Revised: June 4, 1993

CONTROL THEORY METHODS IN PRODUCT QUALITY ASSURANCE: AN ILLUSTRATION OF OPEN LOOP CONTROL

by

Nozer D. Singpurwalla
The George Washington University

Jingxian Chen
Shanghai Second Polytechnic University, and
The George Washington University

and

Prabhakar Murthy
University of Queensland, Australia

DTIC
ELECTE
MAY 0 9 1995
S D
F

GWU/IRRA/TR-89/1
January 10, 1989

This document has been approved for public release and sale; its distribution is unlimited.

The George Washington University
School of Engineering and Applied Science
Institute for Reliability and Risk Analysis

Research Sponsored by

Contract N00014-85-K-0202
Project NR 042-372
Office of Naval Research

and

Grant DAAL03-87-K-0056
U.S. Army Research Office

19950505 141

DTIC QUALITY INSPECTED 5

Revised: June 4, 1993

CONTROL THEORY METHODS IN PRODUCT QUALITY ASSURANCE: AN ILLUSTRATION OF OPEN LOOP CONTROL

by

Nozer D. Singpurwalla
The George Washington University, Washington, DC

Jingxian Chen
Shanghai Second Polytechnic University, and
The George Washington University, Washington, DC

and

Prabhakar Murthy
University of Queensland, Australia

Accesion For
NTIS CRA&I ☑
DTIC TAB ☐
Unannounced ☐
Justification
By per Acq. ltr
Distribution /
Availability Codes
Dist | Avail and / or Special
A-1

ABSTRACT

This is the first of a series of papers that we are hoping to write on the use of control theory methods for controlling the quality of manufactured items. Such methods involve decision making under uncertainty for a system whose dynamics are defined by equations that are suggested by physical or plausible laws. The emphasis of our paper is a single stage production system with open loop control. The output quality can be controlled via an optimal scheduling of machine maintenance and/or a switch to better input quality. We advocate a simple structure for undertaking the above task. Our hope is that future work in this area will involve an improvisation of our theme. The key virtue of the proposed approach is the optimality of decisions; its disadvantage is the need to make realistic assessments of prior distributions and loss functions.

Key Words : Decision Theory, Quality Control, Robust Manufacturing, Taguchi Methods, Loss Functions, Optimal Maintenance.

AMS 1991 Subject Classification: 62N10, 49N40, 49J99, 90B30, 90B90

0. Introduction

A *single-stage production system* is one in which the input to the system is the raw material and its output, the finished product. A *multi-stage production system* is one in which the final product is manufactured in several stages with the output of one stage (except for the last) becoming the input for the next stage. Each stage of a multi-stage system can be regarded as a single stage system and thus it is sufficient to focus attention on the latter.

For any single stage system, the output quality, as measured by say fraction non-defective, is assumed to be influenced by only two factors:

i) the input quality, measured in terms of say fraction non-defective, or some other characteristic, such as material property, and

ii) the state of the machine (or the system), such as its age, its maintenance history, its precision, etc.

The state of the machine generally degrades with age and this has an adverse effect on output quality. One way to slow or to arrest the deterioration of the output quality is to control the rate of degradation of the machine; this can be done via preventive maintenance or overhaul. Another option - when available - is to control the input by switching over to superior (and more expensive) input quality. The latter strategy is not without its criticism; nonetheless it is used in practice and currently goes under the guise of robust manufacturing [Taguchi (1986)]. The question of which strategy to invoke and when to invoke it poses a problem in stochastic control theory which in essence boils down to decision making under uncertainty [cf. Lindley (1989), p.301]. The objective of this paper is to propose and to develop a preliminary structure, albeit an elementary one, which can help address the stated problem.

For convenience, we consider a discrete time formulation of the above scenario, and relate the output quality to the input quality and the state of the machine, by a simple relationship. Similarly, we describe the dynamical behavior of the machine - that is, the manner in which its state changes over time - by another simple relationship. In control theory, it is expected that both these relationships have a scientific basis. However, in the absence of knowledge about any physical laws governing the system's behavior, the relationships proposed by us are to be viewed as first degree approximations.

An outline of this paper is as follows:

In Section 1 we give some preliminaries and establish notation. In Section 2 we propose our model for a single production system with either open or closed loop control. In Section 3 we describe a decision theoretic approach under open loop control for addressing the question of optimally scheduling maintenance. In Section 4 we consider the case of optimally scheduling maintenance and/or switching to superior input quality. In Section 5 we describe a numerical example whose main purpose is to demonstrate the computational feasibility of our approach. The paper concludes with Section 6 which comments on the scope of the development here and prospects for its improvisation.

As a final point, we are unaware of any literature which combines in the modeling and control of machine degradation and the optimal scheduling of machine maintenance with respect to quality control, particularly within the context of control theory. The closest that comes to what we have in mind here is the work of Barlow, Durst and Smiriga (1984).

1. Preliminaries

Let T denote the length of the period within which the single stage manufacturing process is studied. Since we consider a discrete time formulation of the process, we divide the interval [0, T) into ℓ subintervals of duration T/ℓ; assume that ℓ is not too large. The items produced in the i-th subinterval are referred to as batch i. Assume that the manufacturing time of one item is small in comparison to T/ℓ, so that for all intents the manufacture of an item is completed in the subinterval that it is commenced.

Let w_k denote the quality of input (e.g. the fraction of good items), for the production of batch k. Let θ_k denote the quality of the output of batch k; i.e. the proportion of non-defective items. Let $0 \leq \phi_k \leq 1$, an unknown parameter, describe the state of the machine during the production of batch k; ϕ_k is the (unobservable) *state of nature* with small values denoting a large degree of

deterioration of the machine, and $\phi_k = 1$ denoting the fact that the machine is in perfect condition. Let θ^* denote the *minimum acceptable* batch quality; whenever $\theta_k < \theta^*$, the production of batch k is considered unacceptable, so that the system should be shut down. Note that the choice of θ^* is itself a decision problem that we do not address here.

Schematically, it is helpful to think of the single stage production quality control system via Figure 1.1.

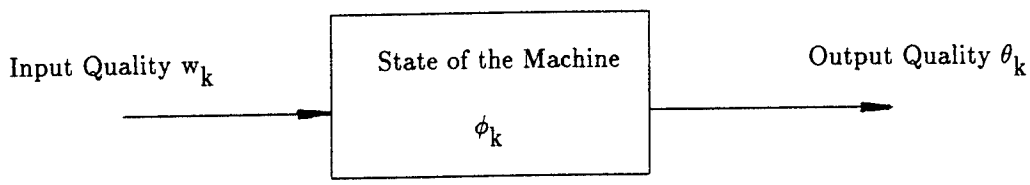

Figure 1.1 Schematic of the Single Stage Production Quality Control System

2. Model Specification

Since the input quality may change stochastically, we describe it by a sequence of random variables $\{w_k\}$, k=1, 2, $\cdots$, ℓ, with each $w_k \in [0, 1]$. If the input quality is to be defined in terms of a characteristic other than proportion non-defective, then w_k may take values in $[0, \infty]$, with larger values of w_k denoting better quality. The elements of $\{w_k\}$ could be correlated, and if the input quality deteriorates over time, the mean of w_k, will decrease with k, and/or its variance will increase with k. Thus, it is reasonable to describe $\{w_k\}$ by a suitable discrete time stochastic process.

Since machines deteriorate over time, it is reasonable to assume that $\phi_{k+1} \leq \phi_k$; if maintenance is undertaken in period k, then $\phi_{k+1} \geq \phi_k$. We assume that $\phi_1 = 1$, and if there is an overhaul of the machine just before batch k is put into production, then $\phi_k = \phi_1 = 1$. Motivated by the fact that the times to failure of complex systems have an exponential distribution, a plausible model for describing machine deterioration, under no maintenance or overhaul is

$$\phi_k = \beta\, \phi_{k-1}\,, \qquad \text{for } k=2, 3, \cdots, \tag{2.1}$$

where β is an *unknown* coefficient taking values in [0, 1]; let a prior distribution $\pi(\beta)$ describe our uncertainty about β.

Since θ_k is assumed to depend on both w_k and ϕ_k, our next task is to specify a relationship of

the form

$$\theta_k = g(w_k, \phi_k), \qquad \text{for } k=1, 2, \cdots, \tag{2.2}$$

where the function g is such that its first order partial derivatives are nonnegative; this ensures that the output quality does not degrade when the input quality and the state of the machine improve. For this, we propose the multiplicative relationship

$$\theta_k = \phi_k \, w_k \, , \quad k=1, \cdots, \ell \, . \tag{2.3}$$

A generalization of (2.1) is to let $\phi_k = \beta\phi_{k-1} + u_k$, $k=2, \cdots, \ell$, where u_k is an innovation. Uncertainty about (2.3) can be accounted for by introducing another innovation term v_k, and writing $\theta_k = \phi_k \, w_k + v_k$, $k=1, \cdots, \ell$, with a probabilistic structure imposed on the sequence $\{v_k\}$, $k=1, \cdots, \ell$. In what follows, we suppose that the u_k's and the v_k's have very small variances, so that the additive terms u_k and v_k can be ignored.

The above set-up could be criticized as being simplistic. In the development that follows there is nothing that precludes one from entertaining extensions and generalizations of (2.1) and (2.3). However, such extensions would result in a consideration of details which tend to obscure the gist of development.

3. Controlling Quality via an Optimal Maintenance Schedule

In open loop control problem there is no in-process monitoring and so there is no new knowledge about β beyond what has been specified via $\pi(\beta)$. Decision making and control in open loop problems are not adaptive (i.e., not based on any in-process data); such problems are encountered when it is inconvenient to interrupt the process for collecting data, or when the cost of collecting data is prohibitive.

For the model of Section 2, we note from (2.1), that between maintenance actions, ϕ_k decreases exponentially in k so that for $k \geq 2$, $\phi_k = \beta^{k-1} \phi_1$. In what follows, we assume that maintenance at any index m, results in a complete overhaul of the machine, so that $\phi_m = \phi_1 = 1$.

Suppose that the input sequence is such that the w_k's are independent and identically distributed with a nonzero mean. Because of (2.1) and (2.3), the output quality degrades exponentially so that $\theta_k = \beta^{k-1} w_k$. Consequently, for some $\tilde{k} > 1$ and $\theta_{\tilde{k}} < \theta^*$, our problem is to find a $k^* < \tilde{k}$ at

which overhaul is to be performed. With open loop control, inference about $\tilde{k}$ will be based on $\pi(\beta)$ and the assumed properties of $\{w_k\}$.

3.1 A Decision – Theoretic Approach for Choosing k^*

Considering the principle of maximization of expected utility [cf. Lindley (1985)], we present a coherent approach of undertaking the desired task in this section.

Let a(i), denote the decision to undertake overhaul at the i-th subinterval, i= 2, 3,$\cdots$, (ℓ+1) . Note that a(ℓ+1) denotes the decision not to schedule maintenance throughout the interval [0, T], and that for any choice a(i), overhaul precedes production in the period i.

The decision problem is depicted via the decision tree of Figure 3.1 with boxes (circles) representing the decision (random) nodes.

Since the machine starts out being perfect, overhaul can not be scheduled at the first subinterval. Open loop means that decision to schedule maintenance must be made at the decision node $\mathfrak{D}$, where, we may choose any one of the ℓ possible actions a(2), a(3), $\cdots$, a(ℓ+1). Each action a(i), i = 2, $\cdots$, $\ell+1$, results in a series of consequences (output qualities) θ_2, $\cdots$, θ_{i-1}, θ_i, where θ_i, the output quality of batch i, equals to θ_1. At the time of taking action a(i), the θ_1, $\cdots$, θ_{i-1} are unknown; let $P_k(\theta|\, a(i))$ denote the probability density of θ_k (at θ), under action a(i), for k= 1, $\cdots$, i-1.

The combination of an action and its consequence result in a loss (or utility); this is in keeping with Taguchi's notions about quality loss [cf. Kackar (1986)]. Let $\mathcal{L}_k(a(i), \theta)$, i = 2, $\cdots$, ℓ+1, k = 1, $\cdots$, i-1, denote the loss incurred when action a(i) results in a consequence $\theta_k = \theta$. In considering the loss functions, an important issue to bear in mind is the potential loss due to a premature overhaul; for any choice a(i), this loss would manifest itself with respect to batch i only. $\mathcal{L}_{1i}(a(i), \tilde{\theta}_1, \tilde{\theta}_i)$, i = 2, $\cdots$, ℓ, denote the above loss, where $\tilde{\theta}_1$ (or $\tilde{\theta}_i$) is the consequence that would have resulted had we (or had we <u>not</u>) scheduled an overhaul at batch i. A consideration of θ_i is necessary for us to evaluate the effect of a premature overhaul. Finally, let $P_{1i}(\tilde{\theta}_1, \tilde{\theta}_i|\, a(i))$ be the joint probability density of θ_1 and θ_i (at $\tilde{\theta}_1$ and $\tilde{\theta}_i$) under the circumstances described above.

A specification of the above ingredients – also indicated at the appropriate branches of the decision tree of Figure 3.1 – is all that is needed to arrive at a decision for the optimal schedule of maintenance. For any action a(i), E(a(i)) its ***expected loss*** (over the production cycle of length (i – 1)), assuming additivity of losses is given as

$$E(a(i)) = \sum_{k=2}^{i-1} \int_{\theta} \mathcal{L}_k(a(i), \theta)\, P_k(\theta \,|\, a(i))\, d\theta + \int_{\tilde{\theta}_1, \tilde{\theta}_i} \mathcal{L}_{1i}(a(i), \tilde{\theta}_1, \tilde{\theta}_i)\, P_{1i}(\tilde{\theta}_1, \tilde{\theta}_i \,|\, a(i))\, d\tilde{\theta}_1 d\tilde{\theta}_i$$

with the last term being deleted when $i = \ell + 1$. The optimum decision is to choose that action for which $E(a(i))$ is a minimum, or to choose that action for which $E(a(i))/(i-1)$ is a minimum, the latter denoting the expected loss per interval of production.

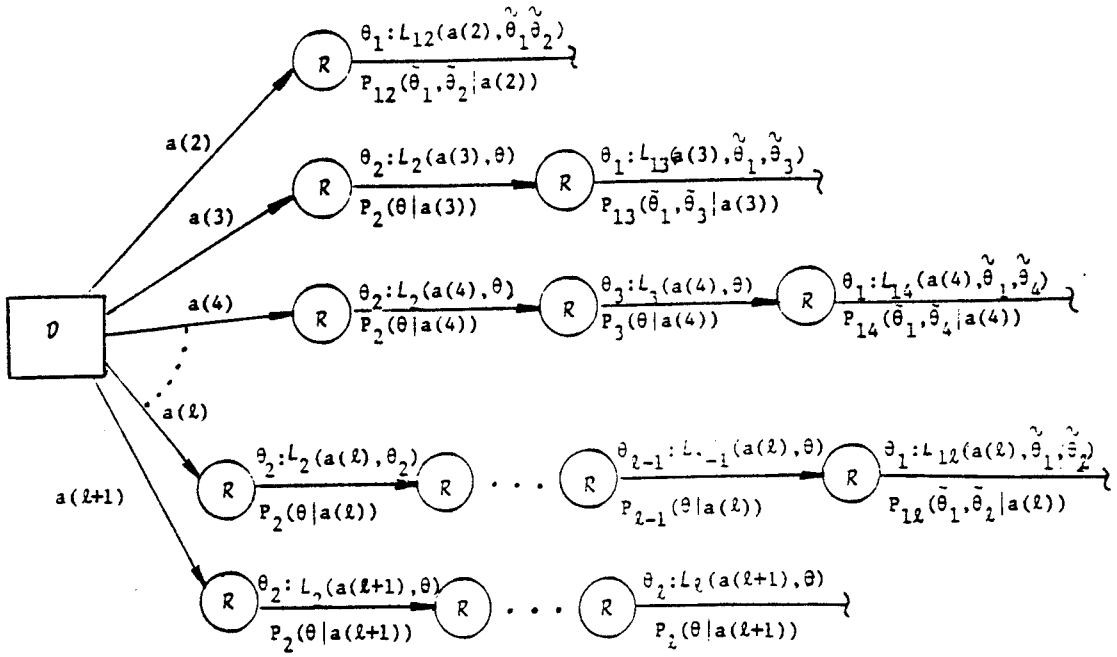

Figure 3.1 Decision Tree for Controlling Quality by Choosing an Optimal Maintenance Action in an Open Loop Control Problem.

3.1.1 Plausible Forms for the Loss Functions and Densities

A realistic evaluation of the above described loss functions and the associated probability densities is an important step in any given application and should be exercised with caution. In what follows, we suggest some plausible forms for these.

Let $c(i)$ $[c'(i)]$ for $i = 2, \cdots, \ell+1$ be the cost of overhaul during interval i [the cost of an unwarranted overhaul in interval i]; it is reasonable to suppose that $\{c(i)\}$ $[\{c'(i)\}]$ is nondecreasing [nonincreasing]. Note that $c'(i)$ would capture the consequences of tampering with production. Finally let an exponent c describe the magnitude of the loss due to defective quality - see the equations below.

Then, motivated by Taguchi's suggestion of quadratic losses, a plausible form for $\mathcal{L}$, for any $k = 2, 3, \cdots, i-1$, and $i = 2, \cdots, \ell+1$, is $\mathcal{L}_k(a(i), \theta) = (\theta^* - \theta)^c$, if $\theta < \theta^*$, and $\mathcal{L}_k(a(i), \theta) = 0$ if $\theta \geq \theta^*$.

When $k = i$, we would have $\mathcal{L}_{1i}(a(i); \tilde{\theta}_1, \tilde{\theta}_i) = c(i) + c_1 \cdot (\theta^* - \tilde{\theta}_1)^c + c_2 \cdot c'(i)$, where $c_1 = 1(0)$ if $\tilde{\theta}_1 < (\geq)\ \theta^*$, and since $c'(i)$ will be incurred only if the consequence of not scheduling maintenance would be deleterious, $c_2 = 1(0)$ if $\tilde{\theta}_i \geq (<)\ \theta^*$.

As far as the specification of the needed probability densities, we note that for any $k = i$, and $i = 2, \cdots, \ell+1$

$$P_k(\theta \mid a(i)) = \frac{d}{d\theta} P(\theta_i \leq \theta \mid a(i)) = \frac{d}{d\theta} P(\phi_i\, w_i \leq \theta \mid a(i)) = \frac{d}{d\theta} P(w_1 \leq \theta \mid a(i)) \stackrel{def}{=} P_1(\theta),$$

since the effect of $a(i)$ is to make $\phi_i = 1$. For any $k < i$, and $k-2, \cdots, i-1$, it can be verified – see A1 in the Appendix – that

$$P_k(\theta \mid a(i)) = \int_{w \geq \theta} \frac{1}{(k-1)w} \left(\frac{\theta}{w}\right)^{\frac{2-k}{k-1}} \cdot \pi\left(\left(\frac{\theta}{w}\right)^{\frac{1}{k-1}}\right) \cdot P_1(w)\, dw, \tag{3.1}$$

where $\pi(\cdot)$ is the prior of β, assumed to be a beta density with parameters p and q. Specifically,

$$\pi(\beta) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \beta^{p-1}(1-\beta)^{q-1}, \qquad 0 < \beta < 1. \tag{3.2}$$

A plausible choice for $P_1(w)$, the marginal distribution of the stationary process $\{w_k\}$ is the uniform density over some interval (a, b), where $0 \leq a \leq b \leq 1$.

To specify $P_{1i}(\tilde{\theta}_1, \tilde{\theta}_i \mid a(i))$, we note that its factorization may be written as $P_{1i}(\tilde{\theta}_1 \mid a(i))\ P_{1i}(\tilde{\theta}_i)$ since given $a(i)$, $\tilde{\theta}_1$ is independent of $\tilde{\theta}_i$ and $\tilde{\theta}_i$ is independent of $a(i)$. Finally, $P_{1i}(\tilde{\theta}_1 \mid a(i))$ is simply $P_1(\theta)$, and $P_{1i}(\tilde{\theta}_i)$ is given by (3.1) with the index i replacing the index k.

4. Controlling Quality by Controlling Input and Maintenance Schedules

In this section we will consider the situation in which the output quality can be controlled by both, a judicious choice of input quality and the selection of an optimal maintenance schedule.

To simplify our development, suppose that the input quality can be described by two stationary stochastic processes, both of which are assumed to be identical in all respects except that one has mean μ and the other has mean μ' with $\mu > \mu'$. Suppose that it costs an additional $\mathcal{C}$ units of

money per batch, to use the input with mean μ. Suppose also, that when a production cycle is begun, the input quality has mean μ'; the idea here is that a good state of machine compensates for the lower quality of input and that maintenance or overhaul will not take place unless the input quality has been switched from μ' to μ. Finally, suppose that the input quality with mean μ is described by the process $\{w_k\}$ and that with mean μ' is described by the process $\{w'_k\}$.

In what follows, we shall adopt the set-up of Section 3, except that now, during the first time period, we neither switch the input quality from $\{w'_k\}$ to $\{w_k\}$ nor undertake maintenance. During the second time period, we are <u>not</u> allowed to schedule maintenance since a switch in input quality must precede maintenance. However, during the second time period, we must take one of $\left(\frac{\ell(\ell-1)}{2}+1\right)$ actions a(s, m), where a(s, m) denotes the action to switch input quality from at the s-th time period, and to schedule maintenance during the m-th time period with $s \leq m$, $s = 2, \cdots, \ell+1$, and $m = 3, \cdots, \ell+1$; note that $s = m$ only when both are $\ell+1$. Also, $a(\ell+1, \ell+1)$ implies that we need not switch input quality nor schedule maintenance throughout the ℓ time intervals. We shall also find it convenient to introduce the notation $a(\tilde{s}, \tilde{m})$, $a(\tilde{s}, m)$ and $a(s, \tilde{m})$, where $\tilde{s}$ and $\tilde{m}$ represent the actions of switching input quality after time interval s and scheduling maintenance after time interval m respectively.

The decision tree for this scenario is given in Figure 4.1; much of the notation used is analogous to that of Figure 3.1. It can be verified, see A2 in the Appendix, that for any $m > s$, and $m = 3, \cdots, \ell+1$, $P_m(\theta \mid a(s, m)) = \frac{d}{d\theta} P(w'_1 \leq \theta) \stackrel{def}{=} P'_1(\theta)$, also, for any $s \leq k < m$, $P_k(\theta \mid a(s, m))$ is given by the right hand side of (3.1), whereas for any $1 < k < s$ ($<$ m), it is given by (3.1) with the last term replaced by $P'_1(w)$. A plausible choice for $P'_1(w)$, is a uniform density over $(a-e, b-e)$, for some $e > 0$, $0 < a-e < b-e \leq 1$. The quantity $P_{ss^*}(\tilde{\theta}_s, \tilde{\theta}^*_s \mid a(s, m))$ denotes the joint density function of θ_s (at $\tilde{\theta}_s$) and θ^*_s – the output quality of batch s had the switch in input quality not occurred at batch s – (at $\tilde{\theta}^*_s$). Similarly, $P_{1m}(\tilde{\theta}_1, \tilde{\theta}_m \mid a(s, m))$ denotes the joint density function of θ_1 (at $\tilde{\theta}_1$) and θ_m – the output quality of batch m were maintenance not undertaken at batch m – (at $\tilde{\theta}_m$).

Finally, following arguments analogous to those at the end of Section 3, we can show that $P_{ss^*}(\tilde{\theta}_s, \tilde{\theta}^*_s \mid a(s, m))$ can be written as

$$\left\{ \int_{w \geq \tilde{\theta}_s} \frac{1}{(s-1)w} \left(\frac{\tilde{\theta}_s}{w}\right)^{\frac{2-s}{s-1}} \cdot \pi\left(\left(\frac{\tilde{\theta}_s}{w}\right)^{\frac{1}{s-1}}\right) \cdot P_1(w)\, dw \right\} \cdot \left\{ \int_{w \geq \tilde{\theta}^*_s} \frac{1}{(s-1)w} \left(\frac{\tilde{\theta}^*_s}{w}\right)^{\frac{2-s}{s-1}} \cdot \pi\left(\left(\frac{\tilde{\theta}^*_s}{w}\right)^{\frac{1}{s-1}}\right) \cdot P'_1(w)\, dw \right\},$$

and $$P_{1m}(\tilde{\theta}_1, \tilde{\theta}_m \mid a(s, m)) = P'_1(\tilde{\theta}_1) \cdot \int_{w \geq \tilde{\theta}_m} \frac{1}{(m-1)w} \left(\frac{\tilde{\theta}_m}{w}\right)^{\frac{2-m}{m-1}} \cdot \pi\left(\left(\frac{\tilde{\theta}_m}{w}\right)^{\frac{1}{m-1}}\right) \cdot P_1(w)\, dw.$$

Our next step is to specify, for each combination of action and consequences, the loss functions. In specifying these, we shall follow the notation and the cost structure of Section 3.1. Thus, for the production during any period prior to the switch in input quality and overhaul, i.e., for $k = 2, \cdots, s-1$, $s = 2, \cdots, \ell+1$, $s \leq m = 3, \cdots, \ell+1$, the loss $\mathcal{L}'$ is given as

$$\mathcal{L}'_k(a(s, m), \theta) = (\theta^* - \theta)^c \text{ if } \theta < \theta^*, \text{ and } \mathcal{L}'_k(a(s, m), \theta) = 0 \text{ if } \theta \geq \theta^*.$$

For the production during any period k at which a switch in input quality occurs, that is when $k = s$, $s = 2, \cdots, \ell$, the loss $\mathcal{L}$ is a function of three arguments, the index s represented via a(s, m), the state of nature $\theta_s = \tilde{\theta}_s$ resulting from the action a(s, m), and the state of nature $\theta^*_s = \tilde{\theta}^*_s$, were action a($\tilde{s}$, m) taken. Note that θ^*_s is needed to capture the cost of an unwarranted switch in input quality. Let $c^*(s)$, $s = 2, \cdots, \ell$, denote the cost of an unwarranted switch in input quality during the interval s, and suppose that $\{c^*(s)\}$ is a *nonincreasing sequence*, then

$$\mathcal{L}_{ss^*}(a(s,m), \tilde{\theta}_s, \tilde{\theta}^*_s) = \mathcal{C} + I(\theta^* - \tilde{\theta}_s) \cdot (\theta^* - \tilde{\theta}_s)^c + I(\tilde{\theta}^*_s - \theta^*) \cdot c^*(s).$$

where I(x) is 1(0) if $x \geq (<)\ 0$. For any k <u>following</u> a switch in input quality but <u>preceding</u> the interval at which overhaul is undertaken, that is for $k = s+1, s+2, \cdots, m-1$, the loss $\mathcal{L}$ is a function of two arguments a(s, m) and θ, a realization of θ_k, $s = 2, \cdots, \ell$, $m = 3, \cdots, \ell+1$, so that

$$\mathcal{L}_k(a(s, m), \theta) = \mathcal{C} + I(\theta^* - \theta) \cdot (\theta^* - \theta)^c.$$

Finally, for any m at which an overhaul occurs, $\mathcal{L}$ is a function of three arguments, the index m represented via the action a(s, m), the state of nature θ_1 resulting from action a(s, m) (in fact, θ_1 is lot quality of batch m) and θ_m, the state of nature were action a(s, $\tilde{m}$) taken. Thus, for any $m = 3, \cdots, \ell+1$, $m \geq s$,

$$\mathcal{L}_{1m}(a(s,m), \tilde{\theta}_1, \tilde{\theta}_m) = c(m) + I(\theta^* - \tilde{\theta}_1) \cdot (\theta^* - \tilde{\theta}_1)^c + I(\tilde{\theta}_m - \theta^*) \cdot c'(m),$$

where c(m) and c'(m) are as described in Section 3.1.1.

The necessary ingredients to compute E(a(s, m)), the expected loss (over the production cycle of length (m − 1)) resulting from action a(s, m) are in place, and the optimum decision is to choose that action for which E(a(s, m)) or E(a(s, m))/(m−1) is a minimum.

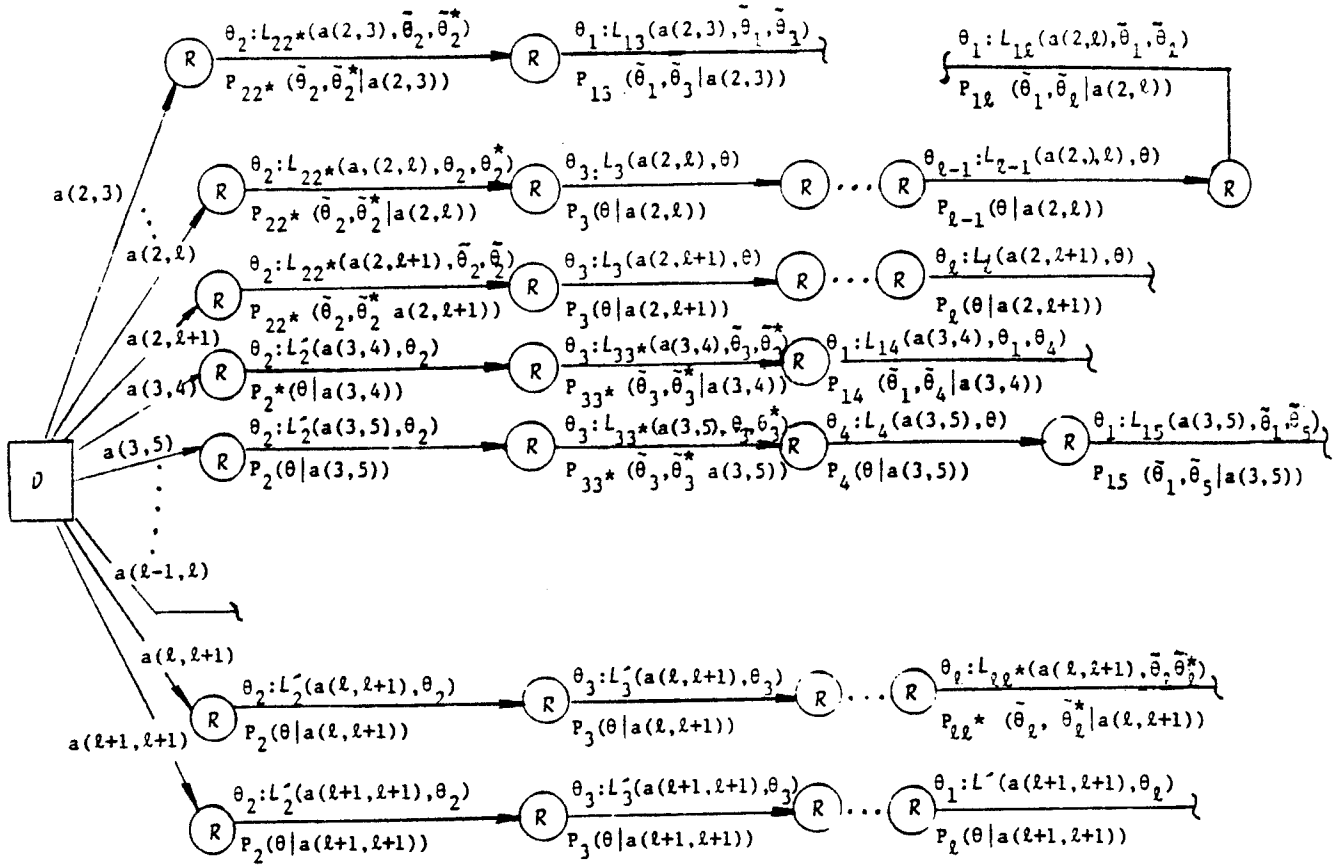

Figure 4.1 Decision Tree for Controlling Quality by Switching Input Quality and Choosing an Optimal Maintenance Action

5. An Illustrative Example

As an illustration of our approach, suppose that the w_k's and the w'_k's of the input quality are uniformly distributed over (a, b) and (a−e, b−e) respectively, with $0 < a-e < a \leq b-e < b \leq 1$; let $a = .9$, $b = 1$ and $e = .1$. Suppose that the coefficient β of (2.1) has the beta distribution (3.2) with $p = 22.5$ and $q = 2.5$. Finally, suppose that ℓ, the number of subintervals is 15 and θ^* the minimum acceptable batch quality is .75.

Recall that ϕ_k is the state of the machine during the k-th interval; let $\pi_k(y)$ denote the probability density of ϕ_k at y, for $0 < y \leq 1$. Then, for $k \neq 1$, it can be shown [see A3 in the Appendix] that

$$\pi_k(y) = \Gamma(p+q) \cdot y^{\frac{p}{k-1}-1} \left(1 - y^{\frac{1}{k-1}}\right)^{q-1} / [(k-1)\Gamma(p)\Gamma(q)].$$

For our choice of c(i), the cost of an overhaul during interval i, we choose $c(i) = E\{(1-\phi_i)^2\}$, where the expectation is taken with respect to $\pi_i(y)$, given above. Note that the chosen form of c(i) captures the deviation of ϕ_i from 1, the perfect state. Regarding our choice of $c'(i)$, the cost of an unwarranted overhaul in interval i, we choose $c'(i) = E\{10 \cdot (\theta_i - \theta^*)^2 \cdot I(\theta_i - \theta^*)\}$, where $I(x) = 1(0)$ if $x \geq (<) 0$.

This choice is reasonable; the constant 10 is an arbitrarily chosen scaling constant. The values of c(i) and $c'(i)$, for $i = 2, 3, \cdots, 16$, are given in Table 1. Also shown there are values of E(a(i)) and $E(a(i))/(i-1)$, the expected loss over the production cycle of length $(i-1)$, and the expected loss per interval of production of action a(i). It is clear from the above that using the latter criterion we would choose to schedule maintenance at the 4-th production cycle.

For the scenario of Section 4, we choose $\mathcal{C} = .75$ and $c^*(s) = \mathcal{C} \cdot P(\theta_s^* > \theta^*)$; the other constants are defined in a manner analogous to those of the scenario of Section 3. The necessary details are shown in Table 2, from which it can be seen that the optimum decision is action a(3, 4), i.e. switch input during period 3 and undertake overhaul during period 4.

6. Conclusions

It is apparent from the above, that despite the simplicity of our model formulation, the ensuing development is detailed and fraught with much notation. All the same, coherence demands that we undertake an analysis of problems of this type in the manner indicated here. Thus it appears that improvisations of our set-up, aimed towards making our model more realistic, would lead to complications that would be computationally burdensome to manage. Nonetheless, it is our hope that such difficulties can be overcome using modern numerical and simulation techniques; these we leave to the ingenuity of others. In the meantime, we are content with accepting the fact that the contribution here is a modest one and that its chief virtue may be the setting up of a framework upon which future work – such as say closed loop control, etc. – can be built upon.

Acknowledgments

We gratefully acknowledge the comments of a referee whose comments have helped improve its quality. Research of the first two authors was supported by the Office of Naval Research Contract N00014-85-K-0202 Project NR 042-372, the Army Research Office Grant DAAH04-93-G-0020 and the Air Force Office of Scientific Research Grant AFOSR-F-49620-92-J-0030.

Table 1 Computations for Choosing an Optimal Maintenance Schedule

Production Cycle i	c(i) Cost of Overhaul during the i-th Cycle	c′(i) Cost of an Unwarranted Overhaul during the i-th Cycle	E(a(i)) Expected Loss in choosing a(i)	E(a(i))/(i-1) Expected Loss per Interval of Production in choosing a(i)
1				
2	.01346159	.1466078	.16006939	.1601
3	.04531247	.06189097	.10952019	.0548
4	.08681184	.03175586	.1516	.0505
5	.132768	.01861726	.2627288	.0657
6	.1800835	.01196662	.4328089	.0866
7	.2269232	.00820868	.6542068	.1090
8	.2722247	.00591129	.9205668	.1315
9	.3154032	.00442179	1.226507	.1533
10	.3561711	.00341007	1.567454	.1742
11	.3944248	.0026873	1.9395	.1940
12	.4301732	.00217836	2.33934	.2127
13	.4634929	.00177852	2.764051	.2303
14	.4944989	.00148416	3.211187	.2470
15	.5233261	.0012476	3.678556	.2628
16	.5501175	.000104743	4.164261	.2776

Table 2 Computations for Choosing an Optimal Input-switching and Maintenance Schedule

s	m	$c^*(s)$ Cost of an Unwarranted Switch in Input Quality during the s-th Cycle	c(m) Cost of Overhaul during the m-th Cycle	$c'(s)$ Cost of an Unwarranted Overhaul during the m-th Cycle	E(a(s,m)) Expected Loss in choosing a(s,m)	E(a(s,m))/m-1 Expected Loss per Interval of Production in choosing a(s,m)
3	4	.02139652	.08681184	.03175586	.26180236	.0873
	5	.02139652	.132768	.01861726	.44803108	.1120
	6	.02139652	.1800835	.01196662	.69301114	.1386
	7	.02139652	.2269232	.008208675	.98940910	.1649
	8	.02139652	.2722247	.005911289	1.33076901	.1901
	9	.02139652	.3154032	.004421795	1.71170882	.2140
	10	.02139652	.3561711	.003410068	2.12765569	.2364
	11	.02139652	.3944248	.0026873	2.57469262	.2575
	12	.02139652	.4301732	.002178363	3.04954208	.2772
	13	.02139652	.4634929	.001778515	3.54895858	.2958
	14	.02139652	.4944989	.001484155	4.07138858	.3132
	15	.02139652	.5233261	.0012476	4.61375822	.3296
	16	.02139652	.5501175	.001047427	5.17446345	.3450
2	4	.04827556	.08681184	.003175586	.32129528	.1071
3	4	.02139652	.08681184	.003175586	.26180236	.0873

REFERENCE

1. Barlow, R. E., Durst, M. J. and Smiriga, N. G. (1984). "A Kalman Filter Model for Determining Block and Trickle SNM Losses". In *Low-Probability/High-Consequence Risk Analysis: Issues, Methods, and Case Study*, Eds. R. A. Waller and V. T. Covello. Plenum Press, New York.
2. Kacker, R. N. (1986). "Taguchi's Quality Philosophy: Analysis and Commentary". *Quality Progress*, Dec. 1986, pp. 21-29.
3. Lindley, D. V. (1985). *Making Decisions*, 2nd Ed. John Wiley, New York.
4. Taguchi, G. (1986). *Introduction to Quality Engineering: Designing Quality into Products and Processes.* White Plains, NY: Kraus International Publication.

APPENDIX

A1 Derivation of the Density Function $P_k(\theta \mid a(i))$

Recall that $P_1(\theta)$ is the density function of the input quality at the point θ. Then, since β and w_k are independent of a(i), for $k = 1, 2, \cdots, i-1$, we have

$$P_k(\theta \mid a(i)) = \frac{d}{d\theta} P(\beta^{k-1} w_k \le \theta \mid a(i)) = \frac{d}{d\theta} P(\beta^{k-1} w_k \le \theta).$$

Note that $P(\beta^{k-1} w_k \le \theta) = \int\limits_{\text{all } w} P(\beta^{k-1} \le \frac{\theta}{w_k} \mid w_k = w) \cdot (\frac{d}{dw} P(w_k \le w))\, dw$. Therefore,

$$P_k(\theta \mid a(i)) = \frac{d}{d\theta} \int\limits_{w \ge \theta} P(\beta \le (\frac{\theta}{w})^{\frac{1}{k-1}})\, P_1(w)\, dw = \int\limits_{w \ge \theta} \frac{1}{(k-1)w} (\frac{\theta}{w})^{\frac{2-k}{k-1}} \cdot \pi((\frac{\theta}{w})^{\frac{1}{k-1}}) \cdot P_1(w)\, dw.$$

A2 Derivation of the Density Function $P_k(\theta \mid a(s, m))$

Obviously, for any $m > s$, and $m=3, \cdots, \ell+1$, $P_m(\theta \mid a(s, m)) = \frac{d}{d\theta} P(w_1' \le \theta) \overset{def}{=} P_1'(\theta)$. Therefore,

$$P_k(\theta \mid a(s, m)) = \int\limits_{w \ge \theta} \frac{1}{(k-1)w} (\frac{\theta}{w})^{\frac{2-k}{k-1}} \cdot \pi((\frac{\theta}{w})^{\frac{1}{k-1}}) \cdot P_1(w)\, dw \quad \text{for any } s \le k < m;$$

$$P_k(\theta \mid a(s, m)) = \int\limits_{w \ge \theta} \frac{1}{(k-1)w} (\frac{\theta}{w})^{\frac{2-k}{k-1}} \cdot \pi[(\frac{\theta}{w})^{\frac{1}{k-1}}] \cdot P_1'(w)\, dw \quad \text{for } 1 < k < s\ (< m).$$

A.3. Derivation of the Formulae in Section 3.3

1) Derivation of the density of ϕ_k

Let $\pi_k(y)$ denote the density of ϕ_k. Then, since $\phi_k = \beta^{k-1}$ and $\pi(\beta)$ is a beta distribution

with parameters p and q, we have, if $k \neq 1$ and $0 < y < 1$,

$$\pi_k(y) = \frac{1}{k-1} \cdot \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)} \cdot y^{\frac{p}{k-1}-1}(1-y^{\frac{1}{k-1}})^{q-1}, \text{ and } \pi_k(y) = 0, \text{ o.w.}$$

When k=1, obviously, $P(\phi_1 = y) = 1$ if $y = 1$, and it is 0 otherwise.

2) Derivation of $P_k(\theta \mid a(i))$

Since $P_1(w)$, the density of w_k, is a Uniform(a, b), $P_k(\theta \mid a(i)) = \frac{d}{d\theta}P(\theta_k \leq \theta \mid a(i))$, $\theta_k = w_k\phi_k$ and the density of ϕ_k is as shown above, it is obvious that $P_1(\theta) \sim$ Uniform(a, b) if $\phi_1 = 1$.

Therefore, for k=2, 3, $\cdots$, $i-1$, with the parameter $p > k-1$

$$P_k(\theta \mid a(i)) = \begin{cases} \displaystyle\int_{\theta/b}^{\theta/a} \frac{1}{b-a} \cdot \frac{\Gamma(p+q)}{(k-1)\cdot\Gamma(p)\,\Gamma(q)}\, y^{\frac{p}{k-1}-2}(1-y^{\frac{1}{k-1}})^{q-1}\, dy, & \text{if } 0 < \theta \leq a, \\ \displaystyle\int_{\theta/b}^{1} \frac{1}{b-a} \cdot \frac{\Gamma(p+q)}{(k\text{-}1)\cdot\Gamma(p)\,\Gamma(q)}\, y^{\frac{p}{k-1}-2}(1-y^{\frac{1}{k-1}})^{q-1}\, dy, & \text{if } a < \theta \leq b\,. \end{cases}$$

3) Derivation of $P_k(\theta \mid a(s, m))$

Since $P_1(w)$, the density of w_k, is uniformly distributed in $(a, b) \subset (0, 1)$, $P_1'(w')$, the density of w_k', is uniformly distributed in $(a-e, b-e) \subset (0, 1)$, $P_k(\theta \mid a(s, m)) = \frac{d}{d\theta}\, P(\theta_k \leq \theta \mid a(s, m))$, and the density of ϕ_k is shown above, by the same procedure for showing the density $P_k(\theta \mid a(i))$ it can be proved that $P_m(\theta \mid a(s, m)) = P_1'(\theta) = P_1(\theta \mid a(s, m))$ is uniformly distributed over $(a-e, b-e) \subset (0,1)$ if $\phi_1 = \phi_m = 1$. For $k = 2, \cdots, s-1$, $s = 2, \cdots, s \leq m$ ($s = m$ holds only if $s = m = \ell+1$), with $p > k-1$,

$$P_k(\theta \mid a(s, m)) = \begin{cases} \displaystyle\int_{\frac{\theta}{b\text{-}e}}^{\frac{\theta}{a\text{-}e}} \frac{1}{b-a} \cdot \frac{\Gamma(p+q)}{(k-1)\Gamma(p)\Gamma(q)}\, y^{\frac{p}{k-1}-2}(1-y^{\frac{1}{k-1}})^{q-1}\, dy, & \text{if } 0 < \theta \leq a-e, \\ \displaystyle\int_{\frac{\theta}{b\text{-}e}}^{1} \frac{1}{b-a} \cdot \frac{\Gamma(p+q)}{(k-1)\Gamma(p)\Gamma(q)}\, y^{\frac{p}{k-1}-2}(1-y^{\frac{1}{k-1}})^{q-1}\, dy, & \text{if } a-e < \theta \leq b-e. \end{cases}$$

Finally, for $k = s, \cdots, m-1$, with $p > k-1$,

$$P_k(\theta \mid a(s, m)) = \begin{cases} \displaystyle\int_{\theta/b}^{\theta/a} \frac{1}{b-a} \cdot \frac{\Gamma(p+q)}{(k-1)\Gamma(p)\Gamma(q)}\, y^{\frac{p}{k-1}-2}(1-y^{\frac{1}{k-1}})^{q-1}\, dy, & \text{if } 0 < \theta \leq a, \\ \displaystyle\int_{\theta/b}^{1} \frac{1}{b-a} \cdot \frac{\Gamma(p+q)}{(k-1)\Gamma(p)\Gamma(q)}\, y^{\frac{p}{k-1}-2}(1-y^{\frac{1}{k-1}})^{q-1}\, dy, & \text{if } a < \theta \leq b. \end{cases}$$