \alpha_2\lambda_2/\lambda_1 & \alpha_2 &= 1/d_{33} \\
L_3 &= -\alpha_4\lambda_3/\lambda_1 & \alpha_3 &= 1/d_{11} \\
L_4 &= -\alpha_1\lambda_2/\lambda_1 & \alpha_4 &= 1/d_{22} \\
L_5 &= \alpha_2\lambda_3/\lambda_1
\end{aligned}
$$

For the E4PR and E4PL elements, the coefficients, p_{ij}, of the stress modes are given by

$$
\begin{aligned}
p_{11} &= N_1 & p_{31} &= N_3 R_1 \\
p_{21} &= -N_2\lambda_2/\lambda_1 & p_{32} &= N_3 R_2 \\
p_{22} &= N_2 & p_{33} &= N_3
\end{aligned}
$$

where

$$
\begin{aligned}
R_1 &= (\lambda_2\lambda_6 - \lambda_3\lambda_4)/(\lambda_4\lambda_1 - \lambda_2^2) \\
R_2 &= (\lambda_2\lambda_3 - \lambda_1\lambda_6)/(\lambda_4\lambda_1 - \lambda_2^2) \\
N_1 &= \lambda_1^{-1/2} \\
N_2 &= (\lambda_4 - \lambda_2^2/\lambda_1)^{-1/2} \\
N_3 &= (\lambda_5 + 2R_1\lambda_6 + 2R_2\lambda_3 + R_1^2\lambda_4 + 2R_1R_2\lambda_2 + R_2^2\lambda_1)^{-1/2}
\end{aligned}
$$

For the E4NL element, the stress mode coefficients, p_{ij}, are given by

$$
\begin{aligned}
p_{11} &= N_1 & p_{31} &= N_3 R_1 \\
p_{21} &= -N_2\lambda_2/\lambda_1 & p_{32} &= N_3 R_2 \\
p_{22} &= N_2 & p_{33} &= N_3
\end{aligned}
$$

where

$$
\begin{aligned}
R_1 &= \lambda_2\lambda_3/(\lambda_5\lambda_1 - \lambda_2^2) \\
R_2 &= -\lambda_3\lambda_5/(\lambda_5\lambda_1 - \lambda_2^2) \\
N_1 &= \lambda_1^{-1/2} \\
N_2 &= (\lambda_5 - \lambda_2^2/\lambda_1)^{-1/2} \\
N_3 &= (\lambda_6 + 2R_2\lambda_3 + R_1^2\lambda_5 + 2R_1R_2\lambda_2 + R_2^2\lambda_1)^{-1/2}
\end{aligned}
$$

For the E8NL$_2$ element, the coefficients, p_{ij}, of the stress modes are given by

$$
\begin{aligned}
p_{11} &= N_1 & p_{42} &= N_4 R_4 & p_{55} &= N_5 & p_{71} &= N_7 R_{12} \\
p_{21} &= -N_2\lambda_2/\lambda_1 & p_{43} &= -N_4 P_2 & p_{61} &= N_6 R_8 & p_{72} &= N_7 R_{13} \\
p_{22} &= N_2 & p_{44} &= N_4 & p_{62} &= N_6 R_9 & p_{73} &= N_7 R_{14} \\
p_{31} &= N_3 R_1 & p_{51} &= N_5 R_5 & p_{63} &= N_6 R_{10} & p_{74} &= N_7 R_{15} \\
p_{32} &= N_3 R_2 & p_{52} &= N_5 R_6 & p_{64} &= N_6 R_{11} & p_{75} &= N_7 R_{16} \\
p_{33} &= N_3 & p_{53} &= N_5 R_7 & p_{65} &= -N_6 P_9 & p_{76} &= -N_7 P_{14} \\
p_{41} &= N_4 R_3 & p_{54} &= -N_5 P_5 & p_{66} &= N_6 & p_{77} &= N_7
\end{aligned}
$$

where

$$
\begin{array}{lcl@{\qquad}lcl}
R_1 & = & [N_2^2\lambda_2(\lambda_5\lambda_1 - \lambda_3\lambda_2) - \lambda_3]/\lambda_1 & P_9 & = & N_5^2(R_5\lambda_6 + R_6\lambda_{11} + R_7\lambda_{17} - P_5\lambda_{15} + \lambda_{22}) \\
R_2 & = & N_2^2(\lambda_3\lambda_2/\lambda_1 - \lambda_5) & R_8 & = & (P_6\lambda_2 - \lambda_6)/\lambda_1 - P_7R_1 - P_8R_3 - P_9R_5 \\
P_1 & = & N_2^2(\lambda_7 - \lambda_2\lambda_4/\lambda_1) & R_9 & = & -P_6 - P_7R_2 - P_8R_4 - P_9R_6 \\
P_2 & = & N_3^2(R_1\lambda_4 + R_2\lambda_7 + \lambda_6) & R_{10} & = & P_8P_2 - P_7 - P_9R_7 \\
R_3 & = & (P_1\lambda_2 - \lambda_4)/\lambda_1 - P_2R_1 & R_{11} & = & P_9P_5 - P_8 \\
R_4 & = & -P_1 - P_2R_2 & P_{10} & = & N_2^2(\lambda_{16} - \lambda_2\lambda_7/\lambda_1) \\
P_3 & = & N_2^2(\lambda_{14} - \lambda_2\lambda_5/\lambda_1) & P_{11} & = & N_3^2(R_1\lambda_7 + R_2\lambda_{16} + \lambda_{11}) \\
P_4 & = & N_3^2(\lambda_5R_1 + R_2\lambda_{14} + \lambda_{12}) & P_{12} & = & N_4^2(R_3\lambda_7 + R_4\lambda_{16} - P_2\lambda_{11} + \lambda_{13}) \\
P_5 & = & N_4^2(\lambda_5 + R_4\lambda_{14} - P_2\lambda_{12} + \lambda_{11}) & P_{13} & = & N_5^2(R_5\lambda_7 + R_6\lambda_{16} + R_7\lambda_{11} - P_5\lambda_{13} + \lambda_{23}) \\
R_5 & = & (P_3\lambda_2 - \lambda_5)/\lambda_1 - P_4R_1 - P_5R_3 & P_{14} & = & N_6^2(R_8\lambda_7 + R_9\lambda_{16} + R_{10}\lambda_{11} + R_{11}\lambda_{13} - P_9\lambda_{23} + \lambda_{21}) \\
R_6 & = & -P_3 - P_4R_2 - P_5R_4 & R_{12} & = & (P_{10}\lambda_2 - \lambda_7)/\lambda_1 - P_{11}R_1 - P_{12}R_3 - P_{13}R_5 - P_{14}R_8 \\
R_7 & = & P_5P_2 - P_4 & R_{13} & = & -P_{10} - P_{11}R_2 - P_{12}R_4 - P_{13}R_6 - P_{14}R_9 \\
P_6 & = & N_2^2(\lambda_{11} - \lambda_2\lambda_6/\lambda_1) & R_{14} & = & P_{12}P_2 - P_{11} - P_{13}R_7 - P_{14}R_{10} \\
P_7 & = & N_3^2(R_1\lambda_6 + R_2\lambda_{11} + \lambda_{17}) & R_{15} & = & P_{13}P_5 - P_{12} - P_{14}R_{11} \\
P_8 & = & N_4^2(R_3\lambda_6 + R_4\lambda_{11} - P_2\lambda_{17} + \lambda_{15}) & R_{16} & = & P_{14}P_9 - P_{13}
\end{array}
$$

and

$$
\begin{array}{lcl}
N_1 & = & \lambda_1^{-1/2} \\
N_2 & = & [\lambda_8 - \lambda_2^2/\lambda_1]^{-1/2} \\
N_3 & = & [R_1^2\lambda_1 + \lambda_9 + R_2^2\lambda_8 + 2(R_1R_2\lambda_2 + R_1\lambda_3 + R_2\lambda_5)]^{-1/2} \\
N_4 & = & [R_3^2\lambda_1 + P_2^2\lambda_9 + R_4^2\lambda_8 + \lambda_{10} + 2(R_3R_4\lambda_2 - R_3P_2\lambda_3 + R_3\lambda_4 - R_4P_2\lambda_5 + R_4\lambda_7 - P_2\lambda_6)]^{-1/2} \\
N_5 & = & [R_5^2\lambda_1 + R_6^2\lambda_8 + R_7^2\lambda_9 + P_5^2\lambda_{10} + \lambda_{18} + 2(R_5R_6\lambda_2 + R_5R_7\lambda_3 - R_5P_5\lambda_4 + (R_5 + R_6R_7)\lambda_5 - \\
 & & R_6P_5\lambda_7 + R_6\lambda_{14} - R_7P_5\lambda_6 + R_7\lambda_{12} - P_5\lambda_{11})]^{-1/2} \\
N_6 & = & [R_8^2\lambda_1 + R_9^2\lambda_8 + R_{10}^2\lambda_9 + R_{11}^2\lambda_{10} + P_9^2\lambda_{18} + \lambda_{19} + 2(R_8R_9\lambda_2 + R_8R_{10}\lambda_3 + R_8R_{11}\lambda_4 - \\
 & & (R_8P_9 + R_9R_{10})\lambda_5 + (R_8 + R_{10}R_{11})\lambda_6 + R_9R_{11}\lambda_7 - R_9P_9\lambda_{14} + (R_9 - R_{11}P_9)\lambda_{11} - \\
 & & R_{10}P_9\lambda_{12} + R_{10}\lambda_{17} + R_{11}\lambda_{15} - P_9\lambda_{22})]^{-1/2} \\
N_7 & = & [R_{12}^2\lambda_1 + R_{13}^2\lambda_8 + R_{14}^2\lambda_9 + R_{15}^2\lambda_{10} + P_{14}^2\lambda_{19} + R_{16}^2\lambda_{18} + \lambda_{20} + 2(R_{12}R_{13}\lambda_2 + R_{12}R_{14}\lambda_3 + \\
 & & R_{12}R_{15}\lambda_4 + (R_{12}R_{16} + R_{13}R_{14})\lambda_5 + (R_{14}R_{15} - P_{12}R_{14})\lambda_6 + (R_{12} + R_{13}R_{15})\lambda_7 + \\
 & & R_{13}R_{16}\lambda_{14} + (R_{15}R_{16} + P_{14} - R_{13}P_{14})\lambda_{11} + R_{13}\lambda_{16} + R_{14}R_{16}\lambda_{12} - R_{14}P_{14}\lambda_{17} - \\
 & & R_{15}P_{14}\lambda_{15} + R_{15}\lambda_{13} - R_{16}P_{14}\lambda_{22} + R_{16}\lambda_{23} - P_{14}\lambda_{21}]^{-1/2}
\end{array}
$$

APPENDIX III

INCOMPATIBLE DISPLACEMENT MODES

The constants computed for the element incompatible displacement modes are based on the approach presented in [12] for identically satisfying the convergence criterion

$$\int_v LM\,dv = 0$$

for any arbitrary element configuration.

The E4PL and E4NL elements

For the E4PL element, the various constants used in the definition of the incompatible displacements are given by

$$f_1 = \frac{a_1b_3 - a_1b_1}{a_1b_2 - a_2b_1} \qquad f_2 = \frac{a_2b_3 - a_3b_2}{a_1b_2 - a_2b_1}$$

and

$$\begin{aligned} h_1 &= 3a_1b_3 + a_2b_3(f_2 - f_1) & h_6 &= 3(a_2f_1 - a_1f_2 - a_3) \\ h_2 &= 3b_1b_3 + b_2b_3(f_2 - f_1) & h_7 &= a_1b_3f_1 - 3a_2b_1 - a_2b_3f_2 \\ h_3 &=. 3(b_3 + b_1f_2 - b_2f_1) & h_8 &= b_1b_3f_1 - 3b_2b_1 - b_2b_3f_2 \\ h_4 &= a_1a_3f_1 - 3a_1a_2 - a_2a_3f_2 & h_9 &= 3a_1a_2 + a_2a_3f_2 - a_1a_3f_1 \\ h_5 &= b_1a_3f_1 - 3b_1a_2 - b_2a_3f_2 & h_{10} &= 3b_1a_2 + b_2a_3f_2 - b_1a_3f_1 \end{aligned}$$

For the E4NL element, f_1 and f_2 are the same as given above. The constants h_i are slightly different and are given by

$$\begin{aligned} h_1 &= 8(b_3 + b_1f_2 - b_2f_1)/3 & h_6 &= -8a_3f_2/9 \\ h_2 &= 8(3b_2 - b_3f_1)/9 & h_7 &= 8b_3f_2/9 \\ h_3 &= 8b_3f_2/9 & h_8 &= -8(3b_1 + b_3f_2)/9 \\ h_4 &= 8(a_2f_1 - a_3 - a_1f_2)/9 & h_9 &= -8a_3f_1/9 \\ h_5 &= 8(a_3f_1 - 3a_2)/9 & h_{10} &= 8(3a_1 + a_3f_2)/9 \end{aligned}$$

The E8PL and E8NL elements

A comment on the approach in [12] of forming incompatible modes has been made in [17] which criticizes the algebraic complexity in the 3-D case. However, with careful manipulation, the procedure is quite tractable and is presented in full detail below.

The basic incompatible displacements are selected as

$$u_\lambda = [\xi^2, \eta^2, \zeta^2]$$

and 'virtual parameters' are taken as

$$u_\lambda = [\xi, \eta, \zeta]$$

The elements p_{ij} of P_* are obtained from the integration of

$$P_* = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} d\xi d\eta d\zeta$$

where $t_{ij} = Adj(J)_{ij}$ and which yields

$$\begin{aligned} p_{11} &= 8(\alpha_{45} + 3\alpha_{23})/3; & p_{21} &= 8(\beta_{45} + 3\beta_{23})/3; & p_{31} &= 8(\delta_{45} + 3\delta_{23})/3 \\ p_{12} &= 8(\alpha_{64} + 3\alpha_{31})/3; & p_{22} &= 8(\beta_{64} + 3\beta_{31})/3; & p_{32} &= 8(\delta_{64} + 3\delta_{31})/3 \\ p_{13} &= 8(\alpha_{56} + 3\alpha_{12})/3; & p_{23} &= 8(\beta_{56} + 3\beta_{12})/3; & p_{33} &= 8(\delta_{56} + 3\delta_{12})/3 \end{aligned}$$

where

$$\begin{aligned} \alpha_{ij} &= b_ic_j - b_jc_i \\ \beta_{ij} &= c_ia_j - c_ja_i \\ \delta_{ij} &= a_ib_j - a_jb_i \end{aligned}$$

The inverse of P_* is given by

$$P_*^{-1} = \frac{1}{|P_*|} \begin{bmatrix} p_{22}p_{33} - p_{23}p_{32} & p_{13}p_{32} - p_{12}p_{33} & p_{12}p_{23} - p_{13}p_{22} \\ p_{23}p_{31} - p_{21}p_{33} & p_{11}p_{33} - p_{13}p_{31} & p_{13}p_{21} - p_{11}p_{23} \\ p_{21}p_{32} - p_{22}p_{31} & p_{12}p_{31} - p_{11}p_{32} & p_{11}p_{22} - p_{12}p_{21} \end{bmatrix} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{21} & \bar{p}_{22} & \bar{p}_{23} \\ \bar{p}_{31} & \bar{p}_{32} & \bar{p}_{33} \end{bmatrix}$$

where

$$|P_*| = p_{11}(p_{22}p_{33} - p_{23}p_{32}) - p_{12}(p_{21}p_{33} - p_{23}p_{31}) + p_{13}(p_{21}p_{32} - p_{22}p_{31})$$

The elements p'_{ij} of P_λ are obtained from the integration of

$$P_\lambda = 2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} \xi t_{11} & \eta t_{12} & \zeta t_{13} \\ \xi t_{21} & \eta t_{22} & \zeta t_{23} \\ \xi t_{31} & \eta t_{32} & \zeta t_{33} \end{bmatrix} d\xi d\eta d\zeta = \begin{bmatrix} p'_{11} & p'_{12} & p'_{13} \\ p'_{21} & p'_{22} & p'_{23} \\ p'_{31} & p'_{32} & p'_{33} \end{bmatrix}$$

which yields

$$
\begin{aligned}
p'_{11} &= 16(\alpha_{25}+\alpha_{43})/3 & p'_{21} &= 16(\beta_{25}+\beta_{43})/3 & p'_{31} &= 16(\delta_{25}+\delta_{43})/3 \\
p'_{12} &= 16(\alpha_{61}+\alpha_{34})/3 & p'_{22} &= 16(\beta_{61}+\beta_{34})/3 & p'_{32} &= 16(\delta_{61}+\delta_{34})/3 \\
p'_{13} &= 16(\alpha_{16}+\alpha_{52})/3 & p'_{23} &= 16(\beta_{16}+\beta_{52})/3 & p'_{33} &= 16(\delta_{16}+\delta_{52})/3
\end{aligned}
$$

The computation of the final form of u_λ is given by

$$u_\lambda = ([\xi^2,\eta^2,\zeta^2] - [\xi,\eta,\zeta]P_*^{-1}P_\lambda)\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}$$

such that

$$u_\lambda = [M_1, M_2, M_3]$$

where

$$\mathbf{u}_\lambda = \begin{pmatrix}u\\ v\\ w\end{pmatrix}_\lambda = \begin{bmatrix} M_1 & M_2 & M_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_1 & M_2 & M_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_1 & M_2 & M_3 \end{bmatrix}$$

and

$$
\begin{aligned}
M_1 &= \xi^2 - f_1\xi - f_2\eta - f_3\zeta \\
M_2 &= \eta^2 - f_4\xi - f_5\eta - f_6\zeta \\
M_3 &= \zeta^2 - f_7\xi - f_8\eta - f_9\zeta
\end{aligned}
$$

The constants f_i in the definition of the incompatible modes are given by

$$
\begin{aligned}
f_1 &= \bar{p}_{13}p'_{31} + \bar{p}_{12}p'_{21} + \bar{p}_{11}p'_{11}; & f_2 &= \bar{p}_{23}p'_{31} + \bar{p}_{22}p'_{21} + \bar{p}_{21}p'_{11}; & f_3 &= \bar{p}_{33}p'_{31} + \bar{p}_{32}p'_{21} + \bar{p}_{31}p'_{11} \\
f_4 &= \bar{p}_{13}p'_{32} + \bar{p}_{12}p'_{22} + \bar{p}_{11}p'_{12}; & f_5 &= \bar{p}_{23}p'_{32} + \bar{p}_{22}p'_{22} + \bar{p}_{21}p'_{12}; & f_6 &= \bar{p}_{33}p'_{32} + \bar{p}_{32}p'_{22} + \bar{p}_{31}p'_{12} \\
f_7 &= \bar{p}_{13}p'_{33} + \bar{p}_{12}p'_{23} + \bar{p}_{11}p'_{13}; & f_8 &= \bar{p}_{23}p'_{33} + \bar{p}_{22}p'_{23} + \bar{p}_{21}p'_{13}; & f_9 &= \bar{p}_{33}p'_{33} + \bar{p}_{32}p'_{23} + \bar{p}_{31}p'_{13}
\end{aligned}
$$

APPENDIX IV

COMPUTATION OF BASIC SCALAR INTEGRALS

In the Gram-Schmidt procedure, the inner product defined by (21) requires the evaluation of scalar integrals in generating the orthonormal basis vectors. For the 2-D elements, using complete linear stress expansions in physical coordinates, the required integrals consist of polynomial orders up to quadratic order defined as

$$[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6] = \int_{-1}^{1}\int_{-1}^{1} |J|[1,x,y,x^2,y^2,xy]d\xi d\eta$$

In computing the above integrals, the determinant of the Jacobian is expressed as

$$|J| = J_0 + J_1\xi + J_2\eta$$

where

$$J_0 = a_1b_2 - a_2b_1 \ ; \quad J_1 = a_1b_3 - a_3b_1 \ ; \quad J_2 = a_3b_2 - a_2b_3$$

which under mapping to isoparametric coordinates by

$$
\begin{aligned}
x &= a_1\xi + a_2\eta + a_3\xi\eta \\
y &= b_1\xi + b_2\eta + b_3\xi\eta
\end{aligned}
$$

yield closed form expressions given by

$$
\begin{aligned}
\lambda_1 &= 4J_0 \\
\lambda_2 &= 4(J_1a_1 + J_2a_2)/3 \\
\lambda_3 &= 4(J_1b_1 + J_2b_2)/3 \\
\lambda_4 &= [4J_0(3a_1^2 + 3a_2^2 + a_3^2) + 8(J_1a_2a_3 + J_2a_1a_3)]/9 \\
\lambda_5 &= [4J_0(3b_1^2 + 3b_2^2 + b_3^2) + 8(J_1b_2b_3 + J_2b_1b_3)]/9 \\
\lambda_6 &= 4[b_3(J_0a_3 + J_1a_2 + J_2a_1) + b_2(3a_2J_0 + a_3J_1) + b_1(3a_1J_0 + a_3J_2)]/9
\end{aligned}
$$

For complete linear stresses expressed in natural coordinates the basic integrals are

$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6] = \int_{-1}^{1}\int_{-1}^{1} |J|[1, \xi, \eta, \xi\eta, \xi^2, \eta^2]d\xi d\eta$$

which yield simply

$$
\begin{aligned}
\lambda_1 &= 4J_0 & \lambda_4 &= 0 \\
\lambda_2 &= 4J_1/3 & \lambda_5 &= 4J_0/3 \\
\lambda_3 &= 4J_2/3 & \lambda_6 &= 4J_0/3
\end{aligned}
$$

In the 3-D case, the determinant of the Jacobian is given by

$$
\begin{aligned}
|J| = {} & r_1 + r_2\xi + r_3\eta + r_4\zeta + r_5\eta\xi + r_6\zeta\xi + r_7\zeta\eta + r_8\xi^2 + r_9\eta^2 + r_{10}\zeta^2 + r_{11}\zeta\eta\xi + r_{12}\eta\xi^2 + \\
& r_{13}\zeta\xi^2 + r_{14}\xi\eta^2 + r_{15}\zeta\eta^2 + r_{16}\xi\zeta^2 + r_{17}\eta\zeta^2 + r_{18}\zeta\eta\xi^2 + r_{19}\zeta\xi\eta^2 + r_{20}\eta\xi\zeta^2
\end{aligned}
$$

where

$$
\begin{aligned}
r_1 &= \varphi_{12}^4 & r_6 &= \varphi_{31}^8 + \varphi_{51}^7 + \varphi_{61}^6 & r_{11} &= 2\varphi_{54}^7 & r_{16} &= \varphi_{35}^8 \\
r_2 &= \varphi_{31}^5 + \varphi_{12}^6 & r_7 &= \varphi_{23}^8 + \varphi_{34}^7 + \varphi_{62}^6 & r_{12} &= \varphi_{14}^8 & r_{17} &= \varphi_{37}^7 \\
r_3 &= \varphi_{23}^5 + \varphi_{12}^7 & r_8 &= \varphi_{14}^6 & r_{13} &= \varphi_{51}^8 & r_{18} &= \varphi_{54}^8 \\
r_4 &= \varphi_{31}^7 + \varphi_{23}^6 & r_9 &= \varphi_{42}^7 & r_{14} &= \varphi_{42}^8 & r_{19} &= \varphi_{46}^8 \\
r_5 &= \varphi_{42}^6 + \varphi_{14}^7 + \varphi_{12}^8 & r_{10} &= \varphi_{35}^7 & r_{15} &= \varphi_{26}^8 & r_{20} &= \varphi_{65}^8
\end{aligned}
$$

and

$$\varphi_{jk}^i = a_i(a_jb_k - a_kb_j) + b_i(b_jc_k - b_kc_j) + c_i(c_ja_k - c_ka_j)$$

With assumed stresses of quadratic order, the basic integrals required in computing the orthonormalized stress modes are defined up to quartic order by

$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |J|[1, x, y, z, xz, yz, xy, x^2, y^2]d\xi d\eta d\zeta$$

$$[\lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |J|[z^2, xyz, xy^2, xz^2, yx^2, yz^2, zx^2]d\xi d\eta d\zeta$$

$$[\lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}, \lambda_{22}, \lambda_{23}] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |J|[zy^2, x^3, y^3, z^3x, y^3, xz^3, yx^3]d\xi d\eta d\zeta$$

$$[\lambda_{24}, \lambda_{25}, \lambda_{26}, \lambda_{27}, \lambda_{28}, \lambda_{29}, \lambda_{30}] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |J|[yz^3, zx^3, zy^3, x^2y^2, y^2z^2, z^2x^2, xyz^2]d\xi d\eta d\zeta$$

$$[\lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{34}, \lambda_{35}] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |J|[xy^2z, x^2yz, x^4, y^4, z^4]d\xi d\eta d\zeta$$

where the isoparametric mapping is given by

$$
\begin{aligned}
x &= a_1\xi + a_2\eta + a_3\zeta + a_4\xi\eta + a_5\xi\zeta + a_6\eta\zeta + a_7\xi\eta\zeta \\
y &= b_1\xi + b_2\eta + b_3\zeta + b_4\xi\eta + b_5\xi\zeta + b_6\eta\zeta + b_7\xi\eta\zeta \\
z &= c_1\xi + c_2\eta + c_3\zeta + c_4\xi\eta + c_5\xi\zeta + c_6\eta\zeta + c_7\xi\eta\zeta
\end{aligned}
$$

Closed form expressions for the integrals may be succintly given for lower orders, however, for arbitrary orders, the integrals of the scalar functions are most expediently evaluated using a standard Gaussian integration scheme.

$$\lambda_n = \sum_{i=1}^{N_r}\sum_{j=1}^{N_s}\sum_{k=1}^{N_p} A_{ijk}|J(\xi_i,\eta_j,\zeta_k)|x^r(\xi_i,\eta_j,\zeta_k)y^s(\xi_i,\eta_j,\zeta_k)z^p(\xi_i,\eta_j,\zeta_k)$$

For stresses assumed in natural coordinates, the basic integrals required in computing orthogonal spanning modes are of simple form and can be evaluated explicitly for any arbitrary powers in terms of coefficients, r_i, of the Jacobian determinant. For the E8NL element, 23 basic integrals are defined by

$$\phi = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[|\mathbf{J}|\begin{Bmatrix} 1 \\ \xi \\ \eta \\ \zeta \\ \xi\eta \\ \eta\zeta \\ \zeta\xi \end{Bmatrix}[1,\xi,\eta,\zeta,\xi\eta,\eta\zeta,\zeta\xi]d\xi d\eta d\zeta = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \\ \lambda_2 & \lambda_8 & \lambda_5 & \lambda_7 & \lambda_{14} & \lambda_{11} & \lambda_{16} \\ \lambda_3 & \lambda_5 & \lambda_9 & \lambda_6 & \lambda_{12} & \lambda_{17} & \lambda_{11} \\ \lambda_4 & \lambda_7 & \lambda_6 & \lambda_{10} & \lambda_{11} & \lambda_{15} & \lambda_{13} \\ \lambda_5 & \lambda_{14} & \lambda_{12} & \lambda_{11} & \lambda_{18} & \lambda_{22} & \lambda_{23} \\ \lambda_6 & \lambda_{11} & \lambda_{17} & \lambda_{15} & \lambda_{22} & \lambda_{19} & \lambda_{21} \\ \lambda_7 & \lambda_{16} & \lambda_{11} & \lambda_{13} & \lambda_{23} & \lambda_{21} & \lambda_{20} \end{bmatrix}$$

and evaluate to

$$\begin{aligned}
\lambda_1 &= 8(3r_1 + r_8 + r_9 + r_{10})/3 & \lambda_{13} &= 8(15r_2 + 5r_{14} + 9r_{16})/135 \\
\lambda_2 &= 8(3r_2 + r_{14} + r_{16})/9 & \lambda_{14} &= 8(15r_3 + 9r_{12} + 5r_{17})/135 \\
\lambda_3 &= 8(3r_3 + r_{12} + r_{17})/9 & \lambda_{15} &= 8(15r_3 + 5r_{12} + 9r_{17})/135 \\
\lambda_4 &= 8(3r_4 + r_{13} + r_{15})/9 & \lambda_{16} &= 8(15r_4 + 9r_{13} + 5r_{15})/135 \\
\lambda_5 &= 8(3r_5 + r_{20})/27 & \lambda_{17} &= 8(15r_4 + 5r_{13} + 9r_{15})/135 \\
\lambda_6 &= 8(3r_7 + r_{18})/27 & \lambda_{18} &= 8(15r_1 + 9r_8 + 9r_9 + 5r_{10})/135 \\
\lambda_7 &= 8(3r_6 + r_{19})/27 & \lambda_{19} &= 8(15r_1 + 5r_8 + 9r_9 + 9r_{10})/135 \\
\lambda_8 &= 8(15r_1 + 9r_8 + 5r_9 + 5r_{10})/45 & \lambda_{20} &= 8(15r_1 + 9r_8 + 5r_9 + 9r_{10})/135 \\
\lambda_9 &= 8(15r_1 + 5r_8 + 9r_9 + 5r_{10})/45 & \lambda_{21} &= 8(5r_5 + 3r_{20})/135 \\
\lambda_{10} &= 8(15r_1 + 5r_8 + 5r_9 + 9r_{10})/45 & \lambda_{22} &= 8(5r_6 + 3r_{19})/135 \\
\lambda_{11} &= 8r_{11}/27 & \lambda_{23} &= 8(5r_7 + 3r_{18})/135 \\
\lambda_{12} &= 8(15r_2 + 9r_{14} + 5r_{16})/135 & &
\end{aligned}$$

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