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Frequency Response of Multipass Shell-and-Tube Heat Exchangers

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Transfer functions are derived for exchangers with one shell pass and 2n tube passes. It is shown how the method of derivation may be generalized to apply to exchangers with an arbitrary number of shell-and-tube passes. Distributed thermal capacity in pipe walls may be introduced as such, lumped, or neglected entirely.

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NOMENCLATURE

The following nomenclature is used in the paper:

$$a_1 = \frac{h_T\ A}{W_T\ C_T} \quad (dl)$$

$$a_2 = \frac{h_s\ A}{W_s\ C_s} \quad (dl)$$

A = heat transfer area per pass (sq ft)

$$b_1 = \frac{h_T\ A}{c_w\ \rho_w\ F_w\ V_R} \quad (dl)$$

$$b_2 = \frac{h_s\ A}{c_w\ \rho_w\ F_w\ V_R} \quad (dl)$$

c = heat capacity (BTU/lb F°)

$C,\ C'$ = arbitrary constants (dl)

d = thermal diffusivity of pipe wall (sq ft/hr)

D = intermediate parameter (dl)

e = 2.7182818...   (dl)

E = intermediate parameter (°F)

f = intermediate parameter (dl)

F = cross sectional area of metal in pipe wall (sq ft)

g = intermediate parameter (dl)

h = heat transfer coefficient (BTU/hr. sq ft F°)

k = thermal conductivity (BTU/hr ft F°)

K = intermediate parameter (dl)

L = tube length per pass (ft)

m = number of shell passes (dl)

M = intermediate parameter   (dl)

n = number of tube passes in each direction per shell pass   (dl)

N = intermediate parameter   (dl)

p = intermediate parameter   (dl)

P = heat transfer perimeter of pipe   (ft)

q = constant of integration   (°F)

$r = \dfrac{V_T}{V_S}$ (dl)

R = intermediate parameter   (dl)

s = Laplace transform variable   (dl)

S = shellside fluid temperature   (°F)

$\bar{S}$ = Laplace transform of S (°F)

t = time   (hr)

T = tubeside fluid temperature   (°F)

$\bar{T}$ = Laplace transform of T (°F)

U = overall heat transfer coefficient   (BTU/hr sq ft F°)

v = fluid velocity   (ft/hr)

V, V' = definite integrals   (°F)

w = fluid flow rate   (lb/hr)

W = pipe wall temperature   (°F)

$\bar{W}$ = Laplace transform of W (°F)

x = axial distance along exchanger   (ft)

y = axial distance along exchanger   (dl)

Z = intermediate parameter   (dl)

$\alpha_1 = \dfrac{U A}{w_T c_T}$ (dl)

$\alpha_2 = \dfrac{U A}{w_S c_S}$ (dl)

$$\beta = \left[\frac{V_R \delta^2 s}{L\, d}\right]^{\frac{1}{2}} \quad (dl)$$

$$\gamma_1 = \frac{\delta\, h_T}{k} \quad (dl)$$

$$\gamma_2 = \frac{\delta\, h_S}{k} \quad (dl)$$

δ = pipe wall thickness (ft)

S = intermediate parameter (dl)

z = root of auxiliary cubic (dl)

θ = time (dl)

λ = radial distance through pipe wall (ft)

ζ = intermediate parameter (dl)

ρ = density (lb/cu ft)

σ = radial distance through pipe wall (dl)

ϕ , ϕ' = dummy variables

ψ = intermediate parameter (dl)

Subscripts:

i = ordinal number of shell pass

j = ordinal number of tube pass within a particular shell pass

Q = summation index

R = reference

S = shellside

T = tubeside

W = pipe wall

As usual the use of an independent variable as a subscript of a dependent variable indicates partial differentiation,

$$\text{as } T_x = \frac{\partial T}{\partial x}.$$

INTRODUCTION

A multipass shell and tube exchanger may be described by a set of partial differential equations similar, in many respects, to those describing a counterflow[1,2][1] or one side lumped[3] exchanger. This set of equations may be thought of as constituting a mathematical model of the exchanger. Models of varying degrees of complexity may be constructed for the same exchanger. In this paper models are first constructed which neglect the heat capacity of tube walls. Refinements are then made which allow the walls to be introduced as either lumped or distributed thermal capacity.

The models presented herein are quite formidable of aspect, involving the definition of many sets of intermediate parameters. It should be borne in mind that the question which must be answered regarding the feasibility of use of a given model is not, "How complicated is the model?" but, "How much does it cost to extract the desired information from the model?". The desired information is here the frequency response characteristics of the exchanger. This information may be particularly easily extracted if an explicit expression is obtainable for the transfer function. Such an expression is obtainable for the models considered. In fact, it is the object of this paper to show how such explicit transfer functions may be obtained.

The evaluation at a particular frequency of a transfer function of the complexity here considered takes on the order of 15 seconds on an intermediate speed digital computer and so

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[1] Numbers in parentheses designate References at end of paper.
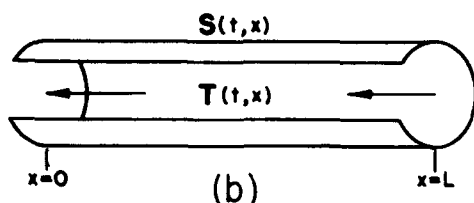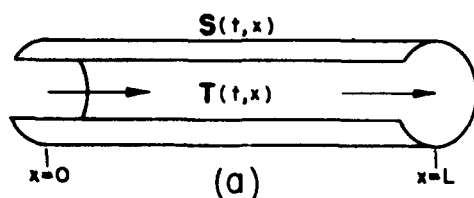
Fig.1 Fluid flow inside pipe.
Direction of flow determines sign
of T_x term in equation.
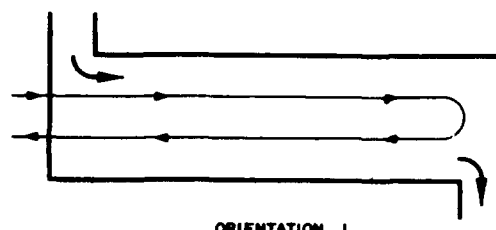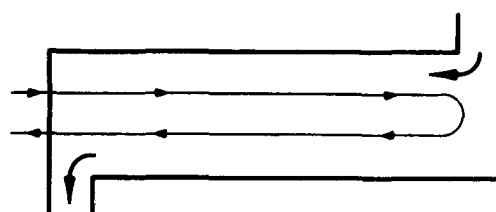
Fig.2 The two possible orientations of
a 1-2 exchanger.

costs about \$0.25. This cost would be cut at least 50 per cent by the use of a high speed computer. Clearly, then, the use of these models even for routine work is definitely feasible.

FUNDAMENTAL EQUATIONS

The partial differential equation describing the temperature changes of a fluid flowing inside a pipe (see figure 1a) is [1] :

$$(1) \qquad T_x + \frac{1}{v} T_t = \frac{U\,P}{W_T\,c_T}\,(S - T)$$

If the direction of flow of the fluid is reversed (figure 1b) this becomes:

$$(2) \qquad -T_x + \frac{1}{v} T_t = \frac{U\,P}{W_T\,c_T}\,(S - T)$$

The assumptions embodied in these equations are essentially those employed to develop the usual expression for the log mean temperature difference[4] and are as follows:

    1. U, w_T , c_T , and v are constant.

2. Plug flow prevails.

3. No temperature gradients exist in the fluid other than in the axial direction.

4. No partial phase changes take place in the system.

Equations (1) and (2) are made dimensionless by the introduction of new space and time variables.

$$(3) \quad y = \frac{x}{L}$$

$$(4) \quad \theta = \frac{t \, v_R}{L}$$

Using (3) and (4), (1) becomes:

$$(5) \quad T_y + \frac{v_R}{v} T_\theta = \alpha_1 (S - T)$$

Each pass of a multipass exchanger is described by an equation similar to (5). These equations together with appropriate boundary conditions constitute one model of the exchanger.

THE 1 - 2n EXCHANGER

Two orientations of a 1-2 exchanger are possible (see figure 2). For each orientation there are four possible temperature forcing transfer functions. Forcing may be applied on either the tube or shell side, and the response may be taken on either side.

Transfer functions will be derived for the case of orientation 1, forcing on the tube side, with response on either side. Disturbances frequently take the path, forcing on the tube side, response on the tube side. Corrective action, however, frequently involves flow forcing which is not here considered. The other six possible temperature

forcing transfer functions may be derived by procedures very similar to the ones to be demonstrated.

The equations for the 1-2n exchanger (see figure 3) are, in dimensionless form:

$$(6) \quad (T_j)_y + (T_j)_\theta = \alpha_1(S - T_j) \qquad j = 1,3,5,\ldots2n-1$$

$$(7) \quad -(T_j)_y + (T_j)_\theta = \alpha_1(S - T_j) \qquad j = 2,4,6,\ldots2n$$

$$(8) \quad S_y + r\, S_\theta = \sum_{j=1}^{2n} \alpha_2 (T_j - S)$$

Note that tubeside velocity is chosen as reference velocity. The additional assumption is made that the area of each pass is the same. Boundary conditions are:

$$(9) \quad T_j(t,0) = T_{j-1}(t,0) \qquad j = 3,5,7,\ldots2n-1$$

$$(10) \quad T_j(t,1) = T_{j-1}(t,1) \qquad j = 2,4,6,\ldots2n$$

$$(11) \quad S(t,0) = 0$$

The solution is carried out in detail in Appendix A.

THE GENERAL MULTIPASS SHELL AND TUBE EXCHANGER (m - 2mn)

Under the additional mild restriction that there be no heat transfer between different shell passes (also a usual steady state assumption[4]) it is possible to derive explicit transfer functions for the general multipass shell and tube exchanger having m shell passes and 2n tube passes per.shell pass. The notation is illustrated by figure 4. When two subscripts are used the first always refers to the number of the shell pass. The equations take the form:
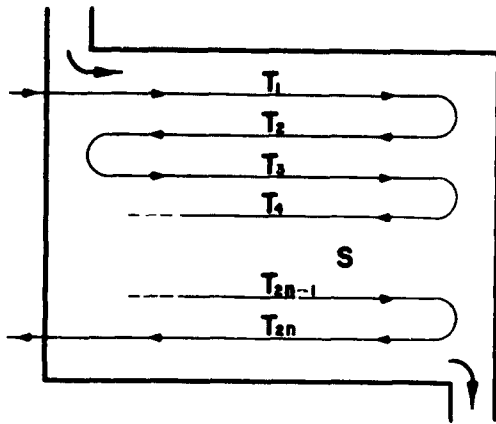
Fig.3 The 1-2n exchanger. One other orientation is possible.
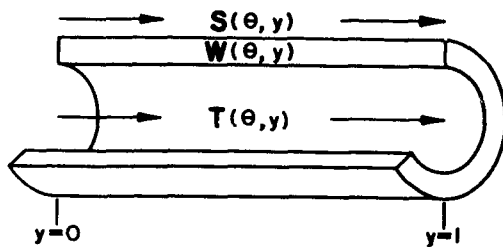
Fig.5 Heat-exchanger model with lumped thermal capacity in wall.
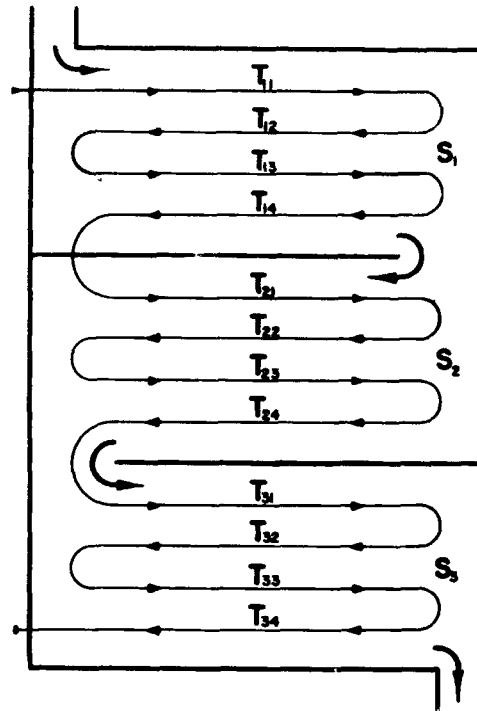
Fig.4 A 3-12 exchanger illustrating the notation used for multipass exchangers. Many other orientations are possible, but all are similarly handled.

$$(12) \quad \pm (T_{ij})_y + (T_{ij})_\theta = \alpha_1 (S_i - T_{ij}) \qquad \begin{bmatrix} \text{odd } j \\ \text{even } j \end{bmatrix}$$

$$(13) \quad \pm (S_i)_y + r (S_i)_\theta = \sum_j \alpha_2 (T_{ij} - S_i) \qquad \begin{bmatrix} \text{odd } i \\ \text{even } i \end{bmatrix}$$

Boundary conditions mirror the continuity of fluid flow, as before. Details of the method are in Appendix B.

LUMPED THERMAL CAPACITY IN WALLS

Thus far models have been considered that have no thermal capacity in the pipe walls. Now it will be shown how lumped thermal capacity may be introduced into the pipe wall between any two fluids. Furthermore it will be shown that the equation describing the more complicated system

may be reduced to the form previously considered.

The system of figure 5 has been described by the equations:

(14) $\quad T_y + T_\theta = \alpha_1(S - T)$

(15) $\quad S_y + r S_\theta = \alpha_2(T - S)$

It may also be described by the set:

(16) $\quad T_y + T_\theta = a_1(W - T)$

(17) $\quad S_y + r S_\theta = a_2(W - S)$

(18) $\quad W_\theta = b_1(T - W) + b_2(S - W)$

Transforming and rearranging (18) one obtains:

(19) $\quad \overline{W} = \dfrac{b_1}{s + b_1 + b_2}\,\overline{T} + \dfrac{b_2}{s + b_1 + b_2}\,\overline{S}$

Using this expression for $\overline{W}$ in the transforms of (16) and (17),

(20) $\quad \overline{T}_y + \left[s + a_1 - \dfrac{a_1\,b_1}{s + b_1 + b_2} \right] \overline{T} = \dfrac{a_1\,b_2}{s + b_1 + b_2}\,\overline{S}$

(21) $\quad \overline{S}_y + \left[rs + a_2 - \dfrac{a_2\,b_2}{s + b_1 + b_2} \right] \overline{S} = \dfrac{a_2\,b_1}{s + b_1 + b_2}\,\overline{T}$

Equations (20) and (21) are of the same form as the transforms of (14) and (15), but with different constants. The solution from this point on is the same as before.

The treatment for the case of a wall in contact with a single fluid (as a shell wall) is similar. It should be noted that in a single exchanger the thermal capacity of

some walls may be neglected, the thermal capacity of other walls lumped, and the capacity of still other walls treated in distributed fashion as will be described below.

DISTRIBUTED THERMAL CAPACITY IN WALLS

The system of figure 6 has been described by the equations:

$$(22) \quad T_x + \frac{1}{v_T} T_t = \frac{U\,P}{W_T\,c_T}\,(S - T)$$

$$(23) \quad S_x + \frac{1}{v_S} S_t = \frac{U\,P}{W_S\,c_S}\,(T - S)$$

It may also be described by the set:

$$(24) \quad T_x + \frac{1}{v_T} T_t = \frac{h_T\,P}{W_T\,c_T}\Big[W(t,x,0) - T \Big]$$

$$(25) \quad S_x + \frac{1}{v_S} S_t = \frac{h_S\,P}{W_S\,c_S}\Big[W(t,x,S) - S \Big]$$

$$(26) \quad W_t = d\,W_{\lambda\lambda}$$
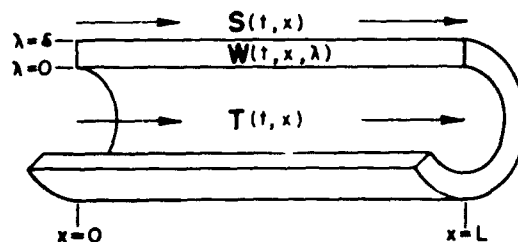
Fig.6 Heat-exchanger model with distributed thermal capacity in wall.

The boundary conditions for (26) are:

$$(27) \quad k\,A\,(W_\lambda)_{\lambda = 0} = h_T\,A\Big[W(t,x,0) - T \Big]$$

$$(28) \quad k\,A\,(W_\lambda)_{\lambda = S} = h_S\,A\Big[S - W(t,x,S) \Big]$$

The independent variables are made dimensionless by the transformations (3), (4), and,

$$(29) \quad \alpha = \frac{\lambda}{S}$$

The reference velocity is v_T. After employing the Laplace transformation and rearranging, the set takes the form:

$$(30) \quad \bar{T}_y + (s + \alpha_1)\,\bar{T} = \alpha_1\,\bar{W}(s,y,0)$$

$$(31) \quad \bar{S}_y + (rs + \alpha_2)\,\bar{S} = \alpha_2\,\bar{W}(s,y,1)$$

$$(32) \quad \bar{W}_{\alpha\alpha} - \beta^2\,\bar{W} = 0$$

$$(33) \quad (\overline{W}_{\alpha})_{\alpha=0} = Y_1\left[\overline{W}(s,y,0) - \overline{T}\right]$$

$$(34) \quad (\overline{W}_{\alpha})_{\alpha=1} = Y_2\left[\overline{S} - \overline{W}(s,y,1)\right]$$

Equation (32) may be solved using boundary conditions (33)
and (34) to obtain $\overline{W}(s,y,\alpha)$ as a linear combination of
$\overline{S}$, $\overline{T}$, $\overline{W}(s,y,0)$, and $\overline{W}(s,y,1)$. If this expression for $\overline{W}(s,y,\alpha)$
is evaluated at $\alpha = 0$ and $\alpha = 1$, one obtains:

$$(35) \quad \zeta_1 \overline{W}(s,y,0) + \zeta_2 \overline{W}(s,y,1) = \zeta_3 \overline{T} + \zeta_4 \overline{S}$$

$$(36) \quad \xi_1 \overline{W}(s,y,0) + \xi_2 \overline{W}(s,y,1) = \xi_3 \overline{T} + \xi_4 \overline{S}$$

where,

$$(37) \quad \zeta_1 = \beta(e^\beta - e^{-\beta}) + Y_1(e^\beta + e^{-\beta})$$

$$(38) \quad \zeta_2 = 2 Y_2$$

$$(39) \quad \zeta_3 = Y_1(e^\beta + e^{-\beta})$$

$$(40) \quad \zeta_4 = 2 Y_2$$

$$(41) \quad \xi_1 = 2 Y_1$$

$$(42) \quad \xi_2 = \beta(e^\beta - e^{-\beta}) + Y_2(e^\beta + e^{-\beta})$$

$$(43) \quad \xi_3 = 2 Y_1$$

$$(44) \quad \xi_4 = Y_2(e^\beta + e^{-\beta})$$

From the linear set (35) and (36) one obtains a solution
of the form:

$$(45) \quad \overline{W}(s,y,0) = D_1 \overline{T} + D_2 \overline{S}$$

$$(46) \quad \overline{W}(s,y,1) = D_3 \overline{T} + D_4 \overline{S}$$

Substitution from (45) and (46) into (30) and (31) reduces
the system of equations to the form previously considered.

Walls which are in contact with a single fluid may be
similarly handled. Any number of walls in an exchanger may
be handled in distributed fashion as shown here.

The reader will have noted that the Fourier heat conduction

equation is used in rectangular coordinates rather than cylindrical, as is strictly required. The error introduced is not large and is outweighed by the convenience of being able to obtain an explicit transfer function in terms of the elementary functions.

It should be noted that models in the literature of counterflow[1,2], parallel flow[1,2], and one side lumped[3] exchangers may be refined and thus undoubtedly brought into better agreement with experimental work by the addition of distributed thermal capacity in the pipe walls, as provided for here.

APPENDIX A

Equations (6), (7), and (8) may be transformed and put into the form:

$$(47) \quad (\overline{T}_j)_y + f_1 \overline{T}_j = g_1 \overline{S} + k_1 \qquad j = 1,3,5,\ldots 2n-1$$

$$(48) \quad -(\overline{T}_j)_y + f_1 \overline{T}_j = g_1 \overline{S} - k_1 \qquad j = 2,4,6,\ldots 2n$$

$$(49) \quad \overline{S}_y + f_2 \overline{S} = g_2 \sum_{j=1}^{2n} \overline{T}_j$$

where,

$$(50) \quad f_1 = s + \alpha_1$$

$$(51) \quad f_2 = 2n\alpha_2 + rs$$

$$(52) \quad g_1 = \alpha_1$$

$$(53) \quad g_2 = \alpha_2$$

The boundary conditions become:

$$(54) \quad \overline{T}_j(s,0) = \overline{T}_{j-1}(s,0) \qquad j = 3,5,7,\ldots 2n-1$$

$$(55) \quad \overline{T}_j(s,1) = \overline{T}_{j-1}(s,1) \qquad j = 2,4,6,\ldots 2n$$

$$(56) \quad \overline{S}(s,0) = 0$$

Equations (47) and (48) possess the well-known solutions :

$$(57) \quad \overline{T}_j = e^{-f_1 y} \int_0^y g_1 \, e^{f_1 y} \cdot \overline{S} \, dy + q_j \, e^{-f_1 y} \quad j = 1,3,\ldots 2n-1$$

$$+ h_1 \, e^{-f_1 y}$$

$$(58) \quad \overline{T}_j = -e^{f_1 y} \int_0^y g_1 \, e^{-f_1 y} \cdot \overline{S} \, dy + q_j \, e^{f_1 y} \quad j = 2,4,\ldots 2n$$

The q_j may be evaluated in order. For instance directly from (57):

$$(59) \quad q_1 = \overline{T}_1 \, (s,0)$$

Then since,

$$(60) \quad \overline{T}_2 \, (s,1) = \overline{T}_1 \, (s,1)$$

q_2 may be obtained, etc. It may be verified that:

$$(61) \quad q_j = V \left[\frac{1 - p^{\frac{j-1}{2}}}{1 - p} \right] + V' \left[\frac{p\left(1 - p^{\frac{j-1}{2}}\right)}{1 - p} \right] + p^{\frac{j-1}{2}} \quad j = 1,3,\ldots 2n-1$$

$$(62) \quad q_j = V \left[\frac{1 - p^{\frac{j}{2}}}{1 - p} \right] + V' \left[\frac{p\left(1 - p^{\frac{j}{2}}\right)}{1 - p} \right] + p^{\frac{j}{2}} \quad j = 2,4,\ldots 2n$$

where $\overline{T}_1 \, (s,0)$ has been set to unity and,

$$(63) \quad p = e^{-2 f_1}$$

$$(64) \quad V = \int_0^1 \overline{S} \, g_1 \, e^{-f_1 y} \, dy$$

$$(65) \quad V' = \int_0^1 \overline{S} \, g_1 \, e^{f_1 y} \, dy$$

For the extension of this method of derivation to the general multipass case it will be important that the q_j are linear combinations of V, V', and 1 with coefficients of the form $\sum_q C_q \, e^{C'_q y}$, where the C_q and C'_q are constants.

The $\overline{T}_j$ may be eliminated from (49) using (57) and (58) and the resulting equation differentiated twice to remove the integrals. One obtains:

(66) $\overline{S}_{yyy} + f_2 \overline{S}_{yy} - f_1^2 \overline{S}_y + (2f_1 \, g_2 \, nf_1 - f_2 \, f_1^2)\overline{S} = 0$

Three initial conditions are needed. $\overline{S}(s,0)$ is obtainable from
(56). $\overline{S}_y(s,0)$ and $\overline{S}_{yy}(s,0)$ are obtainable by setting $y = 0$
before the first and second differentiations, respectively[6] .
$\overline{S}_y(s,0)$ and $\overline{S}_{yy}(s,0)$ turn out to be linear combinations
of the q_i and thus linear combinations of V , V' , and 1 with
coefficients of the same form as before. The algebraic details
will be presented at the end of the appendix so as not to confuse
the development.

   The solution of (66) is of the form:

(67) $\overline{S} = E_1 \, e^{z_1 y} + E_2 \, e^{z_2 y} + E_3 \, e^{z_3 y}$

where z_1 , z_2 , and z_3 are the roots of the auxiliary
cubic. These roots may most simply be found using Cardan's
formula[7] . E_1 , E_2 , and E_3 are related to the initial
conditions on $\overline{S}$ by the linear set:

$$(68) \begin{bmatrix} \overline{S} \ (s,0) \\ \overline{S}_y \ (s,0) \\ \overline{S}_{yy} \ (s,0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

The E are thus also linear combinations of V , V' , and unity.
One may define:

(69) $\quad E_{\substack{1 \\ 2 \\ 3}} = K_{\substack{1 \\ 2 \\ 3}} V + M_{\substack{1 \\ 2 \\ 3}} V' + N_{\substack{1 \\ 2 \\ 3}}$

Equation (67) may now be rewritten:

(70) $\overline{S} = V\left[K_1 \, e^{z_1 y} + K_2 \, e^{z_2 y} + K_3 \, e^{z_3 y}\right]$
$\qquad + V'\left[M_1 \, e^{z_1 y} + M_2 \, e^{z_2 y} + M_3 \, e^{z_3 y}\right]$

$\qquad + \left[N_1 \, e^{z_1 y} + N_2 \, e^{z_2 y} + N_3 \, e^{z_3 y}\right]$

15

ψ_o , ψ , and ψ' are defined from (70) by:

(71) $\bar{S} = \psi_o + \psi \, V + \psi' \, V'$

The ψ are also of the form $\sum_\varphi C_Q \, e^{C_q y}$. All that must now be done before $\bar{S}(s,1)$ (the shellside response) or $\bar{T}_{2n}(s,0)$ (the tubeside response) can be evaluated numerically is to evaluate V and V' . This may be accomplished using (71).

The symbol (φ , φ') is defined by:

(72) $(\varphi , \varphi') = \displaystyle\int_0^1 \varphi \, \varphi' \, dy$

Equation (71) is multiplied by $g_1 \, e^{-f_1 y}$ and integrated with respect to y between 0 and 1. Using the notation defined by (72) this yields:

(73) $V = (\psi_o , g_1 \, e^{-f_1 y}) + V(\psi , g_1 \, e^{-f_1 y}) + V'(\psi' , g_1 \, e^{-f_1 y})$

Similarly, if (71) is multiplied by $g_1 \, e^{f_1 y}$:

(74) $V' = (\psi_o , g_1 \, e^{f_1 y}) + V(\psi , g_1 \, e^{f_1 y}) + V'(\psi' , g_1 \, e^{f_1 y})$

The integrations indicated in (73) and (74) may be easily carried out due to the simple form of the ψ . The second order set of linear algebraic equations defined by (73) and (74) may then be solved numerically for V and V' . With V and V' in hand $\bar{S}(s,1)$ is available numerically from (71) and $\bar{T}_{2n}(s,0)$ from (58) and (62).

All that remains is to present expressions for the ψ as linear combinations of V , V' , and unity; and to show the results of the integrations indicated in (73) and (74). This will be done using several sets of intermediate parameters.

(75) $z_{1\frac{2}{3}} = \dfrac{g_2 \left(\dfrac{h_2}{3} - \dfrac{h_3}{2} \right) \left(\dfrac{h_2}{3} + \dfrac{h_3}{2} + f_2 + f_1 \right)}{(h_1 - h_2)(h_2 - h_3)(h_3 - h_1)}$

Z_1' , Z_2' , and Z_3' are similarly defined with $(-f_1)$ substituted for (f_1).

$$(76) \quad R_1 = \frac{1 - p^n}{1 - p}$$

$$(77) \quad R_2 = \frac{p(1-p^n)}{1 - p}$$

$$(78) \quad R_3 = \frac{n}{1 - p} - \frac{1 - p^n}{(1-p)^2}$$

$$(79) \quad R_4 = \frac{n}{1 - p} - \frac{p(1 - p^n)}{(1 - p)^2}$$

Finally, the K, M, and N defined by (69) are given by:

$$(80) \quad K_{\substack{1 \\ 2 \\ 3}} = Z_1 R_3 + Z_1' R_4$$
$$ {\scriptstyle 2 \atop 3} \qquad {\scriptstyle 2 \atop 3}$$

$$(81) \quad M_{\substack{1 \\ 2 \\ 3}} = Z_1\, pR_3 + Z_1'\, pR_4$$
$$ {\scriptstyle 2 \atop 3} \qquad {\scriptstyle 2 \atop 3}$$

$$(82) \quad N_{\substack{1 \\ 2 \\ 3}} = Z_1 R_1 + Z_2' R_2$$
$$ {\scriptstyle 2 \atop 3} \qquad {\scriptstyle 2 \atop 3}$$

The ψ are now given by (70) and (71). The integrals follow.

$$(83) \quad (\psi_0 , e^{-f_1 y}) = \frac{N_1}{\lambda_1 - f_1}\left[e^{\lambda_1 \cdot f_1} - 1 \right] + \frac{N_2}{\lambda_2 - f_1}\left[e^{\lambda_2 \cdot f_1} - 1 \right]$$
$$+ \frac{N_3}{\lambda_3 - f_1}\left[e^{\lambda_3 \cdot f_1} - 1 \right]$$

$(\psi , e^{-f_1 y})$ and $(\psi' , e^{-f_1 y})$ are similarly evaluated using K_1 , K_2 , K_3 and M_1 , M_2 , M_3 , respectively, for N_1 , N_2 , N_3 .

$(\psi_0 , e^{f_1 y})$, $(\psi , e^{f_1 y})$, and $(\psi' , e^{f_1 y})$ are similarly evaluated with (f_1) substituted for $(-f_1)$.

APPENDIX B

Equations (12) and (13) may be transformed and put into
the form:

(84) $\pm(\overline{T}_{ij})_y + f_1 \overline{T}_{ij} = g_1 \overline{S}_i$ $\begin{bmatrix} \text{odd } j \\ \text{even } j \end{bmatrix}$

(85) $\pm(\overline{S}_i)_y + f_2 \overline{S}_i = g_2 \sum_j \overline{T}_{ij}$ $\begin{bmatrix} \text{odd } i \\ \text{even } i \end{bmatrix}$

Equations (84) possess known solutions[5]   :

(86) $\overline{T}_{ij} = \pm e^{\mp f_1 y} \int_0^y g\, e^{\pm f_1 y}\, \overline{S}_i\, dy + q_{ij}\, e^{\mp f_1 y}$ $\begin{bmatrix} \text{odd } j \\ \text{even } j \end{bmatrix}$

The q_{ij} may again be evaluated in order. They will be
found to be linear combinations of 2m definite integrals
$(V_1\ ,\ V_1'\ ,\ V_2\ ,\ V_2'\ ,\ \ldots V_m\ ,\ V_m'\ )$ and unity.

(87) $V_i = \int_0^1 \overline{S}_i\, g_1\, e^{-f_1 y}\, dy$

(88) $V_i' = \int_0^1 \overline{S}_i\, g_1\, e^{f_1 y}\, dy$

$\overline{T}_{ij}$ from (86) may be substituted into (85) and the resulting
equations differentiated twice to eliminate the integrals. This
yields, for each of the $\overline{S}_i$, a third order differential equation.
The solution may be written:

(89) $\overline{S}_i = E_{i1}\, e^{b_{i1} y} + E_{i2}\, e^{b_{i2} y} + E_{i3}\, e^{b_{i3} y}$

The E_i are not, however, linear combinations of only
$(V_1\ ,\ \ldots V_m'\ ,\ 1)$. Because of the boundary condition

(90) $\overline{S}_i(s,0) = \overline{S}_{i-1}(s,0)$ (odd i , $i \neq 1$)

(91) $\overline{S}_i(s,1) = \overline{S}_{i-1}(s,1)$ (even i),

the E_i are linear combinations also of $\overline{S}_{i-1}(s,0)$ or $\overline{S}_{i-1}(s,1)$

depending on whether i is odd or even respectively. Rearranging (89) as was done to (67) to obtain (71):

$$(92) \quad \bar{S}_i = \widehat{\psi}_{io} + \widehat{\psi}_{i1} V_1 + \widehat{\psi}_{i1}' V_1' + \widehat{\psi}_{i2} V_2 + \widehat{\psi}_{i2}' V_2' + \cdots$$
$$+ \widehat{\psi}_{im} V_m + \widehat{\psi}_{im}' V_m' + \widehat{\psi}_{i0}' \bar{S}_{i-1} \quad (s, \overset{0}{1})$$

The equation for $\bar{S}_1$ has no term in $\bar{S}_{i-1}$ $(s, 0)$. In the equation for $\bar{S}_2$ the term in $\bar{S}_1$ $(s, 1)$ may be eliminated by substitution from the equation for $\bar{S}_1$. Proceeding stepwise the following set is obtained from (92).

$$(93) \quad \bar{S}_i = \psi_{io} + \sum_{j=1}^{m} \left[\psi_{ij} V_j + \psi_{ij}' V_j' \right]$$

Each of the m equations of the set (93) may be multiplied by g_1 $e^{-f_1 y}$ and integrated; and then by g_1 $e^{f_1 y}$ and integrated as was done to (71), to obtain the set of $2m$ equations:

$$(94) \quad V_i = (\psi_{io}, g_1 e^{-f_1 y}) + \sum_{j=1}^{m} \left[V_j (\psi_{ij}, g_1 e^{-f_1 y}) + V_j' (\psi_{ij}', e^{-f_1 y}) \right]$$

$$(95) \quad V_i' = (\psi_{io}, g_1 e^{f_1 y}) + \sum_{j=1}^{m} \left[V_j (\psi_{ij}, g_1 e^{f_1 y}) + V_j' (\psi_{ij}', e^{f_1 y}) \right]$$

This set may be solved for the $2m$ constants $(V_1, \ldots \bar{V}_m')$. The desired responses are then easily evaluated, as before.

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