DISCUSSION OF M. J. BAYARRI AND M. H. DeGROOT'S "GAINING WEIGHT: A BAYESIAN APPROACH"

by

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GWU/IRRA/Serial TR-87/7
17 June 1987

The George Washington University
School of Engineering and Applied Science
Institute for Reliability and Risk Analysis

Research Sponsored by

Contract N00014-85-K-0202
Project NR 042 372
Office of Naval Research

and

Grant DAAL 03-87-K-0056
U. S. Army Research Office

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Abstract

This is an invited discussion of Professors M. J. Bayarri and M. H. DeGroot's paper entitled "Gaining Weight: A Bayesian Approach." Both the paper and the discussion are to appear in *Bayesian Statistics* 3, the Proceedings of the Third Valencia International Meeting on Bayesian Statistics, held in Altea (Spain), on June 1-5, 1987.

Key Words: Pooling Expert Opinion, Incoherence, Linear opinion pool

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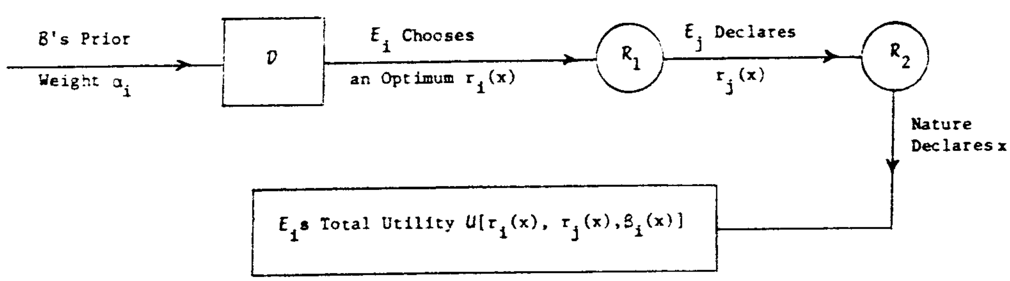
This charming and well written paper addresses an interesting problem, produces elegant results and makes for enjoyable reading. In the sequel, it also advances the state of the art of Bayesian statistics. The issues raised below are intended serve two purposes:

1. To stimulate more work along the avenue of research which Professors Bayarri and DeGroot - henceforth BAD - have initiated, and
2. To argue that the framework for "gaining weight" prescribed here can be used to show that the linear opinion pool - henceforth LOP - leads to *incoherence*.

Regarding 1, one can start off by drawing attention to the criticisms of the LOP [see Genest and Zidek (1986)] and suggest as an alternative, a consideration of the logarithmic opinion pool. The latter has the advantage of being "externally Bayesian" [which Lindley (1985) dismisses as an "adhockery"] but suffers from the disadvantage is that it may lead to cumbersome results.

More substantive avenues of investigation emanate from some extensions of the formulation of this paper. To see these, focus attention on Equation (1.2) in which $\beta_i(x)$, the posterior weight given by a decision maker, the "boss" ($\mathfrak{B}$) to the i-th expert $\mathcal{E}_i$, is specified. Note that $\beta_i(x)$ depends on α_i, the prior weight given by $\mathfrak{B}$ to $\mathcal{E}_i$, and $r_i(x)$, the experts report to $\mathfrak{B}$. BAD assume that $\mathcal{E}_i$'s colleagues are naive and that $\mathcal{E}_i$'s reward is <u>solely</u> based on the magnitude of $\beta_i(x)$. Under the above set-up, $\mathcal{E}_i$ aims to maximize $\beta_i(x)$, and to achieve this $\mathcal{E}_i$ may resort to dishonesty. Observe that $\mathcal{E}_i$'s behavior is influenced solely by the α_i's and the competence of $\mathcal{E}_i$'s colleagues. A more realistic scenario would be one in which $\mathcal{E}_i$ is to be rewarded based upon <u>both,</u> the quality of prediction - as measured by $r_i(x)$ - and also $\beta_i(x)$. For such a generalization, the conditions under which honesty of $\mathcal{E}_i$ is always the best policy needs to be investigated. A formal mechanism for addressing the above is facilitated via the decision tree of Figure 1 in which $\mathfrak{D}$ denotes $\mathcal{E}_i$'s decision node, $\mathcal{R}_1$, denotes a random node pertaining to $r_j(x)$ — the report of $\mathcal{E}_i$'s sole colleague $\mathcal{E}_j$, and $\mathcal{R}_2$ denotes a random node pertaining to the state of nature X taking values x. If $\mathcal{U}[r_i(x), r_j(x), \beta_i(x)]$ denotes $\mathcal{E}_i$'s total utility when $X = x$, $\mathcal{E}_\bullet$ reports $r_\bullet(x)$, $\bullet = i, j$, and the $\mathfrak{B}$ assigns a posterior weight $\beta_i(x)$ to $\mathcal{E}_i$, then $\mathcal{E}_i$ will choose that $r_i(x)$ which maximizes the expectation of $\mathcal{U}$ with respect to $\mathcal{E}_i$'s honest distribution of X (conditional on $r_j(x)$ if $r_j(\cdot)$ is declared by $\mathcal{E}_j$ or averaged over the distribution of $r_j(\cdot)$ — as perceived by $\mathcal{E}_j$).



<u>Figure 1.</u> $\mathcal{E}_i$'s Decision Tree for an Optimal Choice of $r_i(x)$.

Recall that the set-up of BAD assumes that $\mathcal{E}_i$'s colleagues are naive, in the sense that they are not interested in "gaining weight"; thus their reports to the $\mathfrak{B}$ are indeed their honest opinions. If one were to consider the case of a single colleague, say $\mathcal{E}_j$, and assume that $\mathcal{E}_j$ is in competition with $\mathcal{E}_i$ to "gain weight," then a two-person zero sum game would result, for now $\mathcal{E}_j$ also has an incentive to be dishonest. It would be interesting to see if a solution to this two person zero sum game would result in the conclusion that honesty on the part of both $\mathcal{E}_i$ and $\mathcal{E}_j$ would be the best policy.

To see why the framework for "gaining weight" given here provides a vehicle for arguing that the LOP leads to incoherence, consider Equation (2.1). This specifies that for given values of α, s, π and r

$$E(\beta) = \pi \frac{\alpha r}{\alpha r + \bar{\alpha} s} + \bar{\pi} \frac{\alpha \bar{r}}{\alpha \bar{r} + \bar{\alpha}\, \bar{s}}$$

and that the optimal choice of r, obtained by a maximization of $E(\beta)$ is

$$r - \frac{1}{\alpha} \frac{\sqrt{s\pi}}{\sqrt{s\bar{\pi}} + \sqrt{\bar{s}\bar{\pi}}} - \frac{\bar{\alpha}}{\alpha} s, \qquad \bigstar$$

provided that $0 \leq r \leq 1$. The recommendation of BAD that when $\bigstar < (>) \ 0 \ (1)$, the optimal choice of r is $r = 0(1)$, is discomforting. It is an external intervention, analogous to that commonly done by frequentists who set negative estimates of variances and densities arbitrarily equal to zero. In a coherent system involving probability calculations there should be no need for interventions—the probability calculus must automatically lead to admissible answers. A detailed investigation of the behavior of $E(\beta)$ is therefore called for. Specifically, one needs to investigate conditions which ensure that the value of r which gives the global maximum of $E(\beta)$ lies between 0 and 1. Some analysis shows that there are five possible scenarios, three of which are shown in Figure 2. Of these, it is only Scenario A which results in a unique maximizing r which is between 0 and 1. Scenario D, with $0 < A^* < 1$ and $B^* > 1$, where $A^* = -(\bar{\alpha}/\alpha)s$ and $B^* = 1 + (\bar{\alpha}/\alpha)(1\text{-}s)$, is the dual of Scenario C, and is omitted. Both Scenarios C and D yield one global maximum outside [0, 1] and therefore lead to inadmissibility. Omitted also from Figure 2 is Scenario E in which $E(\beta)$ attains a global maximum at A^* and B^*, both of which lie outside the interval [0, 1]. Scenario B yields an admissible answer but it is not unique.

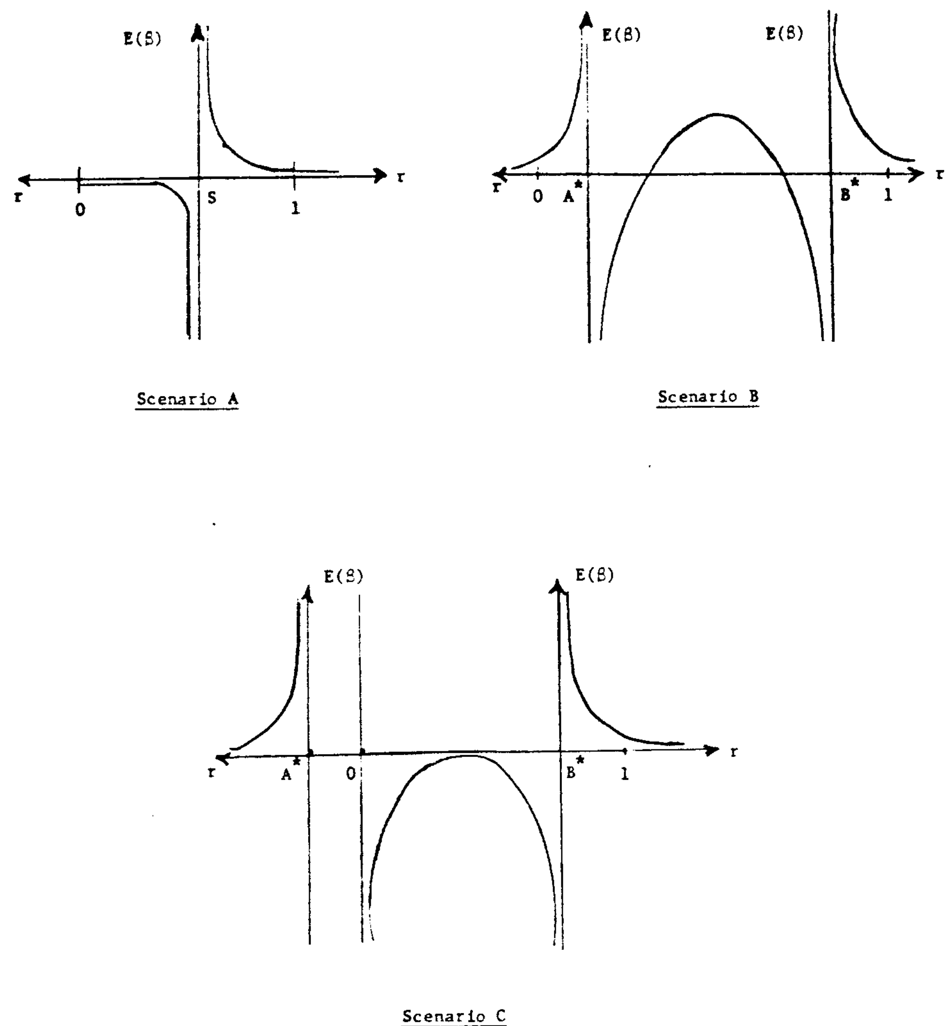


Figure 2. Behavior of the Global Maxima of $E(\beta)$.

Values of α and $\bar{\alpha}$ which lead to each of the above scenarios, are shown in Figure 3.

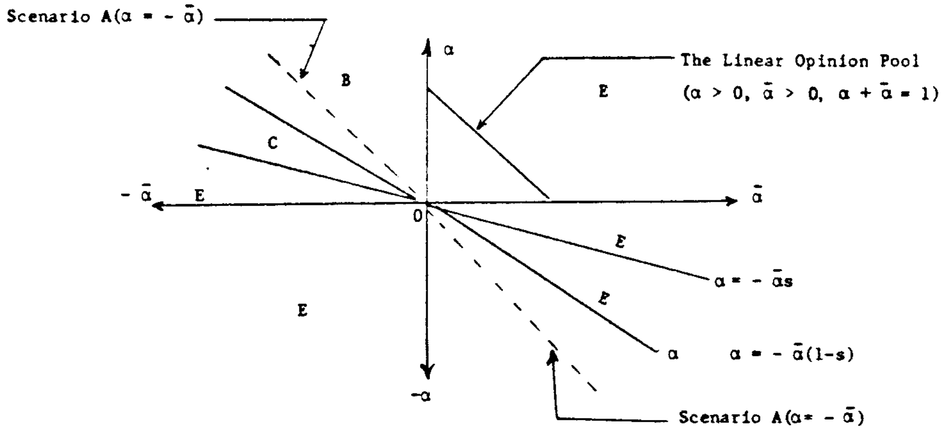


Figure 3. Range of Values of α and $\bar{\alpha}$ leading to Scenarios A, B, C, and E.

It is apparent from Scenario A of Figures 2 and 3, that to obtain a unique value of r which maximizes $E(\beta)$ and which is admissible, α must equal $-\bar{\alpha}$. Furthermore, the maximizing value of r is s^+. Thus in an opinion pool with $\alpha = -\bar{\alpha}$, to maximize one's posterior weight, one must report one's colleagues answer plus a tad more.

In order to be ensured that one can obtain at least one admissible global maximizing value of r, α and $\bar{\alpha}$ must have, as a necessary (but not sufficient) condition opposite signs. This observation supports the result of Genest and Schervish (1985) who prescribe the conditions for the LOP to have a Bayesian justification. Actually Scenario A with its requirement that $\alpha = -\bar{\alpha}$ is a stronger statement than that of above authors.

The LOP requires that $\alpha > 0$, $\bar{\alpha} > 0$ and $\alpha + \bar{\alpha} = 1$; however, these conditions lead us to Scenario E (see the upper right hand quadrant of Figure 3) which implies a violation of an axiom of probability.

By way of a few closing remarks, it is apparent that BAD have unveiled an avenue of research in Bayesian statistics which should generate a flood of new papers around their ideas. It is common to see research in Bayesian statistics center around established themes such as prior to posterior manipulations, approximations, sensitivities, robustification, computations, hierarchicalizations, algorithms with fancy names, and other devices which mimic the frequentists arsenal of techniques. In contrast, the problem addressed by Professors Bayarri and DeGroot is novel, stimulates thought and is fun to work upon. The author has enjoyed the opportunity to read and comment on this work and would like to thank the organizing committee for their contribution to his learning by inviting him to attend the conference and serve as a discussant.

ACKNOWLEDGMENTS

The author also acknowledges the efforts of Mr. Khalid Aboura in connection with the development of Figures 2 and 3.

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