The Sixth Clemson mini-Conference on Discrete Mathematics

Funded by the Office of Naval Research (ONR)

Conference Program and
Invited Talk View Graphs

October 3-4, 1991

Clemson University
Clemson, South Carolina

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Organizers:	S.T. Hedetniemi	Department of Computer Science
	R.C. Laskar	Department of Mathematical Sciences
	R.D. Ringeisen	Department of Mathematical Sciences

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on

Discrete Mathematics

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Schedule of talks
(All talks given in Student Senate Chambers)

Thursday, October 3

1:30 - 2:00 ***Sign-In/Registration***

2:00 - 2:05 ***Welcoming Remarks*** by R.D. Ringeisen, S.T. Hedetniemi, and Dr. Bobby Wixson, Dean of College of Sciences

2:05 - 2:50 ***Marc J. Lipman****, Office of Naval Research*
Mathematical Science Division, Arlington, VA

"Sphere-of-Influence Graphs"

An introduction to a class of geometrically defined objects, sphere-of-influence graphs (SIGs), used by the computer vision community to capture the low-level perceptual structure of a scene, that is, for pattern recognition. The mathematics of SIGs isn't yet mature.

2:55 - 3:35 ***Roger Entringer****, University of New Mexico*
Dept. of Mathematics, Albuquerque, NM

"Two Extremal Problems In Graph Theory"

Two specific instances of the following general problems are addressed:

(i) How many edges can a graph G of order n have if G must have a specific property?

(ii) If G is to have order n and a given number of edges, how are the edges arranged if a specific property must be optimized?

The instance of the second problem involves an attempt to shorten delivery time by the USPS.

3:35 - 4:00 ***BREAK***

4:00 - 4:40

***Edward R. Scheinerman**, Johns Hopkins University*
Department of Mathematical Sciences, Baltimore, MD

"Containment Orders and Planar Graphs"

We will explore interesting relationships between the worlds of partially ordered sets (especially *containment orders*) and planar graphs.

Given any graph $G = (V, E)$, let its *vertex-edge incidence order*, $P(G)$, be the partially ordered set whose ground set is $V \cup E$ together with relations $v < e$ exactly when $v \in V, e \in E$, and v is an end point of e. What *graph* properties of G can we deduce from *poset* properties of $P(G)$?

In this talk we will focus on order theoretic properties of $P(G)$ which turn out to be equivalent to G being planar. We will phrase these properties in terms of geometric containment orders, which we now define.

Given a family Σ of objects, we call a partially ordered set $P = (X, \leq)$ a Σ-order provided we can assign to each $x \in X$, an element $S_x \in \Sigma$, so that $x \leq y$ iff $S_x \subseteq S_y$. In particular, if Σ is the set of disks (circles with their interiors) in the plane, then Σ-orders are also known as *circle orders.*

We will discuss a number of results, all of which have the following flavor.

Theorem. A graph G is planar if and only if its vertex-edge incidence order, $P(G)$, is a circle order.□

Some of the theorems to be presented include joint work with Graham Brightwell, Ann Trenk and Daniel Ullman.

4:45 - 5:25

***Jean R.S. Blair**, University of Tennessee*
Dept. of Computer Science, Knoxville, TN

"On Finding Transmitter-Receiver Matchings"

The problem of finding a maximum transmitter-reciever matching (TRM) in communication networks is addressed. TRM remains NP-complete even for networks whose topologies correspond to chordal graphs. We address the problem for a subclass of chordal graphs, namely those graphs whose clique graphs are acyclic. Using several interesting properties of these graphs, we devise a linear time algorithm to solve the problem.

7:30

Social, Jordan Room

Friday, October 4

8:30 - 9:10

J. Chris Fisher, University of Regina, Canada
(Visiting Clemson University, Dept. of Math. Sci.)

"The Jamison Method in Galois Geometries"

In a fundamental paper Robert E. Jamison showed, among other things, that any subset of the points of AG(2,q) -- the affine plane of order q -- that intersects all lines contains at least 2q-1 points. Here I shall discuss my recent work with Aiden Bruen in which we show that Jamison's method of proof can be applied to several other basic problems in finite geometries of a varied nature. These problems include the celebrated flock theorem and also the characterization of the elements of GF(q) as a set of squares in $GF(q^2)$ with certain properties. This last result, due to A. Blokhuis, settled an important conjecture due to J.H. van Lint and the late J. MacWilliams.

9:15 - 9:55

Fred S. Roberts, Rutgers University
Dept. of Mathematics, Center of Operations Research (RUTCOR), and
Center for Discrete Mathematics and Theoretical Computer Science (DIMACS)
New Brunswick, NJ

"Elementary, Sub-Fibonacci, Regular, Van Lier and Other Interesting Sequences"

In the past five years, problems of the uniqueness of scales of measurement have been giving rise to a variety of interesting sequences of positive integers with fascinating combinatorial properties. Examples of such sequences are all non-decreasing sequences of positive integers $x_1, x_2, ..., x_n$ so that $x_1 = x_2 = 1$. Such a sequence is called ***elementary*** if all $k \le n$, $x_k > 1$ implies that $x_k = x_i + x_j$ for some $i \neq j$. It is called ***sub-Fibonacci*** if $x_k \le x_{k-1} + x_{k-2}$, k = 3, 4, ... It is called ***regular*** if $x_j \le \sum_{i=1}^{j-1} x_i$, j = 3, 4, ... A regular sequence is called ***Van Lier*** if for all $j < k \le n$, there is a subset A of {1, 2, ..., n} with j not in A and $x_k - x_j = \sum_{i \in A} x_i$. We discuss these and other sequences and some of their combinatorial properties.

9:55 - 10:20 ***BREAK***

10:20 - 11:00

Michael S. Jacobson, University of Louisville
Department of Mathematics, Louisville, KY

"Generating k-element Subsets of an n-element Set"

In this talk, a generalization of the idea of De Bruijn graphs will be used to establish sequences which generate all k-element subsets of an n-element set. In the case when n is odd, by using a result of Good, these sequences are shown to exist. When n is even, the technique shown will not generate an appropriate sequence. In fact the generalized De Bruijn graph is disconnected, and by a unique application of Polya's Theorem, the number of components of this graph is calculated.

11:05 - 11:45

E. Rodney Canfield, University of Georgia
Dept. of Computer Science, Athens, GA

"Matchings in the Partition Lattice"

Let $[n]$ be the set $\{1, 2, ..., n\}$. A *partition* of $[n]$ is a set of nonempty, pairwise disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. Partition π_1 is a *refinement* of partition π_2, denoted $\pi_1 \leq \pi_2$, provided each block of π_1 is contained in a block of π_2. Under this ordering the set of partitions P_n forms a lattice. The subcollection of partitions $P_{n,k} \subseteq P_n$ which have exactly k blocks has cardinality $S(n,k)$, the Stirling number of the second kind. The Stirling numbers are unimodal, raising the question of decomposing P_n into disjoint chains, $S(n, K_n)$ in number, $S(n, K_n)$ being $\max_k S(n, k)$. Our topic in this talk: for what k is it possible to find a *matching* of $P_{n,k}$ into $P_{n,k\pm1}$? That is, to find a one-to-one function $\emptyset$ from $P_{n,k}$ into $P_{n, k\pm1}$ with the property that π and $\emptyset(\pi)$ are comparable under the refinement relation "$\leq$".

LUNCH

1:15 - 1:55

Ronald C. Read, University of Waterloo
Dept. of Combinatorics and Optimization, Ontario, Canada

"Algorithms for Small Graphs"

The compilation of an "Atlas" of graph theory - a project that I am working on with R.J. Wilson - has called for the computation of many invariants (girth, connectivity, etc.) and properties (planarity, hamiltonicity, etc.) of large number of graphs; but the graphs themselves are quite small. Thus the usual concern about complexity of the algorithms is largely irrelevant, and the methods that will be used are often quite different from those that would be used for large graphs.

My talk describes some of this work. We shall see what graph theory algorithms look like through the wrong end of the telescope!

2:00 - 2:40

Nathaniel Dean, Bellcore
Morristown, NJ

"Characterization of Generalized Bicritical Graphs"

A recent theorem of Thomas and Yu states that every 4-connected projective planar graph is hamiltonian and, as a corollary, has a 2-factor. We extend this latter result by showing that the deletion of any vertex or two vertices of such a graph leaves a graph with a 2-factor. This result is in fact only an application of results we prove concerning f factors in graphs with removed elements and generalizes several notions in matching theory including bicritical graphs, i.e., where the deletion of any pair of vertices yields a graph with a perfect matching.

2:40 - 3:00 *BREAK*

3:00 - 3:40

Joseph Straight, SUNY at Fredonia
Dept. of Mathematics and Computer Science, Fredonia, NY

"Extremal Problems Involving Neighborhood Numbers and Other Parameters"

Given a simple graph $G = (V,E)$, a subset S of V is called a *neighborhood set* provided G is the union of the subgraphs induced by the closed neighborhoods of the vertices in S. The minimum and maximum cardinalities among all minimal neighborhood sets of G are denoted by $n(G)$ and $N(G)$, respectively; $n(G)$ is call the *neighborhood number* of G. It is known, for instance, that $\gamma(G) \leq n(G) \leq \alpha(G)$, where $\gamma(G)$ and $\alpha(G)$ are the (vertex) domination and covering numbers, respectively.

My colleague, Y.H. Harris Kwong, and I have been investigating the problem of finding the maximum neighborhood number $n(p)$ among all connected graphs of order p. Our work so far has lead us to conjecture that

$$n(p) \leq [9p/13]$$

a result that holds for $2 \leq p \leq 15$. I will report on this work and, as time permits, a number of other extremal problems, including some recent work of David K. Garnick, Kwong, and Felix Lazebnik on the maximum number of edges among all graphs of order p having girth at least 5.

3:45 - 4:25

Andrzej Rucinski, Emory University
Dept. of Mathematics and Computer Science, Atlanta, GA

"Random Graph Processes with Degree Restrictions"

Suppose that a process begins with n isolated vertices, to which edges are added randomly one by one so that the maximum degree of the induced graph is always bounded above by d. We prove that if n approaches infinity with d fixed, then with probability tending to 1, the final result of this process is a graph with [nd/2] edges. For d = 2, the number of 1-cycles in this graph is shown to be asymptotically Poisson (1 > 2).

"Sphere-of-Influence Graphs"

Marc J. Lipman, Office of Naval Research
Mathematical Science Division, Arlington, VA

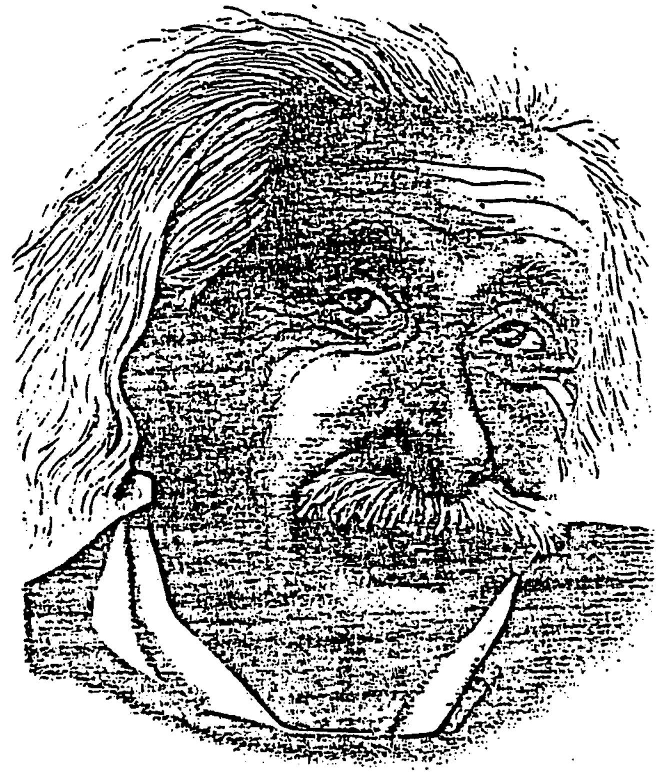
On Abstract Sphere-of-Influence Graphs

Frank Harary

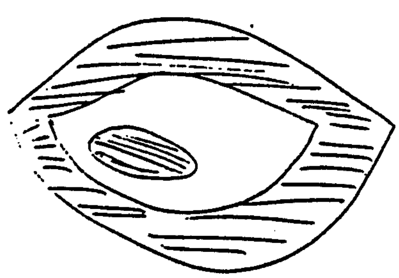
Michael S. Jacobson

Marc J. Lipman

F. R. McMorris



"Perfection of means and confusion of goals seem to characterize our age."



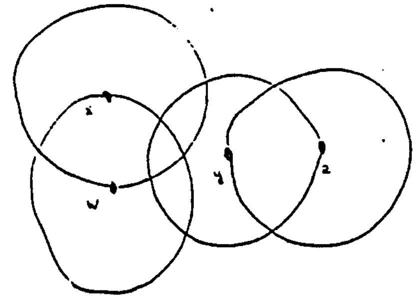
Let S be a finite set of at least two points in the euclidean plane. For each $x \in S$, let r_x be the smallest distance from x to any other point in S.

Let B_x be the open ball of radius r_x centered at x.

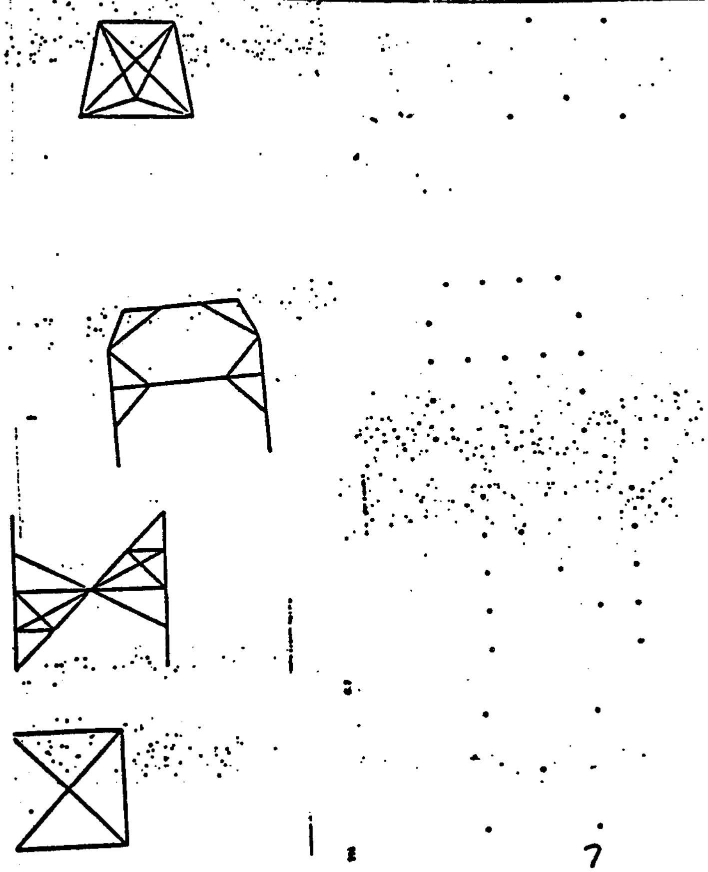
Let A_x be the closed ball of radius r_x centered at x.

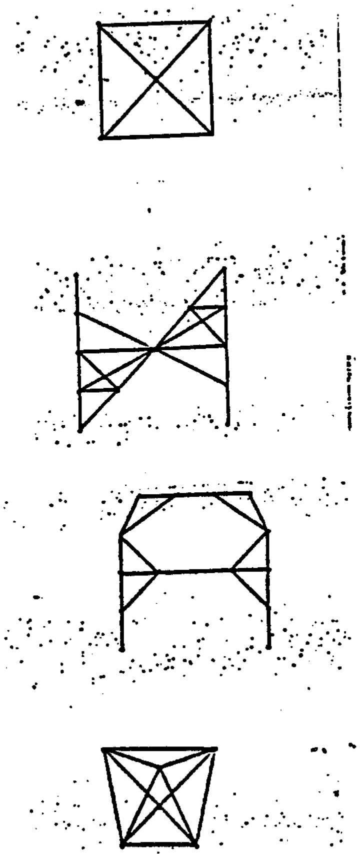
The Sphere-of-Influence Graph of S, $G(S)$, has vertex set S, and for $x, y \in S$, x and y are adjacent in $G(S) \Longleftrightarrow B_x \cap B_y \neq \emptyset$.

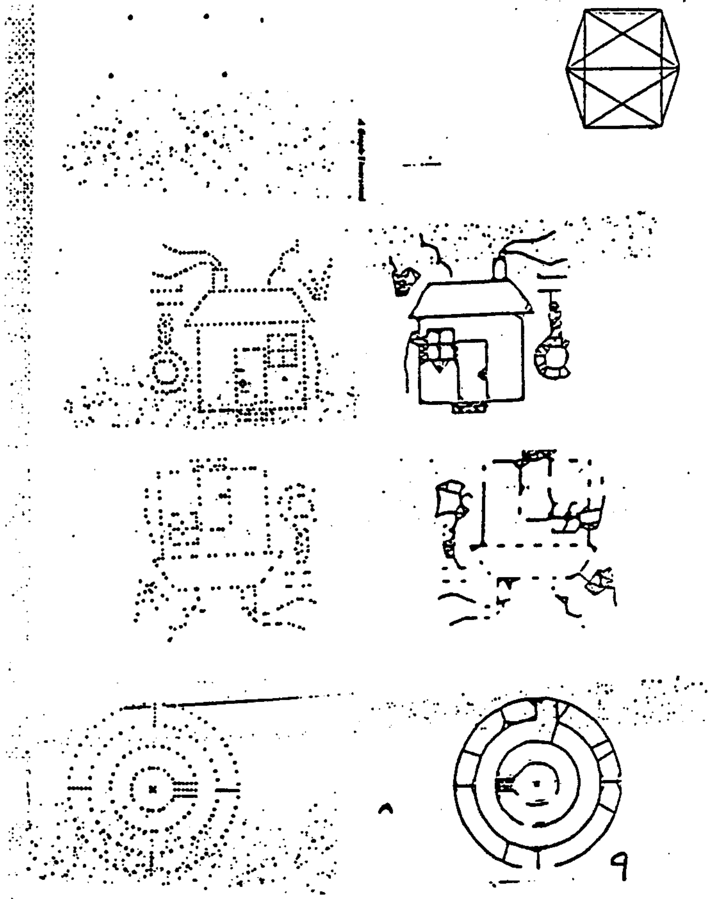
The Closed-Sphere-of-Influence Graph of S, $G^*(S)$, ... $\Longleftrightarrow$ $A_x \cap A_y \neq \emptyset$.



So  is a SIG.
" " " " CSIG.







Thm: The graphs $G(S)$ and $G^*(S)$ for a set S with $|S|=n$ can be computed in time $O(n \log n)$.

So what's the problem?

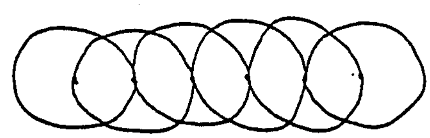
We don't know much about the graphs $G(S)$ and $G^*(S)$. Do they do what they appear to do?

Even more basic: Which graphs are SIGs?

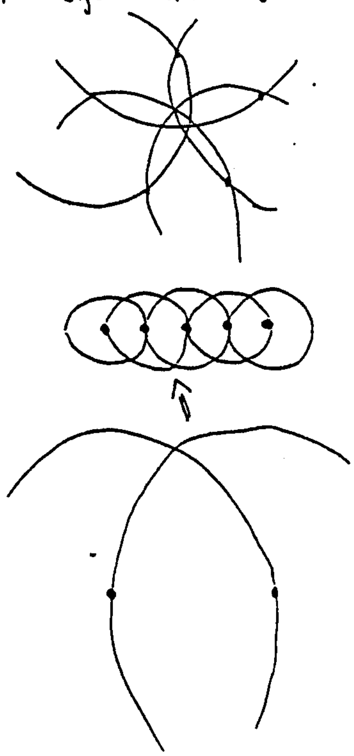
Which graphs are CSIGs?

Easy (important) result: The union of SIGs (CSIGs) is a SIG (CSIG).

Thm: Paths are SIGs.

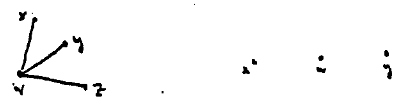


Thm: Cycles are SIGs.



Are there graphs which are not SIGs?

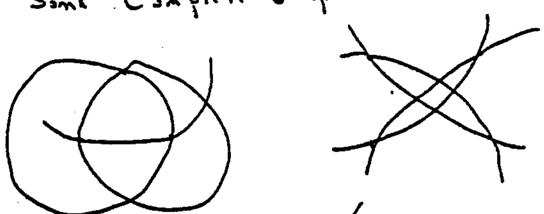
$K_{1,3}$ is not a SIG.

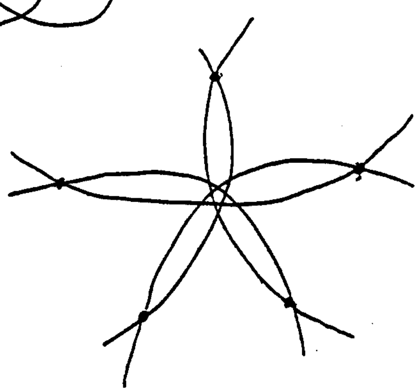


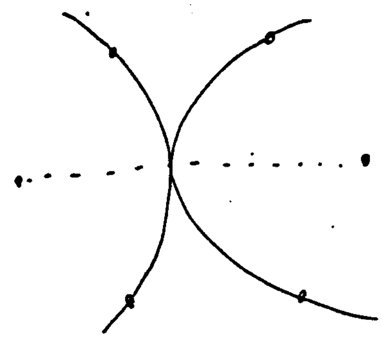
Thm [Erdős + Bateman]: If $|S|=n$, then $G(S)$ has at most $17n$ edges.

So SIGs (and CSIGs) aren't to dense.

Some Complete Graphs are SIGs.







$G(S) \neq K_6$

$G(S) = K_6$

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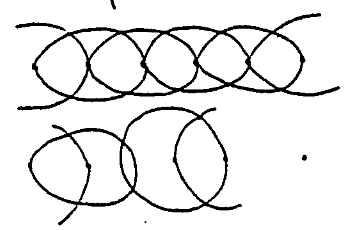
Thm: The path on 3 vertices is not a CSIG.

[If $|S| = 3$, then $G^*(S) \cong K_3$.]

Thm: Even paths are CSIGs. Odd ones aren't.



Thm: The Δ and even cycles are CSIGs. The others aren't.

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☆ The class of SIGs (CSIGs) is NOT closed under taking subgraphs (or even taking induced subgraphs)!

→ There is no forbidden subgraph characterization of SIGs (CSIGs).

Thm: Every tree is the induced subgraph of a SIG.

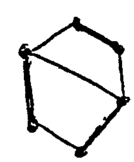
19

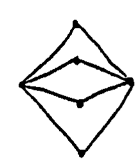
Let us say that "x defines y" if x is a nearest neighbor of y (that is: $r_y = d(x,y)$)

* of course xy is then an edge in the SIG.

Theorem: If G is a CSIG without a Δ, then G has a 1-factor of "defining edges."

(a 1-factor is a set of edges meeting every vertex EXACTLY once)





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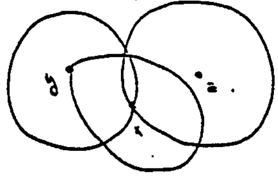
proof: If x defines y and z, then Δ.



Otherwise: spose x_2 defines x_1,
x_3 defines x_2,
⋮
x_1 defines x_k.

Then:
$$r_{x_2} \le d(x_2, x_1) = r_{x_1}$$
$$r_{x_3} \le d(x_3, x_2) = r_{x_2}$$
$$\vdots$$
$$r_{x_1} \le d(x_1, x_k) = r_{x_k}$$

$$\Longrightarrow r_{x_1} = r_{x_2} = \cdots = r_{x_k} = d(x_{i+1}, x_i).$$

If $k > 2$, then x_2 defines x_1 and x_3 !

Therefore, $k = 2$ and x_1 and x_2 <u>define</u> <u>each</u> <u>other</u> (and no other points).

Therefore, the set of <u>all</u> "defining edges" pairs up the points as above — and that is the 1-factor.

* Note: The Theorem merely states that there is a 1-factor. The <u>proof</u> shows that the full set is the 1-factor.

COR: If G is a CSIG without a Δ, then G has an <u>even</u> <u>number</u> of points.

OSIGs are harder: P_3, for instance.

x defines both y and z !

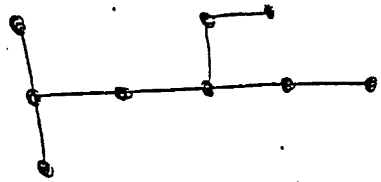
However, if G is a tree, we get a complete characterization:

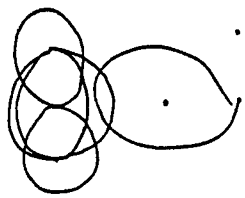
Theorem: Suppose G is a tree. Then

1) G is a CSIG $\Longleftrightarrow$ G has a 1-factor.

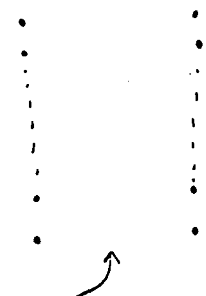
2) G is a OSIG $\Longleftrightarrow$ G has a $\{P_2, P_3\}$-factor.

Ex:





This can't work in general, since



every possible edge between $K_{n,n}$

has too many edges to be a SIG but has a 1-factor and is Δ-free.

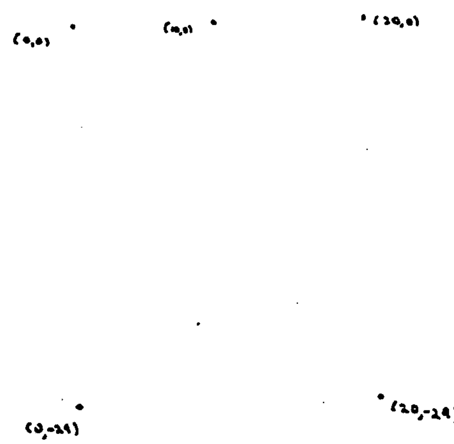
Question:

Since we are interested in computer vision, we really care about SIGs where the points have to show up in pixels.

Which SIGs show up here?

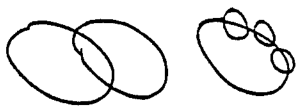
Theorem: If G is a OSIG, then G has a representation in which every point has <u>integer</u> coordinates.

Ex:



C_5

Idea: Spose G has a representation with no circles tangent:



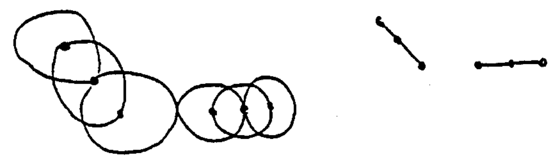
Then $\exists\ \varepsilon > 0$ so that if any circle is moved $< \varepsilon$ no intersections are changed.

Move the circles one at a time to <u>rational</u> points.

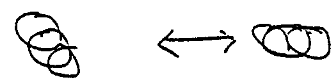
Then "puff up" to integer points.

Oops: Sometimes you have to have tangencies: P_3, P_{odd} for OSIGs.

Fix: Spose only those, that is, no "accidental" tangencies:



$P_3 \cup P_3$



Then: move in "clumps."

If C_x and C_y are tangent at z, (so z defines them!) Then move all 3 circles together until z has rational coordinates.

Then rotate

and move along the xzy-line to fix x and y.

This works IF such a representation exits.

does!

in.

So:

1. We know a little about SIGs and CSIGs.
2. They seem to be useful for object separation in some contexts.
3. They may be useful for object identification in similar contexts.
4. The definitions generalize to three dimensions and different geometries.
5. Our ignorance exceeds our knowledge.

"Two Extremal Problems in Graph Theory"

Roger Entringer, University of New Mexico
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HUNGARY

1

UNAVOIDABLE SUBGRAPHS OF SPARSE GRAPHS

Q. How many edges can a graph of order n have if it doesn't contain a hamilton cycle?

A. (Ore 1961) $\binom{n-1}{2}+1$.

Q. What graphs have this many edges but don't contain a hamilton cycle?

A. K_{n-1} with a pendant vertex, $n \neq 5$.

Let $P(n)$ be a property enjoyed by K_n.

Generic Extremal Problem: *Determine the maximum number of edges, $ex(n;P(n))$, a graph of order n can have if it doesn't satisfy property $P(n)$.*

The graphs of order n that have $ex(n;P(n))$ edges but do not satisfy property $P(n)$ are called the *extremal graphs* for $P(n)$.

2

Q. How many edges can a graph of order n have if it doesn't contain any cycle?

A. $n-1$.

Q. What graphs have this many edges but don't contain a cycle?

A. Trees.

Q. How many edges can a graph of order n have if it doesn't contain a subdivision of K_4?

A. $ex(n;K_4S) = 2n-3$.

Q. How many edges can a graph of order n have if it doesn't contain a subdivision of K_5?

Conjecture (Dirac 1964) $ex(n;K_5S) = 3n-6$.

3

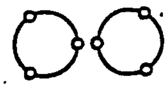
Fix the graph F. We define the *subdivision threshold* of F to be the maximum number of edges, $ex(n;FS)$, a graph of order n can have without containing a subdivision of F as a subgraph.

We denote by $EX(n;FS)$ the family of those graphs of order n that have $ex(n;FS)$ edges and do not contain a subdivision of F.

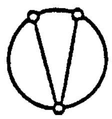
A result of Mader shows that for any graph F there is a constant, c_F, such that $ex(n;FS) \leq c_F n$.

Theorem (Mader 1967). *If a graph has order n and size $2^{\binom{k}{2}-1}kn$ then it contains a subdivision of K_{k+1}.*

4



Theorem. (Erdős and Pósa 1965) $ex(n;FS) = 3n-6$. *G is in* $EX(n;FS)$ *iff* $G = K_3 + \overline{K}_{n-3}$.



Theorem. (BCDES) $ex(n;FS) = \begin{cases} 2n-2, & n \equiv 1 \mod 3 \\ 2n-3, & n \not\equiv 1 \mod 3 \end{cases}$

G is in $EX(n;FSR)$ *iff every block of G, with at most one exception, B, is isomorphic to* K_4 *and* $B = K_2 + \overline{K}$, *or* $B = K_{3,3}$ *or* $B = K_2 \times K_3$ *or B is the nearly 3-regular graph of order 5.*

Problem. *Find all graphs with subdivision threshold less than* $3n-6$.

Properties of F **when** $ex(n;FS) < 3n-6$:

(i) $\Delta(F) \le 6$.

(ii) *F has at most one vertex with degree* ≥ 6.

(iii) *F has at most two vertices with degrees* ≥ 5.

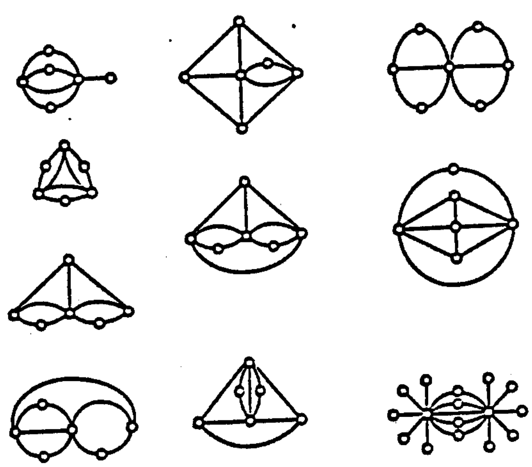
(iv) *F is planar.*

(v) *If F is connected, then it has order* ≤ 7.

(vi) *If F is 2-connected, then it has order* ≤ 6.

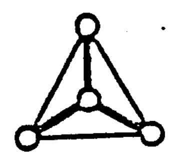
(vii) *F is a subgraph of* $K_3 + \overline{K}_{n-3}$.

Certain subgraphs of

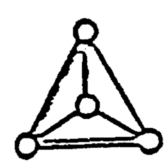


+ pendant vertices.

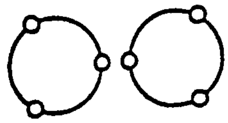
Candidates for graphs G satisfying $ex(n;FS) < 3n-6$.



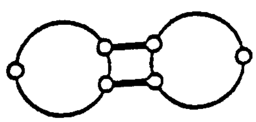
Theorem. (Thomassen 1974) $ex(n;FSR) = 2n-3$. *G is in* $EX(n;FSR)$ *iff G is a* $(*,2)$*-cockade where each member * of the cockade is* K_3 *or* $K_{3,3}$.



Theorem. (Krusenstjerna-Hafstrøm and Toft 1980) $ex(n;FSR) = 2n-3$. *G is in* $EX(n;FSR)$ *iff G is a* $(3,2)$*-cockade.*



Theorem. (Erdős and Pósa 1965) $ex(n;FS) = 3n - 6$. *G is in* $EX(n;FS)$ *iff* $G = K_3 + \overline{K}_{n-3}$.



Theorem. (BCDEKS) $ex(n;FSR) = 3n - 6$. *G is in* $EX(n;FSR)$ *iff* $G = K_3 + \overline{K}_{n-3}$.

9

MINIMUM TRANSMISSION SPANNING TREES

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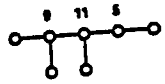
33447

83447 Squirrel, Idaho

33447 Del Ray Beach, Florida

10

The *load*, $L(v)$, of a vertex v of a tree T is the number of paths in T containing vertex v as an internal vertex. This has also been called the *cutting number* by Harary and Ostrand.



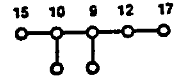
Suppose T has order n and that the branches of T at v have n_i edges, $1 \le i \le k$, (so that $\sum_{i=1}^{k} n_i = n - 1$) then

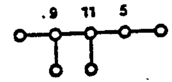
$$L(v) = \sum_{i<j} n_i n_j = \frac{1}{2}\left[(n-1)^2 - \sum_{i=1}^{k} n_i^2\right].$$

The *load*, $L(T)$, of a tree T is the sum of the loads of the vertices.

11

The *transmission*, $\sigma(v)$, of a vertex v in a connected graph G is the sum of the distances from v to each of the remaining vertices of G.





The *transmission of a graph G* is defined by

$$\sigma(G) = \frac{1}{2}\sum_{v \in V(G)} \sigma(v)$$

Observation.

$$L(T) = \sigma(T) - \binom{n}{2}$$

(since a path joining u and v contributes 1 to the load of each of $d(u,v) - 1$ vertices.)

The *transmission center* of a graph is the set of vertices with minimum transmission and consists of one vertex or two adjacent vertices. The transmission center is the centroid (Zelinka).

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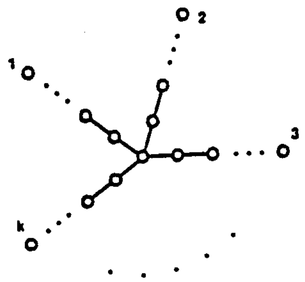
Theorem. *Of all trees of order n with exactly k end vertices, $S(n,k)$ has minimum transmission.*

$S(n,k)$:



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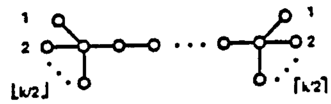
Theorem. *Of all trees of order n with exactly k end vertices, $D(n,k)$ has maximum transmission.*

$D(n,k)$:



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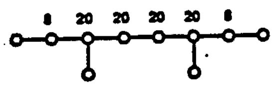
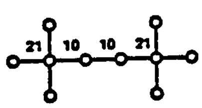
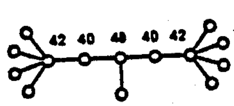


Question. *What fraction of the vertices of a tree of order n can have maximum load? In particular, does this fraction tend to 0?*



Question. *What fraction of the vertices of a tree of order n can have a relatively maximum load?*



15

Problem. *Given a connected graph find a spanning tree with minimum transmission. For example, find a spanning tree of Q_n with minimum transmission.*

Theorem. *Let G be a k-partite graph with smallest part H. The spanning tree of G with minimum transmission contains two adjacent vertices, u in H and v, where u is adjacent to all vertices of G not in H and v is adjacent to all vertices of G in H.*

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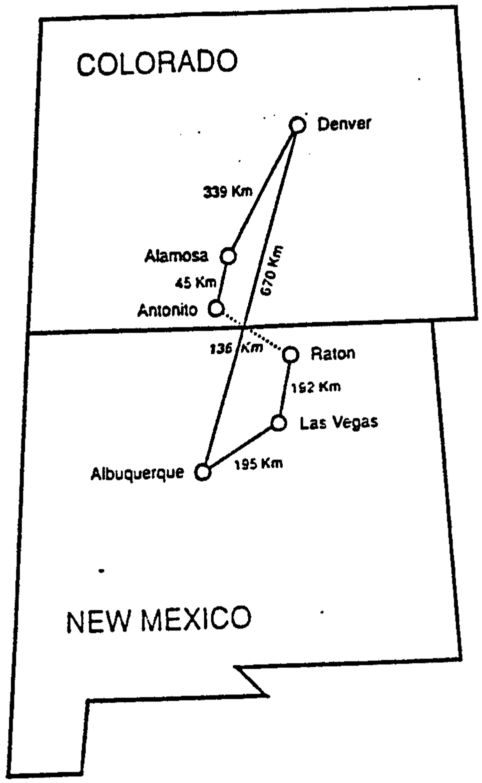

COLORADO
Denver
339 Km
Alamosa
45 Km
Antonito
670 Km
136 Km
Raton
192 Km
Las Vegas
Albuquerque
195 Km
NEW MEXICO

17

"Containment Orders and Planar Graphs"

Edward R. Scheinerman, Johns Hopkins University
Department of Mathematical Sciences, Baltimore, MD

Containment

Planar Graphs

Orders

Edward Scheinerman
Johns Hopkins University

1

Overview

- Circle Orders, and their relatives
- Building Posets from Graphs
- Graph Planarity ⇔ Poset Propertie
  - "Triangle" Orders
  - Circle Orders
  - Point-Halfspace Orders
  - Non-Planar Comments
- Planar Maps & Circle Orders
- The Circle Order Problem

2

Definitions...

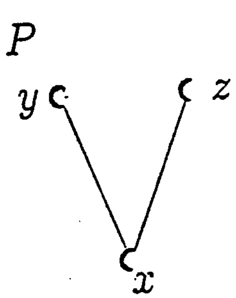
Let P be a finite poset.

We call P a

provided we can assign to each $x \in P$

a

so that x ≤ y iff

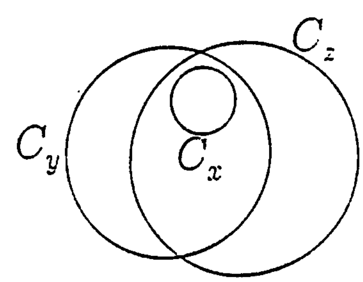


3

circle order

circle C_x

$C_x \subseteq C_y$



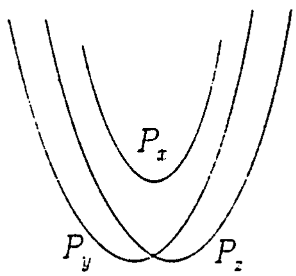
(also: *Sphere Orders* via balls in R^3)

4

parabola order

parabola P_x (upwards, filled)

$$P_x \subseteq P_y$$



RS2PD order*

real, symmetric, 2-by-2 matrix M_x

$M_y - M_x$ is positive

(semi) definite

$$M_x \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_y \quad \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

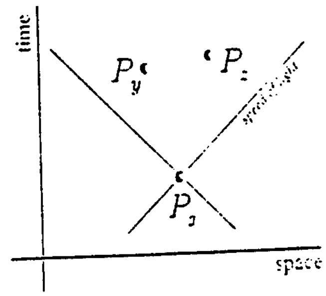
$$M_z \quad \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

* real, symmetric, two-by-two positive definite matrix order
(also, H2PD = complex Hermitian, two-by-two positive definite matrix order

space-time order*

"event" P_x in space-time*

P_x precedes P_y



* two space coordinates, one time coordinate

What's the connection?

Theorem. The following statements about a finite poset *P* are equivalent:

- *P* is a circle order
- *P* is a space*-time
- *P* is a parabola order
- *P* is a RS2PD order

* two space coordinates

Theorem. The following statements about a finite poset *P* are equivalent:

- *P* is a sphere order
- *P* is a space*-time
- *P* is a H2PD order

* three space coordinates

But! Both statements are false for infinite posets. [Felsner & Scheinerman]

Why it works
(Q/L Orders)

Q/L order on $\mathbb{R}^{n+1}$

$$\underbrace{x \le y}_{\in \mathbb{R}^{n+1}} \iff \begin{cases} Q(y-x) \ge 0 \\ L(y-x) \ge 0 \end{cases}$$

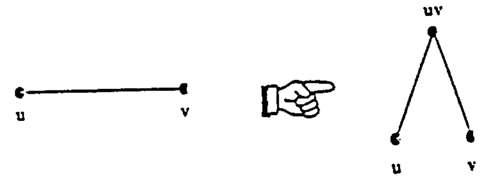
(a quadratic form: Q; a linear form: L)

Theorem. The only possible Q/L orders on $\mathbb{R}^{n+1}$ are:
- anti-chain
- disjoin union of chains
- space*-time order (balls in $\mathbb{R}^n$)

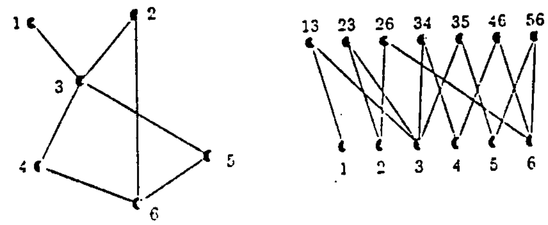
* n space coordinates [Scheinerman]

9

Posets from Graphs



G P(G)



10

What can you deduce about *graph* properties of G from *order* properties of P(G)?

Planarity

Graph Theory World

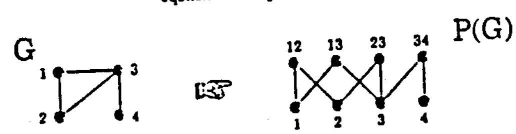
⇕

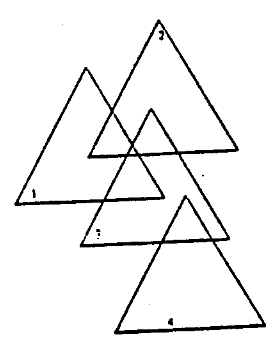
?

Partially Ordered Sets 11

Theorem: A graph G is planar iff P(G) is a triangle* order. [Schnyder]

*equilateral triangles with bottom side parallel to the x-axis





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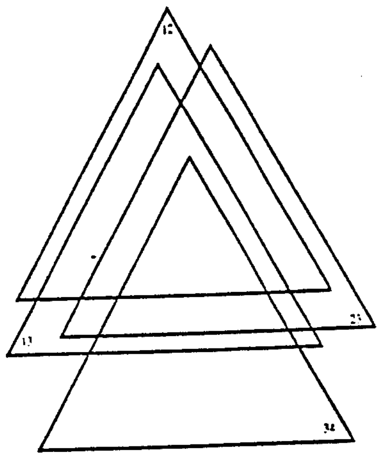

(Actual Statement: G is planar iff dim P(G) ≤ 3.)

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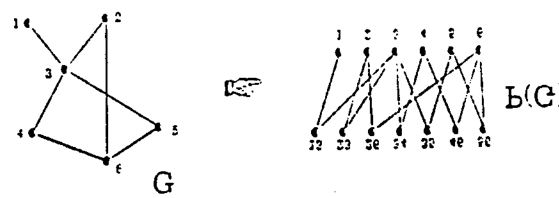
> **Theorem: A graph G is planar iff P(G) is a circle order.** [Scheinerman]

Key Ideas in the Proof

#1 O.K. to work with the dual.



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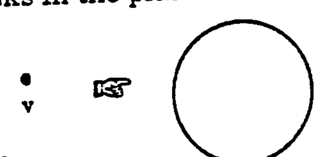
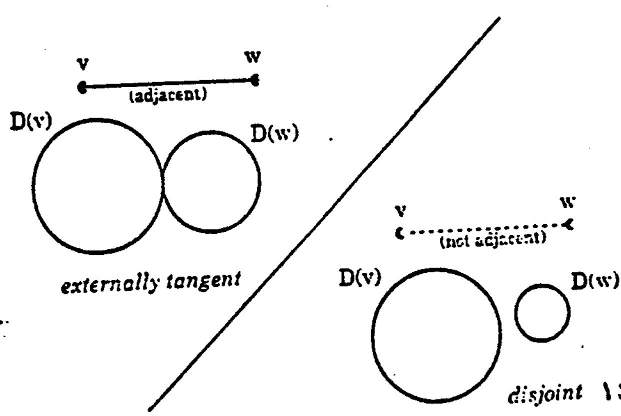
Key Ideas in the Proof
(continued)

#2 Thurston's Theorem.

Every planar graph has a representation by disks in the plane...



such that...



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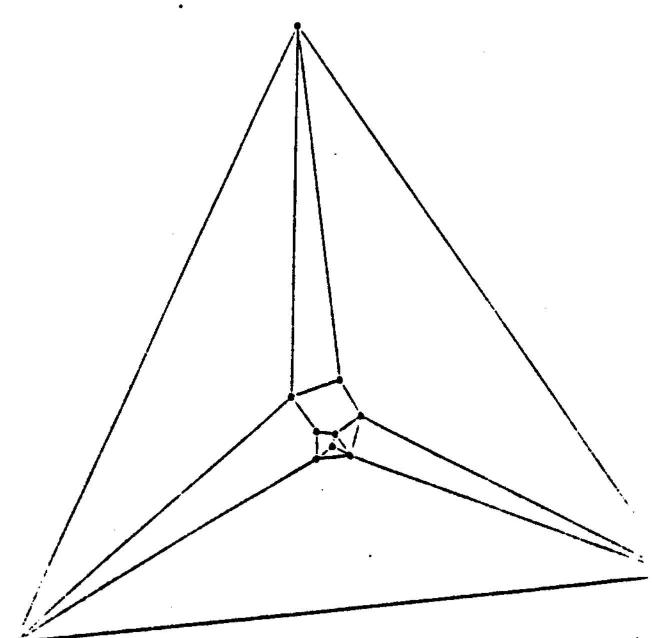

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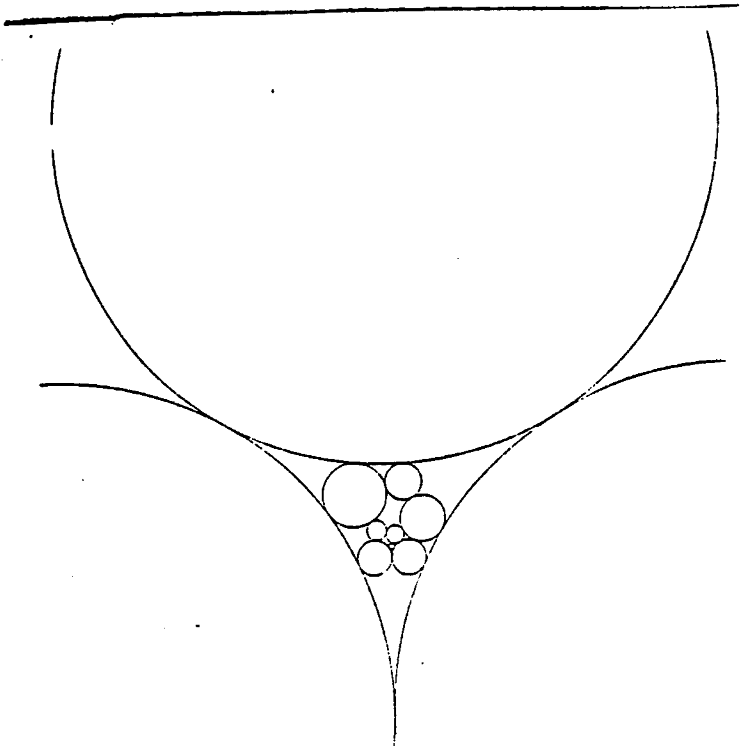

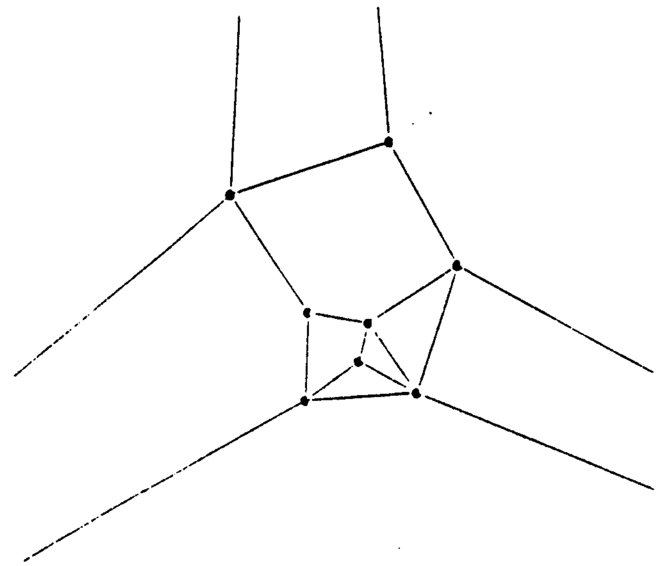

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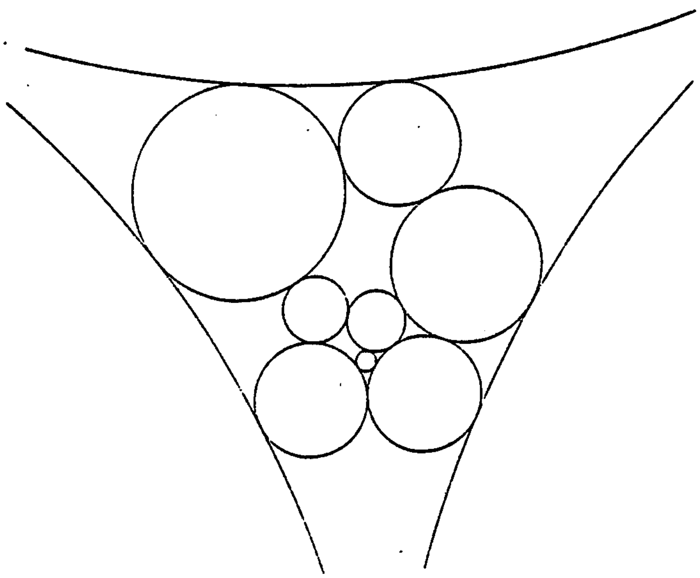
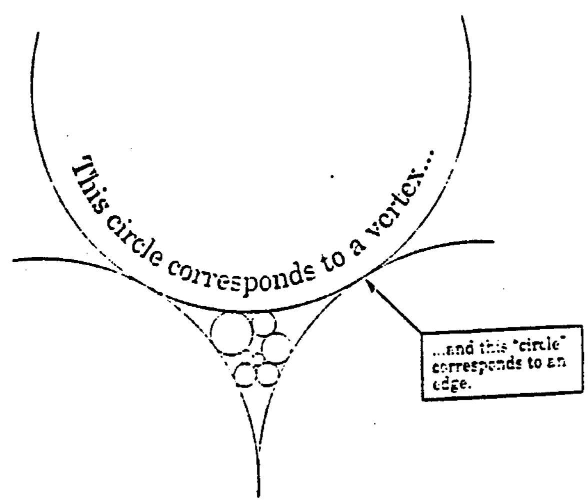


G planar ⇒ P(G) is a circle orde

- Form Thurston circles for G. These will be the circles for V(G).
- Points of tangency will be the circles (of radius 0) for E(G).
- Notice: Every edge circle is contained in exactly its endpoints' circles.



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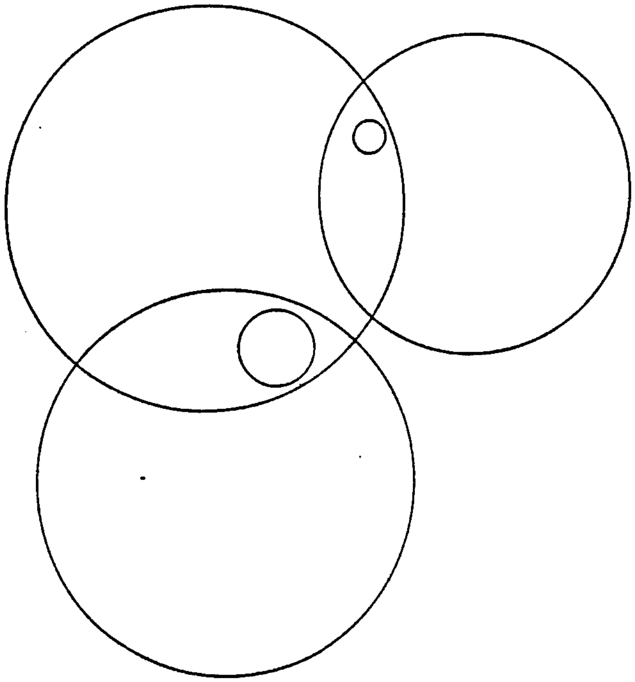
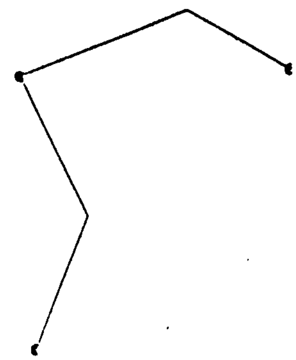
G planar ⇐ P(G) is a circle order

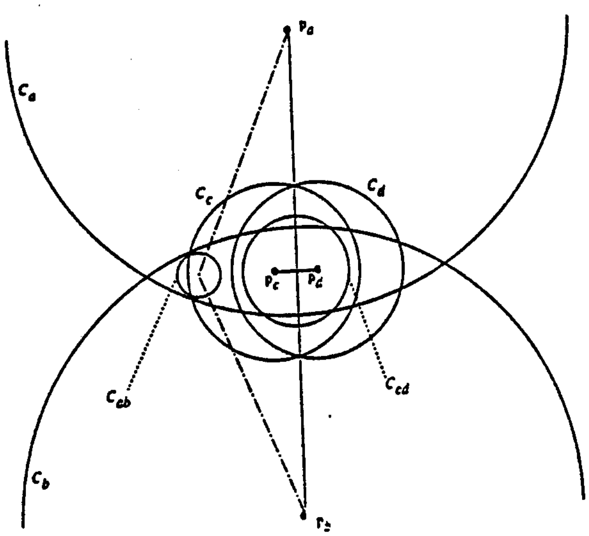
Draw the dual of P(G) as a circle order...





...and this will give an embedding of G in the plane!

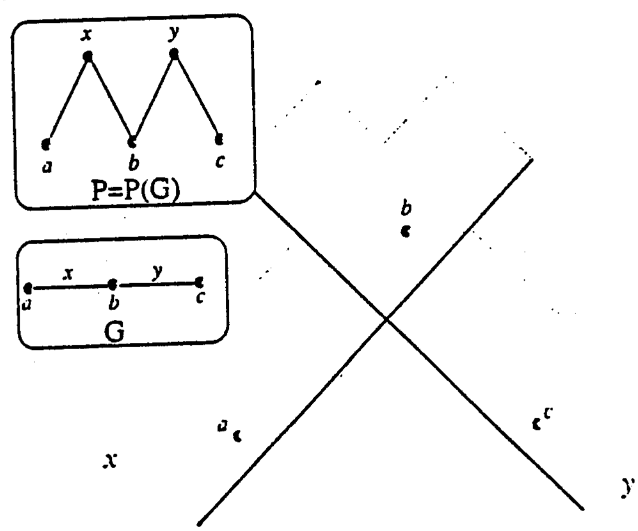
Why 2-step paths might be needed



Circle/Sphere Orders at their extreme...

Point-Halfspace Orders

A "bipartite" poset is called a *point-halfspace order* if...



Theorem. Let G be a graph. P(G) is a point-halfspace order in R^3 if and only if G is planar or K_5.

[Scheinerman, Trenk & Ullman]

Summary

For any graph G...

P(G) is a circle order

G is planar ⟺ P(G) is a triangle* order

P(G) is a point-halfspace order (and G ≠ K_4)

*Equilateral triangles with bottom parallel to x-axis. (Equivalent to dim P ≤ 3.)

What about non-planar graphs?

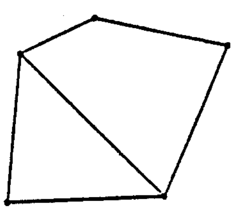
Theorem. If G is *any* graph, then P(G) is a sphere order. [Scheinerman]

Corollary. Let G be a graph. The least d such that G embeds in R^d equals the least d such that P(G) is representable by balls in R^d. [Scheinerman]

Theorem. If G is *any* graph, then P(G) is a point-halfspace order in R^4. [Scheinerman, Trenk & Ullman]

☞ Note: P(G) can have arbitrarily high poset dimension.

Planar Maps
(bounded faces only)

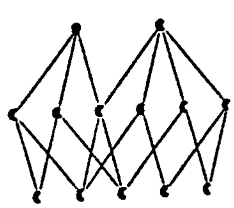


faces

edges

vertices

P(M)



F(M)

E(M)

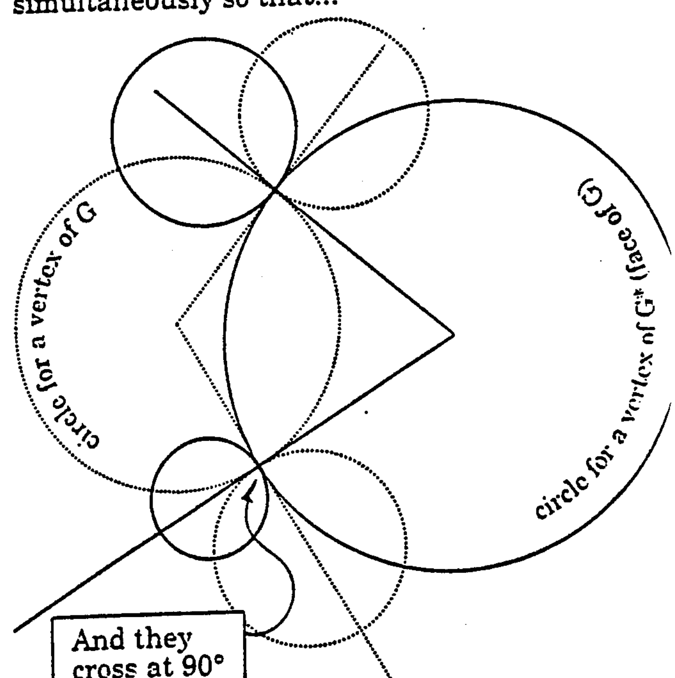
V(M)

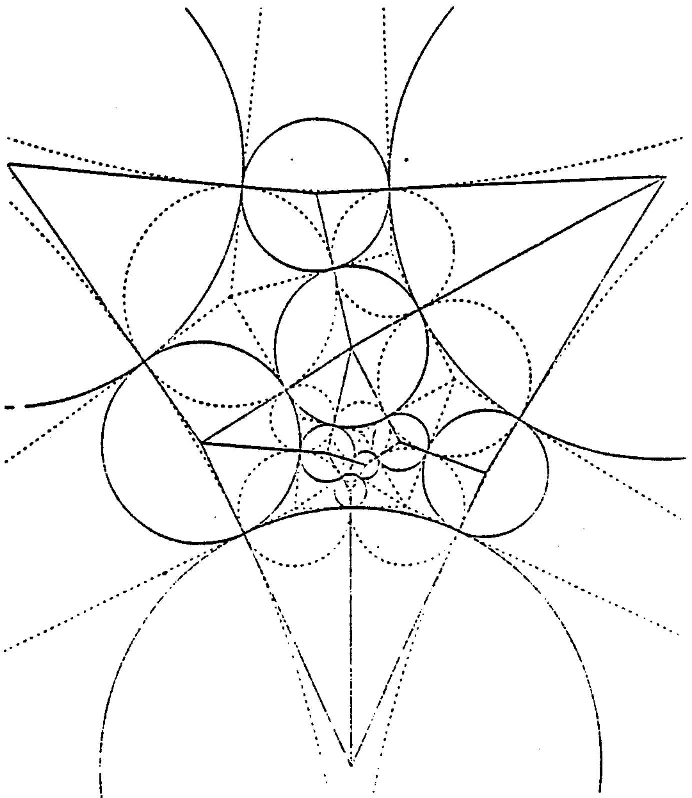
Theorem. For any planar map, dim P(M) ≤ 3. (Brightwell & Trotter)

...but is P(M) a circle order?

Double Thurston Theorem

Theorem. (Brightwell & Scheinerman, Pulleyblank & Rote) If G is a 3-connected planar graph and G* is its dual, then we can make "Thurston Circles" for G and G* simultaneously so that...





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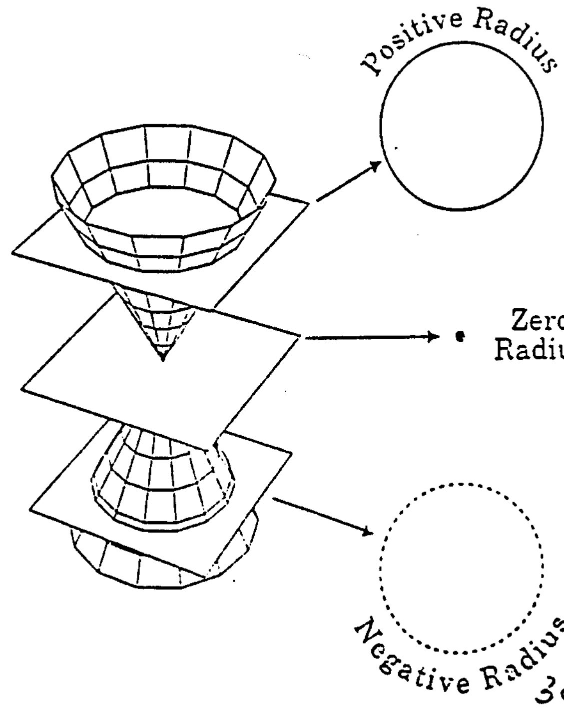
Adjusting Time & Circles of Negative Radius



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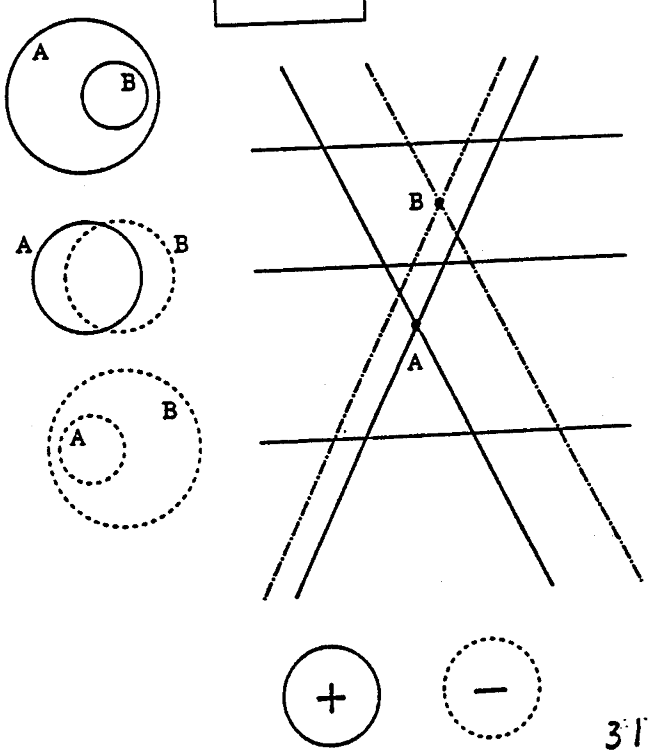
Containment of Positive and Negative Circles

A ⊃ B



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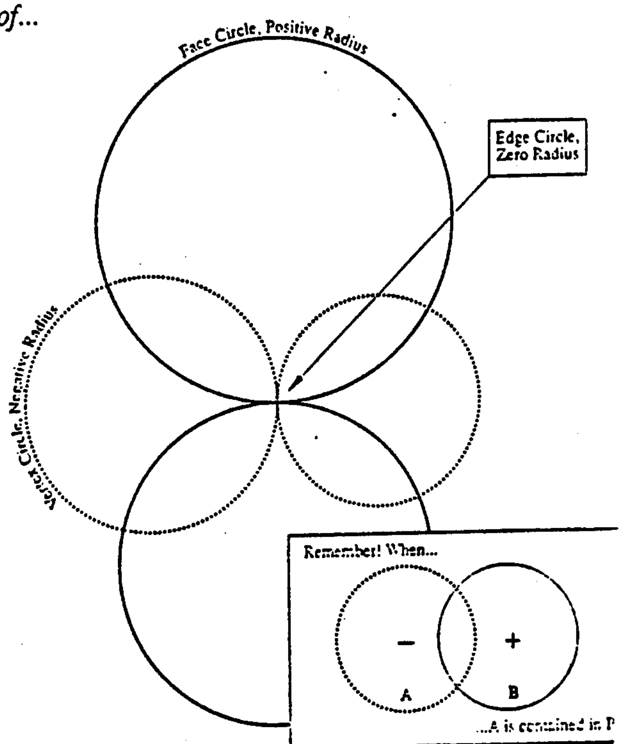
Main Theorem

Theorem. If M is a 3-connected planar map then P(M) is a circle order. (Brightwell & Scheinerman)

Proof...



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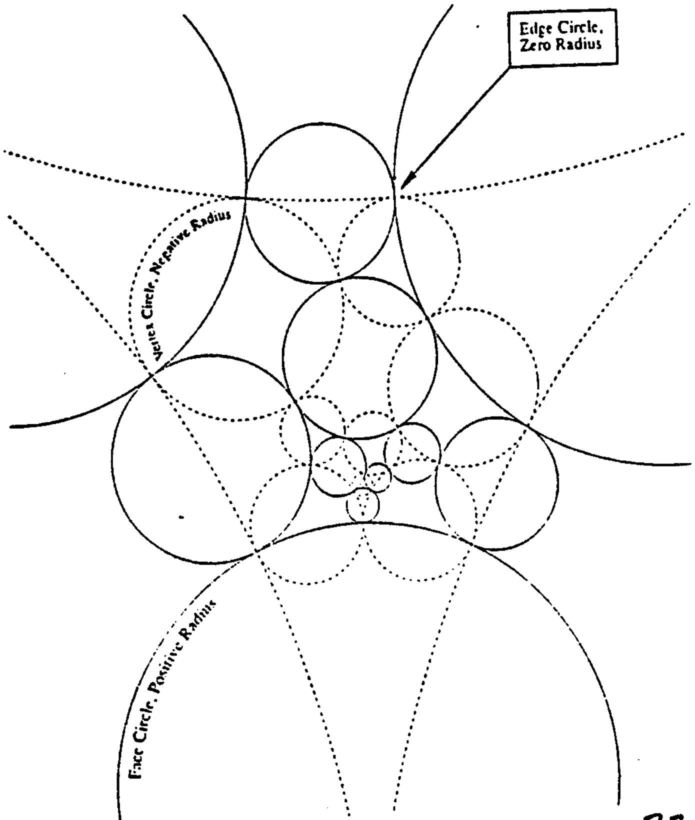
Why P(M) is a circle order



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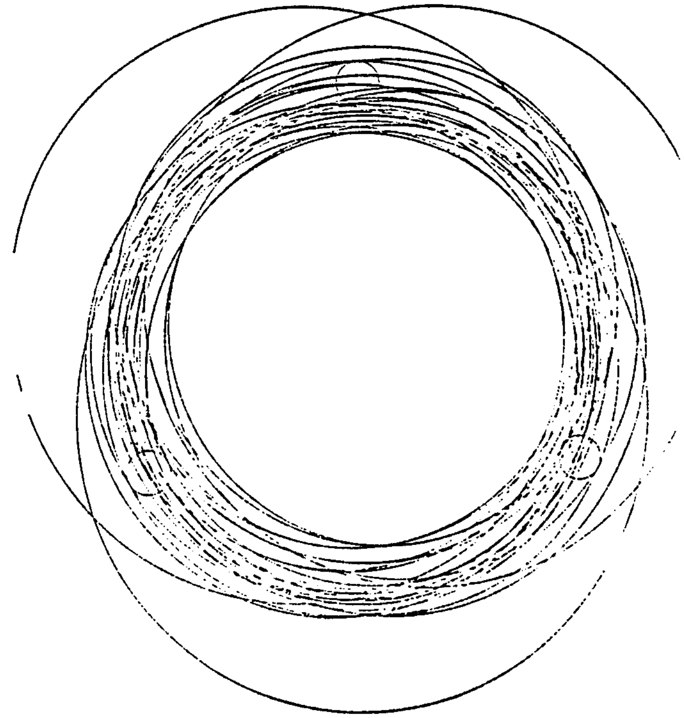

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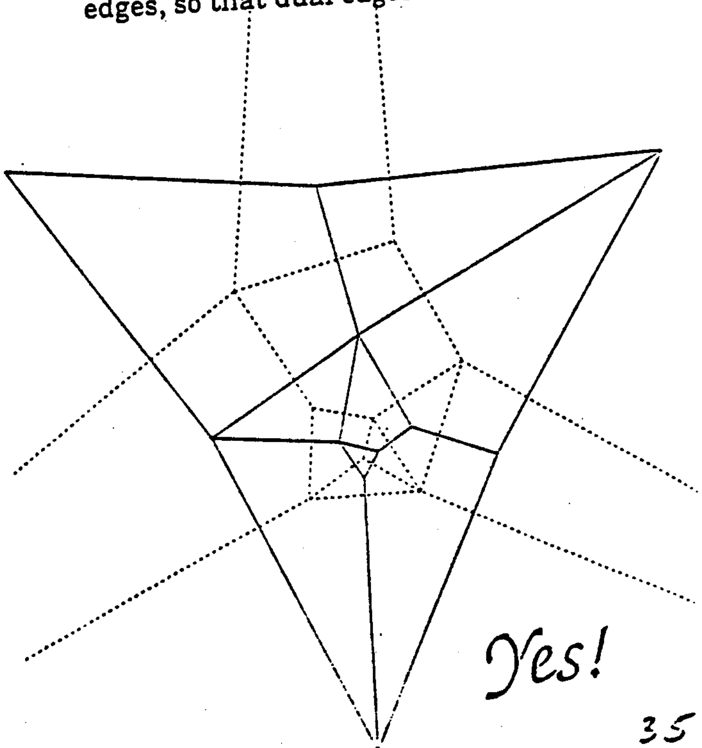
A Conjecture of Tutte...

Let G be a 3-connected planar graph. Can one properly draw G and its dual G* with straight line segments for edges, so that dual edges cross at 90°?



Yes!

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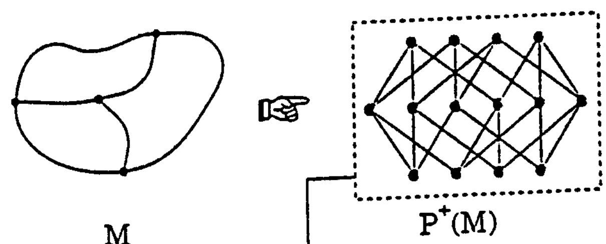
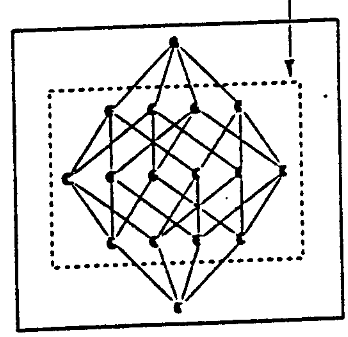
What about the unbounded faces?

For example...

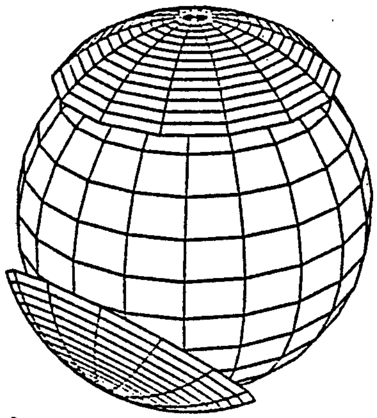




$2^{\{1,2,3,4\}}$ is not a circle order (Jamison), which implies $P^+(M)$ is not a circle order.

but... 36

Theorem. If M is a 3-connected planar map, then $P^{+}(M)$ is a "cap order".
(Brightwell & Scheinerman)



Note: $P^{+}(M)$ is the face lattice of a convex polyhedra in R^{3}.

"On Finding Transmitter-Receiver Matchings"

Jean R.S. Blair, University of Tennessee
Department of Computer Science, Knoxville, TN

On Finding

Transmitter - Receiver Matchings

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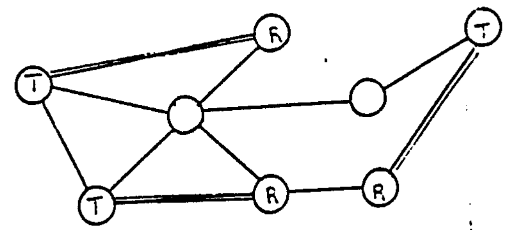
† Portions of this research were performed at the Mathematica Sciences Section of Oak Ridge National Laboratory and partially supported by the Applied Mathematical Sciences Research Program, Office of Energy Research, U.S. Departme of Energy under contract DE-AC05-84OR21400 with Martin Marietta Energy Systems, Inc.

‡ Supported in part by NSF Grants DCI-8603318 and CCR-8905296.

Transmitter - Receiver Matching



Given: a communication network with bidirectional links

Goal: to simultaneously test as many links as possible

Constraints:

(1) a transmitting site sends signals along all of the links emanating from it

(2) a site cannot be both a transmitter and a receiver at the same time

(3) two or more signals reaching a site at the same time interfere

Known Results for different graph structures:

NP-Complete for :

- → Graphs with max. degree 3 [Stockmeyer, Vazirani]
- → Bipartite Graphs [Even, Goldreich, Moran, Tong]
- → Chordal Graphs [this paper]

Linear Time Algorithms for:

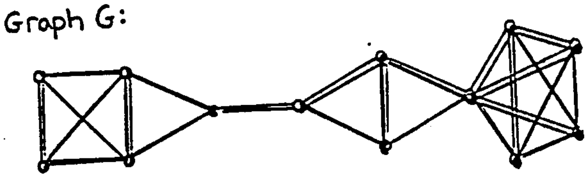
- → Trees [Farley, Proskurowski]
- → 2-Trees [Colbourn, Proskurowski]
- → AC-Graphs [this paper]

Quadratic Time Algorithm for:

- → Interval Graphs [Spinrad]

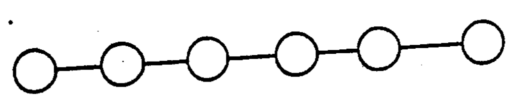
AC Graphs

Graph G:



Clique Graph G_c

- → one node for each maximal clique in G
- → an edge between 2 nodes iff the correspondinc cliques intersect



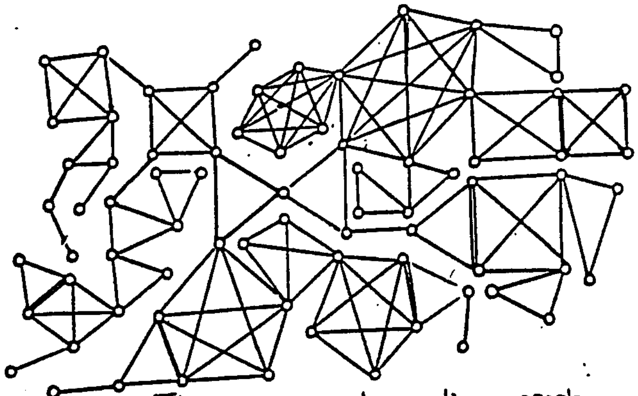
AC Graphs

- → graphs whose clique graphs are acyclic

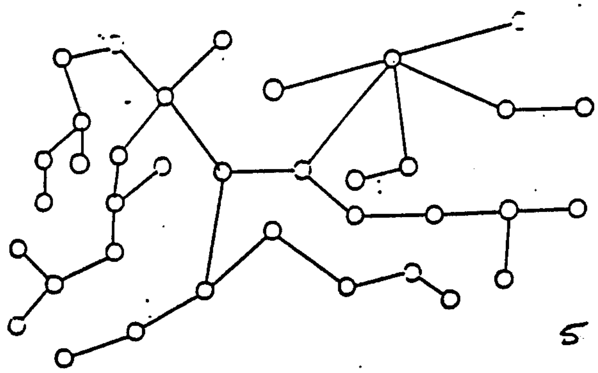
Terminology

- → leaf node of G_c, leaf clique of G, parent clique, simplicial node of G

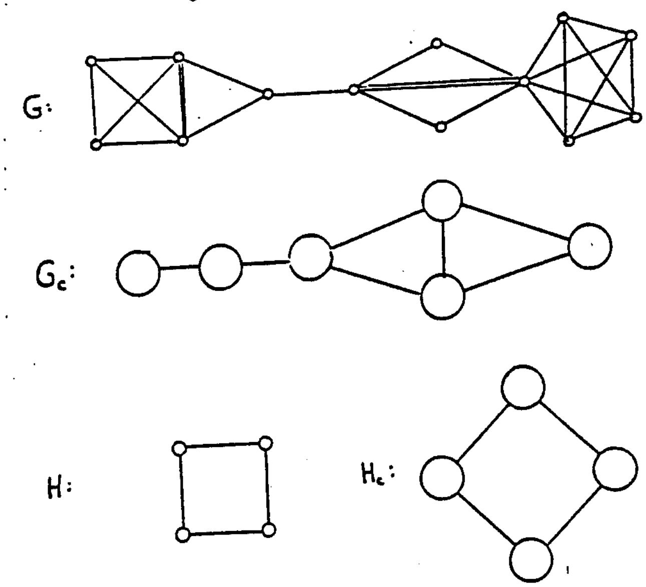
An AC Graph



The corresponding clique graph



Some graphs that are not AC graphs.



Useful Propertys of AC Graphs

Lemma 2.1 -- Any node of an AC Graph, G, belongs to at most two cliques.

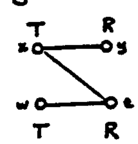
Lemma 2.2 -- Any induced subgraph of an AC Graph is also an AC Graph.

Lemma 2.3 -- Every leaf clique contains at least one simplicial node.

Lemma 2.4 -- Every AC Graph is chordal.

Conflicts

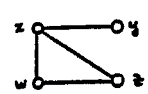
TR-pairs : $\overset{T}{(x},\overset{R}{y)}$ and (w,z) conflict iff at least one of the edges $\{x,z\}$ and $\{y,w\}$ is in G



interference at z

Edges : $\{x,y\}$ and $\{w,z\}$ conflict iff every orientation of the edges results in a c

Lemma 2.5 -- Two distinct edges $\{x,y\}$ and $\{w,$ conflict iff the subgraph of G induc on $\{x,y,w,z\}$ contains a 3-cycle



Orientability

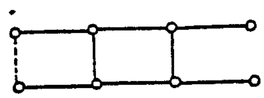
A matching M is orientable iff there exists an orientation of M with no conflicts.

M -- a matching

V_M -- the nodes that comprise M

G_M -- the subgraph of G induced on V_M

Lemma 2.6 -- M is orientable iff G_M does not contain a cycle that uses an edge in M.



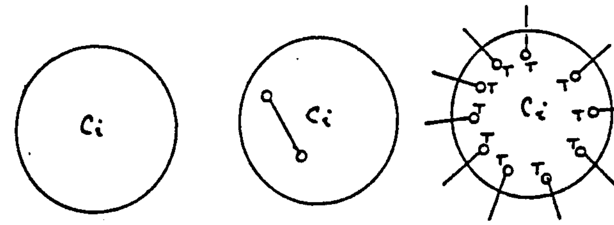
black edges belong to M

Lemma 2.7 -- If no pair of edges in M conflict, then M is orientable.

(Both results require that G be chordal).

Nodes in a Clique and Orientable Matchings

Lemma 3.1 -- Let $I = C_i \cap V_M$ for clique C_i and orientable matching M. Exactly one of following is true:



(a) $I = \emptyset$

(b) for $x, y \in I$ $\{x,y\} \in M$ and $I - \{x,y\} = \emptyset$

(c) nodes in I
single endpoints
pairs in M & in
vaid orientatic
labels are sam

Lemma 3.3 -- If for $x, y \in C_i$ $\nexists \{x,y\} \in M$ then at most one node from each $C_i \cap C_j$, $j \neq i$, can be in

Sketch of the Algorithm

With each clique C_i store PossiblePair(C_i) (initial value is TRUE)

{Find a maximum cardinality orientable matching}

Until all nodes have been removed from G

(1) choose a leaf clique C_ℓ

(2) if PossiblePair(C_ℓ)

choose a "good" pair from C_ℓ

remove from G all nodes tha would conflict with the pai

set PossiblePair (C_p) to FALSE, if necessary

else remove C_ℓ from G

Orient the edges

Choosing a "good pair"

Lemma 3.1(b) ⇒ no matter what pair, no other nodes in C_ℓ can belong to V_M

Lemma 3.1 ⇒ if choose any node from $C_\ell \cap C_p$ then no pairs come from C_p

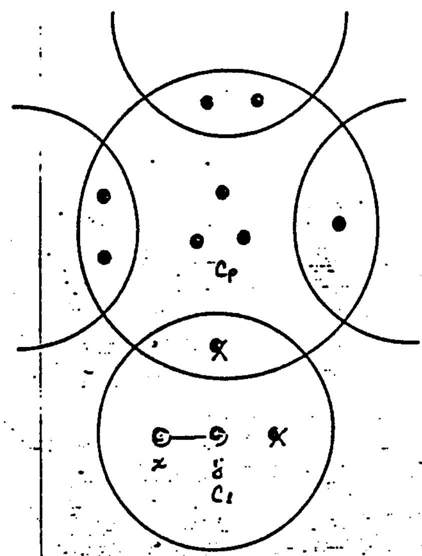
⇒ if choose two nodes from $C_\ell \cap C_p$ then no other nodes come from C_p

Stay away from C_p as much as possible

Lemma 2.3 ⇒ $\exists$ at least one simplicial node in C_ℓ

Example – PossiblePair(C_l) = TRUE

C_l contains >1 simplicial node



choose <x,y> and prune G.

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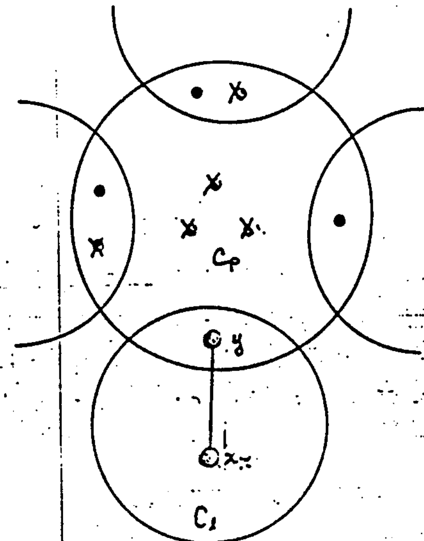
Example -- PossiblePair(C_l) = TRUE

C_l contains one simplicial node



Choose <x,y> and prune G (Lemma 3.3)

PossiblePair (C_p) ⟵ FALSE

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AlgTRM

1. Construct $G_c(V_c, E_c)$ (the clique graph of G)
2. $M \leftarrow \emptyset$
3. for each clique C_i do *PossiblePair* $(C_i) \leftarrow$ true
4. while G_c contains two or more nodes do
   a. $c_l \leftarrow$ a leaf node of G_c
   b. $C_l \leftarrow$ the corresponding leaf clique of G
   c. $C_p \leftarrow$ the clique that intersects C_l
   d. if *PossiblePair* (C_l) then
      i. $x \leftarrow$ a simplicial node in C_l
      ii. if $C_l - \{x\}$ has a simplicial node then
            $y \leftarrow$ a simplicial node in $C_l - \{x\}$
            $P \leftarrow \emptyset$
          else
            $y \leftarrow$ a node in $C_l - \{x\}$
            if *PossiblePair* (C_p) then
               PossiblePair $(C_p) \leftarrow$ false
               $P \leftarrow$ { all nodes in C_p except one from each $C_i \cap C_p$, $i \neq l$ and $i \neq p$ }
            else $P \leftarrow \emptyset$
      iii. Add $\{x, y\}$ to M
      iv. Delete $C_l \cup P$ from G and c_l from G_c
      v. If $|C_p - P| = 1$ then Delete c_p from G_c
   else
      vi. Delete the simplicial nodes of C_l from G
      vii. Delete c_l from G_c
5. If G is not empty then
   a. $x \leftarrow$ a simplicial node in G's remaining clique
   b. $y \leftarrow$ a simplicial node distinct from x in G
   c. Add $\{x, y\}$ to M
6. Orient the edges of M using AlgOrient

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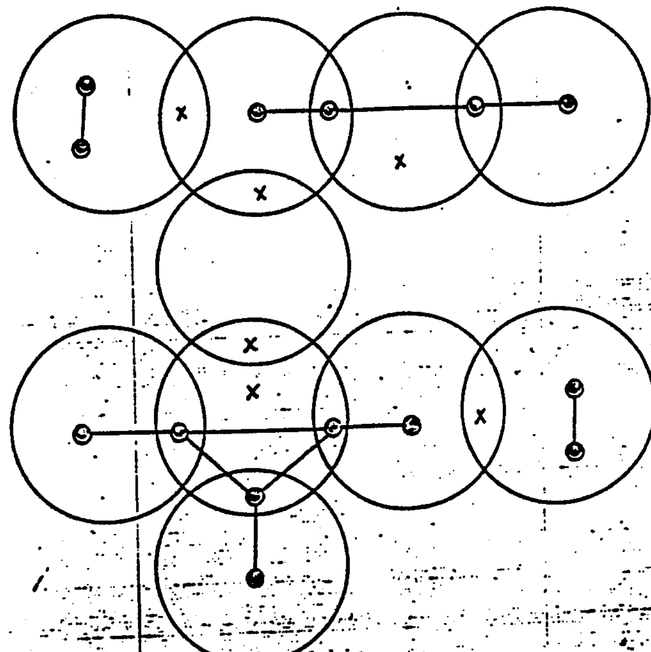
Orientation Step

Given the set M, and G_M



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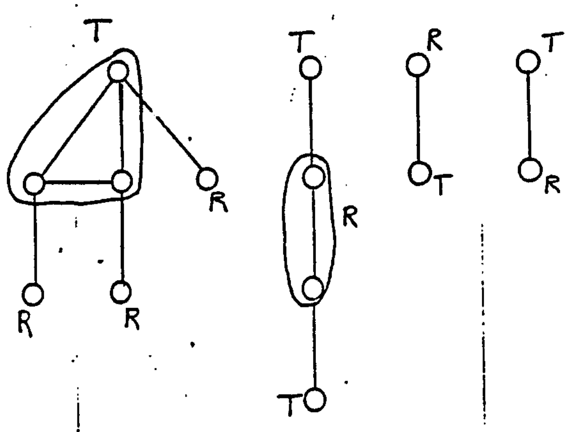
AlgOrient

(1) Find the connected components of G_M
(1a) Shrink non-matching portions of each component
(2) Perform Breadth-First-Search on each component labeling nodes on adjacent levels opposite.

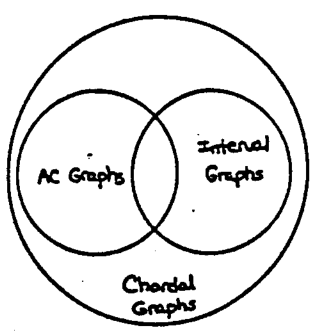


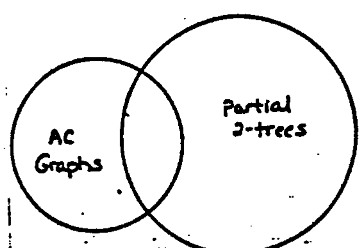
Linear Time of AlgTRM

1. finding the set of maximal cliques -- standard techniques
2. constructing the clique graph -- use $\Theta(|C|^2+|V|)$ space to construct adjacency lists in $\Theta(|V|+|E|)$ time
5. for each leaf clique choose a pair and prune G and G_c
   - each clique is "the leaf clique" at most once
   - each clique is pruned as a parent clique at most once
   - time required to process a leaf clique (or a parent clique) is porportional to the number of nodes in the clique
   - since G is an AC-graph, each node of G belongs to at most two cliques
6. orient the edges in M -- similar to 2-coloring a bipartite graph

Final Remarks

TRM for Chordal Graphs is NP-Complete





Determining if G is an AC-graph

Let G be a chordal graph.

Let G_c be its clique graph.

Property: If there is a cycle in G_c then there is a node in G that belongs to at least 3 cliques.

Lemma 2.1: Any node of an AC graph G belongs to at most 2 cliques.

(1) check for chorality of G
(2) find maximal cliques of G
(3) check to see if any node in G belongs to more than 2 cliques

$\Theta(|V|+|E|)$ time

"The Jamison Method in Galois Geometries"

J. Chris Fisher, University of Regina, Canada
Visiting Clemson University, Department of Mathematical Sciences

JAMISON'S METHOD

Joint work with AIDEN BRUEN

$\Pi = AG(2,q)$ { Affine plane over $GF(q)$ }

Definition.

"S BLOCKS THE LINES OF Π":

For every line $\ell \in \Pi$ there is a $P \in S$ with $P \in \ell$.

THEOREM (JAMISON, 1977)
A BLOCKING SET OF Π MUST CONTAIN AT LEAST $2q-1$ points.

Cf. Projective planes:

$|S| = q+1$ (since each pair of lines meet)

Or (when S contains no line)

$|S| = q + \sqrt{q} + 1$

Facts needed for the proof:

① points: $(x,y) \leftrightarrow z = x + iy \in GF(q^2)$ (elements of $GF(q)$) where $i \in GF(q^2) \setminus GF(q)$

② field automorphism of $GF(q^2)$ of period 2: $\bar{z} = z^q$ (recall $z^{q^2} = z$)

③ line: $\bar{z} + az + b = 0$ $(= z^q + az + b)$

PROOF

suppose to the contrary that $|S| \le 2q-2$

Step 1

REPHRASE THE THEOREM AS A RELATIONSHIP INVOLVING SETS OF points in Π.

original	dual
S contains $2q-2$ points	S^* has $2q-3$ affine lines $\cup \{\ell_\infty\}$
Every $\ell \in \Pi$ contains at least one $P \in S$	Every $P \in \Pi \setminus \{0\}$ lies on at least one $\ell \in S^*$

i.e.

S^* has $2q-3$ affine lines that cover the nonzero points of Π.

STEP 2
FORMULATE THE THEOREM IN TERMS OF POLYNOMIALS OVER $GF(q^2)$

Let S^* CONSIST OF THE LINES

$$\ell_i = z^q + a_i z + b_i = 0 \quad b_i \neq 0$$
$$i = 1, \ldots, 2q-3$$

Assumption:

$$\Pi \setminus \{0\} \subseteq S^*$$

so $z^{q^2-1} - 1$ divides $\prod_{i=1}^{2q-3} \ell_i$

Or

$$\prod_{i=1}^{2q-3} (z^q + a_i z + b_i) = 0 \pmod{z^{q^2-1} - 1}$$

(5)

STEP 3 CALCULATE

$(z^q+a_1z+b_1)(z^q+a_2z+b_2)\cdots(z^q+a_{2q-3}z+b_{2q-3})$

$=(z^q)^{2q-3}+\ldots+(b_1b_2\cdots b_{2q-3})\equiv 0 \pmod{z^{q^2-1}-1}$

key observation:
The coefficient of $z^{q^2-1} = -\prod b_i \neq 0$

Try q z^q's and $q-3$ other factors
$\Rightarrow z^{q^2+?}$ Exponent is too big

Try $q-1$ z^q's and $q-2$ other factors.
$\Rightarrow z^{q^2-q}\times(\;)z^{q-2}=(\;)z^{q^2-2}$ Exponent is Too small

$\Rightarrow c_{q^2-1}=0$
which is a contradiction ⊠

(6)

The Flock Theorem.

GIVEN AN ELLIPTIC QUADRIC IN PG(3,q) AND A SET OF q-1 DISJOINT CONICS PARTITIONING ALL BUT TWO OF ITS POINTS, THEN THE q-1 PLANES OF THOSE CONICS MUST CONTAIN A COMMON LINE THAT MISSES THE QUADRIC.

Blokhuis's Theorem.

FOR q ODD, IF A q-ELEMENT SUBSET OF GF(q^2) CONTAINING 0 AND 1 HAS THE PROPERTY THAT THE DIFFERENCE OF ANY TWO OF ITS ELEMENTS IS A SQUARE OF GF(q^2), THEN IT IS GF(q).

(7)

Defn
A flock is a set of conics that partition all but two points of a sphere.
The flock is linear if the planes of the conics all contain a common line.

The FLOCK THEOREM (1973)
J.A. THAS (for q even)
W.F. Orr (for q odd)

Proof
Step 1 Rephrase.
Project the sphere (from one of the uncovered points) onto Π, sending the other uncovered point to 0
The conics project to circles
$(z-a)\overline{(z-a)}=r$
Then for the $q-1$ circles to be disjoint they must have center 0 (i.e. $a=0$)

(8)

Step 2. Reformulate
Set $c_j = z^{q+1}-a_jz^q-a_j^qz-r_j$ $\quad j=1,\ldots,q-1$
Claim:
$\prod_{j=1}^{q-1} c_j = z^{q^2-1}-1$ iff all $a_j=0$

Step 3. Calculate.
$\prod c_j = \prod_{j\le q-1}(z^{q+1}-a_jz^q) + \{$Terms of degree at most $(q+1)(q-2)+1\}$
Equate terms of degree $>(q+1)(q-2)+1$
$\prod_{j\le q-1}[z^q(z-a_j)] = z^{q^2-1}$ iff all $a_j=0$
i.e. $\prod(z-a_j)=z^{q-1}$ iff all $a_j=0$

But GF(q^2)[z] is a unique factorization domain ⊠

"Elementary, Sub-Fibonacci, Regular, Van Lier and Other Interesting Sequences"

Fred S. Roberts, Rutgers University
Dept. of Mathematics, Center of Operations Research (RUTCOR), and
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New Brunswick, NJ

ELEMENTARY, SUB-FIBONACCI, REGULAR, VAN LIER, AND OTHER INTERESTING SEQUENCES

BY

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MEASUREMENT THEORY:

THE THEORY OF MEASUREMENT IS CONCERNED WITH UNDERSTANDING THE CONDITIONS UNDER WHICH MEASUREMENT PROCESSES TAKE PLACE, WHAT KINDS OF SCALES OF MEASUREMENT ONE GETS, AND WHAT KINDS OF STATEMENTS WE CAN MAKE USING SCALES OF MEASUREMENT.

MEASUREMENT THEORY AND COMBINATORICS:

IN THE PAST FEW YEARS, PROBLEMS OF UNIQUENESS OF SCALES OF MEASUREMENT HAVE BEEN GIVING RISE TO A VARIETY OF INTERESTING SEQUENCES OF POSITIVE INTEGERS WITH FASCINATING COMBINATORIAL PROPERTIES.

THIS TALK:

IN THIS TALK, I MENTION SUCH SEQUENCES AND DISCUSS THEIR COMBINATORIAL PROPERTIES. BECAUSE OF THE SHORTNESS OF TIME, I CANNOT DESCRIBE THE MEASUREMENT THEORY MOTIVATION EXCEPT IN ONE CASE AND I SHALL CONCENTRATE ON JUST ONE COMBINATORIAL PROBLEM: COUNTING THE NUMBER OF SEQUENCES OF DIFFERENT KINDS.

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THE FIBONACCI SEQUENCE

$F_1, F_2, \dots$

$F_1 = F_2 = 1$
$F_K = F_{K-1} + F_{K-2}, \quad K = 3, 4, \dots$

$F_1, F_2, \dots$ IS THE SEQUENCE 1, 1, 2, 3, 5, 8, 13, ...

ELEMENTARY SEQUENCES

MOTIVATION: "EXTENSIVE" MEASUREMENT
TERM INTRODUCED BY FISHBURN AND ROBERTS (1989).
VARIATION ON THE FIBONACCI SEQUENCE

$X_1, X_2, \dots$

POSITIVE, NONDECREASING INTEGER SEQUENCE
$X_1 = X_2 = 1$
$X_K > 1 \longrightarrow X_K = X_I + X_J$, SOME $I \neq J$.

EXAMPLES: 1, 1, 2, 3, 3, 5
1, 1, 1, 2, 2, 4, 6

$\mathcal{E}_N$ = COLLECTION OF ALL ELEMENTARY SEQUENCES OF LENGTH N

SUB-FIBONACCI SEQUENCES

TERM INTRODUCED BY FISHBURN AND ROBERTS (1989)

$X_1, X_2, \dots$

POSITIVE, NONDECREASING INTEGER SEQUENCE
$X_1 = X_2 = 1$
$X_K \le X_{K-1} + X_{K-2}$

$\mathcal{F}_N$ = COLLECTION OF SUB-FIBONACCI SEQUENCES OF LENGTH N.

NOTE: $\mathcal{E}_N \subseteq \mathcal{F}_N$ FOR ALL N.

BY ENUMERATION, $\mathcal{E}_N = \mathcal{F}_N$ FOR $N \le 6$.

THE SMALLEST SUB-FIBONACCI SEQUENCE WHICH IS NOT ELEMENTARY HAS LENGTH 7: 1, 1, 2, 2, 4, 4, 7

THIS IS NOT ELEMENTARY BECAUSE 7 IS NOT THE SUM OF ANY TWO PRECEDING TERMS. ($\mathcal{F}_7$ HAS SEVEN SEQUENCES NOT IN $\mathcal{E}_7$.)

<u>THEOREM (FISHBURN AND ROBERTS 1990)</u>:

$$|\mathcal{F}_N| = \alpha^{N^2(1+o(1))/2}$$

WHERE

$\alpha = (1+\sqrt{5})/2 = 1.61803$ (THE GOLDEN SECTION)

AND $o(1)$ IS A FUNCTION OF N THAT APPROACHES 0 AS N APPROACHES ∞.

<u>COROLLARY</u>: THE SAME ESTIMATE HOLDS FOR $|\mathcal{F}'_N|$, WHERE $\mathcal{F}'_N$ IS THE SUBSET OF $\mathcal{F}_N$ WHOSE ELEMENTS STRICTLY INCREASE FROM K = 2 ON.

<u>OPEN QUESTION</u>: FIND A SIMILAR RESULT FOR $|\mathcal{E}_N|$.
THE FOLLOWING IS KNOWN:

<u>THEOREM (FISHBURN AND ROBERTS 1990)</u>: $|\mathcal{E}_N|/|\mathcal{F}_N| \longrightarrow 0$
AND HENCE $|\mathcal{E}_N|/|\mathcal{F}_N| \longrightarrow 0$ AS $N \longrightarrow \infty$.

<u>SOME COUNTS</u>:

N:	3	4	5	6	7	8	9	10	11
$\lvert\mathcal{E}_N\rvert$	2	4	10	31	120	578	3422	24364	205744
$\lvert\mathcal{F}_N\rvert$	2	4	10	31	127	711	3821	64049	1067599
$\lvert\mathcal{F}'_N\rvert$	1	1	2	6	27	177	1659	23009	455356

SO $|\mathcal{F}'_N|$ DOES NOT EXCEED $|\mathcal{E}_N|$ UNTIL N = 11.

REGULAR SEQUENCES

MOTIVATION: "SUBJECTIVE PROBABILITY" MEASUREMENT
INTRODUCED BY: FISHBURN AND ODLYZKO 1989

$X_1, X_2, \dots$

POSITIVE, NONDECREASING INTEGER SEQUENCE
$X_1 = X_2 = 1$
$X_J \le \sum_{I=1}^{J-1} X_I \quad J = 3, 4, \dots$

EXAMPLES:
N = 2: 1, 1
N = 3: 1, 1, 1; 1, 1, 2
N = 4: 1, 1, 1, 1; 1, 1, 1, 2; 1, 1, 1, 3;
1, 1, 2, 2; 1, 1, 2, 3; 1, 1, 2, 4

$\mathcal{R}_N$ = THE COLLECTION OF REGULAR SEQUENCES OF LENGTH N.

<u>THEOREM (FISHBURN AND ODLYZKO 1989)</u>:

$$|\mathcal{R}_N| = 2^{N^2(1+o(1))/2}$$

<u>SOME COUNTS</u>

N:	2	3	4	5	6	7	8
$\lvert\mathcal{R}_N\rvert$	1	2	6	27	192	2260	47097

THE FOLLOWING THEOREM WILL BE USEFUL LATER.

<u>THEOREM (FISHBURN AND ODLYZKO 1989)</u>: SUPPOSE $X_1, X_2, \ldots$ IS A POSITIVE, NONDECREASING INTEGER SEQUENCE WITH $X_1 = X_2 = 1$. THEN $X_1, X_2, \ldots$ IS A REGULAR SEQUENCE IF AND ONLY IF FOR EVERY J, THERE IS A SET $S \subsetneq \{1, 2, \ldots, N\} - J$ SUCH THAT

(+) $$X_J = \sum_{I \in S} X_I.$$

FOR EXAMPLE, IN 1, 1, 2, 3:
2 IS 1 + 1 AND 3 IS 1 + 2.

IT IS CLEAR THAT CONDITION (+) IMPLIES REGULARITY SINCE IT IMPLIES THAT $X_J \le \sum_{I=1}^{J-1} X_I$. THE CONVERSE IS HARDER.

<u>VAN LIER SEQUENCES</u>

MOTIVATION: "SUBJECTIVE PROBABILITY" MEASUREMENT
TERM INTRODUCED BY VAN LIER (1989), FISHBURN AND ROBERTS (1989), FISHBURN, ROBERTS, AND MARCUS-ROBERTS (1990)

$X_1, X_2, \ldots$

POSITIVE, NONDECREASING INTEGER SEQUENCE
$X_1 = X_2 = 1$
$X_J \le \sum_{I=1}^{J-1} X_I \quad J = 3, 4, \ldots$ (REGULAR SEQUENCE)
AND
$\forall J < K \le N, \ \exists \ A \subsetneq \{1, 2, \ldots, N\}$ S.T. $J \notin A$ AND
$X_K - X_J = \sum_{I \in A} X_I$.

EXAMPLE: 1, 1, 2, 4

4 − 2 = 1 + 1
4 − 1 = 1 + 2
2 − 1 = 1

EXAMPLE: 1, 1, 2, 3, 5, 8 $(= F_1, F_2, F_3, F_4, F_5, F_6)$

8 − 5 = 3, 8 − 3 = 5, 8 − 2 = 1 + 5, 8 − 1 = 2 + 5, 5 − 3 = 2, ETC.

IT IS EASY TO SEE THAT EVERY INITIAL SUBSEQUENCE OF THE FIBONACCI SEQUENCE IS VAN LIER.

THE FOLLOWING RESULT IS QUITE A BIT HARDER:

<u>THEOREM (FISHBURN, ROBERTS, & MARCUS-ROBERTS 1990)</u>: $\mathcal{F}_N \subsetneq \mathcal{V}_N$, I.E., EVERY SUB-FIBONACCI SEQUENCE IS VAN LIER.

NOT EVERY REGULAR SEQUENCE IS VAN LIER. THE SMALLEST REGULAR SEQUENCE WHICH IS NOT VAN LIER IS 1, 1, 2, 4, 5.
5 − 2 IS NOT A SUM OF TERMS FROM {1, 1, 4}.

$\mathcal{V}_N$ = THE COLLECTION OF VAN LIER SEQUENCES OF LENGTH N.

<u>OPEN QUESTION</u>: FIND AN ASYMPTOTIC FORMULA FOR $|\mathcal{V}_N|$.

<u>CONJECTURE (FISHBURN AND ROBERTS 1989)</u>: $|\mathcal{V}_N| / |\mathcal{R}_N| \longrightarrow 0$ AS $N \longrightarrow \infty$.

WE HAVE SHOWN THAT $|\mathcal{V}_N| / |\mathcal{R}_N| \le \lambda$ FOR SOME CONSTANT $\lambda < 1$ ($\lambda = 0.9$ SUFFICES).

<u>SOME COUNTS</u>

N	2	3	4	5	6	7	8
$\lvert\mathcal{V}_N\rvert$	1	2	6	26	164	1529	21439

<u>ASIDE: DISTINCTION BETWEEN REGULAR AND VAN LIER SEQUENCES</u>

A BASIC STRUCTURAL PROPERTY THAT SEPARATES VAN LIER SEQUENCES FROM REGULAR SEQUENCES IS CALLED A GAP.

SUPPOSE $X_1, X_2, \ldots, X_N$ IS A REGULAR SEQUENCE. WE SAY THAT IT HAS A <u>GAP</u> AT X_J IF

$$X_{J+1} > \sum_{I=1}^{J-1} X_I + 1.$$

FOR INSTANCE, CONSIDER 1, 1, 2, 3, 6.

THERE IS A GAP AT $X_4 = 3$ BECAUSE 6 > (1 + 1 + 2) + 1.

<u>THEOREM (FISHBURN, ROBERTS, AND MARCUS-ROBERTS 1990)</u>: EVERY REGULAR SEQUENCE WITHOUT GAPS IS A VAN LIER SEQUENCE.

THEOREM (FISHBURN, ROBERTS, AND MARCUS-ROBERTS 1990): SUPPOSE $X_1, X_2, \ldots, X_N$ IS A VAN LIER SEQUENCE. THEN

(1) EVERY ONE-TERM EXTENSION OF $X_1, X_2, \ldots, X_N$ TO A REGULAR SEQUENCE IS VAN LIER IF AND ONLY IF $X_1, X_2, \ldots, X_N$ HAS NO GAPS.

(2) IF $X_1, X_2, \ldots, X_N$ HAS A GAP AT X_J AND

$$Y = X_J + T + \sum_{I=J+2}^{N} X_I$$

WITH

$$\sum_{I=1}^{J-1} X_I < T < X_{J+1},$$

THEN $X_1, X_2, \ldots, X_N, Y$ IS A ONE-TERM REGULAR EXTENSION WHICH IS NOT VAN LIER.

EXAMPLE: 1, 1, 2, 3, 6 IS VAN LIER AND THERE IS A GAP AT $X_J = 3$. THEN $\sum_{I=J+2}^{N} X_I = 0$. TAKE $T = 5$ AND $Y = X_J + T + \sum_{I=J+2}^{N} X_I = 3 + 5 + 0 = 8$. THE SEQUENCE 1, 1, 2, 3, 6, 8 IS REGULAR BUT NOT VAN LIER. (NOTE THAT $8 - 3$ IS NOT A SUM OF OTHER TERMS.)

ASIDE: CONNECTION TO MEASUREMENT THEORY

ONE OF THE MOST INTERESTING PROBLEMS IN THE THEORY OF MEASUREMENT CONCERNS SUBJECTIVE JUDGEMENTS ABOUT PROBABILITIES. LET $\mathcal{A}_N$ BE THE SET OF ELEMENTS OF THE FINITE BOOLEAN ALGEBRA CONSISTING OF ALL SUBSETS OF $\{1, 2, \ldots, N\}$. SET $\{I\}$ IS CALLED AN ATOM OF $\mathcal{A}_N$.

LET $>$ BE A BINARY RELATION ON $\mathcal{A}_N$, WITH $A > B$ INTERPRETED TO MEAN THAT A IS JUDGED SUBJECTIVELY MORE PROBABLE THAN B. WE SAY A (FINITELY ADDITIVE) PROBABILITY MEASURE P ON $\mathcal{A}_N$ AGREES WITH $>$ IF

$$A > B \text{ IFF } P(A) > P(B)$$

FOR ALL A, B IN $\mathcal{A}_N$. IT IS A VERY OLD QUESTION OF MEASUREMENT THEORY TO UNDERSTAND CONDITIONS ON THE BINARY RELATION $(\mathcal{A}_N, >)$ UNDER WHICH IT AGREES WITH SOME PROBABILITY MEASURE. THE MEASURE IS SAID TO AGREE UNIQUELY IF IT IS THE ONLY AGREEING PROBABILITY MEASURE.

NOTATION: $A \sim B$ MEANS THAT NOT $A > B$ AND NOT $B > A$, I.E., A AND B ARE JUDGED SUBJECTIVELY EQUALLY LIKELY.

EXAMPLE 1: SUPPOSE $N = 4$ AND $>$ IS DEFINED BY THE FOLLOWING:

$$\{3\} \sim \{1,2\}$$
$$\{4\} \sim \{1,3\}$$
$$\{2,3\} \sim \{1,4\}$$

AN AGREEING PROBABILITY MEASURE IS GIVEN BY

(*) $P(\{1\}) = 1/10,\ P(\{2\}) = 2/10,\ P(\{3\}) = 3/10,\ P(\{4\}) = 4/10,$

WITH THE REST OF P DEFINED BY FINITE ADDITIVITY. THIS UNIQUELY AGREES BECAUSE IT IS THE UNIQUE SOLUTION TO THE SYSTEM OF EQUATIONS

$$P(\{3\}) = P(\{1\}) + P(\{2\})$$
$$P(\{4\}) = P(\{1\}) + P(\{3\})$$
$$P(\{2\}) + P(\{3\}) = P(\{1\}) + P(\{4\})$$

EXAMPLE 2: SUPPOSE $N = 3$ AND P IS DEFINED BY

$$\{3\} \sim \{1,2\}.$$

THEN ONE AGREEING PROBABILITY MEASURE IS GIVEN BY

$$P(\{1\}) = 1/6,\ P(\{2\}) = 2/6,\ P(\{3\}) = 3/6.$$

BUT THIS IS NOT UNIQUE, SINCE A SECOND AGREEING PROBABILITY MEASURE IS GIVEN BY

$$P(\{1\}) = 2/10,\ P(\{2\}) = 3/10,\ P(\{3\}) = 5/10.$$

EXAMPLE 3. SUPPOSE $N = 2$ AND P IS DEFINED BY

$$\{1\} > \{2\}.$$

THEN THERE ARE INFINITELY MANY AGREEING PROBABILITY MEASURES P, WITH

$$P(\{1\}) = \alpha,\ P(\{2\}) = 1-\alpha,$$

FOR ANY α WITH $1 > \alpha > 1/2$.

LET US TAKE THE UNIQUE SOLUTION (*) IN THE FIRST EXAMPLE. WE CAN TRANSLATE THIS INTO A SEQUENCE OF POSITIVE INTEGERS 1, 2, 3, 4 (WITH NO COMMON DIVISOR) BY MULTIPLYING BY THE DENOMINATOR 10. CONVERSELY, ANY FINITE SEQUENCE OF POSITIVE INTEGERS CAN BE THOUGHT OF AS A SEQUENCE OF PROBABILITIES BY NORMALIZING, I.E., BY DIVIDING EACH ELEMENT BY THE SUM OF ELEMENTS IN THE SEQUENCE. LET US CALL A NONDECREASING SEQUENCE OF POSITIVE INTEGERS WITH NO COMMON DIVISOR WHICH ARISES FROM A BINARY RELATION > BY FINDING A UNIQUELY AGREEING PROBABILITY MEASURE A <u>UNIQUE PROBABILITY SEQUENCE</u>. THUS, 1, 2, 3, 4 IS A UNIQUE PROBABILITY SEQUENCE WHILE 1, 2, 3 IS NOT.

FISHBURN AND ODLYZKO [1989] PROVE THAT ALL REGULAR SEQUENCES ARE UNIQUE PROBABILITY SEQUENCES. HOWEVER, NOT ALL UNIQUE PROBABILITY SEQUENCES ARE REGULAR. FOR INSTANCE, 1, 2, 3, 4 IS NOT REGULAR: WE DO NOT HAVE $X_2 = 1$.

<u>THEOREM (FISHBURN AND ODLYZKO 1989)</u>: A NONDECREASING SEQUENCE $X_1, X_2, \ldots, X_N$ OF POSITIVE INTEGERS WITH NO COMMON DIVISOR IS A UNIQUE PROBABILITY SEQUENCE IF AND ONLY IF IT IS THE SOLUTION TO N-1 LINEARLY INDEPENDENT EQUATIONS OF THE FORM

$$\sum_{I \in S} X_I = \sum_{J \in T} X_J,$$

WHERE $S, T \subseteq \{1, 2, \ldots N\}$ AND $S \cap T = \phi$.

FOR INSTANCE, THE SEQUENCE 1, 2, 3, 4 IS A UNIQUE PROBABILITY SEQUENCE BECAUSE IT IS THE SOLUTION TO THE 4 - 1 = 3 LINEARLY INDEPENDENT EQUATIONS

$$X_3 = X_1 + X_2$$
$$X_4 = X_1 + X_3$$
$$X_2 + X_3 = X_1 + X_4$$

ALSO, 1, 2, 2, 3 IS A UNIQUE PROBABILITY SEQUENCE SINCE IT IS THE SOLUTION TO THE 4 - 1 LINEARLY INDEPENDENT EQUATIONS

$$X_2 = X_3$$

THIS CORRESPONDS TO THE SUBJECTIVE PROBABILITY CONSTRAINTS

$$P(\{2\}) = P(\{3\})$$
$$P(\{1,2\}) = P(\{4\})$$
$$P(\{1,4\}) = P(\{2,3\})$$

THIS SEQUENCE 1, 2, 2, 3 IS AGAIN IRREGULAR SINCE $X_2 \neq 1$. IT TURNS OUT THAT 1, 2, 3, 4 AND 1, 2, 2, 3 ARE THE ONLY IRREGULAR UNIQUE PROBABILITY SEQUENCES OF LENGTH 4.

HOWEVER, THERE ARE 75 IRREGULAR UNIQUE PROBABILITY SEQUENCES OF LENGTH 5, INCLUDING 1, 1, 3, 3, 5 AND 2, 2, 2, 3, 3. THE FORMER IS INTERESTING. IT IS A UNIQUE PROBABILITY SEQUENCE SINCE IT IS THE SOLUTION TO THE 5 - 1 = 4 LINEARLY INDEPENDENT EQUATIONS

$$X_1 = X_2$$
$$X_3 = X_4$$
$$X_1 + X_2 + X_3 = X_5$$
$$X_3 + X_4 = X_1 + X_5.$$

IT IS NOT REGULAR SINCE 3 IS NOT LESS THAN OR EQUAL TO THE SUM OF THE PREVIOUS TERMS IN THE SEQUENCE, 1 + 1.

LET $\mathcal{U}_N$ BE THE COLLECTION OF UNIQUE PROBABILITY SEQUENCES OF LENGTH N. RECALL THAT $\mathcal{R}_N$ IS THE COLLECTION OF REGULAR SEQUENCES OF LENGTH N.

<u>THEOREM (FISHBURN AND ODLYZKO 1989)</u>: $|\mathcal{R}_N|/|\mathcal{U}_N| \to 0$ AS $N \to \infty$.

WE DO NOT KNOW MUCH ABOUT $|\mathcal{U}_N|$. HOWEVER, WE HAVE THE FOLLOWING UPPER BOUND:

<u>THEOREM (FISHBURN AND ODLYZKO)</u>: $|\mathcal{U}_N| \leq 3^{N^2(1+o(1))}$.

<u>WHERE DO REGULAR AND VAN LIER SEQUENCES ARISE?</u>

THE PROBLEM OF FINDING CONDITIONS UNDER WHICH THERE IS A FINITELY ADDITIVE PROBABILITY MEASURE WHICH AGREES WITH A GIVEN BINARY RELATION "SUBJECTIVELY MORE PROBABLE THAN" $(\mathcal{A}_N, >)$ IS AN OLD PROBLEM. SOME NECESSARY CONDITIONS WERE STATED BY BRUNO DE FINETTI IN 1931. DEFINE $A \geq B$ TO MEAN THAT EITHER $A > B$ OR $A \sim B$.

DE FINETTI AXIOMS

AXIOM A1. $\geq$ IS TRANSITIVE AND COMPLETE ($A \geq B$ OR $B \geq A$ FOR ALL A, B IN $\mathcal{A}_N$)

AXIOM A2. $\{1, 2, ..., N\} > \phi$

AXIOM A3. $A \geq \phi$

Axiom A4. If $(A \cup B) \cap C = \phi$, THEN

$$A \geq B \text{ IFF } A \cup C \geq B \cup C$$

IT IS EASY TO SEE THAT THESE FOUR AXIOMS ARE NECESSARY FOR THE EXISTENCE OF AN AGREEING PROBABLITY MEASURE. IT WAS SHOWN BY KRAFT, PRATT, AND SEIDENBERG IN 1959 THAT THEY ARE NOT SUFFICIENT.

VARIOUS CONDITIONS CAN BE ADDED TO THESE AXIOMS TO GIVE SUFFICIENT CONDITIONS. ONE SIMPLE CONDITION WAS ADDED BY KRAFT, PRATT, AND SEIDENBERG [1959]. IT SAYS THAT IF $A_1, A_2, ..., A_M, B_1, B_2, ..., B_M$ ARE CHOSEN FROM $\mathcal{A}_N$, IF EVERY ATOM IS INCLUDED IN AS MANY A_J AS B_J, AND IF $A_J \geq B_J$ FOR $J = 1, 2, ..., M-1$, THEN $B_M \geq A_M$.

THE NEXT QUESTION IS: UNDER WHAT CONDITIONS IS THERE A UNIQUELY AGREEING PROBABILITY MEASURE, STARTING WITH THE DE FINETTI AXIOMS? RATHER COMPLICATED CONDITIONS FOR UNIQUE AGREEMENT WERE GIVEN BY LUCE [1967] AND BY ROBERTS [1979]. THE FOLLOWING MUCH SIMPLER AXIOM WAS GIVEN BY FISEBURN AND ROBERTS [1989].

AXIOM U1: SUPPOSE X IS AN ATOM SUCH THAT $X > Y > \phi$ FOR SOME ATOM Y. THEN THERE IS AN EVENT $A(X)$ IN $\mathcal{A}_N$ SO THAT $X \sim A(X)$ AND $X > Y$ FOR EVERY ATOM Y IN $A(X)$.

(PUT ANOTHER WAY, THE CONCLUSION OF THIS AXIOM SAYS THAT THERE ARE ATOMS $Y_1, Y_2, ..., Y_K$ SO THAT $X \sim Y_1 \cup Y_2 \cup ... Y_K$ AND $X > Y_I$, $I = 1, 2, ..., K$.)

THIS IS RELATED TO THE FISHBURN-ODLYZKO CHARACTERIZATION OF REGULAR SEQUENCES AS NONDECREASNG SEQUENCES OF POSITIVE INTEGERS SUCH THAT $X_1 = X_2 = 1$ AND SUCH THAT EACH X_J IS A SUM OF OTHER X_I, $I \neq J$.

THEOREM (FISHBURN AND ROBERTS 1989): GIVEN $(\mathcal{A}_N, >)$, THE DE FINETTI AXIOMS PLUS AXIOM U1 IMPLY THAT THERE IS AN AGREEING PROBABILITY MEASURE AND IT IS UNIQUE.

THEOREM (FISHBURN AND ROBERTS 1989): A NONDECREASING SEQUENCE OF POSITIVE INTEGERS WITH NO COMMON DIVISOR DEFINES A REGULAR SEQUENCE IF AND ONLY IF IT IS A UNIQUE PROBABILITY SEQUENCE AGREEING WITH A BINARY RELATION $(\mathcal{A}_N, >)$ WHICH SATISFIES AXIOM U1.

ANOTHER AXIOM IS DUE TO VAN LIER [1989].

AXIOM U2. FOR EVERY $I, J \in \{1, 2, ..., N\}$, IF $\{I\} > \{J\}$, THERE IS $C \in \mathcal{A}_N$ SUCH THAT $\{I\} \sim \{J\} \cup C$.

THEOREM (VAN LIER 1989): GIVEN $(\mathcal{A}_N, >)$, THE DE FINETTI AXIOMS PLUS AXIOM U2 IMPLY THAT THERE IS AN AGREEING PROBABILITY MEASURE AND IT IS UNIQUE.

THEOREM (FISHBURN AND ROBERTS 1989): A REGULAR SEQUENCE IS A VAN LIER SEQUENCE IF AND ONLY IF IT IS A UNIQUE PROBABILITY SEQUENCE AGREEING WITH A BINARY RELATION $(\mathcal{A}_N, >)$ WHICH SATISFIES AXIOM U2.

(TYPE A) TWO-SIDED GENERALIZED FIBONACCI SEQUENCES

MOTIVATION: "DIFFERENCE" MEASUREMENT INTRODUCED BY FISHBURN, MARCUS-ROBERTS, AND ROBERTS (1988) AND FISHBURN, ODLYZKO, AND ROBERTS (1989).

THIS IS A SEQUENCE OF POSITIVE INTEGERS WHICH, IN CONTRAST TO ALL THE TYPES OF SEQUENCES SO FAR, MAY BE DECREASING.

START WITH A PAIR OF ADJACENT 1'S.
CONSTRUCT THE SEQUENCE INSIDE-OUT BY ADDING ONE TERM AT A TIME WHOSE VALUE IS A SUM OF ONE OR MORE CONTIGUOUS TERMS IMMEDIATELY ADJACENT TO THE NEW TERM.

EXAMPLE: 5, 4, 1, 1, 2, 4

THIS IS BUILT UP AS:

1, 1, 2
4, 1, 1, 2
5, 4, 1, 1, 2
5, 4, 1, 1, 2, 4

α_N = THE COLLECTION OF (TYPE A) TWO-SIDED GENERALIZED FIBONACCI SEQUENCES OF LENGTH N.

<u>THEOREM (FISHBURN, MARCUS-ROBERTS, AND ROBERTS 1988 AND FISHBURN, ODLYZKO, AND ROBERTS 1989):</u>

$$|\alpha_N| = N^{N(1+o(1))}.$$

IN FACT, FISHBURN, ODLYZKO AND ROBERTS SHOW THAT

$$\alpha_N \sim \frac{K}{2}\sqrt{\frac{e}{\pi}}\,\frac{e^{2\sqrt{N}}(N-1)!}{N^{1/4}},$$

WHERE

$$K = e^{-1} - \int_0^1 [\exp(\tfrac{1}{1-Y})/(1-Y)]dY = 0.145495...$$

<u>TYPE B TWO-SIDED GENERALIZED FIBONACCI SEQUENCES</u>

MOTIVATION: "DIFFERENCE" MEASUREMENT

SAME AS TYPE A TWO-SIDED GENERALIZED FIBONACCI SEQUENCES WITH THE NEW TERM BEING A SUM OF ONE OR MORE CONTIGUOUS TERMS, BUT NOT NECESSARILY OF TERMS IMMEDIATELY ADJACENT TO THE NEW TERM.

EXAMPLE: 6, 1, 1, 2, 4

THIS IS BUILT UP AS:

1, 1, 2
1, 1, 2, 4
6, 1, 1, 2, 4

THIS IS NOT ATTAINABLE IF WE INSIST THAT EACH NEW TERM IS A SUM OF TERMS IMMEDIATELY ADJACENT TO THE NEW ONE.

β_N = THE COLLECTION OF TYPE B TWO-SIDED GENERALIZED FIBONACCI SEQUENCES OF LENGTH N.

<u>THEOREM (FISHBURN, MARCUS-ROBERTS, AND ROBERTS 1988 AND FISHBURN, ODLYZKO, AND ROBERTS 1989):</u>

$$|\beta_N| = N^{2N(1+o(1))}.$$

<u>TYPE C TWO-SIDED GENERALIZED FIBONACCI SEQUENCES</u>

MOTIVATION: "DIFFERENCE" MEASUREMENT

SAME AS TYPE A TWO-SIDED GENERALIZED FIBONACCI SEQUENCES WITH THE NEW TERM BEING A SUM OF ONE OR MORE PREVIOUS TERMS, BUT NOT NECESSARILY OF CONTIGUOUS TERMS AND NOT NECESSARILY OF TERMS IMMEDIATELY ADJACENT TO THE NEW TERM.

EXAMPLE: 3, 1, 1, 2, 6

THIS IS BUILT UP AS:

1, 1, 2
3, 1, 1, 2
3, 1, 1, 2, 6

IT CANNOT BE BUILT UP BY ADDING CONTIGUOUS TERMS EACH TIME, SINCE 6 CAN ONLY BE OBTAINED AS 3 + 1 + 2.

γ_N = THE COLLECTION OF TYPE C TWO-SIDED GENERALIZED FIBONACCI SEQUENCES OF LENGTH N.

<u>THEOREM (FISHBURN, MARCUS-ROBERTS, & ROBERTS 1988):</u>

$$|\gamma_N| = 2^{(N^2/2)(1+o(1))}$$

THE FOLLOWING COUNTS ARE KNOWN:

N	2	3	4	5	6
$\lvert\alpha_N\rvert$	1	3	14	85	626
$\lvert\beta_N\rvert$	1	3	18	172	2433
$\lvert\gamma_N\rvert$	1	3	18	185	

ALL OTHER VALUES ARE STILL OPEN.

BIREGULAR SEQUENCES

MOTIVATION: "CONJOINT" MEASUREMENT
INTRODUCED BY FISHBURN AND ROBERTS (1988)

THESE ARE TWO-BLOCK SEQUENCES OF POSITIVE INTEGERS
$X_1, X_2, \ldots, X_M$ / $Y_1, Y_2, \ldots, Y_N$

THEY ARE BUILT UP BY STARTING WITH 1 IN EACH BLOCK AND ADDING ONE TERM AT A TIME (TO EITHER BLOCK) THAT IS ADJACENT TO THE TERMS ALREADY SPECIFIED FOR THE BLOCK AND WHOSE VALUE EQUALS A SUM OF TERMS ALREADY SPECIFIED FOR THE OTHER BLOCK.

EXAMPLE: 2, 3, 1, 1, 7 / 6, 1, 2, 10

BUILT UP AS:

1 / 1
1, 1 / 1
1, 1 / 1, 2
3, 1, 1 / 1, 2
2, 3, 1, 1 / 1, 2
2, 3, 1, 1 / 6, 1, 2
2, 3, 1, 1, 7 / 6, 1, 2
2, 3, 1, 1, 7 / 6, 1, 2, 10

$G_B(M,N)$ = THE COLLECTION OF BIREGULAR SEQUENCES OF LENGTHS M AND N.

THEOREM (FISHBURN AND ROBERTS 1988):

$$|G_B(M,N)| = (2^N - 1)^{M(1+o(1))}$$

INTERVAL-RESTRICTED BIREGULAR SEQUENCES

MOTIVATION: "CONJOINT" MEASUREMENT

SAME AS BIREGULAR SEQUENCES BUT ADD THE REQUIREMENT THAT EACH TERM IS A SUM OF CONTIGUOUS TERMS ALEADY SPECIFIED IN THE OTHER BLOCK.

NOTE THAT THE LAST EXAMPLE WORKS UP UNTIL THE LAST STEP. IN THE LAST STEP, 10 CANNOT BE ADDED. HOWEVER, 14 COULD BE, GIVING US

2, 3, 1, 1, 7 / 6, 1, 2, 14

$C_B(M,N)$ = THE COLLECTION OF INTERVAL-RESTRICTED BIREGULAR SEQUENCES OF LENGTHS M AND N.

THEOREM (FISHBURN AND ROBERTS 1988):

$$|C_B(M,N)| = \binom{N+1}{2}^{M(1+o(1))}.$$

NOTE THAT $|G_B(M,1)| = |C_B(M,1)| = 1$ FOR ALL M.

ONE CAN ALSO SEE THAT $|G_B(M,2)| = |C_B(M,2)|$ FOR ALL M. THE FOLLOWING VALUES ARE KNOWN:

M	2	3	4	5	6	7	8
$\lvert G_B(M,2)\rvert$	5	19	69	243	841	2859	9573

$|G_B(9,2)|$ IS ALREADY NOT KNOWN.

"Generating k-element Subsets of an n-element Set"

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Generating k-element subsets of an n-element set with DeBruijn Graphs

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Abstract

1

This tour is brought to you

by

the following problem:

Find an efficient* way to

generate all k-element

subsets of an n-element set.!!

*What does efficient mean?

2

Generate all subsets of an n-element set.

Binary representation of $0 - (2^n - 1)$

yields a "computer understandable"

way to generate these sets.

BUT

An *excessive* amount of work

for the computer to go from

$2^{n-1} - 1 = 011...1$ to $2^{n-1} = 100...0$

Is there a sequence which proceeds from subset to subset without many elements being exchanged?

3

Proceed thru the subsets with exactly one

"bit" changing, either 0 to 1, or 1 to 0.

One element difference from subset to subset.

GRAY CODES

0	0	0	0
0	0	0	1
0	0	1	1
0	0	1	0
0	1	1	0
0	1	0	0
1	1	0	0
1	0	0	0
1	0	0	1
1	0	1	1
1	0	1	0
1	1	1	0
1	1	1	1
1	1	0	1
0	1	0	1
0	1	1	1

4

Utilize the power of the computer!!

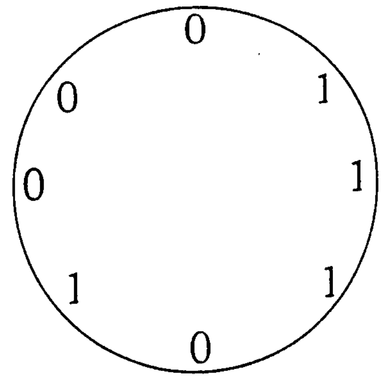
Can we generate all binary sequences

of length n by a "shift" ??

0 0 1 0 1 1 1 0 1
0 1 0 1 1 1 0 1 0
1 0 1 1 1 0 1 0 1
0 1 1 1 0 1 0 1 0
1 1 1 0 1 0 1 0 1
1 1 0 1 0 1 0 1 1
1 0 1 0 1 0 1 1 1
. . .

Does there exist a sequence of length 2^n so that each sequence of length n occurs as a consecutive subsequence exactly once (wrapping allowed) ??

5



6

Efficient way to store all subsets of an n element set.

Efficient way to generate all subsets.

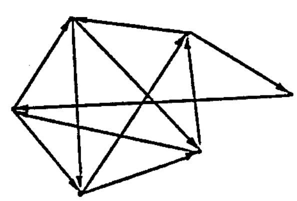
Efficient way to generate all subsets many times.

Efficient way to generate a random subset.

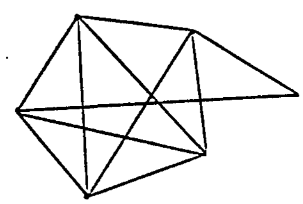
7

Combinatorial Tools

Directed Graph



Graph



Connected

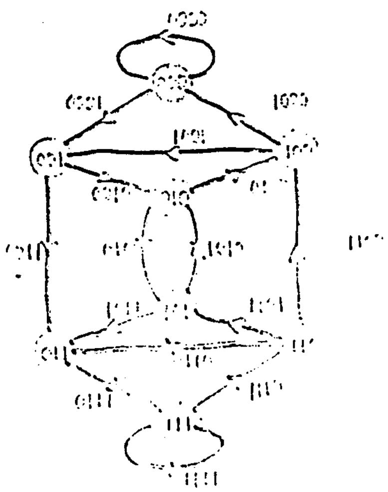
Eulerian Digraph (Graph)

Starting at any point, trace thru the digraph (graph) traversing each edge exactly once.

Euler (1736) Good (1946) If D is a connected digraph with $id(x) = od(x)$ for all vertices x then D is Eulerian.

8

Do there exist circular sequences of length 2^n with each n sequence occurring exactly once?



. . . and now a message from the sponsor

"But we only want the k element subsets!!??"

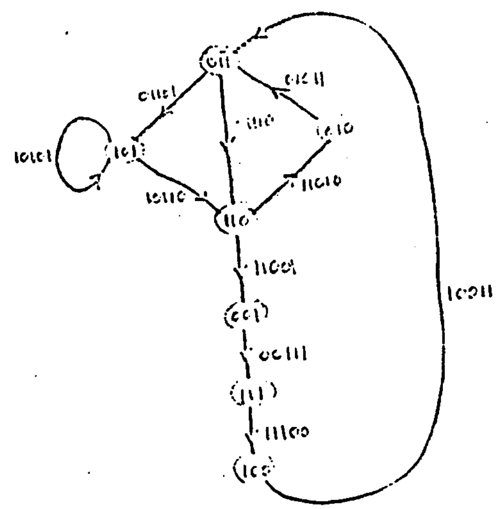
Generate all the subsets, and use only the ones you need.

Can we use shift registers ??

Double Shift Registers

1 1 0 1 0 0 1 0 1 0 0 0
0 1 0 0 1 0 1 0 0 0 1 1
0 0 1 0 1 0 0 0 1 1 0 1
1 0 1 0 0 0 1 1 0 1 0 0
1 0 0 0 1 1 0 1 0 0 0 1
. . . ?

Does there exist a circular sequence of order $2\binom{n}{k}$ so that by double shifting all k element subsets are generated ?



For each vertex x in this digraph, either

id(x) = od(x) = 2 (# 1's is k-1) or

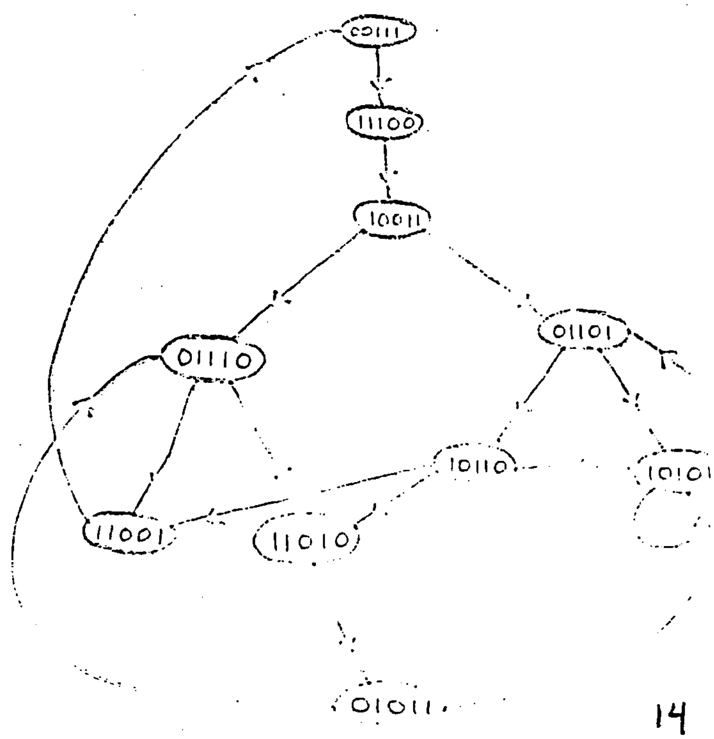
id(x) = od(x) = 1 (#1's is k or k-2).

Euler (1736) Good (1946) If D is a connected digraph with id(x) = od(x) for all vertices x then D is Eulerian.

Good's Thm says Eulerian provided it is connected ...

13

Consider a different graph ...



14

Hamiltonian cycle ⟷ Eulerian circuit

↑

Connected ⟷ Connected

n odd

$a_1 a_2 a_3$ . . . $a_i a_{i+1}$ . . . a_n
$a_3 a_4 a_5$ . . . a_{i+3} . . . a_2

. . .

a_i a_{i+1} . . . $a_n a_1$ . . . a_{i-1}
$a_{i+1} a_{i+2}$ . . . $a_{i-1} a_{i+1} a_i$

. . .

$a_1 a_2 a_3$ . . . $a_{i+1} a_i$ . . . a_n

15

Hence, for n odd, both graphs are connected and the graphs are Eulerian and Hamiltonian, respectively.

n even

$\underline{1}\,\underline{1}$ 0 0 0 0 0 0 0 0
$\underline{0}\,\underline{0}$ 0 0 0 0 0 0 1 1
$\underline{0}\,\underline{0}$ 0 0 0 0 1 1 0 0
$\underline{0}\,\underline{0}$ 0 0 1 1 0 0 0 0
$\underline{0}\,\underline{0}$ 1 1 0 0 0 0 0 0
$\underline{1}\,\underline{1}$ 0 0 0 0 0 0 0 0

16

Cycle thru from component to component

Count the components:

Polya's Thm (Burnside) Let G be a group of permutations acting on A, and let S be the equivalence relation on A induced by G

$$\#\text{E.C.} = \frac{1}{|G|}\sum_{\pi \in G} \text{Inv}(\pi)$$

$G = Z_{\frac{n}{2}}$

A = "<u>family</u>" of k element subsets

with (10 = 01),

Equivalence Classes = Components of G.

17

<u>Example</u>

3-element subsets of an 8-element set.

$a_1 a_2 \quad a_3 a_4 \quad a_5 a_6 \quad a_7 a_8$

One pair = 11 and one pair = 01

or

Three pairs = 01

$|A| = (12 + 4) = 16.$

$\#\text{ Components} = \frac{1}{4}(16 + 0 + 0 + 0) = \underline{4}$

18

A final message from our sponsor . . .

" We <u>need</u> the k - element subsets of an n - element set !! "

When n is odd, find the cycle, and generate the sets...

When n is even, find the k element subsets of an n+1 element set, throw out the subsets with n+1.

19

What's <u>really</u> the story ??

Good's result is an existence Theorem.

How do you find the Eulerian Circuit ??

20

For DeBruijn Sequences ...

Fredrickson has given a "linear" (in n) algorithm to generate the sequence.

For these generalized DeBruijn Sequences,

worst case $o(n^k)$

average case o(logn)

Best Possible ??

UPDATE!!

Hochberg, Hurlbert and Isaak have also discovered the idea of multiple shifting.

Carla Savage uses this idea to generate "new" Grey Codes...

The idea "works" to generate the n! permutations of an n set.

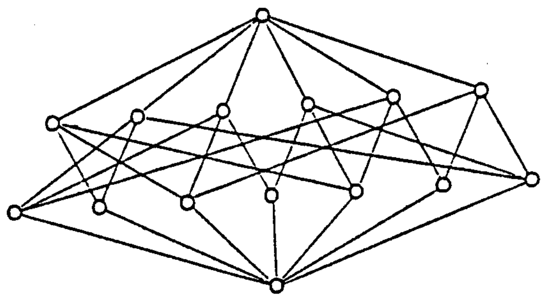
"Matchings in the Partition Lattice"

E. Rodney Canfield, University of Georgia
Department of Computer Science, Athens, GA

Matchings in the Partition Lattice

Rod Canfield

Univ. of Georgia



$[n] = \{1, 2, \ldots, n\}$

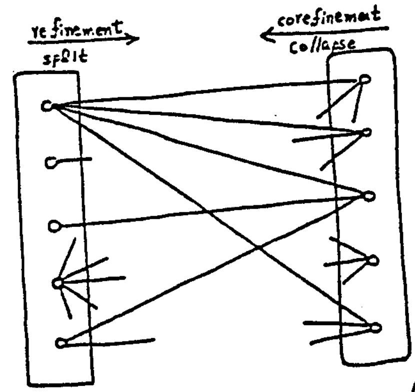
partition $\quad \pi = \{B\} \qquad \Pi_n$

$B_i \cap B_j = \emptyset \qquad \cup B_j = [n]$

$\pi_1 \leq \pi_2$

$\{\{1,5\},\{2\},\{3,8\},\{4,7\},\{6\}\} \leq \{\{1,4,5,7\},\{2\},\{3,6,8\}\}$

$E \subseteq \Pi_{n,k} \times \Pi_{n,k+1}$



$d_E(x) = \sum_{B \in x} (2^{|B|-1} - 1) \qquad d_E(y) = \binom{k+1}{2}$

ϕ — 1 — 1,2 — 1,2,3
2 — 2,3
3 — 1,3

ϕ — 1 — 1,2 — 1,2,3 — 1,2,3,4
2 — 2,3 — 2,3,4
3 — 1,3 — 1,3,4
4 — 1,4 — 1,2,4
2,4
3,4

Chain decompositions in the Boolean lattice B_n, $n = 3, 4$, based on the recursion

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

$$|\Pi_{n,k}| = S(n,k)$$

$$S(n+1,k) = k\,S(n,k) + S(n,k-1)$$

$\{\{1\},\{2,3\}\}$

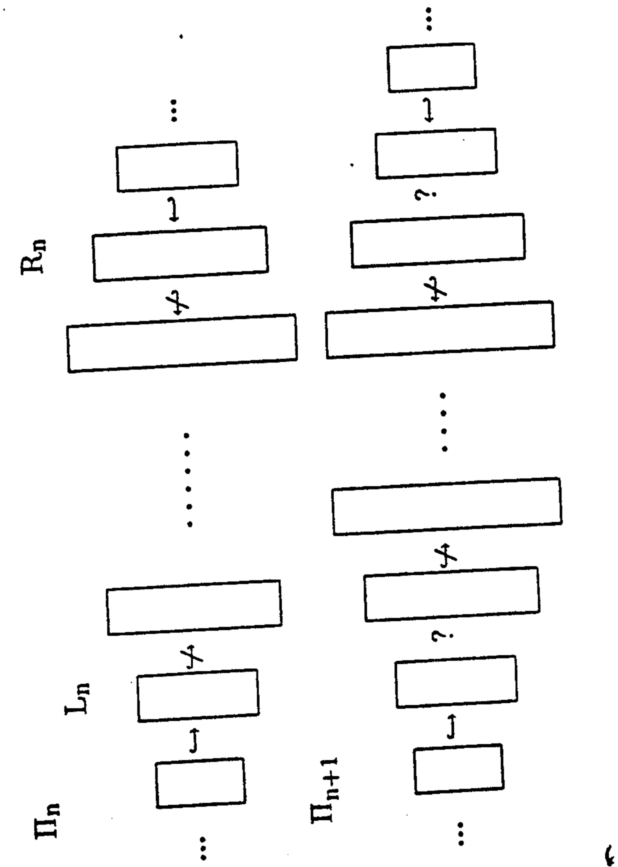
$\{\{1,4\},\{2,3\}\} \quad \{\{1\},\{2,3,4\}\} \quad \{\{1\},\{2,3\},\{\ldots$

log concave : $S(n,k)^2 \geq S(n,k-1)\,S(n,k+1)$

$\therefore$ unimodal

1
1 1
1 3 1
1 7 6 1
1 15 25 10 1
1 31 90 65 15 1
1 63 301 350 140 21 1
1 127 966 1701 1050 266 28 1
1 255 3025 7770 6951 2646 462 36 1
1 511 9330 34105 42525 22827 5880 750 45 1

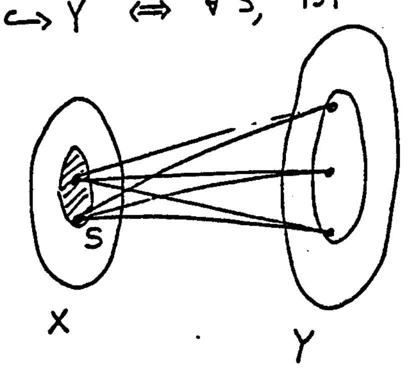
Stirling numbers of the second kind



Construct chains using partition recursion. Claim $\exists$ sharp threshold L_n s.t. $\Pi_{n,k} \hookrightarrow \Pi_{n,k+1} \iff k < L_n$.

First, recall Phillip Hall Criteria

$X \hookrightarrow Y \iff \forall S, \ |S| \le d_E(S)$



$E \subseteq X \times Y$

LMPH $d_E(x) \ge d_E(y) \Rightarrow X \hookrightarrow Y$.

Proof that $L_{n+1} = L_n + \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

$S \subseteq \Pi_{n+1,k} \qquad k < L_n$

$$|S| = \sum_{j=0}^{k} |S_j|,$$

$$S_0 \subseteq \Pi_{n,k-1}$$

$$S_j \subseteq \Pi_{n,k}$$

$$d_E(S) \ge \sum_{j=0}^{k} d_E(S_j)$$

$$\ge \sum_{j=0}^{k} |S_j|$$

$$= |S|.$$

That's half of the proof.

Claim $\exists R_n$ s.t. $\Pi_{n,k} \hookrightarrow \Pi_{n,k-1}$ $\Leftrightarrow k > R_n$.

Proof Let $A \subseteq \Pi_{n+1,k} \cup \Pi_{n+1,k-1}$ be an antichain, w/ $k > R_n + 1$. Want to show $|A| \le S(n, k-1)$.

For each B, $\emptyset \subseteq B \subseteq [n]$, let A_B be those partitions π of $[n]-B$ s.t. $\pi \cup \{B \cup \{n+1\}\} \in A$.

$$A_B \subseteq \Pi_{n-|B|, k-1} \cup \Pi_{n-|B|, k-2}$$

and A_B is an antichain.

Hence, $|A_\emptyset| \le S(n, k-2)$, and

$$|A| = \sum_B |A_B|$$

$$\le S(n,k-2) + \sum_{\emptyset \ne B \subseteq [n]} S(n-|B|, k-2)$$

$$= S(n, k-2) + (k-1)\, S(n, k-1)$$

$$= S(n+1, k-1) \quad \blacksquare$$

Theorem There exist monotone increasing sequences L_n and R_n such that

$\Pi_{n,k} \hookrightarrow \Pi_{n,k+1} \Leftrightarrow k < L_n$

$\Pi_{n,k} \hookrightarrow \Pi_{n,k-1} \Leftrightarrow k > R_n$.

As n increases by 1, each of L_n, R_n grows by at most 1.

Next, want to give bounds for L_n and R_n. What can we find using only LMPH?

Fact 1 For $k \le n \log 2 / \log n$ and $n \ge 5$, $\Pi_{n,k} \hookrightarrow \Pi_{n,k+1}$.

Hence, $L_n > n \log 2 / \log n$, $n \ge 5$.

Proof
$$d_E(x) = \sum_{B \in x} (2^{|B|-1} - 1)$$
$$\ge k\,(2^{\frac{n}{k}-1} - 1)$$
$$\ge k\,(\tfrac{n}{2} - 1)$$
$$\ge \tfrac{1}{2} k\,(n \log 2/\log n + 1), \quad n \ge 6$$
$$\ge \binom{k+1}{2}$$
$$= d_E(y)$$

Fact 2 Fix $\delta > 0$. If n is suff'ly large and $k \ge (1+\delta) n \log 4 / \log n$, then $\Pi_{n,k} \hookrightarrow \Pi_{n,k-1}$. Hence,

$$R_n < (1+\delta)\, n \log 4 / \log n.$$

Proof. Curiously, we throw away edges. Take $\delta \le \frac{1}{2}$; define

$$b = \left\lfloor (1 - \tfrac{\delta}{2}) \frac{\log n}{\log 4} \right\rfloor.$$

Let $E = (y, x) \in \Pi_{n,k-1} \times \Pi_{n,k}$ s.t. x is obtained from y by splitting a block of size $\le 2b$.

Say $x \in \Pi_{n,k}$; since $k\,(b+1) \ge (1+\frac{\delta}{4})\, n$, at least $\left(1 - (1+\frac{\delta}{4})^{-1}\right) k \ge \frac{2\delta}{9} k$ blocks of

x are of size $\le b$. If $\delta k \ge 45$,

$$d_E(x) \ge \frac{1}{45}\, \delta^2 k^2 .$$

On the other hand,

$$d_E(y) \le k\, 2^{2b}$$
$$\le k\, n^{1-\frac{\delta}{2}} .$$

Hence, if

$$\frac{\delta^2}{45}(1+\delta)\log 4 / \log n \ \ge\ n^{-\delta/2},$$

$$d_E(x) \ge d_E(y). \qquad \blacksquare$$

How far from the truth might these two bounds be ?

<u>Thm</u>

$$\frac{L_n}{n/\log n} \longrightarrow \log 2$$

$$\frac{R_n}{n/\log n} \longrightarrow \log 4$$

<u>Further research</u>

1. For what n do we have
$$\big|\{L_n, K_n, R_n\}\big| \ge 2$$
(Let n_0 = smallest)

2. $\max\{|A| : A \subseteq \Pi_n \text{, antichain}\} \sim$?

3. other

g, h polynomials

$$Z = X_1 + \dots + X_\ell + Y_1 + \dots + Y_{k-\ell} \qquad \text{random var.}$$

$$P\{X = j\} = \frac{c_j n^j}{g(n)} \qquad \mu_g = E(X) \qquad \sigma_g^2 = E(X^2) - E(X)^2$$

If $\ell \mu_g + (k-\ell)\mu_h = n$

then
$$[x^n]\, g(x)^\ell h(x)^{k-\ell} = \frac{g(n)^\ell h(n)^{k-\ell}}{n^n}\, P\{Z = n\}$$
$$\approx \frac{g(n)^\ell h(n)^{k-\ell}}{n^n}\, \frac{1}{\sigma\sqrt{2\pi}}$$

by <u>CLT</u>, with $\sigma^2 = \ell\sigma_g^2 + (k-\ell)\sigma_h^2$.

$A \subseteq \Pi_{n,k}$: ℓ blocks of size $1..m$, and $(k-\ell)$ of size $m+1..2m$

$$|A| = \frac{n!}{\ell!\,(k-\ell)!}\,[x^n]\, g(x)^\ell h(x)^{k-\ell}$$

w/ $g(x) = \sum_1^m x^j / j!$

refine $(A) \subseteq C_1 \cup C_2$

- C_2: $\ell+2,\ k-\ell-1$
- C_1: $\ell+1,\ k-1$

Given sequence (n, k) w/ $n \to \infty$,

$k \sim \beta\, n/\log n$, $\beta > \log 2$:

exhibit n, ℓ, m s.t.

(1) estimation procedure works
(2) same r usable for A, C_1, C_2
(3) all three σ's are $\sim$

Find

$$\frac{|C_1| + |C_2|}{|A|} \sim \frac{g(n)}{\ell+1} + \frac{g(n)^2}{(\ell+2)(\ell+1)}\,\frac{(k-\ell)}{h(n)}$$

$$\to 0.$$

Say $k \leq (1-\delta) n \log 4 / \log n$

Let $m = \lfloor (1+\delta) \log n / \log 4 \rfloor$.

$S = \{ \pi \in \Pi_{n,k} :$

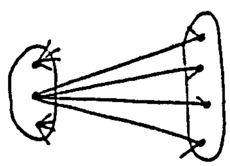
blocksize $(\pi) \subseteq \{m, 2m, 3m, 3m+1, \ldots\}$

$\geq n^{1-\delta/2}$ blocks of size $2m$

$\geq n^{1-\delta/2}$ blocks of size $\geq 3m$ $\}$.

$\frac{1}{2} \binom{2m}{m} n^{1-\delta/2}$ and $\binom{7m}{m}$ are

both $\Omega(n^2)$. Hence,



corefine(S) $\quad S \subseteq \Pi_{n,k}$,

and $\Pi_{n,k} \not\to \Pi_{n,k-1}$.

Knowledge about Π_n

1928 Sperner : The subset lattice is Sperner

1967 Rota : Is Π_n Sperner ?

1967 Harper : Asymptotic normality, $K_n \sim n / \log n$

1968 Lieb : Log concavity

1969 Mullin : K_n is the critical part

1969 Graham, Harper : $\Pi_{12,5} \leftarrow \Pi_{12,6}$ via Phillip Hall quotient reduction

1971 Dilworth, Greene : geometric $\not\Rightarrow$ Sperner (> 60,000 elements)

1971 Kleitman, Edelberg, Lubell : Antichains are symmetric

1974 Harper : Π_n is LYM for $n \leq 19$

1974 Spencer : Π_n is not LYM for $n \geq 20$

1977 Pudlak, Tuma : Every lattice can be embedded in Π_n for some n

1977 Canfield: Π_n is not Sperner, $n_0 \leq$ (?) 6.5×10^{24}

1979 Shearer : Π_n is not Sperner, $n_0 \leq 3.7 \times 10^{9}$

1980 Canfield : $K_n = \ldots$

1980 Shearer : Maximum antichains intersect $\Pi_{n,k}$ for $k \geq (1-\delta) n \log 4 / \log n$

1984 Jiang, Kleitman : Π_n is not Sperner, $n_0 \leq 3.4 \times 10^{6}$

1985 Harper : $a_n \geq$ (?) $(1 - 3\sqrt{3}/5)^{-1/2} S(n, K_n)$

1990 Kung : $\Pi_{n,k} \leftarrow \Pi_{n,k-1}$ for $k > n/2$

1991 Canfield : L_n and R_n

"Algorithms for Small Graphs"

Ronald C. Read, University of Waterloo
Department of Combinatorics and Optimization, Ontario, Canada

Algorithms for small graphs

Ron Read

UNIVERSITY OF WATERLOO

ATLAS OF GRAPHS

Graphs	≤ 7	1252
Trees	≤ 12	987
Line graphs	≤ 8	357
Identity trees	≤ 14	1261
Homeo. irred. trees	≤ 16	568
Unicyclic graphs	≤ 8	253
2-conn. planar	≤ 7	639
3-conn. planar	≤ 8	301
Cubic graphs	≤ 14	621
Quartic "	≤ 11	350
5-reg. "	≤ 10	64
Eulerian graphs	≤ 8	236
Self-comp. "	≤ 9	49
Outerplanar "	≤ 9	372

Digraphs ≤ 4

Tournaments ≤ 7 5

Eulerian digraphs ≤ 5

2-reg. digraphs ≤ 7 1

Acyclic digraphs ≤ 5

Self-comp. digraphs ≤ 5

Signed graphs ≤ 5

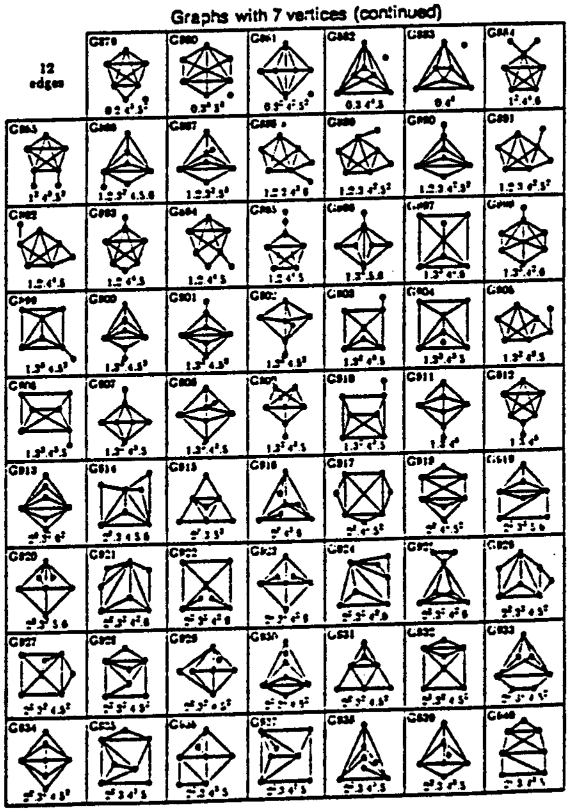
SPECIAL GRAPHS

Platonic
Archimedean
Snarks
Symmetric graphs
Counter-examples
etc.

- 17 -

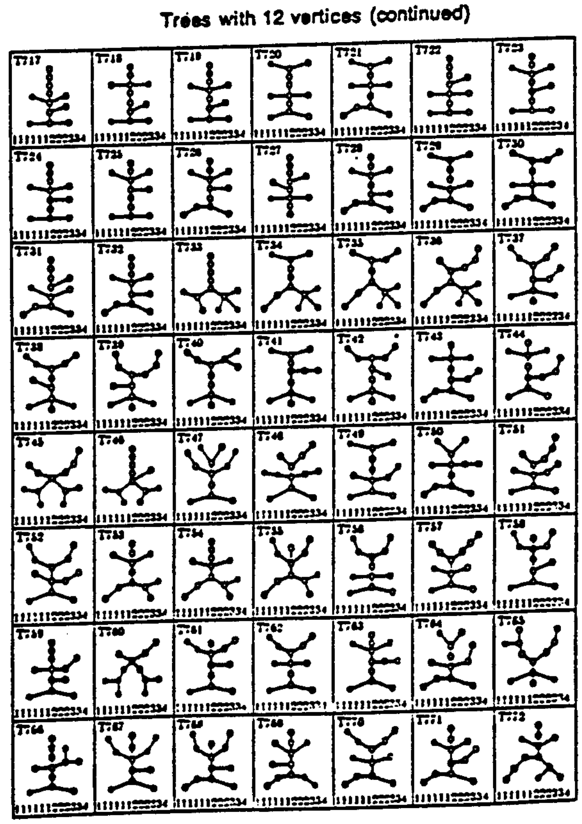
2

Graphs with 7 vertices (continued)

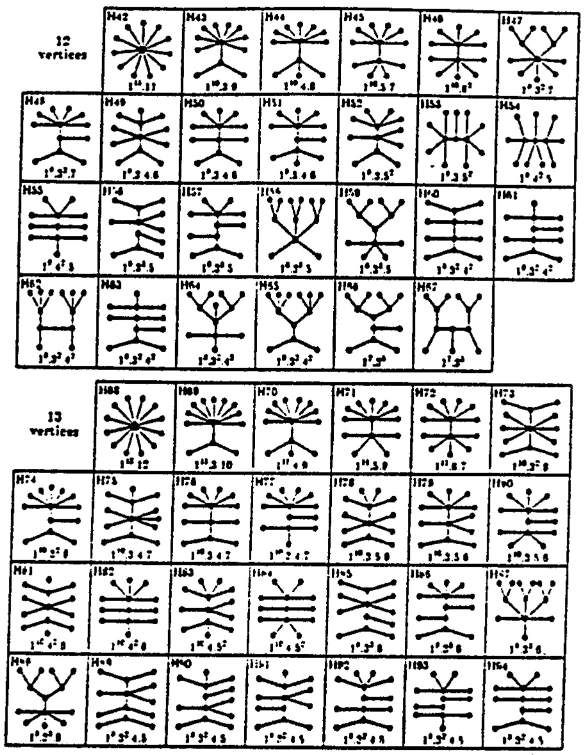


3

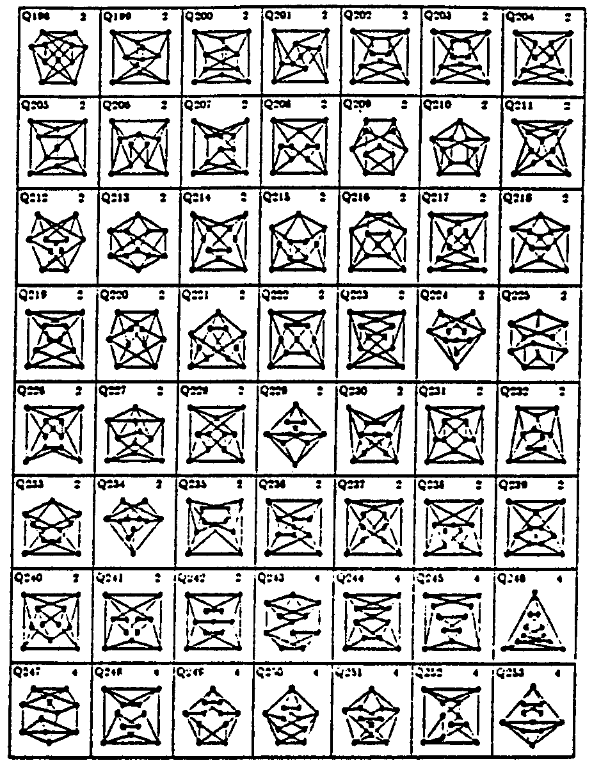
Trees with 12 vertices (continued)



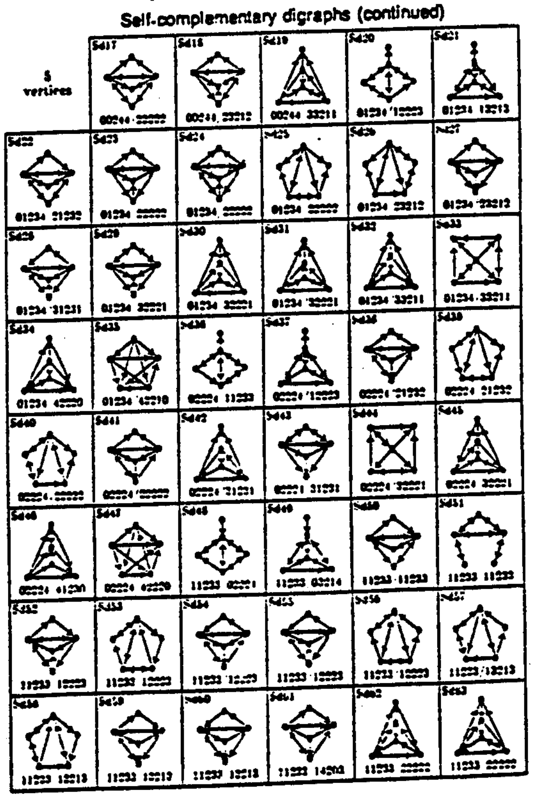
Homeomorphically irreducible trees (continued)



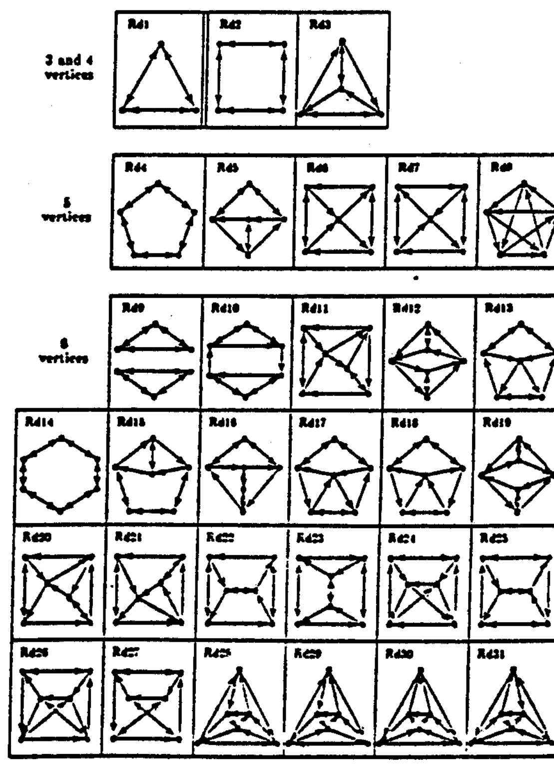
4-regular graphs with 11 vertices (continued)



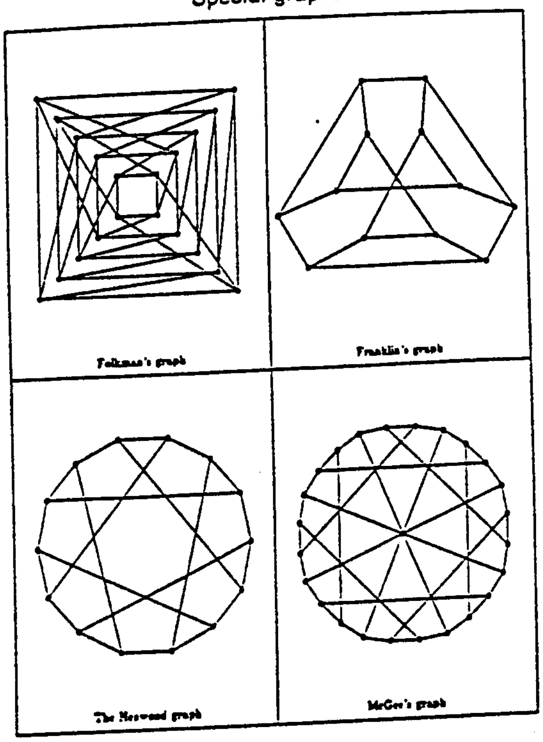
Self-complementary digraphs (continued)



2-regular digraphs



Special graphs. 6



Parameters and Properties.

p = no. of vertices $\quad q$ = no. of edges.
k = no. of components
Degree sequence
g = girth (Girth number)
Circumference
Diameter
κ = vertex connectivity
λ = edge connectivity
Order of automorphism group
Eulerian ?
Hamiltonian ?
Planar ?
Bipartite ?
Tree ?
Chromatic number χ
Edge chromatic number χ'
Chromatic polynomial
Characteristic polynomial
Spectrum

Characteristic polynomial

Adjacency matrix A

$$\phi(x) = |xI - A|.$$

Newton's method.

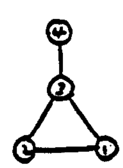
Let q_j = trace A^j

and $\phi(x) = \sum_{k=0}^{n} p_k x^k$.

Then $kp_k = -\sum_{j=0}^{k-1} p_j q_{k-j}$

Example:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$



$q_1 = 0, \; q_2 = 8, \; q_3 = 6, \; q_4 = 28$

$p_0 = 1, \; p_1 = 0, \; p_2 = -4$
$p_3 = -2, \; p_4 = 1$

$$\phi(x) = x^4 - 4x^2 - 2x + 1$$

SPECTRUM

= set of eigenvalues of the adjacency matrix A.

= set of roots of $\phi(x) = 0$

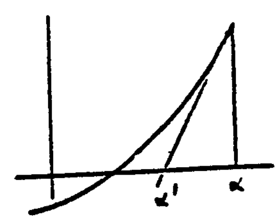
Roots lie between $-(n-1)$ and $n-1$.
All roots are real.

Find integer roots first. Then use Newton's formula, starting with the largest. As each root is found, take out the linear factor.

Newton's formula

If α is an approximate root, then

$$\alpha' = \alpha - \frac{\phi(\alpha)}{\phi'(\alpha)}$$



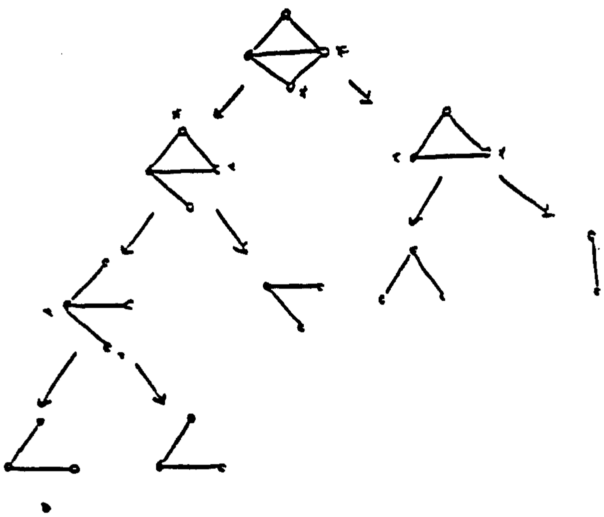
is a better one (under appropriate conditions)

Chromatic Polynomial

$P_G(\lambda)$ = number of ways of coloring G with λ colors available.

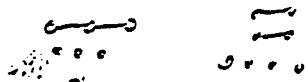
Chromatic reduction :

$P_G(\lambda) = P_{G'}(\lambda) - P_{G''}(\lambda)$



Stop when only 2 edges are left.

This works even if G is not connected.



Properties.

If $P_G(2) > 0$, G is bipartite.

The lowest power of λ with non-zero coefficient is k — the number of components.

If g is the girth of G, the coefficien (in absolute magnitude) are

$1, q, \binom{q}{2}, \binom{q}{3}, \ldots, \binom{q}{g-2}$

and the next coefficient is less than $\binom{q}{g-1}$ by the number of cycles of length g.

If k = 1 and p − q = 1, G is a tree.

$\chi(G)$ is the least integer λ such that $P_G(\lambda) > 0$.

Edge-chromatic number $\chi'(G)$

VIZING'S THEOREM : $\chi'(G) = \Delta$ or $\Delta + 1$.

METHOD Try to color the edges of G in Δ colors. If successful, $\chi'(G) = \Delta$. Otherwise $\chi'(G) = \Delta + 1$.

Start by allocating colors to the edges at a vertex of degree Δ.

Thereafter: For every edge not yet colored there will be a set of allowable colors.

Choose an edge, give it an allowable color.

Continue, backtracking when an edge cannot be colored.

DIAMETER

Diameter = greatest distance between two vertices.

$$= \max_{u,v} \{ \min_{\text{all } u\text{-}v \text{ paths}} (\text{pathlength}) \}$$

Method Find all (u,v)-distances using an algorithm based on a theorem of Warshall.

```
For k = 1 to p
do: for i = 1 to p and ≠ k
    for j = 1 to p and ≠ k
    d(i,j) ← min(d(i,j), d(i,k)+d(k,j))
```

(All d(i,j) initially = ∞)

CIRCUMFERENCE

Circumference = length of a longest cycle

NP-complete (includes the Hamilton cycle problem)

Method: Start at some vertex x. Try to construct a Hamilton cycle, but keep track of the longest cycle found.

If this is p. circumference is p — graph is Hamiltonian

If it is $p-1$ circumference is $p-1$ — graph is not Hamiltonian

Otherwise. Repeat with the graph $G-x$.

<u>Short cuts</u>. Eliminate vertices of degree < 2.
A vertex of degree 2, if included, implies inclusion of its incident edges.

Planarity.

Hopcroft & Tarjan 1974 (linear).
Too elaborate!

Preliminaries

<u>Theorem</u> If $q > 3p-6$ G is non planar.

If
$q-p=-1$ G is a tree
$q-p=0$ G is unicyclic
$q-p=1$ G has two indep. cycles
$q-p=2$ G has 3 " "

— all these must be planar.

The case of $K_{3,3}$ ($p=6, q=9$) shows that $q-p=3$ does not imply planarity.

If $p \le 4$, or $p=5$ and $G \neq K_5$ then G is planar.

The $q>3p-6$ criterion is obscured by the presence of vertices of degree 2.

<u>Method</u> Delete all vertices of degree 1.
"Smooth out" vertices of degree 2.
(continue if <u>possible</u>)

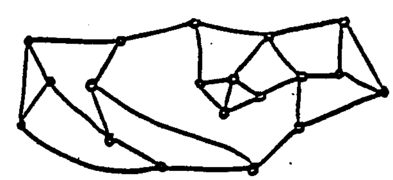
If G is disconnected, keep components with > 5 vertices. (For $p=7$ there will be only one at most)
(Assume G connected).

Perform the preliminary tests.

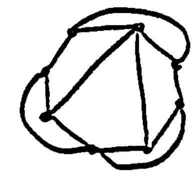
If planarity/nonplanarity is still not determined, perform a planarity test.

Which one?

The Fisher/Wing algorithm.



Special case when G is Hamiltonian



Easy to compute whether the chords are compatible.

What if the graph is not Hamiltonian?

ASK

Connectivity (κ and λ)

Information from the chromatic polynomial.

1. If the coefficient of λ is zero.
   — G is not connected.

2. If $P_G(\lambda)$ is divisible by $(\lambda-1)^2$
   then G has a cut vertex

Otherwise G is at least 2-connected, but there seems to be no short cut to finding the exact value of κ.

The value of λ can be found by using the max-flow-min-cut theorem; but no obvious short cuts.

Number of automorphisms

For graphs with $p \leq 10$ this number was computed and recorded when the graph was generated.

For regular graphs

Example. 4-regular (quartic) graphs, $p = 11$.

These were generated by extracting from the 10-vertex catalog those graphs with degree sequence

4 4 4 4 4 4 3 3 3 3

and joining the four vertices of degree 3 to a new vertex of degree 4.

This gives each required graph at least once.

Now eliminate duplicates

How?

Classify vertices by convenient criteria.

e.g. number of triangles a vertex is on.
(cube of adjacency matrix).
Number of 2-paths between vertices
(square of adjacency matrix).

Run through permutations of vertices which permute sets of equivalent vertices.

"Characterization of Generalized Bicritical Graphs"

Nathaniel Dean, Bellcore
Morristown, NJ

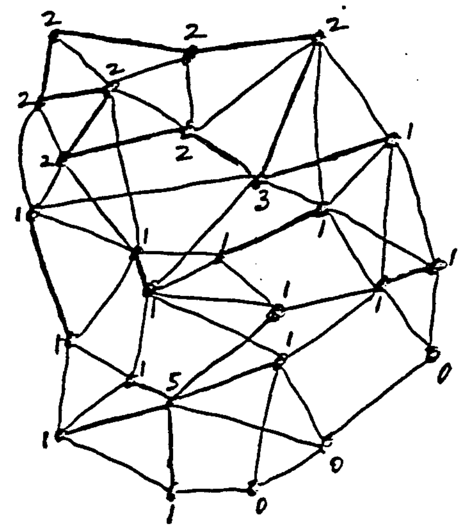
Characterization of Generalized Bicritical Graphs

Nate Dean

Bellcore
Morristown, NJ 07960

1

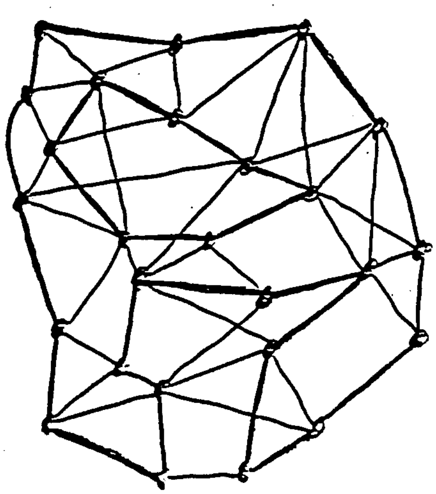
Definitions



f-Factor

2

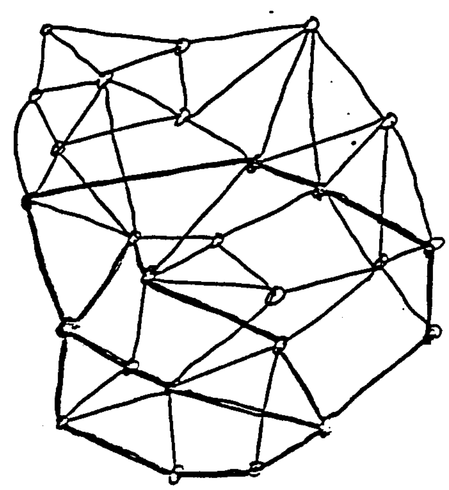
Definitions



2-factor

3

Definitions



4-connectivity

4

Applications

- Lower bound for TSP
- Degree sequences
  - Construct graphs
  - Prove theorems
- Easy proof of Tutte's "4 conn., planar ⇒ Ham." theorem?

5

1-Factor Theorems

Menger - 1927
Konig & Egervary - 1931
Hall - 1935

A bipartite graph $G=(V_1,V_2;E)$ with $|V_1|=|V_2|$ has a 1-factor $\Leftrightarrow |N(S)| \geq |S| \ \forall S \subseteq V_1$.

Tutte - 1947

G has a 1-factor $\Leftrightarrow c_o(G-S) \leq |S| \ \forall S \subseteq V(G)$.

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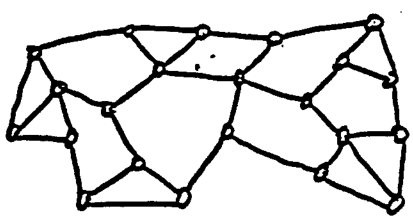
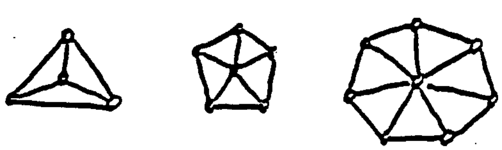
Bicritical Graphs



Even Halin Graphs

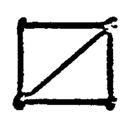


Wheels

7

Special Graphs

- **Elementary Graph:**
  Has a p.m., and the edges contained in a p.m. form a connected subgraph.
- **Bicritical Graph:**
  $G-u-v$ has a p.m. for all vertices u,v.



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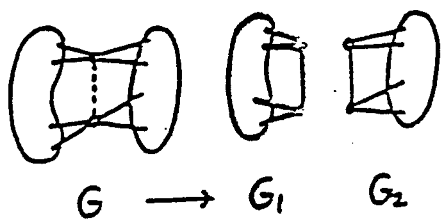
What is a bicritical graph?

- Lovász: G is bicritical iff $S \subseteq V(G)$ and $|S| \geq 2 \Rightarrow c_0(G-S) \leq |S|-2$.
- Bicritical $\Rightarrow$ 2-connected with $\delta \geq 3$.
- G is bicritical iff each split graph G_i of G is bicritical.



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Examples - Lovász and Plummer

- Every even Halin graph is bicritical.
- G is connected, even, vertex-transitive $\Rightarrow$ G is elementary bipartite or bicritical.
- G is cyclically (k+1)-edge connected, even, k-regular $\Rightarrow$ G is elementary bipartite or bicritical.

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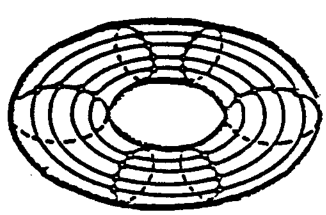
- Corollary (Nishizeki-1978) Every even, 4-connected, projective-planar graph has a perfect matching.
- $\exists$ even, 4-connected, toroidal graphs which are not bicritical. Example: $C_{2m} \times C_{2n}$.



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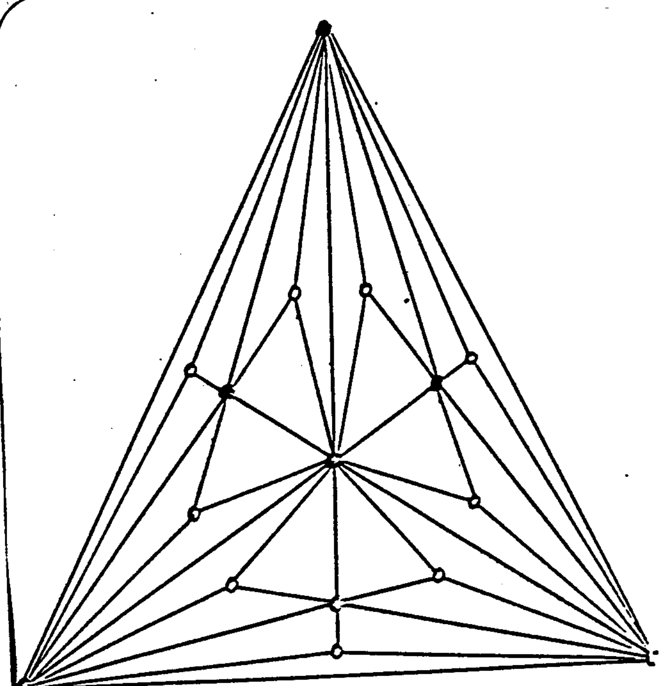

$c_0(G-S) > |S| \Rightarrow$ no p.m.

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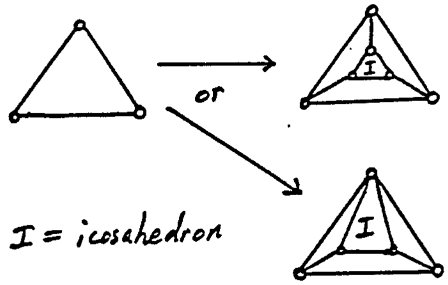

I = icosahedron

3-connected
planar
min degree = 5
$|V(G)|$ even

$\Big\} \not\Rightarrow$ p.m.

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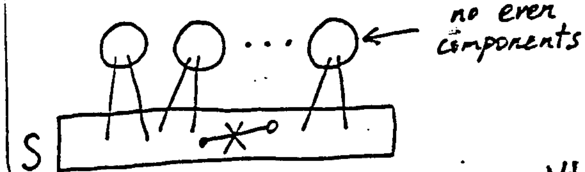
Characterization

Let G be 4-connected and embeddable in the torus or the Klein bottle. Then G is bicritical iff G is even and

(1) G is projective-planar or

(2) G has no set $S \subseteq V(G) \ni$ $G-S$ has no even components and contracting each component of $G-S$ yields a graph G' with $|S| = |V(G')-S|$ and with $G' - E[G[S]]$ 4-regular.



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Matching Extendability

- n-extendable: $p \geq 2n+2$, G has a p.m., and every matching of size n is contained in a p.m.
- Matching extendability $\mu(\Sigma)$ of a surface Σ: smallest integer $n \ni$ no graph embeddable in Σ is n-extendable.
- Plummer: $\mu(\Sigma) = ?$ for orientable Σ.
- Answer: $\Sigma \neq$ sphere $\Rightarrow$

$$\mu(\Sigma) = 2 + \lfloor \sqrt{4 - 2\chi} \rfloor .$$

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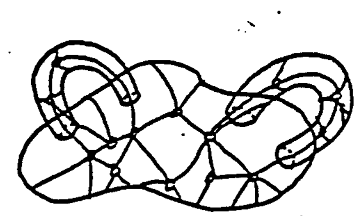
Telecommunications



- Irregularly shaped surface
- Nodes communicate along ≤ 1 edge
- Can matching be extended to one where every node communicates?
- To what extent does the surface obstruct the extension?

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Sufficient Conditions for Hamiltonian Cycle

Whitney (1931)
4-connected plane triangulation

Tutte (1956)
4-connected, planar

Duke (1972)
1) 2-conn., toroidal, $\delta \geq 6$
2) 2-conn., toroidal, $\delta \geq 4$, triangle-free

Thomas & Yu (1991)
4-conn., projective planar

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Conjectures

Grunbaum (1970) & Nash-Williams (1973)
4-conn., toroidal ⇒ Ham.

Molluzzo (1979) — Negami & Ota
6-conn., toroidal ⇒ Ham.-conn.

Grunbaum (1970) — Thomas & Yu
4-conn., projective planar ⇒ Ham.

Dean (1990)
4-conn., proj. planar ⇒ Ham.-conn.
5-conn., toroidal ⇒ Ham.-conn.?

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Tutte's f-Factor Theorem

$$\delta(f,S,T) \triangleq \sum_{x \in S} f(x) + \sum_{x \in T} d_G(x) - e(S,T) - \sum_{x \in T} f(x) - h(f,S,T)$$

where

$h(f,S,T) \triangleq$ number of components C of $G-S-T$ ∋ $\sum_{x \in V(C)} f(x) + e(V(C),T)$ is odd.

G has an f-factor iff $\delta(f,S,T) \geq 0$ $\forall$ disjoint $S,T \subseteq V(G)$.

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f-Bicritical Graphs

Let $f: V(G) \to \mathbb{Z}$.

G is f-bicritical if $\forall u,v \in V(G)$, $G-uv$ has an $f_{u,v}$-factor where $f_{u,v}(x) = f(x)-1$ if $x \in \{u,v\}$ and $f_{u,v}(x) = f(x)$ otherwise.

Note:

- G is f-bicritical ⇒ every edge lies in an f-factor.
- G is 1-bicritical ⇔ G is bicritical (i.e., $\forall u,v \in V(G)$, $G-u-v$ has a p.m.)

Theorem. $\delta(f,S,T) \geq 2$, $\forall S,T \subseteq V(G)$ with $S \cap T = \emptyset$ and $S \cup T \neq \emptyset$ ⇒ f-bicritical.

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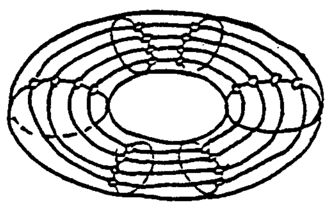
CONJECTURE

Grunbaum - 1970

Nash-Williams - 1973

Every 4-connected, toroidal graph is hamiltonian.



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Surfaces

- Sphere
- Torus, ...
- Projective plane, Klein bottle, ...

(1) G is bipartite and embeddable in the P.P. or the K.B. or torus $\Rightarrow |E(G)| \leq 2|V(G)|$.

(2) $G \rightarrow \Sigma$ are closed under minors.

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Graph Minors Family $\mathfrak{Z}_k$

(1) Closed under minors

(2) Every bipartite member B satisfies $|E(B)| \leq 2|V(B)| - k$.

Examples

- $\mathfrak{Z}_0 \supseteq$ {toroidal graphs}
- $\mathfrak{Z}_0 \supseteq$ {Klein bottle graphs}
- $\mathfrak{Z}_2 \supseteq$ {projective planar graphs}
- $\mathfrak{Z}_4 \supseteq$ {planar graphs}

$$\mathfrak{Z}_0 \supseteq \mathfrak{Z}_1 \supseteq \mathfrak{Z}_2 \supseteq \mathfrak{Z}_3 \supseteq \mathfrak{Z}_4$$

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2-Factors in 4-Conn. G

$G \in \mathfrak{Z}_0 \Rightarrow G$ has a 2-factor.

$G \in \mathfrak{Z}_1 \Rightarrow$

(1) G is 2-bicritical.

(2) $G - u$ has a 2-factor, $\forall u \in V(G)$.

(3) $G - u - v$ has a 2-factor, $\forall u, v \in V(G)$.

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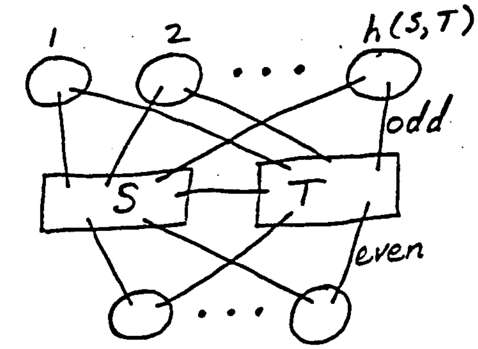
Sample Theorem
(with K. Ota)

G is 4-connected and embeddable in the torus or the Klein bottle $\Rightarrow$ G has a 2-factor.

Best possible: $K_{4,n}$, $n \geq 5$

- 4-connected
- not embeddable in torus
- not embeddable in Klein bottle
- no 2-factor

Tutte's 2-Factor Theorem



$$\delta(S,T) \triangleq 2|S| + \sum_{x \in T} d_G(x) - e(S,T) - 2|T| - h(S,T)$$

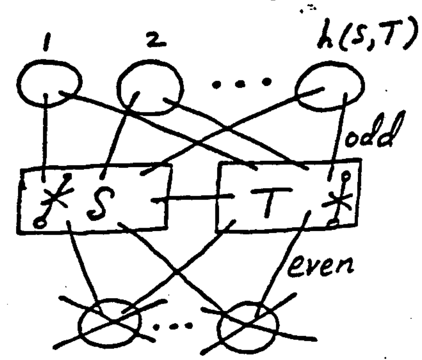
Theorem.
G has a 2-factor $\Leftrightarrow$ $\delta(S,T) \geq 0$ $\forall$ disjoint $S,T \subseteq V(G)$.

Note: $\delta(\emptyset,\emptyset) = 0.$

Proof Strategy

- Reduce G to the essentials
- Case $|S \cup T| \leq 3$
- Case $|S \cup T| \geq 4$
- Collect info on odd components
  - definitions + claims
- Substitute into formula for $\delta(S,T)$ to show ≥ 0

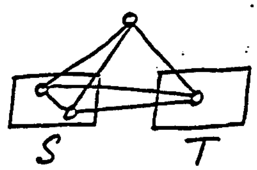
Reductions
$G \rightarrow G_0$



Contract each odd component.

Case $|S \cup T| \le 3$

G_0: (example)



- $h(S,T) \le 1$ since G is 4-conn.
- $\delta(S,T) \ge 2|S| + 4|T| - e(S,T) - 2|T| - 1$

$$= 2|S \cup T| - e(S,T) - 1$$

$$\ge \begin{cases} 4-2-1, & \text{if } |S \cup T| = 2 \text{ or } 3 \\ 2-0-1, & \text{if } |S \cup T| = 1 \end{cases}$$

$$\ge 1$$

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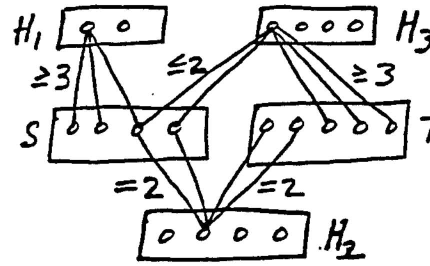
Case $|S \cup T| \ge 4$



$h(S,T) = |H_1| + |H_2| + |H_3|$

Claim.
$\forall u \in H_2 \ \exists x \in T \ni e_G(x, C(u)) \ge 2.$

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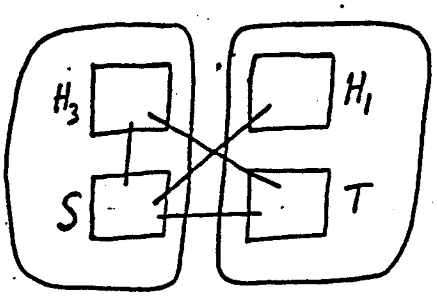
Bipartite Inequality



$= B$

$|E(B)| \ge e_G(S,T) + 3|H_1| + 3|H_3| + |H_2'|$

$|V(B)| = |H_1| + |H_3| + |S| + |T|$

$|E(B)| \le 2|V(B)|$

$e_G(S,T) \le 2|S| + 2|T| - |H_1| - |H_2'| - |H_3|$

$\cdots \Rightarrow \delta(S,T) \ge 0.$

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Connectivity for 2-Factors

$t(\Sigma) \triangleq$ smallest integer $k \ni$ every k-connected graph embeddable in Σ has a 2-factor.

$\chi(S_h) = 2 - 2h$

$\chi(N_k) = 2 - k$

Duke (1972)
$t(\Sigma) \le 3 + \sqrt{9 - 3\chi}$, if $\Sigma \ne S_0$.

New Results:
$t(N_1) = 4 < 5$
$t(S_1) = t(N_2) = 4 < 6$

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"Extremal Problems Involving Neighborhood Numbers and Other Parameters"

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Department of Mathematics and Computer Science, Fredonia, NY

Extremal Problems Involving Neighborhood Numbers and Other Graph Parameters

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ABSTRACT

Given a simple graph $G = (V, E)$, a subset S of V is called a *neighborhood set* provided G is the union of the subgraphs induced by the closed neighborhoods of the vertices in S. The minimum cardinality among all minimal neighborhood sets of G is denoted by $n(G)$ and is called the *neighborhood number* of G. It is known, for instance, that $\gamma(G) \leq n(G) \leq \alpha(G)$ for any G without isolated vertices, where $\gamma(G)$ and $\alpha(G)$ are the (vertex) domination and covering numbers, respectively.

My colleague, Y.H. Harris Kwong, and I have been investigating the problem of finding the maximum neighborhood number $n(p)$ among all connected graphs of order p. Our work so far has lead us to conjecture that

$$n(p) \leq \lfloor 9p/13 \rfloor$$

a result that holds for $2 \leq p \leq 15$. I will report on this work and, as time permits, some recent work of David K. Garnick, Kwong, and Felix Lazebnik on the maximum number of edges among all graphs of order p having girth at least 5.

Observe that if G is the disjoint union of graphs G_1 and G_2, then

$$\gamma(G) = \gamma(G_1) + \gamma(G_2), \quad n(G) = n(G_1) + n(G_2), \quad \alpha(G) = \alpha(G_1) + \alpha(G_2)$$

We thus assume henceforth that $G = (V, E)$ is connected. Now, if u and v are nonadjacent vertices of G, then

$$\gamma(G + uv) \leq \gamma(G) \quad \text{and} \quad \alpha(G + uv) \geq \alpha(G)$$

However,

$$n(G) - 1 \leq n(G + uv) \leq n(G) + 1$$

For example, consider the graph $G = (\{u, v, w, x, y\}, \{uv, vw, wx, xy, uy\})$. Note that G is a 5-cycle, $n(G + uw) = 2$, $n(G) = 3$, and $n(G - uv) = 2$.

This leads us to consider the following extremal problem: find the maximum neighborhood number $n(p)$ among all connected graphs of order p.

Given a simple graph $G = (V, E)$, the set $N(v) = \{w \in V \mid vw \in E\}$ is call the *neighborhood* of v and $N[v] = N(v) \cup \{v\}$ is its *closed neighborhood*. A subset S V is called a *neighborhood set* provided G is the union of the subgraphs induced by t closed neighborhoods of the vertices in S. The minimum cardinality among all minin neighborhood sets of G is denoted by $n(G)$ and is called the *neighborhood number* of and was introduced by E. Sampathkumar and P.S. Neeralagi [The neighborhood numl of a graph, *Indian J. Pure Appl. Math.* 16 (1985), 126–132].

Two related parameters are the (vertex) domination and covering numbers. A sub S of V is a *dominating set* provided every vertex not in S is adjacent to a vertex in it is a *covering set* provided every edge of G has at least one of its incident vertices in Let $\gamma(G)$ denote the minimum cardinality of a dominating set and let $\alpha(G)$ denote t minimum cardinality of a covering set; if G has no isolated vertices, then any covering is also a neighborhood set and any neighborhood set is also a dominating set. Thus, any graph G without isolated vertices,

$$\gamma(G) \leq n(G) \leq \alpha(G)$$

It is natural to wonder whether these parameters are independent, in some sense. is easy to see that $\gamma(G) = 1$ if and only if $n(G) = 1$; the complete graph K_p of order has $\gamma(K_p) = n(K_p) = 1$ and $\alpha(K_p) = p - 1$. It was also observed in the above mention paper that $n(G) = \alpha(G)$ if G is connected and has girth at least 4. However, Jayara Kwong, and Straight [Neighborhood sets in graphs, *Indian J. Pure Appl. Math.*, to appe gave, for positive integers r, s and t with $2 \leq r \leq s \leq t$, an example of a graph G havi $\gamma(G) = r, n(G) = s$, and $\alpha(G) = t$.

To begin, we find that $n(2) = n(3) = 1$ and $n(4) = 2$. Next, $n(5) = 3$, with the uniq extremal graph being the 5-cycle. At this point we make the following observation: l G be formed from the disjoint union of G_1 and G_2 by adding a single edge that joins vertex of G_1 with a vertex of G_2. Then $n(G) \geq n(G_1) + n(G_2)$. As a consequence, let be a rational number between 0 and 1 and suppose there exists a graph G_0 of order having $n(G_0) = rp_0$. Then there exist infinitely many values of p for which $n(p) \geq r$ Since we have a graph of order 5 with neighborhood number 3, we tentatively conjectu that $n(p) \leq 3p/5$ (for all $p > 1$).

Two more useful observations. The first is that

$$n(p + 1) \leq n(p) + 1$$

Secondly, suppose G_1 is a connected graph of order p and G_2 is the complete graph order 2. Form the graph G as above. Then G has order $p + 2$ and $n(G) = n(G_1) + 1$. Th we find that

$$n(p) = n(p + 1) \rightarrow n(p + 2) = n(p + 1) + 1$$

Continuing, we find that $n(6) = 3$ and $n(7) = 4$. But now consider the following gra of order 8:

$$J = (\{s, t, u, v, w, x, y, z\}, \{st, sw, sz, tu, tx, uv, uz, vw, vy, wx, xy, yz\})$$

It is not difficult to show that $n(J) = 5$. It follows that $n(8) = 5$, and we are forced revise our tentative conjecture, namely, we now conjecture that $n(p) \leq 5p/8$.

Let p be fixed and suppose the value of $n(p)$ is known. As a consequence of the observations made on the preceding page, if one claims that $n(p+1) = n(p)+1$, then one must give an example of a graph G of order $p+1$ having $n(G) = n(p)+1$. On the other hand, if one claims that $n(p+1) = n(p)$, then one must prove that every connected graph of order $p+1$ has neighborhood number at most $n(p)$.

We can show that every connected graph of order 9 has neighborhood number at most 5. This, together with $n(8) = 5$, gives us that $n(9) = 5$. Joining two disjoint 5-cycles with an edge yields a connected graph of order 10 having neighborhood number 6; hence, $n(10) = 6$. At this point in our investigation we were able to find a graph of order 11 with neighborhood number 7. This disproves our tentative conjecture that $n(p) \le 5p/8$! However, if one writes down a table of values of $n(p)$ for $2 \le p \le 11$ the following revised conjecture strongly suggests itself.

Conjecture. For $p \ge 2$,

$$n(p) = n(p+1) \to n(p+2) = n(p)+1 \text{ and } n(p+3) = n(p+4) = n(p)+2$$

(Unfortunately?) Harris Kwong found a graph H of order 12 having neighborhood number 8, so the conjecture is false for $p = 8$. The graph H can be used to construct a graph of order 13 with neighborhood number 9. Then it can be shown that $n(14) = 9$, also, so that $n(15) = 10$. This is the state of the problem at this point in time. By the way, I still believe that the above conjecture is true for p sufficiently large, say $p \ge 13$.

4

This paper investigates the values of $f(v)$, the maximum number of edges in a graph of order v and girth at least 5. For small values of v we also enumerate the set of extremal graphs. This problem has been mentioned several times by P. Erdős, who conjectured that $f(v) = (1/2 + o(1))^{3/2} v^{3/2}$.

Given graphs $G_1, G_2, \dots, G_k$, let $ex(v; G_1, G_2, \dots, G_k)$ denote the greatest size of a graph of order v which contains no subgraph isomorphic to some $G_i, 1 \le i \le k$. One of the main classes of problems in extremal graph theory, known as Turan-type problems, is for given $v, G_1, G_2, \dots, G_k$ to determine explicitly the function $ex(v; G_1, G_2, \dots, G_k)$, or to find its asymptotic behavior for large values of v. Thus, the problem we consider in this paper, that of finding the maximum size of a graph of girth at least 5, can be stated as finding the value of $ex(v; C_3, C_4)$.

It is well known that $ex(v; C_3) = \lfloor v^2/4 \rfloor$, and the extremal graph is the complete bipartite graph $K_{\lfloor v/2 \rfloor, \lceil v/2 \rceil}$.

The exact value of $ex(v; C_4)$ is known for all values of v of the form $v = q^2 + q + 1$, where q is a power of 2 [Z. Füredi, Graphs without quadrilaterals, *JCT B* 34 (1983), 187–190], or a prime power exceeding 13 [Füredi, preprint], and it is equal to $q(q+1)^2/2$. The extremal graphs for these values of v are known. For $1 \le v \le 21$, the values of $ex(v; C_4)$ and the corresponding extremal graphs can be found in [C.R.J. Clapham, A. Flockhart, and J. Sheehan, Graphs without four-cycles, *JGT* 13 (1989), 29–47]. Apparently, the same results were obtained by W. McCuaig in 1984, but were not published. It is well known that $ex(v; C_4) = (1/2 + o(1))v^{3/2}$.

It is important to note that attempts to construct extremal graphs for $ex(v; C_3, C_4)$ by destroying all 4-cycles in the extremal graphs for $ex(v; C_3)$, or by destroying all 3-cycles in the extremal graphs for $ex(v; C_4)$, fail; neither method leads to graphs of order v with $f(v)$ edges.

The other results I wish to report on are in the following paper:

Extremal Graphs without Three-Cycles or Four-Cycles

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Abstract

We derive bounds for $f(v)$, the maximum number of edges in a graph on v ver[tices] that contains neither three-cycles nor four-cycles. Also, we give the exact value of $f(v$[)] [for] all v up to 24 and constructive lower bounds for all v up to 200.

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In this section we present some theoretical results about $f(v)$ and the structure of extremal graphs. Many of them will be used in the subsequent sections. We call a gr[aph] G of order v *extremal* if $g(G) \ge 5$ and $e = e(G) = f(v)$. The following statement [gives] some simple facts about extremal graphs.

Proposition 2.1. Let G be an extremal graph of order v. Then

(a) G is connected and the diameter of G is at most 3.

(b) If $d(x) = \delta(G) = 1$, then the graph $G - x$ has diameter at most 2.

It turns out that the extremal graphs of diameter 2 are very rare. In fact, it has [been] shown that the only graphs of order v with no 4-cycles and of diameter 2 are:

The star $K_{1,v-1}$;

Moore graphs: C_5, Petersen graph (the only 3-regular graph of order 10, diame[ter 2] and girth 5), Hoffman-Singleton graph (the only 7-regular graph of order 50, diam[eter] 2 and girth 5), and a 57-regular graph of order 3250, diameter 2 and girth 5, if [it] exists;

Polarity graphs.

Remark: The only graphs from the list above which in addition contain no triangle[s and] are regular are the Moore graphs. It is also known that a graph of diameter $k \ge 1$ [and] girth $2k+1$ must be regular.

We now derive an upper bound on $f(v)$.

Theorem 2.2. Let G be an extremal graph of order $v \ge 3$ and size e. Then

$$f(v) = e \le \frac{1}{2} v \sqrt{v-1}$$

Furthermore, equality holds if and only if G is regular and of diameter 2, i.e. G is a M[oore] graph.

Corollary 2.3. Let G be an extremal graph of order v, size e, and diameter 3. If G [is] regular, then

$$f(v) = e \le \frac{1}{2}\sqrt{v^2(v-1) - \frac{5}{2}v}$$

If, in addition, the average degree of G is an integer, then

$$f(v) = e \le \frac{1}{2}\sqrt{v^2(v-1) - 4v}$$

Now we derive a lower bound for $f(v)$. Let q be a prime power, and let $v_q = q^2+q+1$, and $e_q = (q+1)v_q$. By B_q we denote the point-line incidence bipartite graph of the projective plane $PGL(2,q)$. More precisely, the partite sets of B_q represent the set of points and the set of lines of $PGL(2,q)$, and the edges of B_q correspond to the pairs of incident points and lines. Then B_q is a $(q+1)$-regular bipartite graph of order $2v_q$ and size e_q. Also $g(B_q) \geq 5$: being bipartite B_q has no 3-cycles, and the existence of a 4-cycle in B_q would mean that in $PGL(2,q)$ there are two distinct lines passing through two distinct points.

Theorem 2.4. Let G be an extremal graph of order v and size e. Let q be the largest prime power such that $2v_q \leq v$. Then

$$f(v) = e \geq e_q + 2(v - 2v_q) = 2v + (q-3)v_q$$

Proof. Consider the graph obtained from B_q by adding to $V(B_q)$ a set of $v - 2v_q$ isolated vertices and connecting each of these to two nonadjacent vertices of B_q taken from different partite sets (such two vertices always exist, since B_q is not a complete bipartite graph). If the chosen pairs of vertices of B_q are distinct for distinct isolated vertices, then we obtain a graph H of order v, size $e_q + 2(v-2v_q)$ and girth at least 5. There are q^2v_q pairs of disjoint vertices of B_q in which two vertices in the pair belong to different partite sets. We claim that $q^2v_q > v - 2v_q$. Indeed, if it is not the case and $q > 2$, then $v \geq (q^2+2)v_q > 8v_q$. According to Bertrand's Postulate, for any integer $n \geq 1$, there is at least one prime number p such that $n < p \leq 2n$. Let $n = q$, and let p be a prime satisfying the inequality $q < p \leq 2q$. Then

$$2v_q < 2v_p \leq 8q^2 + 4q + 2 < 8v_q < v$$

which contradicts our choice of q in the statement of the theorem. For $q = 2$, $2v_2 = 14 \leq v < 2v_3 = 26$ implies that $v - 2v_2 = v - 14 < 12 < 28 = 2^2v_2$. Therefore, for all prime powers q, $q^2v_q > v - 2v_q$, and the construction of H described above is possible.

We next define a restricted type of tree; many of the proofs in in the next section rely on the presence of these trees in extremal graphs. Consider a vertex x of maximum degree Δ in a $\{C_3, C_4\}$-free graph G. Let the neighborhood of x be $N(x) = \{x_1, x_2, \ldots, x_\Delta\}$. Clearly $N(x)$ is an independent set of vertices. Furthermore, the sets of vertices $N(x_i) - \{x\}$, $1 \leq i \leq \Delta$, are pairwise disjoint; otherwise there would be a quadrilateral in G. This motivates the notion of an (m,n)-*star* $S_{m,n}$, which is defined to be the tree in which the root (center) has m children, and each of the root's children has n children, all of which are leaves. The subtree containing a child of the root and all its n children is called a *branch* of $S_{m,n}$. We will make use of the following easily established facts: (1) $|V(S_{m,n})| = 1 + m + mn$ and $|E(S_{m,n})| = m + mn$; (2) Every $\{C_3, C_4\}$-free graph G with at least 5 vertices, $\Delta(G) = \Delta$, and $\delta(G) = \delta$, contains $S_{\Delta,\delta-1}$; (3) In any $\{C_3, C_4\}$-free graph G containing an (m,n)-star S, no vertex in $G - S$ can be adjacent to two siblings in S. In fact, every set of siblings in S has a unique common neighbor, namely, their parent.

Proposition 2.6. For all $\{C_3, C_4\}$-free graphs G, $v \geq 1 + \Delta\delta \geq 1 + \delta^2$.

Proposition 2.7. For all $\{C_3, C_4\}$-free graphs G on $v \geq 1$ vertices and e edges, $\delta \geq e - f(v-1)$ and $\Delta \geq \lceil 2e/v \rceil$.

Proposition 2.8. For all $v \geq 1$, we have $v \geq 1 + \lceil 2f(v)/v \rceil (f(v) - f(v-1))$.

Theorem 2.2 states that $f(v) \leq \lfloor v\sqrt{v-1}/2 \rfloor$. For $1 \leq v \leq 10$, we have constructed $\{C_3, C_4\}$-free graphs with $\lfloor v\sqrt{v-1}/2 \rfloor$ edges. These graphs are shown in Figure 1. This yields the following theorem.

Theorem 3.1. For $1 \leq v \leq 10$, $f(v) = \lfloor v\sqrt{v-1}/2 \rfloor$.

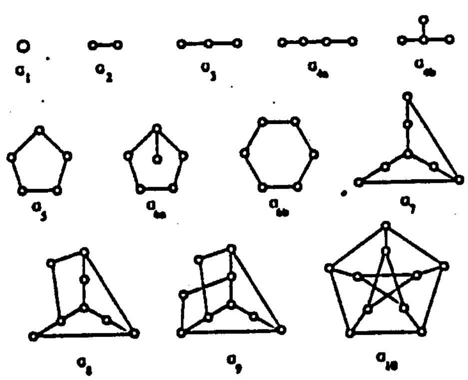
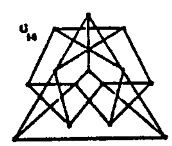


Figure 1: Extremal graphs with $v \leq 10$

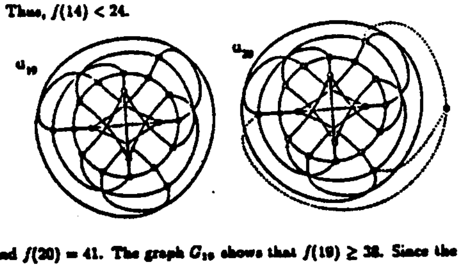
Theorem 3.3. The values of $f(v)$ for $11 \leq v \leq 20$ are as follows:

$f(11) = 16$ $f(12) = 18$ $f(13) = 21$ $f(14) = 23$ $f(15) = 26$
$f(16) = 28$ $f(17) = 31$ $f(18) = 34$ $f(19) = 38$ $f(20) = 41$

Proof. We look at several specific cases to illustrate the techniques involved.



$f(14) = 23$. The graph G_{14} demonstrates that $f(14) \geq 23$. Assume there exists an extremal graph G with 24 edges. Propositions 2.6 and 2.7 imply that $\delta = 3$ and $\Delta = 4$. Thus G has 8 vertices of degree 3 and 6 vertices of degree 4. Further, G cannot contain a pair of adjacent vertices x and y each having degree 3, since $G - \{x, y\}$ would have 19 edges, contradicting $f(12) = 18$. We may then conclude that every edge of G joins a degree 3 vertex with a degree 4 vertex. If for each degree 3 vertex we represent its neighborhood as a triple, then the existence of G implies the existence of a design of 8 triples on 6 elements (the six vertices of degree 4), where each element occurs in 4 triples and each distinct pair of elements occurs in at most 1 triple. But since 6 elements constitute 15 distinct pairs and each triple specifies 3 pairs, such 8 triples will specify $8 \times 3 = 24$ pairs, and therefore cannot exist. Thus, $f(14) < 24$.



$f(19) = 38$ and $f(20) = 41$. The graph G_{19} shows that $f(19) \geq 38$. Since the outermost 3 vertices (the vertices outside the dodecagon) are mutually distance 3 apart, we can construct G_{20} by adding a vertex to G_{19} that is adjacent to the 3 outer vertices. (In the figure, dashed lines are used to show the edges from the 20th vertex to G_{19}.) Thus $f(20) \geq |E(G)| = 41$. The upper bounds are obtained by applying Proposition 2.8.

It can be noted that G_{19} is the Robertson graph – the unique (4, 5)-cage. It follows that G_{19} and G_{20} are the unique extremal graphs of orders 19 and 20, respectively.

Theorem 3.4. The values of $f(v)$ for $21 \leq v \leq 24$ are as follows:

$$f(21) = 44 \quad f(22) = 47 \quad f(23) = 50 \quad f(24) = 54$$

Proof. Again, to illustrate the techniques involved, we give the proof that $f(22) = 47$. First of all, a $\{C_3, C_4\}$-free graph of order 22 and size 47 is constructed, showing that $f(22) \geq 47$. Suppose there exists a $\{C_3, C_4\}$-free graph G with $v = 22$, $e = 48$. By Proposition 2.7, $\delta \geq 4$ and $\Delta \geq 5$. Then $\Delta = 5$, for otherwise there is a (6, 3) star in G, and such a tree has more than 22 vertices; therefore, G contains 14 vertices of degree 4 and 8 vertices of degree 5. This in turn implies that there are at least 8 edges among the degree 4 vertices. Since any graph with 14 vertices and at least 8 edges must contain a path of length 3, there must be at least one degree 4 vertex adjacent to two other degree 4 vertices; thus there is a P_3 in G each of whose vertices has degree 4. However, if there is a path P on 3 vertices of degree 4, then $G - V(P)$ is the Robertson graph. Each of the pendant vertices in P, x and y, has 3 neighbors in the Robertson graph that are mutually distance 3 apart. But the Robertson graph has only one such set of 3 vertices. Therefore, G cannot contain a P_3 of degree 4 vertices, giving us a contradiction. Therefore, $f(22) \leq 47$.

This paper gives exact values of $f(v)$ for $1 \leq v \leq 24$. It is also noted that $f(50) = 175$, the extremal graph being the Hoffman-Singleton graph. The paper also gives constructive lower bounds for $f(v)$ for $25 \leq v \leq 200$; these are found using an algorithm that combines hill-climbing and backtracking techniques. For instance, at one point a $\{C_3, C_4\}$-free graph of order 96 and size 397 was found; adding an isolated vertex gave a $\{C_3, C_4\}$-free graph of order 97 and size 397. Backtracking applied to this graph yielded a $\{C_3, C_4\}$-free graph of the same order with 403 edges. Hill-climbing from that point resulted in the addition of one more edge, giving a $\{C_3, C_4\}$-free graph G of order 97 and size 404. Thus, $f(97) \geq 404$. Further hill-climbing rearranged the edges of G so that some vertex had degree 6; removing this vertex then gave a $\{C_3, C_4\}$-free graph of order 96 and size 398, thereby improving the lower bound for $f(96)$.

"Random Graph Processes with Degree Restrictions"

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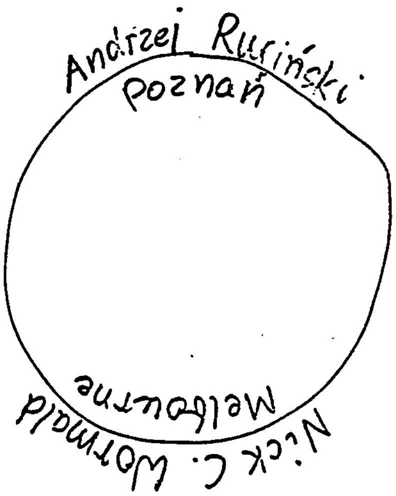
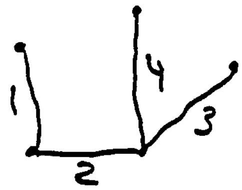
①

Random Graph Processes with degree restrictions



Random graph process

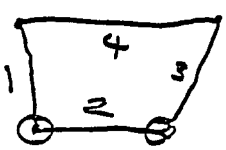


$(E_1, \dots, E_{\binom{n}{2}})$

$\left(\binom{n}{2}!\right)^{-1}$

Degree restriction $\Delta(G) \leqslant d$

$d=2$



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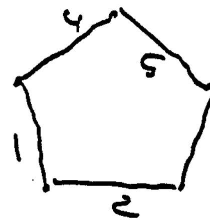

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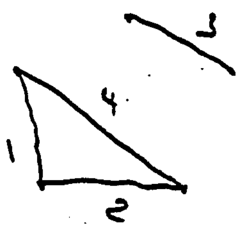


$(E_1, \dots, E_w)$

$\frac{nd}{2} - \frac{d(d+2)}{8} \leqslant w \leqslant \lfloor \frac{dn}{2} \rfloor$

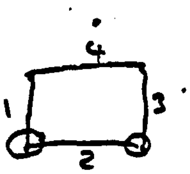
③

3 major differences:

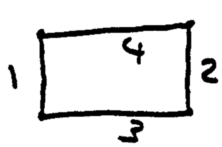
1° many final stages

2° the length varies

3° not an equiprobable space



$\frac{1}{10}\ \frac{1}{9}\ \frac{1}{6}\ \frac{1}{3}$



$\frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{3}$

Focus on final stage:

a maximal graph with $\Delta(G)=d$

d-maximal graph

saturated vts (degree d)

unsaturated vts (degree $<d$)

Unsaturated vts form a clique, so there are $\leqslant d$ of them.

Let $U = U(d,n) = \#$ unsaturated vts at the end

Erdös asked

$\lim_{n\to\infty} P(U=x)$, $x=0,1,\dots,d$?

Nontrivial even for $d=2$

	n=4		n=5	n=500	n=30,000
$P(U=0) =$	$\frac{11}{15}$	$\searrow$	$\frac{17}{27}$ $\nearrow$ .879	$\overset{?}{\nearrow}$	$\approx .9$

$P(U=0) \to 1$?

(5)

Theorem (R., Wormald, 1992)
$\forall d$ the process saturates a.s., i.e. $P(U=0) \to 1$ if nd even
$P(U=1) \to 1$ if nd odd

It contrasts with the equiprobable space of d-maximal graphs, where $\forall d \geq 2$, dn even,

$$P(U=i) \to a_i > 0 \quad i=0,1,2,$$
$$a_0 + a_1 + a_2 = 1$$

Structural results only for $d=2$

$$EC \leq \log n + 3$$

$$\# C_l \overset{\mathcal{D}}{\to} \text{Poisson}$$

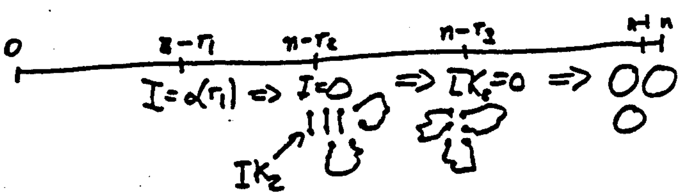
$$E\# C_3 \sim \frac{1}{2}\int_0^\infty \frac{(\log(1+x))^2}{xe^x}dx \approx$$

$$.18873533493577888 30 \neq \frac{1}{6}$$

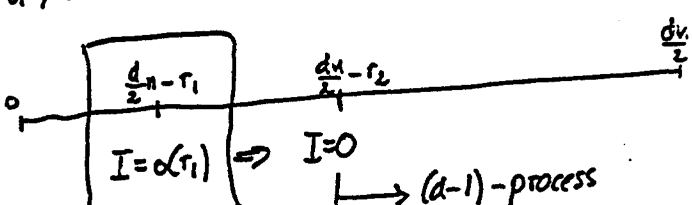
(7)

The idea of proof:
study # isolates I_t

$d=2$



$d \geq 2$ induction on d



Quite messy: 1° forbidden pairs at start
2° nonuniform degree bounds

Lemma $P(u,v)$ – probability that vertex 1 remains isolated until time v, provided it was isolated at time u. Then

$$P(u,v) = O\left(\frac{n/2 - v}{n/2 - u}\right)$$

Proof $P(u,v) = \prod_{t=u}^{v-1} P(t,t+1)$

H_t – the event that 1 is isolated in ...

$$P(t,t+1) = E(E(\mathrm{Ind}(H_{t+1})|G_t)|H_t)$$
$$= E\left(1 - \frac{U_t - 1}{\binom{U_t}{2} - F_t} \Big| H_t\right) \leq$$
$$\leq \exp\left(-\frac{2}{n-2t} + O\left(\frac{1}{n^2}\right)\right)$$

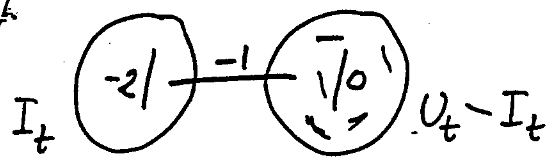
9

Lemma

$$I_{\lfloor dn/2 \rfloor - n^{9/10}} = O\left(\frac{n^{9/10}}{\log n}\right)$$

Outline of proof:

$$E(I_{t+1} - I_t \mid G_t) = ?$$



$$= \frac{-2\binom{I_t}{2} - I_t(U_t - I_t)}{\binom{U_t}{2} - F_t} \sim -\frac{2 I_t}{U_t}$$

But

$$dn - 2t \geq dI_t + (U_t - I_t)$$

$\overset{=}{(d=2)}$

$$U_t \leq dn - 2t - (d-1) I_t$$

$\overset{=}{(d=2)}$

So

$$E(\underline{I_{t+\Delta t}} - I_t \mid G_t) \leq -\frac{2\Delta t \, I_t}{dn - 2t - (d-1) I_t}$$

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Define Doob's martingale

$$X_i = E(I_{t+\Delta t} - I_t \mid G_t, E_{t+1}, \ldots, E_{t+i})$$

$$i = 0, \ldots, \Delta t$$

to show the sharp concentration

$$I_{t+\Delta t} - I_t \sim E(I_{t+\Delta t} - I_t \mid G_t)$$

by Azuma's inequality

11

Define $b = b(x)$, $0 \leq x < \frac{d}{2}$

by

$$(*) \quad b' = \frac{-2b}{d - 2x - (d-1)b}, \quad b(0) = 1$$

$b(x)$ should well approximate an upper bound on $\frac{I_t}{n}$, $x = \frac{t}{n}$

We justify this by partitioning

and using induction

$$\Delta \sim n^{1/4}$$

The asymptotic solution to (*), as $x \to \frac{d}{2}$, is

$$b(x) \sim \frac{-(d-2x)}{(d-1)\log\left(\frac{d}{2}-x\right)}$$

Finally,

$$I_s \leq n\,b\left(\tfrac{s}{n}\right) + \underline{O(n^{8/9})} \sim$$

$$\sim \frac{-n\left(d-2\frac{s}{n}\right)}{(d-1)\log\left(\frac{d}{2}-\frac{s}{n}\right)} = \frac{-2n^{9/10}}{(d-1)\log n^{-1/10}} =$$

$$= \frac{20\, n^{9/10}}{(d-1)\log n}$$ ☒

Open problem

How far apart are max and min of

$$P\left(G_{\lfloor \frac{dn}{2} \rfloor} = G\right)$$

a) over all d-maximal G's

b) over all d-regular G's

?

Sixth Discrete Mathematics mini-Conference

October 3-4, 1991

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