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ASYMPTOTIC PROPERTIES OF A SENSOR ALLOCATION MODEL

by

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Asymptotic Properties of a Sensor Allocation Model

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0. Introduction and Summary

Sensor resources are to be allocated among I non-overlapping but possibly contiguous geographical locations called *nodes*. Each sensor look at a node results in an observation, with error, of the number of *units* that are at the node. In military applications *units* may be individual assets such as tanks or ships, or possibly small groups organized as platoons or companies and in geographical proximity; in ecological applications, they might be animals, singly or in groups, or particular vegetation types. The observations are then to be used to estimate the numbers of units at each node. The problem is to allocate sensor resources so as to minimize a measure of the error of the estimates, and of the estimate of the sum of the numbers at all nodes. It is assumed that the units remain on the nodes and do not migrate, although for some sensor types the units must be in motion on the node at least some of the time in order for detection to take place.

The objective of this paper is to propose and study *adaptive allocation* of sensor effort in such a way as to focus the sensor's attention sequentially and purposefully on nodes so as to pay most attention to those nodes about which the greatest uncertainty, or interest, currently prevails. This problem has features in common with adaptive bin-packing problems (cf. Gaver et al., 1995) and adaptive allocation of customers to servers (cf. Gaver et al., 1993).

Numerical examples show that a properly selected sequentially adaptive rule will provide estimates of improved precision.

A related investigation that has recently come to our attention is that by Thompson and Seber (1994). Their procedures can also be analyzed using the methodology proposed in this paper.

1. The Model

There are I nodes. Assume that node i contains r_i units. Sensor resources are allocated to one of the I nodes at times that occur according to a Poisson process with rate λ. The sensor resource is allocated to node i with probability α_i which is tailored to depend on past allocations in a purposeful way. Let $Z_n(i)$ be the n^{th} observation of node i; a simple model is that $\{Z_n(i);\ n = 1, 2, \ldots\}$ are iid with binomial distribution with r_i trials and known probability of success p_i which is the probability of unit detection at node i. We assume for illustration that the number of units on a node does not change, although an adaptive scheme of the type proposed should effectively follow changes in node population.

Let $N_i(t)$ be the number of times in $[0,t]$ that node i is observed by a sensor. Let

$$V(i;t) = Z_1(i) + \ldots + Z_{N_i(t)}(i)$$

be the sum of all the observations during $[0,t]$ for node i.

Under some conditions $V(i;\ t)$ has a binomial distribution with $N_i(t)r_i$ trials and probability of success p_i, given $N_i(t)$, although this assumption may not be especially accurate in general. Assuming it to be adequate for the moment, an estimate for the number of units at node i based on observations made during $[0,t]$ is

$$\hat{r}_i(t) = \frac{V(i;t)}{N_i(t)p_i} \tag{1.1}$$

with

$$\mathrm{Var}[\hat{r}_i(t)] = \frac{r_i N_i(t) p_i (1-p_i)}{N_i(t)^2 p_i^2} \tag{1.2}$$

$$= \frac{r_i(1-p_i)}{N_i(t)p_i} \approx \frac{\hat{r}_i(t)(1-p_i)}{N_i(t)p_i} = \frac{V(i;t)(1-p_i)}{N_i(t)^2 p_i^2}. \tag{1.3}$$

There has been much statistical attention paid to estimating the number of trials in a binomial distribution, given probability of success; for recent discussion see Hall (1994). Here we use the simplest such estimator.

Now consider the following *adaptive allocation rule*. A sensor resource arriving at time t is allocated to node i with probability

$$\alpha_i(N_i(t), V_i(t); N(t), V(t)) = \frac{h_i(N_i(t), V_i(t))}{\sum_{j=1}^{I} h_j\big(N_j(t), V_j(t)\big)} \tag{1.4}$$

where the h_i are strictly positive sufficiently smooth functions such that $h_j(c^p x, c^p y) = c^p h_j(x, y)$.

One possible form for h_i is

$$h_i(x,v) = \left[\frac{a_i}{x} + \frac{v(1-p_i)}{x^2 p_i^2}\right]^{\gamma} \quad \text{for } \gamma > 0 \text{ and } a_i > 0; \tag{1.5}$$

if it is assumed that given $N_i(t)$, $V_i(t)$ is binomially distributed with mean $N_i(t)p_i$, then from (1.3) it follows that for this function h_i, sensor resources tend to be allocated to those nodes for which the variance, and hence its square root, the standard error of the estimated number of units, is the largest; the probability of allocation to the most uncertain node (by this measure) increases rapidly, approaching unity as γ increases. This tends to bring down that standard error quickly, and to equate standard errors of the estimates across nodes. Clearly, alternative measures of overall sensor performance are feasible, and possibly desirable, such as ones that endeavor to equalize fractional or percent error on nodes, or ones that also respond to an independent measure of importance of the units on a node. Additionally, node contents may be of various types, which can be considered. Thus, the present discussion is of an illustration of an adaptive allocation scheme.

The purposeful allocation of (1.4) introduces dependence between $\{N_i(t);\, t \geq 0\}$ and $\{V_i(t);\, t \geq 0\}$. Asymptotic results for the means $E[N_i(t)]$ and $E[V_i(t)]$ as the rate of the Poisson process $\lambda \to \infty$ are obtained in Section 2. It is shown that the purposeful allocation estimate $\hat{r}_i(t)$ is asymptotically unbiased. Section 3 presents results for the asymptotic means for specific form of function h_i (1.5). Section 4 discusses asymptotic results for the second moments of $\{N_i(t);\, t \geq 0\}$ and $\{V_i(t);\, t \geq 0\}$ and presents approximate expressions for the $Var[\hat{r}_i(t)]$ and $Var\left[\sum_i \hat{r}_i(t)\right]$. Section 5 describes a simpler Poisson approximation. Section 6 presents results from simulation experiments.

2. First-Moment Calculations and Asymptotics

Note that for $h > 0$

$$E\left[N_i(t+h) \middle| N(t), V(t)\right] = N_i(t) + \lambda \alpha_i(N,V)h. \tag{2.1}$$

Thus

$$E\left[N_i(t+h)\right] = E\left[N_i(t)\right] + \lambda E\left[\alpha_i(N,V)h\right]. \tag{2.2}$$

Assuming derivatives exist, we have

$$\frac{d}{dt} E\left[N_i(t)\right] = \lambda E\left[\alpha_i(N(t), V(t))\right]. \tag{2.3}$$

Similarly,

$$E\left[V_i(t+h) \middle| N(t), V(t)\right] = V_i(t) + \lambda h \alpha_i(N,V) r_i p_i. \tag{2.4}$$

Thus,

$$E[V_i(t+h)] = E[V_i(t)] + \lambda h r_i p_i E[\alpha_i(N,V)h]. \tag{2.5}$$

Assuming derivatives exist, we have

$$\frac{d}{dt}E[V_i(t)] = r_i p_i \lambda E[\alpha_i(N,V)]. \tag{2.6}$$

Assume α_i is of the form (1.4) where h_i is sufficiently smooth with $h_i(c^p x, c^p y) = c^p h_i(x, y)$ for constants c and $h_i(x, y) > 0$ for all $x, y > 0$. We have

$$\begin{aligned} \alpha_i(N(t),V(t)) &= \frac{h_i(N_i(t),V_i(t))}{\sum_j h_j(N_j(t),V_j(t))} \\ &= \frac{h_i(N_i(t)/\lambda, V_i(t)/\lambda)}{\sum_j h_j(N_j(t)/\lambda, V_j(t)/\lambda)}. \end{aligned} \tag{2.7}$$

Assume $\lim_{\lambda\to\infty} N_i(t)/\lambda = m_i(t)$, $\lim_{\lambda\to\infty} V_i(t)/\lambda = v_i(t)$, $\lim_{\lambda\to\infty} \frac{d}{dt}E[N_i(t)/\lambda] = \frac{d}{dt}m_i(t)$, and $\lim_{\lambda\to\infty} \frac{d}{dt}E[V_i(t)/\lambda] = \frac{d}{dt}v_i(t)$.

Dividing both sides of (2.3) and (2.6) by λ and letting $\lambda \to \infty$, the bounded convergence theorem yields

$$\frac{d}{dt}m_i(t) = \frac{h_i(m_i(t),v_i(t))}{\sum_j h_j(m_j(t),v_j(t))}. \tag{2.8}$$

and

$$\frac{d}{dt}v_i(t) = r_i p_i \frac{h_i(m_i(t),v_i(t))}{\sum_j h_j(m_j(t),v_j(t))} \tag{2.9}$$

Thus,

$$v_i(t) = r_i p_i m_i(t). \tag{2.10}$$

3. Purposeful Allocation

The equations for $m_i(t)$, (2.8), can be solved for special functions h_i. For example, assume

$$h_i(x,y)=\left[\frac{a_i}{x}+\frac{y(1-p_i)}{x^2p_i^2}\right]^{\gamma} \quad \text{for } \gamma \geq 0 \text{ and } a_i > 0. \tag{3.1}$$

Since

$$\begin{aligned} v_i(t) &= m_i(t)r_ip_i, \\ h_i(m_i(t),v_i(t)) &= \left[\frac{a_i}{m_i(t)}+\frac{r_i(1-p_i)}{m_i(t)p_i}\right]^{\gamma}. \end{aligned} \tag{3.2}$$

In this case (2.8) can be rewritten as

$$\frac{\frac{d}{dt}m_i(t)}{\frac{d}{dt}m_j(t)}=\left[\frac{a_i+r_ic_i}{a_j+r_jc_j}\frac{m_j(t)}{m_i(t)}\right]^{\gamma} \tag{3.3}$$

where $c_i = (1-p_i)/p_i$.

The functions

$$m_i(t)=K(a_i+r_ic_i)^{\gamma/(\gamma+1)}t \tag{3.4}$$

with

$$K=\left[\sum_{i=1}^{I}(a_i+r_ic_i)^{\gamma/(\gamma+1)}\right]^{-1} \tag{3.5}$$

are a solution to (2.8).

Thus, in the limit as $t \to \infty$, the probability that a sensor resource at time t looks at node i is

$$\alpha_i=\frac{(a_i+r_ic_i)^{\gamma/(\gamma+1)}}{\sum_j (a_j+r_jc_j)^{\gamma/(\gamma+1)}}. \tag{3.6}$$

If $\gamma = 0$, then

$$\alpha_i = 1/I \tag{3.7}$$

and the allocation is equally likely.

If $\gamma \to \infty$, then

$$\alpha_i = \frac{a_i + r_i c_i}{\sum_j a_j + r_j c_j}. \tag{3.8}$$

If, further, the probability of detection on node i, p_i, is constant for all the nodes, then the probabilistic allocation for $\gamma \to \infty$ is roughly proportional to the number of units on the node.

4. Scaling and Approximate Variances of Estimators

Let

$$X_j(t) = \frac{N_j(t) - \lambda m_j(t)}{\sqrt{\lambda}} \tag{4.1}$$

$$Y_j(t) = \frac{V_j(t) - \lambda v_j(t)}{\sqrt{\lambda}} \tag{4.2}$$

In Appendix A, moment generating functions are used to show the asymptotic normality of $\{(X_j(t), Y_j(t));\ t \geq 0\}$ as $\lambda \to \infty$. Further, differential equations for their second moments are obtained.

Rewriting

$$N_j(t) = \lambda m_j(t) + \sqrt{\lambda} X_j(t) \tag{4.3}$$

$$V_j(t) = \lambda v_j(t) + \sqrt{\lambda} Y_j(t) \tag{4.4}$$

An approximate variance of the estimator of the number of units on node i,

$$\hat{r}_i(t) = \frac{V_i(t)}{N_i(t)} \frac{1}{p_i} \tag{4.5}$$

can be computed using the "delta method" as follows; cf. Bickel and Doksum (1977). To begin,

$$Var\left[\hat{r}_i(t)\right] = \frac{1}{p_i^2}\ Var\left[\frac{V_i(t)}{N_i(t)}\right]. \tag{4.6}$$

A Taylor expansion yields

$$\begin{aligned} \frac{V_i(t)}{N_i(t)} &= \frac{v_i(t) + Y_i(t)/\sqrt{\lambda}}{m_i(t) + X_i(t)/\sqrt{\lambda}} \\ &= \frac{v_i(t)}{m_i(t)} + \frac{1}{m_i(t)}\left[Y_i(t)/\sqrt{\lambda}\right] - \frac{v_i(t)}{m_i(t)^2}\left(X_i(t)/\sqrt{\lambda}\right) \\ &\quad + O\left(\frac{1}{\lambda}\right). \end{aligned} \tag{4.7}$$

Thus,

$$\begin{aligned} Var\left[\frac{V_i(t)}{N_i(t)}\right] &\approx E\left[\left(\frac{V_i(t)}{N_i(t)} - \frac{v_i(t)}{m_i(t)}\right)^2\right] \\ &\approx \frac{1}{\lambda m_i(t)^2}\left\{E\left[Y_i^2(t)\right] + \left[\frac{v_i(t)}{m_i(t)}\right]^2 E\left[X_i^2(t)\right] - 2\frac{v_i(t)}{m_i(t)}E\left[Y_i(t)X_i(t)\right]\right\} \end{aligned} \tag{4.8}$$

$$= \frac{1}{\lambda m_i(t)^2}\left\{E\left[Y_i^2(t)\right] + (r_i p_i)^2 E\left[X_i^2(t)\right] - 2 r_i p_i E\left[X_i(t)Y_i(t)\right]\right\}. \tag{4.9}$$

Hence, an approximate variance of the estimate of the number of units on node i

$$Var\left[\hat{r}_i(t)\right] \approx \frac{1}{p_i^2}\frac{1}{\lambda m_i(t)^2}\left\{E\left[Y_i^2(t)\right] + (r_i p_i)^2 E\left[X_i^2(t)\right] - 2 r_i p_i E\left[X_i(t)Y_i(t)\right]\right\}. \tag{4.10}$$

To approximate the variance of the sum of the estimates, a Taylor expansion yields

$$
\begin{aligned}
E\left[\frac{V_i(t)}{N_i(t)}\frac{V_j(t)}{N_j(t)}\right] &= E\left[\frac{v_i(t)+Y_i(t)/\sqrt{\lambda}}{m_i(t)+X_i(t)/\sqrt{\lambda}}\,\frac{v_j(t)+Y_j(t)/\sqrt{\lambda}}{m_j(t)+X_j(t)/\sqrt{\lambda}}\right] \\
&= \frac{v_i(t)v_j(t)}{m_i(t)m_j(t)}+\frac{1}{\lambda}\frac{1}{m_i(t)m_j(t)}E\left[Y_i(t)Y_j(t)\right] \qquad (4.11) \\
&\quad -\frac{v_i(t)v_i(t)}{\lambda\left[m_i(t)m_i(t)\right]^2}E\left[X_i(t)X_j(t)\right]+o\left(\frac{1}{\lambda}\right)
\end{aligned}
$$

for $i \neq j$.

Thus,

$$
\begin{aligned}
Cov\left(\hat{r}_i(t),\hat{r}_j(t)\right) &= \frac{1}{p_i p_j}Cov\left(\frac{V_i(t)}{N_i(t)},\frac{V_j(t)}{N_j(t)}\right) \\
&\approx \frac{1}{\lambda}\frac{1}{p_i m_i(t)}\frac{1}{p_j m_j(t)}\left\{E\left[Y_i(t)Y_j(t)\right]-r_i p_i r_j p_j E\left[X_i(t)X_j(t)\right]\right\}.
\end{aligned}
\qquad (4.12)
$$

Expressions (4.10) and (4.12) can be used to approximate $Var\left[\sum_i \hat{r}_i(t)\right]$.

5. A Simpler Poisson Approximation

Assume the probability of allocation α_i is independent of $(V_i(t), N_i(t))$; then the number of looks at node i is a Poisson process with rate $\lambda\alpha_i$ independent of the other nodes. Further

$$E[N_i(t)] = 1 + \alpha_i t \qquad (5.1)$$

$$E\left[V_i(t)\right] = r_i p_i\left[1+\alpha_i t\right] \qquad (5.2)$$

$$Var\left[N_i(t)\right] = \alpha_i t \qquad (5.3)$$

$$Var\left[V_i(t)\right] = r_i p_i(1-p_i)+\alpha_i t\left[r_i p_i(1-p_i)+(r_i p_i)^2\right] \qquad (5.4)$$

$$Cov[N_i(t), V_i(t)] = r_i p_i \alpha_i t \tag{5.5}$$

where we assume that at time 0 each node is looked at once.

In this case (4.10) becomes

$$Var[\hat{r}_i(t)] = \frac{1}{p_i^2} \frac{r_i p_i (1-p_i)}{1+\alpha_i t}. \tag{5.6}$$

and the estimators $\hat{r}_i(t)$ are independent.

Assume purposeful allocation is adapted with function h_i as in (3.1). A simple approximation to $Var\left[\sum_i \hat{r}_i(t)\right]$ can be obtained by neglecting all covariances and assuming the number of looks at node i, $\{N_i(t);\ t \geq 0\}$, is a Poisson process with rate $\lambda\alpha_i$ where α_i is determined by (3.6). In this case

$$Var\left[\sum_i \hat{r}_i(t)\right] \approx \sum_i \frac{1}{p_i} \frac{r_i(1-p_i)}{(1+\alpha_i t)}. \tag{5.7}$$

6. Numerical Examples

Suppose there are 3 nodes with r_j units on node j with $r_1 = 49$, $r_2 = 25$, and $r_3 = 16$. The probabilities of detecting a unit on node j, p_j, are $p_1 = 1/11$, $p_2 = 0.5$, and $p_3 = 10/11$.

The variance of the estimate of the sum of the units on all the nodes under purposeful allocation with h_i as in (3.1) was studied using simulation for $\gamma = 0, 1, 10$. Each replication of the simulation begins with one observation at each node. The times of arrival of a Poisson process with rate 1 are then simulated. A node for observation at time t is randomly chosen using probabilities

$$\alpha_i(t) = K(t)\left[\frac{a_i}{N_i(t)} + \frac{V_i(t)(1-p_i)}{N_i(t)^2 p_i^2}\right]$$

for $i = 1, 2, 3$ with $K(t)$ the normalizing constant. A binomial observation is generated for the node chosen. The simulation has 500 replications.

Table 1 records the sample mean and square root of the sample variance of the sum of the estimates $\sum_{i=1}^{3} \hat{r}_i(t)$ for $t = 5, 10, 20, 50$ where

$$\hat{r}_i(t) = \frac{V_i(t)}{N_i(t)p_i}.$$

Also shown are the sample means of the square root of the sum of estimated approximate variances (1.3)

$$\hat{\sigma}_i^2(t) = \frac{V_i(t)(1-p_i)}{N_i(t)^2 p_i^2}.$$

Table 1

Estimate of Total Number of Units on All Nodes

$r_1 = 49, r_2 = 25, r_3 = 16;\ p_1 = 1/11, p_2 = 0.5, p_3 = 10/11$

		Simulation			Approximation	
		$\sum_i \hat{r}_i(t)$		$\sqrt{\sum_i \hat{\sigma}_i^2(t)}$	Square Root of Differential Equation Approx Variance	Square Root of Poisson Approx Variance
Time	γ	Sample Mean	Square Root of Sample Variance	Sample Mean		
5	0	89.8	18.0	15.0	13.9	13.9
	1	88.1	15.1	10.7	12.3	10.7
	10	89.3	15.2	10.9	12.5	10.4
10	0	90.3	12.4	11.9	10.9	10.9
	1	89.8	9.2	8.2	9.1	8.1
	10	90.1	10.2	8.6	10.2	8.0
20	0	90.0	8.9	8.6	8.2	8.2
	1	89.5	7.1	6.1	6.7	6.0
	10	89.5	8.3	6.5	8.0	6.1
50	0	89.9	5.8	5.5	5.4	5.4
	1	89.9	3.9	3.9	4.2	3.9
	10	89.5	6.2	4.3	5.7	4.2

Table 1 also displays results of the approximate variance $Var\left[\sum_i \hat{r}_i(t)\right]$ obtained using (4.10) and (4.12) and solving the differential equations (2.8) and the second moment equations in Appendix A, (A.27) – (A.32). The equations were solved numerically using the 4th/5th order Runge-Kutta-Fehlberg method as implemented in MATLAB; (cf. Math Works, 1992).

Table 1 also displays the simple Poisson approximation to $Var\left[\sum_i \hat{r}_i(t)\right]$ of (5.7) with the α_i in (3.6).

Recall that the estimate $Var\left[\sum_i \hat{r}_i(t)\right]$ is unbiased for both approximations. The true value of $\sum r_i$ is 90. The differential equation approximation is close to the square root of the sample variance for all values of γ for times 10, 20, and 50. The Poisson approximation is close to the square root of the sum of the estimated approximate variances; both of these approximations are neglecting the covariances induced by the purposeful allocation; these covariances become more pronounced as γ becomes larger. There is no covariance for $\gamma = 0$, equally likely allocation. Note that the Poisson approximation is conservative. However for $\gamma = 1$, the Poisson approximation is within about 10% of the differential equation approximation which incorporates the covariances. However the difference is larger for $\gamma = 10$.

Note that the square root of the sample variance and the differential equation approximation suggest that purposeful allocation with $\gamma = 1$ yields the smallest variance of the estimated sum of the numbers of units on all the nodes. A rationale for this suggestion follows.

Suppose there are a fixed number of looks K that the sensor can take of all nodes and the number of units on each node i, r_i, is known along with the probability of detecting a unit on node i, p_i. Let k_i be the number of looks the sensor gives to node i. If each observation has a binomial distribution

$$Var\left[\sum_{i=1}^{3}\hat{r}_i(t)\right]=\sum_{i=1}^{3}\frac{k_i r_i p_i(1-p_i)}{k_i^2 p_i^2}$$

$$=\sum_{i=1}^{3}\frac{r_i(1-p_i)}{k_i p_i}.$$

Lagrange multipliers can be used to show that the (approximate) k_i, $i = 1, 2, 3$ that minimize $Var\left[\sum_{i=1}^{3}\hat{r}_i(t)\right]$ are $k_i=\left[r_i(1-p_i)/p_i\right]^{\frac{1}{2}}$. This solution corresponds to the α_i of (3.6) with $a_i = 0$ and $\gamma = 1$. Thus, if one is interested in minimizing the estimated variance of the sum of the number of units on all the nodes, then one should look at node i a number of times proportional to $\left[r_i(1-p_i)/p_i\right]^{0.5}$. If one were interested in minimizing the estimated variance of the estimate of the number of units on the node with the greatest number of units then one would allocate all looks to that node; this corresponds to the purposeful allocation policy of $\gamma = \infty$.

Tables 2 and 3 present results of the simulation experiment with $r_1 = 49$, $r_2 = 25$, $r_3 = 16$ and $p_1 = 0.7$, $p_2 = 0.8$, and $p_3 = 0.9$. Table 2 presents the simulation and approximation results for the estimate of the sum of units on all the nodes. Table 3 presents the simulation and approximation results for the number of units on the individual nodes. The differential equation results are close to the simulated values for times $t = 10, 20, 50$. The Poisson approximation also seems to be adequate. The Poisson approximation may be doing better in this case because the probabilities of unit detection are larger. One source of the covariance between the estimators $\hat{r}_i(t)$ is the possibility that $V(i, t)$ may be 0, in which case node i will not be visited very frequently for $\gamma > 0$ for the purposeful allocation with function h_i as in (3.1).

Table 2
Estimate of Total Number of Units on All Nodes

$r_1 = 49, r_2 = 25, r_3 = 16;\ p_1 = 0.7, p_2 = 0.8, p_3 = 0.9$

		Simulation			Approximation	
		$\sum_i \hat{r}_i(t)$		$\sqrt{\sum_i \sigma_i^2(t)}$	Square Root of Differential Equation Approx Variance	Square Root of Poisson Approx Variance
Time	γ	Sample Mean	Square Root of Sample Variance	Sample Mean		
5	0	89.9	5.52	3.70	3.30	3.30
	1	89.4	6.53	3.15	3.11	3.02
	10	88.9	8.55	3.14	3.06	2.98
10	0	90.1	2.75	2.84	2.59	2.59
	1	89.9	2.37	2.45	2.46	2.35
	10	89.7	2.46	2.53	2.51	2.35
20	0	90.1	2.04	2.07	1.95	1.95
	1	90.0	1.72	1.80	1.90	1.76
	10	89.9	1.87	1.88	2.06	1.80
50	0	90.0	1.30	1.32	1.28	1.28
	1	90.0	1.12	1.17	1.29	1.16
	10	89.9	1.26	1.24	1.35	1.21

Table 3
Simulation: 500 replications
Estimate of Number of Units on Node

		Node 1			Node 2			Node 3		
		$\hat{r}_i(t)$		$\sqrt{\hat{\sigma}_i^2(t)}$	$\hat{r}_i(t)$		$\sqrt{\hat{\sigma}_i^2(t)}$	$\hat{r}_i(t)$		$\sqrt{\hat{\sigma}_i^2(t)}$
Time	γ	Mean	Square Root Sample Variance (Diff Eqtn Approx)	Sample Mean (Poisson Approx)	Mean	Square Root Sample Variance (Diff Eqtn Approx)	Sample Mean (Poisson Approx)	Mean	Square Root Sample Variance (Diff Eqtn Approx)	Sample Mean (Poisson Approx)
5	0	49.0	3.89 (2.81)	3.11 (2.81)	24.9	2.20 (1.53)	1.68 (1.53)	16.0	1.17 (0.82)	0.87 (0.82)
	1	48.6	3.95 (2.27)	2.37 (2.38)	24.8	2.38 (1.62)	1.69 (1.59)	16.0	1.54 (1.07)	1.13 (0.99)
	10	48.5	4.85 (2.00)	2.12 (2.17)	24.6	2.94 (1.88)	1.89 (1.70)	15.8	1.97 (1.33)	1.32 (1.13)
10	0	49.1	2.33 (2.20)	2.39 (2.20)	25.0	1.36 (1.20)	1.30 (1.20)	16.0	0.71 (0.64)	0.68 (0.64)
	1	49.0	1.81 (1.74)	1.81 (1.81)	24.9	1.36 (1.27)	1.32 (1.25)	16.0	0.96 (0.89)	0.93 (0.82)
	10	48.9	1.57 (1.53)	1.61 (1.63)	24.9	1.50 (1.44)	1.48 (1.37)	15.9	1.33 (1.29)	1.26 (1.00)
20	0	49.1	1.68 (1.66)	1.75 (1.66)	25.0	0.98 (0.90)	0.95 (0.90)	16.0	0.52 (0.48)	0.50 (0.48)
	1	49.0	1.32 (1.30)	1.32 (1.33)	24.9	0.97 (0.96)	0.98 (0.95)	16.0	0.71 (0.68)	0.71 (0.65)
	10	49.0	1.16 (1.15)	1.19 (1.19)	24.9	1.08 (1.08)	1.10 (1.05)	15.9	1.06 (1.00)	0.95 (0.84)
50	0	49.1	1.08 (1.09)	1.12 (1.09)	25.0	0.63 (0.59)	0.61 (0.59)	16.0	0.33 (0.32)	0.32 (0.32)
	1	49.0	0.88 (0.85)	0.86 (0.86)	25.0	0.64 (0.63)	0.64 (0.63)	16.0	0.45 (0.45)	0.46 (0.44)
	10	49.0	0.77 (0.76)	0.77 (0.77)	24.9	0.72 (0.72)	0.72 (0.70)	16.0	0.71 (0.66)	0.65 (0.61)

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APPENDIX A

In the Appendix we present details of the normal approximation. We follow an analytical approach used in a different context in Gaver and Jacobs (1995), and in Gaver, Morrison, and Silveira (1993). Let the moment-generating function (assumed to exist, otherwise use the characteristic function) be

$$\begin{aligned}\psi(\boldsymbol{\theta},\boldsymbol{\xi};t) &= E\left[\exp\{\boldsymbol{\theta}N(t)+\boldsymbol{\xi}V(t)\}\right] \\ &= E\left[\exp\left\{\sum_{j=1}^{I}\theta_j N_j(t)+\sum_{j=1}^{I}\xi_j V_j(t)\right\}\right]. \end{aligned} \tag{A.1}$$

Condition on $(N_i(t), V_i(t))$, $i \in \{1, 2, \ldots, I\}$ to obtain

$$\begin{aligned} & E\left[\exp\left\{\sum_{j=1}^{I}\theta_j N_j(t+h)+\xi_j V_j(t+h)\right\}\middle| N(t),V(t)\right] \\ &= (1-\lambda h)\exp\{\boldsymbol{\theta}N(t)+\boldsymbol{\xi}V(t)\} \\ &+\lambda h\sum_{i}\alpha_i(N(t),V(t))\exp\{\boldsymbol{\theta}N(t)+\boldsymbol{\xi}V(t)\}\left[e^{\theta_i}\hat{b}(\xi_i)\right] \end{aligned} \tag{A.2}$$

where

$$\hat{b}(\xi_i) = E\left[e^{\xi_i Z(i)}\right]. \tag{A.3}$$

with Z(i) an observation of the number of units on node i.

Let $h \to 0$ to obtain

$$\frac{\partial}{\partial t}\psi(\boldsymbol{\theta},\boldsymbol{\xi};t) = -\lambda\psi(\boldsymbol{\theta},\boldsymbol{\xi};t)+\lambda\sum_{i=1}^{I}E\left[\alpha_i(N(t),V(t))\exp\{\boldsymbol{\theta}N(t)+\boldsymbol{\xi}V(t)\}e^{\theta_i}\hat{b}(\xi_i)\right]. \tag{A.4}$$

Scaling

Let

$$X_j(t) = \frac{N_j(t)-\lambda m_j(t)}{\sqrt{\lambda}} \tag{A.5}$$

$$Y_j(t) = \frac{V_j(t) - \lambda v_j(t)}{\sqrt{\lambda}} \tag{A.6}$$

and let $\lambda \gg 1$.

Let

$$\varphi(\boldsymbol{\theta}, \boldsymbol{\xi}; t) = E\left[\exp\{\boldsymbol{\theta} X(t) + \boldsymbol{\xi} Y(t)\}\right]; \tag{A.7}$$

then

$$\psi\left(\theta/\sqrt{\lambda}, \xi/\sqrt{\lambda}\,; t\right) = \varphi(\boldsymbol{\theta}, \boldsymbol{\xi}; t)\exp\left\{\sqrt{\lambda}\left[\boldsymbol{\theta} m(t) + \boldsymbol{\xi} v(t)\right]\right\}. \tag{A.8}$$

Thus, we have the following equation from (A.8) and (A.4)

$$\begin{aligned}
&\frac{\partial}{\partial t}\psi\left(\theta/\sqrt{\lambda}, \xi/\sqrt{\lambda}\,; t\right) \\
&= \frac{\partial}{\partial t}\varphi(\boldsymbol{\theta}, \boldsymbol{\xi}; t)\exp\left\{\sqrt{\lambda}\left[\boldsymbol{\theta} m(t) + \boldsymbol{\xi} v(t)\right]\right\} \\
&+ \sqrt{\lambda}\varphi(\boldsymbol{\theta}, \boldsymbol{\xi}; t)\exp\left\{\sqrt{\lambda}\left[\boldsymbol{\theta} m(t) + \boldsymbol{\xi} v(t)\right]\right\}\left[\boldsymbol{\theta} m'(t) + \boldsymbol{\xi} v'(t)\right] \\
&= \Bigg[-\lambda\varphi(\boldsymbol{\theta}, \boldsymbol{\xi}; t) \\
&+ \lambda\sum_{i=1}^{I} E\left[\alpha_i\left(m(t) + \frac{1}{\sqrt{\lambda}}X(t), v(t) + \frac{1}{\sqrt{\lambda}}Y(t)\right)\exp\{\boldsymbol{\theta} X(t) + \boldsymbol{\xi} Y(t)\}\right] e^{\theta_i/\sqrt{\lambda}}\hat{b}_i\left(\xi_i/\sqrt{\lambda}\right)\Bigg] \\
&\times \exp\left\{\sqrt{\lambda}\left[\boldsymbol{\theta} m(t) + \boldsymbol{\xi} v(t)\right]\right\}.
\end{aligned} \tag{A.9}$$

Dividing both sides by $\exp\left\{\sqrt{\lambda}\left[\boldsymbol{\theta} m(t) + \boldsymbol{\xi} v(t)\right]\right\}$ we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\varphi(\boldsymbol{\theta}, \boldsymbol{\xi}; t) + \sqrt{\lambda}\,\varphi(\boldsymbol{\theta}, \boldsymbol{\xi}; t)\left[\boldsymbol{\theta} m'(t) + \boldsymbol{\xi} v'(t)\right] \\
&= -\lambda\varphi(\boldsymbol{\theta}, \boldsymbol{\xi}; t) \\
&+ \lambda\sum_{i=1}^{I} E\Bigg[\Bigg\{\alpha_i(m(t), v(t)) + \sum_j \frac{\partial}{\partial m_j}\alpha_i(m(t), v(t))\frac{1}{\sqrt{\lambda}}X_j(t) \\
&+ \sum_j \frac{\partial}{\partial \beta_j}\alpha_i(m(t), v(t))\frac{1}{\sqrt{\lambda}}Y_j(t) + O\left(\frac{1}{\sqrt{\lambda}}\right)\Bigg\}\exp\{\boldsymbol{\theta} X(t) + \boldsymbol{\xi} Y(t)\}\Bigg] e^{\theta_i/\sqrt{\lambda}}\hat{b}_i\left(\xi_i/\sqrt{\lambda}\right)
\end{aligned} \tag{A.10}$$

Let

$$\alpha_i(m(t),v(t)) = \frac{h_i(m_i(t),v_i(t))}{\sum_j h_j(m_j(t),v_j(t))} \tag{A.11}$$

where $h_j(x, y)$ is a sufficiently smooth function with first order partial derivatives and $h_j(\lambda^p x, \lambda^p y) = \lambda^p h_j(x,y)$.

Let

$$H_i(x;t) = \frac{\frac{\partial}{\partial m_i} h_i(m_i(t),v_i(t))}{\sum_j h_j(m_j(t),v_j(t))} \tag{A.12}$$

$$H_i(y;t) = \frac{\frac{\partial}{\partial v_i} h_i(m_i(t),v_i(t))}{\sum_j h_j(m_j(t),v_j(t))} \tag{A.13}$$

$$H_{ik}(x;t) = \frac{h_i(m_i(t),v_i(t))}{\left(\sum_j h_j(m_j(t),v_j(t))\right)^2} \frac{\partial}{\partial m_k} h_k(m_k(t),v_k(t)) \tag{A.14}$$

$$H_{ik}(y;t) = \frac{h_i(m_i(t),v_i(t))}{\left(\sum_j h_j(m_j(t),v_j(t))\right)^2} \frac{\partial}{\partial v_k} h_k(m_k(t),v_k(t)) \tag{A.15}$$

Note that

$$\begin{aligned}
&\alpha_i\left(\lambda m_i(t) + \sqrt{\lambda} X_i(t), \lambda v_i(t) + \sqrt{\lambda} Y_i(t)\right) \\
&= \frac{h_i(m_i(t),v_i(t))}{\sum_{j=1}^{I} h_j(m_j(t),v_j(t))} \\
&+ \frac{1}{\sqrt{\lambda}}\left[H_i(x;t)X_i(t) + H_i(y;t)Y_i(t) + \sum_k H_{ik}(x;t)X_k(t) + \sum_k H_{ik}(y;t)Y_k(t)\right] + O\left(\frac{1}{\lambda}\right).
\end{aligned} \tag{A.16}$$

Since

$$\sum_i \alpha_i\left(\lambda m_i(t)+\sqrt{\lambda}X_i(t),\lambda v_i(t)+\sqrt{\lambda}Y_i(t)\right)=1 \tag{A.17}$$

this implies that the summed coefficients of $1/\sqrt{\lambda}$, $1/\lambda$ etc. must individually be 0.

Expression (A.10) can be rewritten as

$$\begin{aligned}
&\frac{\partial}{\partial t}\varphi(\boldsymbol{\theta},\xi;t)+\sqrt{\lambda}\,\varphi(\boldsymbol{\theta},\xi;t)\left[\boldsymbol{\theta}m'(t)+\xi v'(t)\right] \\
&=-\lambda\varphi(\boldsymbol{\theta},\xi;t) \\
&+\lambda\sum_i\left[1+\frac{\theta_i}{\sqrt{\lambda}}+\frac{b_1(i)\xi_i}{\sqrt{\lambda}}+\frac{1}{2}\frac{\theta_i^2}{\lambda}+\frac{1}{2}\frac{b_2(i)\xi_i^2}{\lambda}\right] \\
&\times\left\{\frac{h_i\left(m_i(t),v_i(t)\right)}{\sum_j h_j\left(m_j(t),v_j(t)\right)}\varphi(\boldsymbol{\theta},\xi;t)\right. \\
&+\frac{1}{\sqrt{\lambda}}\left[H_i(x;t)\frac{\partial}{\partial\theta_i}\varphi(\boldsymbol{\theta},\xi;t)+H_i(y;t)\frac{\partial}{\partial\xi_i}\varphi(\boldsymbol{\theta},\xi;t)\right. \\
&\left.\left.-\sum_{k=1}^{I}\left[H_{ik}(x;t)\frac{\partial}{\partial\theta_k}\varphi(\boldsymbol{\theta},\xi;t)+H_{ik}(y;t)\frac{\partial}{\partial\xi_k}\varphi(\boldsymbol{\theta},\xi;t)\right]\right]\right\}+O\left(\frac{1}{\lambda}\right)
\end{aligned} \tag{A.18}$$

where $b_n(i)=E\left[Z(i)^n\right]$ the n^{th} moment of an observation at node i.

Let

$$\varphi(\boldsymbol{\theta},\xi;t)=\sum_{\ell=0}^{\infty}\varphi_\ell(\boldsymbol{\theta},\xi;t)\lambda^{-\ell/2}. \tag{A.19}$$

Substituting (A.19) into (A.18) results in the following equation for φ_0

$$
\begin{aligned}
&\frac{\partial}{\partial t}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)+\sqrt{\lambda}\,\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\left[\boldsymbol{\theta}m'(t)+\boldsymbol{\xi}v'(t)\right]\\
&=-\lambda\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\\
&+\lambda\sum_i\left[1+\frac{1}{\sqrt{\lambda}}\left(\theta_i+b_1(i)\xi_i\right)+\frac{1}{2}\frac{1}{\lambda}\left(\theta_i^2+b_2(i)\xi_i^2\right)\right]\\
&\times\left\{\frac{h_i(m_i(t),v_i(t))}{\sum_j h_j\left(m_j(t),v_j(t)\right)}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\right.\\
&+\frac{1}{\sqrt{\lambda}}\left[H_i(x;t)\frac{\partial}{\partial\theta_i}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)+H_i(y;t)\frac{\partial}{\partial\xi_i}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\right.\\
&\left.\left.-\sum_{k=1}^{I}\left[H_{ik}(x;t)\frac{\partial}{\partial\theta_k}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)+H_{ik}(y;t)\frac{\partial}{\partial\xi_k}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\right]\right]\right\}+O\left(\frac{1}{\lambda}\right).
\end{aligned}
\tag{A.20}
$$

Equating terms of order $\lambda^{\ell/2}$, the terms of order λ cancel. The terms of order $\sqrt{\lambda}$ result in the equation

$$
\begin{aligned}
&\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\left[\boldsymbol{\theta}m'(t)+\boldsymbol{\xi}v'(t)\right]\\
&=\sum_{i=1}^{I}\left\{\frac{h_i(m_i(t),v_i(t))}{\sum_j h_j\left(m_j(t),v_j(t)\right)}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\left[\theta_i+b_1(i)\xi_i\right]\right.\\
&+H_i(x;t)\frac{\partial}{\partial\theta_i}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)+H_i(y;t)\frac{\partial}{\partial\xi_i}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\\
&\left.+\sum_{k=1}^{I}\left[H_{ik}(x;t)\frac{\partial}{\partial\theta_k}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)+H_{ik}(y;t)\frac{\partial}{\partial\xi_k}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\right]\right\}+O\left(\frac{1}{\lambda}\right).
\end{aligned}
\tag{A.21}
$$

The terms of order $\sqrt{\lambda}$ cancel if

$$\boldsymbol{\theta} m'(t)+\boldsymbol{\xi} v'(t)=\sum_{i=1}^{I}\frac{h_i(m_i(t),v_i(t))}{\sum_j h_j\big(m_j(t),v_j(t)\big)}[\theta_i+b_1(i)\xi_i]. \tag{A.22}$$

In order for this to occur

$$\frac{d}{dt}m_i(t)=\frac{h_i(m_i(t),v_i(t))}{\sum_j h_j\big(m_j(t),v_j(t)\big)} \quad i=1,\ldots I \tag{A.23}$$

and

$$\frac{d}{dt}v_i(t)=b_1(i)\frac{h_i(m_i(t),v_i(t))}{\sum_j h_j\big(m_j(t),v_j(t)\big)} \quad i=1,\ldots I. \tag{A.24}$$

Thus,

$$v_i(t)=b_1(i)m_i(t). \tag{A.25}$$

Next look for terms of order 1 in (A.20).

$$\begin{aligned}
&\frac{\partial}{\partial t}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\\
&=\sum_i\frac{h_i(m_i(t),v_i(t))}{\sum_j h_j\big(m_j(t),v_j(t)\big)}\left[\frac{1}{2}\theta_i^2+\frac{1}{2}b_2(i)\xi_i^2+b_1(i)\theta_i\xi_i\right]\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\\
&+\sum_i\left[H_i(x;t)\frac{\partial}{\partial\theta_i}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)+H_i(y;t)\frac{\partial}{\partial\xi_i}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\right][\theta_i+b_1(i)\xi_i]\\
&+\sum_i\sum_k\left[H_{ik}(x;t)\frac{\partial}{\partial\theta_k}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)+H_{ik}(y;t)\frac{\partial}{\partial\xi_k}\varphi_0(\boldsymbol{\theta},\boldsymbol{\xi};t)\right][\theta_i+b_1(i)\xi_i].
\end{aligned} \tag{A.26}$$

Equations for the joint moments of $\{(X_j(t), Y_j(t)\}$ can be obtained by differentiating (A.26) with respect to $\{\theta_i\}$ and $\{\xi_i\}$ and evaluated at $\boldsymbol{\theta}=\boldsymbol{\xi}=0$. The resulting equations are

$$\begin{aligned}
&\frac{d}{dt}E\left[X_\ell^2(t)\right] \\
&=\frac{h\left(m_\ell(t),v_\ell(t)\right)}{\sum_j h\left(m_j(t),v_j(t)\right)}+2H_\ell(x;t)E\left[X_\ell(t)^2\right]+2H_\ell(y;t)E\left[X_\ell(t)Y_\ell(t)\right] \\
&+2\sum_k H_{\ell k}(x;t)E\left[X_\ell(t)X_k(t)\right]+2\sum_k H_{\ell k}(y;t)E\left[X_\ell(t)Y_k(t)\right]
\end{aligned} \tag{A.27}$$

For $\ell \neq a$

$$\begin{aligned}
&\frac{d}{dt}E\left[X_\ell(t)X_a(t)\right] \\
&=H_a(x;t)E\left[X_a(t)X_\ell(t)\right]+H_a(y;t)E\left[Y_a(t)X_\ell(t)\right] \\
&+H_\ell(x;t)E\left[X_a(t)X_\ell(t)\right]+H_\ell(y;t)E\left[X_a(t)Y_\ell(t)\right] \\
&+\sum_k H_{ak}(x;t)E\left[X_k(t)X_\ell(t)\right]+\sum_k H_{ak}(y;t)E\left[X_\ell(t)Y_k(t)\right] \\
&+\sum_k H_{\ell k}(x;t)E\left[X_a(t)X_k(t)\right]+\sum_k H_{\ell k}(y;t)E\left[X_a(t)Y_k(t)\right].
\end{aligned} \tag{A.28}$$

$$\begin{aligned}
\frac{d}{dt}E\left[Y_\ell^2(t)\right]=&\frac{h\left(m_\ell(t),v_\ell(t)\right)}{\sum_j h\left(m_j(t),v_j(t)\right)}b_2(\ell) \\
&+2b_1(\ell)\left[H_\ell(x;t)E\left[X_\ell(t)Y_\ell(t)\right]+H_\ell(y;t)E\left[Y_\ell^2(t)\right]\right] \\
&+2b_1(\ell)\sum_j H_{\ell j}(x;t)E\left[X_j(t)Y_\ell(t)\right]+H_{\ell j}(y;t)E\left[Y_\ell(t)Y_j(t)\right]
\end{aligned} \tag{A.29}$$

For $j \neq k$

$$\begin{aligned}
&\frac{d}{dt}E\left[Y_k(t)Y_j(t)\right] \\
&=\left\{H_j(x;t)E\left[X_j(t)Y_k(t)\right]+H_j(y;t)E\left[Y_j(t)Y_k(t)\right]\right\}b_1(j) \\
&+\left\{H_k(x;t)E\left[X_k(t)Y_j(t)\right]+H_k(y;t)E\left[Y_j(t)Y_k(t)\right]\right\}b_1(k) \\
&+b_1(j)\sum_{\ell}H_{j\ell}(x;t)E\left[X_\ell(t)Y_k(t)\right]+H_{j\ell}(y;t)E\left[Y_k(t)Y_\ell(t)\right] \\
&+b_1(k)\sum_{\ell}H_{k\ell}(x;t)E\left[X_\ell(t)Y_j(t)\right]+H_{k\ell}(y;t)E\left[Y_\ell(t)Y_j(t)\right]
\end{aligned} \tag{A.30}$$

$$\begin{aligned}
&\frac{d}{dt}E\left[X_k(t)Y_k(t)\right] \\
&=\frac{h_k(m_k(t),v_k(t))}{\sum_j h_j\left(m_j(t),v_j(t)\right)}b_1(k) \\
&+H_k(x;t)\left\{E\left[Y_k(t)X_k(t)\right]+E\left[X_k^2(t)\right]b_1(k)\right\} \\
&+H_k(y;t)\left[E\left[Y_k^2(t)\right]+E\left[X_k(t)Y_k(t)\right]b_1(k)\right] \\
&+\sum_j\left\{H_{kj}(x;t)E\left[X_j(t)Y_k(t)\right]+H_{kj}(y;t)E\left[Y_k(t)Y_j(t)\right]\right\} \\
&+\sum_j\left\{H_{kj}(x;t)E\left[X_j(t)X_k(t)\right]+H_{kj}(y;t)E\left[X_k(t)Y_j(t)\right]\right\}b_1(k)
\end{aligned} \tag{A.31}$$

For $\ell \neq k$

$$\begin{aligned}
&\frac{d}{dt}E\left[X_\ell(t)Y_k(t)\right] \\
&= H_\ell(x;t)E\left[X_\ell(t)Y_k(t)\right] + H_\ell(y;t)E\left[Y_\ell(t)Y_k(t)\right] \\
&+\left\{H_k(x;t)E\left[X_\ell(t)X_k(t)\right] + H_k(y;t)E\left[X_\ell(t)Y_k(t)\right]\right\}b_1(k) \\
&+\sum_j H_{\ell j}(x;t)E\left[X_j(t)Y_k(t)\right] + H_{\ell j}(y;t)E\left[Y_k(t)Y_j(t)\right] \qquad \text{(A.32)} \\
&+\sum_j \left\{H_{kj}(x;t)E\left[X_j(t)X_\ell(t)\right] + H_{kj}(y;t)E\left[Y_j(t)X_\ell(t)\right]\right\}b_1(k)
\end{aligned}$$

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