To appear in Multidimensional Systems and Signal Processing, 1994.

N00014-94-1-0454

jun0193*.tex

An Algorithm for Stability Determination of Two-Dimensional Delta-Operator Formulated Discrete-Time Systems

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Abstract. The recent interest in delta-operator (or, δ-operator) formulated discrete-time systems (or, δ-systems) is due mainly to (a) their superior finite wordlength characteristics as compared to their more conventional shift-operator (or, q-operator) counterparts (or, q-systems), and (b) the possibility of a more unified treatment of both continuous- and discrete-time systems. With such advantages, design, analysis, and implementation of two-dimensional (2-D) discrete-time systems using the δ-operator is indeed warranted. Towards this end, the work in this paper addresses the development of an easily implementable *direct* algorithm for stability checking of 2-D δ-system transfer function models. *Indirect* methods that utilize transformation techniques are not pursued since they can be numerically unreliable. In developing such an algorithm, a tabular form for stability checking of δ-system characteristic polynomials with complex-valued coefficients and certain quantities that may be regarded as their corresponding Schur-Cohn minors are also proposed.

Keywords. Two-dimensional discrete-time systems, two-dimensional digital filters, δ-operator formulated discrete-time systems, bivariate polynomials, Schur-Cohn minors, stability.

UNIVERSITY OF Miami

August 3, 1994

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1. Introduction

The increased interest in δ-systems during the recent years (see [1-6], and references therein) is due mainly to two reasons: (a) δ-systems provide superior finite wordlength properties with respect to roundoff noise propagation [5] and coefficient sensitivity [1], [5], [7], as compared to their q-system counterparts, and (b) the δ-operator yields the differential operator as a limiting case when sampling time approaches zero enabling a unified treatment of both continuous- and discrete-time systems [1].

With such advantages in mind, development of 2-D and multi-dimensional (m-D) δ-system models must clearly be undertaken. Such research can, for example, provide m-D digital filters with superior roundoff error and coefficient sensitivity performance allowing their implementation to be carried out in a shorter wordlength environment. This is especially crucial in real-time applications, such as, in implementing narrow bandwidth filters under high sampling rates (for example, in current wide bandwidth communication system applications) where traditional q-operator implementations perform poorly [8].

In applications mentioned above, and those dealing with high-speed processing of 2-D and m-D data (for instance, in weather, seismic, gravitational photographs, video images, systems with mutliple sampling rates, etc.), ensuring stability is an important consideration (see [9], and references therein). Given the characteristic polynomial of a δ-system, to determine stability, one may first use a variable transformation that yields a more familiar stability region, for instance, the unit bi-circle. Then, an existing technique (see [9-10], and references therein) may be applied. However, such techniques are known to be prone to numerically ill-conditioning [1], [6]. In the 1-D case, direct stability checking methods for δ-system polynomials are in [6] (where a tabular method based on the work in [11] is given) and [12] (where a Hermite-Bieler-like Theorem is utilized). Hence, our purpose here is to develop a ***direct*** easily implementable stability checking technique applicable to m-D δ-systems. As usual, for notational simplicity, we concentrate on the 2-D case, the extension to the m-D case being quite straight-forward.

In checking stability of bivariate characteristic polynomials, two conditions must be satisfied.

(a) Condition I involves a 1-D stability check of a polynomial with real-valued coefficients. One may use the table form in [6]. Alternately, one may utilize an explicit root location scheme.

(b) Condition II involves a stability check of a polynomial with complex-valued coefficients where the latter are dependent on a parameter taking values on a certain circle in the complex plane. Explicit root location schemes are now ineffective, and the value of tabular methods becomes apparent. Note that, in such a situation, compared to Nyquist-like techniques [13], tabular methods are known to provide certain numerical advantages as well [14].

In checking condition II for 2-D q-systems, an effective technique involves checking positive definiteness of the Hermitian Schur-Cohn matrix [15]. This lets one use an important simplification due to Siljak [16]. The tabular form in [15] takes full use of this since it provides the Schur-Cohn minors (that is, the principal minors of the Hermitian Schur-Cohn matrix) directly from its entries [15], [17]. A similar simplification applicable to δ-systems is clearly possible if condition II may be reduced to checking positive definiteness of a Hermitian matrix.

With the above in mind, we develop the following in this paper: (a) Tabular form for stability checking of δ-system characteristic polynomials possessing complex-valued coefficients, (b) Analogs of Schur-Cohn minors and a corresponding Hermitian matrix applicable for such systems, and (c) a direct stability checking algorithm for 2-D δ-system transfer function models.

The paper is organized as follows. Section 2 introduces the notation used throughout and a brief review of previous results. Section 3 develops a tabular form for stability checking of δ-systems with complex-valued coefficients and some important relevant results. Section 4 presents quantities that may be regarded as the analogs of Schur-Cohn minors for δ-systems. The 2-D stability checking algorithm in Section 5 is based on the tabular form for real-valued coefficients [6]. Since only little extra work is needed, results in both

Sections 3 and 4 however are developed for the more general complex-coefficient case. Section 6 presents an example to validate the results. Section 7 contains the conclusion and some final remarks.

2. Preliminaries

2.1. Notation

$\Re$, $\Im$	Real and complex number fields.
$\Re^{p\times q}$, $\Im^{p\times q}$	Set of matrices of size $p\times q$ over $\Re$ and $\Im$, respectively.
$\mathrm{var}\{\cdot\}$	Number of sign changes in the sequence $\{\cdot\}$ of real numbers.
$\mathrm{Re}[\cdot]$, $\mathrm{Im}[\cdot]$	Real part and imaginary part of $[\cdot]\in\Im$.
$\bar{[\cdot]}$	Complex conjugate of $[\cdot]\in\Im$.
A^T, $\bar{A}$, A^*	Transpose, complex conjugate, and complex conjugate transpose of $A\in\Im^{p\times q}$, respectively.
$\Re[w]_n$, $\Im[w]_n$	Set of univariate polynomials of degree n (with respect to the indeterminate $w\in\Im$) over $\Re$ and $\Im$, respectively.
$\Re(w)$	Set of rational univariate polynomials (that is, quotient of univariate polynomials) over $\Re$.
$\Re[w_1]_{n_1}[w_2]_{n_2}$	Set of bivariate polynomials of relative degrees n_1 and n_2 (with respect to the indeterminates $w_1\in\Im$ and $w_2\in\Im$, respectively) over $\Re$.
$\Re(w_1,w_2)$	Set of rational bivariate polynomials over $\Re$.
z, c	Indeterminates of q- and δ-systems, respectively.
τ	Real positive number, usually the sampling time.

The transformation relationship between corresponding q-and δ-systems is

$$\delta=\frac{q-1}{\tau}\Longleftrightarrow c=\frac{z-1}{\tau}. \tag{2.1}$$

$\breve{[\cdot]}$	q-system quantity analogous to its corresponding δ-system quantity $[\cdot]$; for example, transfer function of a given discrete-time system is either $H(c)$ if implemented based on the δ-operator or $\breve{H}(z)$ if implemented based on the q-operator.
$H(c)\vert_{c\to z}$	$H(c)\vert_{c=(z-1)/\tau}$
$G(z)\vert_{z\to c}$	$G(z)\vert_{z=1+\tau c}$
$H(c_1,c_2)\vert_{\mathbf{c}\to\mathbf{z}}$	$H(c_1,c_2)\vert_{c_i=(z_i-1)/\tau,\, i=1,2}$
$G(z_1,z_2)\vert_{\mathbf{z}\to\mathbf{c}}$	$G(z_1,z_2)\vert_{z_i=1+\tau c_i,\, i=1,2}$

Stability studies of 1-D and 2-D q- and δ-systems involve the following regions:

$\mathcal{U}_q, \mathcal{U}_q^2$	$\{z \in \Im : \|z\| < 1\}$, $\{(z_1, z_2) \in \Im^2 : \|z_i\| < 1, i = 1, 2\}$.
$\overline{\mathcal{U}}_q, \overline{\mathcal{U}}_q^2$	$\{z \in \Im : \|z\| \leq 1\}$, $\{(z_1, z_2) \in \Im^2 : \|z_i\| \leq 1, i = 1, 2\}$.
$\mathcal{T}_q, \mathcal{T}_q^2$	$\{z \in \Im : \|z\| = 1\}$, $\{(z_1, z_2) \in \Im^2 : \|z_i\| = 1, i = 1, 2\}$.
$\mathcal{U}_\delta, \mathcal{U}_\delta^2$	$\{c \in \Im : \|c + 1/\tau\| < 1/\tau\}$, $\{(c_1, c_2) \in \Im^2 : \|c_i + 1/\tau\| < 1/\tau, i = 1, 2\}$.
$\overline{\mathcal{U}}_\delta, \overline{\mathcal{U}}_\delta^2$	$\{c \in \Im : \|c + 1/\tau\| \leq 1/\tau\}$, $\{(c_1, c_2) \in \Im^2 : \|c_i + 1/\tau\| \leq 1/\tau, i = 1, 2\}$.
$\mathcal{T}_\delta, \mathcal{T}_\delta^2$	$\{c \in \Im : \|c + 1/\tau\| = 1/\tau\}$, $\{(c_1, c_2) \in \Im^2 : \|c_i + 1/\tau\| = 1/\tau, i = 1, 2\}$.

To avoid unnecessary notational complications, the sampling time in both horizontal and vertical directions is taken to be equal to $\tau > 0$.

To emphasize the degree of $F(w) = \sum_{k=0}^{n} a_k^{(n)} w^k \in \Im[w]_n$, we sometimes denote it as $F(w)_n$ as well.

$\bar{F}(w)$	Conjugate polynomial of $F(w)$, that is, $\sum_{k=0}^{n} \bar{a}_k^{(n)} w^k$
$F^\sharp(z)$	Reciprocal polynomial of $F(z)$, that is, $z^n \bar{F}(1/z)$
$F^\sharp(c)$	Reciprocal polynomial of $F(c)$, that is, $(1 + \tau c)^n \bar{F}\left(\frac{-c}{1+\tau c}\right)$

A q-system polynomial is *q-symmetric* if $F(z) = F^\sharp(z)$. A δ-system polynomial is *δ-symmetric* if $F(c) = F^\sharp(c)$.

Tabular forms of stability checking of a polynomial in $\Im[\omega]_n$ typically employ a sequence of polynomials each of descending order. The first row of such a tabular form is denoted as *row #n*, the second row is *row #n* $- 1$, and so on.

JT, MJT	Jury table [18], modified Jury table [15], [17].
real-q-BT	Bistritz table for q-system polynomials with real-valued coefficients [11].
complex-q-BT	Bistritz table for q-system polynomials with complex-valued coefficients [19].
real-δ-BT	Table form for δ-system polynomials with real-valued coefficients [6].
complex-δ-BT	Table form for δ-system polynomials with complex-valued coefficients (this paper).

A q-system polynomial with all its roots in $\mathcal{U}_q$ (for the 1-D case) or $\mathcal{U}_q^2$ (for the 2-D case) is said to be ***stable***. The corresponding regions for a δ-system polynomial are $\mathcal{U}_\delta$ (for the 1-D case) or $\mathcal{U}_\delta^2$ (for the 2-D case), respectively.

2.2. Review of complex-q-BT

The complex-δ-BT introduced in Section 3 is based on the complex-q-BT, and hence, we briefly review it now. For more details, see [10]. Let the characteristic polynomial of a q-system be

$$\breve{F}(z) = \sum_{k=0}^{n} \breve{a}_k^{(n)} z^k \in \Im[z]_n \quad \text{with} \quad \breve{F}(1) \in \Re \quad \text{and} \quad \breve{F}(1) \neq 0. \tag{2.2}$$

The complex-q-BT is formed using the symmetric polynomial sequence $\{\breve{T}(z)_i\}_{i=0}^n$ where [19]

$$\breve{T}(z)_i = \begin{cases} \breve{F}(z)_n + \breve{F}^\sharp(z)_n, & \text{for } i = n; \\ \dfrac{\breve{F}(z)_n - \breve{F}^\sharp(z)_n}{z-1}, & \text{for } i = n-1; \\ \dfrac{(\breve{\delta}_{i+2} + \bar{\breve{\delta}}_{i+2} z)T(z)_{i+1} - T(z)_{i+2}}{z}, & \text{for } i = n-2, n-3, \ldots, 0, \end{cases} \tag{2.3}$$

where

$$\breve{\delta}_{i+2} = \frac{\breve{T}(0)_{i+2}}{\breve{T}(0)_{i+1}} = \frac{\breve{t}_0^{(i+2)}}{\breve{t}_0^{(i+1)}}, \; i = n-2, n-3, \ldots, 0. \tag{2.4}$$

As in [11] and [19], equating similar powers on either side, we may also get the following determinental rule: For $k = 0, 1, \ldots, i$, and $i = n-2, n-3, \ldots, 0$,

$$\breve{t}_k^{(i)} = \frac{1}{\breve{t}_0^{(i+1)}} \begin{vmatrix} \breve{t}_0^{(i+2)} & \breve{t}_{k+1}^{(i+2)} \\ \breve{t}_0^{(i+1)} & \breve{t}_{k+1}^{(i+1)} \end{vmatrix} + \frac{1}{\breve{t}_{i+1}^{(i+1)}} \begin{vmatrix} \breve{t}_{i+2}^{(i+2)} & \breve{t}_{k+1}^{(i+2)} \\ \breve{t}_{i+1}^{(i+1)} & \breve{t}_k^{(i+1)} \end{vmatrix} + \breve{t}_{k+1}^{(i+2)}. \tag{2.5}$$

Remark. The computational advantage of BT is due to $\breve{T}(z)_i$ being q-symmetric. This implies $\breve{t}_k^{(i)} = \bar{\breve{t}}_{i-k}^{(i)}$, $k = 0, 1, \ldots, i$, and hence, it is necessary to evaluate only half the coefficients of each row.

Using (12-13), (16), and Theorem 6 of [19], we get

THEOREM 2.1. [19] The polynomial $\breve{F}(z) \in \Im[z]_n$ is q-stable iff
I. $\breve{t}_0^{(i)} \neq 0,\ i = n-1, n-2, \ldots, 0$, and
II. $\nu_n = \mathrm{var}\{\breve{T}(1)_n, \breve{T}(1)_{n-1}, \ldots, \breve{T}(1)_0\} = 0$.

2.3. Some results on 2-D stability

Consider the 2-D q-system transfer function

$$\breve{H}(z_1, z_2) = \frac{\breve{E}(z_1, z_2)}{\breve{F}(z_1, z_2)} \in \Re(z_1, z_2) \tag{2.6}$$

where $\breve{E}(z_1, z_2) \in \Re[z_1]_{n_1}[z_2]_{n_2}$ and $\breve{F}(z_1, z_2) \in \Re[z_1]_{n_1}[z_2]_{n_2}$. The 2-D z-transform is taken using positive powers of z_i. For a comprehensive discussion regarding stability of such systems, see [9-10], and references therein. Hence, for reasons of brevity, only some analog results applicable to 2-D δ-systems are provided. It is only necessary to observe that the corresponding δ-system $H(c_1, c_2)$ satisfies

$$H(c_1, c_2) = \frac{E(c_1, c_2)}{F(c_1, c_2)} = \breve{H}(z_1, z_2)|_{\mathbf{z} \to \mathbf{c}} \in \Re(c_1, c_2) \tag{2.7}$$

where $E(c_1, c_2) \in \Re[c_1]_{n_1}[c_2]_{n_2}$ and $F(c_1, c_2) \in \Re[c_1]_{n_1}[c_2]_{n_2}$. In the remainder of this paper, we will only be dealing with transfer functions $H(c_1, c_2)$ that are devoid of nonessential singularities of the second kind on $\mathcal{T}_\delta^2$ and the pair $E(c_1, c_2)$ and $F(c_1, c_2)$ is taken to be coprime. If the 2-D polynomial $F(c_1, c_2) \neq 0,\ \forall (c_1, c_2) \in \overline{\mathcal{U}}_\delta^2$, it is said to be δ-*stable*. After using (2.1), the following result follows directly from [20]:

THEOREM 2.2. The 2-D δ-system in (2.7) is δ-stable iff
I. $F(c_1, -1/\tau) \neq 0,\ \forall c_1 \in \overline{\mathcal{U}}_\delta$, and
II. $F(c_1, c_2) \neq 0,\ \forall c_1 \in \mathcal{T}_\delta,\ \forall c_2 \in \overline{\mathcal{U}}_\delta$.

The following result, which allows one to use the real-δ-BT, is directly from [21-22] after using (2.1):

THEOREM 2.3. The 2-D δ-system in (2.7) is δ-stable iff
I. $F(c_1, -1/\tau) \neq 0,\ \forall c_1 \in \overline{\mathcal{U}}_\delta$, and

II. $G(x, c_2) \neq 0, \ \forall x \in [-2/\tau, 0], \ \forall c_2 \in \overline{\mathcal{U}}_\delta$.

Here $G(x, c_2) = F(c_1, c_2)F(\bar{c}_1, c_2)\Big|_{\substack{c_1 \in \mathcal{T}_\delta \\ x=(c_1+\bar{c}_1)/2}}$.

2.4. Schur-Cohn minors

In stability checking of 2-D q-systems, the following result is important:

THEOREM 2.4. [15], [23-24] The polynomial $\breve{F}(z) \in \Im[z]_n$ is stable iff $\breve{\Delta}_i > 0$, $i = 1, 2, \ldots, n$, where $\breve{\Delta}_i$ is the principal minor of the Hermitian Schur-Cohn matrix $\breve{\Gamma} = \breve{\Gamma}^* = \{\breve{\gamma}_{ij}\} \in \Im^{n \times n}$ defined as

$$\breve{\gamma}_{ij} = \sum_{k=1}^{i} (\breve{a}_{n-i+k}\bar{\breve{a}}_{n-j+k} - \bar{\breve{a}}_{i-k}\breve{a}_{j-k}), \quad \text{for} \quad i \leq j.$$

Stability checking of 2-D q-systems then involves positivity checking of all Schur-Cohn minors $\breve{\Delta}_i(z)$, $\forall i = 1, 2, \ldots, n$, $\forall |z| = 1$. A necessary and sufficient condition for this is positivity of $\breve{\Delta}_i(1)$, $\forall i = 1, 2, \ldots, n$, and $\breve{\Delta}_n(z)$, $\forall |z| = 1$. This is the simplification due to [16] that has been effectively utilized in applying the MJT [15]. The advantage of the latter is that its entries yield the Schur-Cohn minors directly. The fact that complex-q-BT's entries also yield the Schur-Cohn minors was only recently shown.

THEOREM 2.5. [10], [25] The Schur-Cohn minors of $\breve{F}(z)$ are the principal minors of the $(n \times n)$ tridiagonal Hermitian matrix

$$\breve{\Delta} = \begin{bmatrix} \mathrm{Re}[\check{t}_0^{(n)}\check{t}_{n-1}^{(n-1)}] & \frac{1}{2}[\check{t}_{n-1}^{(n-1)}\check{t}_0^{(n-2)}] & 0 & \cdots & 0 & 0 \\ \frac{1}{2}[\check{t}_0^{(n-1)}\check{t}_{n-2}^{(n-2)}] & \mathrm{Re}[\check{t}_0^{(n-1)}\check{t}_{n-2}^{(n-2)}] & \frac{1}{2}[\check{t}_{n-2}^{(n-2)}\check{t}_0^{(n-3)}] & \ddots & 0 & 0 \\ 0 & \frac{1}{2}[\check{t}_0^{(n-2)}\check{t}_{n-3}^{(n-3)}] & \mathrm{Re}[\check{t}_0^{(n-2)}\check{t}_{n-3}^{(n-3)}] & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \mathrm{Re}[\check{t}_0^{(2)}\check{t}_1^{(1)}] & \frac{1}{2}[\check{t}_1^{(1)}\check{t}_0^{(0)}] \\ 0 & 0 & 0 & \cdots & \frac{1}{2}[\check{t}_0^{(1)}\check{t}_0^{(0)}] & \mathrm{Re}[\check{t}_0^{(1)}\check{t}_0^{(0)}] \end{bmatrix}.$$

3. Complex-δ-BT

With no loss of generality, consider the δ-system characteristic polynomial

$$F(c) = \sum_{k=0}^{n} a_k^{(n)} c^k \in \Im[c]_n, \tag{3.1}$$

where

$$a_0^{(n)} \in \Re \quad \text{and} \quad a_0^{(n)} > 0. \tag{3.2}$$

We now construct the complex-δ-BT with the use of the δ-symmetric polynomial sequence $\{T(c)_i\}_{i=0}^{n}$ where

$$T(c)_i = \begin{cases} F(c)_n + F^{\sharp}(c)_n, \; i = n; \\[2mm] \dfrac{F(c)_n - F^{\sharp}(c)_n}{c}, \; i = n-1; \\[2mm] \dfrac{(\delta_{i+2} + \breve{\delta}_{i+2}(1+\tau c))T(c)_{i+1} - T(c)_{i+2}}{1+\tau c}, \; i \le n-2. \end{cases} \tag{3.3}$$

Here

$$\delta_{i+2} = \frac{T(-1/\tau)_{i+2}}{T(-1/\tau)_{i+1}}, \; i = n-2, n-3, \ldots, 0. \tag{3.4}$$

The *normal conditions* required to complete the sequence are

$$T(-1/\tau)_i \neq 0, \; i = 1, 2, \ldots, n-1. \tag{3.5}$$

Remarks.

1. To determine δ-stability of $F(c)$, one may of course first obtain $\breve{F}(z) = F(c)|_{c \to z}$ and then determine its q-stability by applying familiar stability checking algorithms (e.g., BT or MJT). The possible shortcomings of such a scheme are outlined in [1] and [6]. The purpose here is to obtain a direct check for δ-stability.
2. We follow the work in [6] and [19], and hence, for brevity, all details are omitted.
3. The conditions $T(-1/\tau)_i = 0$, for some $i = 1, 2, \ldots n-1$, imply certain singular conditions on the root distribution of $F(c)$ [11], [19]. The equivalent singular conditions for the real-δ-BT is in [6].
4. Using δ-symmetry, it is easy to show that

$$T(-1/\tau)_i = \frac{\bar{t}_i^{(i)}}{\tau^i}, \; i = 0, 1, \ldots, n. \tag{3.6}$$

Therefore

$$\delta_{i+2} = \frac{1}{\tau}\frac{\bar{t}_{i+2}^{(i+2)}}{\bar{t}_{i+1}^{(i+1)}},\ i = n-2, n-3, \dots, 0. \tag{3.7}$$

The normal conditions in (3.5) may now be expressed as

$$t_{i+1}^{(i+1)} \neq 0,\ i = n-2, n-3, \dots, 0. \tag{3.8}$$

Analogous to [6], [11], and [19], we then have

THEOREM 3.1. The polynomial $F(c) \in \Im[c]_n$ is stable iff
I. $t_i^{(i)} \neq 0,\ i = n-1, n-2, \dots, 1$, and
II. $\nu_n = \text{var}\{T(0)_n, T(0)_{n-1}, \dots, T(0)_0\} = 0$.

One of the main advantages of the complex-q-BT is that all computations may be carried out through real arithmetic only [19]. The same holds true for the the complex-δ-BT introduced above as well. To see this, let

$$T(c)_i = S(c)_i + jA(c)_i \quad \text{with} \quad \delta_i = \text{Re}[\delta_i] + j\text{Im}[\delta_i], \tag{3.9}$$

for $i = 2, 3, \dots, n$. It is easy to show that $S(c)_i$'s and $A(c)_i$'s form sequences of δ-symmetric and δ-antisymmetric polynomials, respectively. Now, (3.3) may be expressed as

$$\begin{aligned} S(c)_{i-2} &= \frac{1}{1+\tau c}\left[\text{Re}[\delta_i](2+\tau c)\cdot S(c)_{i-1} + \text{Im}[\delta_i]\tau c \cdot A(c)_{i-1} - S(c)_i\right]; \\ A(c)_{i-2} &= \frac{1}{1+\tau c}\left[-\text{Im}[\delta_i]\tau c \cdot S(c)_{i-1} + (2+\tau c)\text{Re}[\delta_i]\cdot A(c)_{i-1} - A(c)_i\right], \end{aligned} \tag{3.10}$$

for $i = 2, 3, \dots, n$.

Remark. Note that, $T(0)_i = S(0)_i + jA(0)_i = S(0)_i$.

In the real-δ-BT construction, a certain 'scaling' of $\{T(c)_i\}_{i=0}^n$ was useful [6]. We use the same technique in the complex-δ-BT case as well, thus providing the following advantages: (a) Terms containing τ are avoided during construction, (b) δ_i and ν_i may be deduced by simple inspection, and thus (c) computational effort is reduced.

The sequence of polynomials that incorporates 'scaling' is $\{U(\zeta)_i\}_{i=0}^n$ where

$$U(\zeta)_i = \sum_{k=0}^{i} u_k^{(i)}\zeta^k = T(c)_i \Big|_{c=-\zeta/\tau} \Longleftrightarrow u_k^{(i)} = \left(-\frac{1}{\tau}\right)^k t_k^{(i)},\ k = 0,1,\ldots,i, \tag{3.11}$$

for $i = 0,1,\ldots,n$. Thus, from (3.3), we get, for $i = n-2, n-3, \ldots 0$,

$$\begin{aligned} u_0^{(i)} &= (\delta_{i+2} + \bar{\delta}_{i+2})u_0^{(i+1)} - u_0^{(i+2)}; \\ u_k^{(i)} &= (\delta_{i+2} + \bar{\delta}_{i+2})u_k^{(i+1)} - \bar{\delta}_{i+2}u_{k-1}^{(i+1)} - u_k^{(i+2)} + u_{k-1}^{(i)},\ k = 1,2,\ldots i. \end{aligned} \tag{3.12}$$

Note that

$$\delta_{i+2} = \frac{1}{\tau}\frac{\bar{t}_{i+2}^{(i+2)}}{\bar{t}_{i+1}^{(i+1)}} = -\frac{\bar{u}_{i+2}^{(i+2)}}{\bar{u}_{i+1}^{(i+1)}},\ i = n-2, n-3, \ldots, 0, \tag{3.13}$$

and

$$\nu_n = \mathrm{var}\{T(0)_i\}_{i=0}^n = \mathrm{var}\{u_0^{(i)}\}_{i=0}^n. \tag{3.14}$$

Therefore, condition II of Theorem 3.1 may be checked by inspecting the constant coefficients of $\{U(\zeta)_i\}_{i=0}^n$.

Remark. One may use the same 'scaling' strategy in an implementation that uses only real arithmetic.

Relationship between complex-q-BT and complex-δ-BT

As was agreed upon previously, given $F(c)_n \in \Im[z]$, let us use the notation $\breve{F}(z)_n$ to indicate

$$\breve{F}(z)_n = \lambda F(c)_n \Big|_{c\to z} \tag{3.15}$$

where $\lambda \in \Re$ is a possible scaling constant. The establishment of the relationship between the rows of complex-q-BT of $\breve{F}(z)$, i.e., $\{\breve{T}(z)_i\}_{i=0}^n$, and complex-$\delta$-BT of $F(c)$, i.e., $\{T(c)_i\}_{i=0}^n$, which is the subject of this section, is useful later in obtaining the Schur-Cohn minors from the latter.

CLAIM 3.2.

$$\breve{F}^\sharp(z)_n = \lambda F^\sharp(c)_n \Big|_{c\to z}$$

Proof. Note that

$$\breve{F}^{\sharp}(z)_n = z^n \breve{\bar{F}}\left(\frac{1}{z}\right)_n = \lambda z^n \bar{F}(c)_n \Big|_{c\to z} = \lambda z^n \bar{F}\left(\frac{1-z}{\tau z}\right)_n ;$$

$$F^{\sharp}(c)_n \Big|_{c\to z} = (1+\tau c)^n \bar{F}\left(-\frac{c}{1+\tau c}\right)_n \Big|_{c\to z} = z^n \bar{F}\left(\frac{1-z}{\tau z}\right)_n .$$

The claim is thus proven. ∎

THEOREM 3.3. The rows of the complex-q-BT of $\breve{F}(z)$ and the complex-δ-BT of $F(c)$ are related by

$$\breve{T}(z)_i = \begin{cases} \lambda T(c)_i \Big|_{c\to z}, & i = n, n-2, \ldots; \\ \frac{\lambda}{\tau} T(c)_i \Big|_{c\to z}, & i = n-1, n-3, \ldots . \end{cases}$$

Proof. First, using Claim 3.2, note that

$$\breve{T}(z)_n = \lambda T(c)_n \Big|_{c\to z} .$$

Thus, Theorem 3.3 is established for $i = n$. $i = n-1$ may also be established directly. For $i = n-2, n-3, \ldots, 0$, use (2.3) and (3.3). ∎

COROLLARY 3.4.

$$\breve{t}_0^{(i)} = \begin{cases} \frac{\lambda}{\tau^i} t_i^{(i)}, & \text{for } i = n, n-2, \ldots; \\ \frac{\lambda}{\tau^{i+1}} t_i^{(i)}, & \text{for } i = n-1, n-3, \ldots, \end{cases}$$

$$\breve{t}_i^{(i)} = \begin{cases} \frac{\lambda}{\tau^i} \breve{\bar{t}}_i^{(i)}, & \text{for } i = n, n-2, \ldots; \\ \frac{\lambda}{\tau^{i+1}} \breve{\bar{t}}_i^{(i)}, & \text{for } i = n-1, n-3, \ldots . \end{cases}$$

Proof. This follows directly from Theorem 3.3. ∎

4. Schur-Cohn Minors for δ-Systems

We now develop quantities that may be considered the analogs of Schur-Cohn minors for δ-system polynomials.

LEMMA 4.1. The relationship between the complex-δ-BT of $F(c)_n \in \Im[c]_n$ and the Schur-Cohn minors $\breve{\Delta}_i$, $i = 1, 2, \ldots, n$, of $\breve{F}(z)_n \in \Im[z]_n$ is

$$\begin{aligned}\breve{\Delta}_i = \frac{\lambda^2}{2\tau^{2(n-i+1)}} \Big[&(t_{n-i+1}^{(n-i+1)} \bar{t}_{n-i}^{(n-i)} + \bar{t}_{n-i+1}^{(n-i+1)} t_{n-i}^{(n-i)}) \breve{\Delta}_{i-1} \\ &- \frac{\lambda^2}{2\tau^{2(n-i+1)}} |t_{n-i+1}^{(n-i+1)} t_{n-i}^{(n-i)}|^2 \breve{\Delta}_{i-2} \Big], \ \breve{\Delta}_0 = 1, \ \breve{\Delta}_i = 0, \ i < 0.\end{aligned}$$

Proof. Note that, the relationship between the complex-q-BT of $\breve{F}(z)_n$ and its Schur-Cohn minors are given by [25]

$$\breve{\Delta}_i = \frac{1}{2} \left[(\bar{\breve{t}}_0^{(n-i+1)} \bar{\breve{t}}_{n-i}^{(n-i)} + \breve{t}_{n-i+1}^{(n-i+1)} \bar{\breve{t}}_0^{(n-i)}) \breve{\Delta}_{i-1} - \frac{1}{2} |\breve{t}_{n-i+1}^{(n-i+1)} \breve{t}_{n-i}^{(n-i)}|^2 \breve{\Delta}_{i-2} \right],$$

with $\breve{\Delta}_0 = 1$ and $\breve{\Delta}_i = 0$, $i < 0$. Now, the claim follows from Corollary 3.4. ∎

Let

$$D = \operatorname{diag}\left\{ \frac{1}{\tau^n}, \frac{1}{\tau^{n-1}}, \ldots, \frac{1}{\tau} \right\} \in \Re^{n \times n}. \tag{4.1}$$

Then, from Lemma 4.1, $\breve{\Delta}$ in Theorem 2.5 is given by

$$\breve{\Delta} = \lambda^2 \cdot D \cdot \Delta \cdot D \tag{4.2}$$

where

$$\Delta = \begin{bmatrix} \mathrm{Re}[t_n^{(n)} \bar{t}_{n-1}^{(n-1)}] & \frac{\tau}{2}[\bar{t}_{n-1}^{(n-1)} t_{n-2}^{(n-2)}] & 0 & \cdots & 0 \\ \frac{\tau}{2}[t_{n-1}^{(n-1)} \bar{t}_{n-2}^{(n-2)}] & \mathrm{Re}[t_{n-1}^{(n-1)} \bar{t}_{n-2}^{(n-2)}] & \frac{\tau}{2}[\bar{t}_{n-2}^{(n-2)} t_{n-3}^{(n-3)}] & \ddots & 0 \\ 0 & \frac{\tau}{2}[t_{n-2}^{(n-2)} \bar{t}_{n-3}^{(n-3)}] & \mathrm{Re}[t_{n-2}^{(n-2)} \bar{t}_{n-3}^{(n-3)}] & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{Re}[t_1^{(1)} \bar{t}_0^{(0)}] \end{bmatrix}. \tag{4.3}$$

Clearly, positive definiteness of $\breve{\Delta}$ and Δ are equivalent statements. Hence, we may consider the principal minors of Δ to be the Schur-Cohn minors of $F(c)$.

DEFINITION 4.1. The Schur-Cohn minors of $F(c) \in \Im[c]_n$ are the principal minors of the tridiagonal Hermitian matrix Δ in (4.3).

Therefore, from Theorem 2.4, we have

THEOREM 4.2. The polynomial $F(c) \in \Im[c]_n$ is stable iff $\Delta_i > 0,\ i = 1, 2, \ldots, n$, where Δ_i is the $(i \times i)$-principal minor of Δ in (4.3).

Remarks.

1. Tridiagonal Hermitian matrices constitute an important class of matrices that have been extensively investigated in matrix theory literature [26]. See also [10].
2. Since the Schur-Cohn minor $\check{\Delta}_i$ obtained from the complex-q-BT are necessarily proper [10], [25], the Schur-Cohn minors defined above for δ-systems are proper as well.

In terms of the 'scaled' sequence of polynomials $\{U(\zeta)_i\}_{i=0}^{n}$, Theorem 4.2 may be stated as

COROLLARY 4.3. The polynomial $F(c) \in \Im[c]_n$ is stable iff $\tilde{\Delta}_i > 0,\ i = 1, 2, \ldots, n$, where $\tilde{\Delta}_i$ is the $(i \times i)$-principal minor of

$$\tilde{\Delta} = \begin{bmatrix} -\mathrm{Re}[u_n^{(n)} \bar{u}_{n-1}^{(n-1)}] & -\frac{1}{2}[\bar{u}_{n-1}^{(n-1)} u_{n-2}^{(n-2)}] & 0 & \cdots & 0 \\ -\frac{1}{2}[u_{n-1}^{(n-1)} \bar{u}_{n-2}^{(n-2)}] & -\mathrm{Re}[u_{n-1}^{(n-1)} \bar{u}_{n-2}^{(n-2)}] & -\frac{1}{2}[\bar{u}_{n-2}^{(n-2)} u_{n-3}^{(n-3)}] & \ddots & 0 \\ 0 & -\frac{1}{2}[u_{n-2}^{(n-2)} \bar{u}_{n-3}^{(n-3)}] & -\mathrm{Re}[u_{n-2}^{(n-2)} \bar{u}_{n-3}^{(n-3)}] & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\mathrm{Re}[u_1^{(1)} \bar{u}_0^{(0)}] \end{bmatrix}. \tag{4.3}$$

Proof. Using (3.11), and factoring out the appropriate diagonal matrices, the result immediately follows. ■

Remark. Again, notice how the use of the 'scaled' sequence simplifies the entries.

5. Algorithm for Checking Stability of 2-D δ-Systems

To check condition II of Theorem 2.2, we may adopt the following approach:

(a) Express $F(c_1, c_2) \in \Re[c_1]_{n_1}[c_2]_{n_2}$ as a polynomial in $\Im[c_2]_{n_2}$ so that its coefficients, as well as the corresponding Schur-Cohn minors, are parameterized by $c_1 \in \mathcal{T}_\delta$. Here, we have assumed that $n_1 \geq n_2$; otherwise, the roles of n_1 and n_2 may be interchanged.

(b) Check positivity of each of the Schur-Cohn minors, or positive definiteness of the tridiagonal Hermitian matrix $\Delta \in \Im^{n_2 \times n_2}$, for all $c_1 \in \mathcal{T}_\delta$ (see condition II of Theorem 2.2 and Theorem 4.2). These checks may be simplified by applying a direct extension of Siljak's result [16].

However, construction of the complex-δ-BT and the entries of Δ require complex conjugation of certain entries that are functions of $c_1 \in \mathcal{T}_\delta$. This of course complicates the scheme since $\bar{c}_1 = -c_1/(1 + \tau c_1)$, $\forall c_1 \in \mathcal{T}_\delta$. On the other hand, in dealing with 2-D q-system stability, we have $\bar{z}_1 = 1/z_1$, $\forall z_1 \in \mathcal{T}_q$. This simple relationship has led to stability checking schemes that use the complex forms of tabular forms [10] that incorporate the ***polynomial array*** method [27]. To circumvent the above difficulty, the algorithm given below uses the real-δ-BT in order to check Theorem 2.3. In the appendix, an easily implementable algorithm that yields

$$G(x, c_2) = G(x)_{n_1}(c_2)_{2n_2} = F(c_1, c_2)F(\bar{c}_1, c_2)\big|_{\substack{c_1 \in \mathcal{T}_\delta \\ x = (c_1 + \bar{c}_1)/2}} \in \Re[x]_{n_1}[c_2]_{2n_2} \tag{5.1}$$

is provided. Note that

$$c_1 \in \mathcal{T}_\delta \iff x \in [-2/\tau, 0]. \tag{5.2}$$

Before proceeding, however, it is important to note that tabular methods are useful in checking for no roots to be ***outside*** the stability region. However, since in typical 2-D stability studies the 2-D transforms are taken with positive powers [9-10], prior to applying the stability check, the following 'preparation' must be done:

(a) Condition I in Theorem 2.3 may be checked by explicitly finding the roots or applying the real-δ-BT to ensure

$$F^\sharp(c_1)(-1/\tau) \doteq (1 + \tau c_1)^{n_1} \bar{F}\left(\frac{-c_1}{1 + \tau c_1}\right)\left(-\frac{1}{\tau}\right) \neq 0,\ \forall c_1 \in \Im \setminus \mathcal{U}_\delta \tag{5.3}$$

(that is, polynomial is reciprocated with respect to c_1).

(b) First form

$$G(x)_{n_1}(c_2)_{2n_2} = \sum_{\ell=0}^{2n_2} g_\ell^{(2n_2)}(x)c_2^\ell \in \Re[x]_{n_1}[c_2]_{2n_2} \quad \text{where} \quad g_\ell^{(2n_2)}(x) = \sum_{k=0}^{n_1} g_{k\ell}^{(2n_2)} x^k \in \Re[x]_{n_1}. \tag{5.4}$$

Here $x \in [-2/\tau, 0]$. Now, condition II in Theorem 2.3 may be checked by applying the real-δ-BT to ensure

$$\begin{aligned}\tilde{G}(x)_{n_1}(c_2)_{2n_2} &\doteq \sum_{\ell=0}^{2n_2} \tilde{g}_\ell^{(2n_2)}(x)c_2^\ell \quad \text{where} \quad \tilde{g}_\ell^{(2n_2)}(x) = \sum_{k=0}^{n_1} \tilde{g}_{k\ell}^{(2n_2)} x^k \in \Re[x]_{n_1} \\ &\doteq G(x)^\sharp(c_2) \\ &= (1+\tau c_2)^{2n_2} G(x)\left(\frac{-c_2}{1+\tau c_2}\right) \neq 0,\ \forall x \in [-2/\tau, 0],\ \forall c_2 \in \Im \setminus \mathcal{U}_\delta\end{aligned} \tag{5.5}$$

(that is, polynomial is reciprocated with respect to c_2). Again, $x \in [-2/\tau, 0]$.

We will hence implicitly assume that the given 2-D δ-polynomial has already been appropriately 'prepared' as above. In addition, the construction of the real-δ-BT for $\tilde{G}(x)(c_2)$ requires ensuring [11]

$$\tilde{g}_0^{(2n_2)}(x) \neq 0 \quad \text{and} \quad \tilde{g}_{2n_2}^{(2n_2)} > 0, \quad \forall x \in [-2/\tau, 0]. \tag{5.6}$$

Violation of the first condition in (5.6) is equivalent to

$$F(c_1)(0) = 0 \quad \text{for some} \quad c_1 \in \mathcal{T}_\delta. \tag{5.7}$$

Assuming, with no loss of generality, $\tilde{g}_{2n_2}^{(2n_2)} > 0$ for some $x \in [-2/\tau, 0]$, violation of the second condition in (5.6) is equivalent to

$$F(c_1)(-1/\tau) = 0 \quad \text{for some} \quad c_1 \in \mathcal{T}_\delta. \tag{5.8}$$

Therefore, each of these violations imply instability. Verifying condition (5.7) must be included in the algorithm. Condition (5.8) is automatically verified when condition I in Theorem 2.3 is checked (see (5.3)).

Then, we have the following

THEOREM 5.1. The 2-D δ-system in (2.7) is stable iff

I. $F(c_1)(-1/\tau) \neq 0$, $\forall c_1 \in \overline{\mathcal{U}}_\delta$, and

II. $F(c_1)(0) \neq 0$, $\forall c_1 \in \mathcal{T}_\delta$, and

III. $\Delta_i(0) > 0$, $\forall i = 1, 2, \ldots, 2n_2$, and

IV. $\Delta_{2n_2}(x) > 0$, $\forall x \in [-2/\tau, 0]$, which is satisfied whenever $\Delta_{2n_2}(x) \neq 0$, $\forall x \in [-2/\tau, 0]$, together with condition III.

Here, Δ is the Hermitian matrix mentioned in Theorem 4.2 corresponding to $\tilde{G}(x)(c_2)$ where $x \in [-2/\tau, 0]$.

Conditions I and II in Theorem 5.1 are easy to carry out (they may in fact be verified by explicitly finding the roots). Condition III and IV require construction of the real-δ-BT and the Schur-Cohn minors for which we now develop polynomial arrays [27]. We also provide a scaling scheme so that the numerical reliability of the resulting algorithm is enhanced.

5.1. Polynomial array for entries of real-δ-BT

Express $G(x)(c_2)$ as

$$G(x)(c_2) = \mathbf{x}^{(n_1)^T} \cdot \mathbf{G} \cdot \mathbf{c}_2^{(2n_2)} \tag{5.9}$$

where $\mathbf{x}^{(n_1)} = [x^{n_1}, x^{n_1-1}, \ldots, 1]^T$, $\mathbf{c}_2^{(2n_2)} = [c_2^{2n_2}, c_2^{2n_2-1}, \ldots, 1]^T$, and $\mathbf{G} = \{g_{i,j}\} \in \Re^{(n_1+1)\times(2n_2+1)}$ is the coefficient matrix. Then, it is easy to show that [6]

$$\tilde{G}(x)(c_2) = \mathbf{x}^{(n_1)^T} \cdot \tilde{\mathbf{G}} \cdot \mathbf{c}_2^{(2n_2)} \quad \text{where} \quad \tilde{\mathbf{G}} = \mathbf{G}\mathcal{T}^{(2n_2)^{-1}}\mathbf{P}^{(2n_2)}\mathcal{T}^{(2n_2)}. \tag{5.10}$$

Here

$$\begin{aligned} \mathcal{T}^{(2n_2)} &= \operatorname{diag}\{\tau^{2n_2}, \tau^{2n_2-1}, \ldots, 1\} \in \Re^{(2n_2+1)\times(2n_2+1)}; \\ \mathbf{P}^{(2n_2)} &= \{p_{ij}\} \in \Re^{(2n_2+1)\times(2n_2+1)} \quad \text{where} \quad p_{ij} = (-1)^{2n_2+1-i}\rho_{ij}. \end{aligned} \tag{5.11}$$

The elements ρ_{ij}, which in fact are those of the Pascal's triangle, are given by

$$\rho_{ij} = \begin{cases} 0, & \text{for } i < j; \\ 1, & \text{for } i = j; \\ \rho_{i-1,j-1} + \rho_{i-1,j}, & \text{elsewhere.} \end{cases} \tag{5.12}$$

The real-δ-BT is constructed using the 'scaled' polynomial sequence in (3.11-14). Let

$$\begin{aligned} \tilde{H}(y)(\zeta) &= \tilde{G}(x)(c_2)\,\Big|_{\substack{c_2=-\zeta/\tau \\ x=-y/\tau}}; \\ H(y)(\zeta) &= \tilde{G}(x)^{\sharp}(c_2)\,\Big|_{\substack{c_2=-\zeta/\tau \\ x=-y/\tau}} = G(x)(c_2)\,\Big|_{\substack{c_2=-\zeta/\tau \\ x=-y/\tau}}. \end{aligned} \tag{5.13}$$

Note that, $x \in [-2/\tau, 0]$ iff $y \in [0,2]$. Now, using (5.9-12), row #$2n_2$ and $2n_2 - 1$ of the corresponding 'scaled' real-δ-BT are given by

$$\begin{aligned} U(y)(\zeta)_{2n_2} &= \sum_{\ell=0}^{2n_2} u_\ell^{(2n_2)} \zeta^\ell = \tilde{H}(y)(\zeta) + H(y)(\zeta) \\ &= \mathbf{y}^{(n_1)^T} \cdot \hat{\mathcal{T}}^{(n_1)^{-1}} \mathbf{G} \mathcal{T}^{(2n_2)^{-1}} (\hat{I}^{(2n_2)} + \hat{\mathbf{P}}^{(2n_2)}) \cdot \zeta^{(2n_2)}; \\ U(y)(\zeta)_{2n_2-1} &= \sum_{\ell=0}^{2n_2-1} u_\ell^{(2n_2-1)} \zeta^\ell = \frac{\tilde{H}(y)(\zeta) - H(y)(\zeta)}{-\zeta/\tau} \\ &= \mathbf{y}^{(n_1)^T} \cdot \hat{\mathcal{T}}^{(n_1)^{-1}} \tau \mathbf{G} \mathcal{T}^{(2n_2)^{-1}} (\hat{I}^{(2n_2)} - \hat{\mathbf{P}}^{(2n_2)}) \cdot \begin{bmatrix} \zeta^{(2n_2-1)} \\ 0 \end{bmatrix}, \end{aligned} \tag{5.14}$$

where $\zeta^{(2n_2)} = [\zeta^{2n_2}, \zeta^{2n_2-1}, \ldots, 1]^T$, and

$$\begin{aligned} \hat{\mathcal{T}}^{(n_1)} &= \mathrm{diag}\{(-\tau)^{n_1}, (-\tau)^{n_1-1}, \ldots, 1\} \in \Re^{(n_1+1)\times(n_1+1)}; \\ \hat{I}^{(2n_2)} &= \mathrm{diag}\{(-1)^{2n_2}, (-1)^{2n_2-1}, \ldots, 1\} \in \Re^{(2n_2+1)\times(2n_2+1)}; \\ \hat{\mathbf{P}}^{(2n_2)} &= \{\hat{p}_{ij}\} \in \Re^{(2n_2+1)\times(2n_2+1)} \quad \text{where} \quad \hat{p}_{ij} = (-1)^{i+j} \rho_{ij}. \end{aligned} \tag{5.15}$$

Each element of the remaining rows is of the form

$$u_\ell^{(i)}(y) = \frac{n_\ell^{(i)}(y)}{d^{(i)}(y)}, \; \ell = 0, 1, \ldots, i, \; i = 2n_2, 2n_2 - 1, \ldots, 0, \tag{5.16}$$

where $n_\ell^{(i)}(y) \in \Re[y]_{\sigma^{(i)}}$ and $d^{(i)}(y) \in \Re[y]_{\varsigma^{(i)}}$. Substituting in (3.12), it is easy to show that, for $\ell = 0, 1, \ldots, i$,

$$\begin{aligned} n_\ell^{(i)} &= n_{i+2}^{(i+2)}(n_{\ell-1}^{(i+1)} - 2n_\ell^{(i+1)}) - n_{i+1}^{(i+1)} n_\ell^{(i+2)} + n_{\ell-1}^{(i)}, \quad \text{for} \quad i = 2n_2 - 2, \ldots, 0; \\ d^{(i)} &= \begin{cases} 1, & \text{for } i = 2n_2, 2n_2 - 1, \\ d^{(i+2)} n_{i+1}^{(i+1)}, & \text{for } i = 2n_2 - 2, \ldots, 0. \end{cases} \end{aligned} \tag{5.17}$$

Note that $u_\ell^{(2n_2)} = n_\ell^{(2n_2)}$ and $u_\ell^{(2n_2-1)} = n_\ell^{(2n_2-1)}$. Moreover

$$\begin{aligned} \sigma^{(i)} &= \begin{cases} n_1, & \text{for } i = 2n_2, 2n_2 - 1, \\ \sigma^{(i+2)} + \sigma^{(i+1)}, & \text{for } i = 2n_2 - 2, \ldots, 0; \end{cases} \\ \varsigma^{(i)} &= \begin{cases} 0, & \text{for } i = 2n_2, 2n_2 - 1, \\ \sigma^{(i)} - n_1, & \text{for } i = 2n_2 - 2, \ldots, 0. \end{cases} \end{aligned} \tag{5.18}$$

Scaling scheme. Let us scale rows $\#2n_2$ and $\#(2n_2-1)$ so that each coefficient takes values in $[-1,1]$. Correspondingly, for $\ell = 0,1,\ldots,i$; $i = 2n_2, 2n_2-1$, let

$$\begin{aligned} n_\ell^{(i)} &= \lambda^{(i)} \check{n}_\ell^{(i)}; \\ d^{(i)} &= \gamma^{(i)} \check{d}^{(i)}, \end{aligned} \tag{5.19}$$

where $\lambda^{(i)}, \gamma^{(i)} > 0$, $i = 2n_2, 2n_2-1$, are the scaling constants and $\check{[\cdot]}$ denote scaled quantities. Note that

$$\begin{aligned} \frac{n_\ell^{(2n_2)}}{d^{(2n_2)}} &= \frac{\lambda^{(2n_2)}}{\gamma^{(2n_2)}} \frac{\check{n}_\ell^{(2n_2)}}{\check{d}^{(2n_2)}}; \\ \frac{n_\ell^{(2n_2-1)}}{d^{(2n_2-1)}} &= \frac{\lambda^{(2n_2-1)}}{\gamma^{(2n_2-1)}} \frac{\check{n}_\ell^{(2n_2-1)}}{\check{d}^{(2n_2-1)}}. \end{aligned} \tag{5.20}$$

Now, substituting in (5.17-18), we get

$$\begin{aligned} \frac{n_\ell^{(2n_2-2)}}{\lambda^{(2n_2)}\lambda^{(2n_2-1)}} &= \check{n}_{2n_2}^{(2n_2)}(\check{n}_{\ell-1}^{(2n_2-1)} - 2\check{n}_\ell^{(2n_2-1)}) - \check{n}_{2n_2-1}^{(2n_2-1)}\check{n}_\ell^{(2n_2)} + \frac{n_{\ell-1}^{(2n_2-2)}}{\lambda^{(2n_2)}\lambda^{(2n_2-1)}}; \\ \frac{d^{(2n_2-2)}}{\gamma^{(2n_2)}\lambda^{(2n_2-1)}} &= \check{d}^{(2n_2)}\check{n}_{2n_2-1}^{(2n_2-1)}. \end{aligned} \tag{5.21}$$

It can now be seen that, it is only necessary to compute the quantities on the left hand side of (5.21). Then, one may scale these to get

$$\begin{aligned} n_\ell^{(2n_2-2)} &= \lambda^{(2n_2)}\lambda^{(2n_2-1)}\lambda^{(2n_2-2)}\check{n}_\ell^{(2n_2-2)}; \\ d^{(2n_2-2)} &= \gamma^{(2n_2)}\gamma^{(2n_2-2)}\lambda^{(2n_2-1)}\check{d}^{(2n_2-2)}. \end{aligned} \tag{5.22}$$

Note that

$$\frac{n_\ell^{(2n_2-2)}}{d^{(2n_2-2)}} = \frac{\lambda^{(2n_2)}\lambda^{(2n_2-2)}}{\gamma^{(2n_2)}\gamma^{(2n_2-2)}} \frac{\check{n}_\ell^{(2n_2-2)}}{\check{d}^{(2n_2-2)}}. \tag{5.23}$$

Continuing in this manner, the computation of the entries of real-δ-BT may be summarized as follows:

(a) From (5.14), compute $n_\ell^{(i)}, d^{(i)}$, $i = 2n_2, 2n_2-1$.

(b) From (5.19), use scaling constants $\lambda^{(i)}, \gamma^{(i)}$, $i = 2n_2, 2n_2-1$, to get $\check{n}_\ell^{(i)}, \check{d}^{(i)}$, $i = 2n_2, 2n_2-1$.

(c) From (5.21), for $\ell = 0,1,\ldots,i$; $i = 2n_2-2,\ldots,0$, compute

$$\begin{aligned} \frac{n_\ell^{(i)}}{K_n^{(i)}} &= \check{n}_{i+2}^{(i+2)}(\check{n}_{\ell-1}^{(i+1)} - 2\check{n}_\ell^{(i+1)}) - \check{n}_{i+1}^{(i+1)}\check{n}_\ell^{(i+2)} + \frac{n_{\ell-1}^{(i)}}{K_n^{(i)}}; \\ \frac{d^{(i)}}{K_d^{(i)}} &= \check{d}^{(i+2)}\check{n}_{i+1}^{(i+1)}, \end{aligned} \tag{5.24}$$

and use scaling constants $\lambda^{(i)}, \gamma^{(i)}$, $i = 2n_2 - 2, \ldots, 0$, to get $\check{n}_\ell^{(i)}, \check{d}^{(i)}$, $i = 2n_2 - 2, \ldots, 0$. Here, $K_n^{(i)}$ and $K_d^{(i)}$ are constants.

(d) Notice the relationships

$$\frac{n_\ell^{(i)}}{d^{(i)}} = \begin{cases} \frac{\lambda^{(2n_2)}\lambda^{(2n_2-2)}\ldots\lambda^{(i)}}{\gamma^{(2n_2)}\gamma^{(2n_2-2)}\ldots\gamma^{(i)}}\frac{\check{n}_\ell^{(i)}}{\check{d}^{(i)}}, & \text{for } i = 2n_2, 2n_2 - 2, \ldots, 0; \\ \frac{\lambda^{(2n_2-1)}\lambda^{(2n_2-3)}\ldots\lambda^{(i)}}{\gamma^{(2n_2-1)}\gamma^{(2n_2-3)}\ldots\gamma^{(i)}}\frac{\check{n}_\ell^{(i)}}{\check{d}^{(i)}}, & \text{for } i = 2n_2 - 1, 2n_2 - 3, \ldots, 1. \end{cases} \tag{5.25}$$

5.2. Polynomial array for Schur-Cohn minors

Each Schur-Cohn minor obtained from the table, in general, will be of the form

$$\Delta_i(y) = \frac{N^{(i)}(y)}{D^{(i)}(y)} \in \Re(y),\ i = 1, 2, \ldots, 2n_2, \tag{5.26}$$

where $N^{(i)}(y) \in \Re[y]_{\rho^{(i)}}$ and $D^{(i)}(y) \in \Re[y]_{\varrho^{(i)}}$. From Corollary 4.3, we get

$$\Delta_i(y) = -u_{2n_2-i+1}^{(2n_2-i+1)} u_{2n_2-i}^{(2n_2-i)} \Delta_{i-1} - \frac{1}{4} u_{2n_2-i+1}^{(2n_2-i+1)^2} u_{2n_2-i}^{(2n_2-i)^2} \Delta_{i-2},\ i = 1, 2, \ldots, 2n_2, \tag{5.27}$$

where $\Delta_0 \doteq 1$ and $\Delta_i = 0$, $\forall i < 0$.

Remark. Actually, as in [10], one may show that, for stability determination purposes, only the numerator polynomials of Δ_i need be computed. However, to contain the orders of the resulting polynomials, and hence improve numerically conditioning, we do not recommend this scheme.

Scaling scheme. Due to the scaling of entries of the real-δ-BT, computation of Δ_i, $i = 1, 2, \ldots, 2n_2$, may be modified as follows: Let

$$\begin{aligned} \Delta_1 &= -u_{2n_2}^{(2n_2)} u_{2n_2-1}^{(2n_2-1)} \\ &= -\frac{\lambda^{(2n_2)}\lambda^{(2n_2-1)}}{\gamma^{(n2_2)}\gamma^{(2n_2-1)}} \frac{\check{n}_{2n_2}^{(2n_2)}\check{n}_{2n_2-1}^{(2n_2-1)}}{\check{d}^{(2n_2)}\check{d}^{(2n_2-1)}}. \end{aligned} \tag{5.28}$$

Hence, it is only necessary to compute the quantity

$$\check{\Delta}_1 \doteq -\frac{\check{n}_{2n_2}^{(2n_2)}\check{n}_{2n_2-1}^{(2n_2-1)}}{\check{d}^{(2n_2)}\check{d}^{(2n_2-1)}}. \tag{5.29}$$

Continuing in this manner, the computation of the Schur-Cohn minors may be summarized as follows: From (5.27), for $i = 1, 2, \ldots, 2n_2$, compute

$$\check{\Delta}_i = -\frac{\check{n}_{2n_2-i+1}^{(2n_2-i+1)}\check{n}_{2n_2-i}^{(2n_2-i)}}{\check{d}^{(2n_2-i+1)}\check{d}^{(2n_2-i)}}\left[\check{\Delta}_{i-1} + \frac{1}{4}\frac{\lambda^{(2n_2-i)}}{\gamma^{(2n_2-i)}}\frac{\check{n}_{2n_2-i+1}^{(2n_2-i+1)}\check{n}_{2n_2-i}^{(2n_2-i)}}{\check{d}^{(2n_2-i+1)}\check{d}^{(2n_2-i)}}\check{\Delta}_{i-2}\right], \tag{5.30}$$

where $\check{\Delta}_0 \doteq 1$ and $\check{\Delta}_i = 0$, $\forall i < 0$.

Remark. Note that, since $\Delta_i(y)$ is necessarily a proper polynomial (that is, denominator divides numerator properly with no remainder), and not a rational polynomial (see Remark 2 after Theorem 4.2), it is easy to see that $\check{d}^{(2n_2-i+1)}\check{d}^{(2n_2-i)}$ must divide $\check{n}_{2n_2-i+1}^{(2n_2-i+1)}\check{n}_{2n_2-i}^{(2n_2-i)}$ exactly.

5.3. Algorithm

The following result, which is the basis of the stability checking algorithm, is now obvious from [10] and Theorem 5.1:

THEOREM 5.4. The 2-D δ-system in (2.7) is stable iff
I. $F(c_1, -1/\tau) \neq 0$, $\forall c_1 \in \overline{\mathcal{U}}_\delta$, and
II. $F(c_1)(0) \neq 0$, $\forall c_1 \in \mathcal{T}_\delta$, and
III. $\Delta_i(0) > 0$, $\forall i = 1, 2, \ldots, 2n_2$, and
IV. $\Delta_{2n_2}(y) \neq 0$, $\forall y \in [0, 2]$.

The 2-D stability checking algorithm may now be summarized as follows:

GIVEN.
A 2-D δ-polynomial $F(c_1, c_2) \in \Re[c_1]_{n_1}[c_2]_{n_2}$. Without any loss of generality, assume that $n_1 \geq n_2$, and express $F(c_1, c_2)$ as $F(c_1)_{n_1}(c_2)_{n_2}$.

STEP I. Condition I of Theorem 5.4:
Apply an explicit root location procedure. If result is satisfactory, proceed; otherwise, system is unstable.

STEP II. Condition II of Theorem 5.4:

Apply an explicit root location procedure. If result is satisfactory, proceed; otherwise, system is unstable.

STEP III.

Form $G(y)(c_2)$ using the algorithm in the appendix; then form $U(y)(\zeta)_{2n_2}$ and $U(y)(\zeta)_{2n_2-1}$ from (5.14). These yield $n_\ell^{(2n_2)}$ and $n_\ell^{(2n_2-1)}$. Of course, $d^{(2n_2)} = d^{(2n_2-1)} = 1$.

From (5.19), obtain $\check{n}_\ell^{(2n_2)}$, $\check{n}_\ell^{(2n_2-1)}$, and the associated scaling constants $\lambda^{(2n_2)}$ and $\lambda^{(2n_2-1)}$. Of course, $\check{d}^{(2n_2)} = \check{d}^{(2n_2-1)} = 1$ and $\gamma^{(2n_2)} = \gamma^{(2n_2-1)} = 1$.

STEP IV. Condition III of Theorem 5.4:

Form $\check{\Delta}_1(y)$ from (5.30) and check whether $\check{\Delta}_1(0) > 0$.

If result is satisfactory, form $\check{n}_\ell^{(2n_2-2)}$ and $\check{d}^{(2n_2-2)}$ and the associated scaling constants $\lambda^{(2n_2-2)}$ and $\gamma^{(2n_2-2)}$ from (5.24). Form $\check{\Delta}_2(y)$ from (5.30) and check whether $\check{\Delta}_2(0) > 0$. If result is satisfactory, proceed likewise until $\check{\Delta}_{2n_2}(0) > 0$ is checked. Note that, this requires checking of only the constant coefficients. If result is satisfactory, proceed; otherwise, if the check fails at any $i = 1, 2, \ldots, 2n_2$, system is unstable.

STEP V. Condition IV of Theorem 5.4:

Apply an explit root location procedure to check whether $\check{\Delta}_{2n_2}(y) \neq 0$, $\forall y \in [0, 2]$.

Remarks. The possible numerical difficulties that may arise in using explit root location procedures may be avoided as follows: (a) Steps I and II may be verified using the real-δ-BT [6], and (b) step V may be verified by the Sturm sequence method.

6. Example

The stability checking algorithm presented in the previous section is now illustrated through an example. Polynomial entries are denoted using a self-explanatory shorthand notation where the highest degree coefficient is written first. Moreover, only four decimal digital on the mantissa are shown.

Consider the 2-D polynomial

$$F(c_1, c_2) = [c_1^2 \quad c_1 \quad 1] \begin{bmatrix} 1 & 50 & 740 \\ 52 & 2700 & 38480 \\ 740 & 38480 & 547600 \end{bmatrix} \begin{bmatrix} c_2^2 \\ c_2 \\ 1 \end{bmatrix}$$

with the sampling time $\tau = 0.1$ s.

STEP I. Condition I of Theorem 5.4:

By applying an explicit root location procedure, one can show that

$$F(c_1)(-1/\tau) = 340c_1^2 + 16680c_1 + 236800 \neq 0, \ \forall c_1 \in \overline{\mathcal{U}}_\delta.$$

STEP II. Condition II of Theorem 5.4:

By applying an explicit root location procedure, one can show that

$$F(c_1)(0) = 740c_1^2 + 38480c_1 + 547600 \neq 0, \ \forall c_1 \in \mathcal{T}_\delta.$$

STEP III. Using the algorithm in Appendix, we get

$$G(y)(\zeta) = [y^2 \quad y \quad 1] \begin{bmatrix} 1.2800e+03 & 1.2992e+05 & 5.1904e+06 & 9.6141e+07 & 7.0093e+08 \\ 5.2480e+04 & 5.4011e+06 & 2.1662e+08 & 3.9968e+09 & 2.8738e+10 \\ 5.4760e+05 & 5.6950e+07 & 2.2912e+09 & 4.2143e+10 & 2.9987e+11 \end{bmatrix} \begin{bmatrix} \zeta^2 \\ \zeta \\ 1 \end{bmatrix}.$$

After scaling, rows #4 and #3 are computed as follows:

$$\tilde{n}_4^{(4)} = [1.2859e-02, -5.0693e-02, 5.1315e-0$$

$$\tilde{n}_3^{(4)} = [-7.9833e-02, 3.1991e-01, -3.2798e$$

$$\tilde{n}_2^{(4)} = [1.9671e-01, -7.9909e-01, 8.2798e-01];$$

$$\tilde{n}_1^{(4)} = [-2.3375e-01, 9.5836e-01, -1.0000e+00];$$

$$\tilde{n}_0^{(4)} = [1.1687e-01, -4.7918e-01, 5.0000e-01],$$

with $\lambda^{(4)} = 1.1995e + 12$, and

$$\check{n}_3^{(3)} = [-2.4050e-02,\ 9.4053e-02,\ -9.4595e-02];$$
$$\check{n}_2^{(3)} = [1.3044e-01,\ -5.1542e-01,\ 5.2252e-01];$$
$$\check{n}_1^{(3)} = [-2.4703e-01,\ 9.8194e-01,\ -1.0000e+00];$$
$$\check{n}_0^{(3)} = [1.6469e-01,\ -6.5463e-01,\ 6.6667e-01],$$

with $\lambda^{(3)} = 5.3490e + 10$. Of course, $\check{d}^{(4)} = \check{d}^{(3)} = 1$ with $\gamma^{(4)} = \gamma^{(3)} = 1$.

STEP IV. Condition III of Theorem 5.4:

We get

$$\check{\Delta}_1 = [3.0926e-04,\ -2.4286e-03,\ 7.2184e-03,\ -9.6217e-03,\ 4.8541e-03].$$

Clearly, $\check{\Delta}_1(0) = 4.8541e - 03 > 0$.

Now, row #2 is computed as follows:

$$\check{n}_2^{(2)} = [-8.8619e-03,\ 6.8253e-02,\ -1.9920e-01,\ 2.6099e-01,\ -1.2957e-01];$$
$$\check{n}_1^{(2)} = [3.3584e-02,\ -2.5969e-01,\ 7.6068e-01,\ -1.0000e+00,\ 4.9793e-01];$$
$$\check{n}_0^{(2)} = [-3.3584e-02,\ 2.5969e-01,\ -7.6068e-01,\ 1.0000e+00,\ -4.9793e-01],$$

with $\lambda^{(2)} = 4.2420e - 02$. Also,

$$\check{d}^{(2)} = [-2.5424e-01,\ 9.9428e-01,\ -1.0000e+00],$$

with $\gamma^{(2)} = 9.4595e - 02$. We get

$$\check{\Delta}_2 = [1.8046e-07,\ -2.8190e-06,\ 1.9343e-05,\ -7.6148e-05,\ 1.8810e-04,$$
$$-2.9857e-04,\ 2.9737e-04,\ -1.6992e-04,\ 4.2654e-05].$$

Clearly, $\check{\Delta}_2(0) = 4.2654e - 05 > 0$.

Now, row #1 is computed as follows:

$$\check{n}_1^{(1)} = [2.5168e-03,\ -2.8515e-02,\ 1.3555e-01,\ -3.4597e-01,\ 5.0000e-01,$$
$$-3.8792e-01,\ 1.2623e-01];$$
$$\check{n}_0^{(1)} = [-5.0336e-03,\ 5.7031e-02,\ -2.7110e-01,\ 6.9194e-01,\ -1.0000e+00,$$
$$7.7584e-01,\ -2.5246e-01],$$

with $\lambda^{(1)} = 3.0980e-02$. Also,

$$\check{d}^{(1)} = [-3.3954e-02,\ 2.6151e-01,\ -7.6322e-01,\ 1.0000e+00,\ -4.9646e-01],$$

with $\gamma^{(1)} = 2.6099e-01$. We get

$$\begin{aligned}\check{\Delta}_3 = [&4.0500e-10,\ -9.3525e-09,\ 9.9260e-08,\ -6.4020e-07,\ 2.7947e-06,\\ &-8.6990e-06,\ 1.9797e-05,\ -3.3188e-05,\ 4.0679e-05,\ -3.5552e-05,\\ &2.1029e-05,\ -7.5594e-06,\ 1.2489e-06].\end{aligned}$$

Clearly, $\check{\Delta}_3(0) = 1.2489e-06 > 0$.

Now, row #0 is computed as follows:

$$\begin{aligned}\check{n}_0^{(0)} = [&-1.0487e-04,\ 1.9379e-03,\ -1.6174e-02,\ 8.0291e-02,\ -2.6251e-01,\\ &5.9070e-01,\ -9.2642e-01,\ 1.0000e+00,\ -7.1104e-01,\ 3.0076e-01,\\ &-5.7473e-02],\end{aligned}$$

with $\lambda^{(0)} = 4.4719e-02$. Also,

$$\begin{aligned}\check{d}^{(0)} = [&-6.7946e-04,\ 1.0355e-02,\ -6.9373e-02,\ 2.6679e-01,\ -6.4420e-01,\\ &1.0000e+00,\ -9.7458e-01,\ 5.4519e-01,\ -1.3404e-01],\end{aligned}$$

with $\gamma^{(0)} = 9.4174e-01$. We get

$$\begin{aligned}\check{\Delta}_4 = [&4.3531e-12,\ -1.3058e-10,\ 1.8400e-09,\ -1.6166e-08,\ 9.9118e-08,\\ &-4.4970e-07,\ 1.5618e-06,\ -4.2352e-06,\ 9.0628e-06,\ -1.5355e-05,\\ &2.0530e-05,\ -2.1433e-05,\ 1.7129e-05,\ -1.0130e-05,\ 4.1814e-06,\\ &-1.0762e-06,\ 1.3014e-07].\end{aligned}$$

Clearly, $\check{\Delta}_4(0) = 1.3014e-07 > 0$.

STEP V. Condition IV of Theorem 5.4:

By applying an explicit root location procedure, one can show that

$$\check{\Delta}_4(y) \neq 0,\ \forall y \in [0, 2].$$

Thus, we conclude that $F(c_1, c_2)$ is stable.

7. Conclusion and Final Remarks

In this paper, we have developed an efficient stability checking algorithm applicable for 2-D δ-system characteristic polynomials. Our purpose here is to obtain a direct algorithm due to the possible numerical disadvantages associated with indirect methods that utilize transformation techniques.

In arriving at the algorithm, the following contributions have been made: (a) Tabular method of stability checking applicable for δ-system polynomials possibly possessing complex-valued coefficients, (b) quantities that may be regarded as the Schur-Cohn minors applicable for such systems, and (c) polynomial arrays for computing both table entries and Schur-Cohn minors.

The proposed Schur-Cohn minors lets one use a Siljak-like simplification [16] in the stability check. Although the algorithm utilizes only the real-δ-BT, results regarding the Schur-Cohn minors are in fact valid for the more general complex-valued coefficient case as well.

As in [10], it is possible to develop the algorithm such that only the numerator polynomials of the entries of the real-δ-BT and the Schur-Cohn minors are computed. Then, we do not require polynomial division operations. However, our experience has been that such a scheme is prone to be numerically unreliable. This is mainly due to the explosion of polynomial degree especially in computing the Schur-Cohn minors. To avoid these difficulties and enhance numerical reliability, we have (a) introduced a scaling scheme, and (b) used polynomial division to contain the polynomial degree. The latter is not new; in fact, MJT also uses this. If the user is interested in implementing the algorithm using PRO-MATLAB [28], these polynomial division operations may be conveniently performed using the routine **deconv**.

We believe that a suitable scaling strategy can improve the numerical reliability of the MJT as well. The authors are currently looking into this.

The algorithm developed is easily implementable on a computer. The authors have implemented it via a C-language routine that the interested reader may request from the second author.

Acknowledgement

The authors are indebted to Professor Eliahu I. Jury for many helpful discussions and the reviewers for their careful examination of the manuscript and constructive suggestions. The first author's research work was partially supported by the Office of Naval Research (ONR) through the grant N00014-94-1-0454. This support is gratefully acknowledged.

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Appendix. Algorithm to obtain $G(x)_{n_1}(c_2)_{2n_2}$ from $F(c_1)_{n_1}(c_2)_{n_2}$

Given

$$F(c_1)_{n_1}(c_2)_{n_2} = \sum_{\ell=0}^{n_2} f_\ell(c_1) \cdot c_2^\ell \quad \text{where} \quad f_\ell(c_1) = \sum_{k=0}^{n_1} f_{k,\ell} \cdot c_1^k,\ c_1 \in \mathcal{T}_\delta, \tag{a.1}$$

we now develop an algorithm that yields

$$G(x)_{n_1}(c_2)_{2n_2} \doteq \sum_{j=0}^{2n_2} g_j(x) \cdot c_2^j = F(c_1)_{n_1}(c_2)_{n_2} \cdot F(\bar{c}_1)_{n_1}(c_2)_{n_2},\ c_1 \in \mathcal{T}_\delta. \tag{a.2}$$

First, we see

$$\begin{aligned} G(x)(c_2) &= \sum_{\ell=0}^{n_2} \sum_{j=0}^{n_2} f_\ell(c_1) f_j(\bar{c}_1) \cdot c_2^{\ell+j} \\ &= \sum_{\ell=0}^{n_2} \sum_{j=\ell}^{n_2+\ell} f_\ell(c_1) f_{j-\ell}(\bar{c}_1) \cdot c_2^j = \sum_{j=0}^{2n_2} \sum_{\ell=0}^{j} f_\ell(c_1) f_{j-\ell}(\bar{c}_1) \cdot c_2^j \end{aligned} \tag{a.3}$$

(quantities with negative subscripts are taken to be zero). Hence, comparing (a.2-3), we get

$$\begin{aligned} g_j(x) &= \sum_{\ell=0}^{j} f_\ell(c_1) f_{j-\ell}(\bar{c}_1) = \sum_{\ell=0}^{j} \left[\sum_{k=0}^{n_1} \sum_{i=0}^{n_1} f_{k,\ell} f_{i,j-\ell} \cdot c_1^k \bar{c}_1^i \right] \\ &= \sum_{\ell=0}^{j} \left[\sum_{k=0}^{n_1} f_{k,\ell} f_{k,j-\ell} \cdot (c_1 \bar{c}_1)^k + \sum_{k=0}^{n_1} \sum_{\substack{i=0 \\ i \neq k}}^{n_1} f_{k,\ell} f_{i,j-\ell} \cdot c_1^k \bar{c}_1^i \right] \\ &= \sum_{\ell=0}^{j} \sum_{k=0}^{n_1} f_{k,\ell} f_{k,j-\ell} \cdot (c_1 \bar{c}_1)^k + X \end{aligned} \tag{a.4}$$

where

$$\begin{aligned} X &= \sum_{\ell=0}^{j} \sum_{k=0}^{n_1} \left[\sum_{\substack{i=0 \\ i \neq k}}^{k} f_{k,\ell} f_{i,j-\ell} \cdot c_1^k \bar{c}_1^i + \sum_{\substack{i=k \\ i \neq k}}^{n_1} f_{k,\ell} f_{i,j-\ell} \cdot c_1^k \bar{c}_1^i \right] \\ &= \sum_{\ell=0}^{j} \left[\sum_{k=0}^{n_1} \sum_{\substack{i=0 \\ i \neq k}}^{k} f_{k,\ell} f_{i,j-\ell} \cdot c_1^k \bar{c}_1^i + \sum_{k=0}^{n_1} \sum_{\substack{i=0 \\ i \neq k}}^{k} f_{i,\ell} f_{k,j-\ell} \cdot c_1^i \bar{c}_1^k \right] \\ &= \sum_{\ell=0}^{j} \left[\sum_{k=0}^{n_1} \sum_{\substack{i=0 \\ i \neq k}}^{k} \left(f_{k,\ell} f_{i,j-\ell} \cdot c_1^{k-i} + f_{i,\ell} f_{k,j-\ell} \cdot \bar{c}_1^{k-i} \right) \cdot (c_1 \bar{c}_1)^i \right] \\ &= \sum_{k=0}^{n_1} \sum_{i=0}^{k} (c_1 \bar{c}_1)^i \sum_{\ell=0}^{j} \left[f_{k,\ell} f_{i,j-\ell} \cdot c_1^{k-i} + f_{i,j-\ell} f_{k,\ell} \cdot \bar{c}_1^{k-i} \right]. \end{aligned} \tag{a.5}$$

Let us use the notation

$$c_1^{(n)} = \frac{c_1^n + \bar{c}_1^n}{2}, \; c_1 \in \mathcal{T}_\delta, \; n = 0, 1, \ldots. \tag{a.6}$$

Noting that, for $c_1 \in \mathcal{T}_\delta$,

$$\bar{c}_1 = -\frac{c_1}{1 + \tau c_1}, \tag{a.7}$$

it is easy to show that

$$c_1 \bar{c}_1 = -\frac{2}{\tau} c_1^{(1)}. \tag{a.8}$$

Substituting in (a.5), we get

$$X = \sum_{\ell=0}^{j} \left[\sum_{k=0}^{n_1} \sum_{i=0}^{k-1} 2 f_{k,\ell} f_{i,j-\ell} \left(\frac{-2}{\tau} \right)^i \cdot c_1^{(1)^i} c_1^{(k-i)} \right]. \tag{a.9}$$

Substituting in (a.4), we get

$$g_j(x) = \sum_{\ell=0}^{j} \sum_{k=0}^{n_1} \left[f_{k,\ell} f_{k,j-\ell} \left(\frac{-2}{\tau} \right)^k \cdot c_1^{(1)^k} + \sum_{i=0}^{k-1} 2 f_{k,\ell} f_{i,j-\ell} \left(\frac{-2}{\tau} \right)^i \cdot c_1^{(1)^i} c_1^{(k-i)} \right]. \tag{a.10}$$

Now, in order to develop the algorithm, we need a recursive procedure to compute $c_1^{(n)}$, $n = 0, 1, \ldots$. To proceed, we note that

$$\begin{aligned} c_1^{(n)} &= \frac{(c_1 + \bar{c}_1)(c_1^{n-1} + \bar{c}_1^{n-1}) - c_1 \bar{c}_1 (c_1^{n-2} + \bar{c}_1^{n-2})}{2} \\ &= 2 c_1^{(1)} \left(c_1^{(n-1)} + \frac{1}{\tau} c_1^{(n-2)} \right), \; n = 2, 3, \ldots. \end{aligned} \tag{a.11}$$

Let

$$c_1^{(n)} = \sum_{i=0}^{n} c_{1,i}^{(n)} x^i \tag{a.12}$$

where

$$c_1^{(1)} \doteq x. \tag{a.13}$$

Remark. Note that

$$c_1^{(0)} = 1. \tag{a.14}$$

Substituting (a.12) in (a.11), and equating similar coefficients, we get

$$c_{1,i}^{(n)} = 2 \left(c_{1,i-1}^{(n-1)} + \frac{1}{\tau} c_{1,i-1}^{(n-2)} \right), \; i = 0, \ldots, n, \; n = 2, 3, \ldots. \tag{a.15}$$

For instance, $c_1^{(n)}$, $n = 0, 1, \ldots, 5$, may be conveniently obtained from

$$\begin{bmatrix} c_1^{(0)} \\ c_1^{(1)} \\ c_1^{(2)} \\ c_1^{(3)} \\ c_1^{(4)} \\ c_1^{(5)} \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 2/\tau & 2 & & & \\ 0 & 0 & 6/\tau & 4 & & \\ 0 & 0 & 4/\tau^2 & 16/\tau & 8 & \\ 0 & 0 & 0 & 20/\tau^2 & 40/\tau & 16 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix}.$$