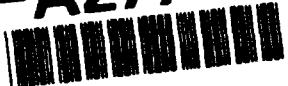


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Clutters that Pack and the
Max Flow Min Cut Property:
A Conjecture

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Clutters that Pack and the Max Flow Min Cut Property: A Conjecture

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A clutter C is a collection $E(C)$ of subsets of a finite set $V(C)$ with the property that $A_1 \not\subseteq A_2$ for all $A_1, A_2 \in E(C)$. Let $M(C)$ be the 0,1 matrix with columns indexed by $V(C)$ whose rows are the incidence vectors of the members of $E(C)$.

Consider the two linear programs

$$\min\{wx : x \geq 0, M(C)x \geq 1\} \quad (1)$$

$$\max\{y1 : y \geq 0, yM(C) \leq w\}. \quad (2)$$

The clutter C has the *Max Flow Min Cut property* if (1) and (2) have integral optimum solutions x and y for all nonnegative integral vectors w . C is *ideal* if (1) has an integral optimum solution x for all nonnegative (integral) vectors w . C *packs* if (1) and (2) have integral optimum solutions x and y when $w = 1$.

For a clutter C , the *deletion* $C \setminus j$ and *contraction* C/j of an element $j \in V(C)$ are clutters defined as follows: $V(C \setminus j) = V(C/j) = V(C) - \{j\}$, $E(C \setminus j) = \{A \in E(C) : j \notin A\}$ and $E(C/j)$ are the minimal members of

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$\{A - \{j\} : A \in E(C)\}$. Any clutter obtained from C by repeated application of the contraction and deletion operations is called a *minor* of C .

We propose the following conjecture.

Conjecture 1 *A clutter C has the Max Flow Min Cut property if and only if C and all its minors pack.*

Remark 1 The “only if” part of the above statement is obvious since saying that a minor of C packs is equivalent to stating that (1) and (2) have integral solution vectors x and y for the objective function $w_j = 0$ for a deleted element j , $w_j = \infty$ for a contracted element j and $w_j = 1$ for the remaining elements.

Remark 2 A weakening of the “if” condition follows from Lehman’s characterization of ideal clutters [2] which implies the following:

If C is not ideal, it contains a minor C' such that the linear program (1) associated with C' has a unique optimum solution x whose components are all fractional when $w = 1$. It follows that if C and all its minors pack, then C is ideal.

Remark 3 A special case where the conjecture holds is when C is a binary clutter. Seymour [4] proved that a binary clutter has the Max Flow Min Cut property if and only if it does have a Q_6 minor, where $V(Q_6) = \{1, 2, 3, 4, 5, 6\}$ and $E(Q_6) = \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$. It is easy to check that the linear program (2) associated with Q_6 has a unique optimum solution vector $y = \frac{1}{2}$ when $w = 1$.

Remark 4 This conjecture is the exact analog of the replication lemma used in the proof of the Fulkerson-Lovász [1] [3] pluperfect graph theorem. Let

$$\max\{wx : x \geq 0, M(C)x \leq 1\} \quad (3)$$

$$\min\{y1 : y \geq 0, yM(C) \geq w\}. \quad (4)$$

The replication lemma shows that (3) and (4) have integral optimum solution vectors x and y for every 0,1 vector w if and only if they have integral optimum solution vectors for every (nonnegative) integral vector w .

The pluperfect graph theorem states that if (4) has an integral optimum solution y for every 0,1 vector w , then (3) and (4) have integral optimum solutions x and y for every (nonnegative) integral vector w .

Remark 5 To prove the "if" part of Conjecture 1, it is sufficient to show the following.

Conjecture 2 (Replication Conjecture) *If C and all its minors pack, then the clutter C_j defined below packs. For $j \in V(C)$, let*

$$V(C_j) = V(C) \cup \{j'\}$$

$$E(C_j) = E(C) \cup \left(\bigcup_{A \ni j} A \setminus \{j\} \cup \{j'\} \right).$$

Indeed, observe that the linear programs (1) and (2) have integral optimum solutions for the vector w such that $w_j = 2$ and $w_i = 1$ for $i \neq j$ if and only if the clutter C_j packs. Therefore, using the replication conjecture recursively, it follows that C has the Max Flow Min Cut property.

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