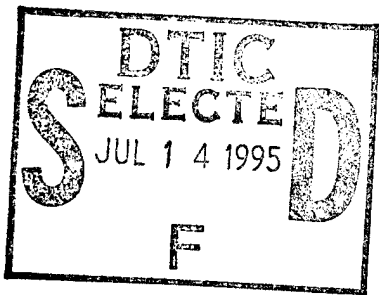


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TECHNICAL REPORT ARCCB-TR-95018

**IMPLEMENTING TENSOR ANALYSIS IN  
*Mathematica* WITH ILLUSTRATIONS  
FROM SCHWARZCHILD GRAVITATION**



L.V. MEISEL

MARCH 1995

	<p><b>US ARMY ARMAMENT RESEARCH, DEVELOPMENT AND ENGINEERING CENTER</b> CLOSE COMBAT ARMAMENTS CENTER BENÉT LABORATORIES WATERVLIET, N.Y. 12189-4050</p>	
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## TABLE OF CONTENTS

INTRODUCTION .....	1
PART I. TENSOR ANALYSIS IN <i>Mathematica</i> .....	1
Covariant Metric Tensor .....	1
Christoffel Symbols .....	2
The Curvature Tensor .....	3
The Ricci Tensor .....	4
PART II. APPLICATION: THE SCHWARZCHILD METRIC .....	6
Conventional Notation .....	6
The Einstein Equations .....	7
Solution of the Einstein Equations .....	7
The Schwarzschild Metric .....	9
REFERENCES .....	12

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## INTRODUCTION .

Andrew and Fleming (ref 1) discussed the calculation of null geodesics for the Schwarzschild, Kerr-Newman, and Winicour metrics. Numerical computation employed FORTRAN code, which was produced by *Mathematica* (ref 2). The *Mathematica* code employed to generate the FORTRAN code for the neutral ( $Q = 0$ ) Kerr-Newman geodesics was presented. Reference 1 provides a convincing demonstration of the utility of symbolic manipulation software for the development of error-free, complex FORTRAN (or C) code.

In this report, a general approach to problems in tensor analysis (ref 3) employing *Mathematica* is described. The standard expressions of tensor calculus are transcribed directly into *Mathematica* modules in Part I. The operation of the modules is illustrated as they are described by application to a simple two-dimensional curved space. The present formulation allows enumeration and simplification of complex tensor equations, employing built-in *Mathematica* functions (such as `Expand[]`, `Together[]`, or `Simplify[]`).

The code is applied to address some simple problems pertinent to Schwarzschild space-time in Part II. Employing the built-in *Mathematica* operator `Simplify[]`, the modules described produced FORTRAN coded Runge-Kutta geodesic equations for Kerr-Newman space-time in less than thirty minutes on a 386-based PC operating at 30 MHz. Of course, if simplification based on more efficient *Mathematica* operations (e.g., combinations of `Apart[]`, `Expand[]`, and `Together[]`) can be achieved, symbolic computation times can be dramatically reduced. Computation times are much shorter for the Schwarzschild geodesic equations.

## PART I. TENSOR ANALYSIS IN *Mathematica*

### Covariant Metric Tensor

The starting point for a typical problem in tensor analysis is the specification of the (covariant) metric tensor.

The operation of the *Mathematica* tensor analysis modules defined here is demonstrated by applications to geometry on the surface of a helicoid in  $R^3$  having

$$g = \|g_{ij}\| = \begin{pmatrix} 1 & 0 \\ 0 & c^2 + x[1][s]^2 \end{pmatrix}$$

where the coordinates are  $x[1][s]$  and  $x[2][s]$  and  $c$  is constant. This  $g$  is referred to as the "example metric" in Part I. One encodes this  $g$  directly into *Mathematica* via

$$g = \text{DiagonalMatrix}\{1, c^2 + x[1][s]^2\};$$

Note that coordinates are given in the form  $x[i][s]$ . For the current analysis, one could simply use  $x[i]$ . However, as in Part II, one is often interested in derivatives with respect to the line element  $s$ , and we choose to include this dependence from the beginning.

## Christoffel Symbols

The first step in a tensor problem is the computation of the Christoffel symbols. Christoffel symbols of the first kind are defined as

$$\Gamma_{i\alpha\beta} = \Gamma_{i\beta\alpha} = \frac{1}{2} \left( \frac{\partial g_{\alpha i}}{\partial x^\beta} + \frac{\partial g_{i\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^i} \right)$$

Christoffel symbols of the second kind are given by

$$\Gamma_{\alpha\beta}^{\delta} = \sum_i g^{\delta i} \Gamma_{i\alpha\beta} = g^{\delta i} \Gamma_{i\alpha\beta}$$

where the last equality introduces "summation convention" which is adopted in this report and the contravariant metric tensor  $g^{ij}$  is the inverse of the covariant metric, i.e.,

$$\delta_{\beta}^{\alpha} = g^{\alpha i} g_{i\beta}$$

The following *Mathematica* module defines the contravariant metric tensor and Christoffel symbols for arbitrary metric tensors in spaces of arbitrary dimension.

```

setup[g_]:=Module[{dim=Length[g]},
  ginv =Together[Inverse[g]];
  GAMMA =Array[g1,{dim,dim,dim}];
  gamma =Array[g2,{dim,dim,dim}];
  Do[ Do[ Do[GAMMA[[i,k,j]]=GAMMA[[i,j,k]]=
    Together[(1/2)(D[g[[i,j]],x[k][s]]+
      D[g[[k,i]],x[j][s]]-D[g[[j,k]],x[i][s]]),
      {i,dim},{k,j,dim},{j,dim}];
  Do[ Do[ Do[gamma[[i,k,j]]=gamma[[i,j,k]]=
    Together[Sum[ginv[[i,l]]GAMMA[[l,j,k]],{l,dim}]],
      {i,dim},{k,j,dim},{j,dim} ] ] ]

```

N. B., the partial derivative of a function  $f[u,v,\dots,x,\dots]$  with respect to the variable  $x$  is computed via  $D[f[u,v,\dots,x,\dots],x]$  in *Mathematica*. N.B., the *Mathematica* `Together[]` operator was effective in simplifying the resulting expressions for a number of "simple problems." Of course, the user could apply an other built-in (e.g., `Simplify[]`) or user-defined *Mathematica* operator.

After `setup[g]` is run, the contravariant metric tensor is returned as `ginv`, Christoffel symbols of the first kind are in the array `GAMMA`, i.e.,  $\Gamma_{ijk} = \text{GAMMA}[[i,j,k]]$ , and Christoffel symbols of the second kind are in the array `gamma`, i.e.,  $\Gamma_{j k}^i = \text{gamma}[[i,j,k]]$ .

To compute the Christoffel symbols and  $g^{-1}$  for the test metric, enter

setup[g].

*Example:* Display gamma for the example metric,

gamma

returns

$$\left\{ \left\{ \left\{ 0, 0 \right\}, \left\{ 0, -x[1][s] \right\} \right\}, \left\{ \left\{ 0, \frac{x[1][s]}{2} \right\}, \left\{ \frac{x[1][s]}{2}, 0 \right\} \right\} \right\},$$

$$\left\{ \left\{ 0, 0 \right\}, \left\{ 0, -x[1][s] \right\} \right\},$$

*Exercise:* Code a *Mathematica* module to compute covariant derivatives.

### The Curvature Tensor

The curvature tensor can be computed directly from the  $\Gamma_{\alpha\beta}^i$

$$B_{\alpha j k}^i = \frac{\partial \Gamma_{\alpha j}^i}{\partial x^k} - \frac{\partial \Gamma_{\alpha k}^i}{\partial x^j} + \Gamma_{\alpha j}^{\beta} \Gamma_{\beta k}^i - \Gamma_{\alpha k}^{\beta} \Gamma_{\beta j}^i$$

Transcription of the expressions for the  $B_{\alpha j k}^i$  is straightforward,

```
curvature[gamma_,i_,a_,j_,k_] := Module[{b,dim=Length[gamma]},
  Together[
    D[gamma[[i,a,j]],x[k][s]]-D[gamma[[i,a,k]],x[j][s]]+
    Sum[gamma[[b,a,j]]gamma[[i,b,k]]-
      gamma[[b,a,k]]gamma[[i,b,j]],{b,dim} ] ]
```

and full curvature tensor is returned by

```
curvature[gamma_] := Module[{dim=Length[gamma]},
  B=Array[r1,{dim,dim,dim,dim}];
  Do[Do[Do[Do[B[[i,a,j,k]]=curvature[gamma,i,a,j,k],
    {i,dim}],{a,dim}],{j,dim}],{k,dim}];B]
```

The covariant form,  $R_{hijk} = g_{ha} B_{i j k}^a$  is called the Riemann-Christoffel curvature tensor.

*Example:* Display the curvature tensor for the example metric

B = curvature[gamma]

returns

$$\left\{ \left\{ \left\{ 0, 0 \right\}, \left\{ 0, 0 \right\} \right\}, \left\{ \left\{ 0, \frac{c^2}{c^2 + x[1][s]^2} \right\}, \left\{ -\frac{c^2}{c^2 + x[1][s]^2}, 0 \right\} \right\} \right\},$$

$$\left\{ \left\{ \left\{ 0, -\frac{c^2}{c^2 + x[1][s]^2} \right\}, \left\{ \frac{c^2}{c^2 + x[1][s]^2}, 0 \right\} \right\}, \left\{ \left\{ 0, 0 \right\}, \left\{ 0, 0 \right\} \right\} \right\}$$

### The Ricci Tensor

The Ricci tensor plays an important role in relativity theory. It is defined as a contraction of the curvature tensor,

$$R_{ij} = B_{iaj}^a = \frac{\partial \Gamma_{i\alpha}^a}{\partial x^j} - \frac{\partial \Gamma_{ij}^a}{\partial x^\alpha} + \Gamma_{i\alpha}^\beta \Gamma_{\beta j}^\alpha - \Gamma_{ij}^\beta \Gamma_{\beta\alpha}^\alpha$$

Thus, to Print[] the components of  $R_{ij} = B_{iaj}^a$ , enter the *Mathematica* statement  
 Do[Do[Print["R(",i,";",j,") = ",Simplify[ Sum[B[[a,i,a,j]],{a,2} ] ],  
 {i,2}], {j,2}]

which yields

$$R(1,1) = \frac{c^2}{c^2 + x[1][s]^2}$$

$$R(2,1) = 0$$

$$R(1,2) = 0$$

$$R(2,2) = \frac{c^2}{c^2 + x[1][s]^2}$$

N.B., Sum[expression,{i,n}] returns the sum of expression evaluated for i-values running from 1 to (the integer) n. Do[expression,{i,n}] works similarly.

We also transcribe separate *Mathematica* modules to compute the Ricci tensor,

```
Ricci[gamma_{i,j}]:=Module[{m,n,dim=Length[gamma]},
Together[Sum[D[gamma[[m,i,m]],x[j][s]]-D[gamma[[m,i,j]],x[m][s]]+
Sum[gamma[[n,i,m]]gamma[[m,n,j]]-
gamma[[n,i,j]]gamma[[m,n,m]],{n,dim}],{m,dim}]]
```

and

```
Ricci[gamma_]:=Module[{dim=Length[gamma]},
R=Array[r1,{dim,dim}];
Do[Do[R[[i,j]]=Ricci[gamma,i,j],{i,dim}],{j,dim}];R]
```

*Example:* Direct computation of Ricci tensor for the example metric.

```
Ricci[gamma]
```

returns

$$\left\{ \left\{ \frac{c}{c^2 + x[1][s]^2}, 0 \right\}, \left\{ 0, \frac{c}{c^2 + x[1][s]^2} \right\} \right\}$$

which is identical to the results obtained by contraction of B, as expected.

*Exercise:* Compute the Christoffel symbols and Ricci tensor for

$$\|g_{ij}\| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

## PART II. APPLICATION: THE SCHWARZCHILD METRIC .

Development of a spherically-symmetric  $\|g_{ij}\|$  consistent with the Einstein equations.  
Assumption:  $\|g_{ij}\|$  is spherically-symmetric. Assume that

$$g = \|g_{ij}\| = \begin{pmatrix} L[r] & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & M[r] \end{pmatrix}$$

$$= \begin{pmatrix} L[x[1][s]] & 0 & 0 & 0 \\ 0 & -x[1][s]^2 & 0 & 0 \\ 0 & 0 & -x[1][s]^2 \sin^2(x[2][s]) & 0 \\ 0 & 0 & 0 & M[x[1][s]] \end{pmatrix}$$

where the functions  $L[r]$  and  $M[r]$  are to be determined. Encode via,

`g=DiagonalMatrix[{L[x[1][s]],-x[1][s]^2,-(x[1][s]Sin[x[2][s]])^2,M[x[1][s]]};`

### Conventional Notation

Direct substitution is achieved in *Mathematica*, via the construction

`expression/.u->v`

which returns *expression* with occurrences of *u* replaced by *v*. Thus, to present output in {r,q,j,t}-form, define the following replacement rules:

`rt={x[1][s]->r,x[2][s]->q,x[3][s]->j,x[4][s]->t};`

N.B., *Mathematica* 2.1 does *not* have Greek fonts. We substitute *j* for *phi* and *q* for *theta* here and in the sequel with a text editor.

Compute the Christoffel symbols, etc.

`setup[g];`

## The Einstein Equations

The Einstein equations require that  $R_{ij} = 0$ . Compute and display the Ricci tensor.

$$R = \text{Simplify}[\text{Ricci}[\text{gamma}]]/.rt$$

returns the Ricci tensor for the spherically-symmetric metric function,

$$\left\{ \left\{ -\frac{L'[r]}{r L[r]} - \frac{L'[r] M'[r]}{4 L[r] M[r]} - \frac{M'[r]^2}{4 M[r]^2} + \frac{M''[r]}{2 M[r]}, 0, 0, 0 \right\}, \right. \\ \left. \left\{ 0, -1 - \frac{1}{L[r]} + \frac{r L'[r]}{2 L[r]^2} + \frac{r M'[r]}{2 L[r] M[r]}, 0, 0 \right\}, \right. \\ \left. \left\{ 0, 0, (\text{Sin}[T])^2 (-2 L[r] M[r] - 2 L[r]^2 M[r]^2 + \right. \right. \\ \left. \left. r M[r] L'[r] - r L[r] M'[r]) / (2 L[r]^2 M[r]), 0 \right\}, \right. \\ \left. \left\{ 0, 0, 0, \frac{M'[r]}{r L[r]} - \frac{L'[r] M'[r]}{2 L[r]^2} - \frac{M'[r]^2}{4 L[r] M[r]} + \frac{M''[r]}{2 L[r]} \right\} \right\}$$

## Solution of the Einstein Equations

We seek functions  $L[r]$  and  $M[r]$  such that the Einstein equations are satisfied.

*Step 1.* Eliminate the  $M''[r]$  terms from the 1,1 and 4,4 parts of  $R_{ij} = R[[i,j]] = 0$ .

$$\text{temp} = ((\text{Simplify}[(R[[1,1]]M[r[s]] - R[[4,4]]L[r[s]])]/.rt)) = 0)$$

returns

$$-\left( \frac{M[r] L'[r] + L[r] M'[r]}{r L[r]} \right) == 0$$

Step 2. Solve for L[r] in terms of M[r]. Use

$$\|g_{ij}\|_{r \rightarrow \infty} \rightarrow \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the *Mathematica* operator DSolve[] to obtain Rule0:

```
Rule0=ExpandAll[Flatten[DSolve[
  {L[Infinity]==-1,M[Infinity]==1,temp},L[r],r]]]
```

*Mathematica* issues a "warning:"

Solve::ifun:

Warning: Inverse functions are being used by Solve, so some solutions may not be found.

then returns the solution of interest

$$\{L[r] \rightarrow -\frac{1}{M[r]}\}$$

Step 3. *Mathematica* will not apply the rule for L[r] to replace L'[r]. Thus, one specifies a separate rule for L'[r] and defines Rule1 as follows:

```
Rule1=Flatten[{Rule0,Solve[temp,L'[r]]/.Rule0}]/.r->x[1][s]
```

which returns

$$\{L[x[1][s]] \rightarrow -\frac{1}{M[x[1][s]]}, L'[x[1][s]] \rightarrow \frac{M'[x[1][s]]}{M[x[1][s]]}\}$$

N.B., Flatten[{{u},{v}}] returns {u,v}, etc. Thus, the subrules comprising Rule1 will be applied consecutively.

Step 4. Use Rule1 and  $R_{22} = 0$  to define the form of M. Express M in a rule (Rule2).

```
Rule2=Flatten[DSolve[
  0==Factor[Expand[R[[2,2]]/.Rule1]]/.rt,M,r]]]
```

which returns the rule governing the function M,

$$\{M \rightarrow (1 + \frac{C[1]}{x[1]^2})\}$$

where C[1] is a constant of integration. The rule on the function works as expected, i.e.,

M[x[1][s]]/.Rule2

returns

$$1 + \frac{C[1]}{x[1][s]^2}$$

### The Schwarzschild Metric

Applying Rule1 and Rule2, one obtains the form of the spherically-symmetric metric tensor consistent with R = 0:

gw=g/.Rule1/.Rule2/.rt

returns

$$\left\{ \left\{ -\frac{1}{1 + \frac{C[1]}{r^2}}, 0, 0, 0 \right\}, \left\{ 0, -r^2, 0, 0 \right\}, \left\{ 0, 0, -r^2 \sin^2[q], 0 \right\}, \left\{ 0, 0, 0, 1 + \frac{C[1]}{r^2} \right\} \right\}$$

As expected, this is the Schwarzschild metric form. The usual form of the metric tensor is obtained by identifying the integration constant C[1] with  $-2 G M/c^2$ , where G is the gravitational constant, M is the central mass, and c is the speed of light.

*Exercise:* Demonstrate that R = 0 for Schwarzschild gravitation.

*Exercise:* The geodesic trajectories are given by solutions of the geodesic equations

$$\frac{d^2 x[i][s]}{ds^2} + \Gamma_{\alpha\beta}^i \frac{dx[\alpha][s]}{ds} \frac{dx[\beta][s]}{ds} = 0$$

Derive the geodesic equations for

$$\|g_{ij}\| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

*Exercise:* Derive the geodesic equations for the "example metric,"

$$\|g_{ij}\| = \begin{vmatrix} 1 & 0 \\ 0 & x[1][s]^2 + c^2 \end{vmatrix}$$

*Exercise:* Planetary orbits.

1. Derive the geodesic equations for the Schwarzschild metric.
2. Demonstrate that if  $q[0] = p/2$  and  $q'[0] = 0$ , then  $q[s] = p/2$ .
  - a. Express the geodesic equations for  $q[s] = p/2$ .
3. Show that  $r[s]^2 j'[s] = h = \text{constant}$ .
4. Show that  $(1 - (2G M/c^2) / r[s]) t'[s] = K = \text{constant}$ .
5. Use the results of parts 1 through 4 and

$$1 = \frac{ds^2}{ds^2} = g_{ij} \frac{dx[i][s]}{ds} \frac{dx[j][s]}{ds}$$

$$= g_{ij} x[i]'[s] x[j]'[s] \rightarrow \sum_{i=1}^4 g_{ij} x[i]'[s]^2$$

to derive the *Mathematica* expression

$$1 == \frac{\frac{K^2}{x[1][s]^2} + \frac{h^2}{x[1][s]^4} + (x[1]'[s])^2}{1 + \frac{C[1]^2}{x[1][s]^2}}$$

6. Make the change of variables,

$$x[1][s] \rightarrow 1/u[x[3][s]],$$

and employ the condition derived in 3, to establish that

$$(x[1])'[s] \rightarrow -(h u'[x[3][s]])$$

and derive the equations:

$$1 = -\left(\frac{2}{h} u[\text{phi}] \right)^2 + \frac{K}{1 + C[1] u[\text{phi}]} - \frac{2}{h} \frac{u'[\text{phi}]^2}{1 + C[1] u[\text{phi}]}$$

$$u'[\text{phi}]^2 + u[\text{phi}]^2 = -h^2 + \frac{2}{h} \frac{K C[1] u[\text{phi}]}{1 + C[1] u[\text{phi}]} - C[1] u[\text{phi}]^3$$

and

$$u''[\text{phi}] + u''[\text{phi}] = -\left(\frac{C[1]}{2}\right) - \frac{3 C[1] u[\text{phi}]}{2}$$

The last equation for  $u[\text{phi}] (= 1/r[j])$  frequently serves as the basis for a discussion of the advance of the perihelion of planetary orbits. See, for example, Crandall (ref 4). How should this approach be modified to treat photon trajectories?

*Exercise:* Compute the Christoffel symbols, curvature tensor, Ricci tensor, and geodesic equations for charge-free (i.e.,  $Q = 0$ ) Kerr-Newman gravitation. Demonstrate that  $R = 0$  for the Kerr-Newman metric.

## REFERENCES

1. Keith Andrew and Charles G. Fleming, *Comp. in Phys*, Vol. 6, 1992, p. 498.
2. Stephen Wolfram, *Mathematica, A System for Doing Mathematics by Computer*, Addison-Wesley, Redwood City, CA, 1991.
3. The expressions presented in this report may be found in any textbook that addresses Riemann geometry. See, for example, Harry Lass, *Vector and Tensor Analysis*, The Maple Press, York, PA, 1950.
4. Richard E. Crandall, *Mathematica for the Sciences*, Addison-Wesley, Redwood City, CA, 1991.

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