DEVELOPMENT OF AN ADVANCED CONTINUUM THEORY FOR COMPOSITE LAMINATES

Final Report

Volume I

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13. ABSTRACT *(Maximum 200 words)*

A continuum theory with a micro-macro structure was developed for composite laminates that satisfies the traction and displacement continuity requirements at interfaces. The interlaminar stresses were included in the model in a natural way and without any ad hoc assumptions. The built-in micro-structure of the continuum model can account for the effects of curvature and geometric nonlinearity. A set of constitutive relations in terms of material properties of individual constituents was developed which is capable of modeling fiber orientation and stacking sequence. The theory was further expanded to include the effects of temperature where a set of coupled thermo-mechanical field equations with corresponding constitutive relations were derived. Development of the theory for cylindrical and spherical geometries was completed and wave motions in flat and curved geometries was studied. A layerwise shear-deformable multi-director C^0 finite element formulation for the theory was developed which accounts for 3-D effects, allows thickness variation, and permits the warping of the deformed normal, while satisfying all continuity requirements.

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VOLUME II

1.0 INTRODUCTION

1.1 Significance of the Problem

Advanced, high performance composite materials are really material systems. The constituent materials interact in such a way that their collective response is more than the linear sum of the response of the constituents. This simple reality provides the technical community with a remarkable opportunity to create composite material systems which are uniquely suited to perform specific engineering tasks. At the same time, this systems aspect of composite materials is a very great challenge to the research community. It introduces complexity, nonlinearity, and scaling problems (to name a few) which require new developments to represent geometry and material behavior, from the standpoint of mechanics.

This challenge is even more formidable when one attempts to find modeling approaches to the representation of the long-term response of composite systems to cyclic mechanical, chemical, and thermal loading. The systems aspect in that context creates the need to represent deformation, degradation, aging, and other processes. These process are "multidisciplinary" in every sense, and the mechanics, chemistry, thermodynamics, and physics of their activity is generally coupled.

The "performance" of a material system is not a material property like stiffness. It depends on the manner in which the environment of mechanical, chemical, and thermal loads are applied, and on the history of that application. The physical events that determine performance are often a "process," with rates and interactions that must be considered and characterized in order to properly describe and anticipate the consequences.

Recent advances in the technologies of manufacturing and materials have enhanced the current application of composite materials from being used as secondary structural elements to becoming primary load-carrying structural components. Consequently thicker and thicker composites structures are being made to carry higher loads. Due to the inherent inhomogeneity and anisotropy of the materials, analysis of these composite structures imposes new challenges to engineers. A widespread and efficient application of composite materials requires detailed and reliable knowledge of their physical properties and, in turn, of their behavior under applied loads. There are a number of important technical problems associated with the mechanics of composite materials. One such problem is the effect of discontinuities (holes and notches) on the strength of composite laminates. This issue is critical for the determination of the load bearing capacity of composite laminates; which is directly applicable to the design of composite panels and the location of fastener holes. Indeed, the manufacture and repair of advanced composite structures have serious problems connected with the placement of fastener holes. This is especially relevant to composite panel repair, both in the field and at the repair facility. At the present time all depots are confronted with these problems. The lack of appropriate data has resulted in new and in-service designs which are often unnecessarily conservative and expensive (both in cost and turn-around-time). Another related problem is the issue of interlaminar response of composite materials which is directly related to delamination and edge effects in composites. In recent time, delamination has become the most feared failure mode in laminated composite structures. It can exhibit unstable crack growth, and while delamination failure itself is not usually a catastrophic event, it can perpetrate such a condition due to its weakening influence on a component in its resistance to subsequent failure modes. Study of delamination is one of the prominent topics in composite mechanics research. Another issue in the engineering application of composite materials is the modeling and study of structures with curved geometries. Because of the complex nature of these structures, present computational capabilities are far behind the engineering developments and only very limited simulations of these systems are feasible presently. All the foregoing problems share one common deficiency, namely, the lack of an adequate and sound theory predicated upon principles of continuum mechanics that could be implemented through an accurate and efficient numerical scheme. Here in Berkeley Applied Science and Engineering Inc. (BASE) we started this research to address these very basic issues associated with application of composites. In particular the following objectives were followed during the course of this research.

1. Development of a thermomechanical theory for composite laminates that has a continuum character. The theory should be able to account for the three-dimensional responses of laminated plates and shells. The theory should also account for effects of micro-structure, anisotropy, and geometric nonlinearities.

2. Formulation of the theory in the context of finite element and numerical implementation of the theory through computationally efficient algorithms suitable for composite applications.

3. Verification of the theory through analysis of a series of benchmark problems.

1.2 Theories of Laminated Composite Plates and Shells

There has been an increasing amount of research activity pertaining to the mechanics of composite laminates and multilayered plate and shell theories. The scope of multilayered shell theories encompasses all the methods used in conjunction with two-dimensional treatments of composite shells. These methods generally lead to a system of partial differential equations in two independent spatial variables, along with a set of boundary/initial conditions compatible with them. As in the case of single-layer isotropic shells, all the different approaches for constructing multilayered shell theories can be viewed as either a single approximation or successive approximations of three-dimensional elasticity models. For a review and a complete list of references, the reader is referred to Noor, A. K., and Burton, W. S. [1990]. The following four general approaches for constructing two-dimensional theories for multilayered shells were identified in this work:

1. method of hypothesis;
2. method of expansion;
3. asymptotic integration technique;

4. iterative methods and methods of successive corrections.

The *first approach* is an extension of the Kirchhoff-Love approach and is based on introducing *a priori* plausible kinematic or static assumptions regarding the variation of displacements, strains and/or stresses in the thickness direction. The simples of these hypotheses is the linear variation of the displacement components used in conjunction with first-order shear deformation theories. Although the method of hypotheses has the advantages of physical clarity and simplicity of applications, it has the drawback of not providing an estimate of the error in the response predictions.

The *second approach* was initiated by Cauchy and Poisson around 1828, and is based on a series expansion, in terms of the thickness coordinate for displacement and/or stresses. For isotropic and anisotropic plates and shells, power series, Legendre polynomials, and trigonometric functions have been employed. The second approach also includes the method of initial functions in which the displacements and stresses are expanded in a Taylor series in the thickness coordinate. The relations between the higher-order derivatives of each of the displacements and stresses and their lower-order derivatives are obtained by successive differentiation of the three-dimensional elasticity relations.

In the *third approach,* appropriate length scales are introduced in the three-dimensional elasticity equations for the different response quantities, followed by parametric (asymptotic) expansions of these quantities in power series in terms of a small thickness parameter. The three-dimensional elasticity equations are thereby reduced to *recursive sets of two-dimensional equations*, governing the interior and edge zone responses of the shell. The edge zone (or boundary layer) is produced by self-equilibrated (in the thickness direction) boundary stresses. The lowest-order system of two-dimensional equations, depending on the choice of the length scales, corresponds to the thin-shell approximation. The higher-order systems introduce thickness correction effects in a systematic and consistent manner. This approach was first applied to isotropic shells by Reissner [1960]. Later, it was extended to anisotropic shells.

The *fourth approach* includes various iterative approximations of the three-dimensional elasticity equations, and predictor-corrector procedures based on a single or successive corrections of the two-dimensional equations.

The following comments on the different approaches of constructing two-dimensional shell theories and the boundary conditions to be used in conjunction with these theories are in order:

1. The state of stress in the shell can be decomposed into an internal state of stress and a boundary layer. The first is generated by external surface forces, and by boundary and reactive stresses, which are not self-equilibrated. The boundary layer is generated by self-equilibrated (in the thickness direction) boundary stresses. The method of hypotheses and the method of expansion can describe well the internal state of stress, but are not suited for describing the boundary layer (because of the complicated nature of the displacement and/or stress distribution through the thickness). By contrast, the asymptotic integration technique is well suited for describing both the internal state of stress and the boundary layer of the shell.

2. If the method of expansion is contrasted with the asymptotic integration approach, the following two major differences can be identified:

(a) No *a priori* assumptions are made regarding the relative magnitudes of the different stress components in the method of expansion. By contrast, in the asymptotic integration approach, such assumptions have to be made either explicitly or implicitly.

(b) Whereas the method of expansion leads to a set of simultaneous equations in all the parameters, the asymptotic integration technique leads to *recursive sets of equations* for both the interior and the edge zone (or boundary layer) of the shell. The lowest-order equations for the interior of the shell correspond to the classical Kirchhoff-Love theory.

3. The aforementioned four approaches are not mutually exclusive. Some of the theories developed can be classified in more than one category. Also, hybrid methods, combining more than one approach, have been proposed. Examples of these hybrid methods are: (a) the use of a three-dimensional model for the core and a two-dimensional model for the facings of a sandwich shell, and (b) the two-step approach based on using a two-dimensional shell theory to evaluate the in-plane stresses and then applying the three-dimensional equilibrium equations to evaluate the transverse shear and normal stresses in laminated composite shells.

4. Although most of the theories developed for laminated composite shells replace the actual shell (or each of its layers) by a smeared ordinary continuum, some microstructural and generalized continuum shell theories have been proposed. In the first class (microstructural theories), the shell is considered to consist of alternating layers of relatively rigid material (with properties representative of fibers) interspersed between flexible layers with properties typical of the matrix material.

5. The derivation of the correct boundary conditions for a particular shear deformation shell theory from prescribed data seems to be important, even for thin shells. Some recent work on isotropic plates and shells indicated that the use of approximate boundary conditions in conjunction with higher-order shell theories can lead to significant errors in the predictions of the shell theory. Therefore, Saint Venant's principle needs to be re-examined when applied, in conjunction with higher-order shear deformation theories, to shell problems.

Extensive research effort has been devoted to the classical laminated theory (C.L.T.) in the past and a huge amount of literature is available on this topic. The classical laminate theory is a direct extension of classical plate theory in which the well known Kirchhoff-Love kinematic hypothesis is enforced. This theory is adequate when the thickness (to side or radius ratio) is small and anisotropy is not pronounced. The range of applicability of the C.L.T. solution has been well established for laminated flat plates. It indicates that a theory which accounts for the transverse shear deformation effects would be adequate to predict only the gross behavior of the laminate.

In order to overcome the deficiencies in C.L.T., refined laminate theories have been proposed. These are single layer theories in which the transverse shear stresses are taken into account. They provide improved global response estimates for deflections, vibration frequencies and buckling loads of moderately thick composites when compared to the classical laminate theory. A Mindlin type first-order transverse shear deformation theory (S.D.T.) was first developed by Whitney and Pagano [1970] for multilayered anisotropic plates, and by Dong and Tso [1972] for multilayered anisotropic shells. Both of these approaches (C.L.T. and S.D.T.) considered all layers as one equivalent single anisotropic layer; thus these approaches are inadequate to model the warpage of cross-sections, that is, the distortion of the deformed normal due to transverse shear stresses. Furthermore, the assumption of nondeformable normal results in incompatible shearing stresses between every two adjacent layers. Also the later approach requires the introduction of an arbitrary shear correction factor which is dependent on the lamination parameters for obtaining accurate results.

The exact analyses performed by Pagano [1989] on the composite flat plates have indicated that the distortion of the deformed normal is dependent not only on the laminate thickness, but also on the orientation and the degree of orthotropy of the individual layers. Therefore the hypothesis of nondeformable normals, while acceptable for isotropic plates and shells is often quite unacceptable for multilayered anisotropic plates and shells with very large ratio of Young's modulus to shear modulus, even if they are relatively thin. Thus a *transverse shear deformation theory which also accounts for distortion of the deformed normal* is required for accurate prediction of the behavior of multilayered anisotropic plates and shells.

Along this line the work of M. Epstein and P. G. Glocker [1977,1979], P. M. Pinsky and K. O. Kim [1986], and J. N. Reddy [1988, 1993] can be mentioned where the theory of multi-director surfaces was used to model multi-layered plates and shells. Pinsky and Kim's work was based on multi-layered shell theories of Epstein and Glockner where the concept of multi-director field defined over one reference surface was employed for the description of the initial geometry and motion of multi-layered shells.

Reddy proposed a displacement based, layerwise shear deformable, C^0 theory which also accounts for the warping of the composite cross section. In his theory, there is a single reference surface and a director is associated with this reference surface. The variable kinematic finite element is developed by superimposing several types of assumed displacement fields within the finite element domain. The underlying foundation of the displacement field is provided by the assumed displacement field of any desired equivalent-single-layer theory and the layerwise displacement field is included as an incremental enhancement to this underlying field. This work has been reported for linear analysis and for flat geometry domain.

Unlike the equivalent single-layer theories, the layerwise theories assume separate displacement field expansions within each material layer, thus providing a kinematically correct representation of the strain field in discrete layer laminates and allowing accurate determination of ply level stresses. During the course of this research, we developed a layerwise shear deformable, multi-director theory which directly address the technical drawbacks present in most of the theories that have been proposed for composite analysis to date. The main features of the theory are summarized as follows:

- The displacement field proposed in this work is continuous in 3-D where as the rotation field is layer-wise continuous (in 2-D) and can be discontinuous across the finite element layers through the thickness direction.
- The displacement field fulfills a priori the static and geometric continuity conditions between contiguous layers.
- The **novel idea** in the assumed displacement field lies in its *capability to model the distortion of the deformed normal*, without increasing the number and order of the par-

theory.

- Another new idea in the theory is its *3-D feature,* thereby modeling the interlaminar conditions and predicting the 3-D edge effects more accurately.
- A salient feature of the proposed theory is that, *at most, only first derivatives of displacement and rotation fields appear in the variational equations*. The practical consequence of this fact is that only C^0 continuity of finite element functions is required which is readily satisfied by the family of Lagrange elements.
- The number of partial differential equations in the resulting system is *independent of the number of plies and their orientations in the composite.*
- Another advantage of the proposed composite shell theory lies in the *greater flexibility in the specification of the boundary conditions.*
- The theory covers a wide range in the sense that in one limit case when there is only one layer of proposed elements through the thickness, one recovers the features of the standard Shear Deformation Theories (S.D.T.). However the added advantage in the present case lies in the 3-D feature of the theory which controls the variation in the thickness via the Poisson terms rather than ad hoc mathematical tricks as done in the literature.
- In another limit case, one can model the composite with one element per ply through the co[illegible]te thickness, a procedure that is typically done while using the standard 3-D anisotropic elasticity elements. The added advantage of the proposed theory in this limit case is that because of the shear deformation capability of the proposed elements, they model the warping of the deformed normal more accurately, thereby improving the bending behavior.
- From a practical design point of view it provides the engineer the freedom to determine the precision in analysis. If a general response of the composite structure is required, the composite can be modeled with one element through the thickness. On

the other hand, the designer can model the thickness with as many layers of the proposed element as deemed necessary to achieve the required accuracy.

- Furthermore, it is feasible to employ this formulation for constructing plate and shell finite elements via the *finite element displacement method*.

Details of this work are discussed in this report and a summary of the content of the report is presented next.

1.3 Summary of the Report

The results of our efforts during the course of this research are presented in section 2 through section 14 of this report. A summary of the contents of these sections is presented in the following.

In section 2 the kinematics of the micro- and macro-structures were examined and the relationship between strain measures at micro- and macro-levels were derived. The field equations for composite laminates were derived through a direct integration of field equations of classical continuum mechanics. The linearized kinematic measures were derived in the context of infinitesimal deformation and the relation of linear strain measures with displacement vector and director displacement vector were obtained. The equations of motion in the linear theory were derived and were presented for both curved and flat geometries.

Section 3 showed the derivation of constitutive relations for composite laminates. A procedure for deriving the relation between composite quantities (i.e., *composite stress tensor* and *composite couple stress tensor*) and strain measures at macro-level were presented. The derivation was performed for a bi-constituent composite laminate and the constitutive relations were expressed in terms of material constants associated with every individual layer.

Section 4 presented the complete theory for linear elastic composite laminates. The relationship between the displacement vector and the director displacement vector was derived based on the geometrical continuity at interfaces. The field equations were derived in terms of displacement vector and it was shown that classical continuum theory can be derived from *Cosserat composite theory* for the case of a single constituent. The theory was further simplified for bi-laminate micro-structure composed of isotropic constituents. Finally the constitutive relations for composite stress tensor, composite couple stress tensor and interlaminar stress vector were derived in terms of the displacement vector, its gradients and material constants of individual constituents.

Section 5 was the extension of the theory for multi-constituent composites. The microstructure or representative element was assumed to be composed of several constituents which repeated themselves in the layering direction. The development of this section is particularly suited for fiber reinforced composites where the fiber direction changes in the stacking sequence of the plies. The theory was simplified for the case of isotropic constituents.

Section 6 presented the extension of the theory from a purely mechanical theory to a thermomechanical theory. In this section composite field quantities corresponding to the heat flux vector, the heat supply and the specific entropy of classical thermo-mechanical theory were introduced and the equation of local balance of energy and the Clausius-Duhem inequality were derived in terms of these composite field quantities.

Section 7 presented the constitutive relations of linear thermoelasticity for composite laminates. These constitutive relations were derived for the composite stress tensor, composite couple stress tensor, entropy, heat flux vector and heat flux couple vector. The developments of this section were parallel to those of section 4 and a set of coupled thermomechanical field equations in terms of the displacement vector and the temperature were presented.

In section 8 a linear theory for cylindrical laminates was presented. Relative kinematic measures for cylindrical geometries were discussed and linearized field equations along with constitutive relations in cylindrical coordinate systems were obtained. The theory was extended to thermoelasticity and explicit thermoelastic constitutive relations for isotropic layers were derived.

Section 9 followed developments parallel to section 8 but for composite laminates with spherical geometry. The theory was extended to thermoelasticity.

Section 10 presented the results of stress analysis of a composite laminate with traction free edges. The problem of a finite-width symmetrically laminated composite plate under uniform one-dimensional stretch was studied and it was shown that the present theory captures the three-dimensional response of the laminate at the free edge boundaries. The predicted results were in agreement with earlier studies of the subject.

Section 11 presented the wave equations in laminated flat composites. Expressions of wave velocities for longitudinal waves, horizontally polarized shear waves and vertically polarized shear waves were derived.

Section 12 followed studies parallel to those of section 11 but for elastic waves in cylindrical and spherical laminates. Expressions for motion of rotary shear waves, axial shear waves and radial waves in cylindrical composite laminates were derived and general solutions in terms of Hankel functions of first and second kind were presented. Similar developments for spherical laminates were followed. It was shown that for isotropic materials the displacement equations of motion for waves with polar symmetry can be recovered.

In Section 13 the finite element formulation of the theory was presented. The approach proposed in this work utilized a displacement field which fulfilled a priori the static and geometric continuity conditions between contiguous layers. The order of the system was the same as in the first-order shear deformation theory. The chief advantage of the assumed displacement field rested on its capability to model the distortion of the deformed normal and to satisfy the continuity requirements without increasing the number and the order of the partial differential equations with respect to the first order transverse shear deformation theory. The theory was used to construct plate and shell elements for composite laminates which accounted for the 3-D effects, through-the-thickness variations of stress and strain measures, and permitted the warping of the deformed normal. These capabilities, in particular for curved geometries, are unique features of the present developments. Based on these developments accurate stress analysis of composite shell structures is no longer a formidable task.

Section 14 presented the results of several finite element modelings. These analyses were performed for both flat and curved geometries. Various fiber orientations were considered and different loading conditions were examined. The study included:

- extension analysis of flat composite laminates with free edges.
- bending analysis of composite plates with different boundary conditions.
- stress analysis of composite laminates with geometric discontinuity in the form of a circular hole.
- bending analysis of cylindrical shell composites with free edge conditions.

These analyses showed the main features of the present theory. In particular, an accurate modeling of discontinuities in composites and analysis of laminates with curved geometries was shown through the application of the proposed theory. These unique enhancements of mechanics of composite materials provide the required computational capabilities for further application of composites.

2.0 MICRO-MACRO CONTINUUM MODEL OF COMPOSITE LAMINATES

2.1 Kinematics of Micro- and Macro-Structures

Let the points of a region $\mathcal{R}$ in a three dimensional Euclidean space be referred to a fixed right-handed rectangular Cartesian coordinate system x^i ($i = 1,2,3$) and let θ^i ($i = 1,2,3$) be a general *convected* curvilinear coordinate system defined by the transformation $x^i = x^i(\theta^j)$. We assume this transformation is nonsingular in $\mathcal{R}$. Furthermore, let ξ represent the coordinate of a micro-structure in the layering direction with $\xi = 0$ corresponding to the bottom surface of the micro-structure. We recall that a convected coordinate system is normally defined in relation to a continuous body and moves continuously with the body throughout the motion of the body from one configuration to another.

Throughout this work, all Latin indices (subscripts or superscripts) take the values 1,2,3; all Greek indices (subscripts or superscripts) take the values 1,2 and the usual summation convention is employed. We will use a comma for partial differentiation with respect to coordinates θ^α and a superposed dot for material time derivative, i.e., differentiation with respect to time holding the material coordinates fixed. Also, we use a vertical bar (|) for covariant differentiation. In what follows, when there is a possibility of confusion, quantities which represent the same physical/geometrical concepts will be denoted by the same symbol but with an added asterisk (*) for classical three dimensional continuum mechanics and no addition for composite laminate (macro-structure). For example, the mass densities of a body in the contexts of the classical continuum mechanics, and the composite laminate (macro-structure) will be denoted by ρ^* and ρ, respectively.

The micro-macro continuum model of a composite laminate is illustrated in Figures 1 and 2. Figure 1 shows a typical composite laminate (only three micro-structures are shown in this figure). Figure 2 shows a shell-like micro-structure with its associated coordinates. This micro-structure is composed of two constitutents and can be generalized for cases of multi-constituent composites.

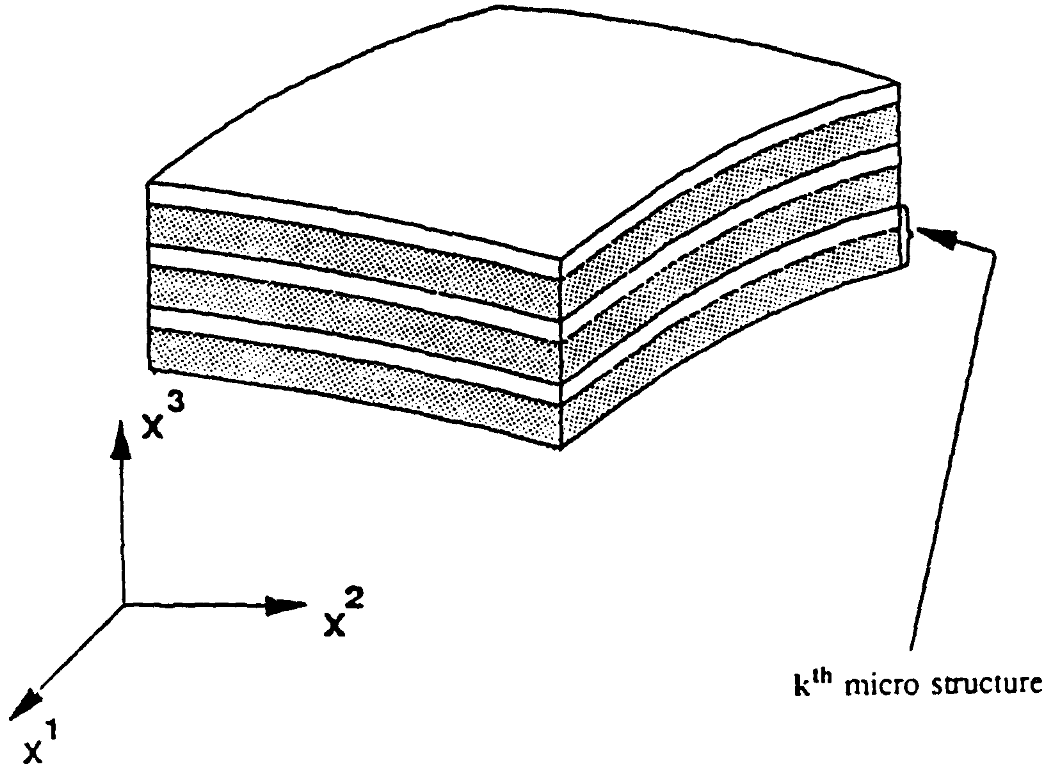

Figure 1

A composite laminate consisting of alternating layers of two materials

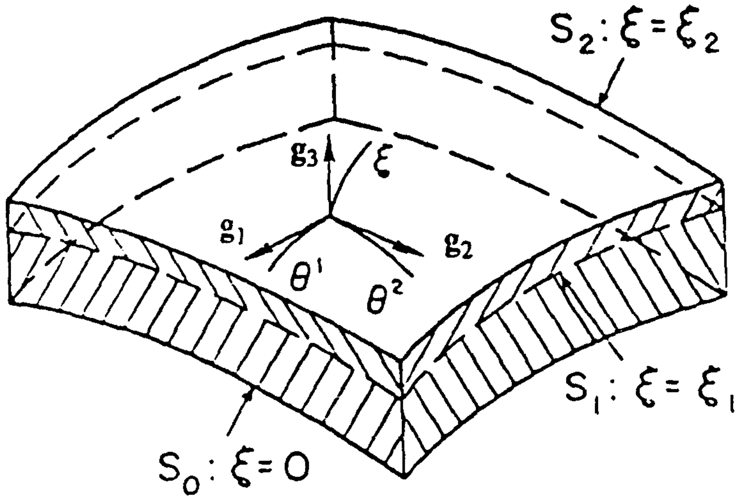

Figure 2

A SHELL-LIKE MICRO-STRUCTURE (REPRESENTATIVE ELEMENT)

We begin the development of the kinematical results by assuming that the position vector of a particle P^* of a representative element (k^{th} micro-structure), i.e., $p^*(\theta^\alpha,\theta^{3(k)},\xi,t)$ in the present configuration has the form

$$p^* = r(\theta^\alpha,\theta^{3(k)},t) + \xi d(\theta^\alpha,\theta^{3(k)},t) \quad (k = 1,\dots,n) \tag{2.1}$$

where r is the position vector for the surface $\xi = 0$ and d is the director field. $\theta^{3(k)}$, at this point, is an identifier for the k^{th} micro-structure. Greek super- or subscripts will assume values of 1 and 2 only. The dual of (2.1) in a reference configuration is given by

$$P^* = R^*(\theta^\alpha,\theta^{3(k)}) + \xi D(\theta^\alpha,\theta^{3(k)}) \tag{2.2}$$

If the reference configuration is taken to be the initial configuration at time t = 0, we obtain

$$\begin{aligned} p^*(\theta^\alpha,\theta^{3(k)},\xi,0) &= r(\theta^\alpha,\theta^{3(k)},0) + \xi d(\theta^\alpha,\theta^{3(k)},0) \\ &= R(\theta^\alpha,\theta^{3(k)}) + \xi D(\theta^\alpha,\theta^{3(k)}) = P^*(\theta^\alpha,\theta^{3(k)},\xi) \end{aligned} \tag{2.3}$$

The velocity vector v^* of the three-dimensional shell-like micro-structure at time t is given by

$$v^* = \frac{\partial p^*(\theta^\alpha,\theta^{3(k)},\xi,t)}{\partial t} = \dot{p}^*(\theta^\alpha,\theta^{3(k)},\xi,t) \tag{2.4}$$

where a superposed dot denotes the material time derivative, holding θ^α and ξ fixed. From (2.1) and (2.4) we obtain

$$v^* = v + \xi w \tag{2.5}$$

where

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{w} = \dot{\mathbf{d}} \tag{2.6}$$

The base vectors for the micro- and macro-structures are denoted by $\mathbf{g}_i^*$ and $\mathbf{g}_i$, respectively, and we have

$$\mathbf{g}_\alpha^* = \frac{\partial \mathbf{p}^*}{\partial \theta^\alpha} \quad , \quad \mathbf{g}_3^* = \frac{\partial \mathbf{p}^*}{\partial \xi}$$
$$\mathbf{g}_\alpha = \frac{\partial \mathbf{p}^*}{\partial \theta^\alpha}\bigg|_{\xi=0} \quad , \quad \mathbf{g}_3 = \frac{\partial \mathbf{p}^*}{\partial \xi}\bigg|_{\xi=0} \tag{2.7}$$

Using (2.1) and (2.7) we obtain the following relations between $\mathbf{g}_i^*$ and $\mathbf{g}_i$

$$\mathbf{g}_\alpha^* = \mathbf{g}_\alpha + \xi \mathbf{d}_{,\alpha}$$
$$\mathbf{g}_3^* = \mathbf{g}_3 = \mathbf{d} \tag{2.8}$$

where $(\ )_{,\alpha}$ denotes partial differentiation with respect to θ^α.

By a smoothing assumption we suggest the existence of continuous vector functions $\mathbf{g}_i(\theta^\alpha,\theta^3)$ for the macro-structure with the following property

$$\mathbf{g}_i(\theta^\alpha,\theta^3)_{|\theta^3=\theta^{3(k)}} = \mathbf{g}_i(\theta^\alpha,\theta^{3(k)}) \tag{2.9}$$

where $\mathbf{g}(\theta^\alpha,\theta^{3(k)})$ are defined according to $(2.7)_2$. A similar smoothing assumption is also made for the director $\mathbf{d}$ which we like to attach to every point of the macro-structure. Based on the smoothing assumptions we can write $(2.8)_1$ as follows

$$\mathbf{g}_\alpha^* = \mathbf{g}_\alpha + \xi \mathbf{g}_k \{_3{}^k{}_\alpha\} \tag{2.10}$$

where { } stands for the Christoffel symbol of the second kind and is defined as

$$\{_3{}^k{}_\alpha\} = g^{kj}[3\alpha,j] = \frac{1}{2}\, g^{kj}\left(\frac{\partial g_{3j}}{\partial \theta^\alpha} + \frac{\partial g_{\alpha j}}{\partial \theta^3} - \frac{\partial g_{3\alpha}}{\partial \theta^j}\right)$$

The following relations can also be derived between the components of metric tensors $g^*_{ij} = g^*_i \cdot g^*_j$ and $g_{ij} = g_i \cdot g_j$

$$g^*_{\alpha\beta} = g_{\alpha\beta} + \xi[\{_3{}^k{}_\alpha\} g_{\beta k} + \{_3{}^k{}_\beta\} g_{\alpha k}] + \xi^2 \{_3{}^k{}_\alpha\}\{_3{}^j{}_\beta\} g_{kj}$$

$$g^*_{\alpha 3} = g_{\alpha 3} + \xi \{_3{}^k{}_\alpha\} g_{k3} \tag{2.11}$$

$$g^*_{33} = g_{33}$$

which after simplification and linearization in terms of ξ reduce to

$$g^*_{\alpha\beta} = g_{\alpha\beta} + \xi g_{\alpha\beta,3}$$

$$g^*_{\alpha 3} = g_{\alpha 3} + \frac{1}{2}\xi g_{33,\alpha} \tag{2.12}$$

$$g^*_{33} = g_{33}$$

The determinants of metric tensors g^*_{ij} and g_{ij} are also related according to the following relation

$$g^* = g + \xi\Delta \tag{2.13}$$

where

$$g^* = \det(g^*_{ij}) \quad , \quad g = \det(g_{ij})$$

$$\Delta = \begin{vmatrix} g_{11,3} & g_{12,3} & g_{33,1} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{vmatrix} + \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{12,3} & g_{22,3} & g_{33,2} \\ g_{13} & g_{23} & g_{33} \end{vmatrix} \tag{2.14}$$

and the final result has been linearized in terms of ξ.

We recall that the director **d** is the same as g_3 and therefore when referred to the base vectors g_i it has only one non-zero component, namely $d^3 = 1$, so we can write

$$\mathbf{d} = d^i \mathbf{g}_i \quad , \quad d^\alpha = 0 \quad , \quad d^3 = 1$$

$$d_i = g_{ij} d^j \quad , \quad d_i = g_{i3} \quad (i=1,2,3) \tag{2.15}$$

where d_i and d^i denote the covariant and contravariant components of $\mathbf{d}$ referred to $\mathbf{g}^i$ and $\mathbf{g}_i$, respectively. The gradient of the director $\mathbf{d}$ may be obtained as follows

$$\mathbf{d}_{,i} = \mathbf{g}_{3,i} = \{_3{}^k{}_i\} \mathbf{g}_k = d^k{}_{|i} \mathbf{g}_k$$

The vertical bar (|) denotes covariant differentiation with respect to g_{ij}. For convenience we introduce the notations

$$\lambda_{ij} = \mathbf{g}_i \cdot \mathbf{d}_{,j} = d_{i|j}$$

$$\lambda^i{}_j = \mathbf{g}^i \cdot \mathbf{d}_{,j} = d^i{}_{|j} \tag{2.17}$$

From (2.17) it is clear that

$$\lambda^i{}_j = g^{ik} \lambda_{kj} \tag{2.18}$$

Making use of (2.17) and (2.16) we have:

$$\lambda^i{}_j = d^i{}_{|j} = \{_3{}^i{}_j\}$$

$$\lambda_{ij} = g_{ki} \lambda^k{}_j = [3j,i] \tag{2.19}$$

Consider now the velocity vector $\mathbf{v}$ which can be written in the form

$$\mathbf{v} = v^i \mathbf{g}_i = v_i \mathbf{g}^i \tag{2.20}$$

Again we make a smoothing assumption for the existence of the vector function $\mathbf{v}(\theta^\alpha,\theta^3)$ such that $\mathbf{v}(\theta^\alpha,\theta^3)_{|\theta^3=\theta^{3(k)}} = \dot{\mathbf{r}}(\theta^\alpha,\theta^{3(k)})$ after which we can define the gradient of the velocity field and we have

$$\mathbf{v}_{,i} = (v^j \mathbf{g}_j)_{,i} = v^j{}_{|i} \mathbf{g}_j \tag{2.21}$$

We now introduce the notations

$$\mathbf{v}_{ij} = \mathbf{g}_i \cdot \mathbf{v}_{,j} = v_{i|j}$$
$$v^i_{\ j} = \mathbf{g}^i \cdot \mathbf{v}_{,j} = v^i_{\ |j} \tag{2.22}$$

From (2.22) it is clear that

$$v^i_{\ j} = g^{ik} v_{kj}$$
$$\mathbf{v}_{,i} = v_{ij}\mathbf{g}^j = v^j_{\ i}\mathbf{g}_j \tag{2.23}$$

We observe that both λ_{ij} and v_{ij} represent the covariant derivative of vector components and hence transform as components of second order covariant tensors.

We may decompose v_{ij} into its symmetric and skew-symmetric parts, i.e.,

$$v_{ij} = v_{(ij)} + v_{[ij]} = \eta_{ij} + \omega_{ij}$$
$$\eta_{ij} = v_{(ij)} = \frac{1}{2}\,(v_{ij}+v_{ji}) = \eta_{ji} \tag{2.24}$$
$$\omega_{ij} = v_{[ij]} = \frac{1}{2}\,(v_{ij}-v_{ji}) = -\,\omega_{ji}$$

Also in view of (2.6), $(2.7)_2$ and $(2.24)_1$ we may express $\dot{\mathbf{g}}_i$ in the form

$$\dot{\mathbf{g}}_\alpha = \mathbf{v}_{,\alpha} = (\eta_{k\alpha} + \omega_{k\alpha})\mathbf{g}^k$$
$$\dot{\mathbf{g}}_3 = \dot{\mathbf{d}} = \mathbf{w} = w_k\mathbf{g}^k = w^k\mathbf{g}_k \tag{2.25}$$

The gradient of the director velocity in θ^α direction is obtained by writing

$$\mathbf{w}_{,\alpha} = \dot{\mathbf{d}}_{,\alpha} = \overline{(\mathbf{d}_{,\alpha})}^{\,\cdot} = \overline{\lambda^k_\alpha \mathbf{g}_k}^{\,\cdot}$$
$$= \overline{\lambda^k_\alpha}^{\,\cdot}\,\mathbf{g}_k + \lambda^k_\alpha\dot{\mathbf{g}}_k = \dot{\lambda}^k_\alpha\mathbf{g}_k + \lambda^\beta_\alpha\dot{\mathbf{g}}_\beta + \lambda^3_\alpha\dot{\mathbf{g}}_3$$

$$= \dot{\lambda}_\alpha^k g_k + \lambda_\alpha^\beta(\eta_{k\beta}+\omega_{k\beta})g^k + \lambda_\alpha^3 w^k g_k$$

$$= [\dot{\lambda}_\alpha^k + \lambda_\alpha^\beta(\eta_\beta^k+\omega_\beta^k) + \lambda_\alpha^3 w^k]g_k \tag{2.26}$$

The dual of the above expressions in the reference configuration can be written easily by substituting appropriate capital letters for small letters.

We now introduce relative kinematic measures γ_{ij} and κ_{ij} such that

$$\gamma_{ij} = \frac{1}{2}(g_{ij} - G_{ij}) = \gamma_{ji} \tag{2.27}$$

$$\kappa_{ij} = \lambda_{ij} - \Lambda_{ij} \tag{2.28}$$

where

$$G_{ij} = G_i \cdot G_j \tag{2.29}$$

$$\Lambda_{ij} = [3j,i] = \frac{1}{2}(G_{3i,j} + G_{ji,3} - G_{3j,i}) \tag{2.30}$$

Making use of (2.12) and similar expressions for the reference configuration we can relate relative kinematic measures $\overset{*}{\gamma}_{ij}$ of the micro-structure as follows

$$\overset{*}{\gamma}_{\alpha\beta} = \overset{*}{\gamma}_{\beta\alpha} = \frac{1}{2}(\overset{*}{g}_{\alpha\beta} - \overset{*}{G}_{\alpha\beta}) = \frac{1}{2}[(g_{\alpha\beta} + \xi\, g_{\alpha\beta,3}) - (G_{\alpha\beta} + \xi\, G_{\alpha\beta,3})]$$

$$= \gamma_{\alpha\beta} + \frac{1}{2}\xi(g_{\alpha\beta,3} - G_{\alpha\beta,3})$$

$$= \gamma_{\alpha\beta} + \frac{1}{2}\xi(\kappa_{\alpha\beta} + \kappa_{\beta\alpha}) \tag{2.31}$$

$$\overset{*}{\gamma}_{\alpha 3} = \overset{*}{\gamma}_{3\alpha} = \frac{1}{2}[(g_{\alpha 3} + \frac{1}{2}\xi\, g_{33,\alpha}) - (G_{\alpha 3} + \frac{1}{2}\xi\, G_{33,\alpha})]$$

$$= \gamma_{\alpha 3} + \frac{1}{2}\,\xi\,\kappa_{3\alpha} \tag{2.32}$$

$$\gamma^{*}_{33} = \gamma_{33} \tag{2.33}$$

In obtaining the above results we have noted that

$$\lambda_{\alpha\beta} = [3\beta,\alpha] = \frac{1}{2}\,(g_{3\alpha,\beta} - g_{3\beta,\alpha}) + \frac{1}{2}\,g_{\alpha\beta,3}$$

$$\lambda_{3\alpha} = [3\alpha,3] = \frac{1}{2}\,g_{33,\alpha} \tag{2.34}$$

$$\kappa_{\alpha\beta} + \kappa_{\beta\alpha} = g_{\alpha\beta,3} - G_{\alpha\beta,3}$$

and we have linearized the result in terms of ξ.

2.2 Basic Field Equations for Micro- and Macro-Structures

The three-dimensional equations of motion in classical continuum mechanics are recorded here for the kth representative element (micro-structure) in the present configuration

$$\dot{\overline{\rho^* g^{*1/2}}} = 0 \tag{2.35}$$

$$T^{*i}_{,i} + \rho^* b^* g^{*1/2} = \rho^* \dot{v}^* g^{*1/2} \tag{2.36}$$

$$g_i^* \times T^{*i} = 0 \tag{2.37}$$

where

$$t^* = g^{*-1/2} T^{*i} n_i^* \quad , \quad T^{*i} = g^{*1/2} \tau^{*ij} g_j^* \tag{2.38}$$

The argument of all starred functions recorded above is $(\theta^\alpha, \theta^{3(k)}, \xi, t)$ and the equations are written for each and every representative element ($k = 1,2,\ldots,n$) which is assumed to repeat itself in the present model.

Now introduce the following quantities for each micro-structure:

Composite Stress Vector T^i:

$$T^i(\theta^\alpha, \theta^{3(k)}, t) \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} T^{*i}(\theta^\alpha, \theta^{3(k)}, \xi, t) d\xi \tag{2.39}$$

Composite Stress Couple Vector S^α:

$$S^\alpha(\theta^\alpha, \theta^{3(k)}, t) \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \xi T^{*\alpha}(\theta^\alpha, \theta^{3(k)}, \xi, t) d\xi \tag{2.40}$$

Composite Mass Density ρ:

$$\rho g^{1/2} \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} d\xi \tag{2.41}$$

$$\rho g^{1/2}(z^{\alpha}) \triangleq \frac{1}{\xi_2}\int_0^{\xi_2} (\xi)^{\alpha}\rho^* g^{*1/2} d\xi \quad (\alpha=1,2) \tag{2.42}$$

Composite Body Force Density **b**:

$$\rho g^{1/2}\mathbf{b} \triangleq \frac{1}{\xi_2}\int_0^{\xi_2} \rho^*\mathbf{b}^* g^{*1/2} d\xi \tag{2.43}$$

Composite Body Couple Density **c**:

$$\rho g^{1/2}\mathbf{c} \triangleq \frac{1}{\xi_2}\int_0^{\xi_2} \rho^*\mathbf{b}^* g^{*1/2}\xi d\xi \tag{2.44}$$

The quantities on the left-hand side of equations (2.39)-(2.44) are discrete in terms of the variable $\theta^{3(k)}$ which are made continuous by smoothing assumptions. The composite mass density ρ_o in the reference configuration is also defined as follows:

$$\rho_o G^{1/2} = \frac{1}{\xi_2}\int_0^{\xi_2} \rho_o^* G^{*1/2} d\xi \tag{2.45}$$

where ρ_o^* is the mass density of the micro-structure in the reference configuration. Since $\rho^* g^{*1/2} = \rho_o^* G^{*1/2}$, the continuity equation for the macro-structure is readily seen to be

$$\rho g^{1/2} = \rho_o G^{1/2} \tag{2.46}$$

Now consider equation (2.36) and first divide it by ξ_2 and then integrate with respect to ξ from 0 to ξ_2 to obtain the equation for balance of linear momentum for the macro-structure

$$\frac{1}{\xi_2}\int_0^{\xi_2} \mathbf{T}^{*\alpha}{}_{,\alpha} d\xi + \frac{1}{\xi_2}\int_0^{\xi_2} \frac{\partial \mathbf{T}^{*3}}{\partial \xi} d\xi + \frac{1}{\xi_2}\int_0^{\xi_2} \rho^*\mathbf{b}^* g^{*1/2} d\xi$$

$$= \frac{1}{\xi_2}\int_0^{\xi_2} \rho^*(\dot{\mathbf{v}} + \xi\mathbf{w}) g^{*1/2} d\xi \tag{2.47}$$

Each term in the above equation can be represented in terms of the quantities defined in (2.39)-(2.44) except the second term which is the difference between interlaminar stresses above and below the representative element divided by its thickness ξ_2 as

$$\frac{1}{\xi_2}\int_0^{\xi_2} \frac{\partial T^{*3}}{\partial \xi}\, d\xi = \frac{1}{\xi_2}\,[T^{*3}(\theta^\alpha,\theta^{3(k+1)},t) - T^{*3}(\theta^\alpha,\theta^{3(k)},t)] \tag{2.43}$$

Now we postulate the existence of the continuous vector function $\sigma(\theta^\alpha,\theta^3,t)$ whose values at $\theta^3 = \theta^{3(k)}$ are the same as interlaminar stresses $T^{*3}(\theta^\alpha,\theta^{3(k)},t)$ and further approximate (2.43) as the gradient of this function in the θ^3 direction. With this in mind we write (2.47) as

$$T^\alpha{}_{,\alpha} + \frac{\partial \sigma}{\partial \theta^3} + \rho b g^{1/2} = \rho g^{1/2}(\dot{v} + z^1 \dot{w}) \tag{2.44}$$

To obtain the equation for balance of director momentum, (2.36) is multiplied by ξ, integrated from 0 to ξ_2 and divided by ξ_2 to get

$$\frac{1}{\xi_2}\int_0^{\xi_2} \xi T^{*\alpha}{}_{,\alpha} d\xi + \frac{1}{\xi_2}\int_0^{\xi_2} \xi \frac{\partial T^{*3}}{\partial \xi}\, d\xi + \frac{1}{\xi_2}\int_0^{\xi_2} \rho^* b^* g^{*1/2} \xi d\xi$$

$$= \frac{1}{\xi_2}\int_0^{\xi_2} \rho^* g^{*1/2}(\xi \dot{v} + \xi^2 \dot{w}) d\xi \tag{2.50}$$

Again the second term in the above equation can be written as

$$\frac{1}{\xi_2}\int_0^{\xi_2} \xi \frac{\partial T^{*3}}{\partial \xi}\, d\xi = \frac{1}{\xi_2}\,[\xi T^{*3}]_0^{\xi_2} - \frac{1}{\xi_2}\int_0^{\xi_2} T^{*3} d\xi = \sigma - T^3 \tag{2.51}$$

As a result we have

$$S^\alpha{}_{,\alpha} + \sigma - T^3 + \rho g^{1/2} c = \rho g^{1/2}(z^1 \dot{v} + z^2 \dot{w}) \tag{2.52}$$

which is the equation for balance of director momentum.

Next, we consider (2.37), divide it by ξ_2 and integrate with respect to ξ from ξ=o to $\xi=\xi_2$ and making use of $(2.8)_{1,2}$ we get

$$\frac{1}{\xi_2}\int_o^{\xi_2}(\mathbf{g}_\alpha^* \times \mathbf{T}^{*\alpha} + \mathbf{g}_3^* \times \mathbf{T}^{*3})d\xi = 0$$

or (2.53)

$$\frac{1}{\xi_2}\int_o^{\xi_2}(\mathbf{g}_\alpha + \xi \mathbf{d}_{,\alpha}) \times \mathbf{T}^{*\alpha} d\xi + \frac{1}{\xi_2}\int_o^{\xi_2} \mathbf{d} \times \mathbf{T}^{*3}\, d\xi = 0$$

and substituting from (2.39) and (2.40) we obtain

$$\mathbf{g}_\alpha \times \mathbf{T}^\alpha + \mathbf{d}_{,\alpha} \times \mathbf{S}^\alpha + \mathbf{d} \times \mathbf{T}^3 = 0 \tag{2.54}$$

which can also be written as

$$\mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,\alpha} \times \mathbf{S}^\alpha = 0 \tag{2.55}$$

This is the balance of moment of momentum for the macro-structure.

Now we proceed to obtain an expression for the specific mechanical energy. Such an expression for each micro-structure can be written as

$$\rho^* g^{*1/2} \dot{\varepsilon}^* = \mathbf{T}^{*i} \cdot \mathbf{v}_{,i}^* \tag{2.56}$$

First, using (2.5) we write this equation as

$$\rho^* g^{*1/2}\dot{\varepsilon}^* = \mathbf{T}^{*\alpha} \cdot (\mathbf{v} + \xi \mathbf{w})_{,\alpha} + \mathbf{T}^{*3} \cdot \frac{\partial}{\partial \xi}(\mathbf{v} + \xi \mathbf{w})$$

$$= \mathbf{T}^{*\alpha} \cdot \mathbf{v}_{,\alpha} + \xi \mathbf{T}^{*\alpha} \cdot \mathbf{w}_{,\alpha} + \mathbf{T}^{*3} \cdot \mathbf{w} \tag{2.57}$$

Dividing (2.57) by ξ_2 and integrating with respect to ξ from $\xi = 0$ to $\xi = \xi_2$ will result in

$$\frac{1}{\xi_2}\int_o^{\xi_2} \rho^* g^{*1/2} \dot{\varepsilon}^* d\xi = \frac{1}{\xi_2}\int_o^{\xi_2} \mathbf{T}^{*\alpha} d\xi \cdot \mathbf{v}_{,\alpha} + \frac{1}{\xi_2}\int_o^{\xi_2} \xi \mathbf{T}^{*\alpha} d\xi \cdot \mathbf{w}_{,\alpha} + \frac{1}{\xi_2}\int_o^{\xi_2} \mathbf{T}^{*3} d\xi \cdot \mathbf{w} \tag{2.58}$$

BASE

We now define ***composite specific mechanical energy*** for the representative element by

$$\rho g^{1/2}\varepsilon = \frac{1}{\xi_2}\int_{0}^{\xi_2} \rho^* g^{*1/2}\varepsilon^* d\xi \tag{2.59}$$

From this definition, the equation of continuity and other definitions (2.39) through (2.44) for composite quantities, (2.58) can be rewritten as

$$\rho g^{1/2}\dot{\varepsilon} = \mathbf{T}^\alpha \cdot \mathbf{v}_{,\alpha} + \mathbf{S}^\alpha \cdot \mathbf{w}_{,\alpha} + \mathbf{T}^3 \cdot \mathbf{w} \tag{2.60}$$

Since $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{v}_{,\alpha} = (\dot{\mathbf{r}})_{,\alpha} = \dot{\overline{\mathbf{v}_{,\alpha}}} = \dot{\mathbf{g}}_\alpha$ and $\mathbf{w} = \dot{\mathbf{d}} = \dot{\mathbf{g}}_3$, we can further reduce (2.60) to

$$\rho g^{1/2}\dot{\varepsilon} = \mathbf{T}^i \cdot \dot{\mathbf{g}}_i + \mathbf{S}^\alpha \cdot \mathbf{w}_{,\alpha} \tag{2.61}$$

which is the appropriate expression for the specific mechanical energy of the macro-structure.

2.3 Field Equations in Component Form

We obtained the following field equations for the macro-structure (balance of mass is not recorded since it is a scalar equation)

$$\mathbf{T}^{\alpha}{}_{,\alpha} + \frac{\partial \boldsymbol{\sigma}}{\partial \theta^3} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + z^1\dot{\mathbf{w}}) \tag{2.62}$$

$$\mathbf{S}^{\alpha}{}_{,\alpha} + \boldsymbol{\sigma} - \mathbf{T}^3 + \rho g^{1/2}\mathbf{c} = \rho g^{1/2}(z^1\dot{\mathbf{v}} + z^3\dot{\mathbf{w}}) \tag{2.63}$$

$$\mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,\alpha} \times \mathbf{S}^{\alpha} = 0 \tag{2.64}$$

And also the following expression was derived for the specific mechanical energy

$$\rho g^{1/2}\dot{\varepsilon} = \mathbf{T}^i \cdot \dot{\mathbf{g}}_i + \mathbf{S}^{\alpha} \cdot \mathbf{w}_{,\alpha} \tag{2.65}$$

By referring various vector quantities to the base $\mathbf{g}_i$ we would like to write the above equations in component form. First write

$$\mathbf{T}^i = g^{1/2}\,\tau^{ij}\mathbf{g}_i \tag{2.66}$$

$$\boldsymbol{\sigma} = \sigma^j \mathbf{g}_j \tag{2.67}$$

$$\mathbf{S}^{\alpha} = g^{1/2}\,S^{\alpha j}\mathbf{g}_j \tag{2.68}$$

$$\mathbf{b} = b^j\mathbf{g}_j \quad , \quad \mathbf{c} = c^j\mathbf{g}_j \tag{2.69}$$

where τ^{ij} and $S^{\alpha j}$ are contravariant components of *composite stress tensor* and *composite couple stress tensor*, respectively, σ^i is the interlaminar stress. Now substitute in (2.62) and obtain

$$(g^{1/2}\,\tau^{\alpha j}\mathbf{g}_j)_{,\alpha} + \frac{\partial}{\partial \theta^3}(\sigma^j\mathbf{g}_j) + \rho g^{1/2}b^j\mathbf{g}_j = \rho g^{1/2}(\dot{v}^j + z^1\dot{w}^j)\mathbf{g}_j$$

$$(g^{1/2}\,\tau^{\alpha j})_{,\alpha}\mathbf{g}_j + g^{1/2}\,\tau^{\alpha j}\left\{{}_j{}^k{}_\alpha\right\}\mathbf{g}_k + \sigma^j{}_{,3}\mathbf{g}_j + \sigma^j\left\{{}_j{}^k{}_3\right\}\mathbf{g}_k + \rho g^{1/2}b^j\mathbf{g}_j$$

$$= \rho g^{1/2}(\dot{v}^j + z^1\dot{w}^j)\mathbf{g}_j$$

or

$$(g^{1/2}\tau^{\alpha j})_{,\alpha} + g^{1/2}\tau^{\alpha k}\{_k{}^j{}_\alpha\} + \sigma^j{}_{,3} + \sigma^k\{_k{}^j{}_3\} + \rho g^{1/2} b^j$$

$$= \rho g^{1/2}(\dot{v}^j + z^1\dot{w}^j) \tag{2.70}$$

Equation (2.63) reduces to

$$(g^{1/2} S^{\alpha j}\mathbf{g}_j)_{,\alpha} + \sigma^j\mathbf{g}_j - g^{1/2}\tau^{3j}\mathbf{g}_j + \rho g^{1/2}c^j\mathbf{g}_j$$

$$= \rho g^{1/2}(z^1\dot{v}^j + z^2\dot{w}^j)\mathbf{g}_j$$

or

$$(g^{1/2} S^{\alpha j})_{,\alpha} + g^{1/2} S^{\alpha k}\{_k{}^j{}_\alpha\} + \sigma^j - g^{1/2}\tau^{3j} + \rho g^{1/2}c^j = \rho g^{1/2}(z^1\dot{v}^j + z^2\dot{w}^j) \tag{2.71}$$

Equation (2.64) can be rewritten as

$$\mathbf{g}_i \times (g^{1/2}\tau^{ij}\mathbf{g}_j) + \lambda^i{}_\alpha\mathbf{g}_i \times (g^{1/2} S^{\alpha j}\mathbf{g}_j) = 0$$

or

$$g^{1/2}(\tau^{ij} + \lambda^i{}_\alpha S^{\alpha j})\mathbf{g}_i \times \mathbf{g}_j = 0 \tag{2.72}$$

since $g \neq 0$, $\mathbf{g}_i \times \mathbf{g}_j = \varepsilon_{ijk}\mathbf{g}^k$ and ε_{ijk} is skew-symmetric we conclude that the quantity in parentheses in (2.72) must be symmetric in i and j. As a result, the conservation of angular momentum in component form is the symmetry of T^{ij} defined by

$$T^{ij} \triangleq \tau^{ij} + \lambda^i{}_\alpha S^{\alpha j} \tag{2.73}$$

$$T^{ij} = T^{ji} \tag{2.74}$$

The expression (2.65) for the specific mechanical energy can also be written as

$$\rho g^{1/2}\dot{\varepsilon} = g^{1/2}\,\tau^{ij}\mathbf{g}_j \cdot \dot{\mathbf{g}}_i + g^{1/2}\,S^{\alpha j}\mathbf{g}_j \cdot \mathbf{w}^i{}_{|\alpha}\mathbf{g}_i$$

$$= g^{1/2}(\tau^{\alpha j}\mathbf{g}_j \cdot \dot{\mathbf{g}}_\alpha + S^{\alpha j}w_{j|\alpha} + \tau^{3j}\mathbf{g}_j \cdot \dot{\mathbf{g}}_3)$$

$$= g^{1/2}(\tau^{\alpha j}\mathbf{g}_j \cdot \mathbf{v}_{,\alpha} + \tau^{3j}\mathbf{g}_j \cdot \mathbf{w} + S^{\alpha j}w_{j|\alpha})$$

$$= g^{1/2}(\tau^{\alpha j}v_{j|\alpha} + \tau^{3j}w_j + S^{\alpha j}w_{j|\alpha})$$

We have now the component form of the expression for mechanical power

$$P \triangleq \rho\dot{\varepsilon} = \tau^{\alpha j}v_{j|\alpha} + \tau^{3j}w_j + S^{\alpha j}w_{j|\alpha} \tag{2.75}$$

An alternative form for mechanical energy expression is derived in which the rates of relative kinematic measures will appear. Using $(2.25)_1$ and (2.26), we rewrite (2.75) as

$$P = \rho\dot{\varepsilon} = \tau^{\alpha j}(\eta_{j\alpha} + \omega_{j\alpha}) + \tau^{3j}w_j + S^{\beta j}[\dot{\lambda}_{j\beta} + \lambda^{\alpha}_{\beta}(\eta_{j\alpha} + \omega_{j\alpha}) + \lambda^{3}_{\beta}w_j]$$

$$= (\tau^{\alpha j} + S^{\beta j}\lambda^{\alpha}_{\beta})\eta_{j\alpha} + (\tau^{\alpha j} + S^{\beta j}\lambda^{\alpha}_{\beta})\omega_{j\alpha} + S^{\beta j}\dot{\lambda}_{j\beta}$$

$$+ (\tau^{3j} + \lambda^{3}_{\beta}S^{\beta j})w_j \tag{2.76}$$

Recalling (2.73) and using symmetry of T^{ij} and skew-symmetry of w_{ij} we can write (2.76) as

$$P = T^{\alpha j}\eta_{j\alpha} + S^{\beta j}\dot{\lambda}_{j\beta} + T^{3j}w_j + T^{\alpha 3}\omega_{3\alpha} \tag{2.77}$$

By $(2.25)_1$ we have

$$\dot{\mathbf{g}}_\alpha \cdot \mathbf{g}_\beta = \eta_{\beta\alpha} + \omega_{\beta\alpha} \quad , \quad \dot{\mathbf{g}}_\beta \cdot \mathbf{g}_\alpha = \eta_{\alpha\beta} + \omega_{\alpha\beta} \tag{2.78}$$

Therefore

$$\dot{\overline{g_{\alpha\beta}}} = \dot{\mathbf{g}}_\alpha \cdot \mathbf{g}_\beta + \mathbf{g}_\alpha \cdot \dot{\mathbf{g}}_\beta = 2\eta_{\alpha\beta} \tag{2.79}$$

$$\eta_{\alpha\beta} = \frac{1}{2}\,\dot{\overline{g_{\alpha\beta}}} = \dot{\gamma}_{\alpha\beta} \tag{2.80}$$

In the last result we have used the definition of $\gamma_{\alpha\beta}$ from (2.27). By $(2.25)_{1,2}$ we have

$$\dot{\mathbf{g}}_\alpha \cdot \mathbf{g}_3 = \eta_{3\alpha} + \omega_{3\alpha} \tag{2.81}$$

$$\dot{\mathbf{g}}_3 \cdot \mathbf{g}_\alpha = w_\alpha \tag{2.82}$$

Therefore

$$\dot{\overline{g_{\alpha 3}}} = \dot{\mathbf{g}}_\alpha \cdot \mathbf{g}_3 + \dot{\mathbf{g}}_3 \cdot \mathbf{g}_\alpha = \eta_{3\alpha} + \omega_{3\alpha} + w_\alpha \tag{2.83}$$

$$\eta_{3\alpha} = \dot{g}_{\alpha 3} - (\omega_{3\alpha} + w_\alpha) = 2\dot{\gamma}_{\alpha 3} - (\omega_{3\alpha} + w_\alpha) \tag{2.84}$$

Again by $(2.25)_2$ and (2.27)

$$\omega_3 = \dot{\mathbf{g}}_3 \cdot \mathbf{g}_3 = \frac{1}{2}\,\dot{g}_{33} \tag{2.85}$$

$$\dot{\gamma}_{33} = \frac{1}{2}\,\dot{g}_{33} = w_3 \tag{2.86}$$

and by (2.28)

$$\dot{\lambda}_{j\beta} = \dot{\kappa}_{j\beta} \tag{2.87}$$

Substituting from (2.80), (2.84), (2.86) and (2.87) in (2.77) we get

$$P = T^{\alpha\beta}\dot{\gamma}_{\alpha\beta} + T^{\alpha 3}\{2\dot{\gamma}_{\alpha 3} - (\omega_{3\alpha} + w_\alpha)\} + S^{\beta j}\dot{\kappa}_{j\beta} + T^{3\alpha}w_\alpha + T^{33}\dot{\gamma}_{33} + T^{\alpha 3}\omega_{3\alpha} \tag{2.88}$$

which is simplified to

$$P = T^{\alpha\beta}\dot{\gamma}_{\alpha\beta} + 2T^{\alpha 3}\dot{\gamma}_{\alpha 3} + T^{33}\dot{\gamma}_{33} + S^{\beta j}\dot{\kappa}_{j\beta} \tag{2.89}$$

If symmetry of T^{ij} and γ_{ij} is considered we can further simplify (2.89) and obtain

$$P = T^{ij}\dot{\gamma}_{ij} + S^{\beta j}\dot{\kappa}_{j\beta} \tag{2.90}$$

or

$$P = (\tau^{ij} + \lambda_{\alpha}^{i} S^{\alpha j})\dot{\gamma}_{ij} + S^{\alpha j}\dot{\kappa}_{j\alpha} \tag{2.91}$$

2.4 General Constitutive Assumption for Elastic Composite

At this point we postulate the existence of specific internal energy in purely mechanical theory which depends on relative kinematic measures γ_{ij} and $\kappa_{j\alpha}$ as defined in (2.27) and (2.28)

$$\psi = \hat{\psi}(\gamma_{ij}, \kappa_{j\alpha}) \tag{2.92}$$

$$P = \rho\dot{\psi} \tag{2.93}$$

By usual procedures we obtain from (2.91), (2.92) and (2.93)

$$\tau^{ij} = \rho\left(\frac{\partial\hat{\psi}}{\partial\gamma_{ij}} - \lambda^{i}_{\alpha}\frac{\partial\hat{\psi}}{\partial\kappa_{j\alpha}}\right) \tag{2.94}$$

$$S^{\alpha j} = \rho\,\frac{\partial\hat{\psi}}{\partial\kappa_{j\alpha}} \tag{2.95}$$

Now the *composite stress vector* $\mathbf{T}^i$ and the *composite couple stress vector* $\mathbf{S}^\alpha$ from (2.66) and (2.68) will be

$$\mathbf{T}^i = \rho g^{1/2}\left(\frac{\partial\hat{\psi}}{\partial\gamma_{ij}} - \lambda^{i}_{\alpha}\frac{\partial\hat{\psi}}{\partial\kappa_{j\alpha}}\right)\mathbf{g}_j \tag{2.96}$$

$$\mathbf{S}^\alpha = \rho g^{1/2}\left(\frac{\partial\hat{\psi}}{\partial\kappa_{j\alpha}}\right)\mathbf{g}_j \tag{2.97}$$

The coefficient $\rho g^{1/2}$ can be replaced by $\rho_o G^{1/2}$ by taking advantage of the continuity equation. Note that by these constitutive relations for $\mathbf{T}^i$ and $\mathbf{S}^\alpha$ the balance of moment of momentum is identically satisfied.

2.5 Linearized Kinematics

For linearized kinematics let

$$\mathbf{r}(\theta^\alpha,\theta^{3(k)},t) = \mathbf{R}(\theta^\alpha,\theta^{3(k)}) + \varepsilon\mathbf{u}(\theta^\alpha,\theta^{3(k)},t) \tag{2.98}$$

$$\mathbf{d}(\theta^\alpha,\theta^{3(k)},t) = \mathbf{D}(\theta^\alpha,\theta^{3(k)}) + \varepsilon\boldsymbol{\delta}(\theta^\alpha,\theta^{3(k)},t) \tag{2.99}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \varepsilon\dot{\mathbf{u}} \quad ,\mathbf{w} = \dot{\mathbf{d}} = \varepsilon\dot{\boldsymbol{\delta}} \tag{2.100}$$

where ε is a non-dimensional parameter. The motion of the macro-structure describes infinitesimal deformation if the magnitude of the gradient of the displacement vector $\varepsilon\mathbf{u}$ and the magnitude of the director displacement vector $\varepsilon\boldsymbol{\delta}$ are of the order of $\varepsilon << 1$ such that in the following developments we can only retain terms which are linear in ε. The base vectors $\mathbf{g}_i$ are found from $(2.7)_2$ as:

$$\mathbf{g}_\alpha = \mathbf{R}_{,\alpha} + \varepsilon\mathbf{u}_{,\alpha} \tag{2.101}$$

$$\mathbf{g}_3 = \mathbf{d} = \mathbf{D} + \varepsilon\boldsymbol{\delta} \tag{2.102}$$

The corresponding vectors in reference configuration are:

$$\mathbf{G}_\alpha = \mathbf{R}_{,\alpha} \quad , \quad \mathbf{G}_3 = \mathbf{D} \tag{2.103}$$

We now proceed to obtain the relative kinematic measures γ_{ij} and $\kappa_{i\alpha}$. Using $(2.103)_2$ and (2.101) together with the definition of $g_{\alpha\beta}$ and $G_{\alpha\beta}$ we write

$$g_{\alpha\beta} = (\mathbf{G}_\alpha + \varepsilon\mathbf{u}_{,\alpha})\cdot(\mathbf{G}_\beta + \varepsilon\mathbf{u}_{,\beta}) = G_{\alpha\beta} + \varepsilon(\mathbf{G}_\alpha\cdot\mathbf{u}_{,\beta} + \mathbf{u}_{,\alpha}\cdot\mathbf{G}_\beta) + O(\varepsilon^2) \tag{2.104}$$

where $O(\varepsilon^2)$ denotes terms of order ε^2 in displacement gradient, where

$$\mathbf{G}_\alpha\cdot\mathbf{u}_{,\beta} + \mathbf{u}_{,\alpha}\cdot\mathbf{G}_\beta = \mathbf{G}_\alpha\cdot u^j{}_{|\beta}\mathbf{g}_j + u^j{}_{|\alpha}\mathbf{g}_j\cdot\mathbf{G}_\beta$$

$$= \mathbf{G}_\alpha\cdot u^\gamma{}_{|\beta}\mathbf{g}_\gamma + \mathbf{G}_\alpha\cdot u^3{}_{|\beta}\mathbf{g}_3 + u^\gamma{}_{|\alpha}\mathbf{g}_\gamma\cdot\mathbf{G}_\beta + u^3{}_{|\alpha}\mathbf{g}_3\cdot\mathbf{G}_\beta$$

$$= u^{\gamma}{}_{|\beta} \mathbf{G}_{\alpha} \cdot (\mathbf{G}_{\gamma} + \varepsilon \mathbf{u}_{,\gamma}) + u^{3}{}_{|\beta} \mathbf{G}_{\alpha} \cdot (\mathbf{D} + \varepsilon \boldsymbol{\delta}) + u^{\gamma}{}_{|\alpha} (\mathbf{G}_{\gamma} + \varepsilon \mathbf{u}_{,\gamma}) \cdot \mathbf{G}_{\beta}$$

$$+ u^{3}{}_{|\alpha} (\mathbf{D} + \varepsilon \boldsymbol{\delta}) \cdot \mathbf{G}_{\beta} \tag{2.105}$$

Retaining terms which are of the order of unity in (2.105) and substituting the result in (2.104) we find

$$\gamma_{\alpha\beta} = \frac{1}{2} (g_{\alpha\beta} - G_{\alpha\beta}) = \frac{1}{2} (u_{\alpha|\beta} + u_{\beta|\alpha}) + \frac{1}{2} (u^{3}_{|\beta} D_{\alpha} + u^{3}_{|\alpha} D_{\beta}) \tag{2.106}$$

Here covariant differentiation is supposed to be performed with respect to the metric G_{ij} of the reference configuration and instead of $\varepsilon \mathbf{u}$ we have used $\mathbf{u}$ with the same assumptions made for linearization. Similarly we can write

$$g_{\alpha 3} = (\mathbf{G}_{\alpha} + \varepsilon \mathbf{u}_{,\alpha}) \cdot (\mathbf{G}_{3} + \varepsilon \boldsymbol{\delta}) = G_{\alpha 3} + \varepsilon (\mathbf{G}_{\alpha} \cdot \boldsymbol{\delta} + \mathbf{u}_{,\alpha} \cdot \mathbf{G}_{3}) + O(\varepsilon^{2})$$

$$= G_{\alpha 3} + \varepsilon (\delta_{\alpha} + u_{3|\alpha}) + O(\varepsilon^{2}) \tag{2.107}$$

Again using $\boldsymbol{\delta}$ instead of $\varepsilon \boldsymbol{\delta}$ with the same interpretation we obtain

$$\gamma_{\alpha 3} = \gamma_{3\alpha} = \frac{1}{2} (g_{\alpha 3} - G_{\alpha 3}) = \frac{1}{2} (\delta_{\alpha} + u_{3|\alpha}) \tag{2.108}$$

To find γ_{33} we write

$$g_{33} = (\mathbf{G}_{3} + \varepsilon \boldsymbol{\delta}) \cdot (\mathbf{G}_{3} + \varepsilon \boldsymbol{\delta}) = G_{33} + 2\varepsilon \delta_{3} + O(\varepsilon^{2}) \tag{2.109}$$

$$\gamma_{33} = \frac{1}{2} (g_{33} - G_{33}) = \delta_{3} \tag{2.110}$$

As for the measures $\kappa_{i\alpha}$ we proceed as follows

$$\lambda_{\alpha\beta} = \mathbf{g}_{\alpha} \cdot \mathbf{d}_{,\beta} = (\mathbf{G}_{\alpha} + \varepsilon \mathbf{u}_{,\alpha}) \cdot (\mathbf{D}_{,\beta} + \varepsilon \boldsymbol{\delta}_{,\beta}) = \Lambda_{\alpha\beta} + \varepsilon (u^{j}{}_{|\alpha} \mathbf{g}_{j} \cdot \mathbf{D}_{,\beta} + \mathbf{G}_{\alpha} \cdot \delta^{j}{}_{|\beta} \mathbf{g}_{j}) + O(\varepsilon^{2}) \tag{2.111}$$

where:

$$u^j{}_{|\alpha}g_i \cdot D_{,\beta} = u^\gamma{}_{|\alpha}(G_\gamma + \varepsilon u_{,\gamma}) \cdot D_{,\beta} + u^3{}_{|\alpha}(G_3 + \varepsilon\delta) \cdot D_{,\beta}$$

$$= u^\gamma{}_{|\alpha}\Lambda_{\gamma\beta} + u^3{}_{|\alpha}\Lambda_{3\beta} + O(\varepsilon)$$

$$G_\alpha \cdot \delta^j{}_{|\beta}g_j = \delta^\gamma{}_{|\beta}G_\alpha \cdot (G_\gamma + \varepsilon u_{,\gamma}) + \delta^3{}_{|\beta}G_\alpha \cdot (G_3 + \varepsilon\delta)$$

$$= \delta_{\alpha|\beta} + \delta^3{}_{|\beta}D_\alpha + O(\varepsilon)$$

Substituting these results in (2.111) and using the definition of $\kappa_{\alpha\beta}$ we get

$$\kappa_{\alpha\beta} = \lambda_{\alpha\beta} - \Lambda_{\alpha\beta} = u^j{}_{|\alpha}\Lambda_{j\beta} + \delta_{\alpha|\beta} + \delta^3{}_{|\beta}D_\alpha \qquad (2.112)$$

Now we obtain an expression for $\kappa_{3\alpha}$

$$\lambda_{3\alpha} = g_3 \cdot d_{,\alpha} = (G_3 + \varepsilon\delta) \cdot (D_{,\alpha} + \varepsilon\delta_{,\alpha})$$

$$\lambda_{3\alpha} = \Lambda_{3\alpha} + \varepsilon(G_3 \cdot \delta_{,\alpha} + \delta \cdot D_{,\alpha}) + O(\varepsilon^2) \qquad (2.113)$$

We simplify each term separately

$$G_3 \cdot \delta_{,\alpha} = G_3 \cdot (\delta^j{}_{|\alpha}g_j) = \delta^\gamma{}_{|\alpha}G_3 \cdot (G_\gamma + \varepsilon u_{,\gamma})$$

$$+ \delta^3{}_{|\alpha}G_3 \cdot (D + \varepsilon\delta) = \delta^\gamma{}_{|\alpha}D_\gamma + \delta^3{}_{|\alpha}D_3 + O(\varepsilon)$$

$$= \delta^j{}_{|\alpha}D_j + O(\varepsilon) \qquad (2.114)$$

$$\delta \cdot D_{,\alpha} = (\delta^j g_j) \cdot (D^k{}_{|\alpha}G_k) = (\delta^j g_j) \cdot D^3{}_{|\alpha}G_3$$

$$= \Lambda^3_\alpha(\delta^\gamma g_\gamma + \delta^3 g_3) \cdot G_3$$

$$= \Lambda^3_\alpha[\delta^\gamma(G_\gamma + \varepsilon u_{,\gamma}) + \delta^3(G_3 + \varepsilon\delta)] \cdot D$$

$$= O(\varepsilon) + \Lambda^3_\alpha\delta^\gamma D_\gamma + \Lambda^3_\alpha\delta_3 D^3 = \Lambda^3_\alpha\delta^j D_j + O(\varepsilon) \qquad (2.115)$$

However, since $D^\alpha = 0$ and $D^3 = 1$

$$\delta^j D_j = \delta_j D^j = \delta_3 \tag{2.116}$$

Substituting from (2.114), (2.115) and (2.116) in (2.113) and using previous notation for $\boldsymbol{\delta}$ we obtain

$$\kappa_{3\alpha} = \lambda_{3\alpha} - \Lambda_{3\alpha} = \delta^j{}_{|\alpha} D_j + \Lambda^3_\alpha \delta_3 = \delta_{3|\alpha} + \Lambda^3_\alpha \delta_3 \tag{2.117}$$

To recapitulate the relative kinematic measures in linear theory are:

$$\gamma_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha} + u^3{}_{|\alpha} D_\beta + u^3{}_{|\beta} D_\alpha)$$

$$\gamma_{\alpha 3} = \gamma_{3\alpha} = \frac{1}{2}\,(u_{3|\alpha} + \delta_\alpha)$$

$$\gamma_{33} = \delta_3 \tag{2.118}$$

$$\kappa_{\alpha\beta} = \Lambda^j_\beta u_{j|\alpha} + \delta_{\alpha|\beta} + \delta^3{}_{|\beta} D_\alpha$$

$$\kappa_{3\alpha} = \delta_{3|\alpha} + \Lambda^3_\alpha \delta_3$$

For a composite with initially flat plates we can always choose our base vectors G_i such that $G_{ij} = G^{ij} = \delta_{ij}$ and as a result $D^\alpha = D_\alpha = 0$ and if we confine ourselves to small deformations, then all Christoffel symbols vanish and covariant differentiations reduce to partial differentiations and equations (2.118) for relative kinematic measures will further reduce to

$$\gamma_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha,\beta} + u_{\beta,\alpha})$$

$$\gamma_{\alpha 3} = \frac{1}{2}\,(u_{3,\alpha} + \delta_\alpha)$$

$$\gamma_{33} = \delta_3 \tag{2.119}$$

$$\kappa_{\alpha\beta} = \delta_{\alpha,\beta} \quad , \quad \kappa_{3\alpha} = \delta_{3,\alpha}$$

In writing these relations it has been noted that $D_\alpha = 0$, $D_3 = 1$ and $\Lambda^j_\alpha \equiv 0$.

It is also desirable to find a relation between g and G, determinants of metric tensors in present and reference configurations, in the linear theory. We recall the following relations

$$g^{1/2} = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 \quad , \quad G^{1/2} = \mathbf{G}_1 \times \mathbf{G}_2 \cdot \mathbf{G}_3$$

$$\mathbf{g}_1 \times \mathbf{g}_2 = (\mathbf{G}_1 + \varepsilon \mathbf{u}_{,1}) \times (\mathbf{G}_2 + \varepsilon \mathbf{u}_{,2})$$

$$= \mathbf{G}_1 \times \mathbf{G}_2 + \varepsilon(\mathbf{u}_1 \times \mathbf{G}_2 + \mathbf{G}_1 \times \mathbf{u}_{,2}) + O(\varepsilon^2)$$

$$[\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3] = [\mathbf{G}_1 \times \mathbf{G}_2 + \varepsilon(\mathbf{u}_1 \times \mathbf{G}_2 + \mathbf{G}_1 \times \mathbf{u}_{,2})] \cdot [\mathbf{G}_3 + \varepsilon \boldsymbol{\delta}] + O(\varepsilon^2)$$

$$= G^{1/2} + \varepsilon[\mathbf{G}_1 \times \mathbf{G}_2 \cdot \boldsymbol{\delta} + \mathbf{u}_1 \cdot \mathbf{G}_2 \times \mathbf{G}_3 + \mathbf{G}_3 \times \mathbf{G}_1 \cdot \mathbf{u}_{,2}] + O(\varepsilon^2)$$

$$= G^{1/2} + \varepsilon G^{1/2}[\mathbf{G}^3 \cdot \boldsymbol{\delta} + \mathbf{u}_{,1} \cdot \mathbf{G}^1 + \mathbf{G}^2 \cdot \mathbf{u}_{,2}] + O(\varepsilon^2)$$

Retaining terms which are of the order of ε and using the previous notations for $\mathbf{u}$ and $\boldsymbol{\delta}$ we get

$$\left(\frac{g}{G}\right)^{1/2} = 1 + \delta^3 + u^\alpha{}_{|\alpha} \tag{2.120}$$

Now the equation for balance of mass will readily reduce to

$$\rho_o = \rho\left(\frac{g}{G}\right)^{1/2} = \rho(1 + \delta^3 + u^\alpha{}_{|\alpha}) \tag{2.121}$$

and since in linear theory displacement vector $\mathbf{u}$ and director displacement $\boldsymbol{\delta}$ satisfy linearity assumptions we obtain

$$\rho = \rho_o(1 - \delta^3 - u^\alpha{}_{|\alpha}) \tag{2.122}$$

2.6 Linearized Field Equations

We use pertinent results from linear kinematics and usual procedures for linearization to write the field equations in linear theory. It should be recalled in such analysis that g is replaced by G, ρ by ρ_o and Christoffel symbols are calculated with respect to G_{ij}. By omitting the details, the linear version of the equations of motion are recorded here

$$(G^{1/2}\,\tau^{\alpha j})_{,\alpha} + G^{1/2}\,\tau^{\alpha k}\,\{_k{}^j{}_\alpha\} + \sigma^j_{,3} + \sigma^k\,\{_k{}^j{}_3\} + \rho_o\,G^{1/2}\,b^j = \rho_o\,G^{1/2}\,(\ddot{u}^j + z^1\ddot{\delta}^j) \tag{2.123}$$

$$(G^{1/2}\,S^{\alpha j})_{,\alpha} + G^{1/2}\,S^{\alpha k}\,\{_k{}^j{}_\alpha\} + \sigma^j - G^{1/2}\,\tau^{3j} + \rho_o\,G^{1/2}\,c^j = = \rho_o\,G^{1/2}\,(z^1\ddot{u}^j + z^2\ddot{\delta}^j) \tag{2.124}$$

$$T^{ij} = \tau^{ij} + \Lambda^i_\alpha S^{\alpha j} = T^{ji} \tag{2.125}$$

For a composite with initially flat plies further simplification can be made. As mentioned earlier, G = 1 and all Christoffel symbols vanish identically. The resulting balance equations for such a situation will be

$$\tau^{\alpha j}{}_{,\alpha} + \sigma^j{}_{,3} + \rho_o b^j = \rho_o(\ddot{u}^j + z^1\ddot{\delta}^j) \tag{2.126}$$

$$S^{\alpha j}{}_{,\alpha} + \sigma^j - \tau^{3j} + \rho_o c^j = \rho_o(z^1\ddot{u}^j + z^2\ddot{\delta}^j) \tag{2.127}$$

$$T^{ij} = \tau^{ij} = \tau^{ji} \tag{2.128}$$

3.0 CONSTITUTIVE RELATIONS FOR LINEAR ELASTICITY

For the representative micro-structure let

$$\tau^{*ij}_{(\alpha)} = c^{ijkl}_{(\alpha)}\gamma^*_{kl} \quad , \quad \alpha = 1,2 \tag{3.1}$$

where

$$c^{ijkl}_{(\alpha)} = \begin{cases} c^{ijkl}_{(1)} & 0 < \xi < \xi_1 \\ c^{ijkl}_{(2)} & \xi_1 < \xi < \xi_2 \end{cases} \tag{3.2}$$

and $c^{ijkl}_{(\alpha}$ (α = 1,2) are material constants in associated layers. Now we proceed to calculate T^i and S^α defined in (2.39) and (2.40). First we recall that $T^{*i} = g^{*1/2}\tau^{*ij}g^*_j$, $g^{*1/2} = g^{1/2}(1 + \frac{\xi}{2}\frac{\Delta}{g})$, $g^*_\gamma = g_\gamma + \xi d_{,\gamma}$, $g^*_3 = g_3$ and for brevity we omit the index α in relations (3.1) and (3.2)

$$T^i = \frac{1}{\xi_2}\int_0^{\xi_2} T^{*i}d\xi = \frac{1}{\xi_2}\int_0^{\xi_2} g^{*1/2}\tau^{*ij}g^*_j d\xi$$

$$= \frac{1}{\xi_2}\int_0^{\xi_2} g^{*1/2} c^{ijkl}\gamma^*_{kl}g^*_j d\xi = \frac{1}{\xi_2}\int_0^{\xi_2} g^{*1/2}(c^{ij\alpha\beta}\gamma^*_{\alpha\beta} + 2c^{ij\alpha3}\gamma^*_{\alpha3} + c^{ij33}\gamma^*_{33})g^*_j d\xi$$

Substitute from (2.31), (2.32) and (2.33) in the above relations and get

$$T^i = \frac{1}{\xi_2}\int_0^{\xi_2} g^{*1/2}(c^{ij\alpha\beta}\gamma_{\alpha\beta} + 2c^{ij\alpha3}\gamma_{\alpha3} + c^{ij33}\gamma_{33})g^*_j d\xi$$

$$+ \frac{1}{\xi_2}\int_0^{\xi_2} g^{*1/2}[(\frac{\kappa_{\alpha\beta}+\kappa_{\beta\alpha}}{2})c^{ij\alpha\beta} + \kappa_{3\alpha}c^{ij\alpha3}]\xi g^*_j d\xi$$

$$= \gamma_{kl}\frac{1}{\xi_2}\int_0^{\xi_2} c^{ijkl}g^{*1/2}g^*_j d\xi + \frac{1}{2}(\kappa_{\alpha\beta} + \kappa_{\beta\alpha})\frac{1}{\xi_2}\int_0^{\xi_2} c^{ij\alpha\beta}g^{*1/2}g^*_j \xi d\xi$$

$$+ \kappa_{3\alpha} \frac{1}{\xi_2} \int_0^{\xi_2} c^{ij\alpha 3} g^{*1/2} g_j^* \xi \, d\xi \tag{3.3}$$

We calculate each integral separately, noting that

$$g^{*1/2} g_\gamma^* = g^{1/2} [g_\gamma + \xi(\frac{\Delta}{2g} g_\gamma + d_{,\gamma}) + \frac{\Delta}{2g} d_{,\gamma}\xi^2] \tag{3.4}$$

$$g^{*1/2} g_3^* = g^{1/2} (g_3 + \frac{\Delta}{2g} \xi g_3) = g^{1/2} (1 + \frac{\Delta}{2g} \xi) g_3 \tag{3.5}$$

The first integral in (3.3) is

$$\frac{1}{\xi_2} \int_0^{\xi_2} c^{ijkl} g^{*1/2} g_j^* \, d\xi = \frac{1}{\xi_2} \int_0^{\xi_2} (c^{i\gamma kl} g^{*1/2} g_\gamma^* + c^{i3kl} g^{*1/2} g_3^*) d\xi \tag{3.6}$$

The first term of (3.6) from (3.4) is equal to

$$g^{1/2} g_\gamma \frac{1}{\xi_2} \int_0^{\xi_2} c^{i\gamma kl} d\xi + g^{1/2} (\frac{\Delta}{2g} g_\gamma + d_{,\gamma}) \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{i\gamma kl} d\xi + \frac{\Delta}{2g^{1/2}} \frac{d_{,\gamma}}{\xi_2} \int_0^{\xi_2} \xi^2 c^{i\gamma kl} d\xi \tag{3.7}$$

and its second term can be written as, from (3.5),

$$g^{1/2} g_3 (\frac{1}{\xi_2} \int_0^{\xi_2} c^{i3kl} d\xi + \frac{\Delta}{2g} \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{i3kl} d\xi) \tag{3.8}$$

Combining (3.7) and (3.8) we rewrite (3.6) as

$$g^{1/2} g_j \frac{1}{\xi_2} \int_0^{\xi_2} c^{ijkl} d\xi + \frac{\Delta}{2g^{1/2}} g_j \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{ijkl} d\xi + g^{1/2} d_{,\gamma} \frac{1}{\xi_2} \int_0^{\xi_2} \xi c^{i\gamma kl} d\xi$$

$$+ \frac{\Delta}{2g^{1/2}} d_{,\gamma} \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 c^{i\gamma kl} d\xi \tag{3.9}$$

The second integral in (3.3) is

$$\frac{1}{\xi_2}\int_0^{\xi_2} c^{ij\alpha\beta}\, g^{*1/2}\, g_j^*\, \xi\, d\xi = \frac{1}{\xi_2}\int_0^{\xi_2} (\xi c^{i\gamma\alpha\beta}\, g^{*1/2}\, g_\gamma^* + \xi c^{i3\alpha\beta}\, g^{*1/2} g_3^*) d\xi \tag{3.10}$$

The first term of (3.10) from (3.4) is

$$g^{1/2}\, g_\gamma \frac{1}{\xi_2}\int_0^{\xi_2} \xi c^{i\gamma\alpha\beta} d\xi + g^{1/2}(\frac{\Delta}{2g}\, g_\gamma + d_{,\gamma})\, \frac{1}{\xi_2}\int_0^{\xi_2} \xi^2 c^{i\gamma\alpha\beta} d\xi$$

$$+ \frac{\Delta}{2g^{1/2}}\, \frac{d_\gamma}{\xi_2}\int_0^{\xi_2} \xi^3 c^{i\gamma\alpha\beta} d\xi \tag{3.11}$$

and its second term from (3.5) is

$$g^{1/2}\, g_3(\frac{1}{\xi_2}\int_0^{\xi_2} \xi c^{i3\alpha\beta} d\xi + \frac{\Delta}{2g}\, \frac{1}{\xi_2}\int_0^{\xi_2} \xi^2 c^{i3\alpha\beta} d\xi) \tag{3.12}$$

Combining (3.11) and (3.12) we rewrite (3.10) as

$$g^{1/2}\, g_j \frac{1}{\xi_2}\int_0^{\xi_2} \xi c^{ij\alpha\beta} d\xi + \frac{\Delta}{2g^{1/2}}\, g_j \frac{1}{\xi_2}\int_0^{\xi_2} \xi^2\, c^{ij\alpha\beta} d\xi + g^{1/2}\, d_{,\gamma} \frac{1}{\xi_2}\int_0^{\xi_2} \xi^2 c^{i\gamma\alpha\beta} d\xi$$

$$+ \frac{\Delta}{2g^{1/2}}\, d_{,\gamma} \frac{1}{\xi_2}\int_0^{\xi_2} \xi^3\, c^{i\gamma\alpha\beta}\, d\xi \tag{3.13}$$

The third integral in (3.3) is

$$\frac{1}{\xi_2}\int_0^{\xi_2} \xi c^{ij\alpha 3} g^{*1/2}\, g_j^* d\xi = g^{1/2}\, g_j \frac{1}{\xi_2}\int_0^{\xi_2} \xi c^{ij\alpha 3} d\xi + \frac{\Delta}{2g^{1/2}}\, g_j \frac{1}{\xi_2}\int_0^{\xi_2} \xi^2\, c^{ij\alpha 3} d\xi$$

$$+ g^{1/2}\, d_{,\gamma} \frac{1}{\xi_2}\int_0^{\xi_2} \xi^2 c^{i\gamma\alpha 3} d\xi + \frac{\Delta}{2g^{1/2}}\, d_{,\gamma} \frac{1}{\xi_2}\int_0^{\xi_2} \xi^3 c^{i\gamma\alpha 3} d\xi \tag{3.14}$$

The last result was written by noting the development in (2.13). The results in (3.9), (3.13) and (3.14) can further be simplified by recalling $(2.9)_3$, namely $d_{,\gamma} = \lambda_\gamma^j g_j$ and using the following definitions and results.

Define

$$m = \xi_1/\xi_2 < 1 \tag{3.15}$$

$$a = \begin{cases} a_1 & 0 < \xi < \xi_1 \\ a_2 & \xi_1 < \xi < \xi_2 \end{cases} \tag{3.16}$$

Then

$$\frac{1}{\xi_2}\int_0^{\xi_2} \xi^k\, a\, d\xi = \frac{\xi_1^k}{k+1}\,[ma_1 + (\frac{1}{m^k} - m)a_2] \quad k \neq -1 \tag{3.17}$$

Now (3.9) is equal to

$$g^{1/2}\, g_j\{mc_1^{ijkl} + (1-m)c_2^{ijkl} + \frac{\xi_1\Delta}{4g}\,[mc_1^{ijkl} + (\frac{1}{m} - m)c_2^{ijkl}]$$

$$+ \lambda_\gamma^j\, \frac{\xi_1}{2}\,[mc_1^{i\gamma kl} + (\frac{1}{m} - m)c_2^{i\gamma kl}] + \frac{\xi_1^2\Delta}{6g}\,\lambda_\gamma^j[mc_1^{i\gamma kl} + (\frac{1}{m^2} - m)c_2^{i\gamma kl}]\}$$

$$= g^{1/2}\, g_j\{(1 + \frac{\xi_1\Delta}{4g})mc_1^{ijkl} + \frac{\xi_1}{2}\,\lambda_\gamma^j(1 + \frac{\xi_1\Delta}{3g})mc_1^{i\gamma kl}$$

$$+ (1-m)(1 + \frac{1+m}{4m}\,\frac{\xi_1\Delta}{g})c_2^{ijkl} + \frac{\xi_1(1-m)}{2m}\,\lambda_\gamma^j(1 + m$$

$$+ \frac{\xi_1\Delta}{3g}\,\frac{1+m+m^2}{m})c_2^{i\gamma kl}\} \tag{3.18}$$

(3.13) can also be written as

$$g^{1/2}\, g_j\{[mc_1^{ij\alpha\beta} + (\frac{1}{m} - m)c_2^{ij\alpha\beta}]\,\frac{\xi_1}{2} + \frac{\Delta}{2g}\,\frac{\xi_1^2}{3}\,[mc_1^{ij\alpha\beta} + (\frac{1}{m^2} - m)c_2^{ij\alpha\beta}]$$

$$+ \lambda_\gamma^j\,\frac{\xi_1^2}{3}\,[mc_1^{i\gamma\alpha\beta} + (\frac{1}{m^2} - m)c_2^{i\gamma\alpha\beta}] + \frac{\Delta}{2g}\,\lambda_\gamma^j\,\frac{\xi_1^3}{4}\,[mc_1^{i\gamma\alpha\beta} + (\frac{1}{m^3} - m)c_2^{i\gamma\alpha\beta}]\}$$

$$= g^{1/2} g_j \{ (\frac{\xi_1}{2} + \frac{\xi_1^2 \Delta}{6g}) m c_1^{ij\alpha\beta} + \frac{\xi_1^2}{3} \lambda_\gamma^j (1 + \frac{3}{8} \frac{\xi_1 \Delta}{g}) m c_1^{i\gamma\alpha\beta}$$

$$+ \frac{\xi_1(1-m)}{2m} (1 + m + \frac{\xi_1 \Delta}{3g} \frac{1+m+m^2}{m}) c_2^{ij\alpha\beta} + \frac{\xi_1^2 \lambda_\gamma^j}{3} [\frac{1}{m^2} - m$$

$$+ \frac{3}{8} \frac{\xi_1 \Delta}{g} (\frac{1}{m^3} - m)] c_2^{j\alpha\beta}]\} \tag{3.19}$$

The expression for (3.14) is exactly the same as (3.19) except that (α,β) in (3.19) should be replaced by $(\alpha,3)$. These results should be incorporated in (3.3) to find an expression for T^i. Due to the presence of the factor $g^{1/2} g_j$ in all these expressions and also the equality (2.66), we can find the constitutive relation for τ^{ij}. However before doing so we exploit the symmetry of c^{ijkl} to further simplify (3.3). Since $c^{ij\alpha\beta} = c^{ij\beta\alpha}$ we can write

$$c^{ij\alpha\beta} \kappa_{\beta\alpha} = c^{ij\beta\alpha} \kappa_{\beta\alpha} = c^{ij\alpha\beta} \kappa_{\alpha\beta} \tag{3.20}$$

Therefore,

$$c^{ij\alpha\beta} \kappa_{\beta\alpha} = \frac{1}{2} c^{ij\alpha\beta} (\kappa_{\alpha\beta} + \kappa_{\beta\alpha}) \tag{3.21}$$

In view of (3.21), now we write (3.3) as

$$T^i = \gamma_{kl} \frac{1}{\xi_2} \int_0^{\xi_2} c^{ijkl} g^{*1/2} g_j^* d\xi + \kappa_{l\alpha} \frac{1}{\xi_2} \int_0^{\xi_2} c^{ijl\alpha} g^{*1/2} g_j^* \xi \, d\xi \tag{3.22}$$

With the explanation presented above, now we write the constitutive relation for τ^{ij}:

$$\tau^{ij} = \{ (1 + \frac{\xi_1 \Delta}{4g}) m c_1^{ijkl} + \frac{\xi_1}{2} \lambda_\gamma^j (1 + \frac{\xi_1 \Delta}{3g}) m c_1^{i\gamma kl}$$

$$+ (1-m)(1 + \frac{1+m}{4m} \frac{\xi_1 \Delta}{g}) c_2^{ijkl} + \frac{\xi_1(1-m)}{2m} \lambda_\gamma^j (1 + m + \frac{\xi_1 \Delta}{3g} \frac{1+m+m^2}{m}) c_2^{i\gamma kl} \} \gamma_{kl}$$

$$+ \{ (1 + \frac{\xi_1 \Delta}{3g}) \frac{m \xi_1}{2} c_1^{ijl\alpha} + \frac{\xi_1^2}{3} \lambda_\gamma^j (1 + \frac{3}{8} \frac{\xi_1 \Delta}{g}) m c_1^{i\gamma l\alpha}$$

$$+\frac{\xi_1(1-m)}{2m}\,(1+m+\frac{\xi_1\Delta}{3g}\,\frac{1+m+m^2}{m})c_2^{ijl\alpha}+\frac{\xi_1^2\lambda_\gamma^j}{3}\,[\frac{1}{m^2}-m$$

$$+\frac{3}{8}\,\frac{\xi_1\Delta}{g}\,(\frac{1}{m^3}-m)]c_2^{i\gamma l\alpha}\}\kappa_{l\alpha} \tag{3.23}$$

If we further restrict ourselves to small deformations of a composite with initially flat plies, equation (3.23) can be simplified to

$$\tau^{ij}=\{mc_1^{ijkl}+(1-m)c_2^{ijkl}\}\gamma_{kl}+\frac{\xi_1}{2}\,\{mc_1^{ijl\alpha}+\frac{1-m^2}{m}\,c_2^{ijl\alpha}\}\kappa_{l\alpha} \tag{3.24}$$

Of course in the above equations no distinction should be made between covariant and contravariant components of tensors. Using (2.114), equation (3.24) can be written in terms of displacement vector **u** and director displacement vector **δ**, hence

$$\tau_{ij}=\{mc_{ijkl}^{(1)}+(1-m)c_{ijkl}^{(2)}\}\gamma_{kl}+\frac{\xi_1}{2}\,\{mc_{ijl\alpha}^{(1)}+\frac{1-m^2}{m}\,c_{ijl\alpha}^{(2)}\}\kappa_{l\alpha}$$

$$=\{mc_{ij\alpha\beta}^{(1)}+(1-m)c_{ij\alpha\beta}^{(2)}\}\,\frac{1}{2}\,(u_{\alpha,\beta}+u_{\beta,\alpha})+2\{mc_{ij\alpha3}^{(1)}+(1-m)c_{ij\alpha3}^{(2)}\}(u_{3,\alpha}+\delta_\alpha)/2$$

$$+\{mc_{ij33}^{(1)}+(1-m)c_{ij33}^{(2)}\}\delta_3+\frac{\xi_1}{2}\,\{mc_{ijl\alpha}^{(1)}+\frac{1-m^2}{m}\,c_{ijl\alpha}^{(2)}\}\delta_{l,\alpha} \tag{3.25}$$

Using the symmetry of c_{ijkl}'s (3.25) can be written as

$$\tau_{ij}=\{mc_{ij\alpha\beta}^{(1)}+(1-m)c_{ij\alpha\beta}^{(2)}\}u_{\alpha,\beta}+\{mc_{ij\alpha3}+(1-m)c_{ij\alpha3}^{(2)}\}u_{3,\alpha}$$

$$+\{mc_{ij\alpha3}^{(1)}+(1-m)c_{ij\alpha3}^{(2)}\}\delta_\alpha+\{mc_{ij33}^{(1)}+(1-m)c_{ij33}^{(2)}\}\delta_3$$

$$+\frac{\xi_1}{2}\,\{mc_{ijl\alpha}^{(1)}+\frac{(1-m^2)}{m}\,c_{ijl\alpha}^{(2)}\}\delta_{l,\alpha}$$

$$=\{mc_{ij\alpha k}^{(1)}+(1-m)c_{ij\alpha k}^{(2)}\}u_{k,\alpha}+\{mc_{ijk3}^{(1)}+(1-m)c_{ijk3}^{(2)}\}\delta_k$$

$$+\frac{\xi_1}{2}\{mc^{(1)}_{ij\ell\alpha}+\frac{(1-m^2)}{m}c^{(2)}_{ij\ell\alpha}\}\delta_{\ell\alpha} \tag{3.26}$$

The same steps can be followed to calculate S^α and we record the procedure here KS

$$S^\alpha=\frac{1}{\xi_2}\int_0^{\xi_2}\xi\, T^{*\alpha}d\xi=\frac{1}{\xi_2}\int_0^{\xi_2}\xi g^{*1/2}\tau^{*\alpha j}g^*_j d\xi$$

$$=\frac{1}{\xi_2}\int_0^{\xi_2}\xi g^{*1/2}c^{\alpha jk\ell}\gamma^*_{k\ell}g^*_j\,d\xi=\frac{1}{\xi_2}\int_0^{\xi_2}\xi\, g^{*1/2}(c^{\alpha j\lambda\beta}\gamma^*_{\lambda\beta}+2c^{\alpha j\lambda 3}\gamma^*_{\lambda 3}$$

$$+c^{\alpha j33}\gamma^*_{33})g^*_j d\xi=\gamma_{k\ell}\frac{1}{\xi_2}\int_0^{\xi_2}\xi g^{*1/2}c^{\alpha jk\ell}g^*_j\,d\xi+\kappa_{\ell\beta}\frac{1}{\xi_2}\int_{\xi_1}^{\xi_2}\xi^2 g^{*1/2}c^{\alpha j\ell\beta}g^*_j d\xi \tag{3.27}$$

This is basically the same result as (3.22) except that i has been replaced by α and the integrand has been multiplied by ξ. The first integral can be reduced to the following by referring to (3.18)

$$g^{1/2}g_j\{\frac{\xi_1}{2}[mc_1^{\alpha jk\ell}+(\frac{1}{m}-m)c_2^{\alpha jk\ell}]+\frac{\xi_1^2\Delta}{6g}[mc_1^{\alpha jk\ell}+(\frac{1}{m^2}-m)c_2^{\alpha jk\ell}]+$$

$$\lambda^j_\gamma\frac{\xi_1^2}{3}[mc_1^{\alpha\gamma k\ell}+(\frac{1}{m^2}-m)c_2^{\alpha\gamma k\ell}]+\frac{\xi_1^3\Delta}{8g}\lambda^j_\gamma[mc_1^{\alpha\gamma k\ell}+(\frac{1}{m^3}-m)c_2^{\alpha\gamma k\ell}]\} \tag{3.28}$$

Similarly the second integral is simplified and by reference to (3.19) the result is

$$g^{1/2}g_j\{\frac{\xi_1^2}{3}[mc_1^{\alpha j\ell\beta}+(\frac{1}{m^2}-m)c_2^{\alpha j\ell\beta}]+\frac{\xi_1^3\Delta}{8g}[mc_1^{\alpha j\ell\beta}+(\frac{1}{m^3}-m)c_2^{\alpha j\ell\beta}]$$

$$+\lambda^j_\gamma\frac{\xi_1^3}{4}[mc_1^{\alpha\gamma\ell\beta}+(\frac{1}{m^3}-m)c_2^{\alpha\gamma\ell\beta}]+\lambda^j_\gamma\frac{\xi_1^4\Delta}{10g}[mc_1^{\alpha\gamma\ell\beta}+(\frac{1}{m^4}-m)c_2^{\alpha\gamma\ell\beta}]\}$$

$$=\xi_1^2g^{1/2}g_j\{(\frac{m}{3}+\frac{m\Delta\xi_1}{8g})c_1^{\alpha j\ell\beta}+\lambda^j_\gamma\xi_1(\frac{1}{4}+\frac{\xi_1\Delta}{10g})mc_1^{\alpha\gamma\ell\beta}$$

$$+[\frac{1}{3}(\frac{1-m^3}{m^2})+\frac{\xi_1\Delta}{8g}\frac{1-m^4}{m^3}]c_2^{\alpha j\ell\beta}+\lambda^j_\gamma\xi_1(\frac{1-m^4}{4m^3}+\frac{\xi_1\Delta}{10g}\frac{1-m^5}{m^4})c_2^{\alpha\gamma\ell\beta}\} \tag{3.29}$$

Again it is seen that because of the factor $g^{1/2}\,\mathbf{g}_j$ and the relation $S^{\alpha} = S^{\alpha j}\, g^{1/2}\, \mathbf{g}_j$ we can readily write the constitutive relation for $S^{\alpha j}$, the result is

$$S^{\alpha j} = \xi_1 \{ \frac{1}{2}\,(mc_1^{\alpha jk\ell})(1 + \frac{\xi_1 \Delta}{3g}) + \xi_1 \lambda_\gamma^j (\frac{1}{3} + \frac{\xi_1 \Delta}{8g}) mc_1^{\alpha\gamma k\ell}$$

$$+ \frac{1}{2}\,(\frac{1-m^2}{m} + \frac{1-m^3}{m^2}\,\frac{\xi_1 \Delta}{3g}) c_2^{\alpha jk\ell} + \lambda_\gamma^j\, \xi_1 (\frac{1-m^3}{3m^2} + \frac{1-m^4}{m^3}\,\frac{\xi_1 \Delta}{8g}) c_2^{\alpha\gamma k\ell} \} \gamma_{k\ell}$$

$$+ \xi_1^2 \{ (\frac{1}{3} + \frac{\xi_1 \Delta}{8g}) mc_1^{\alpha j\ell\beta} + \xi_1 \lambda_\gamma^j (\frac{1}{4} + \frac{\xi_1 \Delta}{10g}) mc_1^{\alpha\gamma\ell\beta} + (\frac{1-m^3}{3m^2} + \frac{\xi_1 \Delta}{8g}\,\frac{1-m^4}{m^3}) c_2^{\alpha j\ell\beta}$$

$$+ \xi_1\, \lambda_\gamma^j (\frac{1-m^4}{4m^3} + \frac{\xi_1 \Delta}{10g}\,\frac{1-m^5}{m^4}) c_2^{\alpha\gamma\ell\beta} \} \kappa_{\ell\beta} \tag{3.30}$$

This is the general constitutive relation for $S^{\alpha j}$ in linear elasticity. If, as before, we confine ourselves to small deformations of a composite with initially flat plies (3.30) can be simplified to

$$S^{\alpha j} = \frac{1}{2}\,\xi_1\,(mc_1^{\alpha jk\ell} + \frac{1-m^2}{m}\,c_2^{\alpha jk\ell}) \gamma_{k\ell} + \frac{1}{3}\,\xi_1^2 (mc_1^{\alpha j\ell\beta} + \frac{1-m^3}{m^2}\,c_2^{\alpha j\ell\beta}) \kappa_{\ell\beta} \tag{3.31}$$

with no distinction between covariant and contravariant tensors. Once written in terms of displacement vector and director displacement and simplified as done in obtaining (3.26) we get

$$S^{\alpha j} = \frac{1}{2}\,\xi_1 (mc_1^{\alpha j\beta k} + \frac{1-m^2}{m}\,c_2^{\alpha j\beta k}) u_{k,\beta} + \frac{1}{2}\,\xi_1 (mc_1^{\alpha jk3} + \frac{1-m^2}{m}\,c_2^{\alpha jk3}) \delta_k$$

$$+ \frac{1}{3}\,\xi_1^2 (mc_1^{\alpha j\ell\beta} + \frac{1-m^3}{m^2}\,c_2^{\alpha j\ell\beta}) \delta_{\ell,\beta} \tag{3.32}$$

As the results of this section indicate, even in the simplest cases of small deformations of an initially flat composite (composites with flat plies) higher gradients of displacement vector become significant and they appear in the constitutive relations for *composite stress* and *composite stress couple*. As defined by (3.15) m is of the order of unity, ξ_1 and ξ_2 are usually small lengths; however their products with components of $c_{ijk\ell}$ and even the product of their higher powers with elastic constants may be indeed significant quantities, in which case we get contributions to τ_{ij} and $S_{\alpha j}$. In the trivial case $m = 1$, $\xi_1 = \xi_2 \to 0$ we get $\tau_{ij} = c_{ijkl}\gamma_{kl}$, $S_{\alpha j} \equiv 0$ and the equations of linear elasticity are recovered.

4.0 COMPLETE THEORY FOR LINEAR ELASTIC COMPOSITE LAMINATES

The results of sections (2) and (3) are combined to obtain the complete equations for a linear elastic composite laminate. However, before doing so we should derive appropriate expression for ρ_o, z^1 and z^2. As before, we assume that the representative micro-structure is composed of two homogeneous layers with respective densities ρ_1 and ρ_2 in the reference configuration (ρ_1 and ρ_2 are constants). Recalling equations (2.45) and (2.46) we write

$$\rho g^{1/2} = \rho_o G^{1/2} = \frac{1}{\xi_2} \int_o^{\xi_2} \rho_o^* G^{*1/2} d\xi \tag{4.1}$$

Now by (2.13) we have

$$G^{*1/2} = G^{1/2}(1 + \frac{\xi \Delta}{2G}) \tag{4.2*}$$

where Δ is understood to be the sum of two determinants similar to those expressed in (2.14) except for substituting g_{ij} by G_{ij}. We have

$$\rho_o^* = \begin{cases} \rho_1 & 0 < \xi < \xi_1 \\ \rho_2 & \xi_1 < \xi < \xi_2 \end{cases} \tag{4.3}$$

Substituting (4.2) and (4.3) in (4.1) and using (3.7) we set

$$\rho_o = (1 + \frac{\xi_1 \Delta}{4G}) m\rho_1 + (1 + \frac{1+m}{4mG} \xi_1 \Delta)(1-m)\rho_2 \tag{4.4}$$

Of course the *composite mass density* ρ is related to ρ_o through the equation (2.122).

We proceed similarly to calculate z^1 and z^2 using their definitions in (2.42)

$$\rho g^{1/2} z^1 = \rho_o G^{1/2} z^1 = \frac{1}{\xi_2} \int_o^{\xi_2} \xi \rho_o^* G^{*1/2} d\xi \tag{4.5}$$

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* Here again the result has been linearized in terms of ξ.

$$\rho g^{1/2} z^2 = \rho_o G^{1/2} z^2 = \frac{1}{\xi_2} \int_0^{\xi_2} \xi^2 \rho_o^* G^{*1/2} d\xi \tag{4.6}$$

After substituting from (4.2) and (4.3) in (4.5) and (4.6) and using (3.17) we get

$$\rho_o z^1 = \frac{\xi_1}{2} \left[\left(1 + \frac{\xi_1 \Delta}{3G}\right) m\rho_1 + \left(\frac{1-m^2}{m} + \frac{\xi_1 \Delta}{3G} \frac{1-m^3}{m^2}\right)\rho_2 \right] \tag{4.7}$$

$$\rho_o z^2 = \frac{\xi_1^2}{3} \left[\left(1 + \frac{3\xi_1 \Delta}{8G}\right) m\rho_1 + \left(\frac{1-m^3}{m^2} + \frac{3\xi_1 \Delta}{8G} \frac{1-m^4}{m^3}\right)\rho_2 \right] \tag{4.8}$$

For a composite with initially flat plies (4.4), (4.7) and (4.8) are reduced respectively to

$$\rho_o = m\rho_1 + (1-m)\rho_2 \tag{4.9}$$

$$\rho_o z^1 = \frac{\xi_1}{2} \left(m\rho_1 + \frac{1-m^2}{m} \rho_2\right) \tag{4.10}$$

$$\rho_o z^2 = \frac{\xi_1^2}{3} \left(m\rho_1 + \frac{1-m^3}{m^2} \rho_2\right) \tag{4.11}$$

To formulate the complete theory it is also worthwhile to derive a relation between the director displacement $\boldsymbol{\delta}$ and the gradient of displacement vector $\mathbf{u}$ in the θ^3-direction. In order to derive such a relation we enforce the continuity of the position vectors $\mathbf{p}^*$ and $\mathbf{P}^*$ between two adjacent micro-structures. Recalling (2.1) and (2.2), we have the following relations for the k^{th} micro-structure:

$$\mathbf{p}^*(\theta^\alpha, \theta^{3(k)}, \xi, t) = \mathbf{r}(\theta^\alpha, \theta^{3(k)}, t) + \xi \mathbf{d}(\theta^\alpha, \theta^{3(k)}, t) \tag{4.12}$$

$$\mathbf{P}^*(\theta^\alpha, \theta^{3(k)}, \xi) = \mathbf{R}(\theta^\alpha, \theta^{3(k)}) + \xi \mathbf{D}(\theta^\alpha, \theta^{3(k)}) \tag{4.13}$$

Now in order that position vectors $\mathbf{p}^*$ and $\mathbf{P}^*$ be continuous on the common surface between k^{th} and $(k+1)^{st}$ micro-structures we should have

$$\mathbf{P}^*(\theta^\alpha, \theta^{3(k)}, \xi_2) = \mathbf{P}^*(\theta^\alpha, \theta^{3(k+1)}, 0) \tag{4.14}$$

$$p^*(\theta^\alpha,\theta^{3(k)},\xi_2,t) = p^*(\theta^\alpha,\theta^{3(k+1)},0,t) \tag{4.15}$$

Using (4.12) and (4.13) we can write (4.14) and (4.15) as

$$\mathbf{R}(\theta^\alpha,\theta^{3(k+1)}) = \mathbf{R}(\theta^\alpha,\theta^{3(k)}) + \xi_2\mathbf{D}(\theta^\alpha,\theta^{3(k)}) \tag{4.16}$$

$$\mathbf{r}(\theta^\alpha,\theta^{3(k+1)},t) = \mathbf{r}(\theta^\alpha,\theta^{3(k)},t) + \xi_2\mathbf{d}(\theta^\alpha,\theta^{3(k)},t) \tag{4.17}$$

Recalling (2.98) and (2.99) and identifying $\varepsilon\mathbf{u}$ and $\varepsilon\boldsymbol{\delta}$ with $\mathbf{u}$ and $\boldsymbol{\delta}$ as before we conclude from (4.16) and (4.17) the following

$$\mathbf{u}(\theta^\alpha,\theta^{3(k+1)},t) = \mathbf{u}(\theta^\alpha,\theta^{3(k)},t) + \xi_2\boldsymbol{\delta}(\theta^\alpha,\theta^{3(k)},t) \tag{4.18}$$

or

$$\boldsymbol{\delta}(\theta^\alpha,\theta^{3(k)},t) = \frac{1}{\xi_2}\,\{\mathbf{u}(\theta^\alpha,\theta^{3(k+1)},t) - \mathbf{u}(\theta^\alpha,\theta^{3(k)},t)] \tag{4.19}$$

By smoothing assumptions and approximating the right-hand side of equation (4.19) as the gradient of the displacement vector in the θ^3 direction we have

$$\boldsymbol{\delta}(\theta^\alpha,\theta^3,t) = \frac{\partial\mathbf{u}(\theta^\alpha,\theta^3)}{\partial\theta^3} \tag{4.20}$$

In component form we have

$$\boldsymbol{\delta} = \delta^j\mathbf{g}_j = (u^j\mathbf{g}_j)_{,3} = u^j{}_{|3}\mathbf{g}_j \tag{4.21}$$

or

$$\delta^j = u^j{}_{|3} \quad , \quad \delta_j = u_{j|3} \tag{4.22}$$

For a composite with initially flat plies the equation (4.22) reduces to

$$\delta_j = u_{j,3} \tag{4.23}$$

With this simplification, equations (2.119) reduce to

$$\gamma_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{4.24}$$

$$\kappa_{j\alpha} = u_{j,\alpha 3} \tag{4.25}$$

Using (4.22), equations (2.121) and (2.125) are also reduced to

$$\rho_o = \rho(\frac{g}{G})^{1/2} = \rho(1+u^j{}_{|j}) \tag{4.26}$$

$$\rho = \rho_o(1-u^j{}_{|j}) \tag{4.27}$$

The constitutive relations (3.26) and (3.32) for τ_{ij} and $S_{\alpha j}$ for a flat composite are also further simplified by using (4.23)

$$\tau_{ij} = \{mc^{(1)}_{ijkl} + (1-m)c^{(2)}_{ijkl}\}u_{k,l} + \{mc^{(1)}_{ijk\alpha} + \frac{1-m^2}{m}\,c^{(2)}_{ijk\alpha}\}\,\frac{\xi_1}{2}\,u_{k,\alpha 3} \tag{4.28}$$

and

$$S_{\alpha j} = \frac{1}{2}\,\xi_1\{mc^{(1)}_{\alpha jkl} + \frac{1-m^2}{m}\,c^{(2)}_{\alpha jkl}\}u_{k,l} + \frac{1}{3}\,\xi_1^2\{mc^{(1)}_{\alpha jk\beta} + \frac{1-m^3}{m^2}\,c^{(2)}_{\alpha jk\beta}\}u_{k,\beta 3} \tag{4.29}$$

Now we can write the field equations (2.113) and (2.124) in terms of the displacement vector u and its gradients. It should be recalled that the resulting equations are the linearized field equation for small deformations of a composite with initially flat plies. These are the counterpart of the Navier-Cauchy equations in linear elasticity. The appropriate equations for a general composite will be derived in later chapters. Using (4.9)-(4.11), (4.23), (4.28) and (4.29) we write (2.123) and (2.124) as

$$\{mc^{(1)}_{\alpha jkl} + (1-m)c^{(2)}_{\alpha jkl}\}u_{k,l\alpha} + \{mc^{(1)}_{\alpha jk\beta} + \frac{1-m^2}{m}\, c^{(2)}_{\alpha jk\beta}\}\, \frac{\xi_1}{2}\, u_{k,\beta 3\alpha}$$

$$+ \sigma_{j,3} + [m\rho_1 + (1-m)\rho_2]b_j = [m\rho_1 + (1-m)\rho_2]\ddot{u}_j$$

$$+ \frac{1}{2}\,\xi_1(m\rho_1 + \frac{1-m^2}{m}\,\rho_2)\ddot{u}_{j,3} \tag{4.30}$$

and

$$\frac{1}{2}\,\xi_1\{mc^{(1)}_{\alpha jkl} + \frac{1-m^2}{m}\, c^{(2)}_{\alpha jkl}\}u_{k,l\alpha} + \frac{\xi_1^2}{3}\,\{mc^{(1)}_{\alpha jk\beta} + \frac{1-m^3}{m^2}\, c^{(1)}_{\alpha jk\beta}\}u_{k,\beta 3\alpha}$$

$$+ \sigma_j - [mc^{(1)}_{3jkl} + (1-m)c^{(2)}_{3jkl}\}u_{k,l} - \frac{\xi_1}{2}\,\{mc^{(1)}_{3jk\alpha} + \frac{1-m^2}{m}\, c^{(2)}_{3jk\alpha}\}u_{k,\alpha 3}$$

$$+ [m\rho_1 + (1-m)\rho_o]c_j = \frac{1}{2}\,\xi_1(m\rho_1 + \frac{1-m^2}{m}\,\rho_2)\ddot{u}_j$$

$$+ \frac{1}{3}\,\xi_1^2(m\rho_1 + \frac{1-m^3}{m^2}\,\rho_2)\ddot{u}_{j,3} \tag{4.31}$$

At this point we may notice that an ordinary continuum (a single material continuum) can be regarded as the limiting case of a composite laminate when $\xi_1 = \xi_2 \to 0$. Therefore, we may anticipate to derive the equations of linear elasticity by letting m = 1 and $\xi_1 \to 0$ in equations (4.30) and (4.31). Doing so, equation (4.30) reduces to

$$c_{\alpha jkl}u_{k,l\alpha} + \sigma_{j,3} + \rho b_j = \rho\ddot{u}_j \tag{4.32}$$

where subscript and superscript 1 are dropped because we have only one material. To simplify equation (4.31), first we recall the definition of c in equation (2.44) and notice that by the mean-value theorem, $c \to 0$ as $\xi_2 \to 0$, hence

$$\sigma_j - c_{3jkl}u_{k,l} = 0 \tag{4.33}$$

Substituting for σ_j from (4.33) in (4.32) we get

$$c_{\alpha jkl} u_{k,l\alpha} + c_{3jkl} u_{k,l3} + \rho b_j = \rho \ddot{u}_j \tag{4.34}$$

and combining the first and the second terms we get

$$c_{ijkl} u_{k,li} + \rho b_j = \rho \ddot{u}_j \tag{4.35}$$

For a completely isotropic continuum

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{4.36}$$

where λ and μ are Lame constants and (4.35) reduces to:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} + \rho b_i = \rho \ddot{u}_i \tag{4.37}$$

which are the equations of motion for an isotropic media.

For the case of the composite laminate we can also eliminate σ_j between equations (4.30) and (4.31) to obtain the appropriate equations for displacement vector **u**. First we do this for a static problem with no body force. For such a case we let

$$\mathbf{b} = \mathbf{c} = \ddot{\mathbf{u}} = 0 \tag{4.38}$$

in equations (4.30) and (4.31), hence

$$\{m c^{(1)}_{\alpha jkl} + (1-m) c^{(2)}_{\alpha jkl}\} u_{k,l\alpha} + \{m c^{(1)}_{\alpha jk\beta} + \frac{1-m^2}{m} c^{(2)}_{\alpha jk\beta}\} \frac{\xi_1}{2} u_{k,\beta 3\alpha} + \sigma_{j,3} = 0$$

and

$$\frac{\xi_1}{2}\{mc^{(1)}_{\alpha jkl}+\frac{1-m^2}{m}c^{(2)}_{\alpha jkl}\}u_{k,l\alpha}+\frac{\xi_1^2}{3}\{mc^{(1)}_{\alpha jk\beta}+\frac{1-m^3}{m^2}c^{(2)}_{\alpha jk\beta}\}u_{k,\beta 3\alpha}$$

$$+\sigma_j-\{mc^{(1)}_{3jkl}+(1-m)c^{(2)}_{3jkl}\}u_{k,l}-\frac{\xi_1}{2}\{mc^{(1)}_{3jk\alpha}+\frac{1-m^2}{m}c^{(2)}_{3jk\alpha}\}u_{k,\alpha 3}=0$$

Eliminating σ_j between these equations, we get

$$\{mc^{(1)}_{\alpha jkl}+(1-m)c^{(2)}_{\alpha jkl}\}u_{k,l\alpha}+\{mc^{(1)}_{\alpha jk\beta}+\frac{1-m^2}{m}c^{(2)}_{\alpha jk\beta}\}\frac{\xi_1}{2}u_{k,3\beta\alpha}$$

$$+\{mc^{(1)}_{3jkl}+(1-m)c^{(2)}_{3jkl}\}u_{k,l3}+\frac{\xi_1}{2}\{mc^{(1)}_{3jk\alpha}+\frac{1-m^2}{m}c^{(2)}_{3jk\alpha}\}u_{k,\alpha 33}$$

$$-\frac{\xi_1}{2}\{mc^{(1)}_{\alpha jkl}+\frac{1-m^2}{m}c^{(2)}_{\alpha jkl}\}u_{k,l\alpha 3}-\frac{\xi_1^2}{3}\{mc^{(1)}_{\alpha jk\beta}+\frac{1-m^3}{m^2}c^{(2)}_{\alpha jk\beta}\}u_{k,\beta\alpha 33}=0 \qquad (4.39)$$

By combining the first and third and also the second and fourth terms of the equation (4.39), we get

$$\{mc^{(1)}_{ijkl}+(1-m)c^{(2)}_{ijkl}\}u_{k,li}+\frac{\xi_1}{2}\{mc^{(1)}_{ijk\alpha}+\frac{1-m^2}{m}c^{(2)}_{ijk\alpha}\}u_{k,i\alpha 3}$$

$$-\frac{\xi_1}{2}\{mc^{(1)}_{\alpha jkl}+\frac{1-m^2}{m}c^{(2)}_{\alpha jkl}\}u_{k,l\alpha 3}-\frac{\xi_1^2}{3}\{mc^{(1)}_{\alpha jk\beta}+\frac{1-m^3}{m^3}c^{(2)}_{\alpha jk\beta}\}u_{k,\beta\alpha 33}=0 \qquad (4.40)$$

This is a fourth order partial differential equation for displacement vector **u**. Now we apply this equation to a composite laminate whose micro-structure is composed of two isotropic layers. For such a case we can write

$$c^{(1)}_{ijkl}=\lambda_1\delta_{ij}\delta_{kl}+\mu_1(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \qquad (4.41)$$

$$c^{(2)}_{ijkl}=\lambda_2\delta_{ij}\delta_{kl}+\mu_2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \qquad (4.42)$$

where λ_i and μ_i (i = 1,2) are Lame's constants for the respective layers. Introducing equations (4.41) and (4.42) in (4.40) we get the following for each term:

$$\{mc^{(1)}_{ijkl} + (1-m)c^{(1)}_{ijkl}\}u_{k,li} = m\{\lambda_1 u_{k,kj} + \mu_1(u_{k,jk} + u_{j,ll})\}$$

$$+ (1-m)\{\lambda_2 u_{k,kj} + \mu_2(u_{k,jk} + u_{j,ll})\}$$

$$= \{m(\lambda_1 + \mu_1) + (1-m)(\lambda_2+\mu_2)\}u_{k,kj} + \{m\mu_1 + (1-m)\mu_2\}u_{j,ll} \tag{4.43}$$

$$\{mc^{(1)}_{ijk\alpha} + \frac{(1-m^2)}{m}\,c^{(2)}_{ijk\alpha}\}u_{k,i\alpha 3} = m\{\lambda_1\delta_{ij}\delta_{k\alpha} + \mu_1(\delta_{ik}\delta_{j\alpha} + \delta_{i\alpha}\delta_{jk})\}u_{k,i\alpha 3}$$

$$+ \frac{(1-m^2)}{m}\,\{\lambda_2\delta_{ij}\delta_{k\alpha} + \mu_2(\delta_{ik}\delta_{j\alpha} + \delta_{i\alpha}\delta_{jk})\}u_{k,i\alpha 3}$$

$$= m\{\lambda_1\delta_{ij}\delta_{\beta\alpha}u_{\beta,i\alpha 3}\} + m\mu_1\{\delta_{ik}\delta_{j\alpha}u_{k,i\alpha 3} + \delta_{\beta\alpha}\delta_{jk}u_{k,\beta\alpha 3}\}$$

$$+ \frac{(1-m^2)}{m}\,\lambda_2(\delta_{ij}\delta_{\beta\alpha}u_{\beta,i\alpha 3}) + \frac{1-m^2}{m}\,\mu_2(\delta_{ik}\delta_{j\alpha}u_{k,i\alpha 3} + \delta_{\beta\alpha}\delta_{jk}u_{k,i\alpha 3})$$

$$= \{m\lambda_1 + \frac{(1-m^2)}{m}\,\lambda_2\}u_{\alpha,j\alpha 3} + \{m\mu_1 + \frac{(1-m^2)}{m}\,\mu_2\}\delta_{j\alpha}u_{k,k\alpha 3}$$

$$+ \{m\mu_1 + \frac{(1-m^2)}{m}\,\mu_2\}u_{j,\alpha\alpha 3} \tag{4.44}$$

$$\{mc^{(1)}_{\alpha jkl} + \frac{(1-m^2)}{m}\,c^{(2)}_{\alpha jkl}\}u_{k,l\alpha 3} = m\{\lambda_1\delta_{\alpha j}\delta_{kl} + \mu_1(\delta_{\alpha k}\delta_{jl} + \delta_{\alpha l}\delta_{jk}\}u_{k,l\alpha 3}$$

$$+ \frac{(1-m^2)}{m}\,\{\lambda_2\delta_{\alpha j}\delta_{kl} + \mu_2(\delta_{\alpha k}\delta_{jl} + \delta_{\alpha l}\delta_{jk})\}u_{k,l\alpha 3}$$

$$= m\lambda_1(\delta_{\alpha j}u_{k,k\alpha 3}) + m\mu_1(u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3})$$

$$+ \frac{(1-m^2)}{m}\,\lambda_2(\delta_{\alpha j}u_{k,k\alpha 3}) + \frac{(1-m^2)}{m}\,\mu_2(u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3})$$

$$= (m\lambda_1 + \frac{(1-m^2)}{m}\,\lambda_2)\delta_{\alpha j}u_{k,k\alpha 3} + (m\mu_1 + \frac{(1-m^2)}{m}\,\mu_2)(u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3}) \tag{4.45}$$

$$\{mc^{(1)}_{\alpha jk\beta} + \frac{1-m^3}{m^2} c^{(2)}_{\alpha jk\beta}\} u_{k,\beta\alpha 33} = m\{\lambda_1\delta_{\alpha j}\delta_{k\beta} + \mu_1(\delta_{k\alpha}\delta_{j\beta} + \delta_{\alpha\beta}\delta_{jk})\} u_{k,\beta\alpha 33}$$

$$+ \frac{1-m^3}{m^2} \{\lambda_2\delta_{\alpha j}\delta_{k\beta} + \mu_2(\delta_{\alpha k}\delta_{j\beta} + \delta_{\alpha\beta}\delta_{jk})\} u_{k,\beta\alpha 33}$$

$$= m\lambda_1\delta_{\alpha j}u_{\beta,\beta\alpha 33} + m\mu_1(\delta_{j\beta}u_{\alpha,\beta\alpha 33} + u_{j,\alpha\alpha 33})$$

$$+ \frac{1-m^3}{m^2} \lambda_2\delta_{\alpha j}u_{\beta,\beta\alpha 33} + \frac{1-m^3}{m^2} \mu_2(\delta_{j\beta}u_{\alpha,\beta\alpha 33} + u_{j,\alpha\alpha 33})$$

$$= (m\lambda_1 + \frac{1-m^3}{m^2} \lambda_2)\delta_{\alpha j}u_{\beta,\beta\alpha 33} + (m\mu_1 + \frac{1-m^3}{m^2} \mu_2)(\delta_{j\beta}u_{\alpha,\beta\alpha 33} + u_{j,\alpha\alpha 33}) \qquad (4.46)$$

Substituting (4.43)-(4.46) in (4.40) we obtain

$$\frac{\xi_1}{2} \{m\lambda_1 + \frac{(1-m^2)}{m} \lambda_2\} u_{\alpha,j\alpha 3} + \{m\mu_1 + \frac{(1-m^2)}{m} \mu_2\}(\delta_{j\alpha}u_{k,k\alpha 3} + u_{j,\alpha\alpha 3}) \frac{\xi_1}{2}$$

$$+ \{m(\lambda_1 + \mu_1) + (1-m)(\lambda_2 + \mu_2)\} u_{k,kj} + \{m\mu_1 + (1-m)\mu_2\} u_{j,ll}$$

$$- \frac{\xi_1}{2} (m\lambda_1 + \frac{(1-m^2)}{m} \lambda_2)\delta_{\alpha j}u_{k,k\alpha 3} - \frac{\xi_1}{2} (m\mu_1 + \frac{(1-m^2)}{m} \mu_2)(u_{\alpha,j\alpha 3} + u_{j,\alpha\alpha 3})$$

$$- \frac{\xi_1^2}{3} (m\lambda_1 + \frac{1-m^3}{m^2} \lambda_2)\delta_{\alpha j}u_{\beta,\beta\alpha 33}$$

$$- \frac{\xi_1^2}{3} (m\mu_1 + \frac{1-m^3}{m^2} \mu_2)(\delta_{j\beta}u_{\alpha,\beta\alpha 33} + u_{j,\alpha\alpha 33}) = 0$$

Now we introduce the following definitions in the last equation:

$$\lambda_{12} = m\lambda_1 + (1-m)\lambda_2$$

$$\mu_{12} = m\mu_1 + (1-m)\mu_2$$

$$\bar{\lambda}_{12} = m\lambda_1 + \frac{1-m^2}{m}\lambda_2$$

$$\bar{\mu}_{12} = m\mu_1 + \frac{1-m^2}{m}\mu_2 \tag{4.47}$$

$$\bar{\bar{\lambda}}_{12} = m\lambda_1 + \frac{1-m^3}{m^2}\lambda_2$$

$$\bar{\bar{\mu}}_{12} = m\mu_1 + \frac{1-m^3}{m^2}\mu_2$$

The result will be

$$(\lambda_{12} + \mu_{12})u_{k,kj} + \mu_{12}u_{j,ll} + \frac{\xi_1}{2}\{\bar{\lambda}_{12}u_{\alpha,j\alpha3} + \bar{\mu}_{12}(\delta_{j\alpha}u_{k,k\alpha3} + u_{j,\alpha\alpha3})\}$$

$$-\frac{\xi_1}{2}\bar{\lambda}_{12}\,\delta_{\alpha j}u_{k,k\alpha3} - \frac{\xi_1}{2}\bar{\mu}_{12}(u_{\alpha,j\alpha3} + u_{j,\alpha\alpha3})$$

$$-\frac{\xi_1^2}{3}\bar{\bar{\lambda}}_{12}\delta_{\alpha j}u_{\beta,\beta\alpha33} - \frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}(\delta_{j\beta}u_{\alpha,\beta\alpha33} + u_{j,\alpha\alpha33} = 0 \tag{4.48}$$

The above equations are counterparts of the classical equations for linear elasto-static problems in the absence of body forces. We need to write these equations in the expanded form. The result would be three scalar equations as follows:

$$(\lambda_{12}+\mu_{12})\frac{\partial}{\partial\theta^1}\left(\frac{\partial u_1}{\partial\theta^1}+\frac{\partial u_2}{\partial\theta^2}+\frac{\partial u_3}{\partial\theta^3}\right)+\mu_{12}\left(\frac{\partial^2 u_1}{\partial\theta_1^2}+\frac{\partial^2 u_1}{\partial\theta_2^2}+\frac{\partial^2 u_1}{\partial\theta_3^2}\right)$$

$$+\frac{\xi_1}{2}\bar{\lambda}_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_3}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}\right)+\frac{\xi_1}{2}\bar{\mu}_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_3}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}+\frac{\partial u_3}{\partial\theta_3}\right)$$

$$+\frac{\xi_1}{2}\bar{\mu}_{12}\frac{\partial}{\partial\theta_3}\left(\frac{\partial^3 u_1}{\partial\theta_1^2}+\frac{\partial^2 u_1}{\partial\theta_2^2}\right)-\frac{\xi_1}{2}\bar{\lambda}_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_3}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}+\frac{\partial u_3}{\partial\theta_3}\right)$$

$$-\frac{\xi_1}{2}\bar{\mu}_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_3}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}\right)-\frac{\xi_1}{2}\bar{\mu}_{12}\frac{\partial}{\partial\theta_3}\left(\frac{\partial^2 u_1}{\partial\theta_1^2}+\frac{\partial^2 u_1}{\partial\theta_2^2}\right)$$

$$-\frac{\xi_1^2}{3}\bar{\bar{\lambda}}_{12}\frac{\partial^3}{\partial\theta_1\partial\theta_3^2}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}\right)-\frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}\frac{\partial^3}{\partial\theta_1\partial\theta_3^2}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta^2}\right)$$

$$-\frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}\frac{\partial^2}{\partial\theta_3^2}\left(\frac{\partial^2 u_1}{\partial\theta_1^2}+\frac{\partial^2 u_1}{\partial\theta_2^2}\right)=0 \tag{4.49}$$

This is the first equation for $j = 1$ which can further be simplified as

$$(\lambda_{12}+\mu_{12})\frac{\partial}{\partial\theta_1}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}+\frac{\partial u_3}{\partial\theta_3}\right)+\mu_{12}\left(\frac{\partial^2 u_1}{\partial\theta_1^2}+\frac{\partial^2 u_1}{\partial\theta_2^2}+\frac{\partial^2 u_1}{\partial\theta_3^2}\right)$$

$$+\frac{\xi_1}{2}(\bar{\mu}_{12}-\bar{\lambda}_{12})\frac{\partial^3 u_3}{\partial\theta_1\partial\theta_3^2}$$

$$-\frac{\xi_1^2}{3}(\bar{\bar{\lambda}}_{12}+\bar{\bar{\mu}}_{12})\frac{\partial^3}{\partial\theta_1\partial\theta_3^2}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}\right)$$

$$-\frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}\frac{\partial^2}{\partial\theta_3^2}\left(\frac{\partial^2 u_1}{\partial\theta_1^2}+\frac{\partial^2 u_1}{\partial\theta_1^2}\right)=0 \tag{4.50}$$

The second equation will be

$$(\lambda_{12}+\mu_{12})\,\frac{\partial}{\partial\theta_2}\,(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta^2}+\frac{\partial u_3}{\partial\theta_3})+\mu_{12}\,(\frac{\partial^2 u_2}{\partial\theta_1^2}+\frac{\partial^2 u_2}{\partial\theta_2^2}+\frac{\partial^2 u_2}{\partial\theta_3^2})$$

$$+\frac{\xi_1}{2}\,(\bar{\mu}_{12}-\bar{\lambda}_{12})\,\frac{\partial^3 u_3}{\partial\theta_2\partial\theta_3^2}$$

$$-\frac{\xi_1^2}{3}\,(\bar{\bar{\lambda}}_{12}+\bar{\bar{\mu}}_{12})\,\frac{\partial^3}{\partial\theta_2\partial\theta_3^2}\,(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2})$$

$$-\frac{\xi_1^2}{3}\,\bar{\bar{\mu}}_{12}\,\frac{\partial^2}{\partial\theta_3^2}\,(\frac{\partial^2 u_2}{\partial\theta_1^2}+\frac{\partial^2 u_2}{\partial\theta_2^2})=0 \qquad (4.51)$$

And the third equation is

$$(\lambda_{12}+\mu_{12})\,\frac{\partial}{\partial\theta_3}(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta^2}+\frac{\partial u_3}{\partial\theta_3})+\mu_{12}(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial\theta_2^2}+\frac{\partial^2 u_3}{\partial\theta_3^2})$$

$$+\frac{\xi_1}{2}\,\{\bar{\lambda}_{12}\,\frac{\partial^2}{\partial\theta_3^2}\,(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2})+\bar{\mu}_{12}\,\frac{\partial}{\partial\theta_3}\,(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial\theta_2^2})\}$$

$$-\frac{\xi_1}{2}\,\bar{\mu}_{12}\,\{\frac{\partial^2}{\partial\theta_3^2}\,(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta^2})+\frac{\partial}{\partial\theta_3}\,(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial\theta_2^2})\}$$

$$-\frac{\xi_1^2}{3}\,\bar{\bar{\mu}}_{12}\,\frac{\partial^2}{\partial\theta_3^2}\,(\frac{\partial^2 u_3}{\partial\theta_1^3}+\frac{\partial^2 u_3}{\partial\theta_2^3})=0$$

or

$$(\lambda_{12}+\mu_{12})\frac{\partial}{\partial\theta_3}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}+\frac{\partial u_3}{\partial\theta_3}\right)+\mu_{12}\left(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial\theta_2^2}+\frac{\partial^2 u_3}{\partial\theta_3^2}\right)$$

$$+\frac{\xi_1}{2}(\bar{\lambda}_{12}-\bar{\mu}_{12})\frac{\partial^2}{\partial\theta_3^2}\left(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}\right)$$

$$-\frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}\frac{\partial^2}{\partial\theta_3^2}\left(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial_2^2}\right)=0 \tag{4.52}$$

For future reference we will also calculate σ_j for static problem in the absence of body forces. The preceding results are used in conjunction with equation (4.31) which has been simplified for such a case and reproduced before the equation (4.39). From (4.45) and (4.46) and (4.47) we have

$$\{mc^{(1)}_{\alpha jkl}+\frac{(1-m^2)}{m}c^{(2)}_{\alpha jkl}\}u_{k,l\alpha}=\bar{\lambda}_{12}\,\delta_{\alpha j}u_{k,k\alpha}+\bar{\mu}_{12}(u_{\alpha,j\alpha}+u_{j,\alpha\alpha}) \tag{4.53}$$

$$\{mc^{(1)}_{\alpha jk\beta}+\frac{1-m^3}{m^2}c^{(2)}_{\alpha jk\beta}\}u_{k,\beta3\alpha}=\bar{\bar{\lambda}}_{12}\,\delta_{\alpha j}u_{\beta,\beta\alpha3}+\bar{\bar{\mu}}_{12}(\delta_{j\beta}u_{\alpha,\beta\alpha3}+u_{j,\alpha\alpha3}) \tag{4.54}$$

Using (4.36) we get

$$\{mc^{(1)}_{3jkl}+(1-m)c^{(2)}_{3jkl}\}u_{k,l}=[m\lambda_1+(1-m)\lambda_2]\delta_{3j}\delta_{kl}u_{k,l}$$

$$+[m\mu_1+(1-m)\mu_2][\delta_{3k}\delta_{jl}+\delta_{3l}\delta_{jk}]u_{k,l}=\lambda_{12}\delta_{3j}u_{k,k}+\mu_{12}(u_{3,j}+u_{j,3}) \tag{4.55}$$

$$\{mc^{(1)}_{3jk\alpha}+\frac{(1-m^2)}{m}c^{(2)}_{3jk\alpha}\}u_{k,\alpha3}=\bar{\lambda}_{12}\delta_{3j}\delta_{k\alpha}u_{k,\alpha3}+\bar{\mu}_{12}(\delta_{3k}\delta_{j\alpha}+\delta_{3\alpha}\delta_{jk})u_{k,\alpha3}$$

$$=\bar{\lambda}_{12}\delta_{3j}u_{\alpha,\alpha3}+\bar{\mu}_{12}\delta_{j\alpha}u_{3,\alpha3} \tag{4.56}$$

Substituting (4.53)-(4.56) in relation for σ_j we obtain

$$\sigma_j = \lambda_{12}\delta_{3j}u_{k,k} + \mu_{12}(u_{3,j} + u_{j,3}) + \frac{\xi_1}{2}[\bar{\lambda}_{12}\delta_{3j}u_{\alpha,\alpha3} + \bar{\mu}_{12}\delta_{j\alpha}u_{3,\alpha3}]$$

$$-\frac{\xi_1}{2}[\bar{\lambda}_2\delta_{\alpha j}u_{k,k\alpha} + \bar{\mu}_{12}(u_{\alpha,j\alpha} + u_{j,\alpha\alpha})]$$

$$-\frac{1}{3}\xi_1^2[\bar{\bar{\lambda}}_{12}\delta_{\alpha j}u_{\beta,\beta\alpha3} + \bar{\bar{\mu}}_{12}(\delta_{j\beta}u_{\alpha,\beta\alpha3} + u_{j,\alpha\alpha3})] \tag{4.57}$$

Writing down the components of σ_j separately we get

$$\sigma_1 = \mu_{12}(\frac{\partial u_3}{\partial\theta_1} + \frac{\partial u_1}{\partial\theta_3}) + \frac{\xi_1}{2}\bar{\mu}_{12}\frac{\partial^2 u_3}{\partial\theta_1\partial\theta_3} - \frac{\xi_1}{2}\bar{\lambda}_{12}\frac{\partial}{\partial\theta_1}(\frac{\partial u_1}{\partial\theta_1} + \frac{\partial u_2}{\partial\theta_2} + \frac{\partial u_3}{\partial\theta_3})$$

$$-\frac{\xi_1}{2}\bar{\mu}_{12}\frac{\partial}{\partial\theta_1}(\frac{\partial u_1}{\partial\theta_1} + \frac{\partial u_2}{\partial\theta_2}) - \frac{\xi_1}{2}\bar{\mu}_{12}(\frac{\partial^2 u_1}{\partial\theta_1^2} + \frac{\partial^2 u_1}{\partial\theta_2^2})$$

$$-\frac{\xi_1^2}{3}\bar{\bar{\lambda}}_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_3}(\frac{\partial u_1}{\partial\theta_1} + \frac{\partial u_2}{\partial\theta_2})$$

$$-\frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}\frac{\partial^2}{\partial\theta_1\partial\theta_3}(\frac{\partial u_1}{\partial\theta_1} + \frac{\partial u_2}{\partial\theta_2}) - \frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}\frac{\partial}{\partial\theta_3}(\frac{\partial^2 u_1}{\partial\theta_1^2} + \frac{\partial^2 u_1}{\partial\theta_2^2})$$

$$= \mu_{12}(\frac{\partial u_1}{\partial\theta_3} + \frac{\partial u_3}{\partial\theta_1}) + \frac{\xi_1}{2}(\bar{\mu}_{12} - \bar{\lambda}_{12})\frac{\partial^2 u_3}{\partial\theta_1\partial\theta_3} - \frac{\xi_1}{2}(\bar{\lambda}_{12} + \bar{\mu}_{12})\frac{\partial}{\partial\theta_1}(\frac{\partial u_1}{\partial\theta_1} + \frac{\partial u_2}{\partial\theta_2})$$

$$-\frac{\xi_1}{2}\bar{\mu}_{12}(\frac{\partial^2 u_1}{\partial\theta_1^2} + \frac{\partial^2 u_1}{\partial\theta_2^2}) - \frac{\xi_1^2}{3}(\bar{\bar{\lambda}}_{12} + \bar{\bar{\mu}}_{12})\frac{\partial^2}{\partial\theta_1\partial\theta_3}(\frac{\partial u_1}{\partial\theta_1} + \frac{\partial u_2}{\partial\theta_2})$$

$$-\frac{\xi_1^2}{3}\frac{\partial}{\partial\theta_3}(\frac{\partial_2 u_1}{\partial\theta_1^2} + \frac{\partial_2 u_1}{\partial\theta_2^2})\bar{\bar{\mu}}_{12} \tag{4.58}$$

$$\sigma_2 = \mu_{12}(\frac{\partial u_2}{\partial\theta_3} + \frac{\partial u_3}{\partial\theta_2}) + \frac{\xi_1}{2}(\bar{\mu}_{12} - \bar{\lambda}_{12})\frac{\partial^2 u_3}{\partial\theta_2\partial\theta_3} - \frac{\xi_1}{2}(\bar{\lambda}_{12} + \bar{\mu}_{12})\frac{\partial}{\partial\theta_2}(\frac{\partial u_1}{\partial\theta_2} + \frac{\partial u_2}{\partial\theta_2})$$

$$-\frac{\xi_1}{2}\bar{\mu}_{12}(\frac{\partial^2 u_2}{\partial\theta_1^2}+\frac{\partial^2 u_2}{\partial\theta_2^2})-\frac{\xi_1^2}{3}(\bar{\bar{\lambda}}_{12}+\bar{\bar{\mu}}_{12})\frac{\partial^2}{\partial\theta_2\partial\theta_3}(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2})$$

$$-\frac{\xi_1^2}{3}\frac{\partial}{\partial\theta_3}(\frac{\partial^2 u_2}{\partial\theta_1^2}+\frac{\partial^2 u_2}{\partial\theta_2^2})\bar{\bar{\mu}}_{12} \tag{4.59}$$

$$\sigma_3=\lambda_{12}u_{k,k}+2\mu_{12}u_{3,3}+\frac{\xi_1}{2}(\bar{\lambda}_{12}u_{\alpha,\alpha 3})-\frac{\xi_1}{2}\bar{\mu}_{12}(u_{\alpha,3\alpha}+u_{3,\alpha\alpha})-\frac{\xi_1^3}{3}\bar{\bar{\mu}}_{12}u_{3,\alpha\alpha 3}$$

$$=\lambda_{12}(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2}+\frac{\partial u_3}{\partial\theta_3})+2\mu_{12}\frac{\partial u_3}{\partial\theta_3}+\frac{\xi_1}{2}\bar{\lambda}_{12}\frac{\partial}{\partial\theta_3}(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2})$$

$$-\frac{\xi_1}{2}\bar{\mu}_{12}\frac{\partial}{\partial\theta_3}(\frac{\partial u_1}{\partial\theta_2}+\frac{\partial u_2}{\partial\theta_2})-\frac{\xi_1}{2}\bar{\mu}_{12}(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial\theta_2^2})$$

$$-\frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial\theta_2^2})_{,3}$$

$$\sigma_3=\lambda_{12}(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2})+(\lambda_{12}+2\mu_{12})\frac{\partial u_3}{\partial\theta_3}+\frac{\xi_1}{2}(\bar{\lambda}_{12}-\bar{\mu}_{12})\frac{\partial}{\partial\theta_3}(\frac{\partial u_1}{\partial\theta_1}+\frac{\partial u_2}{\partial\theta_2})$$

$$-\frac{\xi_1}{2}\bar{\mu}_{12}(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial_2^2})-\frac{\xi_1^2}{3}\bar{\bar{\mu}}_{12}\frac{\partial}{\partial\theta_3}(\frac{\partial^2 u_3}{\partial\theta_1^2}+\frac{\partial^2 u_3}{\partial\theta_2^2}) \tag{4.60}$$

The constitutive relations (4.28) and (4.29) for τ_{ij} and $S_{\alpha j}$ for a flat composite are also simplified here for a composite laminate whose micro-structure is composed of two isotropic layers. Using (4.36) first we simplify different terms of the expressions (4.28) and (4.29)

$$\{mc_{ijkl}^{(1)}+(1-m)c_{ijkl}^{(2)}\}u_{k,l}=\{m\lambda_1+(1-m)\lambda_2\}\delta_{ij}\delta_{kl}u_{k,l}$$

$$+\{m\mu_1+(1-m)\mu_2\}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jl}+u_{k,l}$$

$$=\lambda_{12}\delta_{ij}u_{k,k}+\mu_{12}(u_{i,j}+u_{j,i}) \tag{4.61}$$

where $(30.47)_{1,2}$ are used.

$$\{mc^{(1)}_{\alpha jkl} + \frac{(1-m^2)}{m}\, c^{(2)}_{\alpha jkl}\}u_{k,l} = \bar{\lambda}_{12}\delta_{\alpha j}\delta_{kl}u_{k,l} + \bar{\mu}_{12}(\delta_{\alpha k}\delta_{jl} + \delta_{\alpha l}\delta_{jk})u_{k,l}$$

$$= \bar{\lambda}_{12}\delta_{\alpha j}u_{k,k} + \bar{\mu}_{12}(u_{\alpha,j} + u_{j,\alpha}) \qquad (4.62)$$

$$\{mc^{(1)}_{ijk\alpha} + \frac{(1-m^2)}{m}\, c^{(2)}_{ijk\alpha}\}u_{k,\alpha 3} = \bar{\lambda}_{12}\delta_{ij}\delta_{k\alpha}u_{k,\alpha 3} + \bar{\mu}_{12}(\delta_{ik}\delta_{j\alpha} + \delta_{i\alpha}\delta_{jk})u_{k,\alpha 3}$$

$$= \bar{\lambda}_{12}\delta_{ij}u_{\alpha,\alpha 3} + \bar{\mu}_{12}(\delta_{j\alpha}u_{i,\alpha 3} + \delta_{i\alpha}u_{j,\alpha 3}) \qquad (4.63)$$

$$\{mc^{(1)}_{\alpha jk\beta} + \frac{1-m^3}{m^2}\, c^{(2)}_{\alpha jk\beta}\}u_{k,\beta 3} = \bar{\bar{\lambda}}_{12}\delta_{\alpha j}\delta_{k\beta}u_{k,\beta 3} + \bar{\bar{\mu}}_{12}(\delta_{\alpha k}\delta_{j\beta} + \delta_{\alpha\beta}\delta_{jk})u_{k,\beta 3}$$

$$= \bar{\bar{\lambda}}_{12}\delta_{\alpha j}u_{\beta,\beta 3} + \bar{\bar{\mu}}_{12}(\delta_{j\beta}u_{\alpha,\beta 3} + u_{j,\alpha 3}) \qquad (4.64)$$

Substituting (4.61)-(4.64) in (4.28) and (4.29) we obtain

$$\tau_{ij} = \lambda_{12}\delta_{ij}u_{k,k} + \mu_{12}(u_{i,j} + u_{j,i})$$

$$+ \frac{\xi_1}{2}\,[\bar{\lambda}_{12}\delta_{ij}u_{\alpha,\alpha 3} + \bar{\mu}_{12}(\delta_{j\alpha}u_{i,\alpha 3} + \delta_{i\alpha}u_{j,\alpha 3})] \qquad (4.65)$$

$$S_{\alpha j} = \frac{\xi_1}{2}\,[\bar{\lambda}_{12}\delta_{\alpha j}u_{k,k} + \bar{\mu}_{12}(u_{\alpha,j} + u_{j,\alpha})]$$

$$+ \frac{1}{3}\,\xi_1^2\,[\bar{\bar{\lambda}}_{12}\delta_{\alpha j}u_{\beta,\beta 3} + \bar{\bar{\mu}}_{12}(\delta_{j\beta}u_{\alpha,\beta 3} + u_{j,\alpha 3})] \qquad (4.66)$$

5.0 LINEAR CONSTITUTIVE RELATIONS FOR A MULTI-CONSTITUENT COMPOSITE

In this section we assume that the representative micro-structure is composed of n layers with different constituents. For such a micro-structure we let

$$\tau^{*ij}_{(\alpha)} = c^{ijkl}_{(\alpha)} \gamma^{*}_{kl} \quad (\alpha = 1,2,\ldots,n) \tag{5.1}$$

where $c^{ijkl}_{(\alpha)}$ $(\alpha = 1,\ldots,n)$ are material constants in the associated layers. As before the variable ξ is designated to change across the micro-structure whose thickness is assumed to be ξ_n. It should be noted that although the micro-structure is composed of n layers, ξ_n is still supposed to be a very small number. The range of variation of ξ in the l^{th} layer of the microstructure is from ξ_{l-1} to ξ_l where $l = 1,\ldots,n$ and $\xi_o = 0$. This convention is adopted due to its agreement with the special case of a two-layered micro-structure which was studied before. We further define (n-1) constants $m_1, \ldots, m_{n-1}$ according to the following relations

$$\xi_l = m_l \xi_n \quad (l = 1,\ldots,n-1) \tag{5.2}$$

As a result of this definition the thickness of the l^{th} layer of the micro-structure is equal to $(m_l - m_{l-1})\xi_n$ where $l = 1,\ldots,n$ and $m_o = 0$, $m_n = 1$. The *composite stress vector* T^i and the *composite couple stress* S^α and other quantities are obviously defined over the whole thickness of the micro-structure. For example we have

$$T^i = \frac{1}{\xi_n} \int_0^{\xi_n} T^{*i} d\xi \tag{5.3}$$

$$S^\alpha = \frac{1}{\xi_n} \int_0^{\xi_n} \xi \, T^{*\alpha} d\xi \tag{5.4}$$

In order to derive appropriate constitutive relations we make the following definition. Let

$$a = a_l \quad \text{for} \quad \xi_{l-1} < \xi < \xi_l \quad (l = 1,2,\ldots,n) \tag{5.5}$$

where a_l's are constants and $\xi_o = 0$ as noted earlier. The function a is piecewise continuous for $\xi \in (0, \xi_n)$ and we can evaluate the following integral

$$\frac{1}{\xi_n}\int_0^{\xi_n} \xi^k a\, d\xi = \frac{1}{\xi_n}\sum_{l=1}^{n}\int_{\xi_{l-1}}^{\xi_l} \xi^k a_l d\xi = \frac{1}{\xi_n}\sum_{l=1}^{n}\frac{a_l}{k+1}(\xi_l^{k+1} - \xi_{l-1}^{k+1}) \quad (k \neq -1) \tag{5.6}$$

Using definitions (5.2) we further simplify (5.6)

$$\frac{1}{\xi_n}\int_0^{\xi_n} \xi^k a d\xi = \frac{1}{\xi_n}\frac{1}{k+1}\sum_{l=1}^{n} a_l[m_l^{k+1} - m_{l-1}^{k+1}]\xi_n^{k+1} = \frac{\xi_n^k}{k+1}\sum_{l=1}^{n} a_l(m_l^{k+1} - m_{l-1}^{k+1}) \quad (k \neq -1) \tag{5.7}$$

where $m_o = 0$ and $m_n = 1$. To simplify the final results in constitutive relations we first notice that the integrals which appear in these equations are the weighted averages of the constitutive coefficients. So we adopt the following definition

$$I^{(k)pqrs} \triangleq \frac{1}{\xi_n}\int_0^{\xi_n} \xi^k c^{pqrs} d\xi \tag{5.8}$$

which by (5.7) is seen to be equal to

$$I^{(k)pqrs} = \frac{1}{k+1}\,\xi_n^k \sum_{l=1}^{n} c_{(l)}^{pqrs}(m_l^{k+1} - m_{l-1}^{k+1}) \tag{5.9}$$

We use the same contravariant or covariant index notations for I and c. However, the weighting number k is always written as a superscript in parentheses. Whenever the covariant components of constitutive coefficients are used, the layer index (l) is also written as a superscript in parentheses. Recall (3.22) which for the present situation can be written as

$$T^i = \gamma_{kl}\frac{1}{\xi_n}\int_0^{\xi_n} c^{ijkl}\, g^{*1/2}\, g_j^*\, d\xi + \kappa_{k\alpha}\frac{1}{\xi_n}\int_0^{\xi_n} \xi c^{ijk\alpha}\, g^{*1/2}\, g_j^* d\xi \tag{5.10}$$

Combining (3.4) and $(2.9)_3$ we obtain

$$g^{*1/2}\, g^*_\beta = g^{1/2}\,(1 + \frac{\xi\Delta}{2g})(g_\beta + \xi\lambda^j_\beta g_j) \tag{5.11}$$

and (3.5) reads

$$g^{*1/2}\, g^*_3 = g^{1/2}\,(1 + \frac{\xi\Delta}{2g})g_3 \tag{5.12}$$

The first integral in relation (5.10) is simplified using (5.11) and (5.12). We have

$$\frac{1}{\xi_n}\int_0^{\xi_n} c^{ijkl}\, g^{*1/2}\, g^*_j\, d\xi = \frac{1}{\xi_n}\int_0^{\xi_n} (c^{i\beta kl}\, g^{*1/2}\, g^*_\beta + c^{i3kl}\, g^{*1/2}\, g^*_3)d\xi$$

$$= g^{1/2}\, g_\beta\, \frac{1}{\xi_n}\int_0^{\xi_n} (1 + \frac{\xi\Delta}{2g})c^{i\beta kl}d\xi + g^{1/2}\,\lambda^j_\beta g_j\, \frac{1}{\xi_n}\int_0^{\xi_n}\xi(1 + \frac{\xi\Delta}{2g})c^{i\beta kl}d\xi$$

$$+ g^{1/2}\, g_3\, \frac{1}{\xi_n}\int_0^{\xi_n} (1 + \frac{\xi\Delta}{2g})c^{i3kl}d\xi$$

$$= g^{1/2}\, g_j\{\frac{1}{\xi_n}\int_0^{\xi_n} (1 + \frac{\xi\Delta}{2g})c^{ijkl}d\xi + \lambda^j_\beta\, \frac{1}{\xi_n}\int_0^{\xi_n}\xi(1 + \frac{\xi\Delta}{2g})c^{i\beta kl}d\xi\} \tag{5.13}$$

Now we use definition (5.8) to simplify (5.13). The following expression would be the result

$$1^{st} \text{ term in } (5.10) = g^{1/2}\, g_j\{I^{(0)ijkl} + \frac{\Delta}{2g}\, I^{(1)ijkl} + \lambda^j_\beta(I^{(1)i\beta kl} + \frac{\Delta}{2g}\, I^{(2)i\beta kl})] \tag{5.14}$$

The second integral in (5.10) is also simplified similarly

$$\frac{1}{\xi_n}\int_0^{\xi_n} \xi c^{ijl\alpha}\, g^{*1/2}\, g^*_j\, d\xi = \frac{1}{\xi_n}\int_0^{\xi_n} \xi\, g^{*1/2}\,(c^{i\beta l\alpha} g^*_\beta + c^{i3l\alpha} g^*_3)d\xi$$

$$= \frac{1}{\xi_n}\int_0^{\xi_n} \xi\, g^{1/2}\,(1 + \frac{\xi\Delta}{2g})c^{i\beta l\alpha} g_\beta d\xi + \frac{1}{\xi_n}\int_0^{\xi_n} \xi^2\, g^{1/2}\,(1 + \frac{\xi\Delta}{2g})\lambda^j_\beta\, c^{i\beta l\alpha} g_j d\xi$$

$$+ \frac{1}{\xi_n}\int_0^{\xi_n} \xi\, g^{1/2}\,(1 + \frac{\xi\Delta}{2g})c^{i3l\alpha} g_3 d\xi$$

$$= g^{1/2}\, g_j \{ \frac{1}{\xi_n} \int_0^{\xi_n} \xi (1 + \frac{\xi\Delta}{2g}) c^{ijl\alpha} d\xi + \lambda^j_\beta \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 (1 + \frac{\xi\Delta}{2g}) c^{i\beta l\alpha} d\xi \} \tag{5.15}$$

which again by using (5.8) is simplified to

$$2^{nd} \text{ term in (5.10)} = g^{1/2}\, g_j \{ I^{(1)ijl\alpha} + \frac{\Delta}{2g} I^{(1)ijl\alpha} + \lambda^j_\beta I^{(2)i\beta l\alpha} + \lambda^j_\beta \frac{\Delta}{2g} I^{(3)i\beta l\alpha} \} \tag{5.16}$$

Substituting (5.14) and (5.16) in (5.10) we get the following constitutive relation for T^i

$$T^i = \{ [I^{(0)ijkl} + \frac{\Delta}{2g} I^{(1)ijkl} + \lambda^j_\beta (I^{(1)i\beta kl} + \frac{\Delta}{2g} I^{(2)i\beta kl})] \gamma_{kl}$$

$$+ [I^{(1)ijl\alpha} + \frac{\Delta}{2g} I^{(2)ijl\alpha} + \lambda^j_\beta (I^{(2)i\beta l\alpha} + \frac{\Delta}{2g} I^{(3)i\beta l\alpha})] \kappa_{l\alpha} \} g^{1/2}\, g_j \tag{5.17}$$

The expression inside the bracket is obviously the constitutive relation for τ^{ij}. Hence

$$\tau^{ij} = [I^{(0)ijkl} + \frac{\Delta}{2g} I^{(1)ijkl} + \lambda^j_\beta (I^{(1)i\beta kl} + \frac{\Delta}{2g} I^{(2)i\beta kl})] \gamma_{kl}$$

$$+ [I^{(1)ijl\alpha} + \frac{\Delta}{2g} I^{(2)ijl\alpha} + \lambda^j_\beta (I^{(2)i\beta l\alpha} + \frac{\Delta}{2g} I^{(3)i\beta l\alpha})] \kappa_{l\alpha} \tag{5.18}$$

The same steps are followed to derive the constitutive relation for the *composite couple stress* S^α. By (3.27) we have

$$S^\alpha = \gamma_{kl} \frac{1}{\xi_n} \int_0^{\xi_n} \xi\, g^{*1/2} c^{\alpha jkl} g_j^* d\xi + \kappa_{l\beta} \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 c^{\alpha jl\beta} g^{*1/2} g_j^* d\xi \tag{5.19}$$

which can be reduced to the following form by exactly using the same procedure

$$S^\alpha = \{ [I^{(1)\alpha jkl} + \frac{\Delta}{2g} I^{(2)\alpha jkl} + \lambda^j_\gamma (I^{(2)\alpha\gamma kl} + \frac{\Delta}{2g} I^{(3)\alpha\gamma kl})] \gamma_{kl}$$

$$+ [I^{(2)\alpha jl\beta} + \frac{\Delta}{2g} I^{(3)\alpha jl\beta} + \lambda^j_\gamma (I^{(3)\alpha\gamma l\beta} + \frac{\Delta}{2g} I^{(4)\alpha\gamma l\beta})] \kappa_{l\beta} \}\, g^{1/2}\, g_j \tag{5.20}$$

Subsequently the constitutive relation for $S^{\alpha j}$ would be

$$S^{\alpha j} = [I^{(1)\alpha jkl} + \frac{\Delta}{2g} I^{(2)\alpha jkl} + \lambda^j_\gamma (I^{(2)\alpha\gamma kl} + \frac{\Delta}{2g} I^{(3)\alpha\gamma kl})]\gamma_{kl}$$

$$+ [I^{(2)\alpha jl\beta} + \frac{\Delta}{2g} I^{(3)\alpha jl\beta} + \lambda^j_\gamma (I^{(3)\alpha\gamma l\beta} + \frac{\Delta}{2g} I^{(4)\alpha\gamma l\beta})]\kappa_{l\beta} \tag{5.21}$$

For small deformations of a composite with initially flat plies, the foregoing equations are further simplified. The resulting relations are recorded here:

$$\tau^{ij} = I^{(0)ijkl}\gamma_{kl} + I^{(1)ijl\alpha}\kappa_{l\alpha} \tag{5.22}$$

$$S^{\alpha j} = I^{(1)\alpha jkl}\gamma_{kl} + I^{(2)\alpha jl\beta}\kappa_{l\beta} \tag{5.23}$$

with no distinction between contravariant and covariant components. In terms of displacement vector **u** and its gradients these equations can be written as follows:

$$\tau_{ij} = I^{(0)}_{ijkl}u_{k,l} + I^{(1)}_{ijl\alpha}u_{l,\alpha 3} \tag{5.24}$$

$$S_{\alpha j} = I^{(1)}_{\alpha jkl}u_{k,l} + I^{(2)}_{\alpha jk\beta}u_{l,\beta 3} \tag{5.25}$$

Using (5.9) the constitutive coefficients are written in the expanded form

$$I^{(0)}_{ijkl} = \sum_{r=1}^{n} c^{(r)}_{ijkl}(m_r - m_{r-1})$$

$$I^{(1)}_{ijkl} = \frac{1}{2}\,\xi_n \sum_{r=1}^{n} c^{(r)}_{ijkl}(m_r^2 - m_{r-1}^2) \tag{5.26}$$

$$I^{(2)}_{ijkl} = \frac{1}{2}\,\xi_n^2 \sum_{r=1}^{n} c^{(r)}_{ijkl}(m_r^3 - m_{r-1}^3)$$

To recapitulate, $d^{(r)}_{ijkl}$ ($r = 1,\ldots,n$) are the constitutive coefficients of the micro-structure layers and m_r's ($r = 1,\ldots,n-1$) are dimensionless constants related to the thicknesses of different layers with $m_o = 0$ and $m_n = 1$.

If the micro-structure is composed of n isotropic layers, we can write (5.26) in terms of the Lame's constants of various layers. In fact, we have

$$c^{(r)}_{ijkl} = \lambda_{(r)}\delta_{ij}\delta_{kl} + \mu_{(r)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad r = 1,\dots,n \tag{5.27}$$

For such a case, the relations (5.24) and (5.25) are written in expanded forms as follows

$$\tau_{ij} = u_{k,k}\delta_{ij}\sum_{r=1}^{n}\lambda_{(r)}(m_r - m_{r-1}) + (u_{i,j}+u_{j,i})\sum_{r=1}^{n}\mu_{(r)}(m_1 - m_{r-1})$$

$$+\frac{\xi_n}{2}[\delta_{ij}\delta_{l\alpha}u_{l,\alpha 3}\sum_{r=1}^{n}\lambda_{(r)}(m_r^2 - m_{r-1}^2) + (\delta_{il}\delta_{j\alpha}+\delta_{i\alpha}\delta_{jl})$$

$$\times u_{l,\alpha 3}\sum_{r=1}^{n}\mu_{(r)}(m_r^2 - m_{r-1}^2)]$$

or

$$\tau_{ij} = u_{k,k}\,\delta_{ij}\sum_{r=1}^{n}\lambda_{(r)}\,\Delta m_r + (u_{i,j}+u_{j,i})\sum_{r=1}^{n}\mu_{(r)}\Delta m_r + \frac{\xi_n}{2}\delta_{ij}u_{\alpha,\alpha 3}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^2$$

$$+\frac{\xi_n}{2}(u_{i,\alpha 3}\delta_{j\alpha}+u_{j,\alpha 3}\delta_{i\alpha})\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^2 \tag{5.28}$$

$$S_{\alpha j} = \frac{1}{2}\xi_n[\delta_{\alpha j}\delta_{kl}u_{k,l}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^2 + (\delta_{\alpha k}\delta_{jl} + \delta_{\alpha l}\delta_{jk})u_{k,l}\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^2]$$

$$+\frac{1}{3}\xi_n^2[\delta_{\alpha j}\delta_{l\beta}u_{l,\beta 3}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^3 + (\delta_{\alpha l}\delta_{j\beta} + \delta_{\alpha\beta}\delta_{jl})u_{l,\beta 3}\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^3]$$

or

$$S_{\alpha j} = \frac{1}{2}\xi_n[\delta_{\alpha j}u_{k,k}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^2 + (u_{\alpha,j}+u_{j,\alpha})\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^2]$$

$$+\frac{1}{3}\xi_n^2[\delta_{\alpha j}u_{\beta,\beta 3}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^3 + (\delta_{j\beta}u_{\alpha,\beta 3} + u_{j,\alpha 3})\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^3] \tag{5.29}$$

where for brevity we have introduced

$$\Delta m_r^p = m_r^p - m_{r-1}^p \qquad r = 1,\dots,n \tag{5.30}$$

and in the above relations p = 1,2,3.

In order to obtain the complete field equations for a linear elastic composite whose microstructures comprise n layers we should substitute the constitutive relations (5.24) and (5.25) in the equations of motion (2.123) and (2.124). However before that we should obtain appropriate expressions for ρ_o, $\rho_o z^1$ and $\rho_o z^2$. We assume that the micro-structure layers are homogeneous with densities $\rho_o^{(r)}$ (r = 1,...,n) in the reference configuration. Recalling (2.45) and (2.46) we can write

$$\rho g^{1/2} = \rho_o G^{1/2} = \frac{1}{\xi_n}\int_0^{\xi_n} \rho_o^* G^{*1/2} d\xi \tag{5.31}$$

where

$$\rho_o^* = \rho_o^{(r)} \quad \text{for} \quad \xi_{1-1} < \xi < \xi_r \quad (r = 1,\dots,n)$$

and (5.32)

$$\xi_o = 0$$

Using (4.2), (5.32) and (5.7) in (5.31) we conclude

$$\rho_o = \frac{1}{\xi_n}\int_0^{\xi_n} (1 + \frac{\xi\Delta}{2G})\rho_o^* d\xi = \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r + \frac{\Delta\xi_n}{4G}\sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r^2 \tag{5.33}$$

We will proceed similarly to calculate $\rho_o z^1$ and $\rho_o z^2$ using their definitions for the present situation. By (2.42) we have

$$\rho g^{1/2} z^1 = \rho_o G^{1/2} z^1 = \frac{1}{\xi_n}\int_0^{\xi_n} \xi\rho_o^* G^{*1/2} d\xi \tag{5.34}$$

$$\rho g^{1/2} z^2 = \rho_o G^{1/2} z^2 = \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 \rho_o^* G^{*1/2} d\xi \tag{5.35}$$

Similar to what was done in the derivation of (5.33) we write

$$\rho_o z^1 = \frac{1}{\xi_n} \int_0^{\xi_n} \xi (1 + \frac{\xi \Delta}{2G}) \rho_o^* d\xi = \frac{1}{2} \xi_n \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r^2 + \frac{\Delta \xi_n^2}{6G} \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r^3 \tag{5.36}$$

and

$$\rho_o z^2 = \frac{1}{\xi_n} \int_0^{\xi_n} \xi^2 (1 + \frac{\xi \Delta}{2G}) \rho_o^* d\xi = \frac{1}{3} \xi_n^2 \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r^3 + \frac{\Delta \xi_n^3}{8G} \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r^4 \tag{5.37}$$

For a composite with initially flat plies, equations (5.35), (5.36) and (5.37) are reduced respectively to

$$\rho_o = \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r \tag{5.38}$$

$$\rho_o z^1 = \frac{1}{2} \xi_n \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r^2 \tag{5.39}$$

$$\rho_o z^2 = \frac{1}{3} \xi_n^2 \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r^3 \tag{5.40}$$

Now we can substitute (5.24), (5.25), (5.38)-(5.40) and (4.23) in (2.123) and (2.124) to derive the linear equations of motion for a flat composite. The resulting equations are recorded below:

$$I^{(0)}_{\alpha jk\ell} u_{k,\ell\alpha} + I^{(1)}_{\alpha j\ell\beta} u_{\ell,\alpha\beta 3} + b_j \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r + \sigma_{j,3}$$

$$= \ddot{u}_j \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r + \frac{1}{2} \xi_n \ddot{u}_{j,3} \sum_{r=1}^{n} \rho_o^{(r)} \Delta m_r^2 \tag{5.41}$$

BASE

$$I^{(1)}_{\alpha jkl}u_{k,l\alpha} + I^{(2)}_{\alpha jl\beta}u_{l,\alpha\beta 3} + \sigma_j - I^{(0)}_{3jkl}u_{k,l} - I^{(1)}_{3jl\beta}u_{l,\beta 3}$$

$$+ c_j \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r = \frac{1}{2}\,\xi_n \ddot{u}_j \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r^2 + \frac{1}{3}\,\xi_n^2 \ddot{u}_{j,3} \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r^3 \tag{5.42}$$

The appropriate differential equation for the displacement vector **u** is obtained by eliminating σ_j from equations (5.41) and (5.42). For static problems with no body force these equations reduce to

$$I^{(0)}_{\alpha jkl}u_{k,l\alpha} + I^{(1)}_{\alpha jk\beta}u_{k,\alpha 3\beta} + \sigma_{j,3} = 0 \tag{5.43}$$

$$I^{(1)}_{\alpha jkl}u_{k,l\alpha} + I^{(2)}_{\alpha jk\beta}u_{k,\alpha\beta 3} + \sigma_j - I^{(0)}_{3jkl}u_{k,l} - I^{(1)}_{3jk\beta}u_{k\beta 3} = 0 \tag{5.44}$$

Eliminating σ_j between these equations, we get

$$I^{(0)}_{\alpha jkl}u_{k,l\alpha} + I^{(1)}_{\alpha jk\beta}u_{k,\alpha\beta 3} + I^{(0)}_{3jkl}u_{k,l3} + I^{(1)}_{3jk\beta}u_{k,\beta 33} - I^{(1)}_{\alpha jkl}u_{k,l\alpha 3} - I^{(2)}_{\alpha jk\beta}u_{k,\alpha\beta 33} = 0$$

or

$$I^{(0)}_{ijkl}u_{k,li} + I^{(1)}_{ijk\beta}u_{k,i\beta 3} - I^{(1)}_{\alpha jkl}u_{k,l\alpha 3} - I^{(2)}_{\alpha jk\beta}u_{k,\alpha\beta 33} = 0 \tag{5.45}$$

This is a fourth order partial differential equation for the displacement vector **u**. The constitutive coefficients $I^{(r)}_{ijkl}$ ($r = 0,1,2$) are already written in expanded forms in Eqs. (5.26).

For a composite laminate whose micro-structure is composed of n isotropic layers we use (5.27) and (5.26) to rewrite (5.45). The result is

$$\delta_{ij}\delta_{kl}u_{k,li} \sum_{r=1}^{n} \lambda_{(r)}\Delta m_r + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})u_{k,li} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r$$

$$+ \frac{1}{2}\,\xi_n[\delta_{ij}\delta_{k\beta}u_{k,i\beta 3} \sum_{r=1}^{n} \lambda_{(r)}\Delta m_r^2 + (\delta_{ik}\delta_{j\beta} + \delta_{i\beta}\delta_{jk})u_{k,i\beta 3} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r^2]$$

$$- \frac{1}{2}\,\xi_n[\delta_{\alpha j}\delta_{kl}u_{k,l\alpha 3} \sum_{r=1}^{n} \lambda_{(r)}\Delta m_r^2 + (\delta_{\alpha k}\delta_{jl} + \delta_{\alpha l}\delta_{jk})u_{k,l\alpha 3} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r^2]$$

$$-\frac{1}{3}\xi_n^2[\delta_{\alpha j}\delta_{k\beta}u_{k,\alpha\beta 33}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^3+(\delta_{\alpha k}\delta_{j\beta}+\delta_{\alpha\beta}\delta_{jk})u_{k,\alpha\beta 33}\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^3]=0$$

or

$$u_{i,ji}(\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r+\sum_{r=1}^{n}\mu_{(r)}\Delta m_r)+u_{j,ii}\sum_{r=1}^{n}\mu_{(r)}\Delta m_r$$

$$+\frac{1}{2}\xi_n[u_{\beta,j\beta 3}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^2+(\delta_{j\beta}u_{i,i\beta 3}+u_{j,\beta\beta 3})\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^2]$$

$$-\frac{1}{2}\xi_n[\delta_{\alpha j}u_{k,k\alpha 3}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^2+(u_{\alpha,j\alpha 3}+u_{j,\alpha\alpha 3})\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^2]$$

$$-\frac{1}{3}\xi_n^2[\delta_{\alpha j}u_{\beta,\alpha\beta 33}\sum_{r=1}^{n}\lambda_{(r)}\Delta m_r^3+(\delta_{j\beta}u_{\alpha,\alpha\beta 33}+u_{j,\alpha\alpha 33})\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^3]$$

$$=u_{i,ji}\sum_{r=1}^{n}(\lambda_{(r)}+\mu_{(r)})\Delta m_r+u_{j,ii}\sum_{r=1}^{n}\mu_{(r)}\Delta m_r$$

$$+\frac{1}{2}\xi_n\{(u_{\alpha,j\alpha 3}-\delta_{\alpha j}u_{i,i\alpha 3})\sum_{r=1}^{n}[\lambda_{(r)}-\mu_{(r)}]\Delta m_r^2\}$$

$$-\frac{1}{3}\xi_n^2\{\delta_{\alpha j}u_{\beta,\alpha\beta 33}\sum_{r=1}^{n}[\lambda_{(r)}+\mu_{(r)}]\Delta m_r^3+u_{j,\alpha\alpha 33}\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^3\}=0 \qquad (5.46)$$

There are three partial differential equations of fourth order for displacement vector **u** and we can write them in two separate sets, one for $j=\gamma$ and the other for $j=3$. For $j=\gamma$ we obtain

$$u_{i,\gamma i}\sum_{r=1}^{n}(\lambda_{(r)}+\mu_{(r)})\Delta m_r+u_{\gamma,ii}\sum_{r=1}^{n}\mu_{(r)}\Delta m_r$$

$$-\frac{1}{2}\xi_n\,u_{3,\gamma 33}\sum_{r=1}^{n}(\lambda_{(r)}-\mu_{(r)})\Delta m_r^2$$

$$-\frac{1}{3}\xi_n^2\{u_{\beta,\gamma\beta 33}\sum_{r=1}^{n}(\lambda_{(r)}+\mu_{(r)})\Delta m_r^3+u_{\gamma,\alpha\alpha 33}\sum_{r=1}^{n}\mu_{(r)}\Delta m_r^3\}=0 \qquad (5.47)$$

and for $j=3$ we get

$$u_{i,i3} \sum_{r=1}^{n} (\lambda_{(r)} + \mu_{(r)})\Delta m_r + u_{3,ii} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r + \frac{1}{2} \xi_n u_{\alpha,\alpha 33} \sum_{r=1}^{n} (\lambda_{(r)} - \mu_{(.)})\Delta m_r^2$$

$$- \frac{1}{3} \xi_n^2 u_{3,\alpha\alpha 33} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r^3 = 0 \qquad (5.48)$$

For the special case where n = 2 these equations reduce to the equations (4.50)-(4.52) derived before for a bi-laminate composite. From (5.44) we can also calculate σ_j

$$\sigma_j = I_{3jkl}^{(0)} u_{k,l} - I_{\alpha jkl}^{(1)} u_{k,l\alpha} + I_{3jk\beta}^{(1)} u_{k,\beta 3} - I_{\alpha jk\beta}^{(2)} u_{k,\alpha\beta 3} \qquad (5.49)$$

which for the isotropic case reduces to

$$\sigma_j = (\delta_{3j}\delta_{kl}u_{k,l}) \sum_{r=1}^{n} \lambda_{(r)}\Delta m_r + (\delta_{3k}\delta_{jl} + \delta_{3l}\delta_{jk})u_{k,l} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r$$

$$- \frac{\xi_n}{2} [\delta_{\alpha j}\delta_{kl}u_{k,l\alpha} \sum_{r=1}^{n} \lambda_{(r)}\Delta m_r^2 + (\delta_{\alpha k}\delta_{jl} + \delta_{\alpha l}\delta_{jk})u_{k,l\alpha} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r^2]$$

$$+ \frac{\xi_n}{2} [\delta_{3j}\delta_{k\beta}u_{k,\beta 3} \sum_{r=1}^{n} \lambda_{(r)}\Delta m_r^2 + (\delta_{3\beta}\delta_{jk} + \delta_{3k}\delta_{j\beta})u_{k,\beta 3} \sum_{r=1}^{n} \mu_{(1)}\Delta m_r^2]$$

$$- \frac{1}{3} \xi_n^2[\delta_{\alpha j}\delta_{k\beta}k_{k,\alpha\beta 3} \sum_{r=1}^{n} \lambda_{(r)}\Delta m_r^3 + (\delta_{\alpha k}\delta_{j\beta} + \delta_{\alpha\beta}\delta_{jk})u_{k,\alpha\beta 3} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r^3] \qquad (5.50)$$

The equations (5.50) are written in two separate sets. For $j = \gamma$ we get

$$\sigma_\gamma = (u_{\gamma,3} + u_{3,\gamma}) \sum_{r=1}^{n} \mu_{(r)}\Delta m_r + \frac{\xi_n}{2} u_{3,\gamma 3} \sum_{r=1}^{n} [\mu_{(r)} - \lambda_{(r)}]\Delta m_r^2$$

$$- \frac{\xi_n}{2} [u_{\alpha,\alpha\gamma} \sum_{r=1}^{n} (\lambda_{(r)} + \mu_{(r)})\Delta m_r^2 + u_{\gamma,\alpha\alpha} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r^2]$$

$$- \frac{1}{3} \xi_n^2[u_{\alpha,\alpha\gamma 3} \sum_{r=1}^{n} (\lambda_{(r)} + \mu_{(r)})\Delta m_r^3 + u_{\gamma,\alpha\alpha 3} \sum_{r=1}^{n} \mu_{(r)}\Delta m_r^3] \qquad (5.51)$$

and for $j = 3$ we obtain

$$\sigma_3 = u_{k,k} \sum_{r=1}^{n} \lambda_{(r)} \Delta m_r + 2u_{3,3} \sum_{r=1}^{n} \mu_{(r)} \Delta m_r$$

$$- \frac{\xi_n}{2} (u_{\alpha,3\alpha} + u_{3,\alpha\alpha}) \sum_{r=1}^{n} \mu_{(r)} \Delta m_r^2 + \frac{\xi_n}{2} u_{\alpha,\alpha 3} \sum_{r=1}^{n} \lambda_{(r)} \Delta m_r^2$$

$$- \frac{1}{3} \xi_n^2 u_{3,\alpha\alpha 3} \sum_{r=1}^{n} \mu_{(r)} \Delta m_r^3 \qquad (5.52)$$

For the special case of a laminate with two layer micro-structure (n=2) these equations reduce to (4.58)-(4.60).

6.0 THERMOMECHANICAL THEORY OF COMPOSITE LAMINATES

In order to develop a thermo-mechanical theory for the composite laminates, we begin by writing down the local balance of energy and the Clausius-Duhem inequality for the k^{th} representative micro-structure. First we introduce the following additional five quantities which we associate with a motion of the micro-structure:

The specific internal energy $\varepsilon^* = \varepsilon^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$

The heat flux vector $\mathbf{q}^* = \mathbf{q}^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$

The heat supply or heat absorption $r^* = r^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$

The specific entropy $\eta^* = \eta^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$ and,

The local temperature $\theta^* = \theta^*(\theta^\alpha, \theta^{3(k)}, \xi, t)$ which is assumed to be always positive. The equation for the local balance of energy — the first law of thermodynamics — can be written in the following form

$$\rho^* r^* - \rho^* \dot{\varepsilon}^* + \tau^{*ij} \dot{\gamma}^*_{ij} - q^{*k}{}_{|k} = 0 \tag{6.1}$$

where ρ^* is the density of the micro-structure, q^{*k} and $\dot{\gamma}^*_{ij}$ are defined by

$$\mathbf{q}^* = q^{*k} \mathbf{g}^*_k \quad , \quad \dot{\gamma}^*_{ij} = \frac{1}{2} \dot{g}^*_{ij} \tag{6.2}$$

and covariant differentiation is performed with repsect to the metric tensor g^*_{ij} of the micro-structure. Recalling the relations

$$\dot{\mathbf{g}}^*_i = \mathbf{v}^*_{,i}$$

$$g^*_{ij} = \mathbf{g}^*_i \cdot \mathbf{g}^*_j \tag{6.3}$$

$$\mathbf{T}^{*i} = g^{*1/2} \tau^{*ij} \mathbf{g}^*_j$$

$$T^{*i} = g^{*1/2}\tau^{*ij}g_j^*$$

We can write

$$\dot{\gamma}_{ij}^{*} = \frac{1}{2}\,\dot{g}_{ij}^{*} = \frac{1}{2}\,(v_{,i}^{*}\cdot g_j^{*} + v_{,j}^{*}\cdot g_i^{*}) \tag{6.4}$$

Using $(6.3)_3$ and the symmetry of τ^{*ij}, we can write

$$\tau^{*ij}\dot{\gamma}_{ij}^{*} = \frac{1}{2}\,(v_{,i}^{*}\cdot \tau^{*ij}g_j^{*} + v_{,j}^{*}\cdot \tau^{*ij}g_i^{*})$$

$$= \frac{1}{2}\,(v_{,i}^{*}\cdot g^{*-1/2}T^{*i} + v_{,j}^{*}\cdot g^{*-1/2}T^{*j}) = g^{*-1/2}T^{*i}\cdot v_{,i}^{*} \tag{6.5}$$

As for the divergence of the heat flux vector, we have

$$\mathrm{div}\,q^{*} = q^{*k}{}_{|k} = \frac{1}{g^{*1/2}}\,(g^{*1/2}\,q^{*k})_{,k} \tag{6.6}$$

Introducing the results (6.5) and (6.6) in (6.1), we can write the local energy equation in the following alternative form

$$\rho^{*}r^{*} - \rho^{*}\dot{\varepsilon}^{*} + g^{*-1/2}[T^{*i}\cdot v_{,i}^{*} - (g^{*1/2}q^{*k})_{,k}] = 0 \tag{6.7}$$

The energy equation can also be written in terms of the Helmholtz free energy function defined by

$$\psi^{*} = \varepsilon^{*} - \theta^{*}\eta^{*} \tag{6.8}$$

The Clausius-Duhem inequality as a statement for second law of thermodynamics has the following local form for the representative micro-structure

$$\rho^{*}\theta^{*}\dot{\eta}^{*} - \rho^{*}r^{*} + \theta^{*}g^{*-1/2}\left(\frac{g^{*1/2}q^{*k}}{\theta^{*}}\right)_{,k} \geq 0 \tag{6.9}$$

By combining (6.9) and (6.1) and using (6.6) we have the inequality

$$\rho^*(\theta^*\dot{\eta}^* - \dot{\varepsilon}^*) + \tau^{*ij}\dot{\gamma}_{ij}^* - \frac{1}{\theta^*} q^{*k}\theta_{,k}^* \geq 0 \tag{6.10}$$

which in terms of the Helmholtz free energy ψ^* defined in (6.8) becomes

$$-\rho^*(\dot{\psi}^* + \eta^*\dot{\theta}^*) + \tau^{*ij}\dot{\gamma}_{ij}^* - \frac{1}{\theta^*} q^{*k}\theta_{,k}^* \geq 0 \tag{6.11}$$

Now for elastic materials the constitutive relations for Helmholtz free energy, the specific entropy and the stress tensor can be expressed in the following forms

$$\psi^* = \tilde{\psi}^*(\gamma_{ij}^*, \theta^*) \tag{6.12}$$

$$\eta^* = -\frac{\partial\tilde{\psi}^*}{\partial\theta^*} \tag{6.13}$$

$$\tau^{*ij} = \rho^* \frac{\partial\tilde{\psi}^*}{\partial\gamma_{ij}^*} \tag{6.14}$$

where the partial derivative with respect to the symmetric tensor γ_{ij}^* is understood to have the following symmetric form

$$\frac{1}{2}\left(\frac{\partial\tilde{\psi}^*}{\partial\gamma_{ij}^*} + \frac{\partial\tilde{\psi}^*}{\partial\gamma_{ji}^*}\right)$$

The constitutive relation for the heat flux vector has the form

$$q^{*k} = \tilde{q}^{*k}(\gamma_{ij}^*, \theta^*, \theta_{,m}^*) \tag{6.15}$$

and the response function $\tilde{q}^{*k}$ in the light of the Clausius-Duhem inequality is seen to be restricted by the inequality

$$-\tilde{q}^{*k}\theta_{,k}^* \geq 0 \tag{6.16}$$

With the help of (6.13) and (6.14) the energy equation (6.1) is reduced to the following form

$$\rho^* r^* - q^{*k}{}_{|k} - \rho^* \theta^* \dot{\eta}^* = 0 \tag{6.17}$$

where we have used the definition (6.8) in order to calculate $\dot{\varepsilon}^*$ in terms of $\dot{\psi}^*$ and then used the relations (6.13) and (6.14) to further simplify the energy equation. It should be recalled that the argument of different functions in the energy equation (6.17) is $(\theta^\alpha, \theta^{3(k)}, \xi, t)$ and this equation is written for each and every representative element ($k = 1,2,\ldots,n$) which repeats itself in our model and $n \to \infty$. For a bi-laminate representative micro-structure with thickness ξ_2 we introduce the following composite quantities. These relations can be generalized for a multi-constituent micro-structure without any difficulty (see definitions 5.3 and 5.4)

$$\begin{aligned} \rho g^{1/2} r &\triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} r^* d\xi \\ \rho g^{1/2} r_1 &\triangleq \int_0^{\xi_2} \rho^* g^{*1/2} r^* \xi \, d\xi \end{aligned} \tag{6.18}$$

$$\begin{aligned} g^{1/2} q^i &\triangleq \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} q^{*i} d\xi \\ g^{1/2} q_1^\alpha &\triangleq \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} q^{*\alpha} \xi \, d\xi \end{aligned} \tag{6.19}$$

$$\rho g^{1/2} \eta_{(m)} \triangleq \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \eta^* \xi^m d\xi \quad (m = 0,1,2) \tag{6.20}$$

We further assume that the variation of temperature θ^* across the micro-structure is a linear function of ξ, hence

$$\theta^*(\theta^\alpha, \theta^{3(k)}, \xi, t) = \phi_o(\theta^\alpha, \theta^{3(k)}, t) + \xi \phi_1(\theta^\alpha, \theta^{3(k)}, t) \tag{6.21}$$

In order to derive the appropriate form of the energy equation for the composite laminate, first we write (6.17) in the following form

$$\rho^* g^{*1/2} \gamma^* - (g^{*1/2} q^{*k})_{,k} = \rho^* g^{*1/2} \dot{\eta}^* \theta^* \tag{6.22}$$

which after using (6.21) reduces to

$$\rho^* g^{*1/2} \gamma^* - (g^{*1/2} q^{*k})_{,k} = \rho^* g^{*1/2} \dot{\eta}^* (\phi_o + \xi \phi_1) \tag{6.23}$$

Now divide (6.23) by ξ_2 and integrate with respect to ξ from 0 to ξ_2, the result is

$$\frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \gamma^* d\xi - \frac{1}{\xi_2} \int_0^{\xi_2} (g^{*1/2} q^{*\alpha})_{,\alpha} d\xi - \frac{1}{\xi_2} \int_0^{\xi_2} \frac{\partial}{\partial \xi} (g^{*1/2} q^{*3}) d\xi$$

$$= \frac{\phi_o}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \dot{\eta}^* d\xi + \frac{\phi_1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \dot{\eta}^* \xi \, d\xi \tag{6.24}$$

Each term in the above equation can be written in terms of the composite quantities introduced in (6.18)-(6.20) except the third term which is the difference between the values of $g^{*1/2} q^{*3}$ above and below the representative element divided by its thickness ξ_2, namely

$$\frac{1}{\xi_2} \int_0^{\xi_2} \frac{\partial}{\partial \xi} (g^{*1/2} q^{*3}) d\xi = \frac{1}{\xi_2} [g^{*1/2} q^{*3} (\theta^\alpha, \theta^{3(k+1)}, t) - g^{*1/2} q^{*3} (\theta^\alpha, \theta^{3(k)}, t)]$$

Now we assume the existence of the continuous function $h(\theta^\alpha, \theta^3, t)$ which coincides with $q^{*3}(\theta^\alpha, \theta^{3(k)}, t)$ at $\theta^3 = \theta^{3(k)}$, and further approximate the right side of the above equation as the gradient of this function multiplied by $g^{1/2}$ in the θ^3 direction, i.e., $\partial(g^{1/2} h)/\partial\theta^3$. As a result, (6.24) can be written as

$$\rho g^{1/2} \gamma - (g^{1/2} q^\alpha)_{,\alpha} - \frac{\partial}{\partial \theta^3} (g^{1/2} h) = \rho g^{1/2} (\phi_o \dot{\eta}_o + \phi_1 \dot{\eta}_1) \tag{6.25}$$

In writing (6.25) we have also made use of the balance of mass equation.

Next we multiply (6.23) by ξ, integrate with respect to ξ from 0 to ξ_2 and divide it by ξ_2 to get

$$\frac{1}{\xi_2}\Big[\int_0^{\xi_2} \rho^* g^{*1/2} r^* \xi \, d\xi - \int_0^{\xi_2} \xi (g^{*1/2} q^{*\alpha})_{,\alpha} - \int_0^{\xi_2} \xi \frac{\partial}{\partial \xi} (g^{*1/2} q^{*3}) d\xi\Big]$$

$$= \frac{1}{\xi_2} \int_0^{\xi_2} \rho^* g^{*1/2} \dot{\eta}^* \xi (\phi_o + \xi \phi_1) d\xi \tag{6.26}$$

Using integration by part, the third term on the left-hand side of (6.26) can be written as

$$\frac{1}{\xi_2} \int_0^{\xi_2} \xi \frac{\partial}{\partial \xi} (g^{*1/2} q^{*3}) d\xi = \frac{1}{\xi_2} [\xi g^{*1/2} q^{*3}]_0^{\xi_2} - \frac{1}{\xi_2} \int_0^{\xi_2} g^{*1/2} q^{*3} d\xi = g^{1/2} (h - q^3) \tag{6.27}$$

which in writing the last term we have used $(6.19)_1$ and the definition of h given above. Using this result together with the relations $(6.18)_2$, $(6.19)_2$ and (6.20) we can write (6.26) in the following form

$$\rho g^{1/2} r_1 - (g^{1/2} q_1^{\alpha})_{,\alpha} - g^{1/2} (h - q^3) = \rho g^{1/2} (\phi_o \dot{\eta}_1 + \phi_1 \dot{\eta}_2) \tag{6.28}$$

To determine the appropriate form of constraints on the composite heat flux vectors, first we write the Clausius-Duhem inequality (6.16) in the following form

$$g^{*1/2} \tilde{q}^{*k} \theta^*_{,k} \leq 0$$

which by (6.21) reduces to

$$g^{*1/2} \tilde{q}^{*\alpha} (\phi_{o,\alpha} + \xi \phi_{1,\alpha}) + g^{*1/2} \tilde{q}^{*3} \phi_1 \leq 0 \tag{6.29}$$

Next we divide (6.29) by ξ_2 and integrate with respect to ξ from 0 to ξ_2 which after using (6.19) can be written as

$$g^{1/2} \tilde{q}^{\alpha} \phi_{o,\alpha} + g^{1/2} \tilde{q}_1^{\alpha} \phi_{1,\alpha} + g^{1/2} \tilde{q}^3 \phi_1 \leq 0$$

or

$$q^{\alpha} \phi_{o,\alpha} + q_1^{\alpha} \phi_{1,\alpha} + q^3 \phi_1 \leq 0 \tag{6.30}$$

which is the appropriate form of Clausius-Duhem inequality for the elastic composite laminates.

When the rate of heat supply or absorption is zero ($r = r^1 = 0$) the energy equations for the composite reduce to

$$(g^{1/2} q^{\alpha})_{,\alpha} + \frac{\partial}{\partial \theta^3}(g^{1/2} h) + \rho\, g^{1/2} (\phi_o \dot{\eta}_o + \phi_1 \dot{\eta}_1) = 0 \tag{6.31}$$

$$(g^{1/2} q_1^{\alpha})_{,\alpha} + g^{1/2} (h - q^3) + \rho\, g^{1/2} (\phi_o \dot{\eta}_1 + \phi_1 \dot{\eta}_2) = 0 \tag{6.32}$$

For small deformations of composites with initially flat plies the energy equations (6.31) and (6.32) further reduce to

$$\begin{gathered} q^{\alpha}_{,\alpha} + \frac{\partial h}{\partial \theta^3} + \rho(\phi_o \dot{\eta}_o + \phi_1 \dot{\eta}_1) = 0 \\ q^{\alpha}_{1,\alpha} + h - q^3 + \rho(\phi_o \dot{\eta}_1 + \phi_1 \dot{\eta}_2) = 0 \end{gathered} \tag{6.33}$$

where obviously no distinction should be made between contravariant and covariant components of heat flux vectors.

The derivation of energy equations (6.25)-(6.28) and the Clausius-Duhem inequality (6.30) for the composite laminates is not affected by the number of layers (or constituents) in the representative micro-structure. The only necessary modification in the case of a multi-constituent composite is the replacement of ξ_2 by ξ_n in definitions (6.18)-(6.20). Here ξ_n is the thickness of the representative element which is supposed to consist of n layers. Of course ξ_n is still supposed to be a very small number.

To recapitulate, for a composite whose micro-structure is composed of n layers with a total thickness of ξ_n we have the following relations for energy balance and the Clausius-Duhem inequality

$$\rho r - g^{-1/2}[(g^{1/2} q^{\alpha})_{,\alpha} + \frac{\partial}{\partial \theta^3}(g^{1/2} h)] = \rho(\phi_o \dot{\eta}_o + \phi_1 \dot{\eta}_1)$$

$$\rho r_1 + q^3 - h - g^{-1/2}(g^{1/2}q_1^{\alpha})_{,\alpha} = \rho(\phi_o\dot{\eta}_1 + \phi_1\dot{\eta}_2) \tag{6.34}$$

$$q^{\alpha}\phi_{o,\alpha} + q_1^{\alpha}\phi_{1,\alpha} + q^3\phi_1 \geq 0$$

where the composite quantities are defined as follows

$$\rho g^{1/2} r = \frac{1}{\xi_n}\int_0^{\xi_n} \rho^* g^{*1/2} r^* d\xi$$

$$\rho g^{1/2} r_1 = \frac{1}{\xi_n}\int_0^{\xi_n} \rho^* g^{*1/2} r^* \xi \, d\xi$$

$$\rho g^{1/2}\eta_{(m)} = \frac{1}{\xi_n}\int_0^{\xi_n} \rho^* g^{*1/2}\eta^* \xi^m d\xi \quad (m = 0,1,2) \tag{6.35}$$

$$g^{1/2} q^i = \frac{1}{\xi_n}\int_0^{\xi_n} g^{*1/2} q^{*i} d\xi$$

$$g^{1/2} q_1^{\alpha} = \frac{1}{\xi_n}\int_0^{\xi_n} g^{*1/2} q^{*\alpha}\xi \, d\xi$$

7.0 CONSTITUTIVE RELATIONS FOR LINEAR THERMO-ELASTICITY

For a composite laminate whose micro-structure is composed of n layers with different linear thermo-elastic constituents, we recall the following constitutive equations for the stress tensor τ^{*ij}, entropy η^* and the heat flux vector q^*

$$\tau^{*ij}_{(\alpha)} = c^{ijkl}_{(\alpha)}\gamma^*_{kl} - c^{ij}_{(\alpha)}\theta^* \tag{7.1}$$

$$(\rho^*\eta^*)_{(\alpha)} = c^{ij}_{(\alpha)}\gamma^*_{ij} + (\rho^* c)_{(\alpha)}\theta^* \tag{7.2}$$

$$q^{*i}_{(\alpha)} = -k^{ij}_{(\alpha)}\theta^*_{,j} \tag{7.3}$$

where $c^{ijkl}_{(\alpha)}$, $c^{ij}_{(\alpha)}$, $c_{(\alpha)}$ and $k^{ij}_{(\alpha)}$ ($\alpha = 1,2,\dots,n$) are constants in the associated layers. Moreover, we have the following symmetries

$$c^{ijkl}_{(\alpha)} = c^{jikl}_{(\alpha)} = c^{ijlk}_{(\alpha)} = c^{klij}_{(\alpha)} \tag{7.4}$$

$$c^{ij}_{(\alpha)} = c^{ji}_{(\alpha)} \tag{7.5}$$

Now we proceed to calculate the appropriate constitutive relations for *composite stress vector* T^i, *composite couple stress* S^α, *composite entropy* $\eta_{(m)}$ ($m = 0,1,2$), and *composite heat flux vectors* q^i and q_1^α. The contribution of the first part of (7.1) to the constitutive relations for T^i and S^α (and consequently τ^{ij} and $S^{\alpha j}$) has already been calculated (see section 5). Therefore we need to find out the effect of the second part of (7.1) in the constitutive relations for T^i and S^α. Similar to what was done in section 5 we adopt the following definitions for the weighted averages of various quantities

$$J^{(k)ij} = \frac{1}{\xi_n}\int_0^{\xi_n} \xi^k c^{ij}_{(\alpha)} d\xi \tag{7.6}$$

$$K^{(k)} = \frac{1}{\xi_n}\int_0^{\xi_n} \xi^k (\rho^* c)_{(\alpha)} d\xi \tag{7.7}$$

BASE

$$L^{(k)ij} = \frac{1}{\xi_n}\int_0^{\xi_n} \xi^k k^{ij}_{(\alpha)} d\xi \tag{7.8}$$

Now recalling (5.11), (5.12), (6.21) and (7.6) we can write

$$\frac{1}{\xi_n}\int_0^{\xi_n} c^{ij}_{(\alpha)}\,\theta^*\, g^{*1/2}\, g_j^* d\xi = \frac{\phi_o}{\xi_n}\int_0^{\xi_n} c^{ij}_{(\alpha)}\, g^{*1/2}\, g_j^*\, d\xi$$

$$+\frac{\phi_1}{\xi_n}\int_0^{\xi_n} \xi c^{ij}_{(\alpha)}\, g^{*1/2}\, g_j^*\, d\xi = \frac{\phi_o}{\xi_n}\int_0^{\xi_n} (c^{i\beta} g_\beta^* + c^{i3} g_3^*)\, g^{*1/2}\, d\xi$$

$$+\frac{\phi_1}{\xi_n}\int_0^{\xi_n} \xi g^{*1/2}\,(c^{i\beta} g_\beta^* + c^{i3} g_3^*) d\xi$$

$$=\phi_o g^{1/2}\, g_\beta\, \frac{1}{\xi_n}\int_0^{\xi_n} (1+\frac{\xi\Delta}{2g})c^{i\beta} d\xi + \phi_o\, g^{1/2}\,\lambda^j_\beta g_j\, \frac{1}{\xi_n}\int_0^{\xi_n} \xi(1+\frac{\xi\Delta}{2g})c^{i\beta} d\xi$$

$$+\phi_o\, g^{1/2}\, g_3\, \frac{1}{\xi_n}\int_0^{\xi_n} (1+\frac{\xi\Delta}{2g})c^{i3} d\xi + \frac{\phi_1 g^{1/2}}{\xi_n}\, g_\beta \int_0^{\xi_n} \xi(1+\frac{\xi\Delta}{2g})c^{i\beta} d\xi$$

$$+\phi_1\, g^{1/2}\,\lambda^j_\beta\, g_j\, \frac{1}{\xi_n}\int_0^{\xi_n} \xi^2(1+\frac{\xi\Delta}{2g})c^{i\beta} d\xi + \frac{\phi_1 g^{1/2}\, g_3}{\xi_n}\int_0^{\xi_n} \xi(1+\frac{\xi\Delta}{2g})c^{i3} d\xi$$

$$=\phi_o\, g^{1/2}\, g_j\{\frac{1}{\xi_n}\int_0^{\xi_n} (1+\frac{\xi\Delta}{2g})c^{ij} d\xi + \lambda^j_\beta\, \frac{1}{\xi_n}\int_0^{\xi_n} \xi(1+\frac{\xi\Delta}{2g})c^{i\beta} d\xi\}$$

$$+\phi_1\, g^{1/2}\, g_j\{\, \frac{1}{\xi_n}\int_0^{\xi_n} \xi(1+\frac{\xi\Delta}{2g})c^{ij} d\xi + \lambda^j_\beta\, \frac{1}{\xi_n}\int_0^{\xi_n} \xi^2(1+\frac{\xi\Delta}{2g})c^{i\beta} d\xi\}$$

$$=\phi_o\, g^{1/2}\, g_j\{J^{(0)ij} + \frac{\Delta}{2g}\, J^{(1)ij} + \lambda^j_\beta(J^{(1)i\beta} + \frac{\Delta}{2g}\, J^{(2)i\beta})\}$$

$$+\phi_1\, g^{1/2}\, g_j\{J^{(1)ij} + \frac{\Delta}{2g}\, J^{(2)ij} + \lambda^j_\beta(J^{(2)i\beta} + \frac{\Delta}{2g}\, J^{(3)i\beta})\} \tag{7.9}$$

By combining the results (7.9) and (5.17) we can obtain the response function for the *composite stress vector* $\mathbf{T}^i$. As before if we disregard the factor $g^{1/2}\, \mathbf{g}_j$, what remains is the constitutive relation for τ^{ij} which is recorded below

$$\tau^{ij} = \{I^{(0)ijkl} + \frac{\Delta}{2g} I^{(1)ijkl} + \lambda^j_\beta(I^{(1)i\beta kl} + \frac{\Delta}{2g} I^{(2)i\beta kl})\}\gamma_{kl}$$

$$+ \{I^{(1)ijl\alpha} + \frac{\Delta}{2g} I^{(2)ijl\alpha} + \lambda^j_\beta(I^{(2)i\beta l\alpha} + \frac{\Delta}{2g} I^{(3)i\beta l\alpha})\}\kappa_{l\alpha}$$

$$- \phi_o\{J^{(0)ij} + \frac{\Delta}{2g} J^{(1)ij} + \lambda^j_\beta(J^{(1)i\beta} + \frac{\Delta}{2g} J^{(2)i\beta})\}$$

$$- \phi_1\{J^{(1)ij} + \frac{\Delta}{2g} J^{(2)ij} + \lambda^j_\beta(J^{(2)i\beta} + \frac{\Delta}{2g} J^{(3)i\beta})\} \tag{7.10}$$

Similar steps are followed to find the constitutive equations for S^α and $S^{\alpha j}$. The contribution of the thermal term is

$$\frac{1}{\xi_n}\int_0^{\xi_n} \xi\, c^{\alpha j}_{(\alpha)}\, \theta^*\, g^{*1/2}\, \mathbf{g}^*_j d\xi = \phi_o\, g^{1/2}\, \mathbf{g}_j\, \{J^{(1)\alpha j} + \frac{\Delta}{2g} J^{(2)\alpha j} + \lambda^j_\beta(J^{(2)\alpha\beta} + \frac{\Delta}{2g} J^{(3)\alpha\beta})\}$$

$$+ \phi_1\, g^{1/2}\, \mathbf{g}_j\, \{J^{(2)\alpha j} + \frac{\Delta}{2g} J^{(3)\alpha j} + \lambda^j_\beta(J^{(3)\alpha\beta} + \frac{\Delta}{2g} J^{(4)\alpha\beta})\} \tag{7.11}$$

Again combining (7.11) and (5.20) we find the constitutive relation for S^α. Dropping the common factor $g^{1/2}\, \mathbf{g}_j$ would result in the constitutive relation for $S^{\alpha j}$ which is recorded below.

$$S^{\alpha j} = \gamma_{kl}\{I^{(1)\alpha jkl} + \frac{\Delta}{2g} I^{(2)\alpha jkl} + \lambda^j_\gamma(I^{(2)\alpha\gamma kl} + \frac{\Delta}{2g} I^{(3)\alpha\gamma kl})\}$$

$$+ \kappa_{l\beta}\{I^{(2)\alpha jl\beta} + \frac{\Delta}{2g} I^{(3)\alpha jl\beta} + \lambda^j_\gamma(I^{(3)\alpha\gamma l\beta} + \frac{\Delta}{2g} I^{(4)\alpha\gamma l\beta})\}$$

$$- \phi_o\{J^{(1)\alpha j} + \frac{\Delta}{2g} J^{(2)\alpha j} + \lambda^j_\beta(J^{(1)\alpha\beta} + \frac{\Delta}{2g} J^{(3)\alpha\beta})\}$$

$$- \phi_1 \{ J^{(2)\alpha j} + \frac{\Delta}{2g} J^{(3)\alpha j} + \lambda^j_\beta (J^{(3)\alpha\beta} + \frac{\Delta}{2g} J^{(4)\alpha\beta}) \} \tag{7.12}$$

The next step is to find the constitutive relation for the specific entropies $\eta_{(m)}$ (m = 0,1,2). We calculate the contribution of each term of the relation (7.2) separately. By $(6.35)_3$ we have

$$\rho g^{1/2} \eta_{(m)} = \frac{1}{\xi_n} \int_0^{\xi_n} \rho^* \eta^* g^{*1/2} \xi^m d\xi$$

$$= \frac{1}{\xi_n} \int_0^{\xi_n} c^{ij}_{(\alpha)} \gamma^*_{ij} g^{*1/2} \xi^m d\xi + \frac{1}{\xi_n} \int_{\xi_1}^{\xi_2} (\rho^* c)_{(\alpha)} \theta^* g^{*1/2} \xi^m d\xi \tag{7.13}$$

Since $g^{*1/2} = g^{1/2} (1 + \frac{\xi\Delta}{2g})$ we can write the first part of (7.13) as

$$\frac{1}{\xi_n} \int_0^{\xi_n} c^{ij}_{(\alpha)} \gamma^*_{ij} g^{*1/2} \xi^m d\xi = \frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^m (1 + \frac{\xi\Delta}{2g}) c^{ij}_{(\alpha)} \gamma^*_{ij} d\xi \tag{7.14}$$

which by (2.31)-(2.34) reduces to

$$= [\frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^m (1 + \frac{\xi\Delta}{2g}) c^{ij}_{(\alpha)} d\xi] \gamma_{ij} + [\frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^{m+1} (1 + \frac{\xi\Delta}{2g}) c^{i\beta}_{(\alpha)} d\xi] \kappa_{i\beta}$$

$$= g^{1/2} \{ \gamma_{ij} [J^{(m)ij} + \frac{\Delta}{2g} J^{(m+1)ij}] + \kappa_{i\beta} [J^{(m+1)i\beta} + \frac{\Delta}{2g} J^{(m+2)i\beta}] \} \tag{7.15}$$

The second part of (7.13), by (6.21) and (7.7), is written as

$$\frac{1}{\xi_n} \int_0^{\xi_n} (\rho^* c)_{(\alpha)} \theta^* g^{*1/2} \xi^m d\xi = \frac{g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^m (1 + \frac{\xi\Delta}{2g}) (\phi_o + \xi\phi_1) (\rho^* c)_{(\alpha)} d\xi$$

$$= \frac{\phi_o g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^m (1 + \frac{\xi\Delta}{2g}) (\rho^* c)_{(\alpha)} d\xi$$

$$+ \frac{\phi_1 g^{1/2}}{\xi_n} \int_0^{\xi_n} \xi^{m+1} (1 + \frac{\xi\Delta}{2g}) (\rho^* c)_{(\alpha)} d\xi$$

$$= \phi_o\, g^{1/2}\, \{K^{(m)}$$

$$+ \frac{\Delta}{2g} K^{(m+1)}\} + \phi_1\, g^{1/2}\, \{K^{(m+1)} + \frac{\Delta}{2g} K^{(m+2)}\} \qquad (7.16)$$

Substituting (7.15) and (7.16) in (7.13) we obtain

$$\rho g^{1/2}\eta_{(m)} = g^{1/2}\, \{\gamma_{ij}(J^{(m)ij} + \frac{\Delta}{2g} J^{(m+1)ij})\} + g^{1/2}\, \kappa_{i\beta}(J^{(m+1)i\beta} + \frac{\Delta}{2g} J^{(m+2)i\beta})$$

$$+ \phi_o\, g^{1/2}\, (K^{(m)} + \frac{\Delta}{2g} K^{(m+1)}) + \phi_1\, g^{1/2}\, (K^{(m+1)} + \frac{\Delta}{2g} K^{(m+2)})$$

or

$$\rho\eta_{(m)} = \gamma_{ij}(J^{(m)ij} + \frac{\Delta}{2g} J^{(m+1)ij}) + \kappa_{i\beta}(J^{(m+1)i\beta} + \frac{\Delta}{2g} J^{(m+2)i\beta})$$

$$+ \phi_o(K^{(m)} + \frac{\Delta}{2g} K^{(m+1)}) + \phi_1(K^{(m+1)} + \frac{\Delta}{2g} K^{(m+2)}) \quad (m = 0,1,2) \qquad (7.17)$$

We start with $(6.35)_4$ to find the constitutive relation for q^i. Substituting from (7.3) in $(6.35)_4$ we get

$$g^{1/2}q^i = \frac{1}{\xi_n}\int_0^{\xi_n} g^{*1/2} q^{*i} d\xi = -\frac{1}{\xi_n}\int_0^{\xi_2} g^{*1/2}\, k^{ij}_{(\alpha)}\theta^*_{,j} d\xi$$

$$= -\frac{g^{1/2}}{\xi_n}\int_0^{\xi_n} (1 + \frac{\xi\Delta}{2g})(k^{i\beta}\theta^*_{,\beta} + k^{i3}\theta^*_{,3}) d\xi \qquad (7.18)$$

By (6.21)

$$\theta^*_{,\beta} = \phi_{o,\beta} + \xi\phi_{1,\beta} \qquad (7.19)$$

$$\theta^*_{,3} = \phi_1 \qquad (7.20)$$

Noting these results we simplify (7.18). Hence

$$g^{1/2}q^i = -\frac{g^{1/2}}{\xi_n}\int_0^{\xi_n}(1+\frac{\xi\Delta}{2g})k^{i\beta}(\phi_{0,\beta}+\xi\phi_{1,\beta})d\xi$$

$$-\frac{g^{1/2}}{\xi_n}\int_0^{\xi_n}(1+\frac{\xi\Delta}{2g})k^{i3}\phi_1 d\xi$$

$$= -g^{1/2}\{\frac{\phi_{o,\beta}}{\xi_n}\int_0^{\xi_n}(1+\frac{\xi\Delta}{2g})k^{i\beta}d\xi + \frac{\phi_{1,\beta}}{\xi_n}\int_0^{\xi_n}\xi(1+\frac{\xi\Delta}{2g})k^{i\beta}d\xi$$

$$+\frac{\phi_1}{\xi_n}\int_0^{\xi_n}(1+\frac{\xi\Delta}{2g})k^{i3}d\xi\} \tag{7.21}$$

Using (7.8) we get

$$g^{1/2}q^i = -g^{1/2}\{\phi_{o,\beta}[L^{(0)i\beta}+\frac{\Delta}{2g}L^{(1)i\beta}]+\phi_{1,\beta}[L^{(1)i\beta}+\frac{\Delta}{2g}L^{(2)i\beta}]$$

$$+\phi_1[L^{(0)i3}+\frac{\Delta}{2g}L^{(1)i3}]\}$$

or

$$q^i = -(L^{(0)i\beta}+\frac{\Delta}{2g}L^{(1)i\beta})\phi_{o,\beta}-(L^{(1)i\beta}+\frac{\Delta}{2g}L^{(2)i\beta})\phi_{1,\beta}$$

$$-(L^{(0)i3}+\frac{\Delta}{2g}L^{(1)i3})\phi_1 \tag{7.22}$$

Finally we use $(6.35)_5$ to derive the constitutive equation for q_1^α. Similar to the above development, we write

$$g^{1/2}q_1^\alpha = \frac{1}{\xi_n}\int_0^{\xi_n}g^{*1/2}q^{*\alpha}\xi\,d\xi = -\frac{1}{\xi_n}\int_0^{\xi_n}\xi\,g^{*1/2}\,k^{\alpha j}\theta^*_{,j}d\xi$$

$$= -\frac{g^{1/2}}{\xi_n}\int_0^{\xi_n}\xi(1+\frac{\xi\Delta}{2g})(k^{\alpha\beta}\theta^*_{,\beta}+k^{\alpha 3}\theta^*_{,3})d\xi$$

$$= -\frac{g^{1/2}}{\xi_n}\int_0^{\xi_n}\xi(1+\frac{\xi\Delta}{2g})k^{\alpha\beta}(\phi_{o,\beta}+\xi\phi_{1,\beta})d\xi - \frac{g^{1/2}}{\xi_n}\int_0^{\xi_n}\xi(1+\frac{\xi\Delta}{2g})k^{\alpha 3}\phi_1 d\xi \tag{7.23}$$

which by (7.8) reduces to

$$q_1^{\alpha} = -(L^{(1)\alpha\beta}+\frac{\Delta}{2g}L^{(2)\alpha\beta})\phi_{o,\beta} - (L^{(2)\alpha\beta}+\frac{\Delta}{2g}L^{(3)\alpha\beta})\phi_{1,\beta} - (L^{(1)\alpha 3}+\frac{\Delta}{2g}L^{(2)\alpha 3})\phi_1 \tag{7.24}$$

This concludes our derivation of linear thermo-elastic constitutive relations for composite laminates.

For small deformations of a composite with initially flat plies the foregoing equations are simplified to the following constitutive relations:

$$\tau_{ij} = I_{ijkl}^{(0)}u_{k,l} + I_{ijl\alpha}^{(1)}u_{l,\alpha 3} - J_{ij}^{(0)}\phi_o - J_{ij}^{(1)}\phi_1 \tag{7.25}$$

$$S_{\alpha j} = I_{\alpha jkl}^{(1)}u_{k,l} + I_{\alpha jl\beta}^{(2)}u_{l,\beta 3} - J_{\alpha j}^{(1)}\phi_o - J_{\alpha j}^{(2)}\phi_1 \tag{7.26}$$

$$\rho\eta_{(m)} = J_{ij}^{(m)}u_{i,j} + J_{i\beta}^{(m+1)}u_{i,\beta 3} + K^{(m)}\phi_o + K^{(m+1)}\phi_1 \quad (m=0,1,2) \tag{7.27}$$

$$q_i = -L_{i\beta}^{(0)}\phi_{o,\beta} - L_{i\beta}^{(1)}\phi_{1,\beta} - L_{i3}^{(0)}\phi_1 \tag{7.28}$$

$$q_1^{\alpha} = -L^{(1)\alpha\beta}\phi_{o,\beta} - L^{(2)\alpha\beta}\phi_{1,\beta} - L^{(1)\alpha 3}\phi_1 \tag{7.29}$$

The constitutive coefficients $I^{(0)}$, $I^{(1)}$ and $I^{(2)}$ have already been calculated and recorded in equations (5.26). As for the other constitutive coefficients we use the results of section 5. Comparing definitions (7.6)-(7.8) with (5.8) and using the results (5.4) and (5.30) we can write

$$J^{(k)ij} = \frac{1}{k+1}\xi_n^k\sum_{r=1}^{n}c_{(r)}^{ij}\,\Delta m_r^{k+1} \tag{7.30}$$

$$K^{(k)} = \frac{1}{k+1}\xi_n^k\sum_{r=1}^{n}(\rho^* c)_{(r)}\Delta m_r^{k+1} \tag{7.31}$$

$$L^{(k)ij} = \frac{1}{k+1}\xi_n^k\sum_{r=1}^{n}k_{(r)}^{ij}\,\Delta m_r^{k+1} \tag{7.32}$$

It should be noted that $k^{ij}_{(r)}$ in equation (7.32) are the coefficients of thermal conductivity of different layers of the representative micro-structure and are not to be confused with the superscript k which assumes non-negative integer values.

If the micro-structure is composed of isotropic layers, the coefficients of thermal stress $c^{ij}_{(r)}$ and thermal conductivity $k^{ij}_{(r)}$ can be written in terms of only one constant for each layer. For such case we write

$$c_{ij}^{(r)} = \beta_{(r)}\delta_{ij} \tag{7.33}$$

$$k_{ij}^{(r)} = k_{(r)}\delta_{ij} \tag{7.34}$$

Taking note of these relations and relations (7.30) and (7.32) we obtain

$$\begin{aligned}
J_{ij}^{(0)} &= \delta_{ij} \sum_{r=1}^{n} \beta_{(r)}\Delta m_r \\
J_{ij}^{(1)} &= \frac{1}{2}\, \xi_n \delta_{ij} \sum_{r=1}^{n} \beta_{(r)}\Delta m_r^2 \\
J_{ij}^{(2)} &= \frac{1}{3}\, \xi_n^2 \delta_{ij} \sum_{r=1}^{n} \beta_{(r)}\Delta m_r^3 \\
J_{ij}^{(3)} &= \frac{1}{4}\, \xi_n^3 \delta_{ij} \sum_{r=1}^{n} \beta_{(r)}\Delta m_r^4
\end{aligned} \tag{7.35}$$

$$L_{ij}^{(0)} = \delta_{ij} \sum_{r=1}^{n} k_{(r)}\Delta m_r$$

$$\begin{aligned}
L_{ij}^{(1)} &= \frac{1}{2}\, \xi_n \delta_{ij} \sum_{r=1}^{n} k_{(r)}\Delta m_r^2 \\
L_{ij}^{(2)} &= \frac{1}{3}\, \xi_n^2 \delta_{ij} \sum_{r=1}^{n} k_{(r)}\Delta m_r^3
\end{aligned} \tag{7.36}$$

Consequently the constitutive equations (7.25)-(7.29) reduce

$$\tau_{ij} = \delta_{ij} \sum_{r=1}^{n} (\lambda_{(r)} u_{k,k} - \beta_{(r)} \phi_o)\Delta m_r + (u_{i,j} + u_{j,i}) \sum_{r=1}^{n} \mu_{(r)} \Delta m_r$$

$$+ \frac{1}{2} \xi_n \delta_{ij} \sum_{r=1}^{n} (\lambda_{(r)} u_{\alpha,\alpha 3} - \beta_{(r)} \phi_1)\Delta m_r^2$$

$$+ \frac{\xi_n}{2} (u_{i,\alpha 3} \delta_{j\alpha} + u_{j,\alpha 3} \delta_{i\alpha}) \sum_{r=1}^{n} \mu_{(r)} \Delta m_r^2 \tag{7.37}$$

$$S_{\alpha j} = \frac{1}{2} \xi_n \{ \delta_{\alpha j} \sum_{r=1}^{n} (\lambda_{(r)} u_{k,k} - \beta_{(r)} \phi_o)\Delta m_r^2 + (u_{\alpha,j} + u_{j,\alpha}) \sum_{r=1}^{n} \mu_{(r)} \Delta m_r^2 \}$$

$$+ \frac{1}{3} \xi_n^2 \{ \delta_{\alpha j} \sum_{r=1}^{n} (\lambda_{(r)} u_{\beta,\beta 3} - \beta_{(r)} \phi_1)\Delta m_r^3$$

$$+ (\delta_{j\beta} u_{\alpha,\beta 3} + u_{j,\alpha 3} \sum_{r=1}^{n} \mu_{(r)} \Delta m_r^3 \} \tag{7.38}$$

$$\rho\eta_{(0)} = \sum_{r=1}^{n} [\beta_{(r)} u_{i,i} + (\rho^* c)_{(r)} \phi_o]\Delta m_r$$

$$+ \frac{1}{2} \xi_n \sum_{r=1}^{n} [\beta_{(r)} u_{\beta,\beta 3} + (\rho^* c)_{(r)} \phi_1]\Delta m_r^2 \tag{7.39}$$

$$\rho\eta_{(1)} = \frac{1}{2} \xi_n \sum_{r=1}^{n} [\beta_{(r)} u_{i,i} + (\rho^* c)_{(r)} \phi_o]\Delta m_r^2$$

$$+ \frac{1}{3} \xi_n^2 \sum_{r=1}^{n} [\beta_{(r)} u_{\beta,\beta 3} + (\rho^* c)_{(r)} \phi_1]\Delta m_r^3 \tag{7.40}$$

$$\rho\eta_{(2)} = \frac{1}{3} \xi_n^2 \sum_{r=1}^{n} [\beta_{(r)} u_{i,i} + (\rho^* c)_{(r)} \phi_o]\Delta m_r^3$$

$$+ \frac{1}{4} \xi_n^3 \sum_{r=1}^{n} [\beta_{(r)} u_{\beta,\beta 3} + (\rho^* c)_{(r)} \phi_1]\Delta m_r^4 \tag{7.41}$$

$$q_\alpha = -\phi_{o,\alpha} \sum_{r=1}^{n} k_{(r)}\Delta m_r - \frac{1}{2}\xi_n\phi_{1,\alpha} \sum_{r=1}^{n} k_{(r)}\Delta m_r^2 \tag{7.42}$$

$$q_3 = -\phi_1 \sum_{r=1}^{n} k_{(r)}\Delta m_r \tag{7.43}$$

$$q_1^\alpha = -\frac{1}{2}\xi_n\phi_{o,\alpha} \sum_{r=1}^{n} k_{(r)}\Delta m_r^2 - \frac{1}{3}\xi_n^2\phi_{1,\alpha} \sum_{r=1}^{n} k_{(r)}\Delta m_r^3 \tag{7.44}$$

In relation (7.44), α is written as a superscript only for convenience and does not signify the contravariant position.

The linear equations of motion and balance of energy for a composite with initially flat plies are derived by substituting (7.25)-(7.29) in (2.123), (2.124) and (6.33). Using the results (5.41) and (5.42) in conjunction with (7.25) and (7.26) we have the following equations of motion in linear thermo-elastic theory

$$I^{(0)}_{\alpha jkl}u_{k,l\alpha} + I^{(1)}_{\alpha jl\beta}u_{l,\alpha\beta 3} - J^{(0)}_{\alpha j}\phi_{0,\alpha} - J^{(1)}_{\alpha j}\phi_{1,\alpha} + b_j \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r + \sigma_{j,3}$$

$$= \ddot{u}_j \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r + \frac{1}{2}\xi_n\ddot{u}_{j,3} \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r^2 \tag{7.45}$$

$$I^{(1)}_{\alpha jkl}u_{k,l\alpha} + I^{(2)}_{\alpha jl\beta}u_{l,\alpha\beta 3} - J^{(1)}_{\alpha j}\phi_{0,\alpha} - J^{(2)}_{\alpha j}\phi_{1,\alpha} + \sigma_j - I^{(0)}_{3jkl}u_{k,l}$$

$$- I^{(1)}_{3jl\beta}u_{l,\beta 3} + J^{(0)}_{3j}\phi_o + J^{(1)}_{3j}\phi_1 + c_j \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r$$

$$= \frac{1}{2}\xi_n\ddot{u}_j \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r^2 + \frac{1}{3}\xi_n^2\ddot{u}_{j,3} \sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r^3 \tag{7.46}$$

The energy equations when the rate of heat supply or absorption is zero are recorded in relations (6.33) for small deformations of thermo-elastic composites with initially flat plies. Substituting the constitutive relations (7.27)-(7.29) in (6.33) we find the following coupled differential equations for displacement and temperature fields

$$(\phi_o J_{ij}^{(0)} + \phi_1 J_{ij}^{(1)})\dot{u}_{i,j} + (\phi_o J_{i\beta}^{(1)} + \phi_1 J_{i\beta}^{(2)})\dot{u}_{i,\beta 3} + (\phi_o K^{(0)} + \phi_1 K^{(1)})\dot{\phi}_o$$

$$+ (\phi_o K^{(1)} + \phi_1 K^{(2)})\dot{\phi}_1 + \frac{\partial h}{\partial \theta^3} - L_{\alpha\beta}^{(0)}\phi_{0,\alpha\beta} - L_{\alpha\beta}^{(1)}\phi_{1,\alpha\beta} - L_{\alpha 3}^{(0)}\phi_{1,\alpha} = 0 \tag{7.47}$$

$$(\phi_o J_{ij}^{(1)} + \phi_1 J_{ij}^{(2)})\dot{u}_{i,j} + (\phi_o J_{i\beta}^{(2)} + \phi_1 J_{i\beta}^{(3)})\dot{u}_{i,\beta 3} + (\phi_o K^{(1)} + \phi_1 K^{(2)})\dot{\phi}_o$$

$$+ (\phi_o K^{(2)} + \phi_1 K^{(3)})\dot{\phi}_1 + h + L_{3\beta}^{(0)}\phi_{o,\beta} + L_{3\beta}^{(1)}\phi_{1,\beta} + L_{33}^{(0)}\phi_1$$

$$- L_{\alpha\beta}^{(1)}\phi_{o,\alpha\beta} - L_{\alpha\beta}^{(2)}\phi_{1,\alpha\beta} - L_{\alpha 3}^{(1)}\phi_{1,\alpha} = 0 \tag{7.48}$$

For static problems in the absence of body force and heat supply, the foregoing equations are further reduced to

$$I_{\alpha jkl}^{(0)}u_{k,l\alpha} + I_{\alpha jl\beta}^{(1)}u_{l,\beta 3\alpha} - J_{\alpha j}^{(0)}\phi_{o,\alpha} - J_{\alpha j}^{(1)}\phi_{1,\alpha} + \sigma_{j,3} = 0 \tag{7.49}$$

$$I_{\alpha jkl}^{(1)}u_{k,l\alpha} + I_{\alpha jl\beta}^{(2)}u_{l,\alpha\beta 3} - J_{\alpha j}^{(1)}\phi_{o,\alpha} - J_{\alpha j}^{(2)}\phi_{1,\alpha} + \sigma_j - I_{3jkl}^{(0)}u_{k,l}$$

$$- I_{3jl\beta}^{(1)}u_{l,\beta 3} + J_{3j}^{(0)}\phi_o + J_{3j}^{(1)}\phi_1 = 0 \tag{7.50}$$

$$\frac{\partial h}{\partial \theta^3} - L_{\alpha\beta}^{(0)}\phi_{0,\alpha\beta} - L_{\alpha\beta}^{(1)}\phi_{1,\alpha\beta} - L_{\alpha 3}^{(0)}\phi_{1,\alpha} = 0 \tag{7.51}$$

$$h + L_{3\beta}^{(0)}\phi_{0,\beta} + L_{3\beta}^{(1)}\phi_{1,\beta} + L_{33}^{(0)}\phi_1 - L_{\alpha\beta}^{(1)}\phi_{0,\alpha\beta} - L_{\alpha\beta}^{(2)}\phi_{1,\alpha\beta} - L_{\alpha 3}^{(1)}\phi_{1,\alpha} = 0 \tag{7.52}$$

Similar to what was done previously in order to find a relation between the director displacement and the gradient of displacement vector, we enforce the continuity of the temperature field across two adjacent micro-structures to derive an analogous relation between ϕ_o and ϕ_1 defined in equation (6.21). In order that the temperature field be continuous on the common surface between k^{th} and $(k+1)^{st}$ micro-structures we should have

$$\theta^*(\theta^\alpha, \theta^{3(k+1)}, 0, t) = \theta^*(\theta^\alpha, \theta^{3(k)}, \xi_n, t) \tag{7.53}$$

Now by (6.21) we have

$$\theta^*(\theta^\alpha,\theta^{3(k+1)},0,t) = \phi_o(\theta^\alpha,\theta^{3(k+1)},t) \tag{7.54}$$

$$\theta^*(\theta^\alpha,\theta^{3(k)},\xi_n,t) = \phi_o(\theta^\alpha,\theta^{3(k)},t) + \xi_n\phi_1(\theta^\alpha,\theta^{3(k)},t) \tag{7.55}$$

Substituting from (7.54) and (7.55) in (7.53) we get

$$\phi_o(\theta^\alpha,\theta^{3(k+1)},t) = \phi_o(\theta^\alpha,\theta^{3(k)},t) + \xi_n\phi_1(\theta^\alpha,\theta^{3(k)},t)$$

or

$$\phi_1(\theta^\alpha,\theta^{3(k)},t) = \frac{1}{\xi_n}\{\phi_o(\theta^\alpha,\theta^{3(k+1)},t) - \phi_o(\theta^\alpha,\theta^{3(k)},t)\} \tag{7.56}$$

By smoothing assumptions and noting the smallness of ξ_n we approximate the right-hand side of (7.56) as the gradient of ϕ_o in the θ^3 direction. So we obtain

$$\phi_1(\theta^\alpha,\theta^3,t) = \frac{\partial}{\partial\theta^3}\,\phi_o(\theta^\alpha,\theta^3,t) \tag{7.57}$$

This conclusion is used in various field equations. In particular, equation (7.49)-(7.52) reduce to

$$I^{(0)}_{\alpha jkl}u_{k,l\alpha} + I^{(1)}_{\alpha jl\beta}u_{l,\alpha\beta 3} - J^{(0)}_{\alpha j}\phi_{0,\alpha} - J^{(1)}_{\alpha j}\phi_{0,\alpha 3} + \sigma_{j,3} = 0 \tag{7.58}$$

$$\begin{aligned} I^{(1)}_{\alpha jkl}u_{k,l\alpha} + I^{(2)}_{\alpha jl\beta}u_{l,\alpha\beta 3} - J^{(1)}_{\alpha j}\phi_{0,\alpha} - J^{(2)}_{\alpha j}\phi_{0,\alpha 3} + \sigma_j - I^{(0)}_{3jkl}u_{k,l} \\ - I^{(1)}_{3jl\beta}u_{l,\beta 3} + J^{(0)}_{3j}\phi_o + J^{(1)}_{3j}\phi_{0,3} = 0 \end{aligned} \tag{7.59}$$

$$h_{,3} - L^{(0)}_{\alpha,\beta}\phi_{0,\alpha\beta} - L^{(1)}_{\alpha\beta}\phi_{0,\alpha\beta 3} - L^{(0)}_{\alpha 3}\phi_{0,\alpha 3} = 0 \tag{7.60}$$

$$h + L^{(0)}_{3\beta}\phi_{0,\beta} + L^{(1)}_{3\beta}\phi_{0,\beta 3} + L^{(0)}_{33}\phi_{0,3} - L^{(1)}_{\alpha\beta}\phi_{0,\alpha\beta} - L^{(2)}_{\alpha\beta}\phi_{0,\alpha\beta 3} - L^{(1)}_{\alpha 3}\phi_{0,\alpha 3} = 0 \tag{7.61}$$

Eliminating σ_j between (7.58) and (7.59), and h between (7.60) and (7.61) we find the following coupled differential equations for displacement and temperature fields. Since we are investigating the static problems in the present derivation the equation for temperature, i.e., the equation resulting from the energy equations is independent of the displacement field. Recalling (5.45), the displacement equation becomes

$$I^{(0)}_{ijkl}u_{k,li} + I^{(1)}_{ijk\beta}u_{k,i\beta 3} - I^{(1)}_{\alpha jkl}u_{k,l\alpha 3} - I^{(2)}_{\alpha jk\beta}u_{k,\alpha\beta 33}$$

$$- J^{(0)}_{\alpha i}\phi_{0,i} - J^{(1)}_{ij}\phi_{0,i3} + J^{(1)}_{\alpha j}\phi_{0,\alpha 3} + J^{(2)}_{\alpha j}\phi_{0,\alpha 33} = 0 \qquad (7.62)$$

and the temperature equation by (7.60) and (7.61) is

$$L^{(0)}_{ij}\phi_{0,ij} + (L^{(1)}_{3\alpha} - L^{(1)}_{\alpha 3})\phi_{0,\alpha 33} - L^{(2)}_{\alpha\beta}\phi_{0,\alpha\beta 33} = 0 \qquad (7.63)$$

Having determined the displacement and the temperature fields, the interlaminar stresses σ_j and heat flux h can be determined from (7.59) and (7.61), respectively. The results are

$$\sigma_j = I^{(0)}_{3jkl}u_{k,k} + I^{(1)}_{3jk\beta}u_{k,\beta 3} - I^{(1)}_{\alpha jkl}u_{k,l\alpha} - I^{(2)}_{\alpha jl\beta}u_{l,\alpha\beta 3}$$

$$+ J^{(1)}_{\alpha j}\phi_{0,\alpha} + J^{(2)}_{\alpha j}\phi_{0,\alpha 3} - J^{(0)}_{3j}\phi_0 - J^{(1)}_{3j}\phi_{0,3} \qquad (7.64)$$

$$h = L^{(1)}_{\alpha\beta}\phi_{0,\alpha\beta} + L^{(2)}_{\alpha\beta}\phi_{0,\alpha\beta 3} + L^{(1)}_{\alpha 3}\phi_{0,\alpha 3} - L^{(0)}_{3\alpha}\phi_{0,\alpha} - L^{(1)}_{3\alpha}\phi_{0,\alpha 3} - L^{(2)}_{33}\phi_{0,3}$$

$$= L^{(1)}_{\alpha\beta}\phi_{0,\alpha\beta} + L^{(2)}_{\alpha\beta}\phi_{0,\alpha\beta 3} + (L^{(1)}_{\alpha 3} - L^{(1)}_{3\alpha})\phi_{0,\alpha 3} - L^{(0)}_{3j}\phi_{0,j} \qquad (7.65)$$

8.0 LINEAR THEORY OF INITIALLY CYLINDRICAL LAMINATES

For an initially cylindrical laminate, we choose the usual cylindrical coordinates (r,θ,z) which are related to the coordinates $(\theta^1,\theta^2,\theta^3)$ according to the following relations

$$\theta^1=\theta \quad , \quad \theta^2=z \quad , \quad \theta^3=r \tag{8.1}$$

The choice of coordinate r for θ^3 direction is natural, since the cylindrical laminates are piled up in r direction. The metric tensor G_{ij} and its conjugate G^{ij} and $G = \det(G_{ij})$ are given by the following relations

$$(G_{ij})=\begin{bmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (G^{ij})=\begin{bmatrix} 1/r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.2}$$

$$G=\det(G_{ij})=r^2 \tag{8.3}$$

Since we are working within the realm of linear theory, the reference metric tensor G_{ij} is appropriate for calculating the covariant derivatives of various quantities. We further recall if v_i and T_{ij} are covariant components of arbitrary vectors and tensors, their physical components $V_{<i>}$ and $T_{<ij>}$ are given by

$$V_{<i>}=\frac{V_i}{\sqrt{G_{ii}}} \quad , \quad T_{<ij>}=\frac{T_{ij}}{\sqrt{G_{ii}}\sqrt{G_{jj}}} \text{ (no sum)} \tag{8.4}$$

and in terms of covariant components

$$V_{<i>}=V^i\sqrt{G_{ii}} \quad , \quad T_{<ij>}=T^{ij}\sqrt{G_{ii}}\sqrt{G_{jj}} \text{ (no sum)} \tag{8.5}$$

For the cylindrical coordinates defined in (8.1), the non-vanishing Christoffel symbols of first and second kind are as follows:

$$[13,1]=[31,1]=r \quad , \quad [11,3]=-r \tag{8.6}$$

$$\{_1{}^1{}_3\} = \{_3{}^1{}_1\} = \frac{1}{r} \quad , \quad \{_1{}^3{}_1\} = -r \tag{8.7}$$

For the subsequent analysis we also need an expression for the quantity Δ defined in (2.14):

$$\Delta = \begin{bmatrix} 2r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{bmatrix} + \begin{bmatrix} r^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2r \tag{8.8}$$

This quantity relates the determinants of the metric tensors of micro- and macro-structures through relation (2.13).

8.1 Relative Kinematic Measures

The relative kinematic measures γ_{ij} and $\kappa_{j\alpha}$ were given by (2.118) which for cylindrical coordinates since $D_\alpha = G_{\alpha 3} = 0$, by (2.103) and (8.2), are simplified to

$$\gamma_{ij} = \frac{1}{2}\,(u_{i|j} + u_{j|i}) \tag{8.9}$$

$$\kappa_{\alpha\beta} = \Lambda^j_\beta u_{j|\alpha} + \delta_{\alpha|\beta} \quad , \quad \kappa_{3\alpha} = \delta_{3|\alpha} \tag{8.10}$$

In writing the above relations, we have also used the resuts (4.22) and (2.30) while noting in cylindrical coordinates we can write

$$\Lambda^3_\alpha = \{_3{}^3{}_\alpha\} \equiv 0$$

By straightforward calculations, covariant derivatives of displacement vector **u** and director displacement **δ** are found and substituted in (8.9) and (8.10) in order to obtain the covariant components of the relative kinematic measures. These results together with physical components of each tensor are recorded below:

$$\gamma_{11} = \frac{\partial u_1}{\partial \theta} + r u_3 \quad , \quad \gamma_{\theta\theta} = \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\gamma_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial \theta}\right) \;,\; \gamma_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

$$\gamma_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial r} + \frac{\partial u_3}{\partial \theta}\right) - \frac{u_1}{r} \;,\; \gamma_{\theta r} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)$$

$$\gamma_{22} = \frac{\partial u_2}{\partial z} \;,\; \gamma_{zz} = \frac{\partial u_z}{\partial z} \tag{8.12}$$

$$\gamma_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial r} + \frac{\partial u_3}{\partial z}\right) \;,\; \gamma_{rz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

$$\gamma_{33} = \frac{\partial u_3}{\partial r} \;,\; \gamma_{rr} = \frac{\partial u_r}{\partial r}$$

$$\kappa_{11} = \frac{\partial}{\partial r}\left(\frac{\partial u_1}{\partial \theta}\right) + \frac{\partial}{\partial r}(r u_3) \;,\; \kappa_{\theta\theta} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}(r u_r)$$

$$\kappa_{12} = \frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial r} - \frac{u_1}{r}\right) \;,\; \kappa_{\theta z} = \frac{\partial^2 u_\theta}{\partial z \partial r}$$

$$\kappa_{21} = \frac{1}{r}\frac{\partial u_1}{\partial z} + \frac{\partial^2 u_z}{\partial \theta \partial r} \;,\; \kappa_{z\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial z} + \frac{\partial^2 u_z}{\partial \theta \partial r}\right)$$

$$\kappa_{22} = \frac{\partial^2 u_2}{\partial z \partial r} \;,\; \kappa_{zz} = \frac{\partial^2 u_z}{\partial z \partial r} \tag{8.13}$$

$$\kappa_{31} = \frac{\partial^2 u_3}{\partial \theta \partial r} - \frac{u_1}{r} \;,\; \kappa_{r\theta} = \frac{1}{r}\left(\frac{\partial^2 u_r}{\partial \theta \partial r} - \frac{\partial u_\theta}{\partial r}\right)$$

$$\kappa_{32} = \frac{\partial^2 u_3}{\partial z \partial r} \;,\; \kappa_{rz} = \frac{\partial^2 u_r}{\partial z \partial r}$$

The equation for balance of mass is also obtained by using (2.121) and (2.122). The result is as follows

$$\rho_o = \rho\left(1 + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right)$$

$$\rho = \rho_o(1 - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{\partial u_z}{\partial z} - \frac{\partial u_r}{\partial r} - \frac{u_r}{r}) \tag{8.14}$$

8.2 Linearized field equations

These equations have already been derived for a general composite laminate and are recorded in (2.123)-(2.125). In cylindrical coordinates and in terms of physical components of different tensors, these equations are reduced to the following forms

$$\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{\tau_{\theta r}}{r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial r} + \rho_o b_\theta = \rho_o(\ddot{u}_\theta + z^1\ddot{\delta}_\theta)$$

$$\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\frac{\partial \sigma_z}{\partial r} + \rho_o b_z = \rho_o(\ddot{u}_z + z^1\ddot{\delta}_z) \tag{8.15}$$

$$\frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_r}{\partial r} + \rho_o b_r = \rho_o(\ddot{u}_r + z^1\ddot{\delta}_r)$$

and

$$\frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{z\theta}}{\partial z} + \frac{S_{\theta r}}{r} + \frac{\sigma_\theta}{r} - \tau_{r\theta} + \rho_o c_\theta = \rho_o(z^1\ddot{u}_\theta + z^2\ddot{\delta}_\theta)$$

$$\frac{1}{r}\frac{\partial S_{\theta z}}{\partial z} + \frac{\partial s_{zz}}{\partial z} + \frac{\sigma_z}{r} - \tau_{rz} + \rho_o c_z = \rho_o(z^1\ddot{u}_z + z^2\ddot{\delta}_z) \tag{8.16}$$

$$\frac{1}{r}\frac{\partial s_{\theta r}}{\partial \theta} + \frac{\partial s_{zr}}{\partial z} - \frac{s_{\theta\theta}}{r} + \frac{\sigma_r}{r} - \tau_{rr} + \rho_o c_r = \rho_o(z^1\ddot{u}_r + z^2\ddot{\delta}_r)$$

and

$$\tau_{z\theta} - \tau_{\theta z} = \frac{1}{r}s_{\theta z}$$

$$\tau_{r\theta} - \tau_{\theta r} = \frac{1}{r}s_{\theta r} \tag{8.17}$$

$$\tau_{rz} = \tau_{zr}$$

As noted earlier in (1.125) the composite stress τ^{ij} is not in general symmetric and in particular in cylindrical coordinates the asymmetry is represented by equations (8.17).

8.3 Constitutive relations

We assume that the representative micro-structure of the initially cylindrical composite is composed of n layers with different linear constitution. The results developed in section 5 are used to derive the appropriate constitutive relations. For the present case it is noted that

$$\frac{\Delta}{2G} = \frac{1}{r} \tag{8.18}$$

and the only non-vanishing component of Λ^j_β by noting (8.7) is

$$\Lambda^1_1 = \{_3{}^1{}_1\} = \frac{1}{r} \tag{8.19}$$

Using these results, we simplify (5.18) and (5.21) and derive the following contravariant forms for the constitutive relations in cylindrical coordinates

$$\tau^{i1} = (I^{(0)i1kl} + \frac{2}{r} I^{(1)i1kl} + \frac{1}{r^2} I^{(2)i1kl})\gamma_{kl} + (I^{(1)i1l\alpha} + \frac{2}{r} I^{(2)i1l\alpha} + \frac{1}{r^2} I^{(3)i1l\alpha})\kappa_{l\alpha} \tag{8.20}$$

$$\tau^{i2} = (I^{(0)i2kl} + \frac{1}{r} I^{(1)i2kl})\gamma_{kl} + (I^{(1)i2l\alpha} + \frac{1}{r} I^{(2)i2l\alpha})\kappa_{l\alpha} \tag{8.21}$$

$$\tau^{i3} = (I^{(0)i3kl} + \frac{1}{r} I^{(1)i3kl})\gamma_{kl} + (I^{(1)i3l\alpha} + \frac{1}{r} I^{(2)i3l\alpha})\kappa_{l\alpha} \tag{8.22}$$

$$S^{\alpha 1} = (I^{(1)\alpha 1kl} + \frac{2}{r} I^{(2)\alpha 1kl} + \frac{1}{r^2} I^{(3)\alpha 1kl})\gamma_{kl} + (I^{(2)\alpha 1l\beta} + \frac{2}{r} I^{(3)\alpha 1l\beta} + \frac{1}{r^2} I^{(4)\alpha 1l\beta})\kappa_{l\beta} \tag{8.23}$$

$$S^{\alpha 2} = (I^{(1)\alpha 2kl} + \frac{1}{r} I^{(2)\alpha 2kl})\gamma_{kl} + (I^{(2)\alpha 2l\beta} + \frac{1}{r} I^{(3)\alpha 2l\beta})\kappa_{l\beta} \tag{8.24}$$

$$S^{\alpha 3} = (\Gamma^{(1)\alpha 3kl} + \frac{1}{r}\Gamma^{(2)\alpha 3kl})\gamma_{kl} + (\Gamma^{(2)\alpha 3l\beta} + \frac{1}{r}\Gamma^{(3)\alpha 3l\beta})\kappa_{l\beta} \tag{8.25}$$

where the covariant components of the relative kinematic measures γ_{ij} and $\kappa_{j\alpha}$ have already been recorded in (8.12). The composite constitutive coefficients $\Gamma^{(k)pqrs}$ depend on the material behavior of every constituent of the composite laminte. For the special case where the micro-structure is composed of n isotropic layers with different elastic constants, we can write

$$C_{(l)}^{pqrs} = \lambda_{(l)}G^{pq}G^{rs} + \mu_{(l)}(G^{pr}G^{qs} + G^{ps}G^{qr}) \tag{8.26}$$

where $\lambda_{(l)}$ and $\mu_{(l)}$ (l = 1,...,n) are the Lame's constants of each layer in the micro-structure. By (8.2) the non-vanishing constitutive coefficients are as follows:

$$\begin{aligned}
C^{1111} &= \frac{1}{r^4}(\lambda + 2\mu) \\
C^{1122} &= \frac{\lambda}{r^2} \\
C^{1133} &= \frac{\lambda}{r^2} \\
C^{2222} &= \lambda + 2\mu \\
C^{2233} &= \lambda \\
C^{3333} &= \lambda + 2\mu \\
C^{2323} &= \mu \\
C^{1313} &= \frac{\mu}{r^2} \\
C^{1212} &= \frac{\mu}{r^2}
\end{aligned} \tag{8.27}$$

where subscript (l) is dropped for simplicity. It should be recalled that following symmetries of the constitutive coefficients must be remembered when the expressions involving the constitutive coefficients are expanded

$$C^{ijkl} = C^{jikl} = C^{ijlk} = C^{klij} \tag{8.28}$$

By (5.9) the non-zero components of the composite constitutive coefficients have the same superscript as (8.27). Substituting from (8.27) in (5.9) and using the results in (8.20)-(8.25) we find the following constitutive relations for the physical components of the composite stresses and composite stress couples. The summations in these relations extend over the micro-structure from $l = 1$ to $l = n$

$$\begin{aligned}\tau_{\theta\theta} = {} & \gamma_{\theta\theta}\sum (\lambda_l + 2\mu_l)(\Delta m_l + \frac{\xi_n}{r}\Delta m_l^2 + \frac{\xi_n^2}{3r^2}\Delta m_l^3) \\ & + (\gamma_{zz} + \gamma_{rr})\sum \lambda_l(\Delta m_l + \frac{\xi_n}{r}\Delta m_l^2 + \frac{\xi_n^2}{3r^2}\Delta m_l^3) \\ & + \kappa_{\theta\theta}\sum (\lambda_l + 2\mu_l)(\frac{\xi_n}{2}\Delta m_l^2 + \frac{2\xi_n^2}{3r}\Delta m_l^3 + \frac{\xi_n^3}{4r^2}\Delta m_l^4) \\ & + \kappa_{zz}\sum \lambda_l(\frac{\xi_n}{2}\Delta ml^2 + \frac{2\xi_n^2}{3r}\Delta m_l^3 + \frac{\xi_n^3}{4r^2}\Delta m_l^4)\end{aligned} \tag{8.29}$$

$$\begin{aligned}\tau_{z\theta} = {} & 2\gamma_{z\theta}\sum \mu_l(\Delta m_l + \frac{\xi_n}{r}\Delta m_l^2 + \frac{\xi_n^2}{3r^2}\Delta m_l^3) + (\kappa_{z\theta} \\ & + \kappa_{\theta z})\sum \mu_l(\frac{\xi_n}{2}\Delta m_l^3 + \frac{2\xi_n^2}{3r}\Delta m_l^3 + \frac{\xi_n^3}{4r^2}\Delta m_l^4)\end{aligned} \tag{8.30}$$

$$\begin{aligned}\tau_{r\theta} = {} & 2\gamma_{r\theta}\sum \mu_l(\Delta m_l + \frac{\xi_n}{r}\Delta m_l^2 + \frac{\xi_n^2}{3r^2}\Delta m_l^3) \\ & + \kappa_{r\theta}\sum \mu_l(\frac{\xi_n}{2}\Delta m_l^2 + \frac{2\xi_n^2}{3r}\Delta m_l^3 + \frac{\xi_n^3}{4r^2}\Delta m_l^4)\end{aligned} \tag{8.31}$$

$$\tau_{\theta z} = 2\gamma_{\theta z} \sum \mu_l(\Delta m_l + \frac{\xi_n}{2r} \Delta m_l^2) + (\kappa_{z\theta} + \kappa_{\theta z}) \sum \mu_l(\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{3r} \Delta m_l^3) \qquad (8.32)$$

$$\tau_{zz} = (\gamma_{\theta\theta}+\gamma_{rr})\sum \lambda_l(\Delta m_l + \frac{\xi_n}{2r} \Delta m_l^2) + \gamma_{zz} \sum (\lambda_l + 2\mu_l)(\Delta m_l + \frac{\xi_n}{2r} \Delta m_l^2) \qquad (8.33)$$

$$+ \kappa_{\theta\theta} \sum \lambda_l(\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{3r} \Delta m_l^2) + \kappa_{zz} \sum (\lambda_l + 2\mu_l)(\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{3r} \Delta m_l^3)$$

$$\tau_{rz} = 2\gamma_{rz} \sum \mu_l(\Delta m_l + \frac{\xi_n}{2r} \Delta m_l^2) + \kappa_{rz} \sum \mu_l (\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{3r} \Delta m_l^3) \qquad (8.34)$$

$$\tau_{\theta r} = 2\gamma_{\theta r} \sum \mu_l(\Delta m_l + \frac{\xi_n}{2r} \Delta m_l^2) + \kappa_{r\theta} \sum \mu_l(\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{3r} \Delta m_l^3) \qquad (8.35)$$

$$\tau_{zr} = 2\gamma_{zr} \sum \mu_l(\Delta m_l + \frac{\xi_n}{2r} \Delta m_l^3) + \kappa_{rz} \sum \mu_l(\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{3r} \Delta m_l^3) \qquad (8.36)$$

$$\tau_{rr} = (\gamma_{\theta\theta}+\gamma_{zz}+\gamma_{rr}) \sum \lambda_l(\Delta m_l + \frac{\xi_n}{2r} \Delta m_l^2) + 2\gamma_{rr} \sum \mu_l(\Delta m_l + \frac{\xi_n}{2r} \Delta m_l^2) \qquad (8.37)$$

$$+ (\kappa_{\theta\theta}+\kappa_{zz}) \sum \lambda_l(\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{3r} \Delta m_l^3)$$

$$S_{\theta\theta} = \xi_n \gamma_{\theta\theta} \sum (\lambda_l + 2\mu_l)(\frac{1}{2} \Delta m_l^2 + \frac{2}{3r} \xi_n \Delta m_l^3 + \frac{1}{4r^2} \xi_n^2 \Delta m_l^4) \qquad (8.38)$$

$$+ \xi_n(\gamma_{zz}+\gamma_{rr})\sum \lambda_l(\frac{1}{2} \Delta m_l^2 + \frac{2}{3r} \xi_n \Delta m_l^3 + \frac{1}{4r^2} \xi_n^2 \Delta m_l^4)$$

$$+ \xi_n^2\kappa_{\theta\theta} \sum (\lambda_l+2\mu_l)(\frac{1}{3} \Delta m_l^3 + \frac{1}{2r} \xi_n \Delta m_l^4 + \frac{1}{5r^2} \xi_n^2 \Delta m_l^5)$$

$$+ \xi_n^2\kappa_{zz} \sum \lambda_l (\frac{1}{3} \Delta m_l^3 + \frac{1}{2r} \xi_n \Delta m_l^4 + \frac{1}{5r^2} \xi_n^2 \Delta m_l^5)$$

$$S_{z\theta} = 2\xi_n\gamma_{\theta z} \sum \mu_l(\frac{1}{2} \Delta m_l^2 + \frac{2\xi_n}{3r} \Delta m_l^3 + \frac{\xi_n^2}{4r^2} \Delta m_l^4) + \xi_n^2(\kappa_{\theta z} \qquad (8.39)$$

$$+ \kappa_{z\theta})\sum \mu_l(\frac{1}{3}\Delta m_l^3 + \frac{\xi_n}{4r}\Delta m_l^4 + \frac{\xi_n^2}{5r^2}\Delta m_l^5)$$

$$S_{\theta z} = 2\gamma_{\theta z}\,\xi_n \sum \mu_l(\frac{1}{2}\Delta m_l^2 + \frac{\xi_n}{3r}\Delta m_l^3) + \xi_n^2(\kappa_{\theta z} \tag{8.40}$$

$$+ \kappa_{z\theta})\sum \mu_l(\frac{1}{3}\Delta m_l^3 + \frac{\xi_n}{4r}\Delta m_l^4)$$

$$S_{zz} = \xi_n \gamma_{zz} \sum (\lambda_l + 2\mu_l)(\frac{1}{2}\Delta m_l^2 + \frac{\xi_n}{3r}\Delta m_l^3) \tag{8.41}$$

$$+ \xi_n^2\,\kappa_{zz} \sum (\lambda_l + 2\mu_l)(\frac{1}{3}\Delta m_l^3 + \frac{\xi_n}{4r}\Delta m_l^4)$$

$$S_{\theta r} = 2\xi_n\,\gamma_{\theta r} \sum \mu_l(\frac{1}{2}\Delta m_l^2 + \frac{\xi_n}{3r}\Delta m_l^3) + \xi_n^2\,\kappa_{r\theta} \sum \mu_l(\frac{1}{3}\Delta m_l^3 + \frac{\xi_n}{4r}\Delta m_l^4) \tag{8.42}$$

$$S_{zr} = 2\xi_n\,\gamma_{zr} \sum \mu_l(\frac{1}{2}\Delta m_l^2 + \frac{\xi_n}{3r}\Delta m_l^3) + \xi_n^2\,\kappa_{rz} \sum \mu_l(\frac{1}{3}\Delta m_l^3 + \frac{\xi_n}{4r}\Delta m_l^4) \tag{8.43}$$

Using (5.33), (5.36) and (5.37), the composite mass density ρ_o and the composite mass moments $\rho_o z^1$ and $\rho_o z^2$ which appear in the equations of motion are also calculated for an initially cylindrical laminate

$$\rho_o = \sum \rho_{o(l)}\Delta m_l + \frac{\xi_n}{2r}\sum \rho_{o(l)}\Delta m_l^2 \tag{8.44}$$

$$\rho_o z^1 = \frac{1}{2}\,\xi_n \sum \rho_{o(l)}\Delta m_l^2 + \frac{\xi_n^2}{3r}\sum \rho_{o(l)}\Delta m_l^3 \tag{8.45}$$

$$\rho_o z^2 = \frac{1}{3}\,\xi_n^2 \sum \rho_{o(l)}\Delta m_l^3 + \frac{\xi_n^3}{4r}\sum \rho_{o(l)}\Delta m_l^4 \tag{8.46}$$

8.4 Energy equations and constitutive relations in linear thermoelasticity

Energy equations $(6.34)_{1,2}$ in the absence of heat supply or heat absorption reduce to the following forms when written in the cylindrical coordinates (8.1)

$$\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}+\frac{1}{r}\frac{\partial (rh)}{\partial r}+\rho_o(\phi_o\dot{\eta}_o+\phi_1\dot{\eta}_1)=0 \tag{8.47}$$

$$\frac{1}{r}\frac{\partial q_{1\theta}}{\partial \theta}+\frac{\partial q_{1z}}{\partial z}+h-q_r+\rho_o(\phi_o\dot{\eta}_1+\phi_1\dot{\eta}_2)=0 \tag{8.48}$$

where q_θ, q_z, q_r are the physical components of heat flux vector for the component laminate; $q_{1\theta}, q_{1z}$ are the physical components of the composite heat flux moment, and other quantities have the same meaning as section 6. The constitutive relations for various composite quantities in thermoelasticity can be written with reference to the development in section 7. The mechanical parts of the constitutive equations for composite stress and stress moments were derived in section 8.3. Therefore in what follows we record the thermal contributions to these quantities. The complete constitutive relations in thermo-mechanical theory is obviously obtained by superimposing these distinct parts.

From (7.10) we have

$$\tau^{ij}(\text{thermal})=-\phi_o\{J^{(0)ij}+\frac{1}{r}J^{(1)ij}+\lambda^j_\beta(J^{(1)i\beta}+\frac{1}{r}J^{(2)i\beta})\} \tag{8.49}$$

$$-\phi_1\{J^{(1)ij}+\frac{1}{r}J^{(2)ij}+\lambda^j_\beta(J^{(2)i\beta}+\frac{1}{r}J^{(3)i\beta})\}$$

Using (8.19) this reduces to

$$\tau^{i1}=-\phi_o\{J^{(0)i1}+\frac{2}{r}J^{(1)i1}+\frac{1}{r^2}J^{(2)i1}\}-\phi_1\{J^{(1)i1}+\frac{2}{r}J^{(2)i1}+\frac{1}{r^2}J^{(3)i1}\} \tag{8.50}$$

$$\tau^{i2}=-\phi_o[J^{(0)i2}+\frac{1}{r}J^{(1)i2}]-\phi_1[J^{(1)i2}+\frac{1}{r}J^{(2)i2}] \tag{8.51}$$

$$\tau^{i3} = -\phi_o[J^{(0)i3} + \frac{1}{r} J^{(1)i3}] - \phi_1[J^{(1)i3} + \frac{1}{r} J^{(2)i3}] \tag{8.52}$$

where the word thermal is dropped for brevity. Similarly from (7.12) we find the thermal contributions to stress moments as follows:

$$S^{\alpha 1} = -\phi_o\{J^{(1)\alpha 1} + \frac{2}{r} J^{(2)\alpha 1} + \frac{1}{r^2} J^{(3)\alpha 1}\} - \phi_1\{J^{(2)\alpha 1} + \frac{2}{r} J^{(3)\alpha 1} + \frac{1}{r^2} J^{(4)\alpha 1}\} \tag{8.53}$$

$$S^{\alpha 2} = -\phi_o[J^{(1)\alpha 2} + \frac{1}{r} J^{(2)\alpha 2}] - \phi_1[J^{(2)\alpha 2} + \frac{1}{r} J^{(3)\alpha 2}] \tag{8.54}$$

$$S^{\alpha 3} = -\phi_o[J^{(1)\alpha 3} + \frac{1}{r} J^{(2)\alpha 3}] - \phi_1[J^{(2)\alpha 3} + \frac{1}{r} J^{(3)\alpha 3}] \tag{8.55}$$

The constitutive relations for composite entropy and its moments η_m, heat flux vector **q** and its moment q_1 are also derived from equations (7.17), (7.22) and (7.24). Using (8.18) we have

$$\rho_o\eta_m = \gamma_{ij}[J^{(m)ij} + \frac{1}{r} J^{(m+1)ij}] + \kappa_{i\beta}[J^{(m+1)i\beta} + \frac{1}{r} J^{(m+2)i\beta}]$$

$$+ \phi_o[K^{(m)} + \frac{1}{r} K^{(m+1)}] + \phi_1[K^{(m+1)} + \frac{1}{r} K^{(m+2)}] \quad (m = 0,1,2) \tag{8.56}$$

$$q^i = -[L^{(0)i\beta} + \frac{1}{r} L^{(1)i\beta}]\phi_{o,\beta} - [L^{(1)i\beta} + \frac{1}{r} l^{(2)i\beta}]\phi_{1,\beta} - [L^{(0)i3} + \frac{1}{r} L^{(1)i3}]\phi_1 \tag{8.57}$$

$$q_1^\alpha = -[L^{(1)\alpha\beta} + \frac{1}{r} L^{(2)\alpha\beta}]\phi_{o,\beta} - [L^{(2)\alpha\beta} + \frac{1}{r} L^{(3)\alpha\beta}]\phi_{1,\beta}$$

$$- [L^{(1)\alpha 3} + \frac{1}{r} L^{(2)\alpha 3}]\phi_1 \tag{8.58}$$

where the thermal constitutive coefficients $J^{(k)ij}, K^{(k)}$ and $L^{(k)ij}$ for the composite laminate are given in (7.6)-(7.2) and calculated in expanded form in (7.30)-(7.32).

If the micro-structure is composed of isotropic layers the coefficients of thermal stress and thermal conductivity of each layer can be represented in terms of only one constant. For the present case of cylindrical laminates we can write

$$C^{ij}_{(l)} = G^{ij}\beta_{(l)} \quad (l = 1,\ldots,n) \tag{8.59}$$

$$k^{ij}_{(l)} = G^{ij}k_{(l)} \tag{8.60}$$

Substituting from (8.59) and (8.60) in (7.30) and (7.32) we have

$$J^{(k)11} = \frac{1}{k+1}\, \xi_n^k/r^2 \sum \beta_{(l)}\Delta m_l^{k+1} \tag{8.61}$$

$$J^{(k)22} = J^{(k)33} = \frac{1}{k+1}\, \xi_n^k \sum \beta_{(l)}\Delta m_l^{k+1} \tag{8.62}$$

$$L^{(k)11} = \frac{1}{k+1}\, \frac{\xi_n^k}{r^2} \sum k_{(l)}\Delta m_l^{k+1} \tag{8.63}$$

$$L^{(k)22} = L^{(k)33} = \frac{1}{k+1}\, \xi_n^k \sum k_{(l)}\Delta m_l^{k+1} \tag{8.64}$$

These are the only non-vanishing components of the composite thermal coefficients and $K^{(k)}$'s are scalar quantities independent of the coordinate system. Substituting from (8.61) and (8.62) in (8.50)-(8.55) we get the following contributions to the thermal parts of the composite stress tensors and moments:

$$\tau_{\theta\theta}(\text{thermal}) = -\phi_o \sum \beta_l\left(\Delta m_l + \frac{\xi_n}{r}\, \Delta m_l^2 + \frac{\xi_n^2}{3r^2}\, \Delta m_l^3\right)$$

$$-\phi_1 \sum \beta_l \left(\frac{\xi_n}{2}\, \Delta m_l^2 + \frac{2\xi_n^2}{3r}\, \Delta m_l^3 + \frac{\xi_n^3}{4r^2}\, \Delta m_l^4\right) \tag{8.65}$$

$$\tau_{z\theta}(\text{thermal}) = \tau_{r\theta}(\text{thermal}) = 0 \tag{8.66}$$

$$\tau_{\theta z}(\text{thermal}) = \tau_{rz}(\text{thermal}) = 0 \tag{8.67}$$

$$\tau_{\theta r}(\text{thermal}) = \tau_{zr}(\text{thermal}) = 0 \tag{8.68}$$

$$\tau_{zz}(\text{thermal}) = \tau_{rr}(\text{thermal}) = -\phi \sum \beta_l(\Delta m_l + \frac{\xi_{sn}}{2r}\Delta m_l^2)$$

$$-\phi_1 \sum \beta_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3) \tag{8.69}$$

$$S_{\theta\theta}(\text{thermal}) = -\phi_o \sum \beta_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{2\xi_{sn}^2}{3r}\Delta m_l^3 + \frac{\xi_{sn}^3}{4r^2}\Delta m_l^4)$$

$$-\phi_1 \sum \beta_l(\frac{\xi_{sn}^2}{3}\Delta m_l^3 + \frac{\xi_{sn}^3}{2r}\Delta m_l^4 + \frac{\xi_{sn}^4}{5r^2}\Delta m_l^5) \tag{8.70}$$

$$S_{zz}(\text{thermal}) = -\phi_o \sum \beta_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3)$$

$$-\phi_1 \sum \beta_l(\frac{\xi_{sn}^2}{3}\Delta m_l^3 + \frac{\xi_{sn}^3}{4r}\Delta m_l^4) \tag{8.71}$$

The thermal contributions to other components of S is zero. As for the entropy and its moments, heat flux and its moment, we substitute from (8.61)-(8.64) in (8.56)-(8.58) and get the following results

$$\rho_o\eta_{(o)} = (\gamma_{\theta\theta} + \gamma_{zz} + \gamma_{rr}) \sum \beta_l(\Delta m_l + \frac{\xi_{sn}}{2r}\Delta m_l^2)$$

$$+ (\kappa_{\theta\theta} + \kappa_{zz}) \sum \beta_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3)$$

$$+ \phi_o \sum (\rho c)_l(\Delta m_l + \frac{\xi_{sn}}{2r}\Delta m_l^3) + \phi_1 \sum (\rho c)_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3) \tag{8.72}$$

$$\rho_o\eta_{(1)} = (\gamma_{\theta\theta} + \gamma_{zz} + \gamma_{rr}) \sum \beta_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3)$$

$$+ (\kappa_{\theta\theta} + \kappa_{zz}) \sum \beta_l(\frac{\xi_{sn}^2}{3}\Delta m_l^3 + \frac{\xi_{sn}^3}{4r}\Delta m_l^4)$$

$$+\phi_o \sum (\rho c)_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3)$$

$$+\phi_1 \sum (\rho c)_l(\frac{\xi_{sn}^2}{3}\Delta m_l^3 + \frac{\xi_{sn}^3}{4r}\Delta m_l^4) \tag{8.73}$$

$$\rho_o\eta_{(2)} = (\gamma_{\theta\theta} + \gamma_{zz} + \gamma_{rr}) \sum \beta_l(\frac{\xi_{sn}^2}{3}\Delta m_l^3 + \frac{\xi_{sn}^3}{4r}\Delta m_l^4)$$

$$+ (\kappa_{\theta\theta} + \kappa_{zz}) \sum \beta_l(\frac{\xi_{sn}^3}{4}\Delta m_l^4 + \frac{\xi_{sn}^4}{5r}\Delta m_l^5)$$

$$+\phi_o \sum(\rho c)_l(\frac{\xi_{sn}^2}{3}\Delta m_l^3 + \frac{\xi_{sn}^3}{4r}\Delta m_l^4)$$

$$+\phi_1 \sum(\rho c)_l(\frac{\xi_{sn}^3}{4}\Delta m_l^4 + \frac{\xi_{sn}^4}{5r}\Delta m_l^5) \tag{8.74}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi_o}{\partial\theta}\sum k_l(\Delta m_l + \frac{\xi_{sn}}{2r}\Delta m_l^2) - \frac{1}{r}\frac{\partial\phi_1}{\partial\theta}\sum k_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3) \tag{8.75}$$

$$q_z = -\frac{\partial\phi_o}{\partial z}\sum k_l(\Delta m_l + \frac{\xi_{sn}}{2r}\Delta m_l^2) - \frac{\partial\phi_1}{\partial z}\sum k_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3) \tag{8.76}$$

$$q_r = -\phi_1 \sum k_l(\Delta m_l + \frac{\xi_{sn}}{2r}\Delta m_l^2) \tag{8.77}$$

$$q_{1\theta} = -\frac{1}{r}\frac{\partial\phi_o}{\partial\theta}\sum k_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3) - \frac{1}{r}\frac{\partial\phi_1}{\partial\theta}\sum k_l(\frac{\xi_{sn}^2}{3}\Delta m_l^3 + \frac{\xi_{sn}^3}{4r}\Delta m_l^4) \tag{8.78}$$

$$q_{1z} = -\frac{\partial\phi_o}{\partial z}\sum k_l(\frac{\xi_{sn}}{2}\Delta m_l^2 + \frac{\xi_{sn}^2}{3r}\Delta m_l^3) - \frac{\partial\phi_1}{\partial z}\sum k_l(\frac{\xi_{sn}^2}{3}\Delta m_l^3 + \frac{\xi_{sn}^3}{4r}\Delta m_l^4) \tag{8.79}$$

It is worthwhile to recall that in the above relations, ϕ_1 is the gradient of ϕ_o in $\theta^3 = r$-direction.

9.0 LINEAR THEORY OF INITIALLY SPHERICAL LAMINATES

For an initially spherical laminate, we choose the usual spherical coordinates (r,ϕ,θ) which are related to the coordinates $(\theta^1,\theta^2,\theta^3)$ according to the following relations

$$\theta^1 = \phi \quad , \quad \theta^2 = \theta \quad , \quad \theta^3 = r \tag{9.1}$$

The choice of coordinate r for θ^3 direction is natural, since here similar to the cylindrical case, the spherical laminates are piled up in the r direction. The metric tensor G_{ij} and its conjugate G^{ij} and $G = \det(G_{ij})$ are given by the following relations

$$(G_{ij}) = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r^2\sin^2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (G^{ij}) = \begin{bmatrix} 1/r^2 & 0 & 0 \\ 0 & 1/r^2\sin^2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.2}$$

$$G = \det(G_{ij}) = r^4 \sin^2\phi \tag{9.3}$$

For the spherical coordinates defined in (9.1), the non-vanishing Christoffel symbols of first and second kind are as follows

$$\begin{aligned}
[11,3] &= -\frac{1}{2}\frac{\partial G_{11}}{\partial \theta^3} = -\theta^3 = -r \\
[12,2] = [21,2] &= \frac{1}{2}\frac{\partial G_{22}}{\partial \theta^1} = (\theta^3)^2 \sin\theta^1 \cos\theta^1 = r^2 \sin\phi\cos\phi \\
[13,1] = [31,1] &= \frac{1}{2}\frac{\partial G_{11}}{\partial \theta^3} = \theta^3 = r \\
[23,2] = [32,2] &= \frac{1}{2}\frac{\partial G_{22}}{\partial \theta^3} = \theta^3 \sin^2\theta^1 = r\sin^2\phi \\
[22,1] &= -\frac{1}{2}\frac{\partial G_{22}}{\partial \theta^1} = -(\theta^3)^2 \sin\theta^1\cos\theta^1 = -r^2\sin\phi\cos\phi \\
[22,3] &= -\frac{1}{2}\frac{\partial G_{22}}{\partial \theta^3} = -\theta^3\sin^2\theta^1 = -r\sin^2\phi
\end{aligned} \tag{9.4}$$

$$\{_1{}^3{}_1\} = -\theta^3 = -r$$

$$\{_1{}^2{}_2\} = \{_2{}^2{}_1\} = \cot\theta^1 = \cot\phi$$

$$\{_1{}^1{}_3\} = \{_1{}^3{}_1\} = \frac{1}{\theta^3} = \frac{1}{r}$$

$$\{_3{}^2{}_2\} = \{_2{}^2{}_3\} = \frac{1}{\theta^3} = \frac{1}{r} \tag{9.5}$$

$$\{_2{}^1{}_2\} = -\sin\phi\,\theta^1\cos\theta^1 = -\sin\phi\cos\phi$$

$$\{_2{}^3{}_2\} = -\theta^3\sin^2\theta^1 = -r\sin^2\phi$$

From (9.2) and (2.14) we find the following expression for Δ

$$\Delta = \begin{vmatrix} 2r & 0 & 0 \\ 0 & r^2\sin^2\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} r^2 & 0 & 0 \\ 0 & 2r\sin^2\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4r^3\sin^2\phi \tag{9.6}$$

9.1 Relative Kinematic Measures

The relative kinematic measures γ_{ij} were given by $(2.118)_{1,2,3}$, and since $D_\alpha = G_{\alpha 3} = 0$ — by (2.103) and (9.2) — and $\delta_j = u_{j|3}$ — by (4.22) — we can write

$$\gamma_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i}) \tag{9.7}$$

By (2.14) and (9.5) the non-vanishing components of Λ_β^j are

$$\Lambda_1^1 = \Lambda_2^2 = \frac{1}{r} \tag{9.8}$$

As a result we have the following expressions for $\kappa_{j\alpha}$ given in $(2.118)_{4,3}$

$$\kappa_{\alpha\beta} = \frac{1}{r}u_{\beta|\alpha} + \delta_{\alpha|\beta} \tag{9.9}$$

$$\kappa_{3\alpha} = \delta_{3|\alpha}$$

The components of director displacement δ_j by (4.22) and (9.5) are

$$\delta_\alpha = u_{\alpha,3} - \frac{1}{r} u_\alpha$$

$$\delta_3 = u_{3,3} \tag{9.10}$$

Now we proceed to calculate the covariant components of γ_{ij} in terms of the covariant components of the displacement vector u and its partial derivatives. From (9.7) we can write

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \{_i{}^k{}_j\}\, u_k \tag{9.11}$$

Using (9.3) we get

$$\gamma_{11} = u_{1,1} + r u_3$$

$$\gamma_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) - \cot\phi\, u_2$$

$$\gamma_{13} = \frac{1}{2}(u_{1,3} + u_{3,1}) - \frac{1}{r} u_1$$

$$\gamma_{22} = u_{2,2} + \sin\phi\cos\phi\, u_1 + r\sin^2\phi\, u_3 \tag{9.12}$$

$$\gamma_{23} = \frac{1}{2}(u_{2,3} + u_{3,1}) - \frac{1}{r} u_2$$

$$\gamma_{33} = u_{3,3}$$

These results can be written in terms of the physical components by using (8.4) and (9.2). The appropriate expressions after simplificiation are as follows

$$\gamma_{\phi\phi} = \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \phi} + u_r\right)$$

$$\gamma_{\phi\theta} = \frac{1}{2}\left(\frac{1}{r\sin\phi}\frac{\partial u_\phi}{\partial\theta} + \frac{1}{r}\frac{\partial u_\theta}{\partial\phi} - \frac{\cot\phi}{r}u_\theta\right)$$

$$\gamma_{\phi r} = \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r}\right)$$

$$\gamma_{\theta\theta} = \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta} + \frac{\cot\phi}{r}u_\phi + \frac{u_r}{r} \tag{9.13}$$

$$\gamma_{\theta r} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r\sin\phi}\frac{\partial u_r}{\partial\theta}\right)$$

$$\gamma_{rr} = \frac{\partial u_r}{\partial r}$$

The relations for $\kappa_{\alpha\beta}$ and $\kappa_{3\alpha}$ given in (9.9), after using (9.10) are reduced to

$$\kappa_{\alpha\beta} = \frac{1}{r}(u_{\beta,\alpha} - u_{\alpha,\beta}) - \frac{1}{r}\{{}_{\beta}{}^{k}{}_{\alpha}\}u_k + u_{\alpha,3\beta} - \{{}_{\alpha}{}^{m}{}_{\beta}\}u_{m,3} + \{{}_{\alpha}{}^{m}{}_{\beta}\}\{{}_{m}{}^{k}{}_{3}\}u_k$$

$$\kappa_{3\alpha} = u_{3,3\alpha} - \{{}_{3}{}^{k}{}_{\alpha}\}u_{k,3} + \{{}_{3}{}^{k}{}_{\alpha}\}\{{}_{k}{}^{j}{}_{3}\}u_j$$

In expanded form these relations are simplified to

$$\kappa_{11} = u_3 + u_{1,31} + ru_{3,3}$$

$$\kappa_{12} = \frac{1}{r}(u_{2,1} - u_{1,2}) + u_{1,32} - \cot\phi\, u_{2,3}$$

$$\kappa_{21} = \frac{1}{r}(u_{1,2} - u_{2,1}) + u_{2,31} - \cot\phi\, u_{2,3}$$

$$\kappa_{22} = \sin^2\phi\, u_3 + u_{2,32} + \sin\phi\cos\phi\, u_{1,3} + r\sin^2\phi\, u_{3,3} \tag{9.14}$$

$$\kappa_{31} = u_{3,31} - \frac{1}{r}u_{1,3} + \frac{1}{r^2}u_1$$

$$\kappa_{32} = u_{3,32} - \frac{1}{r}u_{2,3} + \frac{1}{r^2}u_2$$

Using (8.4) and (9.2), we can write the physical components of these relative kinematic measures. The results after simplification are as follows

$$\kappa_{\phi\phi} = \frac{1}{r^2}\frac{\partial}{\partial r}[r(u_r + \frac{\partial u_\phi}{\partial \phi})]$$

$$\kappa_{\phi\theta} = \frac{1}{r^2}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial^2 u_\phi}{\partial\theta\partial r} - \frac{\cot\phi}{r}\frac{\partial u_\theta}{\partial r}$$

$$\kappa_{\theta\phi} = \frac{1}{r^2\sin\phi}\frac{\partial u_\phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial\phi\partial r} - \frac{\cot\phi}{r^2}u_\theta \tag{9.15}$$

$$\kappa_{\theta\theta} = \frac{1}{r^2}\frac{\partial}{\partial r}[r(u_r + \cot\phi\, u_\phi + \frac{1}{\sin\phi}\frac{\partial u_\theta}{\partial\theta})]$$

$$\kappa_{r\phi} = \frac{1}{r}\frac{\partial}{\partial r}(\frac{\partial u_r}{\partial\phi} - u_\phi)$$

$$\kappa_{r\theta} = \frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{\sin\phi}\frac{\partial u_r}{\partial\theta} - u_\theta)$$

The equations for balance of mass are also obtained by substituting covariant derivatives of displacement vector in spherical coordinates in the expressions (2.121) and (2.122). The simplified results in terms of physical components of the displacement vector and its derivatives are as follows:

$$\rho_o = \rho(1 + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_r}{\partial r} + \cot\phi\frac{u_\phi}{r} + \frac{2}{r}u_r)$$

$$\rho = \rho_o(1 - \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} - \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta} - \frac{\partial u_r}{\partial r} - \frac{\cot\phi}{r}u_\phi - \frac{2}{r}u_r) \tag{9.16}$$

9.2 Linearized field equations

In order to derive these equations, it is sufficient to substitute from (9.3) and (9.5) in (2.123)-(2.125). The results after simplficiation are written in terms of the physical components of various tensors by using (9.3) and are recorded below.

Balance of linear momentum:

$$\frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\,\tau_{\phi\phi})+\frac{1}{r\sin\phi}\frac{\partial\tau_{\theta\phi}}{\partial\theta}+\frac{\tau_{\phi r}}{r}-\frac{\cot\phi}{r}\tau_{\theta\theta}+\frac{1}{r^2\sin\phi}\frac{\partial\sigma_\phi}{\partial r}+\rho_o b_\phi$$

$$=\rho_o(\ddot{u}_\phi+z^1\,\ddot{\delta}_\phi)$$

$$\frac{1}{r}\frac{\partial\tau_{\phi\theta}}{\partial\phi}+\frac{1}{r\sin\phi}\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{\tau_{\theta r}}{r}+\frac{\cot\phi}{r}(\tau_{\phi\theta}+\tau_{\theta\phi})+\frac{1}{r^2\sin\phi}\frac{\partial\sigma_z}{\partial r}+\rho_o b_\theta$$

$$=\rho_o(\ddot{u}_\theta+z^1\ddot{\delta}_\theta) \tag{9.17}$$

$$\frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\,\tau_{\phi r})+\frac{1}{r\sin\phi}\frac{\partial\tau_{\theta r}}{\partial\theta}-\frac{\tau_{\phi\phi}+\tau_{\theta\theta}}{r}+\frac{1}{r^2\sin\phi}\frac{\partial\sigma_r}{\partial r}+\rho_o b_r$$

$$=\rho_o(\ddot{u}_r+z^1\ddot{\delta}_r)$$

Balance of director momentum:

$$\frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, s_{\phi\phi}) + \frac{1}{r\sin\phi}\frac{\partial s_{\theta\phi}}{\partial\theta} + \frac{s_{\phi r}}{r} - \frac{\cot\phi}{r}s_{\theta\theta} + \frac{\sigma_\phi}{r^2\sin\phi} - \tau_{r\phi} + \rho_o c_\phi$$

$$= \rho_o(z^1\ddot{u}_\phi + z^2\ddot{\delta}_\phi)$$

$$\frac{1}{r}\frac{\partial s_{\phi\theta}}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial s_{\theta\theta}}{\partial\theta} + \frac{\cot\phi}{r}(s_{\phi\theta}+s_{\theta\phi}) + \frac{s_{\theta r}}{r} + \frac{\sigma_\theta}{r^2\sin\phi} - \tau_{r\theta} + \rho_o c_\theta$$

$$= \rho_o(z^1\ddot{u}_\theta + z^2\ddot{\delta}_\theta) \tag{9.18}$$

$$\frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, s_{\phi r}) + \frac{1}{r\sin\phi}\frac{\partial s_{\theta r}}{\partial\theta} - \frac{s_{\theta\theta}}{r} + \frac{\sigma_r}{r^2\sin\phi} - \tau_{rr} + \rho_o c_r$$

$$= \rho_o(z^1\ddot{z}_r + z^2\ddot{\delta}_r)$$

Balance of moment of momentum:

$$\tau_{\phi\theta} - \tau_{\theta\phi} = \frac{1}{r}(s_{\theta\phi} - s_{\phi\theta})$$

$$\tau_{r\phi} - \tau_{\phi r} = \frac{1}{r}s_{\phi r} \tag{9.19}$$

$$\tau_{r\theta} - \tau_{\theta r} = \frac{1}{r}s_{\theta r}$$

Here again as in cylindrical coordinates, the composite stress tensor τ^{ij} is not symmetric and the asymmetry is represented in equations (9.19).

9.3 Constitutive relations

We follow exactly the same procedure as in Section 8.3. For quantities in spherical coordinates, by (9.6) and (9.3), we get

$$\frac{\Delta}{2G} = \frac{4r^3\sin^2\phi}{2r^4\sin^2\phi} = \frac{2}{r} \tag{9.20}$$

and by (9.8)

$$\Lambda_1^1 = \Lambda_2^2 = \frac{1}{r}$$

Using these results in (5.18) and (5.21) we obtain the following contravariant forms of the constitutive relations in spherical coordinates. These constitutive equations are obviously written for a purely mechanical theory.

$$\tau^{i\alpha} = [I^{(0)i\alpha kl} + \frac{3}{r} I^{(1)i\alpha kl} + \frac{2}{r^2} I^{(2)i\alpha kl}]\gamma_{kl} + [I^{(1)i\alpha l\beta} + \frac{3}{r} I^{(2)i\alpha l\beta} + \frac{2}{r^2} I^{(3)i\alpha l\beta}]\kappa_{l\beta} \tag{9.21}$$

$$\tau^{i3} = [I^{(0)i3kl} + \frac{2}{r} I^{(1)i3kl}]\gamma_{kl} + [I^{(2)i3l\alpha} + \frac{2}{r} I^{(2)i3l\alpha}]\kappa_{l\alpha} \tag{9.22}$$

$$S^{\alpha\beta} = [I^{(1)\alpha\beta kl} + \frac{3}{r} I^{(2)\alpha\beta kl} + \frac{1}{r^2} I^{(3)\alpha\beta kl}]\gamma_{kl} + [I^{(2)\alpha\beta l\gamma} + \frac{3}{r} I^{(3)\alpha\beta l\gamma} + \frac{1}{r^2} I^{(4)\alpha\beta l\gamma}]\kappa_{l\gamma} \tag{9.23}$$

$$S^{\alpha 3} = [I^{(1)\alpha 3kl} + \frac{2}{r} i^{(2)\alpha 3kl}]\gamma_{kl} + [I^{(2)\alpha 3l\beta} + \frac{2}{r} I^{(3)\alpha 3l\beta}]\kappa_{l\beta} \tag{9.24}$$

The covariant components of the relative kinematic measures γ_{kl} and $\kappa_{l\alpha}$ were calculated and are recorded in (9.12) and (9.14). The composite constitutive coefficients $I^{(k)pqrs}$ depend on the constitution of the laminates. For the special case where the micro-structure is composed of n isotropic layers with different elastic constants we have

$$C_l^{pqrs} = \lambda_{(l)} G^{pq} G^{rs} + \mu_{(l)} (G^{pr} G^{qs} + G^{ps} G^{qr})$$

$$(l = 1,\ldots,n)$$

where $\lambda_{(l)}$ and $\mu_{(l)}$ are the Lame constants of each layer in the micro-structure. By $(9.2)_2$ the non-vanishing constitutive coefficients are

$$
\begin{aligned}
C^{1111} &= (\lambda+2\mu)/r^4 \\
C^{1122} &= \lambda/r^4\sin^2\phi \\
C^{1133} &= \lambda/r^4 \\
C^{2222} &= (\lambda+2\mu)/r^4\sin^4\phi \\
C^{2233} &= \lambda/r^2\sin^2\phi \\
C^{3333} &= \lambda + 2\mu \\
C^{2323} &= \mu/r^2\sin^2\phi \\
C^{1313} &= \mu/r^2 \\
C^{1212} &= \mu/r^4\sin^2\phi
\end{aligned}
\tag{9.25}
$$

where the subscript (l) is dropped for brevity. Of course, the symmetries of the constitutive coefficients C^{ijkl} as expressed in (8.28) must be recalled when the expressions involving the constitutive coefficients are to be expanded. Substituting from (9.25) in (5.9) and using the results in (9.21)-(9.24) we find the following constitutive relations for the physical components of the composite stresses and composite couples. The summations in these relations extend over the micro-structure from $l = 1$ to $l = n$.

$$\tau_{\phi\phi} = \gamma_{\phi\phi} \sum(\lambda_l + 2\mu_l)(\Delta m_l + \frac{3\xi_n}{2r}\Delta m_l^2 + \frac{2\xi_n^2}{3r^2}\Delta m_l^3)$$

$$+ (\gamma_{\theta\theta}+\gamma_{rr}) \sum \lambda_l(\Delta m_l + \frac{3\xi_n}{2r}\Delta m_l^2 + \frac{2\xi_n^2}{3r^2}\Delta m_l^3)$$

$$+ \kappa_{\phi\phi} \sum (\lambda_l + 2\mu_l)(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}^2}{r} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r^2} \Delta m_l^4)$$

$$+ \kappa_{\theta\theta} \sum \lambda_l (\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}}{2r} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r^2} \Delta m_l^4) \tag{9.26}$$

$$\tau_{\phi\theta} = 2\gamma_{\phi\theta} \sum \mu_l (\Delta m_l^2 + \frac{3\xi_{sn}}{2r} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r^2} \Delta m_l^3)$$

$$+ (\kappa_{\phi\theta} + \kappa_{\theta\phi}) \sum \mu_l (\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}^2}{r} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r^2} \Delta m_l^4) \tag{9.27}$$

$$\tau_{\phi r} = 2\gamma_{\phi r} \sum \mu_l (\Delta m_l + \frac{\xi_{sn}}{r} \Delta m_l^2) + \kappa_{r\phi} \sum \mu_l (\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r} \Delta m_l^3) \tag{9.28}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} \tag{9.29}$$

$$\tau_{\theta\theta} = (\gamma_{\phi\phi} + \gamma_{rr}) \sum \lambda_l (\Delta m_l + \frac{3\xi_{sn}}{2r} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r^2} \Delta m_l^3)$$

$$+ \gamma_{\theta\theta} \sum (\lambda_l + 2\mu_l)(\Delta m_l + \frac{3\xi_{sn}}{2r} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r^2} \Delta m_l^3)$$

$$+ \kappa_{\phi\phi} \sum \lambda_l (\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}}{2r} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r^2} \Delta m_l^4)$$

$$+ \kappa_{\theta\theta} \sum (\lambda_l + 2\mu_l)(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}^2}{r} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r^2} \Delta m_l^4) \tag{9.30}$$

$$\tau_{\theta r} = 2\gamma_{\theta r} \sum \mu_l (\Delta m_l + \frac{\xi_{sn}}{r} \Delta m_l^2)$$

$$+ \kappa_{r\theta} \sum \mu_l (\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r} \Delta m_l^3) \tag{9.31}$$

$$\tau_{r\phi} = 2\gamma_{r\phi} \sum \mu_l(\Delta m_l + \frac{3\xi_{sn}}{2r} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r^2} \Delta m_l^3)$$

$$+ \kappa_{r\phi} \sum \mu_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}^2}{r} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r^2} \Delta m_l^4) \qquad (9.32)$$

$$\tau_{r\theta} = 2\gamma_{r\theta} \sum \mu_l(\Delta m_l + \frac{3\xi_{sn}}{2r} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r^2} \Delta m_l^3)$$

$$+ \kappa_{r\theta} \sum \mu_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}^2}{r} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r^2} \Delta m_l^4) \qquad (9.33)$$

$$\tau_{rr} = (\gamma_{\phi\phi} + \gamma_{\theta\theta}) \sum \lambda_l(\Delta m_l + \frac{\xi_{sn}}{r} \Delta m_l^2)$$

$$+ \gamma_{rr} \sum (\lambda_l+2\mu_l)(\Delta m_l + \frac{\xi_{sn}}{r} \Delta m_l^2)$$

$$+ (\kappa_{\phi\phi}+\kappa_{\theta\theta}) \sum \lambda_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r} \Delta m_l^3) \qquad (9.34)$$

$$S_{\phi\phi} = \gamma_{\phi\phi} \sum (\lambda_l+2\mu_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}^2}{r} \Delta m_l^3 + \frac{\xi_{sn}^3}{4r^2} \Delta m_l^4)$$

$$+ (\gamma_{\theta\theta}+\gamma_{rr}) \sum \lambda_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}^2}{r} \Delta m_l^3 + \frac{\xi_{sn}^3}{4r^2} \Delta m_l^4)$$

$$+ \kappa_{\phi\phi} \sum (\lambda_l+2\mu_l)(\frac{\xi_{sn}^2}{3} \Delta m_l^3 + \frac{3\xi_{sn}^3}{4r} \Delta m_l^4 + \frac{\xi_{sn}^4}{5r^2} \Delta m_l^5)$$

$$+ \kappa_{\theta\theta} \sum \lambda_l(\frac{\xi_{sn}^2}{3} \Delta m_l^3 + \frac{3\xi_{sn}^3}{4r} \Delta m_l^4 + \frac{\xi_{sn}^4}{5r^2} \Delta m_l^5) \qquad (9.35)$$

$$S_{\phi\theta} = 2\gamma_{\phi\theta} \sum \mu_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{\xi_{sn}^2}{r} \Delta m_l^3 + \frac{\xi_{sn}^3}{4r^2} \Delta m_l^4)$$

$$+ (\kappa_{\phi\theta}+\kappa_{\theta\phi}) \sum \mu_l (\frac{\xi_n^2}{3} \Delta m_l^3 + \frac{3\xi_n^3}{4r} \Delta m_l^4 + \frac{\xi_n^4}{5r^2} \Delta m_l^5)$$

$$S_{\phi r} = 2\gamma_{\phi r} \sum \mu_l (\frac{\xi_n}{2} \Delta m_l^2 + \frac{2\xi_n^2}{3r} \Delta m_l^3)$$

$$+ \kappa_{r\phi} \sum \mu_l (\frac{\xi_n^2}{3} \Delta m_l^3 + \frac{\xi_n^3}{2r} \Delta m_l^4) \qquad (9.37)$$

$$S_{\theta\phi} = S_{\phi\theta} \qquad (9.38)$$

$$S_{\theta\theta} = (\gamma_{\phi\phi}+\gamma_{rr}) \sum \lambda_l (\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{r} \Delta m_l^3 + \frac{\xi_n^3}{4r^2} \Delta m_l^4)$$

$$+ \gamma_{\theta\theta} \sum (\lambda_l+2\mu_l)(\frac{\xi_n}{2} \Delta m_l^2 + \frac{\xi_n^2}{r} \Delta m_l^3 + \frac{\xi_n^3}{4r^2} \Delta m_l^4)$$

$$+ \kappa_{\phi\phi} \sum \lambda_l (\frac{\xi_n^2}{3} \Delta m_l^3 + \frac{3\xi_n^3}{4r} \Delta m_l^4 + \frac{\xi_n^4}{5r^2} \Delta m_l^5)$$

$$+ \kappa_{\theta\theta} \sum (\lambda_l+2\mu_l)(\frac{\xi_n^2}{3} \Delta m_l^3 + \frac{3\xi_n^3}{4r} \Delta m_l^4 + \frac{\xi_n^4}{5r^2} \Delta m_l^5) \qquad (9.39)$$

$$S_{\theta r} = 2\gamma_{\theta r} \sum \mu_l (\frac{\xi_n}{2} \Delta m_l^2 + \frac{2\xi_n^2}{3r} \Delta m_l^3)$$

$$+ \kappa_{r\theta} \sum \mu_l (\frac{\xi_n^2}{3} \Delta m_l^3 + \frac{\xi_n^3}{2r} \Delta m_l^4) \qquad (9.40)$$

Using (5.33), (5.36) and (5.37), the composite mass density ρ_o and the composite mass moments $\rho_o z^1$ and $\rho_o z^2$ which appear in the equations of motion are also calculated for an initially spherical laminate

$$\rho_o = \sum \rho_{o(l)} \Delta m_l + \frac{\xi_n}{r} \sum \rho_{o(l)} \Delta m_l^2 \qquad (9.41)$$

$$\rho_o z^1 = \frac{1}{2}\,\xi_n \sum \rho_{\alpha(l)} \Delta m_l^2 + \frac{2\xi_n^2}{3r} \sum \rho_{\alpha(l)} \Delta m_l^3 \tag{9.42}$$

$$\rho_o z^2 = \frac{1}{3}\,\xi_n^2 \sum \rho_{\alpha(l)} \Delta m_l^3 + \frac{\xi_n^3}{2r} \sum \rho_{\alpha(l)} \Delta m_l^4 \tag{9.43}$$

9.4 Energy equations and constitutive relations of linear thermoelasticity for an initially spherical composite

In the absence of heat supply or heat absorption, the energy equations $(6.34)_{1,2}$ reduce to the following forms when written in spherical coordinates as defined by (9.1)

$$\frac{1}{r\sin\phi}\,[\frac{\partial}{\partial\phi}(\sin\phi\, q_\phi) + \frac{\partial q_\theta}{\partial\theta}] + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 h) + \rho_o(\phi_o\dot{\eta}_o + \phi_1\dot{\eta}_1) = 0 \tag{9.44}$$

$$\frac{1}{r\sin\phi}\,[\frac{\partial}{\partial\phi}(\sin\phi\, q_{1\phi}) + \frac{\partial q_{1\theta}}{\partial\theta}] + h - q_r + \rho_o(\phi_o\dot{\eta}_1 + \phi_1\dot{\eta}_2) = 0 \tag{9.45}$$

where q_ϕ, q_θ and q_r are the physical components of the heat flux vector for the composite laminate; $q_{1\phi}$ and $q_{1\theta}$ are the physical components of the composite heat flux moment, and other quantities have the same meaning as section 6. The constitutive relations for various composite quantities in thermoelasticity can be written similar to what was done in section 8.4 for initially cylindrical laminates. The mechanical parts of such constitutive equations were deived in section 9.3. Therefore in what follows we record only the thermal parts of these equations. The complete constitutive relations in linear thermoelasticity are obtained by adding these distinct parts. Using (7.10), (9.8) and (9.20), the thermal part of the contravariant stress tensor τ^{ij} is

$$\tau^{ij}(\text{thermal}) = -\phi_o\{J^{(0)ij} + \frac{2}{r}\,J^{(1)ij} + \Lambda^j_\beta(J^{(1)i\beta} + \frac{2}{r}\,J^{(2)i\beta})\}$$

$$-\phi_1\{J^{(1)ij} + \frac{2}{r}\,J^{(2)ij} + \Lambda^j_\beta(J^{(2)i\beta} + \frac{2}{r}\,J^{(3)i\beta})\} \tag{9.46}$$

or

BASE

$$\tau^{i1} = -\phi_o(J^{(0)i1} + \frac{3}{r} J^{(1)i1} + \frac{2}{r^2} J^{(2)i1}) - \phi_1(J^{(1)i1} + \frac{3}{r} J^{(2)i1} + \frac{2}{r^2} J^{(3)i1}) \qquad (9.47)$$

$$\tau^{i2} = -\phi_o(J^{(0)i2} + \frac{3}{r} J^{(1)i2} + \frac{2}{r^2} J^{(2)i2}) - \phi_1(J^{(1)i2} + \frac{3}{r} J^{(2)i2} + \frac{2}{r^2} J^{(3)i2}) \qquad (9.48)$$

$$\tau^{i3} = -\phi_o(J^{(0)i3} + \frac{2}{r} J^{(1)i3}) - \phi_1(J^{(1)i3} + \frac{2}{r} J^{(2)i3}) \qquad (9.49)$$

where the word (thermal) is dropped for brevity. Similarly the thermal parts of the composite couple stress are written by using (7.12), (9.8) and (9.20)

$$S_{\alpha\beta} = -\phi_o(J^{(1)\alpha\beta} + \frac{3}{r} J^{(2)\alpha\beta} + \frac{2}{r^2} J^{(3)\alpha\beta}) - \phi_1(J^{(2)\alpha\beta} + \frac{3}{r} J^{(3)\alpha\beta} + \frac{2}{r^2} J^{(4)\alpha\beta}) \qquad (9.50)$$

$$S^{\alpha 3} = -\phi_o(J^{(1)\alpha 3} + \frac{2}{r} J^{(2)\alpha 3}) - \phi_1(J^{(2)\alpha 3} + \frac{2}{r} J^{(3)\alpha 3}) \qquad (9.51)$$

The constitutive relations for composite entropy and its moments η_m, the heat flux vector $\mathbf{q}$ and its moment $\mathbf{q}_1$ are also derived by substituting from (9.8) and (9.20) in (7.17), (7.22) and (7.24)

$$\rho_o\eta_{(m)} = \gamma_{ij}(J^{(\alpha)ij} + \frac{2}{r} J^{(m+1)ij}) + \kappa_{i\beta}(J^{(m+1)i\beta} + \frac{2}{r} J^{(m+2)i\beta})$$

$$+ \phi_o(K^{(m)} + \frac{2}{r} K^{(m+1)}) + \phi_1(K^{(m+1)} + \frac{2}{r} K^{(m+2)}) \quad (m = 0,1,2) \qquad (9.52)$$

$$q^i = -(L^{(0)i\beta} + \frac{2}{r} L^{(1)i\beta})\phi_{o,\beta} - (L^{(1)i\beta} + \frac{2}{r} L^{(2)i\beta})\phi_{1,\beta} - (L^{(0)i3} + \frac{2}{r} L^{(1)i3})\phi_1 \qquad (9.53)$$

$$q_1^\alpha = -(L^{(1)\alpha\beta} + \frac{2}{r} L^{(2)\alpha\beta})\phi_{o,\beta} - {}^{(2)\alpha\beta} + \frac{2}{r} L^{(3)\alpha\beta})\phi_{1,\beta} - (L^{(1)\alpha 3} + \frac{2}{r} L^{(2)\alpha 3})\phi_1 \qquad (9.54)$$

where the thermal constitutive coefficients $J^{(k)ij}$, $K^{(k)}$ and $L^{(k)ij}$ are given in (7.6)-(7.8) and also in (7.30)-(7.32).

If the micro-structure is composed of isotropic spherical shells, the coefficients of thermal stress and thermal conductivity of each layer can be represented in terms of only one constant.

Writing (8.59) and (8.60) in spherical coordinates and then substituting in (7.30) and (7.32) we get the following results for the non-vanishing components of $J^{(k)ij}$ and $L^{(k)ij}$.

$$J^{(k)11} = \frac{1}{k+1}\,\frac{\xi_n^k}{r^2}\sum \beta_{(l)}\Delta m_l^{k+1} \tag{9.55}$$

$$J^{(k)22} = \frac{1}{k+1}\,\frac{\xi_n^k}{r^2\sin^2\phi}\sum \beta_{(l)}\Delta m_l^{k+1} \tag{9.56}$$

$$J^{(k)33} = \frac{1}{k+1}\,\xi_n^k\sum \beta_{(l)}\Delta m_l^{k+1} \tag{9.57}$$

$$L^{(k)11} = \frac{1}{k+1}\,\frac{\xi_n^k}{r^2}\sum k_{(l)}\Delta m_l^{k+1} \tag{9.58}$$

$$L^{(k)22} = \frac{1}{k+1}\,\frac{\xi_n^k}{r^2\sin^2\phi}\sum k_{(l)}\Delta m_l^{k+1} \tag{9.59}$$

$$L^{(k)33} = \frac{1}{k+1}\,\xi_n^k\sum k_{(l)}\Delta m_l^{k+1} \tag{9.60}$$

$\beta_{(l)}$'s and $k_{(l)}$'s are the coefficients of thermal stress and thermal conductivity of each layer and the summations are all extended over the micro-structure from $l = 1$ to $l = n$. Substituting from (9.55)-(9.57) in (9.47)-(9.49) we get the following expressions for the thermal parts of the physical components of the composite stress tensor.

$$\begin{aligned}\tau_{\phi\phi} = {}& -\phi_o \sum \beta_l\Big(\Delta m_l + \frac{3\xi_n}{2r}\,\Delta m_l^2 + \frac{2\xi_n^2}{3r^2}\,\Delta m_l^3\Big) \\ & -\phi_1 \sum \beta_l\Big(\frac{\xi_n}{2}\,\Delta m_l^2 + \frac{\xi_n}{2r}\,\Delta m_l^3 + \frac{\xi_n^3}{2r^2}\,\Delta m_l^4\Big)\end{aligned} \tag{9.61}$$

$$\tau_{\theta\phi} = \tau_{r\phi} = 0 \tag{9.62}$$

$$\tau_{\phi\theta} = \tau_{r\theta} = 0 \tag{9.63}$$

$$\tau_{\theta\theta} = -\phi_o \sum \beta_l(\Delta m_l + \frac{3\xi_n}{2r}\Delta m_l^2 + \frac{2\xi_n^2}{3r^2}\Delta m_l^3)$$

$$-\phi_1 \sum \beta_l(\frac{\xi_n}{2}\Delta m_l^2 + \frac{\xi_n}{2r}\Delta m_l^3 + \frac{\xi_n^3}{2r^2}\Delta m_l^4) = \tau_{\phi\phi} \qquad (9.64)$$

$$\tau_{\phi r} = \tau_{\theta r} = 0 \qquad (9.65)$$

$$\tau_{rr} = -\phi_o \sum \beta_l(\Delta m_l + \frac{\xi_n}{r}\Delta m_l^2) - \phi_1 \sum \beta_l(\frac{\xi_n}{2}\Delta m_l^2 + \frac{2\xi_n^2}{3r}\Delta m_l^3) \qquad (9.66)$$

The thermal part of the composite stress moment is also similarly calculated by substituting from (9.55)-(9.56) in (9.50) and (9.51).

$$S_{\phi\phi} = -\phi_o\, \beta_l(\frac{\xi_n}{2}\Delta m_l^2 + \frac{\xi_n}{2r}\Delta m_l^3 + \frac{\xi_n^3}{2r^2}\Delta m_l^4)$$

$$-\phi_1 \sum \beta_l(\frac{\xi_n^2}{3}\Delta m_l^3 + \frac{3\xi_n^3}{4r}\Delta m_l^4 + \frac{2\xi_n^4}{5r^2}\Delta m_l^5) \qquad (9.67)$$

$$S_{\phi\theta} = S_{\theta\phi} = 0 \qquad (9.68)$$

$$S_{\theta\theta} = S_{\phi\phi} \qquad (9.69)$$

$$S_{\phi r} = S_{\theta r} = 0 \qquad (9.70)$$

In order to find the appropriate forms of the constitutive equations for the entropy and its moments, heat flux and its moments we substitute from (9.55)-(9.57) and (9.58)-(9.60) and also (7.31) in (9.52)-(9.54) and obtain the following results

$$\rho_o\eta_{(0)} = (\gamma_{\phi\phi}+\gamma_{\theta\theta}+\gamma_{rr}) \sum \beta_l(\Delta m_l + \frac{\xi_n}{r}\Delta m_l^2)$$

$$+ (\kappa_{\phi\phi}+\kappa_{\theta\theta}) \sum \beta_l(\frac{\xi_n}{2}\Delta m_l^2 + \frac{2\xi_n^2}{3r}\Delta m_l^3)$$

$$+ \phi_o \sum (\rho c)_l(\Delta m_l + \frac{\xi_{sn}}{r} \Delta m_l^2) + \phi_1 \sum (\rho c)_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r} \Delta m_l^3) \quad (9.71)$$

$$\rho_o \eta_{(1)} = (\gamma_{\phi\phi}+\gamma_{\theta\theta}+\gamma_{rr}) \sum \beta_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r} \Delta m_l^3)$$

$$+ (\kappa_{\phi\phi}+\kappa_{\theta\theta}) \sum \beta_l(\frac{\xi_{sn}^2}{3} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r} \Delta m_l^4)$$

$$+ \phi_o \sum (\rho c)_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r} \Delta m_l^3) + \phi_1 \sum (\rho c)_l(\frac{\xi_{sn}^2}{3} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r} \Delta m_l^4) \quad (9.72)$$

$$\rho_o \eta_{(2)} = (\gamma_{\phi\phi}+\gamma_{\theta\theta}+\gamma_{rr}) \sum \beta_l(\frac{\xi_{sn}^2}{3} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r} \Delta m_l^4)$$

$$+ (\kappa_{\phi\phi}+\kappa_{\theta\theta}) \sum \beta_l(\frac{\xi_{sn}^3}{4} \Delta m_l^4 + \frac{2\xi_{sn}^4}{5r} \Delta m_l^5)$$

$$+ \phi_o \sum (\rho c)_l(\frac{\xi_{sn}^2}{3} \Delta m_l^3 + \frac{\xi_{sn}^3}{2r} \Delta m_l^4) + \phi_1 \sum (\rho c)_l(\frac{\xi_{sn}^3}{4} \Delta m_l^4 + \frac{2\xi_{sn}^4}{5r} \Delta m_l^5) \quad (9.73)$$

$$q_\phi = -\frac{1}{r} \frac{\partial \phi_o}{\partial \phi} \sum k_l(\Delta m_l + \frac{\xi_{sn}}{r} \Delta m_l^2)$$

$$-\frac{1}{r} \frac{\partial \phi_1}{\partial \phi} \sum k_l(\Delta m_l^2 \frac{\xi_{sn}}{2} + \frac{2\xi_{sn}^2}{3r} \Delta m_l^3) \quad (9.74)$$

$$q_\theta = -\frac{1}{r \sin \phi} \frac{\partial \phi_o}{\partial \theta} \sum k_l(\Delta m_l + \frac{\xi_{sn}}{r} \Delta m_l^2)$$

$$-\frac{1}{r \sin \phi} \frac{\partial \phi_1}{\partial \theta} \sum k_l(\frac{\xi_{sn}}{2} \Delta m_l^2 + \frac{2\xi_{sn}^2}{3r} \Delta m_l^3) \quad (9.75)$$

$$q_r = -\phi_1 \sum k_l(\Delta m_l + \frac{\xi_{sn}}{r} \Delta m_l^2) \quad (9.76)$$

$$q_{1\phi} = -\frac{1}{r}\frac{\partial\phi_o}{\partial\phi}\sum k_l\left(\frac{\xi_n}{2}\Delta m_l^2 + \frac{2\xi_n^2}{3r}\Delta m_l^3\right)$$

$$-\frac{1}{r}\frac{\partial\phi_1}{\partial\phi}\sum k_l\left(\frac{\xi_n^2}{3}\Delta m_l^3 + \frac{\xi_n^3}{2r}\Delta m_l^4\right) \qquad (9.77)$$

$$q_{1\theta} = -\frac{1}{r\sin\phi}\frac{\partial\phi_o}{\partial\theta}\sum k_l\left(\frac{\xi_n}{2}\Delta m_l^2 + \frac{2\xi_n^2}{3r}\Delta m_l^3\right)$$

$$-\frac{1}{r\sin\phi}\frac{\partial\phi_1}{\partial\theta}\sum k_l\left(\frac{\xi_n^2}{3}\Delta m_l^3 + \frac{\xi_n^3}{2r}\Delta m_l^4\right) \qquad (9.78)$$

It should be recalled that in the above relations, ϕ_1 is the gradient of ϕ_o in the θ^3, i.e., r-direction.

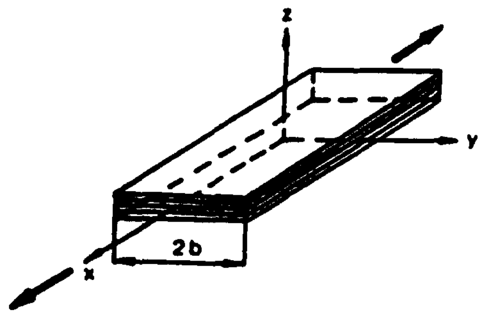
10.0 ANALYSIS OF COMPOSITE LAMINATES FOR IN-PLANE LOADING

10.1 Introduction

In the previous sections a complete thermomechanical theory of composite laminates was developed. In this section the results of stress analysis of composite laminates with traction-free edges are presented. Composite laminates with traction-free edges are known to develop inter-laminar stress concentrations near the edge region. The problem of a finite width, symmetrically laminated composite plate under uniform one-dimensional stretch has been studied by many authors, see Section 1. Pagano and Pipes (1970, 1973) showed that free edge effects on the interlaminar stresses are important issues in determining the failure and the strength of such laminates. Their analytical work was based on linear elastic, generalized plane-strain formulation and numerical solutions were obtained using a finite different procedure. Their study revealed that certain interlaminar stresses rise in magnitude near the free-edge region. It was suggested that a possible stress singularity exists at the free edge. A. S. D. Wang (1977) followed a finite element scheme to investigate the same problem with emphasis placed on assessing in detail the stress field closest to the ply interfaces and laminate's free edge, where stress singularity is suspected. In this section the analysis of the same problem based on the theory developed in previous sections is presented. A finite difference scheme was adopted for the solution of governing partial differential equations. The objective of this numerical modeling study was to examine the three-dimensional state of stress at the free edges of composite laminates and to show that the proposed theory is in agreement with recorded experimental data.

10.2 Free Edge Boundary Value Problem

Consider a prismatic symmetric laminate shown in Figure 10.1.



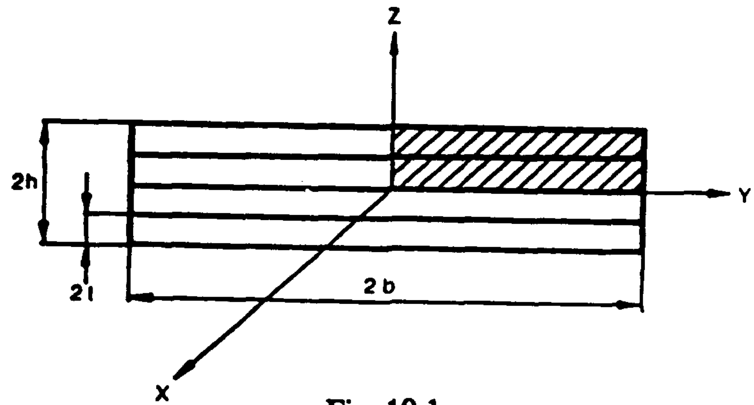


Fig. 10.1

The corresponding boundary-value problem for a uniform strain field in x-direction and traction-free edges at $y = \pm b$, and top and bottom surfaces ($z = \pm h$) were derived based on the linearized field equations (2.126) and (2.127) and constitutive relations (3.24) and (3.31). It was assumed that representative elements are made of orthotropic plies. The stress-strain relations of each ply in a coordinate system with major axis along fiber direction is:

$$\tau^* = C\gamma \tag{10.1}$$

where

$$\tau_1^* = \tau_{11}^* \quad , \quad \tau_4^* = \tau_{23}^*$$

$$\tau_2^* = \tau_{22}^* \quad , \quad \tau_5^* = \tau_{13}^*$$

$$\tau_3^* = \tau_{33}^* \quad , \quad \tau_6^* = \tau_{12}^*$$

$$\gamma_1 = u_{1,1} \quad , \quad \gamma_4 = u_{2,3} + u_{3,2}$$

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$$\gamma_2 = u_{2,2} \quad , \quad \gamma_5 = u_{1,3} + u_{3,1}$$

$$\gamma_3 = u_{3,3} \quad , \quad \gamma_6 = u_{1,2} + u_{2,1}$$

and C is a (6 × 6) matrix with elements related to the nine material constants of each ply, i.e., extensional elastic moduli in three directions, shear moduli and Poisson ratios in corresponding directions as shown in [Whitney, 1989]. The constitutive relations for composite stress and composite stress couple were derived by rotation of coordinate system in (10.1) to coincide with the direction of axial loading and the integration of these relations across the thickness of the representative element as shown in equation (3.24) and (3.31). For small deformations of flat composites, the constraint relation (4.20) was employed, i.e.,

$$\delta(\theta^\alpha,\theta^3,t) = \frac{\partial u(\theta^\alpha,\theta^3)}{\partial \theta^3} \tag{10.2}$$

The final form of constitutive relations for the composite stress and the composite stress couple assumed the following presentation:

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{C}\boldsymbol{\gamma} + \mathbf{D}\boldsymbol{\kappa} \\ \mathbf{S} &= \mathbf{D}\boldsymbol{\gamma} + \mathbf{F}\boldsymbol{\kappa} \end{aligned} \tag{10.3}$$

where

$$\begin{aligned} \mathbf{C} &= m\mathbf{C}_{(1)} + (1-m)\mathbf{D}_{(2)} \\ \mathbf{D} &= \frac{\xi_2}{2}\,[m^2\mathbf{C}_{(1)} + (1-m^2)\mathbf{C}_{(2)}] \\ \mathbf{F} &= \frac{\xi_2^2}{3}\,[m^3\mathbf{C}_{(1)} + (1-m^3)\mathbf{C}_{(2)}] \end{aligned} \tag{10.4}$$

$$\kappa_1 = u_{1,13} \quad , \quad \kappa_4 = u_{3,23}$$

$$\kappa_2 = u_{2,23} \quad , \quad \kappa_5 = u_{3,13}$$

$$\kappa_3 = 0 \quad , \quad \kappa_6 = u_{1,23} + u_{2,13}$$

In (10.4), $C_{(1)}$ and $C_{(2)}$ are corresponding C matrices for each constituent (or for each fiber direction) present in the representative element and ξ_2 is the thickness of representative element and m is defined in (3.15). The linearized field equations (2.126) and (2.127) for a static loading and in the absence of body force are:

$$\begin{aligned} &\tau^{\alpha j}{}_{,\alpha} + \sigma^j{}_{,3} = 0 \\ &S^{\alpha j}{}_{,\alpha} + \sigma^j - \tau^{3j} = 0 \end{aligned} \tag{10.5}$$

Elimination of the interlaminar stress vector σ^j from these equations resulted in:

$$\tau^{ij}{}_{,i} - S^{\alpha j}{}_{,\alpha 3} = 0 \tag{10.6}$$

For an axially loaded strip, the stress and stress couple components were taken to be independent of the axial direction. Consequently the general form of the displacement field was assumed as:

$$\begin{aligned} u_1 &= \alpha x_1 + U(x_2,x_3) \\ u_2 &= V(x_2,x_3) \\ u_3 &= W(x_1,x_3) \end{aligned} \tag{10.7}$$

Identifying direction 1 with x, direction 2 with y and direction 3 with z, the field equation (10.6) reduces to the following set of partial differential equations:

$$\begin{aligned} &C_{66}U_{,yy} + C_{55}U_{,zz} + C_{26}V_{,yy} + C_{45}V_{,zz} + (C_{36}+C_{45})W_{,yz} \\ &\quad + (D_{45}-D_{36})W_{,yzz} - F_{66}U_{,yyzz} - F_{26}V_{,yyzz} = 0 \\ &C_{26}U_{,yy} + C_{45}U_{,zz} + C_{22}V_{,yy} + C_{44}V_{,zz} + (C_{23}+C_{44})W_{,yz} \\ &\quad + (D_{44}-D_{23})W_{,yzz} - F_{26}U_{,yyzz} - F_{66}V_{,yyzz} = 0 \end{aligned} \tag{10.8}$$

$$(C_{45}+C_{36})U_{,yz} + (C_{44}+C_{23})V_{,yz} + C_{44}W_{,yy} + C_{33}W_{,zz}$$

$$+ (D_{36}-D_{45})U_{,yzz} + (D_{23}-D_{44})V_{,yzz}$$

$$- F_{44}W_{,yyzz} = 0$$

where C_{ij}, D_{ij} and F_{ij} are the components of C, D and F matrices. The domain and the boundary conditions for these equations are shown in Figure 10.2.

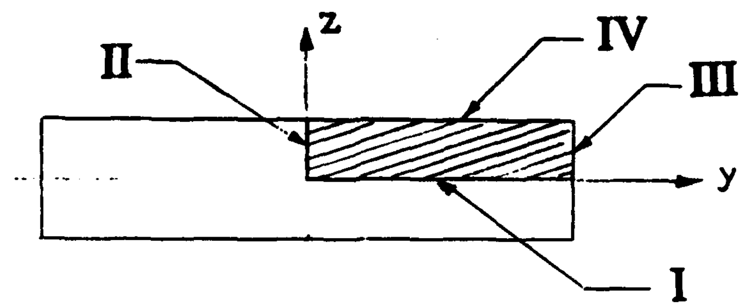

Figure 10.2

$$\text{For I: } \begin{cases} U_{,z} = 0 \\ V_{,z} = 0 \\ W=0 \end{cases} \quad \text{For II: } \begin{cases} U_{,y} = 0 \\ W_{,y} = 0 \\ V=0 \end{cases}$$

$$\text{For III: } \begin{cases} \tau_{21} = 0 \\ \tau_{22} = 0 \\ \tau_{23} = 0 \end{cases} \quad \text{For IV: } \begin{cases} \sigma_1 = 0 \\ \sigma_2 = 0 \\ \sigma_3 = 0 \end{cases}$$

The following material properties were used for the analysis:

$$E_x = 48 \times 10^4 \text{psi}$$

$$E_y = E_z = 5 \times 10^4 \text{psi}$$

$$G_{xy} = G_{yz} = G_{zx} = 2 \times 10^4 \text{psi}$$

$$\nu_{xy} = \nu_{zy} = \nu_{xz} = 0.21$$

A rectangular mesh of 41 × 11 nodes was used for the finite difference discretization and a four layer symmetry laminate under a uniform axial strain $\gamma_x = 0.01$ was considered. Each layer itself could be a collection of thin plies repeated in a consistent pattern. The presented results are for the case that all plies in the top and bottom layers are in the $+\theta$ direction and all plies in the two middle layers are in the $-\theta$ direction providing a $[\pm\theta]_s$ laminate. Complete stress and displacement results were obtained for various values of θ. These results are presented in Figures 10.3 through 10.13 and summarized in the following.

10.3 Results of Finite Difference Simulation

The purpose of this simulation was to examine the response of composite laminates under uniaxial extension and to show that the proposed theory reflects the complex three-dimensional response of free edge problem in composite laminates as recorded in the literature. Following this verification, a systematic discretization technique in the context of finite element method was developed and extensive analyses simulating various flat and curved composite laminates under in-plane and out-of-plane loading were performed.

In Figures 10.3 through 10.9 various components of stress tensor and interlaminar stress vector for $[\pm30]_s$, $[\pm45]_s$ and $[\pm60]_s$ laminates are plotted along the symmetry line of the laminate. Figure 10.3 is the axial stress which shows a decrease at the free edge. For $[\pm30]_s$ lay-up this decrease is about 50% of stress at the centerline $y = 0$. Figure 10.4 is the composite shear stress τ_{xy} which assumes its maximum for $\theta = 30°$ and approaches zero at the free edge.

Figure 10.5 is the normal in-plane composite stress in the y direction, perpendicular to the loading axis. The value of this stress component is negligible as compared to the axial stress, about 0.3% of the axial stress.

Figure 10.6 shows the interlaminar normal stress in z direction. The value of this stress component assumes its maximum on the free-edge boundary.

Figures 10.7 and 10.8 are the interlaminar shear stress components. The magnitude of the interlaminar shear stress along the x-direction is in the same order as normal interlaminar stress and it increases with a high gradient as it approaches the free edge. The magnitude of the yz component of shear stress at the centerline is negligible compared to other components.

Figure 10.10 shows the variation of interlaminal shear stress τ_{xz} for various values of fiber direction θ.

Figures 10.11 through 10.13 present the various stress components across the thickness of the laminate. The shear stress τ_{xy} shows very small variation across the thickness. The normal interlaminar stress assumes its maximum along the symmetry line and it approaches zero on the top and bottom surfaces of the laminate. The normal stress τ_{yy} assumes its pick on the top and bottom layers. Its value at the symmetry surface is not zero but it is considerably smaller than those values at top and bottom surfaces.

A more detailed study of this problem is presented in Chapter 14 based on a finite element scheme. In particular it is discussed that even for a symmetric laminate, the problem of in-plane loading of a finite width strip is a three-dimensional problem and should be modeled accordingly.

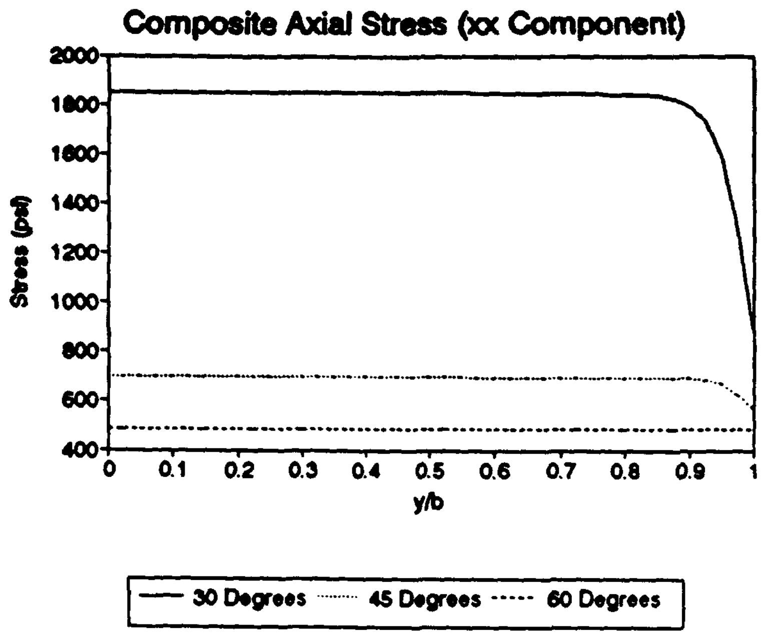

Figure 10.3
Extension Analysis — Axial Stress (τ_{xx})

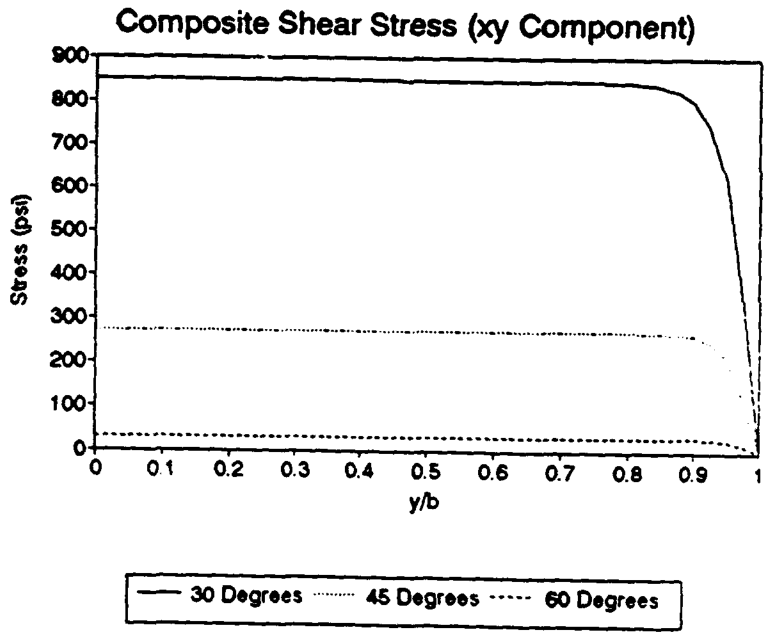

Figure 10.4
Extension Analysis — Shear Stress (τ_{xy})

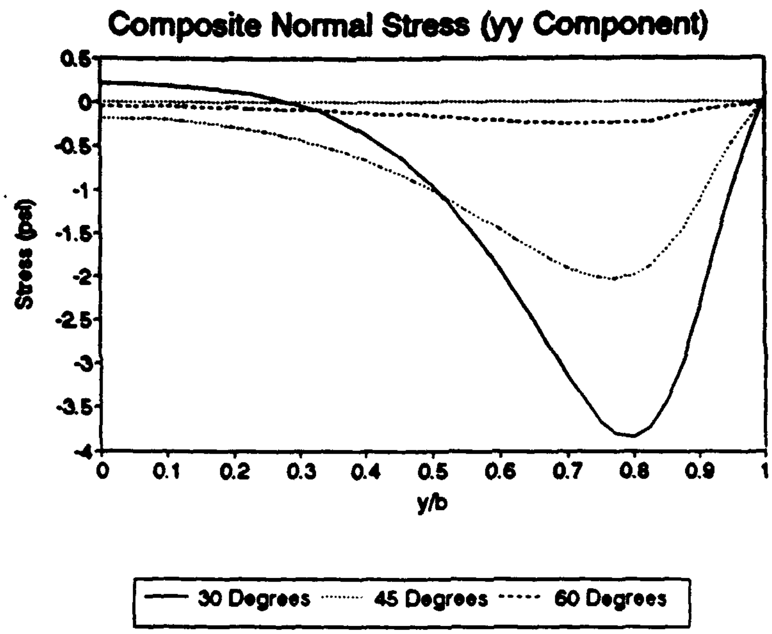

Figure 10.5
Extension Analysis — Normal Stress (τ_{yy})

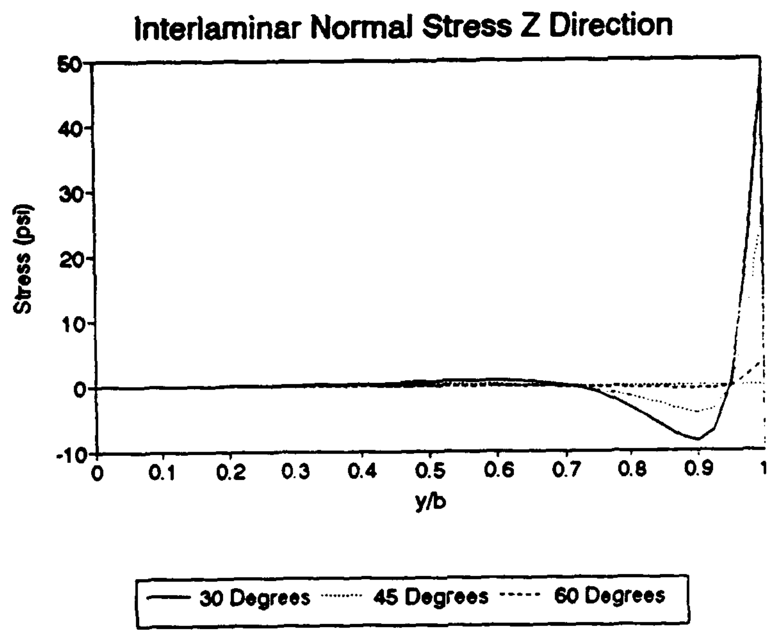

Figure 10.6
Extension Analysis — Interlaminar Normal Stress (σ_3)

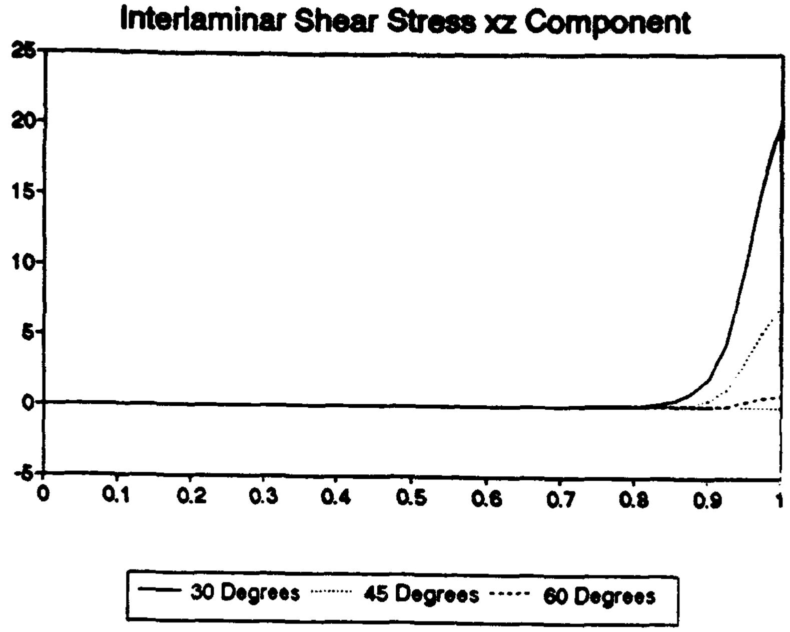

Figure 10.7
Extension Analysis — Interlaminar Shear (σ_1)

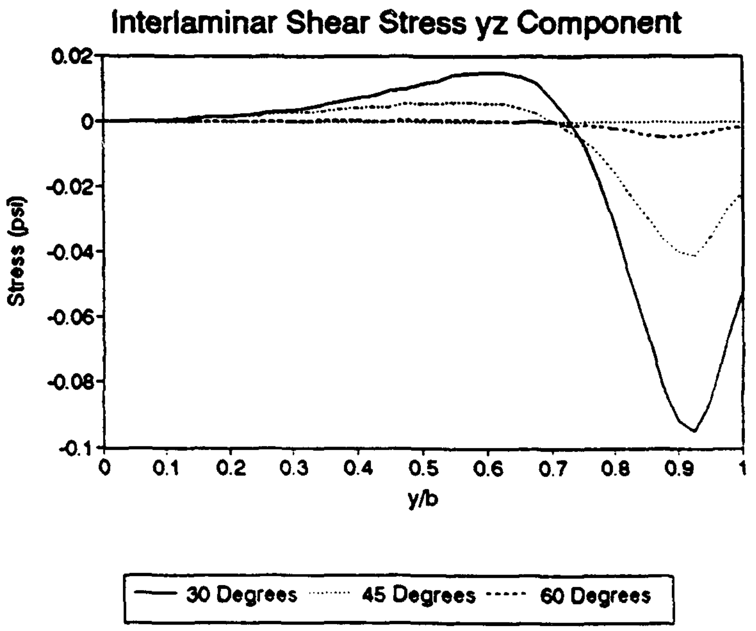

Figure 10.8
Extension Analysis — Interlaminar Shear (σ_2)

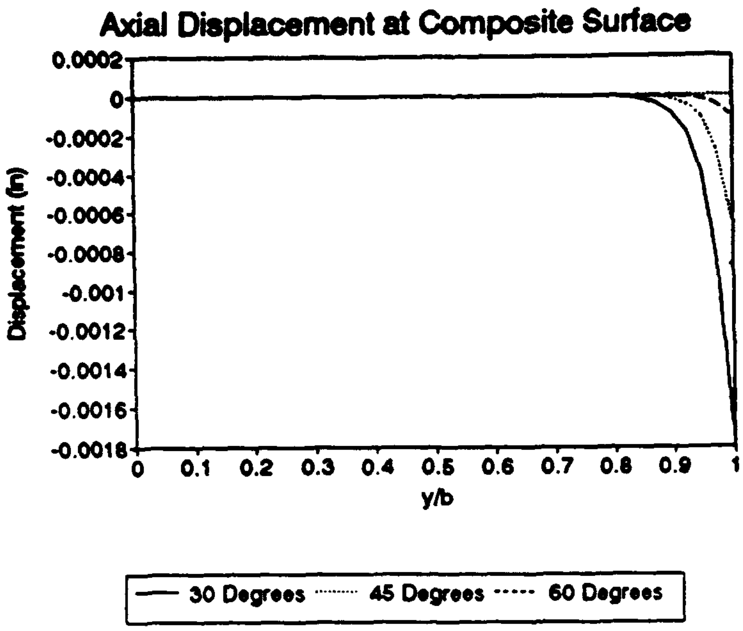

Figure 10.9
Extension Analysis — U(y,z)

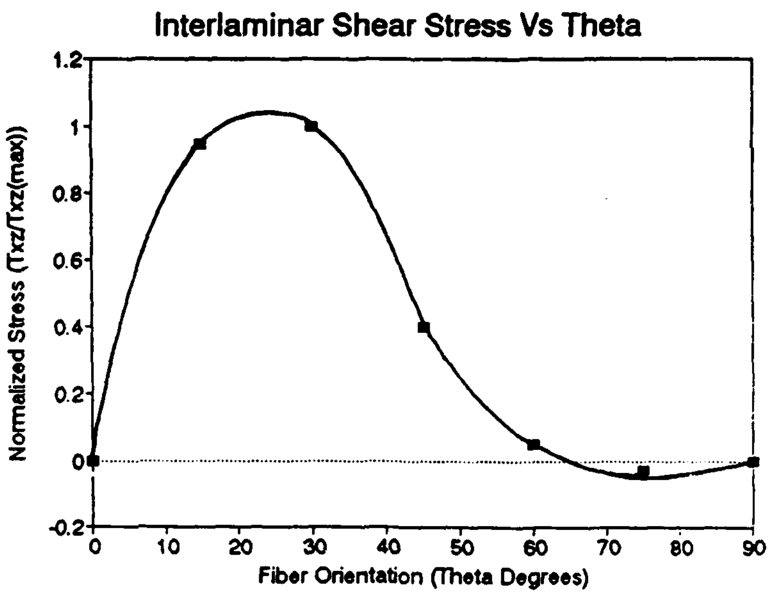

Figure 10.10
Effect of Fiber Orientation

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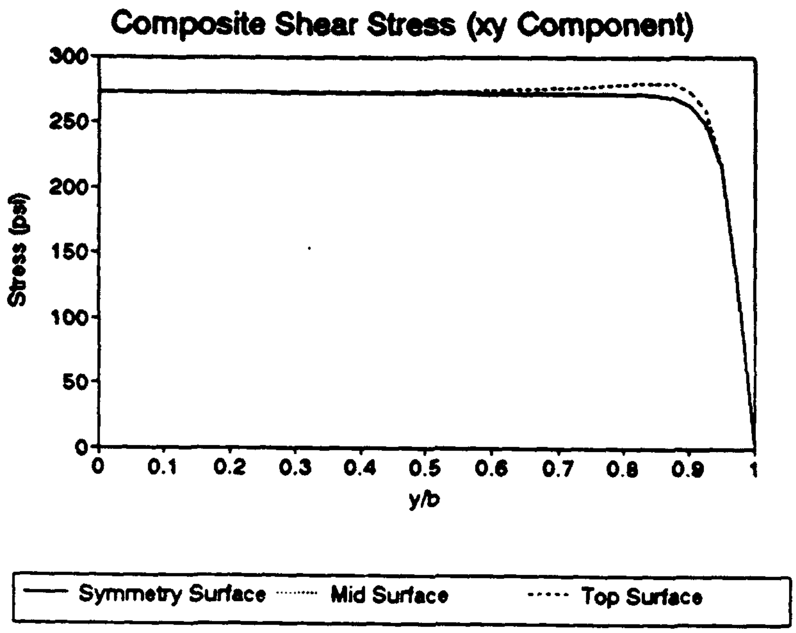

Figure 10.11
Variation through-the-thickness (τ_{xy})

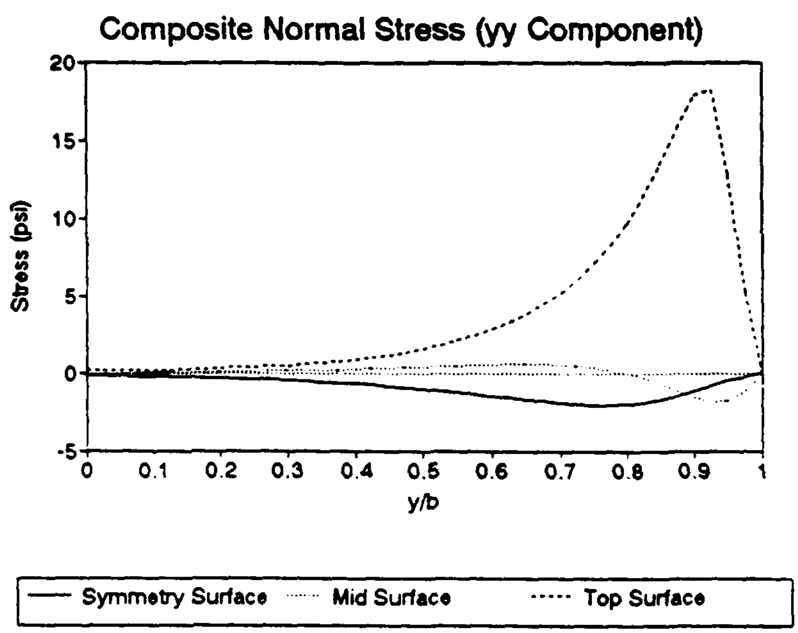

Figure 10.12
Variation through-the-thickness (τ_{yy})

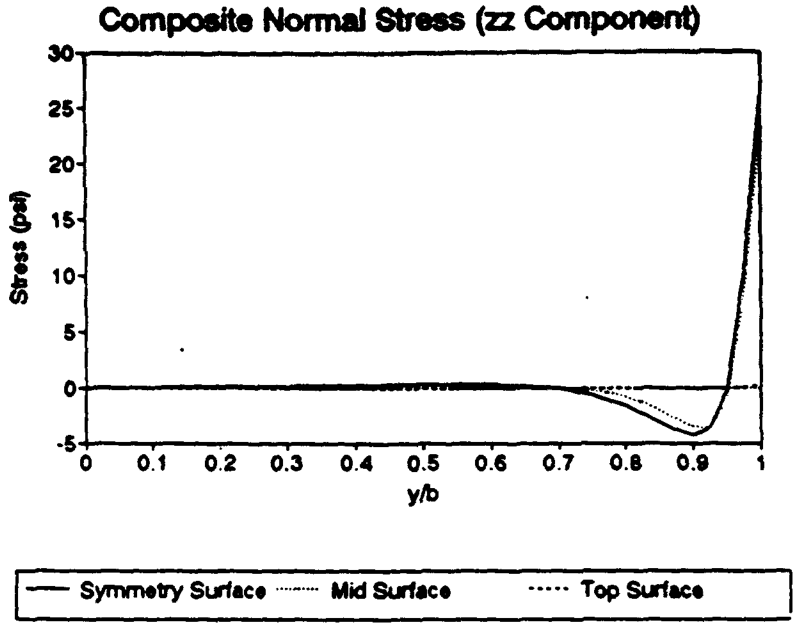

Figure 10.13
Variation through-the-thickness (τ_{zz})

11.0 WAVE MOTIONS IN LAMINATED FLAT COMPOSITES

Using relations (5.38)-(5.42), the linear equations of motion for a laminated composite with initially flat plies, in the absence of body forces, can be written as

$$I^{(0)}_{\alpha jkl}u_{k,l\alpha} + I^{(1)}_{\alpha jl\beta}u_{l,\alpha\beta 3} + \sigma_{j,3} = \rho_o\ddot{u}_j + \rho_o z^1\ddot{u}_{j,3} \tag{11.1}$$

$$I^{(1)}_{\alpha jkl}u_{k,l\alpha} + I^{(2)}_{\alpha jl\beta}u_{l,\alpha\beta 3} + \sigma_j - I^{(0)}_{3jkl}u_{k,l} - I^{(1)}_{3jl\beta}u_{l,\beta 3} = \rho_o z^1\ddot{u}_j + \rho_o z^2\ddot{u}_{j,3} \tag{11.2}$$

Eliminating σ_j between these equations we get

$$I^{(0)}_{ijkl}u_{k,li} + I^{(1)}_{ijk\beta}u_{k,i\beta 3} - I^{(1)}_{\alpha jkl}u_{k,l\alpha 3} - I^{(2)}_{\alpha jk\beta}u_{k,\alpha\beta 33} = \rho_o\ddot{u}_j - \rho_o z^2\ddot{u}_{j,33} \tag{11.3}$$

The equations (11.3) are the differential equations for the displacement vector **u** in elasto-dynamical problems. These equations are now used to investigate the propagation of small amplitude harmonic waves in a laminated composte. In the following special cases that we examine, k is the wave number and c the phase velocity of the appropriate wave

(a) Longitudinal Waves in the x_1-direction

For waves of this type the non-zero displacement component is u_1 and we have

$$u_1 = A_1 \exp[ik(x_1 - ct)] \tag{11.4}$$

where A_1 is the constant wave amplitude and assumed to be small. Differentiating (11.4) with respect to x_1 and t we get

$$\ddot{u}_1 = -k^2c^2u_1 \quad , \quad u_{1,11} = -k^2u_1 \tag{11.5}$$

Substituting (11.5) in (11.3) we obtain

$$c^2 = \frac{I^{(0)}_{1111}}{\rho_o} \tag{11.6}$$

For a composite whose micro-structure is composed of n isotropic layers, by (5.27) and $(5.26)_1$

$(5.26)_1$ we have

$$I^{(0)}_{1111} = \sum_{r=1}^{n} (\lambda_{(r)} + 2\mu_{(r)})\Delta m_r \tag{11.7}$$

Substituting this result together with (5.38) in (11.6), the wave speed c for a longitudinal wave would be

$$c^2 = \frac{\sum_{r=1}^{n}(\lambda_{(r)}+2\mu_{(r)})\Delta m_r}{\sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r} \tag{11.8}$$

(b) Horizontally polarized shear waves in the x_1-direction

For this type of waves the non-zero displacement component is u_2 and we have

$$u_2 = A_L \exp[ik(x_1-ct] \tag{11.9}$$

Substituting from (11.9) in (11.3) we get the following expression for the wave velocity

$$c^2 = \frac{I^{(0)}_{1212}}{\rho_o} \tag{11.10}$$

which for the special case of isotropic laminates reduces to

$$c^2 = \frac{\sum_{r=1}^{n} \mu_{(r)}\Delta m_r}{\sum_{r=1}^{n} \rho_o^{(r)}\Delta m_r} \tag{11.11}$$

(c) Vertically polarized shear waves in the x_1-direction

The non-zero displacement component for this wave is u_3 and we have

$$u_3 = A_3 \exp[ik(x_1 - ct)] \tag{11.12}$$

The wave velocity for this case, similar to the above cases, is found and we obtain

$$c^2 = \frac{I^{(0)}_{1313}}{\rho_o} \tag{11.13}$$

Again for the special case of isotropic laminates we get

$$c^2 = \frac{\sum_{r=1}^{n} \mu_{(r)} \Delta m_r}{\sum_{r=1}^{n} \rho_o^{(1)} \Delta m_r} \tag{11.14}$$

(d) Longitudinal waves in the x_3-direction

In this case the non-zero displacement component is u_3 and we have

$$u_3 = B_3 \exp[ik(x_3 - ct)] \tag{11.15}$$

These waves, unlike the above three cases, are dispersive and the wave speed depends on frequency. The non-zero space and time derivatives of (11.13) which are relevant to (11.3) are

$$\ddot{u}_3 = -k^7 c^2 u_3 \quad , \quad u_{3,33} = -k^7 u_3 \quad , \quad \ddot{u}_{3,33} = k^4 c^2 u_3 \tag{11.16}$$

Substituting from (11.16) in (11.3) we get

$$I^{(0)}_{3333} = \rho_o c^2 + \rho_o z^2 c^2 k^2 \tag{11.17}$$

If we introduce the wave frequency $\omega = ck$ in (11.17) we get

$$c^2 = \frac{I^{(0)}_{3333} - \rho_o z^2 \omega^2}{\rho_o} \tag{11.18}$$

For the case of a composite whose micro-structure is composed of n isotropic layers, this relation reduces to

$$c^2 = \frac{\sum_{r=1}^{n}(\lambda_{(r)}+2\mu_{(r)})\Delta m_r - \frac{(\omega\xi_n)^2}{3}\sum_{r=1}^{n}\rho_o^{(r)}\Delta m_r^3}{\sum_{r=1}^{n}\rho_o^{(r)}\Delta m_r} \qquad (11.19)$$

(e) Transverse shear waves in the x_3-direction

In this case we consider a transverse shear wave propagating normal to the laminates with its amplitude in the x_1-direction. Consequently the only non-zero displacement component will be u_1 and we have

$$u_1 = B_1 \exp[ik(x_3 - ct)] \qquad (11.20)$$

Here again, the phase velocity c is obtained similar to the case (d). The result is

$$c^2 = \frac{I_{1313}^{(0)} - \rho_o z^2 \omega^2}{\rho_o} \qquad (11.21)$$

which shows the dependence of c on frequency ω. For the case of isotropic laminates (11.21) can be written as

$$c^2 = \frac{\sum_{r=1}^{n}\mu_{(r)}\Delta m_r - \frac{(\omega\xi_n)^2}{3}\sum_{r=1}^{n}\rho_o^{(r)}\Delta m_r^3}{\sum_{r=1}^{n}\rho_o^{(r)}\Delta m_r} \qquad (11.22)$$

It should be noted that the phase velocity of a shear wave propagating in the x_3-direction with amplitude in the x_1-direction can be obtained by substituting $I_{2323}^{(0)}$ in place of $I_{1313}^{(0)}$ in the relation (11.21). The general solution of a transverse shear wave propagating normal to the laminates is the sum of these solutions.

12.0 WAVE MOTIONS IN CYLINDRICAL AND SPHERICAL LAMINATES

The results of sections 8 and 9 are used to derive equations of motion in terms of displacement vector **u** for the cylindrical and spherical laminates. In each of the two cases the micro-structure is supposed to consist of n isotropic layers with different elastic constants. Using the same notations as previous sections we define the constitutive coefficients $\lambda^{(r)}$ and $\mu^{(r)}$ according to the following relations:

$$\lambda^{(r)} = \xi_n^{r-1} \sum \lambda_l \Delta m_l^r \qquad (12.1)$$

$$(r = 1,2,\ldots)$$

$$\mu^{(r)} = \xi_n^{r-1} \sum \mu_l \Delta m_l^r \qquad (12.2)$$

where summations extend over the micro-structure from $l = 1$ to $l = n$. For future reference we further introduce the quantities $\rho_o^{(r)}$ related to the densities of different layers according to the following relations

$$\rho_o^{(r)} = \xi_n^{r-1} \sum \rho_{o(l)} \Delta m_l^r \quad (r = 1,2,\ldots) \qquad (12.3)$$

where summation again extends over the micro-structure from $l = 1$ to $l = n$. The quantities $\lambda^{(r)}$, $\mu^{(r)}$ and $\rho_o^{(r)}$ defined in (12.1)-(12.3) are known a priori for each composite laminate.

12.1 Governing Equations for Cylindrical Laminates

Using the results of section 8, the equations of motion are derived for axial symmetry. In other words we will study motions which are independent of the axial coordinate z and the angular coordinate θ. With these specifications we will not have variations in $\theta^1 = \theta$ and $\theta^3 = z$ directions. From relations (8.12) and (8.13) we calculate the physical components of the relative kinematic measures:

$$\gamma_{\theta\theta} = \frac{u_r}{r} \ , \ \gamma_{\theta z} = 0 \ , \ \gamma_{\theta r} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\gamma_{zz} = 0 \quad , \quad \gamma_{rz} = \frac{1}{2}\frac{\partial u_z}{\partial r} \quad , \quad \gamma_{rr} = \frac{\partial u_r}{\partial r}$$

$$\kappa_{\theta\theta} = \frac{1}{r^2}\frac{\partial}{\partial r}(ru_r) \quad , \quad \kappa_{\theta z} = 0 \quad , \quad \kappa_{z\theta} = 0 \tag{12.4}$$

$$\kappa_{zz} = 0 \quad , \quad \kappa_{r\theta} = -\frac{1}{r}\frac{\partial u_\theta}{\partial r} \quad , \quad \kappa_{rz} = 0$$

The equation of motion in terms of the physical components of the composite stress tensor and the composite stress couple are derived using relations (8.15) and (8.16). These equations are written in the absence of body force and body couple.

$$\frac{1}{r}\tau_{\theta r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial r} = \rho_o(\ddot{u}_\theta + z^1\ddot{\delta}_\theta)$$

$$\frac{1}{r}\frac{\partial \sigma_z}{\partial r} = \rho_o(\ddot{u}_z + z^1\ddot{\delta}_z)$$

$$-\frac{1}{r}\tau_{\theta\theta} + \frac{1}{r}\frac{\partial \sigma_r}{\partial r} = \rho_o(\ddot{u}_r + z^1\ddot{\delta}_r)$$

$$\frac{1}{r}s_{\theta r} + \frac{1}{r}\sigma_\theta - \tau_{r\theta} = \rho_o(z^1\ddot{u}_\theta + z^2\ddot{\delta}_\theta) \tag{12.5}$$

$$\frac{1}{r}\sigma_z - \tau_{rz} = \rho_o(z^1\ddot{u}_z + z^2\ddot{\delta}_z)$$

$$-\frac{1}{r}s_{\theta\theta} + \frac{1}{r}\sigma_r - \tau_{rr} = \rho_o(z^1\ddot{u}_r + z^2\ddot{\delta}_r)$$

The constitutive relations for various components of the composite stress tensor and the composite stress couple are derived using relations (8.29)-(8.43) along with definitions (12.1) and (12.2)

$$\tau_{\theta\theta} = \{\lambda^{(1)} + 2\mu^{(1)} + \frac{1}{r}(\lambda^{(2)} + 2\mu^{(2)}) + \frac{1}{3r^2}(\lambda^{(3)} + 2\mu^{(3)})\}\gamma_{\theta\theta}$$

$$+ (\lambda^{(1)} + \frac{1}{r}\lambda^{(2)} + \frac{1}{3r^2}\lambda^{(3)})\gamma_{rr}$$

$$+ \{\frac{1}{2}(\lambda^{(2)} + \mu^{(2)}) + \frac{2}{3r}(\lambda^{(3)} + 2\mu^{(3)}) + \frac{1}{4r^2}(\lambda^{(4)} + 2\mu^{(4)})\}\kappa_{\theta\theta} \tag{12.6}$$

$$\tau_{r\theta} = 2(\mu^{(1)} + \frac{1}{r}\mu^{(2)} + \frac{1}{3r^2}\mu^{(3)})\gamma_{r\theta} + (\frac{1}{2}\mu^{(2)} + \frac{2}{3r}\mu^{(3)} + \frac{1}{4r^2}\mu^{(4)})\kappa_{r\theta} \tag{12.7}$$

$$\tau_{zz} = (\lambda^{(1)} + \frac{1}{2r}\lambda^{(2)})(\gamma_{rr} + \gamma_{\theta\theta}) + (\frac{1}{2}\lambda^{(2)} + \frac{1}{3r}\lambda^{(3)})\kappa_{\theta\theta} \tag{12.8}$$

$$\tau_{rz} = 2(\mu^{(1)} + \frac{1}{2r}\mu^{(2)})\gamma_{rz} \tag{12.9}$$

$$\tau_{\theta r} = 2(\mu^{(1)} + \frac{1}{2r}\mu^{(2)})\gamma_{\theta r} + (\frac{1}{2}\mu^{(2)} + \frac{1}{3r}\mu^{(3)})\kappa_{r\theta} \tag{12.10}$$

$$\tau_{zr} = 2(\mu^{(1)} + \frac{1}{2r}\mu^{(2)})\gamma_{zr} \tag{12.11}$$

$$\tau_{rr} = \{\lambda^{(1)} + 2\mu^{(1)} + \frac{1}{2r}(\lambda^{(2)} + 2\mu^{(2)})\}\gamma_{rr}$$

$$+ (\lambda^{(1)} + \frac{1}{2r}\lambda^{(2)})\gamma_{\theta\theta} + (\frac{1}{2}\lambda^{(2)} + \frac{1}{3r}\lambda^{(3)})\kappa_{\theta\theta} \tag{12.12}$$

$$s_{\theta\theta} = \{\frac{1}{2}(\lambda^{(2)} + 2\mu^{(2)}) + \frac{2}{3r}(\lambda^{(3)} + 2\mu^{(3)}) + \frac{1}{4r^2}(\lambda^{(4)} + 2\mu^{(4)})\}\gamma_{\theta\theta}$$

$$+ (\frac{1}{2}\lambda^{(2)} + \frac{2}{3r}\lambda^{(3)} + \frac{1}{4r^2}\lambda^{(4)})\gamma_{rr}$$

$$+ \{\frac{1}{3}(\lambda^{(3)} + 2\mu^{(3)}) + \frac{1}{2r}(\lambda^{(4)} + 2\mu^{(4)}) + \frac{1}{5r^2}(\lambda^{(5)} + 2\mu^{(5)})\}\kappa_{\theta\theta} \tag{12.13}$$

$$s_{\theta r} = 2(\frac{1}{2}\,\mu^{(2)} + \frac{1}{3r}\,\mu^{(3)})\gamma_{\theta r} + (\frac{1}{3}\,\mu^{(3)} + \frac{1}{4r}\,\mu^{(4)})\kappa_{r\theta} \tag{12.14}$$

$$s_{zz} = (\frac{1}{2}\,\lambda^{(2)} + \frac{1}{3r}\,\lambda^{(3)})(\gamma_{\theta\theta} + \gamma_{rr}) + (\frac{1}{3}\,\lambda^{(3)} + \frac{1}{4r}\,\lambda^{(4)})\kappa_{\theta\theta} \tag{12.15}$$

$$s_{zr} = 2(\frac{1}{2}\,\mu^{(2)} + \frac{1}{3r}\mu^{(3)})\gamma_{rz} \tag{12.16}$$

$$\tau_{z\theta} = \tau_{\theta z} = s_{z\theta} = s_{\theta z} = 0 \tag{12.17}$$

It should be mentioned that with the constitutive relations (12.6)-(12.17), the equations (8.17) for balance of the angular momentum are identically satisfied.

The composite mass density ρ_o and the composite mass moments $\rho_o z^1$ and $\rho_o z^2$ are calculated using relations (8.44)-(8.46) together with the definitions (12.3)

$$\rho_o = \rho_o^{(1)} + \frac{1}{2r}\,\rho_o^{(2)}$$

$$\rho_o z^1 = \frac{1}{2}\,\rho_o^{(2)} + \frac{1}{3r}\,\rho_o^{(3)} \tag{12.18}$$

$$\rho_o z^2 = \frac{1}{3}\,\rho_o^{(3)} + \frac{1}{4r}\,\rho_o^{(4)}$$

As for the physical components of the director vector $\boldsymbol{\delta}$, using (4.22) in conjunction with (8.1), (8.2), (8.5) and (8.7) we find the following results

$$\delta_\theta = \frac{\partial u_\theta}{\partial r} \quad , \quad \delta_z = \frac{\partial u_z}{\partial r} \quad , \quad \delta_r = \frac{\partial u_r}{\partial r} \tag{12.19}$$

In order to substitute the constitutive relations (12.6)-(12.17) in the equations of motion, first we eliminate the interlaminar stress components σ_θ, σ_z and σ_r from the set of six equations (12.5). The result would be the following set of three equations of motion. In these equations, as previous similar equations, double dot denotes second partial derivative with respect to time t.

$$\frac{\partial}{\partial r}\tau_{r\theta} + \frac{1}{r}(\tau_{r\theta} + \tau_{\theta r}) - \frac{1}{r}\frac{\partial}{\partial r}s_{\theta r} =$$

$$= \{\rho_o^{(1)} - \frac{1}{3r}\rho_o^{(3)}\frac{\partial}{\partial r} - (\frac{1}{3}\rho_o^{(3)} + \frac{1}{4r}\rho_o^{(4)})\frac{\partial^2}{\partial r^2}\}\ddot{u}_\theta$$

$$\frac{\partial}{\partial r}\tau_{rz} + \frac{1}{r}\tau_{rz} = \{\rho_o^{(1)} - \frac{1}{3r}\rho_o^{(3)}\frac{\partial}{\partial r} - (\frac{1}{3}\rho_o^{(3)} + \frac{1}{4r}\rho_o^{(4)})\frac{\partial^2}{\partial r^2}\}\ddot{u}_z \qquad (12.20)$$

$$\frac{\partial}{\partial r}\tau_{rr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + \frac{1}{r}\frac{\partial}{\partial r}s_{\theta\theta} =$$

$$= \{\rho_o^{(1)} - \frac{1}{3r}\rho_o^{(3)}\frac{\partial}{\partial r} - (\frac{1}{3}\rho_o^{(3)} + \frac{1}{4r}\rho_o^{(4)})\frac{\partial^2}{\partial r^2}\}\ddot{u}_r$$

In deriving these equations we have also used the results (12.18) and (12.19). Now the relevant constitutive relations are substituted in equations (12.20). After straightforward calculations and some simplifications we get the following results for the left-hand sides of these equations:

$$\frac{\partial}{\partial r}\tau_{r\theta} + \frac{1}{r}(\tau_{r\theta} + \tau_{\theta r}) - \frac{1}{r}\frac{\partial}{\partial r}s_{\theta r} = 2(\frac{2}{r}\mu^{(1)} + \frac{1}{2r^2}\mu^{(2)})\gamma_{r\theta}$$

$$+ 2(\mu^{(1)} + \frac{1}{2r}\mu^{(2)})\frac{\partial\gamma_{r\theta}}{\partial r} + (\frac{1}{r}\mu^{(2)} + \frac{1}{3r^2}\mu^{(3)})\kappa_{r\theta} + (\frac{1}{2}\mu^{(2)} + \frac{1}{3r}\mu^{(3)})\frac{\partial\kappa_{r\theta}}{\partial r} ,$$

$$\frac{\partial}{\partial r}\tau_{rz} + \frac{1}{r}\tau_{rz} = 2(\mu^{(1)} + \frac{1}{2r}\mu^{(2)})\frac{\partial\gamma_{rz}}{\partial r} - \frac{1}{r^2}\mu^{(2)}\gamma_{rz} \qquad (12.21)$$

$$\frac{\partial}{\partial r}\tau_{rr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + \frac{1}{r}\frac{\partial}{\partial r}s_{\theta\theta} = \frac{1}{r}(2\mu^{(1)} - \frac{1}{r}\lambda^{(2)} - \frac{1}{r^2}\lambda^{(3)} - \frac{1}{2r^3}\lambda^{(4)})\gamma_{rr}$$

$$+ \{\lambda^{(1)} + 2\mu^{(1)} + \frac{1}{r}(\lambda^{(2)} + \mu^{(2)}) + \frac{2}{3r^2}\lambda^{(3)} + \frac{1}{4r^3}\lambda^{(4)}\}\frac{\partial}{\partial r}\gamma_{rr}$$

$$- \frac{1}{r}\{2\mu^{(1)} + \frac{1}{r}(\lambda^{(2)} + 2\mu^{(2)}) + \frac{1}{r^2}(\lambda^{(3)} + 2\mu^{(3)}) + \frac{1}{2r^3}(\lambda^{(4)} + 2\mu^{(4)}\}\gamma_{\theta\theta}$$

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$$+\frac{4}{5r^4}(\lambda^{(5)}+2\mu^{(5)})\}u_r$$

$$=\{\rho_o^{(1)}-\frac{1}{3r}\rho_o^{(3)}\frac{\partial}{\partial r}-(\frac{1}{3}\rho_o^{(3)}+\frac{1}{4r}\rho_o^{(4)})\frac{\partial^2}{\partial r^2}\}\ddot{u}_r \tag{12.24}$$

It is interesting to note that these equations are uncoupled in terms of the displacement components and represent three types of transient wave motions with axial symmetry. Equation (12.22) is the governing equation for rotary shear motions. Equation (12.23) represents axial shear waves, and finally equation (12.24) is the dynamical equation for radial waves. Since the present theory is designed in such a way that the classical theory for a homogeneous continuum can be derived through a limiting procedure in which the thickness of the micro-structure approaches zero, we expect to reconstruct the equations for wave motions with axial symmetry in a homogeneous isotropic medium by letting ξ_n -> 0 in equations (12.22)-(12.24). Doing so, the only non-vanishing constants in the coefficients of these equations are those with superscript (1) and we obtain the following results.

Rotary shear waves:

$$\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}=\frac{1}{c_T^2}\frac{\partial^2 u_\theta}{\partial t^2} \tag{12.25}$$

Axial shear waves:

$$\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}=\frac{1}{c_T^2}\frac{\partial^2 u_z}{\partial t^2} \tag{12.26}$$

Radial waves:

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}=\frac{1}{c_L^2}\frac{\partial^2 u_r}{\partial t^2} \tag{12.27}$$

where

$$+ \{\lambda^{(1)} + \frac{1}{r}(\lambda^{(1)} + \mu^{(2)}) + \frac{2}{3r^2}(\lambda^{(3)} + 2\mu^{(3)}) + \frac{1}{4r^2}(\lambda^{(4)} + 2\mu^{(4)})\} \frac{\partial}{\partial r}\gamma_{\theta\theta}$$

$$- \frac{1}{r}\{\mu^{(2)} + \frac{2}{3r}(\lambda^{(3)} + 2\mu^{(3)}) + \frac{3}{4r^2}(\lambda^{(4)} + 2\mu^{(4)}) + \frac{2}{5r^3}(\lambda^{(5)} + 2\mu^{(5)})\}\kappa_{\theta\theta}$$

$$+ \{\frac{1}{2}\lambda^{(2)} + \frac{2}{3r}(\lambda^{(3)} + \mu^{(3)}) + \frac{1}{2r^2}(\lambda^{(4)} + 2\mu^{(4)}) + \frac{1}{5r^3}(\lambda^{(5)} + 2\mu^{(5)}) \frac{\partial\kappa_{\theta\theta}}{\partial r}$$

Finally we substitute for the relative kinematic measures from (12.4) and obtain the following equations for the components of the displacement vector **u**:

$$(\mu^{(1)} - \frac{1}{3r^2}\mu^{(3)}) \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}(\mu^{(1)} + \frac{1}{r}\mu^{(2)}) \frac{\partial u_\theta}{\partial r} - \frac{1}{r^2}\mu^{(1)}u_\theta$$

$$= \{\rho_o^{(1)} - \frac{1}{3r}\rho_o^{(3)}\frac{\partial}{\partial r} - (\frac{1}{3}\rho_o^{(3)} + \frac{1}{4r}\rho_o^{(4)})\frac{\partial^2}{\partial r^2}\}\ddot{u}_\theta \qquad (12.22)$$

$$(\mu^{(1)} + \frac{1}{2r}\mu^{(2)}) \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\mu^{(1)}\frac{\partial u_z}{\partial r}$$

$$= \{\rho_o^{(1)} - \frac{1}{3r}\rho_o^{(3)} - (\frac{1}{3}\rho_o^{(3)} + \frac{1}{4r}\rho_o^{(4)})\frac{\partial^2}{\partial r^2}\}\ddot{u}_z \qquad (12.23)$$

$$\{\lambda^{(1)} + 2\mu^{(1)} + \frac{1}{2r}(3\lambda^{(2)} + 2\mu^{(2)}) + \frac{2}{3r^2}(2\lambda^{(3)} + \mu^{(3)}) + \frac{1}{4r^3}(3\lambda^{(4)} + 4\mu^{(4)})$$

$$+ \frac{1}{5r^4}(\lambda^{(5)} + \mu^{(5)})\} \frac{\partial^2 u_r}{\partial r^2}$$

$$+ \frac{1}{r}\{(\lambda^{(1)} + 2\mu^{(1)} - \frac{1}{r^2}\lambda^{(3)} - \frac{1}{r^3}(\lambda^{(4)} + \mu^{(4)}) - \frac{2}{5r^4}(\lambda^{(5)} + 2\mu^{(5)})\} \frac{\partial u_r}{\partial r}$$

$$- \frac{1}{r^2}\{\lambda^{(1)} + 2\mu^{(1)} + \frac{1}{r}(3\lambda^{(2)} + 4\mu^{(2)}) + \frac{11}{3r^2}(\lambda^{(3)} + 2\mu^{(3)}) + \frac{5}{2r^3}(\lambda^{(4)} + 2\mu^{(4)})$$

$$C_L^2 = \frac{\lambda+2\mu}{\rho_o} \quad , \quad c_T^2 = \frac{\mu}{\rho_o} \tag{12.28}$$

and we have deleted the superscript (1) for simplicity. These are the familiar equations for wave motions with axial symmetry in cylindrical coordinates. Equations (12.25) and (12.27) are of the same type and the space part of their solutions can be represented in terms of Hankel functions of the first order. The space part of the general solution of equation (12.26) can be written in terms of Hankel function of zero order. For each case considering a solution of the general form $F(r)e^{i\omega t}$ we get the following results:

$$u_\theta(r,t) = \{C_1H_1^{(1)}(\frac{\omega}{C_T} r) + C_2H_1^{(2)}(\frac{\omega}{C_T} r)\}e^{i\omega t} \tag{12.29}$$

$$u_z(r,t) = \{C_3H_o^{(1)}(\frac{\omega}{C_T} r) + C_4H_o^{(2)}(\frac{\omega}{C_T} r)\}e^{i\omega t} \tag{12.30}$$

$$u_r(r,t) = \{C_5H_1^{(1)}(\frac{\omega}{C_T} r) + C_6H_1^{(2)}(\frac{\omega}{C_T} r)\}e^{i\omega t} \tag{12.31}$$

where $H_n^{(1)}$ and $H_n^{(2)}$ $(n = 0,1)$ are Hankel functions of first and second kind.

12.2 Investigating Wave Motions in Cylindrical Laminates

Returning back to the equations (12.22)-(12.24) we can again use the technique of separation of variables to get the appropriate differential equations for the spatial part of the wave motions.

For the rotary shear motions if we let

$$u_\theta(r,t) = F(r)e^{i\omega t} \tag{12.32}$$

in the equation (12.22) we will obtain the following second order ordinary differential equation for F(r)

$$(\mu^{(1)} - \frac{1}{3}\,\omega^2\rho_o^{(3)} - \frac{1}{4r}\,\omega^2\rho_o^{(4)} - \frac{1}{3r^2}\,\mu^{(3)})F''(r) \tag{12.33}$$

$$\frac{1}{r}\,(\mu^{(1)} - \frac{1}{3}\,\omega^2\rho_o^{(3)} + \frac{1}{r}\,\mu^{(2)})F'(r) + (\omega^2\rho_o^{(1)} - \frac{1}{r^2}\,\mu^{(1)})F(r) = 0$$

This equation can be written in the following form:

$$F''(r) + \frac{\alpha r+\mu^{(2)}}{\alpha r^2+\beta r+\gamma}\,F'(r) + \frac{\omega^2\rho_o^{(1)}r^2-\mu^{(1)}}{\alpha r^2+\beta r+\gamma}\,F(r) = 0 \tag{12.34}$$

where

$$\alpha = \mu^{(1)} - \frac{1}{3}\,\omega^2\rho_o^{(3)} \ , \ \beta = -\frac{\omega^2\rho_o^{(4)}}{4} \ , \ \gamma = -\frac{1}{3}\,\mu^{(3)} \tag{12.35}$$

The origin $r = 0$ is an ordinary point for this differential equation; therefore, the solution for $F(r)$ is the neighborhood of $r = 0$ can be written in the form of an infinite power series of r. The radius of convergence of this series depends on the frequency ω of the wave motion and is approximately equal to $(|\gamma|/\alpha)^{1/2}$ for small frequencies. At the critical high frequency $\omega_{cr} = (3\mu^{(1)}/\rho_o^{(3)})^{1/2}$, $\alpha = 0$ and the equation (12.33) can be written as

$$(\frac{3}{4}\,\frac{\mu^{(1)}\rho_o^{(4)}}{\rho_o^{(3)}}\,r + \frac{1}{3}\,\mu^{(3)})F''(r) - \mu^{(2)}F'(r) - (\frac{3\mu^{(1)}\rho_o^{(1)}}{\rho_o^{(3)}}\,r^2 - \mu^{(1)})F(r) = 0 \tag{12.36}$$

Substituting $F(r) = \sum_{n=0}^{\infty} a_n r^n$ in this equation, the coefficients a_n can be calculated. This is a convergent seires and its radius of convergence is given below:

$$R = \frac{4}{9}\,\frac{\mu^{(3)}\rho_o^{(3)}}{\mu^{(1)}\rho_o^{(4)}} \tag{12.37}$$

For the axial shear waves if we substitute

$$u_z(r,t) = G(r)e^{i\omega t} \tag{12.38}$$

in the equation (12.23) we will get the following differential equation for G(r)

$$G''(r) + \frac{\mu^{(1)} - \frac{\omega^2}{3}\rho_o^{(3)}}{(\mu^{(1)} - \frac{1}{3}\omega^2\rho_o^{(3)})r - \frac{1}{4}\omega^2\rho_o^{(4)}} G'(r) + \frac{\omega^2\rho_o^{(1)} r}{(\mu^{(1)} - \frac{1}{3}\omega^2\rho_o^{(3)})r - \frac{1}{4}\omega^2\rho_o^{(4)}} G(r) = 0 \tag{12.39}$$

The origin r = 0 is again an ordinary point for this differential equation and its solution in the neighborhood of r = 0 can be written in the form of $G(r) = \sum_{n=0}^{\infty} a_n r^n$. At the critical high frequency $\omega_{cr} = (3\mu^{(1)}/\rho_o^{(3)})^{1/2}$ the equation (12.39) adopts the simpler form

$$G''(r) - \frac{4\rho_o^{(1)}}{\rho_o^{(4)}} r\, G(r) = 0 \tag{12.40}$$

This is the Airy differential equation and its solution can be represented in terms of the Airy functions of the first and second kind

$$G(r) = C_1 A_i\left(2\sqrt{\frac{\rho_o^{(1)}}{\rho_o^{(4)}}}\, r\right) + C_2 B_i\left(2\sqrt{\frac{\rho_o^{(1)}}{\rho_o^{(4)}}}\, r\right) \tag{12.41}$$

Due to our initial assumption that the thickness of the micro-structure is very small and also by definition (12.3) we conclude that the coefficient of r in the arguments of the Airy functions (12.41), i.e., $2\sqrt{\frac{\rho_o^{(1)}}{\rho_o^{(4)}}}$, is a large number. Therefore, we can employ the following asymptotic representations for the Airy functions

$$A_i(\alpha r) \approx \frac{1}{2\sqrt{\pi}} (\alpha r)^{-1/4} e^{-\frac{2}{3}(\alpha r)^{3/2}}$$

$$B_i(\alpha r) \approx \frac{1}{2\sqrt{\pi}} (\alpha r)^{-1/4} e^{\frac{2}{3}(\alpha r)^{3/2}} \tag{12.42}$$

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For the radial waves we have the more involved equation (12.24) and if we let

$$u_r(r,t) = H(r)e^{i\omega t} \tag{12.43}$$

we will obtain the following differential equation for H(r)

$$\left(\alpha_o + \frac{\alpha_1}{r} + \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r^3} + \frac{\alpha_4}{r^4}\right)H''(r) + \frac{1}{r}\left(\alpha_o + \frac{\beta_2}{r^2} + \frac{\beta_3}{r^3} + \frac{\beta_4}{r^4}\right)H'(r)$$

$$+ \frac{1}{r^2}\left(\omega^2\rho_o^{(1)}r^2 - \frac{1}{3}\,\omega^3\rho_o^{(3)} - \left(\alpha_o + \frac{\gamma_1}{r} + \frac{\gamma_2}{r^2} + \frac{\gamma_3}{r^3} + \frac{\gamma_4}{r^4}\right)\right)H(r) = 0 \tag{12.44}$$

where

$$\alpha_o = \lambda^{(1)} + 2\mu^{(1)} - \frac{1}{3}\,\omega^2\rho_o^{(3)}$$

$$\alpha_1 = \frac{3}{2}\,\lambda^{(2)} + \mu^{(2)} - \frac{1}{4}\,\omega^2\rho_o^{(4)}$$

$$\alpha_2 = \frac{2}{3}\,(2\lambda^{(3)} + \mu^{(3)}) \tag{12.45}$$

$$\alpha_3 = \frac{3}{4}\,\lambda^{(4)} + \mu^{(4)}$$

$$\alpha_4 = \frac{1}{5}\,(\lambda^{(5)} + 2\mu^{(5)})$$

$$\beta_2 = -\lambda^{(3)} \quad , \quad \beta_3 = -(\lambda^{(4)} + \mu^{(4)})$$

$$\beta_4 = -\frac{2}{5}\,(\lambda^{(5)} + 2\mu^{(5)}) = -2\alpha_4 \tag{12.46}$$

$$\gamma_1 = 3\lambda^{(2)} + 4\mu^{(2)}$$

$$\gamma_2 = \frac{11}{3}\,(\lambda^{(3)} + 2\mu^{(3)})$$

(12.49)

$$\gamma_3 = \frac{5}{2}\,(\lambda^{(4)} + 2\mu^{(4)})$$

$$\gamma_4 = \frac{4}{5}\,(\lambda^{(5)} + 2\mu^{(5)}) = 4\alpha_4$$

Equation (12.44) can be written in the following form

$$H''(r) + \frac{1}{r}\,p(r)H'(r) - \frac{1}{r^2}\,q(r)H(r) = 0 \tag{12.48}$$

where

$$p(r) = f(r)/h(r) \quad , \quad q(r) = g(r)/h(r) \tag{12.49}$$

are rational functions of r and

$$f(r) = \beta_4 + \beta_3 r + \beta_2 r^2 + \alpha_o r^4$$

$$g(r) = \gamma_4 + \gamma_3 r + \gamma_2 r^2 + \gamma_1 r^3 + (\alpha_o + \frac{1}{3}\,\omega^2\rho_o^{(3)})r^4 - \omega^2\rho_o^{(1)}r^6 \tag{12.50}$$

$$h(r) = \alpha_4 + \alpha_3 r + \alpha_2 r^2 + \alpha_1 r^3 + \alpha_o r^4$$

Unlike the two previous cases the origin $r = 0$ is not an ordinary point for the differential equation of the radial waves, but it is instead a regular singular point. However, since the indicial equation at $r = 0$ is

$$r^2 + (p(0)-1)r - q(0) = 0 \tag{12.51}$$

and by (12.49), (12.50), $(12.46)_3$ and $(12.47)_4$; $p(0) = -2$ and $q(0) = 4$ we always have an analytical solution at $r = 0$. The other independent solution has a singularity at the origin and can be found by the familiar techniques of series solution. The nature of the general solution at origin, as far as the analyticity and singularity of the solution is concerned, is independent of the frequency of the radial waves.

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12.3 Governing Equations for Spherical Laminates

In this section we use the results of section 9 to derive the differential equation of wave motions with polar symmetry. With such an assumption the relevant field variables depend on r and t only. Consequently, we do not have variations in $\theta^1 = \phi$ and $\theta^2 = \theta$ directions and the only non-vanishing component of the displacement vector **u** is u_r which we will denote from now on by u. The physical components of the relative kinematic measures are calculated from (9.13) and (9.15) and the results are recorded below

$$\gamma_{\phi\phi} = \gamma_{\theta\theta} = \frac{u}{r} \quad , \quad \gamma_{rr} = \frac{\partial u}{\partial r}$$
$$\gamma_{\phi\theta} = \gamma_{\phi r} = \gamma_{\theta r} = 0 \tag{12.52}$$

$$\kappa_{\phi\phi} = \kappa_{\theta\theta} = \frac{1}{r^2}\frac{\partial}{\partial r}(ru)$$
$$\kappa_{\phi\theta} = \kappa_{\theta\phi} = \kappa_{r\phi} = \kappa_{r\theta} = 0 \tag{12.53}$$

The constitutive relations for various components of the composite stress tensor and the composite stress couple are derived using relations (9.26)-(9.40) together with definitions (12.1) and (12.2). We also substitute in these constitutive relations the results (12.52) and (12.53) for the relative kinematic measures. After some simplifications we have

$$\tau_{\phi\phi} = \{2(\lambda^{(1)}+\mu^{(1)}) + \frac{4}{r}(\lambda^{(2)}+\mu^{(2)}) + \frac{10}{3r^2}(\lambda^{(3)}+\mu^{(3)}) + \frac{1}{r^3}(\lambda^{(4)}+\mu^{(4)})\}\frac{u}{r}$$
$$+ \{\lambda^{(1)} + \frac{1}{r}(\frac{5}{2}\lambda^{(2)}+\mu^{(2)}) + \frac{2}{r^2}(\frac{4}{3}\lambda^{(3)}+\mu^{(3)}) + \frac{1}{r^3}(\lambda^{(4)}+\mu^{(4)})\}\frac{\partial u}{\partial r} \tag{12.54}$$

$$\tau_{\theta\theta} = \{2(\lambda^{(1)}+\mu^{(1)}) + \frac{4}{r}(\lambda^{(2)}+\mu^{(2)}) + \frac{10}{3r^2}(\lambda^{(3)}+\mu^{(3)}) + \frac{1}{r^3}(\lambda^{(4)}+\mu^{(4)})\}\frac{u}{r}$$

$$+\{\lambda^{(1)}+\frac{1}{r}(\frac{5}{2}\lambda^{(2)}+\mu^{(2)})+\frac{2}{r^2}(\frac{4}{3}\lambda^{(3)}+\mu^{(3)})+\frac{1}{r^3}(\lambda^{(4)}+\mu^{(4)})\}\frac{\partial u}{\partial r} \qquad (12.55)$$

$$\tau_{rr}=(\frac{2}{r}\lambda^{(1)}+\frac{5}{2r^2}\lambda^{(2)}+\frac{2}{3r^3}\lambda^{(3)})u$$

$$+\{\lambda^{(1)}+2\mu^{(1)}+\frac{1}{r}(\frac{3}{2}\lambda^{(2)}+2\mu^{(3)})+\frac{2}{3r^2}\lambda^{(3)}\}\frac{\partial u}{\partial r} \qquad (12.56)$$

$$\tau_{\phi\theta}=\tau_{\phi r}=\tau_{\theta\phi}=\tau_{\theta r}=\tau_{r\phi}=\tau_{r\theta}=0 \qquad (12.57)$$

$$s_{\phi\phi}=\{\lambda^{(2)}+\mu^{(2)}+\frac{8}{3r}(\lambda^{(3)}+\mu^{(3)})+\frac{2}{r^2}(\lambda^{(4)}+\mu^{(4)})+\frac{2}{5r^3}(\lambda^{(5)}+\mu^{(5)})\}\frac{u}{r}$$

$$+\{\frac{1}{2}\lambda^{(2)}+\frac{1}{3r}(5\lambda^{(3)}+2\mu^{(3)})+\frac{1}{4r^2}(7\lambda^{(4)}+6\mu^{(4)})+\frac{2}{5r^3}(\lambda^{(5)}+\mu^{(5)})\}\frac{\partial u}{\partial r} \qquad (12.58)$$

$$s_{\theta\theta}=\{\lambda^{(2)}+\mu^{(2)}+\frac{8}{3r}(\lambda^{(3)}+\mu^{(3)})+\frac{2}{r^2}(\lambda^{(4)}+\mu^{(4)})+\frac{2}{5r^3}(\lambda^{(5)}+\mu^{(5)})\}\frac{u}{r}$$

$$+\{\frac{1}{2}\lambda^{(2)}+\frac{1}{3r}(5\lambda^{(3)}+2\mu^{(3)})+\frac{1}{4r^2}(7\lambda^{(4)}+6\mu^{(4)})+\frac{2}{5r^3}(\lambda^{(5)}+\mu^{(5)})\}\frac{\partial u}{\partial r} \qquad (12.59)$$

$$s_{\phi\theta}=s_{\theta\phi}=s_{\phi r}=s_{\theta r}=0 \qquad (12.60)$$

The equations for balance of linear momentum and director momentum are written in terms of the relevant components of the composite stress tensor and the composite stress couple by using relations (9.17) and (9.18). In the absence of body force and body couple and recalling the assumption stated at the beginning of section 12.1 for polar symmetry we have

$$\frac{\cot\phi}{r}(\tau_{\phi\phi}-\tau_{\theta\theta})+\frac{1}{r^2\sin\phi}\frac{\partial\sigma_\phi}{\partial r}=0 \qquad (12.61)$$

$$\frac{1}{r^2\sin\phi}\frac{\partial\sigma_\theta}{\partial r}=0 \qquad (12.62)$$

$$-\frac{1}{r}(\tau_{\phi\phi}+\tau_{\theta\theta})+\frac{1}{r^2\sin\phi}\frac{\partial\sigma_r}{\partial r}=\rho_o(\ddot{u}+z^1\frac{\partial\ddot{u}}{\partial r}) \tag{12.63}$$

$$\frac{\cot\phi}{r}(s_{\phi\phi}-s_{\theta\theta})+\frac{1}{r^2\sin\phi}\sigma_\phi=0 \tag{12.64}$$

$$\frac{1}{r^2\sin\phi}\sigma_\theta=0 \tag{12.65}$$

$$-\frac{s_{\theta\theta}}{r}+\frac{1}{r^2\sin\phi}\sigma_r-\tau_{rr}=\rho_o(z^1\ddot{u}+z^2\frac{\partial\ddot{u}}{\partial r}) \tag{12.66}$$

In writing down these equations we have also taken notice of the fact that the only non-vanishing component of the director vector is $\delta_3=\delta_r=\frac{\partial u}{\partial r}$. The equations (9.19) for balance of moment of momentum are identically satisfied due to the constitutive relations (12.57) and (12.60). We also notice from (12.54), (12.55), (12.58) and (12.59) that

$$\tau_{\phi\phi}=\tau_{\theta\theta}\quad,\quad s_{\phi\phi}=s_{\theta\theta} \tag{12.67}$$

Using these results together with the equations of motion (12.61), (12.62), (12.64) and (12.65), we conclude that the first two components of interlaminar stress vector are zero, namely

$$\sigma_\phi=\sigma_\theta=0 \tag{12.68}$$

So we have only one equation of motion which should be derived by eliminating σ_r between equations (12.63) and (12.66). In order to write down this equation, first we derive appropriate expressions for the composite mass density ρ_o and the composite mass moments $\rho_o z^1$ and $\rho_o z^2$ by using relations (9.41)-(9.43) and (12.3). The results are as follows:

$$\rho_o=\rho_o^{(1)}+\frac{1}{r}\rho_o^{(2)}$$

$$\rho_o z^1=\frac{1}{2}\rho_o^{(2)}+\frac{2}{3r}\rho_o^{(3)} \tag{12.69}$$

$$\rho_o z^2 = \frac{1}{3}\,\rho_o^{(3)} + \frac{1}{2r}\,\rho_o^{(4)}$$

Next we differentiate (12.66) with respect to r and substitute for $\frac{\partial\sigma_r}{\partial r}$ in (12.63) while making use of (12.69). The resulting equation of motion in terms of the composite stress tensor and the composite stress couple components are as follows:

$$\frac{1}{r^2}\,\frac{\partial}{\partial r}\,(r^2\tau_{rr}+rs_{\theta\theta}) - \frac{2}{r}\,\tau_{\theta\theta} = (\rho_o^{(1)} - \frac{2}{3r^2}\,\rho_o^{(3)})\ddot{u}$$

$$- (\frac{2}{3r}\,\rho_o^{(3)} + \frac{1}{2r^2}\rho_o^{(4)})\,\frac{\partial\ddot{u}}{\partial r} - (\frac{1}{3}\,\rho_o^{(3)} + \frac{1}{2r}\,\rho_o^{(4)})\,\frac{\partial^2\ddot{u}}{\partial r^2} \tag{12.70}$$

Now we substitute the constitutive relations (12.55), (12.56) and (12.59) in the equation of motion (12.70) to derive the displacement equation. After simplification we get the following result

$$\{\lambda^{(1)}+2\mu^{(1)} + \frac{2}{r}\,(\lambda^{(2)}+\mu^{(2)}) + \frac{1}{3r^2}\,(7\lambda^{(3)}+2\mu^{(3)}) + \frac{1}{4r^3}\,(7\lambda^{(4)}+6\mu^{(4)}) + \frac{2}{5r^4}\,(\lambda^{(5)} + \mu^{(5)})\}\,\frac{\partial^2 u}{\partial r^2}$$

$$+ \{2(\lambda^{(1)}+2\mu^{(1)}) + \frac{1}{2r}\,\lambda^{(2)} - \frac{2}{r^2}\,(\lambda^{(3)} + \frac{2}{3}\,\mu^{(3)}) - \frac{1}{4r^3}\,(7\lambda^{(4)}+6\mu^{(4)})$$

$$- \frac{2}{5r^4}\,(\lambda^{(5)}+\mu^{(5)})\}\,\frac{1}{r}\,\frac{\partial u}{\partial r}$$

$$- \{2(\lambda^{(1)}+2\mu^{(1)}) + \frac{8}{r}\,(\lambda^{(2)}+\mu^{(2)}) + \frac{1}{3r^2}\,(31\lambda^{(3)}+28\mu^{(3)}) + \frac{6}{r^3}\,(\lambda^{(4)}+\mu^{(4)})$$

$$+ \frac{6}{5r^4}\,(\lambda^{(5)}+\mu^{(5)})\}\,\frac{u}{r^2}$$

$$= (\rho_o^{(1)} - \frac{2}{3r^2}\,\rho_o^{(3)})\ddot{u} - (\frac{2}{3r}\,\rho_o^{(3)} + \frac{1}{2r^2}\,\rho_o^{(4)})\,\frac{\partial\ddot{u}}{\partial r}$$

$$- (\frac{1}{3}\,\rho_o^{(3)} + \frac{1}{2r}\,\rho_o^{(4)})\,\frac{\partial^2\ddot{u}}{\partial r^2} \tag{12.71}$$

BASE

Again, as in the case of the cylindrical laminates, we expect to recover the classical theory of wave motions with polar symmetry through a limiting procedure in which the thickness of the micro-structure, ξ_n, approaches zero. Recalling (12.1)-(12.3) and suppressing the superscript (1) at the limit, the equation (12.71) reduces to the following form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} = \frac{1}{C_L^2}\frac{\partial^2 u}{\partial t^2} \tag{12.72}$$

where

$$C_L^2 = \frac{\lambda+2\mu}{\rho_o} \tag{12.73}$$

This is the familar displacement equation of motion for waves with polar symmetry. In order to find the general solution of this equation it is convenient to express the radial displacement u(r,t) in terms of a potential function $\phi(r,t)$ through the relation

$$u = \frac{\partial \phi}{\partial r} \tag{12.74}$$

Now the second derivaqive of the product $r\phi$ with respect to r is

$$\frac{\partial^2}{\partial r^2}(r\phi) = r\frac{\partial^2 \phi}{\partial r^2} + 2\frac{\partial \phi}{\partial r} = r\frac{\partial u}{\partial r} + 2u$$

If we rearrange the above result and differentiate again with respect to r, we get

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{2}{r}u\right) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)\right]$$

which reduces by (12.72) to

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)\right] = \frac{1}{C_L^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{C_L^2}\frac{\partial}{\partial r}\left(\frac{\partial^2 \phi}{\partial t^2}\right)$$

Therefore, we conclude that if the product $r\phi$ satisfies the following one-dimensional wave equation

$$\frac{\partial^2}{\partial r^2}(r\phi) = \frac{1}{C_L^2}\frac{\partial^2}{\partial t^2}(r\phi) \tag{12.75}$$

then u(r,t) as defined by (12.74) will satisfy the displacement equation (12.72).

The general solution of (12.75) is

$$\phi(r,t) = \frac{1}{r} f(t-r/C_L) + \frac{1}{r} g(t+r/C_L) \tag{12.76}$$

where all the two terms represent waves diverging from the origin r = 0 and converging to r = 0, respectively.

The displacement equation (12.71) for waves with polar symmetry in spherical laminates can be investigated by substituting a separated solution of the form

$$u(r,t) = F(r)e^{i\omega t} \tag{12.77}$$

The differential equation for the spatial part of the solution, F(r), can be written in the following form

$$F''(r) + \frac{1}{r} P(r)F'(r) + \frac{1}{r^2} Q(r)F(r) = 0 \tag{12.78}$$

where the rational functions P(r) and Q(r) are given by

$$P(r) = N_1(r)/D(r) \quad , \quad Q(r) = N_2(r)/D(r) \tag{12.79}$$

and $N_1(r)$, $N_2(r)$ and D(r) are the following polynomials

$$N_1(r) = 2\alpha_o r^4 + \beta_1 r^3 + \beta_2 r^2 - \alpha_3 r - \alpha_4$$

$$N_2(r) = \omega^2 \rho_o^{(1)} r^6 - \alpha'_o r^4 - 4\alpha_1 r^3 - \gamma_2 r^2 - \gamma_3 r - 3\alpha_4 \tag{12.80}$$

$$D(r) = \alpha_o r^4 + \alpha_1 r^3 + \alpha_2 r^2 + \alpha_3 r + \alpha_4$$

where

$$\alpha_o = \lambda^{(1)} + 2\mu^{(1)} - \frac{1}{3}\,\omega^2\rho_o^{(3)}$$

$$\alpha'_o = 2(\alpha_o + \frac{2}{3}\,\omega^2\rho_o^{(3)})$$

$$\alpha_1 = 2(\lambda^{(2)} + \mu^{(2)}) - \frac{1}{2}\,\omega^2\rho_o^{(4)}$$

$$\alpha_2 = \frac{1}{3}\,(7\lambda^{(3)} + 2\mu^{(3)})$$

$$\alpha_3 = \frac{1}{4}\,(7\lambda^{(4)} + 6\mu^{(4)})$$

$$\alpha_4 = \frac{2}{5}\,(\lambda^{(5)} + \mu^{(5)}) \tag{12.81}$$

$$\beta_1 = \frac{1}{2}\,\lambda^{(2)} - \frac{1}{2}\,\omega^2\rho_o^{(4)}$$

$$\beta_2 = -2(\lambda^{(3)} + \frac{2}{3}\,\mu^{(3)})$$

$$\gamma_2 = \frac{1}{3}\,(31\lambda^{(3)} + 28\mu^{(3)})$$

$$\gamma_3 = 6(\lambda^{(4)} + \mu^{(4)})$$

The origin $r = 0$ is a regular singular point for the equation (12.78). The indicial equation at this point irrespective of the material constants is

$$r^2 - 2r - 3 = 0 \tag{12.82}$$

It is obvious from (12.82) that we have always an analytical solution at $r = 0$ and the other independent solution has a simple pole at this point. By (12.81), the radius of convergence of the series solution depends on the frequency of the wave motion.