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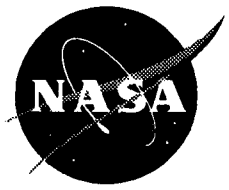

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THE ALGEBRAIC MULTIGRID PROJECTION FOR EIGENVALUE PROBLEMS; BACKROTATIONS AND MULTIGRID FIXED POINTS

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The Algebraic Multigrid Projection for Eigenvalue Problems; Backrotations and Multigrid Fixed Points *

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ABSTRACT

The proofs of the theorem for the algebraic multigrid projection (MGP) for eigenvalue problems, and of the multigrid fixed point theorem for multigrid cycles combining MGP with backrotations, are presented. The MGP and the backrotations are central eigenvector separation techniques for multigrid eigenvalue algorithms. They allow computation on coarse levels of eigenvalues of a given eigenvalue problem, and are efficient tools in the computation of eigenvectors.

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1 Introduction

In [1], [2] we present multigrid (MG) eigenvalue algorithms for linear and nonlinear eigenvalue problems, whose robustness and efficiency rely much on the multigrid projection (MGP) and on the backrotations, introduced in [3], [4], [5]. The applications were for Schrödinger and electromagnetism eigenvalue problems. MGP allows computation on coarse levels of the eigenvalues of a fine level eigenvalue problem. This is stated by the MGP theorem whose proof is given in a general algebraic setting, in Section 4. The MGP coupled with backrotations, relaxations, and inter-level transfers form the MG cycles, which are used in the computation of eigenvectors. A central requirement for MG eigenvalue algorithms is that the MG cycles should have as fixed points the fine level solutions. In Section 6 we prove that this requirement is met by the MG cycles presented in Section 5. The MGP coupled with backrotations also improve the efficiency and robustness of MG algorithms. Computational examples which motivate the need of backrotations and illustrate the efficiency of the presented techniques are shown in Section 7.

For MG techniques and more on MG eigenvalue algorithms, we refer to [6], [7], [8], [9]. For algorithms and theory on algebraic eigenvalue problems we refer to [10], [11], [12], [13].

2 Generalized Rayleigh Ritz Projections

Assume that A, U, T are given matrices, A of dimensions $n \times n$, and U and T of dimension $n \times q$. From now on, if not otherwise specified, a *pair* (E, Λ) refers to $q \times q$ matrices E and Λ where E is invertible and Λ is diagonal. It is said that a pair (E, Λ) is a solution for (A, U, T) if:

$$AUE = UE\Lambda + TE \tag{1}$$

The separation problem requires to find solutions (E, Λ) of (1) for given (A, U, T). In a single level approach this problem is treated by Rayleigh Ritz type projections, while in a multigrid approach the problem is treated by multigrid projections.

If $T = 0$ and (E, Λ) is a solution for (A, U, T) then UE are eigenvectors of A associated to the eigenvalues of Λ. There may be no solutions for (A, U, T). For example, if $T = 0$ or $T = U$, and $q = 1$ there is no solution unless U is an eigenvector.

The pairs (E, Λ) and (F, Θ) will be called equivalent if

$$E\Lambda E^{-1} = F\Theta F^{-1} \tag{2}$$

Lemma 1 *If the columns of U are linearly independent then any two solutions for (A, U, T) are equivalent.*

Proof From (1) results

$$AU - T = UE\Lambda E^{-1} = UF\Theta F^{-1} \tag{3}$$

and since the columns of U are independent results (2).

Lemma 2 *If the pairs (E, Λ) and (F, Θ) are equivalent and (F, Θ) is a solution for (A, U, T) then (E, Λ) is a solution for (A, U, T).*

Proof By (2) the relation (3) is obtained and from here (1)

If the columns of U are not linearly independent then nonequivalent solutions for (A, U, T) may exist. For instance let $q = 3$, $T = 0$ and consider a matrix A which has the eigenvectors U_1, U_2 with respective eigenvalues 1 and 2. Let $U = (U_1, U_2, U_2)$. One solution (E, Λ) has $\Lambda = diag(1, 2, 2)$ with E the identity, and a second solution (F, Θ) has $\Theta = diag(1, 2, 0)$ with f having the first two columns the corresponding columns of the identity matrix, and the third column $(0, 1, -1)^T$. The two solutions are not equivalent since the sets of eigenvalues in the two pairs are different, while in (2) they are obviously the same.

Since the problem of finding a pair (E, Λ) which satisfies (1) may not have solutions, it is reduced to

$$Y^T(AUE - UE\Lambda - TE) = 0 \tag{4}$$

i.e., the projection of the residuals of (1) on the space spanned by the columns of Y, should be minimal. Solutions (E, Λ) to (4) can be obtained from the smaller $q \times q$ generalized eigenvalue problem:

$$(Y^T(AU - T))\ E = (Y^T U)\ E\Lambda \tag{5}$$

One may choose Y in different ways, e.g., either $Y = U$, or, in an MG setting, Y may be the transfer of the solution from another level, or Y may consist of approximations of left eigenvectors of A. Finding a pair (E, Λ) which satisfies (5) is referred as Generalized Rayleigh-Ritz Projection (GRRP).

Remarks The T term makes the difference between the usual Rayleigh-Ritz projection, (where $T = 0$), [11], [13], and the projection presented here. T is introduced by transfers in MG algorithms, as shown in Section 5. The T changes the usual linear eigenvalue problem for U, $AU = U\Lambda$, into a nonlinear problem $AU = U\Lambda + T$. Before GRRP, the U can be multiplied UF by an invertible matrix F, e.g., to orthonormalize the columns of U. The previous solutions E will become $F^{-1}E$.

3 Backrotations

Backrotations [5], [2] are central separation techniques in MG algorithms for eigenvalue problems. A separation matrix E obtained from a projection may combine arbitrarily or even overlap the eigenvectors in a degenerate eigenspace, (as was often observed in tests [2]). (This results in the observation that if F commutes with Λ from (1), and if E is a solution of (1) then EF is a solution also.) Difficulties may occur for close eigenvalues and when the eigenspaces are not well enough approximated. E may permute or rescale the columns of U. The role of backrotations is to correct these in MG algorithms. If U is the solution of the problem then E should be the identity. When U is close to the solution then an E obtained by the projection should be brought close to the identity. In the next backrotation algorithm, it is desired that the eigenvalues be ordered, (e.g., by modulus), except in close clusters where ordering of eigenvalues is not important but the vectors of UE are desired to be *close* to the vectors of U.

Backrotation

Input (E, Λ)

1) Sort the eigenvalues of Λ *and*
permute the columns of E *accordingly*

2) Determine the clusters of eigenvalues of Λ

3) For each diagonal block in E
associated with a nondegenerate cluster do:
bring to the diagonal the dominant elements of the block,
permuting the columns of E
and the diagonal of Λ *correspondingly.*

4) Let F *be a block diagonal matrix*
whose diagonal blocks are the diagonal blocks of E
corresponding to the determined clusters.
Replace each diagonal block which does not correspond
to a degenerate cluster by the corresponding identity matrix

5) Set $E = EF^{-1}$.

6) Change the signs of columns of E
to get positive elements on diagonal.

7) Normalize the columns of E.

Output (E, Λ)

Assume that $Y = U$ in the GRRP. Denote by Id the identity matrix. The projection and

backrotation should provide the solution (Id, Λ) if the equation $AU = U\Lambda + T$ is satisfied, (i.e., if (Id, Λ) is one of the solutions), U has linearly independent columns, and the eigenvalues in Λ are sorted.

This is assured by the following lemma:

Lemma 3 *If U has independent columns, the eigenvalues on the diagonal of Λ are sorted and*

$$AU = U\Lambda + T \tag{6}$$

then the solution of the projection and backrotation is (Id, Λ).

Proof Assume that (E, Λ') is a solution obtained by the projection, with E nonsingular. Since U has independent columns and (Id, Λ) is a solution for (A, U, T), it follows from Lemma 1 that

$$E\Lambda' E^{-1} = \Lambda \tag{7}$$

Thus Λ and Λ' have the same eigenvalues. Then the sorting at Step 1) in the backrotation algorithm will make $\Lambda' = \Lambda$. If all eigenvalues in Λ are different then the initial E is just a permutation and the sorting of Λ' and permutations of columns of E will bring E to diagonal form. The Steps 5) and 6) will impose $E = Id$ while the other steps will let (E, Λ) unchanged. It is sufficient to consider the case when Λ has on diagonal only degenerate clusters. Then after the permutations at 1), E results block diagonal with the blocks corresponding to degenerate clusters. In this case $F = E^{-1}$, thus by Step 5) $E = Id$.

4 The Algebraic Multigrid Projection

The solutions (E, Λ) of (1), obtained by a GRRP on a single level may be obtained by an MG projection, transferring the problem to another level, (e.g., to a coarser level). The problems on the initial level (1) and on the new level have the same form, and the same solutions.

Assume that the problem (1) for (A, U, T):

$$AUE = UE\Lambda + TE \tag{8}$$

is transfered into a problem for (A', U', T'):

$$A'U'E = U'E\Lambda + T'E \tag{9}$$

such that:

(A1) There exists a matrix J with $U' = JU$

(A2) U' has independent columns

(A3) T' is defined by

$$T' = A'U' + J(T - AU) \tag{10}$$

The next result provides the theoretical basis of the MG projection:

Theorem (The Two Level Projection Theorem)
If assumptions (A1) to (A3) hold and (A,U,T) has solutions, then the sets of solutions for (A,U,T) and (A',U',T') coincide and all solutions are equivalent.

Proof Let (E,Λ) be a solution for (A,U,T).Then by assumptions (A1), (A3) and (8) it follows:

$$A'U'E - U'E\Lambda - T'E = J(AUE - UE\Lambda - TE) = 0 \tag{11}$$

Thus (E,Λ) is a solution for (A',U',T') (9). Assume that (F,Θ) is another solution for (A',U',T'). Then by (A2) and Lemma 1 the pairs (F,Θ) and (E,Λ) are equivalent. By Lemma 2 it follows that (F,Θ) is a solution for (A,U,T).

Problems (8) and (9) have the same form. To find a solution for (8), the two level projection transfers U and the residual $T - AU$ by J and computes a solution of (9) by a GRRP. This is described by the algorithm:

Two-Level-Projection
1) Transfer $U' = JU$
2) Transfer $T' = A'U' + J(T - AU)$
3) Get (E,Λ) **for** (A',U',T') **by a GRRP**

Observe the freedom in choosing A' and J in the above theorem and algorithm. In particular the dimension of A' may be much smaller than the dimension of A. The A' can also be used in relaxations in MG cycles to approximate the eigenvectors U.

The Multigrid-Projection (MGP) algorithm is a straightforward generalization of the Two-Level-Projection, in which a sequence of problems $A_iU_iE = U_iE\Lambda + T_iE$ for $i = 1,...,l$ is defined using transfers satisfying (A1), (A2), (A3). For $i = l$ one has the initial problem, whereas for $i = 1$ the final one to which the GRRP is applied. The J_i denotes a transfer matrix from level $i+1$ to level i.

MGP
For $i = l-1$ **to** 1 **do:**
1) Transfer $U_i = J_iU_{i+1}$
2) Transfer $T_i = A_iU_i + J_i(T_{i+1} - A_{i+1}U_{i+1})$
end
3) Get (E,Λ) **for** (A_1,U_1,T_1) **by a GRRP**

Corollary (The MGP Theorem)
If in the MGP algorithm, the U_i for $i = 1, \ldots, l$ have the same number of columns and their columns are independent, and if (A_l, U_l, T_l) has solutions, then the sets of solutions of (A_l, U_l, T_l) and of (A_1, U_1, T_1) coincide, and all solutions are equivalent.
The proof goes by induction by the number of levels l using the Two Level Projection Theorem.

Remarks Assume that in the two level projection the matrices A, A', U have the dimensions $n \times n$, $m \times m$, $n \times q$, and that $n \gg m \geq q$. The work required to compute a pair (E, Λ) by a GRRP on the initial level is of order $O(q^2 n C_A)$ due to the scalar products ($U^T AU$ and $U^T U$), where nC_A is the amount of work required to multiply A with a vector, (the diagonalization process, of order q^3, is considered negligible in comparison with $q^2 n$). The amount of work to obtain a pair (E, Λ) by the two level projection is of order $O(qnC_A + qnC_J + q^2 mC_{A'})$, which may be regarded in many cases as $O(qnC_A)$. This may suggest a reduction of work from order $q^2 n$ to qn. One is interested not only in E and Λ but also in the separated solution UE. To compute UE alone requires $q^2 n$ operations, which is the order of work of a fine level GRRP. Instead of the expensive direct computation of UE, one may use other levels and the computed Λ to approximate the q vectors of UE efficiently. Moreover, this approach allows one to improve the accuracy of the eigenspace spanned by the columns of U.

5 Multigrid Combined Cycles and FMG Algorithms

This section presents an MG cycle which consists of a combination of the MGP with the usual MG cycles. The combined cycles are incorporated in an FMG algorithm. This algorithm is used to compute both the eigenvalues Λ and the eigenvectors U. For simplicity assume first that there are only two levels. The notations from the previous section are used. Consider the single level subspace iteration type algorithm [11], [13], used to obtain the largest eigenvalues Λ and the corresponding eigenvectors U of the problem:

$$AU = U\Lambda \tag{12}$$

Subspace-Iteration

1) Relax $AU = U\Lambda$

2) Update (U, Λ) **by** (E, Λ) **using a GRRP**

The steps 1) and 2) may be improved using another level. Assume next $T = 0$. At step 1) the relaxation can be replaced by a Two-Level-Solver-Cycle, having the same transfers as in the MGP. The approximate solutions U are corrected with U' by step 5), where I is a given matrix, e.g., $I = J^T$.

Two-Level-Solver-Cycle

1) Relax $AU = U\Lambda + T$

2) Transfer $U' = JU$

3) Transfer $T' = A'U' + J(T - AU)$

4) Relax $A'U' = U'\Lambda + T'$

5) Correct: $U = U + I(U' - JU)$

6) Relax $AU = U\Lambda + T$

Step 2) in the Subspace-Iteration algorithm can be replaced by a Two-Level-Projection. The Two-Level-Solver-Cycle can be combined with the Two-Level-Projection in a Two-Level-Combined-Cycle:

Two-Level-Combined-Cycle

1) Relax $AU = U\Lambda + T$

2) Transfer $U' = JU$,

3) Transfer $T' = A'U' + J(T - AU)$

4) Get (E, Λ') **for** (A', U', T') **by a GRRP**

5) Backrotate (E, Λ')

6) Separate $U' = U'E, \quad T' = TE, \quad \Lambda = \Lambda'$

7) Relax $A'U' = U'\Lambda + T'$

8) Correct $U = U + I(U' - JU)$

9) Relax $AU = U\Lambda + T$

Denote by I_i a transfer matrix from level $i - 1$ to level i. The Two-Level-Combined-Cycle can be directly extended to an MG Cycle:

MG-Combined-Cycle (l)

For $i = l$ **to** 2 **do:**

1) Relax $A_iU_i = U_i\Lambda + T_i$

2) Transfer $U_{i-1} = J_{i-1}U_i$

3) Transfer $T_{i-1} = A_{i-1}U_{i-1} + J_{i-1}(T_i - A_iU_i)$

end

4) Get (E, Λ') **for** (A_1, U_1, T_1) **by a GRRP**

5) Backrotate (E, Λ')

6) Separate $U_1 = U_1E, \quad T_1 = T_1E, \quad \Lambda = \Lambda'$

7) Relax $A_1U_1 = U_1\Lambda + T_1$

For $i = 2$ **to** l **do:**

8) Correct $U_i = U_i + I_i(U_{i-1} - J_{i-1}U_i)$

9) Relax $A_iU_i = U_i\Lambda + T_i$

end

An FMG algorithm can be defined by:

FMG (l)

For $i = 1$ **to** l **do:**

If $i = 1$ **solve the problem by Subspace-Iterations**

Else

1) Interpolate $U_i = I_iU_{i-1}$, **set** $T_i = 0$

2) Do few MG-Combined-Cycles (i)

Endif

The transfers at steps 3) and 8) in the MG-Combined-Cycle are FAS type transfers [6]. Usually the FMG (from Full MG, [6]) starts by approximating the solution (U_1, Λ) on a coarse level, $i = 1$, then interpolates the solution to successively finer levels. On each level the solution (U_i, Λ) is improved by a few MG cycles. In the general algebraic setting no assumption is done on the levels being coarse or fine.

6 The Fixed Point Theorem for MG Cycles

Usually relaxations do not change the exact solutions of (13):

$$AU - U\Lambda = T \tag{13}$$

Examples of such relaxations are:

$$U_{n+1} = U_n + M(T - AU_n - U_n\Lambda) \tag{14}$$

e.g., Jacobi type or relaxations which have locally the above form, e.g., Gauss-Seidel or Kaczmarz. Power iterations change the exact solutions if they are not coupled with normalizations. One may consider relaxations which are more general procedures, e.g., including rescalings, orthogonalizations, or projections in clusters. The next theorem assumes that: 1) an exact solution U of (13) has linearly independent columns and the eigenvalues in Λ are sorted; 2) the transfer J in the Two-Level-Combined-Cycle maps the solution U into $U' = JU$ with linearly independent columns and with the same number of columns; 3) on both levels, the relaxations don't change the exact solutions of the respective levels. These are natural assumptions met generally by MG procedures. For simplicity, as in MG algorithms, in the Two-Level-Combined-Cycle and in the MG-Combined-Cycle, the solutions of the problem on the level on which the cycles start, i.e., the (U, Λ) respective (U_l, Λ_l), will be called fine level solutions.

Theorem (The Two-Level Fixed Point Theorem)

The Two-Level-Combined-Cycle has as fixed points the exact fine level solutions.

Proof Suppose that (U, Λ) is a solution of (13). The relaxation in 1) in the Two-Level-Combined-Cycle, does not change (U, Λ). Due to 2) and 3) the transfers of the fine level solutions U, Λ are coarse level solutions, U', Λ of

$$A'U' - U'\Lambda = T'. \tag{15}$$

The result of the projection and backrotation is (Id, Λ), by Lemma 3. Thus step 6) leaves (U', Λ) unchanged and the same holds for the relaxation 7). Since $U' = JU$ initially and U' is not changed by the coarse level steps, the correction 8) leaves U unchanged.

By induction one gets the

Corollary (The MG Fixed Point Theorem)

The MG-Combined-Cycle has as fixed points the exact fine level solutions.

Generally, a fine level solution would not be a fixed point of an MG cycle without backrotations, since the projection may permute the solutions for example. Moreover, a fixed point solution would not be a stable solution without using backrotations, in the case of clustered eigenvalues, (e.g., due to relaxations and projections). This is very important for the convergence and the robustness

of the algorithms. Other MG cycles can be defined where the GRRP and the backrotations are performed on any intermediate level. It is easy to see that the above theorem holds for these cycles too.

7 Computational Results

The next examples illustrate the efficiency of the FMG and MG cycles for a Schrödinger eigenvalue problem:

$$(\Delta - V)u = \lambda u \tag{16}$$

discretized in a box with the edge $a = 2\pi/10$. Periodic boundary conditions are assumed. The problems are in 2-D. In the notation $k - FMG - V(\nu_1, \nu_2)$, the constant k is the number of MG cycles performed on each level; the ν_1, (ν_2) is the number of relaxations performed in the MG cycle, on each level in the path from fine to coarse, (coarse to fine). Such a V cycle will be denoted $V(\nu_1, \nu_2)$.

The relaxation is Red-Black Gauss-Seidel. The coarsest grid has 4×4 points. The projection is performed on level 1, the coarsest grid. On the second level two relaxations are performed, showing in many tests a cheap and substantial improvement of the V cycles convergence rate. The coarse level linear systems were solved exactly, using the eigenvalues obtained by the projection, this not being necessary. The eigenvalues are updated by Rayleigh Quotients after the cubic interpolation of the solutions to the current finest level during the FMG. The eigenvectors are normalized to 1 on the current finest levels, and to the FAS norm on the coarsest level, after projection.

The results in Table 1 show that a second order scheme was obtained by an 1-FMG-V(1,1) algorithm. The potential was $V(x, y) = 2 + 0.1sin(10x + 10y)$, chosen to produce a splitting of the first cluster of four equal eigenvalues into two degenerate clusters with very close eigenvalues. Observe the first 3 equal digits of the close clustered eigenvalues and the 13 equal digits of the degenerate eigenvalues, on all levels. On level 5, 8 cycles were performed to show the constant convergence rate per cycle of 1/10 (see cycles 3 and 8 where the convergence rate is exactly 1/10). The eigenvectors came out precisely orthogonal, even in the cluster and degenerate eigenspaces, although the projection was performed only on the coarsest level, (see the scalar products of order 10^{-13} of the eigenvectors on level 5, in Table 2). In the first V cycle on all levels, a convergence rate of about 1/100 was obtained. The next example, see Table 3, shows that without the backrotations the eigenvectors fail to converge and the residuals remain stuck at order 1 on all levels except the coarsest level. A 5-FMG-V(1,1) algorithm was used with no backrotations (but sorting the eigenvalues, which is a first step in backrotations). Even on level 2, the algorithm gets stuck. The eigenvectors just keep rotating in the cluster, rendering the coarse level corrections ineffective. A few observations: 1) The cluster is well separated from the first eigenvalue due to the relaxation and

coarse level approximations of eigenspaces. Two clusters also may become well separated. 2) The eigenvalues are accurately approximated (compare with Table 1) showing that the eigenspace is well approximated. A single MG projection at the end may be sufficient to separate the degenerate eigenspaces. 3) Orthogonality of eigenvectors is also lost even for the eigenvectors with different eigenvalues.

Table 1
1-FMG-V(1,1), 2-D, 5-EV, 5-LEV

cycle	vector	first res	last res	eigenvalue
L E V E L 1				
10	1	0.28E-13	0.89E-13	-0.19999866430865E+01
	2	0.84E-13	0.73E-13	-0.83024036274094E+02
	3	0.56E-13	0.45E-13	-0.83024036274094E+02
	4	0.11E-12	0.10E-12	-0.83089844196732E+02
	5	0.64E-13	0.52E-13	-0.83089844196733E+02
L E V E L 2				
1	1	0.21E-01	0.67E-03	-0.19999793397309E+01
	2	0.30E+02	0.35E+00	-0.97190595196867E+02
	3	0.30E+02	0.35E+00	-0.97190595196868E+02
	4	0.30E+02	0.35E+00	-0.97281226562455E+02
	5	0.30E+02	0.35E+00	-0.97281226562455E+02
L E V E L 3				
1	1	0.64E-02	0.42E-03	-0.19999761538938E+01
	2	0.11E+02	0.50E-01	-0.10069661549360E+03
	3	0.11E+02	0.50E-01	-0.10069661549360E+03
	4	0.11E+02	0.50E-01	-0.10079424742753E+03
	5	0.11E+02	0.50E-01	-0.10079424742753E+03
L E V E L 4				
1	1	0.18E-02	0.13E-03	-0.19999752449715E+01
	2	0.30E+01	0.43E-02	-0.10162979203934E+03
	3	0.30E+01	0.43E-02	-0.10162979203934E+03
	4	0.30E+01	0.43E-02	-0.10172931140738E+03
	5	0.30E+01	0.43E-02	-0.10172931140738E+03
L E V E L 5				
1	1	0.46E-03	0.36E-04	-0.19999750026388E+01
	2	0.76E+00	0.40E-03	-0.10186970728937E+03
	3	0.76E+00	0.40E-03	-0.10186970728937E+03
	4	0.76E+00	0.40E-03	-0.10196970729590E+03
	5	0.76E+00	0.40E-03	-0.10196970729590E+03
3	1	0.35E-05	0.33E-06	-0.19999749801202E+01
	2	0.27E-04	0.26E-05	-0.10186970049930E+03
	3	0.27E-04	0.26E-05	-0.10186970049930E+03
	4	0.27E-04	0.26E-05	-0.10196970049780E+03
	5	0.27E-04	0.26E-05	-0.10196970049780E+03
8	1	0.97E-11	0.26E-10	-0.19999749799142E+01
	2	0.29E-09	0.31E-10	-0.10186970048459E+03
	3	0.29E-09	0.31E-10	-0.10186970048459E+03
	4	0.29E-09	0.31E-10	-0.10196970048302E+03
	5	0.29E-09	0.31E-10	-0.10196970048302E+03

Table 2
Scalar products between eigenvectors on level 5

Vector 1	Vector 2	Scalar Product
1	1	0.10E+01
1	2	-0.19E-13
1	3	-0.16E-14
1	4	-0.19E-13
1	5	0.43E-14
2	2	0.10E+01
2	3	0.60E-14
2	4	0.54E-13
2	5	0.12E-12
3	3	0.10E+01
3	4	-0.17E-12
3	5	0.24E-12
4	4	0.10E+01
4	5	0.63E-14
5	5	0.10E+01

Table 3
5 -FMG-V(1,1), 2-D, 5-EV, 5-LEV,

cycle	vector	first res	last res	eigenvalue
		L E V E L 1		
10	1	0.28E-13	0.89E-13	-0.19999866430865E+01
	2	0.84E-13	0.73E-13	-0.83024036274094E+02
	3	0.56E-13	0.45E-13	-0.83024036274094E+02
	4	0.11E-12	0.10E-12	-0.83089844196732E+02
	5	0.64E-13	0.52E-13	-0.83089844196733E+02
		L E V E L 2		
1	1	0.21E-01	0.67E-03	-0.19999793397309E+01
	2	0.30E+02	0.17E+01	-0.97190595196867E+02
	3	0.30E+02	0.17E+01	-0.97190595196868E+02
	4	0.30E+02	0.11E+01	-0.97281226562455E+02
	5	0.30E+02	0.11E+01	-0.97281226562455E+02
5	1	0.78E-08	0.16E-09	-0.19999785210023E+01
	2	0.16E+01	0.17E+01	-0.96920257120980E+02
	3	0.16E+01	0.17E+01	-0.96920257120980E+02
	4	0.35E+01	0.59E+01	-0.97015473866025E+02
	5	0.35E+01	0.59E+01	-0.97015473866025E+02
		L E V E L 3		
1	1	0.61E-02	0.36E-03	-0.19999761161604E+01
	2	0.11E+02	0.14E+01	-0.10069706261680E+03
	3	0.11E+02	0.14E+01	-0.10069706261680E+03
	4	0.14E+02	0.25E+01	-0.10080438766161E+03
	5	0.14E+02	0.25E+01	-0.10080438766161E+03
		L E V E L 4		
1	1	0.16E-02	0.94E-04	-0.19999752199391E+01
	2	0.34E+01	0.77E+00	-0.10162995680857E+03
	3	0.34E+01	0.77E+00	-0.10162995680857E+03
	4	0.31E+01	0.25E+01	-0.10172932150242E+03
	5	0.31E+01	0.25E+01	-0.10172932150242E+03
		L E V E L 5		
1	1	0.40E-03	0.23E-04	-0.19999749946246E+01
	2	0.13E+01	0.47E+00	-0.10186978359913E+03
	3	0.13E+01	0.47E+00	-0.10186978359913E+03
	4	0.27E+01	0.41E+00	-0.10197015834270E+03
	5	0.27E+01	0.41E+00	-0.10197015834270E+03
5	1	0.21E-07	0.20E-08	-0.19999749799150E+01
	2	0.54E+00	0.27E+00	-0.10186974707079E+03
	3	0.54E+00	0.27E+00	-0.10186974707079E+03
	4	0.35E+01	0.31E+01	-0.10197059714773E+03
	5	0.35E+01	0.31E+01	-0.10197059714773E+03

8 Conclusions

Proofs for the algebraic multigrid projection theorem and for the MG fixed point theorem for cycles coupling relaxations, projections, backrotations and inter-level transfers, are presented. Important computational work savings can be obtained, for example, when the transferred problem has significantly smaller size than the initial problem. The MGP and backrotations can be coupled in MG cycles, in which the fine level exact solutions are not changed by the MG cycle. Thus fine level solutions are fixed points for the MG cycles. This is important for the efficiency and robustness of the MG eigenvalue algorithms. Computational examples which show the failure of the algorithms in absence of backrotations, and the difficulties to be overcome by the MG cycles are presented. For the same problems, the efficiency of the MGP and backrotations is illustrated. These problems present special computational difficulties such as very close and equal eigenvalues.

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13. ABSTRACT *(Maximum 200 words)*
The proofs of the theorem for the algebraic multigrid projection (MGP) for eigenvalue problems, and of the multigrid fixed point theorem for multigrid cycles combining MGP with backrotations, are presented. The MGP and the backrotations are central eigenvector separation techniques for multigrid eigenvalue algorithms. They allow computation on coarse levels of eigenvalues of a given eigenvalue problem, and are efficient tools in the computation of eigenvectors.

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