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NUMERICAL SOLUTION OF A SINGULAR INTEGRAL EQUATION ARISING FROM A SEQUENTIAL PROBABILITY RATIO TEST

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Integral equations arise naturally in problems in engineering and physics, and their numerical solutions can be obtained directly using popular commercial software packages. The same cannot be said of singular integral equations. This report proposes two numerical methods for solving a singular integral equation that arises in the computation of critical values for a sequential probability ratio test.

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NUMERICAL SOLUTION OF A SINGULAR INTEGRAL EQUATION ARISING FROM A SEQUENTIAL PROBABILITY RATIO TEST

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STATEMENT OF PROBLEM

In a recently derived sequential probability ratio test (SPRT) [1] to discriminate between the two simple hypotheses: $H_0 : \sigma = \sigma_0$ and $H_1 : \sigma = \sigma_1$ $(0 < \sigma_0 < \sigma_1)$ the relevant probability distribution is given by

$$g_d(z;\sigma) = \begin{cases} \frac{1}{\sqrt{2\pi c}}\frac{1}{\sqrt{z+d}} e^{-\frac{z+d}{2c}}, & \text{for } z > -d \\ 0, & \text{otherwise.} \end{cases}$$

where $d = \ln \frac{\sigma_1}{\sigma_0} > 0$, and $c = c(\sigma) = \frac{\sigma^2}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) > 0$. The operating characteristic (OC), q, satisfies the integral equation [2]

$$q(-b;k,\sigma) = \int_{-\infty}^{b} g_d(z;\sigma)\,dz + \int_0^k g_d(z+b;\sigma)q(z;k,\sigma)\,dz, \qquad 0 < -b < k$$

which in this case, with $y = -b$, reduces to

$$q(y;k,\sigma) = f_0(y;d,c) + c_1 \int_{k_0(y,d)}^{k} \frac{1}{\sqrt{z+d-y}} e^{-\frac{z+d-y}{2c}} q(z;k,\sigma)\,dz, \; 0 < y < k \tag{1}$$

where $k_0(y,d) = \max\{0, y-d\}$, $c_1 = c_1(\sigma) = \frac{1}{\sqrt{2\pi c(\sigma)}}$, and

$$f_0(y;d,c) = \begin{cases} 2\Phi\left(\sqrt{\frac{d-y}{c}}\right) - 1, & \text{for } y < d, \\ 0, & y \geq d. \end{cases}$$

Here Φ is the distribution function of the standard normal distribution. The integral is singular for those values of y satisfying $k > y \geq d$.

For given values of α, β $(0 < \alpha, \beta < 1), \sigma_0$, and σ_1 $(0 < \sigma_0 < \sigma_1)$, the problem is to find (unique) values (k^*, b^*) of (k, b) for which $q(-b^*; k^*, \sigma_0) = 1 - \alpha$, and $q(-b^*; k^*, \sigma_1) = \beta$.

METHOD OF SOLUTION

For given values of α, β $(0 < \alpha, \beta < 1), \sigma_0$, and σ_1 $(0 < \sigma_0 < \sigma_1)$, let $b = b_{\alpha,\sigma_0}(k)$ denote the solution of $q(-b; k, \sigma_0) = 1 - \alpha$, and $b = b_{\beta,\sigma_1}(k)$ the solution of $q(-b, k, \sigma_1) = \beta$. Then k^* will merely be the solution of $b_{\alpha,\sigma_0}(k) = b_{\beta,\sigma_1}(k)$. Once k^* is found, b^* is immediately given by $b^* = b_{\alpha,\sigma_0}(k^*)$. Thus, the basic problem is to solve Equation (1) for $q(y)$ for each given k and σ.

By defining $\tilde{q}$ as $qe^{-\frac{y}{2c}}$, $\tilde{f}_0$ as $f_0 e^{-\frac{y}{2c}}$, and $c_2 = c_2(\sigma)$ as $c_1(\sigma)e^{-\frac{d}{2c(\sigma)}}$, Equation (1) becomes

$$\tilde{q}(y; k, \sigma) = \tilde{f}_0(y; d, c) + c_2 \int_{k_0(y,d)}^{k} \frac{\tilde{q}(z; k, \sigma)}{\sqrt{z + d - y}} \, dz, \; 0 < y < k \tag{2}$$

With the change of variable $s = k - z$ in the integral, Equation (2) becomes

$$\tilde{q}(y; k, \sigma) = \tilde{f}_0(y; d, c) + c_2 \int_{0}^{k - k_0(y,d)} \frac{\tilde{q}(k - s; k, \sigma)}{\sqrt{k + d - y - s}} \, ds, \; 0 < y < k \tag{3}$$

Or,

$$\tilde{q}(y; k, \sigma) = \tilde{f}_0(y; d, c) + c_2 \begin{cases} \int_0^k \frac{\tilde{q}(k-s;k,\sigma)}{\sqrt{k+d-y-s}} \, ds, & 0 < y < \min\{d, k\} \\ \int_0^{k+d-y} \frac{\tilde{q}(k-s;k,\sigma)}{\sqrt{k+d-y-s}} \, ds, & d \leq y < k \end{cases} \tag{4}$$

This integral equation is solved numerically. We have used two different methods. The first method solves the problem by interpolating the numerator in Equation (4) on each appropriate sub-interval by a quadratic polynomial and then integrating the resulting integrand to obtain a set of weights for the numerical scheme. Details of this method can be found in the Appendix. This method is relatively easy to implement and does not rely on any special software routines other than a solver for a system of linear equations.

The second method makes indirect use of special software routines available commercially. Here the sought function $\tilde{q}(y)$ is discretized on the interval $[0, k]$ to yield the approximations $\hat{q}_n$ of $\tilde{q}(nh), (n = 0, 1, \ldots, N)$, where h is the (uniform) mesh size and $k = Nh$. Because the integral equation (4) is only (weakly) singular when $d \leq y < k$, the manner in which it is discretized depends on y. For $y < \min\{d, k\}$, and using Adams method and Backward Differentiation Formulae method [3], one obtains a set of $N_1 (= \min\{n_d = d/h, N\})$ equations of the form

$$\hat{q}_i = \tilde{f}_0(ih) + c_2 h \{ \sum_{j=0}^{p_1 - 1} \frac{W_{N,j} \hat{q}_{N-j}}{\sqrt{(N - i + n_d - j)h}} + \sum_{j=p_1}^{N} \frac{\omega_{N-j} \hat{q}_{N-j}}{\sqrt{(N - i + n_d - j)h}} \} \quad (5)$$

for $i = 0, 1, \ldots, N_1$. Here W and ω are the appropriate starting and convolution weights of the method respectively [3], and p_1 is the order of the method which we took to be 4.

For $y \geq d$ and using a different Backward Differentiation Formulae method [4], one obtains a set of $N - N_1$ equations of the form

$$\hat{q}_i = \tilde{f}_0(ih) + c_2 \sqrt{\pi h} \{ \sum_{j=0}^{2p_2 - 2} W^a_{N+n_d-i,j} \hat{q}_{N-j} + \sum_{j=2p_2-1}^{N+n_d-i} \omega^a_{N+n_d-i-j} \hat{q}_{N-j} \} \quad (6)$$

for $i = N_1 + 1, \ldots, N$. Here W^a and ω^a are appropriate fractional starting and fractional convolution weights of the method respectively [4], and p_2 is the order of the BDF method which we took to be 4. The weights in both Equation (5) and Equation (6) are obtained from appropriate NAG routines. The two sets of equations constitute a set of $N + 1$ linear equations in the $N+1$ unknowns $\{\hat{q}_i\}_{i=0}^{N}$. It can be solved using any number of standard linear equation solvers. We used one of IMSL's routines [5].

RESULTS

On the test problem (to be described) as well as the actual problem, the two numerical methods have yielded similar results. Thus we will not distinguish the two in the following discussion.

We tested the numerical method above on a simple problem whose analytical solution is known. The problem consists of solving Equation 1 with a new f_0 given by

$$f_0(y; d, c) = e^{\frac{y}{2c}}\{1 - 2c_2(\sqrt{k + d - y} - \sqrt{\max\{0, y - d\} - (y - d)})\} \quad (7)$$

The analytical solution for this problem is simply

$$q(y; k, \sigma) = e^{\frac{y}{2c}},$$

or,

$$\tilde{q}(y; k, \sigma) \equiv 1$$

The numerical result of solving this simple problem is displayed in Figure 1. Here $\tilde{q}$ is plotted against y. In this particular test, the parameters are $k = 4, \sigma_0 = 1, \sigma_1 = 2$. $\tilde{q}(y; k, \sigma)$ was calculated twice, once for $\sigma = 1$ and once for $\sigma = 2$. Both of these are shown in Figure 1. As the analytical solution of $\tilde{q}$ does not depend on σ, they are exactly the same, and represent faithfully the anayltyical solution $\tilde{q} \equiv 1$. Note, however, that if we had plotted $q = \tilde{q}e^{\frac{y}{2c}} = e^{\frac{y}{2c}}$ instead, the two would definitely be different, since $c = c(\sigma)$ is a function of σ.

To illustrate how this method solves the real problem (Equation 1), we pick $\sigma_0 = 1$ in the null hypothesis H_0 and $\sigma_1 = 1.2$ in the alternative hypothesis H_1. Assuming $\alpha = 0.05$ and $\beta = 0.1$, we calculated the curves (in the $k - y$ plane) $q(y; k, \sigma_0) = 1 - \alpha$ and $q(y; k, \sigma_1) = \beta$. These are displayed in Figure 2. These curves intersect at $(k^*, y^*) = (4.75, 2.18)$. Finally, in Figure 3, we plotted $q(y; k^*, \sigma_0)$ and $q(y; k^*, \sigma_1)$. Indeed, when $y = y^*$, $q(y, k^*, \sigma_0) = 0.95$ and $q(y, k^*, \sigma_1) = 0.10$.

CONCLUSION

In this paper, we have proposed two numerical methods to solve a singular integral equation which arose in the study of a certain Sequential Probability Ratio Test. One method uses an easily-implemented scheme and the other method involves casting the singular portion of the problem into a weakly singular equation of Abel type and making use of proven software for this type of equation ([6] as implemented by NAG [4]) to solve the problem. Both methods have yielded similar results when tested (successfully) on a simple problem and when applied to the real problem.

References

[1] Chou, Y. M., Anderson, M., and Samn, S. (In preparation) 1994.

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[3] NAG Fortran Library Routine D05BWF.

[4] NAG Fortran Library Routine D05BYF.

[5] IMSL MATH/LIBRARY, Routine LSARG.

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APPENDIX

A Simple Method for Solving Weakly Singular Integrals

In solving Equation (4), the main difficulty is solving a weakly singular integral of the form

$$\int_a^b \frac{f(x)}{(x-c)^\alpha}\, dx \tag{A.1}$$

where $\alpha < 1$ and $c \le a < b$. For the problem in this paper $\alpha = \frac{1}{2}$. If we interpolate $f(x)$ with a (the) polynomial $P(x) = \sum_{k=0}^n a_k x^k$ of degree n at the points $a = x_0 < x_1 < \cdots < x_n = b$, then the integral in (A.1) can be approximated by

$$\int_a^b \frac{\sum_{k=0}^n a_k x^k}{(x-c)^\alpha}\, dx = \sum_{k=0}^n a_k \int_a^b \frac{x^k}{(x-c)^\alpha}\, dx = \sum_{k=0}^n a_k w_k = \mathbf{w}^T \mathbf{a}, \tag{A.2}$$

where

$$w_k = \int_a^b \frac{x^k}{(x-c)^\alpha}\, dx,$$

$\mathbf{w} = (w_0, \ldots, w_n)^T$, and $\mathbf{a} = (a_0, \ldots, a_n)^T$. It is straight forward to show that

$$w_k = Q_k(b,c) - Q_k(a,c),$$

where for all $y \le x$,

$$Q_k(x,y) = \sum_{i=0}^k \frac{\begin{pmatrix} k \\ i \end{pmatrix}}{i+1-\alpha} y^{k-i}(x-y)^{i+1-\alpha}.$$

Since the polynomial $P(x) = \sum_{k=0}^n a_k x^k$ interpolates $f(x)$ at the $n+1$ points $x_0, \ldots, x_n$, each a_k depends only on $\{x_i\}_{i=0}^n$ and $\{f(x_i)\}_{i=0}^n$. In fact, if $\mathbf{V}$ is the Vandermonde matrix corresponding to $\{x_i\}_{i=0}^n$, then

$$\mathbf{Va} = \mathbf{f}, \tag{A.3}$$

where $\mathbf{f} = (f(x_0), \ldots, f(x_n))^T$. Recall the Vandermonde matrix corresponding to $\{x_i\}_{i=0}^n$ is

$$\mathbf{V} = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}$$

Now by combining Equations (A.2) and (A.3), an approximation to the integral in (A.1) can be written compactly as

$$\mathbf{w}^T\mathbf{V}^{-1}\mathbf{f} \tag{A.4}$$

In solving Equation 4, we have chosen the degree n of the interpolating polynomial $P(x)$ to be 3. We could have easily chosen it to be any other larger integer. By repeatedly using Equation (A.4) on successive sets of $n+1$ mesh points in a chosen partition $\{0 = x_0, x_1, \ldots, x_N = k\}$ of the interval $[0, k]$ (see Equation (4)), we immediately obtain a system of $N+1$ linear equations in the $N+1$ unknowns $\{f(x_0), f(x_1), \ldots, f(x_N)\}$. This system can now be solved by using any standard system of linear equations solver.

Figure 1. Test Results

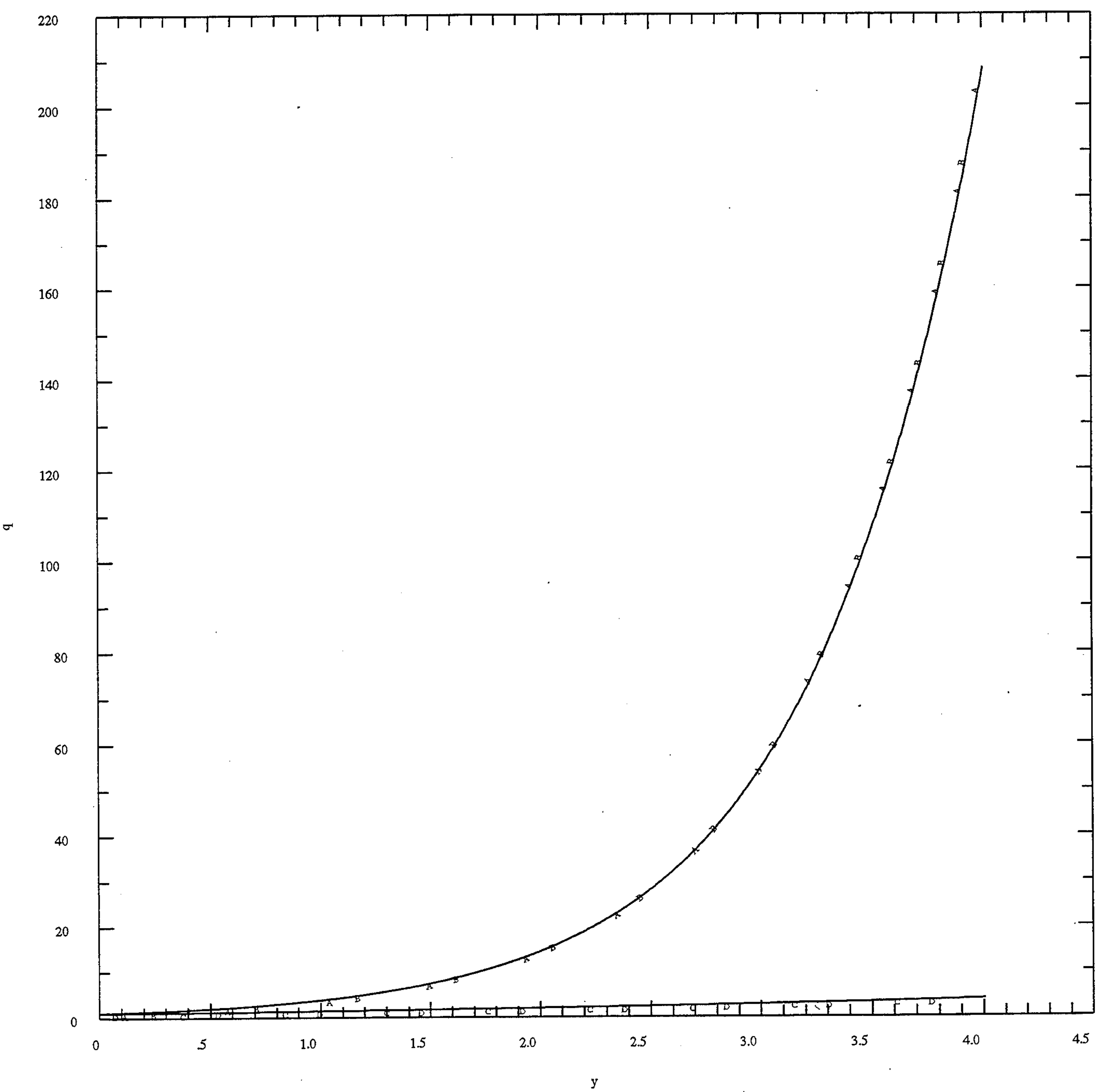

Figure 2. Determination of k and -b

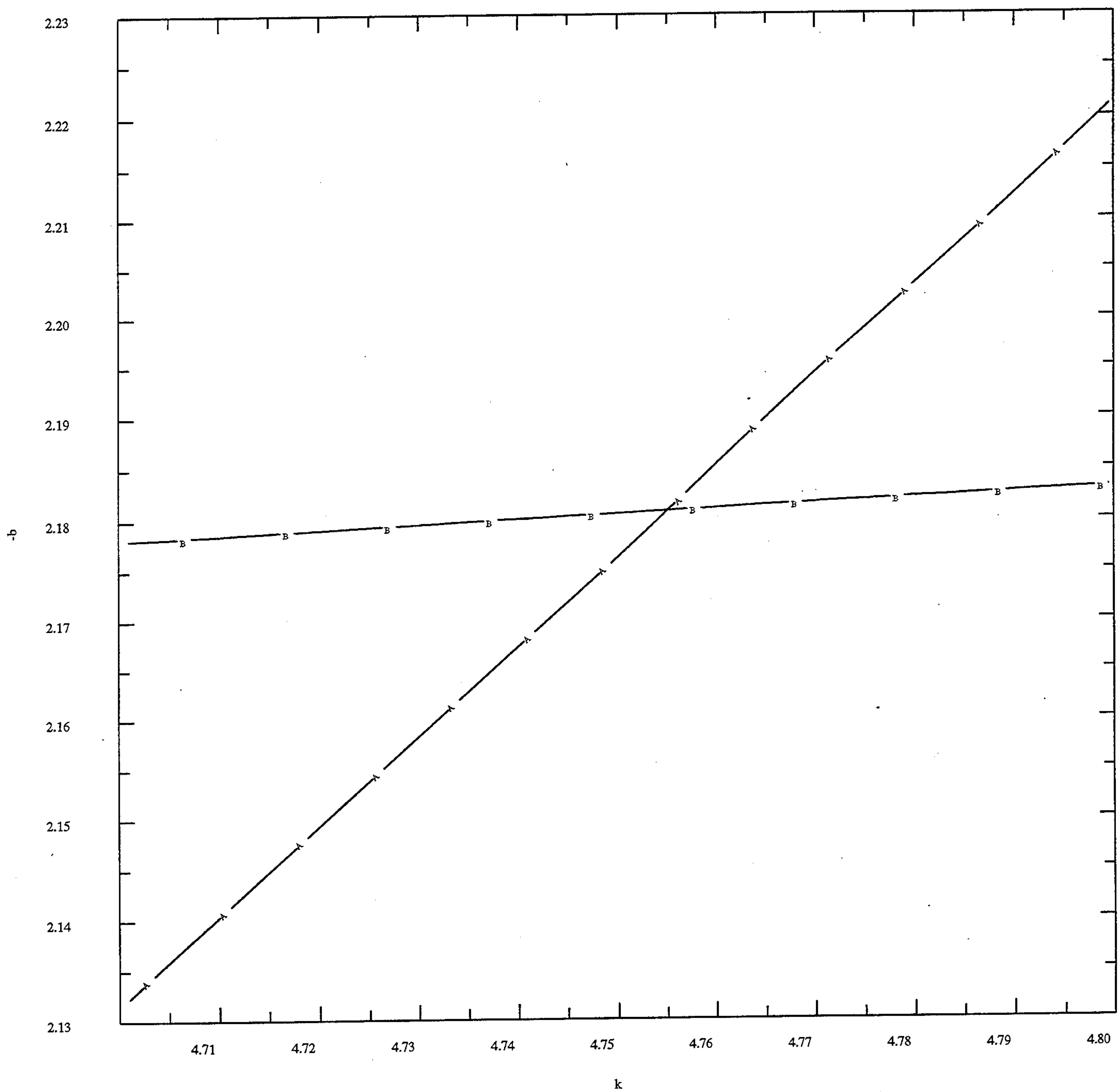

Figure 3. q(y) at k* for two sigmas