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WINDABLE QUASI-GEODESIC PATHS ON SURFACES OF REVOLUTION

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If r is the profile or radius function for a surface of revolution and r_o is the polar radius function, a quasi-geodesic path on the surface can be defined by the generalized Clairaut relation $rSinw=r_o$, where w is the meridional angle. An inequality involving r,r',r'',r_o, and r_o' is derived. The global satisfaction of this inequality guarantees the windability of the path on a convex ($r''<0$) surface by a filament winding machine. If the surface is concave anywhere ($r''>0$) and a more well known "clinging" inequality is also satisfied, windability is also guaranteed. By "windable" we mean that the winding data produced from the path represents a single-valued function and that the wound filament does not bridge. In addition to this new windability criterion, simplified methods for generating quasi-geodesic paths and properly scaled winding data are also presented.

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INTRODUCTION

The off-line programming of a two-axis filament winding machine consists, in principle, of the following four general steps:

1. Establishment of the mandrel geometry by the profile or radius function $r(x)$, where x measures distance along the axis of the surface of revolution.

2. Establishment of the nominal winding path or quasi-geodesic by the polar radius function $r_o(x)$.

3. Computation of T as a single-valued, periodic function of R, where T is the x position of the filament delivery point and R is the aggregate rotational displacement of the filament delivery point around the axis of the mandrel. Actually, the mandrel rotates and the filament delivery point merely translates back and forth parallel to the mandrel axis, but it seems more useful, conceptually, to think of the mandrel as being fixed in space and the filament delivery point as translating (T) and rotating (R).

4. Proper scaling of the (R,T) winding data so as to produce a perfectly uniform wrap which properly covers the surface.

This report is mainly concerned with the proper accomplishment of step 2 so that no problems will arise with the accomplishment of step 3. What problems may arise while doing step 3? Well, if r_o is not defined properly, the (R,T) winding data may exhibit loops, cusps, or at least nonunique T values for a given R value. That is, T will not be a single-valued function of R. Even convex surfaces are not immune to this effect.

MONOTONE WINDING CONDITION

Step 3 is accomplished in practice by computing R and T as parametric functions of some parameter such as angular displacement (of a point on the quasi-geodesic path) θ. We obtain the winding function $T=f(R)$ indirectly by computing $R=g(\theta)$ and $T=h(\theta)$. The function f will be single-valued if and only if g is a monotone increasing function of θ. Hence, we will have "monotone winding" if $dR/d\theta>0$. In what follows, a prime denotes differentiation with respect to x and a dot denotes differentiation with respect to θ.

WINDABILITY

We refer to a path on the surface or the corresponding polar radius function r_o as being admissible if $r=r_o$ at the two turning points and $r_o<r$ at all points in between.

If

$$p = rr''(r_o^2 - r^2) + r_o^2(1 + r'^2)$$

and

$$q = r^3 r' r_o^{-1} r_o'(1 + r'^2)$$

a band of filament laid along an admissible path will not experience bridging or lifting from the surface when the filament is under tension provided $p>0$ for all x. This inequality is derived in References 1 and 2. Note that an admissible path on a convex surface can never experience bridging.

If the inequality $p+q>0$ holds for all x, the winding data will embody a single-valued function (f). Now, either, both, or neither of these inequalities may hold at any given point. For an admissible path to be acceptable, however, they must both hold everywhere. We refer to $p>0$ and $p+q>0$ as the "clinging" and "monotone" inequalities or conditions, respectively.

Assuming the clinging condition holds, the only way in which an admissible path may fail to be windable is when r' and r_o' are of opposite sign and their product is sufficiently negative at some point. Note that at turning points ($r=r_o$), $p+q>0$ reduces to $1+r'r_o'>0$. So, if clinging holds and the product of these derivatives is nonnegative, we are theoretically guaranteed single-valued winding data. Also, if clinging does not hold in some region and the derivative product is nonpositive anywhere in that region, then we are guaranteed multi-valued winding data. However, if the derivative product is positive throughout the region, we might escape multi-valued winding data even though we would still have bridging.

PATH GENERATION

In this section we derive a simplified algorithm for generating quasi-geodesic paths on surfaces of revolution. The development is more straightforward than that of Reference 1. Results of this section are also used in deriving the monotone winding condition in the next section.

A point on the path on the surface is given by $P=ix+jy+kz$ where $y=r\mathrm{Sin}\theta$, $z=r\mathrm{Cos}\theta$, x measures distance along the axis of the surface, and θ is the angular displacement of P around the axis.

From $y=rSin\theta$ and $z=rCos\theta$, we have

$$dy = dr\,Sin\theta + r\,Cos\theta d\theta$$

and

$$dz = dr\,Cos\theta - r\,Sin\theta d\theta$$

If s measures arc length along the path, we have

$$ds^2 = dx^2 + dy^2 + dz^2 = dx^2 + dr^2 + r^2 d\theta^2$$

Now, if ω is the angle that the path makes with a meridian (θ=const), we have

$$ds\,Sin\omega = r d\theta$$

But the generalized Clairaut relation defining a quasi-geodesic path is

$$r\,Sin\omega = r_o$$

Hence

$$d\theta = \frac{r_o}{r^2} ds$$

and therefore

$$ds^2 = dx^2 + r'^2 dx^2 + \frac{r_o^2}{r^2} ds^2$$

From which we conclude that

$$dx^2 = \frac{1}{r^2}\left(\frac{r^2 - r_o^2}{1+r'^2}\right) ds^2 = \frac{r^2}{r_o^2}\left(\frac{r^2 - r_o^2}{1+r'^2}\right) d\theta^2$$

Hence, we have that

$$\dot{x} = \frac{\pm r}{r_o}\left(\frac{r^2 - r_o^2}{1+r'^2}\right)^{\frac{1}{2}} = F(x)$$

where $dx/d\theta > 0$ for increasing x and $dx/d\theta < 0$ for decreasing x. Using a three-term Taylor expansion, we have

$$x(\theta) \approx x(\theta_o) + \dot{x}(\theta_o)(\theta-\theta_o) + \frac{1}{2}\ddot{x}(\theta_o)(\theta-\theta_o)^2$$

Letting

$$A = 1 + r'^2$$

and differentiating the following

$$\mp F(x) = A^{-\frac{1}{2}}(r^4 r_o^{-2} - r^2)^{\frac{1}{2}}$$

We have

$$\mp F'(x) = -r'r''A^{-\frac{3}{2}}(r^4 r_o^{-2} - r^2)^{\frac{1}{2}}$$
$$+ A^{-\frac{1}{2}}(r^4 r_o^{-2} - r^2)^{-\frac{1}{2}}(2r^3 r' r_o^{-2} - r^4 r_o^{-3} r_o' - rr')$$

Since

$$\ddot{x} = F(x)F'(x)$$

We finally have

$$\ddot{x} = \frac{r}{r_o^2 A}[r'(2r^2 - r_o^2) - r^3 r_o^{-1} r_o' - rr'r''(r^2 - r_o^2)A^{-1}]$$

Therefore, we have a simple second order method for moving from point to point along a quasi-geodesic path:

$$A = 1 + r'^2$$

$$B = r'(2r^2 - r_o^2) - r^3 r_o^{-1} r_o'$$

$$C = rr'r''(r^2 - r_o^2)$$

$$\dot{x} = \frac{\pm r}{r_o}\sqrt{\frac{r^2 - r_o^2}{A}}$$

$$\ddot{x} = \frac{r}{r_o^2 A}\left(B - \frac{C}{A}\right)$$

$$\Delta x = \dot{x}\,\Delta\theta + \frac{1}{2}\ddot{x}\,\Delta\theta^2 + O(\Delta\theta^3)$$

Note that these formulas are valid in the vicinity of turning points. This algorithm in conjunction with References 1 and 3 can be used to generate winding data with uniform error.

MONOTONE INEQUALITY DERIVATION

In this section, we derive the test condition which tells us whether or not an admissible polar radius function or path is capable of ultimately producing single-valued winding data.

Let P be a point on the path on the surface and let the taut filament be tangent to P and to the path. Let Q be the other end of the taut filament, called the filament delivery point, residing some constant distance H from the axis of the surface. This configuration implies a two-axis filament winding machine, but our resulting inequality applies to a three-axis machine with variable H as well.

Now,

$$Q = P + \alpha t$$

where t is the unit tangent vector to the path and α is some function of θ. Since Q resides at distance H from the axis, we have

$$H^2 = Q_\perp \cdot Q_\perp$$

where

$$v_\perp = \textit{component of vector } v \text{ } \textit{perpendicular to axis}$$

but

$$Q_\perp = P_\perp + \alpha t_\perp$$

Therefore

$$H^2 = P_\perp \cdot P_\perp + 2\alpha P_\perp \cdot t_\perp + \alpha^2 t_\perp \cdot t_\perp$$

or

$$1 = \frac{P_\perp \cdot P_\perp}{H^2} + \frac{2\alpha}{H}\frac{P_\perp \cdot t_\perp}{H} + \left(\frac{\alpha}{H}\right)^2 t_\perp \cdot t_\perp$$

Knowing P, t, and H, we could easily solve this equation for α to obtain Q, but our intent here is not to actually compute winding data, but rather to obtain a condition under which single-valued winding data is guaranteed to exist. So, if the path can be wound properly for a given H, then it must be windable for any H! We can therefore achieve considerable mathematical simplification by simply letting H become infinite. From the previous equation, we have

$$\frac{\alpha}{H} = \frac{1}{\|t_\perp\|} \quad as\ H \to \infty$$

where

$$\|v\| = \sqrt{v \cdot v}$$

Now, if λ is the angular displacement by which Q leads P, and R is the angular displacement of Q, we have

$$R = \theta + \lambda$$

and

$$P_\perp \cdot Q_\perp = \|P_\perp\|\|Q_\perp\| Cos\,\lambda = rH\,Cos\,\lambda$$

but

$$Q_\perp = P_\perp + \alpha t_\perp$$

Hence,

$$P_\perp \cdot (P_\perp + \alpha t_\perp) = r H \, Cos\,\lambda$$

or

$$Cos\,\lambda = \frac{r}{H} + \frac{\alpha}{H}\frac{P_\perp \cdot t_\perp}{r}$$

So,

$$Cos\,\lambda = \bar{P}_\perp \cdot \bar{t}_\perp = \phi \quad as\ H \rightarrow \infty$$

where

$$\bar{v} = \frac{v}{\|v\|}$$

Now,

$$t = \frac{dP}{ds} = \frac{\dot{P}}{\dot{s}}$$

but

$$P = ix + jy + kz$$

Therefore

$$P_\perp = jy + kz$$
$$t_\perp = \frac{(j\dot{y} + k\dot{z})}{\dot{s}}$$

so that

$$\bar{P}_\perp = \frac{jy + kz}{\sqrt{y^2 + z^2}} = \frac{jy + kz}{r}$$

$$\bar{t}_\perp = \frac{j\dot{y} + k\dot{z}}{\sqrt{\dot{y}^2 + \dot{z}^2}}$$

We therefore have that

$$\phi = Cos\,\lambda = \bar{P}_{\perp} \cdot \bar{t}_{\perp} = \frac{y\dot{y} + z\dot{z}}{r\sqrt{\dot{y}^2 + \dot{z}^2}}$$

But since

$$y = r\,Sin\,\theta,\; z = r\,Cos\,\theta$$

$$\dot{y} = \dot{r}\,Sin\,\theta + r\,Cos\,\theta$$

$$\dot{z} = \dot{r}\,Cos\,\theta - r\,Sin\,\theta$$

it easily follows that

$$y\dot{y} + z\dot{z} = r\dot{r}$$

$$\dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2$$

and that

$$\phi = \frac{\dot{r}}{\sqrt{\dot{r}^2 + r^2}}$$

Now, since we wish to find a condition equivalent to $dR/d\theta > 0$, we will need $d\lambda/d\theta$. Since

$$\phi = Cos\,\lambda$$

we have

$$\dot{\phi} = -Sin\,\lambda\,\dot{\lambda}$$

$$\dot{\lambda} = \frac{-\dot{\phi}}{\sqrt{1-\phi^2}}$$

$$1 - \phi^2 = \frac{r^2}{\dot{r}^2 + r^2}$$

$$\dot{\lambda} = \frac{-\dot{\phi}\sqrt{\dot{r}^2 + r^2}}{r} = \frac{-\dot{r}\,\dot{\phi}}{r\,\phi}$$

Differentiating

$$\phi = \dot{r}(\dot{r}^2 + r^2)^{-\frac{1}{2}}$$

we get

$$\dot{\phi}=\frac{\ddot{r}\phi}{\dot{r}}-\frac{\phi^3(\ddot{r}+r)}{\dot{r}}$$

and

$$\dot{\lambda}=\frac{\dot{r}^2-r\ddot{r}}{\dot{r}^2+r^2}$$

but

$$\dot{r}=\dot{x}r'$$

$$\ddot{r}=\ddot{x}r'+\dot{x}^2r''$$

therefore

$$\dot{\lambda}=\frac{\dot{x}^2r'^2-r(\ddot{x}r'+\dot{x}^2r'')}{\dot{x}^2r'^2+r^2}$$

Since we want $dR/d\theta>0$ and $R=\theta+\lambda$, we need

$$1+\dot{\lambda}>0$$

Now,

$$1+\dot{\lambda}=\frac{\dot{x}^2(2r'^2-rr'')+r^2-rr'\ddot{x}}{\dot{x}^2r'^2+r^2}$$

and since the denominator of this fraction is always positive, we need

$$\dot{x}^2(2r'^2-rr'')+r^2-rr'\ddot{x}>0$$

Now, recalling from the last section that

$$\dot{x}=\frac{\pm r}{r_o}\sqrt{\frac{r^2-r_o^2}{A}}$$

$$\ddot{x}=\frac{r}{r_o^2A}\left(B-\frac{C}{A}\right)$$

where

$$A = 1 + r'^2$$

$$B = r'(2r^2 - r_o^2) - r^3 r_o^{-1} r_o'$$

$$C = r r' r''(r^2 - r_o^2)$$

we have

$$\frac{r^2}{r_o^2}\left(\frac{r^2 - r_o^2}{A}\right)(2r'^2 - r r'') + r^2 - \frac{r^2 r'}{r_o^2 A}\left(B - \frac{C}{A}\right) > 0$$

Multiplying this inequality by

$$\frac{A^2 r_o^2}{r^2}$$

gives us

$$A(r^2 - r_o^2)(2r'^2 - r r'') + A^2 r_o^2 - r'(AB - C) > 0$$

but

$$AB - C = A[r'(2r^2 - r_o^2) - r^3 r_o^{-1} r_o'] - r r' r''(r^2 - r_o^2)$$

Therefore

$$\begin{gathered} A(r^2 - r_o^2)(2r'^2 - r r'') + A^2 r_o^2 \\ - r'A(r'(2r^2 - r_o^2) - r^3 r_o^{-1} r_o') \\ + r r'^2 r''(r^2 - r_o^2) > 0 \end{gathered}$$

Grouping terms involving r'' and factoring out A yields

$$\begin{gathered} -r r''(r^2 - r_o^2)(A - r'^2) \\ + A[2r'^2(r^2 - r_o^2) + A r_o^2 - r'^2(2r^2 - r_o^2) + r^3 r' r_o^{-1} r_o'] > 0 \end{gathered}$$

which simplifies to

$$
\begin{gathered}
-rr''(r^2-r_o^2) \\
+A[r'^2(2r^2-2r_o^2+r_o^2-2r^2+r_o^2)+r_o^2+r^3r'r_o^{-1}r_o'] > 0
\end{gathered}
$$

immediately yielding

$$rr''(r_o^2-r^2)+(1+r'^2)(r_o^2+r^3r'r_o^{-1}r_o') > 0$$

as the monotone winding inequality. Global satisfaction of this condition guarantees the existence of single-valued winding data.

WINDING DATA GENERATION

In this section we derive the equations for generating the nominal winding data that a two-axis winding machine would need to actually wrap a mandrel. The development here is not new, but we include it for the sake of completeness, following the notation developed thus far.

Recalling that

$$\alpha^2 t_\perp \cdot t_\perp + 2\alpha P_\perp \cdot t_\perp + r^2 - H^2 = 0$$

we solve for α, getting

$$\alpha = -\frac{P_\perp \cdot t_\perp}{t_\perp \cdot t_\perp} + \sqrt{\left(\frac{P_\perp \cdot t_\perp}{t_\perp \cdot t_\perp}\right)^2 + \frac{H^2-r^2}{t_\perp \cdot t_\perp}} > 0$$

but

$$P = ix+jy+kz, \quad t = \frac{i\dot{x}+j\dot{y}+k\dot{z}}{\dot{s}}$$

Therefore

$$P_\perp \cdot t_\perp = \frac{y\dot{y}+z\dot{z}}{\dot{s}} = \frac{r\dot{r}}{\dot{s}}$$

$$t_\perp \cdot t_\perp = \frac{\dot{y}^2+\dot{z}^2}{\dot{s}^2} = \frac{\dot{r}^2+r^2}{\dot{s}^2}$$

but

$$\dot{s}=\frac{r^2}{r_o}$$

and letting

$$K=\frac{1}{\dot{r}^2+r^2}$$

we have

$$P_\perp\cdot t_\perp=\frac{\dot{r}r_o}{r}$$

$$t_\perp\cdot t_\perp=\frac{r_o^2}{Kr^4}$$

$$\frac{P_\perp\cdot t_\perp}{t_\perp\cdot t_\perp}=\frac{Kr^3\dot{r}}{r_o}$$

So,

$$\alpha=-\frac{Kr^3\dot{r}}{r_o}+\sqrt{\frac{K^2r^6\dot{r}^2}{r_o^2}+\frac{Kr^4(H^2-r^2)}{r_o^2}}$$

$$=\frac{Kr^2}{r_o}\left(-r\dot{r}+\sqrt{r^2\dot{r}^2+\frac{H^2-r^2}{K}}\right)$$

Letting

$$\beta=r\dot{r},\ \gamma=\frac{H^2-r^2}{K},$$

and

$$u=-\beta+\sqrt{\beta^2+\gamma}=\frac{\gamma}{\beta+\sqrt{\beta^2+\gamma}}$$

we have

$$\alpha=\frac{Kr^2u}{r_o}$$

Recalling the general definition of λ, we have

$$Cos\,\lambda=\frac{r}{H}+\frac{\alpha P_{\perp}\cdot t_{\perp}}{rH}=\frac{r+Ku\dot{r}}{H}$$

Also,

$$T=x+\frac{\alpha\dot{x}}{\dot{s}}=x+\frac{\alpha r_{o}\dot{x}}{r^{2}}=x+Ku\dot{x}$$

We can therefore summarize the equations for generating the nominal winding data.

$$\dot{r}=\dot{x}r'$$

$$K=\frac{1}{\dot{r}^{2}+r^{2}}$$

$$\beta=r\dot{r},\ \gamma=\frac{H^{2}-r^{2}}{K}$$

$$u=-\beta+\sqrt{\beta^{2}+\gamma}\ \ if\ \beta<0$$

$$u=\frac{\gamma}{\beta+\sqrt{\beta^{2}+\gamma}}\ \ if\ \beta>0$$

$$\lambda=Cos^{-1}\left(\frac{r+Ku\dot{r}}{H}\right)$$

$$R=\theta+\lambda$$

$$T=x+Ku\dot{x}$$

SCALING OF WINDING DATA

In order to produce uniform spacing between windings and ensure complete covering of the surface, it is necessary to lengthen the nominal range of rotation for one circuit slightly and to compute the number of circuits and revolutions needed to cover the surface. We will try to do this in a more straightforward manner than that of Reference 1.

If the width of the band of filaments is b, the circumferential cover afforded by the band is given approximately by

$$b = c\,Cos\,\omega = c\sqrt{1-\frac{r_o^2}{r^2}}$$

Now, a circuit is defined as the action of winding from one polar parallel to the other polar parallel and back again. So, if we wind n circuits to get two full layers of coverage, we want

$$\frac{nc}{2} \geq 2\pi r \quad for\ all\ x$$

This is equivalent to

$$n \geq \frac{4\pi}{b}\sqrt{r^2-r_o^2}$$

But we want this inequality to hold for all x, so we must have

$$n \geq \frac{4\pi\sqrt{M}}{b}$$

where

$$M = Max\ r^2-r_o^2$$

Hence, we define the nominal minimum number of circuits necessary for complete double layer coverage as

$$n_o = ceiling\left(\frac{4\pi\sqrt{M}}{b}\right)$$

This number of circuits used with our nominal winding data has no chance of producing a uniform wrap, however.

Suppose the net angular displacement of the filament delivery point, or the point on the path, or the mandrel itself for a complete circuit is R_o. Let i, j, and k be integers here. We have

$$R_o = 2\pi i + \rho$$

where

$$i = floor\left(\frac{R_o}{2\pi}\right)$$

Now define

$$R_1 = 2\pi\left(i + \frac{j}{k}\right)$$

where R_1 is slightly greater than R_o

$$R_1 \geq R_o$$

implies

$$j \geq \frac{k\rho}{2\pi}$$

Therefore define

$$j = ceiling\left(\frac{k\rho}{2\pi}\right)$$

It is necessary that j and k be relatively prime, i.e., $GCF(j,k)=1$. If this is not the case for a given k, we simply do not use that value of k. The k value is the number of winding groups that appear on the surface and finally coalesce as a layer is completed. If we scaled the nominal winding data using R_1, our winding pattern would close after only k circuits, so we define

$$R_2 = R_1 + \epsilon$$

We want ϵ to be defined such that after we wind n circuits, we will have returned to the very same point on the surface at which we started, completing two layers of uniform wrap. That is, we want

$$nR_2 = nR_1 + n\epsilon = 2\pi n\left(i + \frac{j}{k}\right) + n\epsilon = 2\pi m$$

where m is the total number of revolutions of the surface needed to wind a double layer. Hence, we want

$$m = n\left(i + \frac{j}{k}\right) + \frac{n\epsilon}{2\pi}$$

to be an integer. Moreover, we want $GCF(m,n)=1$. If we wind k circuits, our net angular displacement will be

$$kR_2 = 2\pi(ki + j) + k\epsilon$$

and the filament will begin to lay down alongside previous windings. Hence,

$$\delta = k\epsilon$$

will be the angular displacement between adjacent windings in a single layer. But, irrespective of the j and k values, we must have

$$\frac{n\delta}{2}=2\pi$$

if the winding pattern is to close exactly, for the first time, after exactly n circuits have been wound, completing a double layer. Therefore,

$$\frac{n\epsilon}{2\pi}=\frac{nk\epsilon}{2\pi k}=\frac{n\delta}{2\pi k}=\frac{4\pi}{2\pi k}=\frac{2}{k}$$

and we have

$$m=ni+\frac{nj+2}{k}$$

Now, if m is to be an integer, k must divide $nj+2$ exactly. Is there any guarantee that we can find an $n>=n_o$ such that this quotient is an integer? A fundamental theorem of number theory states that given positive integers j and k, there exist other integers u and v such that $uj+vk=GCF(j,k)$. In our case, $GCF(j,k)=1$. Therefore, we are guaranteed integers u and v such that

$$\frac{-2uj+2}{k}=2v$$

So, if we add l to $-2u$, we have

$$\frac{(-2u+l)j+2}{k}=w=2v+\frac{jl}{k}$$

and if we pick l to be a sufficiently large multiple of k, w will be a positive integer. Therefore, there are an infinite number of values of $n>=n_0$ such that k divides $nj+2$. We only need the smallest one. The reason that we have defined n as the number of circuits necessary to lay down a **double** layer is that we want the windings of the second layer to be staggered exactly half a bandwidth from the windings of the first layer in order to most efficiently cover the joins between adjacent windings of the first layer. This staggering of the windings of alternate layers by winding through exactly $2\pi m/n$ radians per circuit should maximize ultimate strength. Our goal then is to find values of j, k, m, and n such that the surface is completely, uniformly, and efficiently covered for minimal m.

Now, the angular displacement between adjacent winding groups will be about $2\pi/k$ initially, so we will need

$$\frac{2\pi}{k} > \delta \quad or \quad k < \frac{n}{2}$$

Therefore, in the algorithm to follow, we will require that

$$k_{\max} \leq floor\left(\frac{n_o}{2}\right)$$

The larger the bandwidth, the smaller n_o will be. Hence, no acceptable k may exist without lowering the bandwidth. This situation is rather unlikely to occur, however. The following algorithm is suggested.

$$m := k_{\max} n_o (i+1), \; n := m$$

$$for \; 2 \leq k \leq k_{\max}$$

$$j := ceiling\left(\frac{k\rho}{2\pi}\right)$$

$$if \; j > 0 \wedge j < k \wedge GCF(j,k) = 1$$

$$then \; find \; smallest \; N \geq n_o \; such \; that \; k \; divides \; Nj+2$$

$$M := Ni + \frac{Nj+2}{k}$$

$$if \; GCF(M,N) = 1 \wedge [M < m \vee (M = m \wedge N < n)]$$

$$then \; n := N, \; m := M \quad etc.$$

When this algorithm terminates, we can define the scaling factor S by

$$S = \frac{2\pi m}{n R_o}$$

where

$$R_o = R_{final} - R_{initial}$$

The scaled R values are therefore given by

$$R_s = R_{initial} + S(R - R_{initial})$$

MODIFIED POLAR RADIUS FUNCTION

The larger the bandwidth b is relative to the mandrel diameter, the more important it becomes to raise the polar radius function slightly thereby bringing the turning points in slightly from their nominal end positions. This is necessary because, ideally, we want to generate a path for the midline of the band.

Let s denote meridional length and a dot denote differentiation with respect to s here. For a meridian, we have

$$ds^2 = dx^2 + dr^2 = dx^2 + r'(x)^2 dx^2$$

Hence

$$\dot{x} = (1 + r'(x)^2)^{-\frac{1}{2}}$$

$$\ddot{x} = -r'(x)r''(x)(1 + r'(x)^2)^{-2}$$

A three-term Taylor expansion for x as a function of s is given by

$$x(s) \approx x(a) + \dot{x}(a)(s-a) + \frac{1}{2}\ddot{x}(a)(s-a)^2$$

$$= x(a) + (1 + r'(x(a))^2)^{-\frac{1}{2}}(s-a) - \frac{1}{2}r'(x(a))r''(x(a))(1 + r'(x(a))^2)^{-2}(s-a)^2$$

Let the length of the mandrel be L and let M be the total length of the meridian between nominal turning points at $x=0$ and $x=L$. The new left turning point ($s=b/2$, $a=0$, $x(a)=0$) will then be

$$x_l = x\left(\frac{b}{2}\right) = (1 + r'(0)^2)^{-\frac{1}{2}}\frac{b}{2} - \frac{1}{2}r'(0)r''(0)(1 + r'(0)^2)^{-2}\frac{b^2}{4}$$

$$= \frac{b}{2\sqrt{1 + r'(0)^2}} - \frac{b^2 r'(0)r''(0)}{8(1 + r'(0)^2)^2}$$

and the new right turning point ($s=M-b/2$, $a=M$, $x(a)=L$) will be

$$x_r = x\left(M - \frac{b}{2}\right) = L + (1 + r'(L)^2)^{-\frac{1}{2}}\left(-\frac{b}{2}\right) - \frac{1}{2}r'(L)r''(L)(1 + r'(L)^2)^{-2}\frac{b^2}{4}$$

$$= L - \frac{b}{2\sqrt{1 + r'(L)^2}} - \frac{b^2 r'(L)r''(L)}{8(1 + r'(L)^2)^2}$$

These approximations to the new turning points should be adequate for most cases. Now we must raise the polar radius function so that it agrees with the radius function at the new turning points.

Define

$$\delta_l = r(x_l) - r_o(x_l), \quad \delta_r = r(x_r) - r_o(x_r)$$

$$\delta(x) = \delta_l + \frac{(x-x_l)(\delta_r - \delta_l)}{x_r - x_l}$$

If we denote the modified polar radius function by r_{oo}, we then have

$$r_{oo}(x) = r_o(x) + \delta(x)$$

$$r'_{oo}(x) = r'_o(x) + \frac{\delta_r - \delta_l}{x_r - x_l}$$

and

$$r_{oo}(x_l) = r(x_l), \quad r_{oo}(x_r) = r(x_r)$$

We would therefore use r_{oo} in place of r_o in the equations of all sections previous to this one.

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