Strong Restricted-Orientation Convexity

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Abstract

Strong $\mathcal{O}$-convexity is a generalization of standard convexity, defined with respect to a fixed set $\mathcal{O}$ of hyperplanar orientations. We explore the properties of strongly $\mathcal{O}$-convex sets in two and more dimensions and develop a mathematical foundation of strong convexity. We characterize strongly $\mathcal{O}$-convex polytopes, flats, and halfspaces, establish the strong $\mathcal{O}$-convexity of the affine hull of a strongly $\mathcal{O}$-convex set, and describe conditions under which two orientation sets yield the same collection of strongly $\mathcal{O}$-convex sets (orientation equivalence).

We identify some of the major properties of standard convex sets that hold for strong $\mathcal{O}$-convexity. In particular, we establish the following results:

- The intersection of a collection of strongly $\mathcal{O}$-convex sets is strongly $\mathcal{O}$-convex
- For every point in the boundary of a strongly $\mathcal{O}$-convex set, there is a supporting strongly $\mathcal{O}$-convex hyperplane through it
- A closed set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if it is the intersection of the strongly $\mathcal{O}$-convex halfspaces that contain it

1 Introduction

Convex sets are a comparatively recent yet fruitful concept in geometry, which has applications in optimization, statistics, geometric number theory, functional analysis, and combinatorics [Klee, 1971, Preparata and Shamos, 1985], as well as in more practical areas, such as VLSI design, computer graphics, architectural databases, and geographic databases. For example, the convex hull of a geometric object is often used as an approximation of the object. As another example, decomposing a polygon into convex subpolygons makes polygonal processing easier to handle.

Researchers have studied many notions of nontraditional convexity along with standard convexity, such as orthogonal convexity [Montuno and Fournier, 1982, Nicholl *et al.*, 1983, Ottmann *et al.*, 1984], finitely oriented convexity [Güting, 1983b, Widmayer *et al.*, 1987, Rawlins and Wood, 1987], restricted-orientation convexity [Rawlins, 1987, Rawlins and Wood, 1991, Schuierer, 1991], NESW convexity [Lipski and Papadimitriou, 1981, Soisalon-Soininen and Wood, 1984, Widmayer *et al.*, 1987], and link convexity [Bruckner and Bruckner, 1962, Valentine, 1965, Schuierer, 1991].

Rawlins introduced the notion of planar strong convexity during his investigation of restricted-orientation visibility [Rawlins, 1987]. This notion is stronger than standard convexity, hence the name. Rawlins and Wood [Rawlins and Wood, 1988, Rawlins and Wood, 1991] studied the properties of strongly convex sets in two dimensions and demonstrated that strong convexity generalizes not only standard convexity but also the notions of orthorectangles (that is, rectangles whose edges are parallel to the coordinate axes) and C-oriented polygons [Güting, 1983a, Güting, 1984]. The work on strong convexity adds to our understanding of convexity in general and may help us to develop simpler and more efficient convexity algorithms.

The research on nontraditional notions of convexity has so far been restricted to two dimensions. The work reported here is the first step in exploring nontraditional convexity in higher dimensions. In this first paper in a series [Fink and Wood, 1995a, Fink and Wood, 1995b], we extend the notion of strong convexity to higher dimensions. This extension is a generalization of planar strong convexity and of standard multidimensional convexity.

We explore the properties of strong convexity in higher dimensions and demonstrate that these properties are much richer than the properties of planar strongly convex sets. We establish analogs of the following basic properties of convex sets:

Visibility For every two points of a convex set, the straight segment joining them is wholly contained in the set.

Intersection The intersection of a collection of convex sets is a convex set. This property is the basis for the definition of the convex hull of a given set, which is the smallest convex set containing the given set.

Supporting planes For every point in the boundary of a convex set, there is a hyperplane through it that supports the set.

Halfspace intersection A closed convex set is the intersection of the halfspaces that contain it.

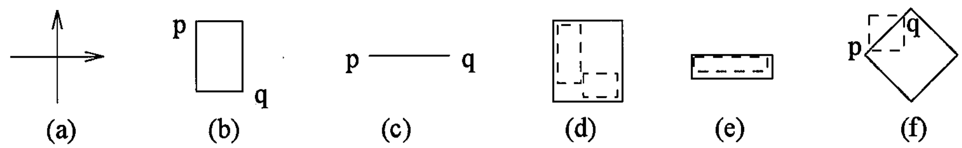

Figure 1: Strong ortho-convexity.

Except for the intersection property, these properties are defining characteristics of convex sets.

We also characterize strongly $\mathcal{O}$-convex polytopes, flats, and halfspaces, establish the strong $\mathcal{O}$-convexity of the affine hull of a strongly $\mathcal{O}$-convex set, and describe conditions under which two orientation sets yield the same collection of strongly $\mathcal{O}$-convex sets (orientation equivalence).

The article is organized as follows. In Section 2, we briefly describe the notion of strong convexity in two dimensions and give basic properties of planar strongly convex sets. In Section 3, we generalize the notion of strong convexity to higher dimensions. In Section 4, we present basic properties of higher-dimensional strongly convex sets. In Section 5, we explore properties of strongly convex flats. In Section 6, we describe strongly convex halfspaces and present analogs of the supporting-planes and halfspace-intersection properties for strongly convex sets. Finally, we conclude, in Section 7, with a summary of the results and a discussion of future work.

2 Strong convexity in two dimensions

We begin by reviewing the notion of strong convexity in two dimensions [Rawlins, 1987] and exploring the basic properties of planar strongly convex sets. Rawlins introduced this notion as part of his research on restricted-orientation visibility. He defined strong convexity through a generalized visibility, by analogy with standard convexity.

We can describe convex sets in terms of visibility: a set is convex if every two points of the set are visible to each other. In other words, for every two points of a convex set, the straight segment joining these points is wholly contained in the set. We introduce a new type of visibility by replacing straight segments with different type of objects, called *blocks*, and define strong convexity in terms of this new visibility.

We first present the notions of ortho-rectangles, ortho-blocks, and strong ortho-visibility. An *ortho-rectangle* is a rectangle whose sides are parallel to the coordinate axes [Güting, 1983a]. The *ortho-block* of two points p and q is the minimal ortho-rectangle that contains p and q (note that p and q are opposite vertices of this ortho-rectangle; see Figure 1b). If p and q are on the same vertical or horizontal line, then the ortho-block of p and q is just the straight segment joining p and q (Figure 1c).

We define strong ortho-convexity using *ortho-block visibility:* a set is strongly ortho-convex if, for every two points of the set, the ortho-block of these two points is wholly contained in the set. For example, the rectangles in Figures 1(d) and 1(e) are strongly ortho-convex (some ortho-blocks contained in these rectangles are shown by dashed lines). On the other hand, the square in Figure 1(f) is *not* strongly ortho-convex, because the dashed

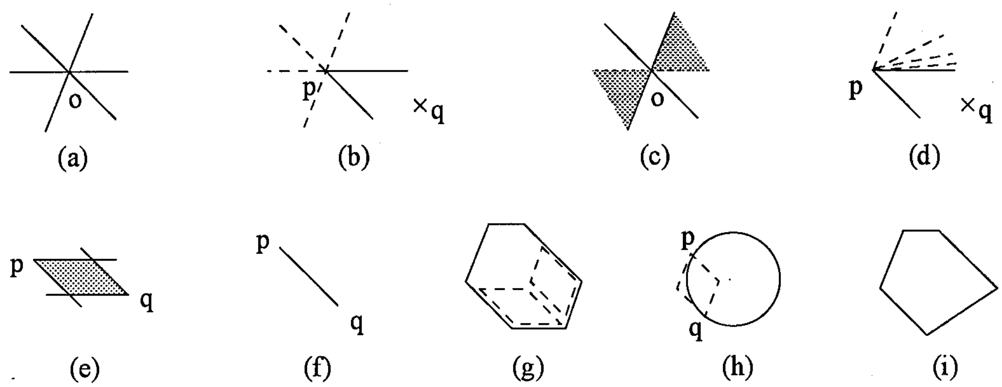

Figure 2: Planar strong convexity.

ortho-block is not in the square.

The following two properties of strongly ortho-convex sets are straightforward to prove:

Lemma 1

1. *A set is strongly ortho-convex if and only if it is an ortho-rectangle.*
2. *The intersection of a collection of strongly ortho-convex sets is strongly ortho-convex.*

Thus, after all, strongly ortho-convex sets are quite simple objects. These objects inherit two important properties of convex sets: visibility and intersection. Ortho-convex sets can be defined in terms of visibility and the intersection of a collection of ortho-convex sets is always an ortho-convex set.

Strong $\mathcal{O}$-convexity is a generalization of strong ortho-convexity. We obtain this generalization by replacing the two coordinate axes with a (finite or infinite) set of lines through a fixed point o. We denote this set of lines by $\mathcal{O}$ and call it an *orientation set.* An example of a finite orientation set is shown in Figure 2(a). A straight line parallel to one of the lines of $\mathcal{O}$ is called an *$\mathcal{O}$-oriented* line.

We now define the *$\mathcal{O}$-block* of two points p and q, which generalizes the notion of the ortho-block. Let us draw all $\mathcal{O}$-oriented rays with endpoint p and choose the two of them closest to q (see Figure 2b). The two selected rays, with the common endpoint p, are the boundary of an angle with vertex p; this angle contains q.

If $\mathcal{O}$ is an infinite set, it may not be closed and, hence, we may not be able to choose the ray closest to q. For example, consider the orientation set in Figure 2(c). All lines in the shaded area are elements of $\mathcal{O}$ and the dotted horizontal line is not in $\mathcal{O}$; this orientation set is not closed. If $\mathcal{O}$ is not closed, we have to use a limit in selecting the two rays. We choose two rays with common endpoint p such that, for each of the two selected rays, (1) there is a sequence of $\mathcal{O}$-oriented rays convergent to this ray and (2) there are no $\mathcal{O}$-oriented rays with endpoint p between this ray and the point q (see Figure 2d). The two selected rays are again the boundary of an angle with vertex p; this angle contains q.

Similarly, we draw the $\mathcal{O}$-oriented rays from q closest to p and obtain the angle with vertex q whose boundary is formed by these rays (Figure 2e). The *$\mathcal{O}$-block* of p and q is the intersection of these two angles (the shaded parallelogram in Figure 2e). As a special case, if the line through p and q is $\mathcal{O}$-oriented, then the $\mathcal{O}$-block of p and q is just the straight segment joining p and q (Figure 2f).

We define strong $\mathcal{O}$-convexity much in the same way as strong ortho-convexity, using $\mathcal{O}$-blocks instead of ortho-blocks.

Definition 1 (Strong $\mathcal{O}$-convexity) *A set is* strongly $\mathcal{O}$-convex *if, for every two points of the set, their $\mathcal{O}$-block is contained in the set.*

Let us denote the orientation set in Figure 2(a) by $\mathcal{O}_a$ and the orientation set in Figure 2(c) by $\mathcal{O}_c$. Then, the polygon in Figure 2(g) is strongly $\mathcal{O}_a$-convex and strongly $\mathcal{O}_c$-convex (two $\mathcal{O}_a$-blocks contained in this polygon are shown by dashed lines). On the other hand, the circle in Figure 2(h) is neither strongly $\mathcal{O}_a$-convex nor strongly $\mathcal{O}_c$-convex, since the block shown by dashed lines, which is an $\mathcal{O}_a$-block as well as $\mathcal{O}_c$-block, is not in the circle. Finally, the polygon in Figure 2(i) is strongly $\mathcal{O}_c$-convex, but is not strongly $\mathcal{O}_a$-convex.

The following properties of strongly $\mathcal{O}$-convex sets readily follow from the definition (Properties 1–4 and 6 were stated by Rawlins [Rawlins, 1987]).

Lemma 2

1. *Every translation of a strongly $\mathcal{O}$-convex set is strongly $\mathcal{O}$-convex.*
2. **(Intersection)** *If C is a collection of strongly $\mathcal{O}$-convex sets, then the intersection $\bigcap C$ of this collection is also strongly $\mathcal{O}$-convex. This property is the basis of the definition of the unique strongly $\mathcal{O}$-convex hull of a given set, which is the smallest strongly $\mathcal{O}$-convex set containing the given set.*
3. *For every orientation set $\mathcal{O}$, every strongly $\mathcal{O}$-convex set is standard convex.*
4. *If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every strongly $\mathcal{O}_1$-convex set is strongly $\mathcal{O}_2$-convex.*
5. *For two orientation sets, $\mathcal{O}_1$ and $\mathcal{O}_2$, through the same point o, strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity if and only if* $\text{Closure}(\mathcal{O}_1) = \text{Closure}(\mathcal{O}_2)$.
6. *For a closed orientation set $\mathcal{O}$, a polygon is strongly $\mathcal{O}$-convex if and only if it is convex and its edges are $\mathcal{O}$-oriented.*

Proof.

(1) By definition, a translation of an $\mathcal{O}$-oriented line is an $\mathcal{O}$-oriented line. Therefore, translations of $\mathcal{O}$-blocks are $\mathcal{O}$-blocks and translations of strongly $\mathcal{O}$-convex sets are strongly $\mathcal{O}$-convex sets.

(2) If C is a collection of strongly $\mathcal{O}$-convex sets, then, for every two points p and q of the intersection $\bigcap C$, the $\mathcal{O}$-block of p and q is a subset of every element of C and, hence, $\mathcal{O}$-block(p,q) is a subset of $\bigcap C$.

(3) For every two points p and q, the straight segment joining them is contained in $\mathcal{O}$-block(p,q). Therefore, for every two points of a strongly $\mathcal{O}$-convex set, the segment joining them is contained in the set.

(4) Suppose that $\mathcal{O}_1 \subseteq \mathcal{O}_2$. We readily conclude from the definition of $\mathcal{O}$-blocks that, for every two points p and q, $\mathcal{O}_2$-block$(p,q) \subseteq \mathcal{O}_1$-block$(p,q)$. Therefore, if P is strongly

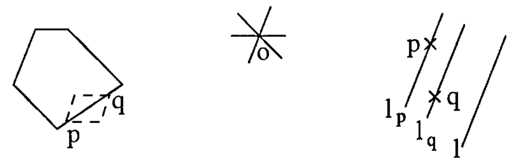

Figure 3: Proof of Lemma 2.

$\mathcal{O}_1$-convex, then, for every two points of P, the $\mathcal{O}_2$-block of these points is in P and, hence, P is strongly $\mathcal{O}_2$-convex.

(5) Let $\mathcal{O}_{1\text{-cl}}$ be the closure of $\mathcal{O}_1$ and $\mathcal{O}_{2\text{-cl}}$ be the closure of $\mathcal{O}_2$. By definition, the notions of $\mathcal{O}_1$-blocks and $\mathcal{O}_{1\text{-cl}}$-blocks are equivalent; therefore, strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_{1\text{-cl}}$-convexity. Similarly, strong $\mathcal{O}_2$-convexity is equivalent to strong $\mathcal{O}_{2\text{-cl}}$-convexity. If $\mathcal{O}_{1\text{-cl}} = \mathcal{O}_{2\text{-cl}}$, then strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity.

Suppose, conversely, that $\mathcal{O}_{1\text{-cl}} \neq \mathcal{O}_{2\text{-cl}}$. Without loss of generality, we assume that $\mathcal{O}_{1\text{-cl}}$ is *not* a subset of $\mathcal{O}_{2\text{-cl}}$. Let p and q be two points such that the line through them is an $\mathcal{O}_{1\text{-cl}}$-line and not an $\mathcal{O}_{2\text{-cl}}$-line. Then, the segment joining p and q is strongly $\mathcal{O}_1$-convex but not strongly $\mathcal{O}_2$-convex.

(6) If a polygon P is *not* convex, then it is not strongly $\mathcal{O}$-convex by Part 3 of the proof. If some edge of P is *not* $\mathcal{O}$-oriented, then, for any two distinct points p and q of this edge, the $\mathcal{O}$-block of p and q is not in P (see Figure 3a) and, hence, we again conclude that P is not strongly $\mathcal{O}$-convex.

Now suppose that P is a convex polygon with $\mathcal{O}$-oriented edges. Then, P is the intersection of several halfplanes whose boundaries are $\mathcal{O}$-oriented lines. We prove that P is strongly $\mathcal{O}$-convex by demonstrating that each of these halfplanes is strongly $\mathcal{O}$-convex: we show that, for every halfplane with $\mathcal{O}$-oriented boundary l and every two points p and q of the halfplane, the $\mathcal{O}$-block of p and q is in the halfplane. Let l_p be the line through p parallel to l and l_q be the line through q parallel to l (see Figure 3b). Since l_p and l_q are $\mathcal{O}$-oriented, $\mathcal{O}$-block(p, q) is contained in the "strip" between l_p and l_q; therefore, $\mathcal{O}$-block(p, q) is in the halfplane. We conclude that P is the intersection of several strongly $\mathcal{O}$-convex halfplanes; therefore, P is strongly $\mathcal{O}$-convex by Part 2 of the proof. □

3 Strong convexity in higher dimensions

We now extend the notion of strong convexity to d-dimensional space $\mathcal{R}^d$. We assume that the space $\mathcal{R}^d$ is fixed; however, all the results are independent of the particular value of d. We introduce a set $\mathcal{O}$ of hyperplanes through a fixed point o, define $\mathcal{O}$-blocks in d dimensions, and use $\mathcal{O}$-blocks to define strongly $\mathcal{O}$-convex sets.

A *hyperplane* in d dimensions is a subset of $\mathcal{R}^d$ that is a $(d-1)$-dimensional space. For example, hyperplanes in three dimensions are the usual planes. Analytically, a hyperplane is the set of points satisfying a linear equation, $a_1x_1 + a_2x_2 + \ldots + a_dx_d = b$, in Cartesian coordinates. Two hyperplanes are *parallel* if they are translations of each other. Analytically, two hyperpanes are parallel if their equations differ only by the value of b.

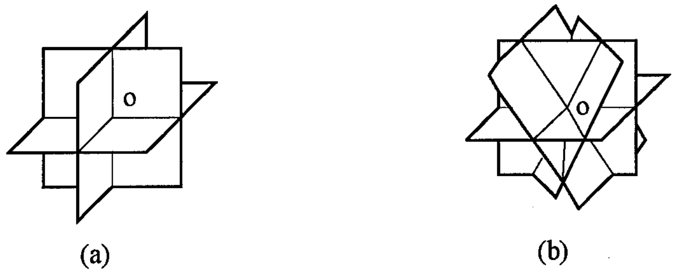

Figure 4: Orientation sets in three dimensions.

Definition 2 (Orientation sets and $\mathcal{O}$-oriented hyperplanes) *An* orientation set $\mathcal{O}$ *in d dimensions is a set of hyperplanes through a fixed point o. A hyperplane parallel to one of the elements of $\mathcal{O}$ is called an* $\mathcal{O}$-oriented hyperplane.

Note that every translation of an $\mathcal{O}$-oriented hyperplane is an $\mathcal{O}$-oriented hyperplane and a particular choice of the point o is not important. When we speak of several different orientation sets in $\mathcal{R}^d$, we always assume that the elements of all these sets are through the same common point o.

In Figure 4, we show two examples of finite orientation sets in three dimensions. The first set contains three mutually orthogonal planes; it gives rise to the three-dimensional analog of strong ortho-convexity. The second orientation set consists of four planes.

The definition of $\mathcal{O}$-blocks in higher dimensions is more complex than the definition of planar $\mathcal{O}$-blocks. First, we define the notion of a *layer* of two points, p and q. Let $\mathcal{H}$ be a hyperplane from the orientation set $\mathcal{O}$, $\mathcal{H}_p$ be the hyperplane through p parallel to $\mathcal{H}$, and $\mathcal{H}_q$ be the hyperplane through q parallel to $\mathcal{H}$. The "layer" of space between the planes $\mathcal{H}_p$ and $\mathcal{H}_q$ is called the *$\mathcal{H}$-layer* of p and q. Analytically, the layer can be defined as follows. Suppose that $\mathcal{H}_p$ is described by equation $a_1x_1 + a_2x_2 + ... + a_dx_d = b_p$ and $\mathcal{H}_q$ is described by equation $a_1x_1 + a_2x_2 + ... + a_dx_d = b_q$ (since $\mathcal{H}_p$ and $\mathcal{H}_q$ are parallel, all coefficients are identical). For simplicity, we assume that $b_p \le b_q$. Then, the *$\mathcal{H}$-layer* of p and q is described by the inequality

$$b_p \le a_1x_1 + a_2x_2 + ... + a_dx_d \le b_q.$$

The *$\mathcal{O}$-block* of p and q is the intersection of all the $\mathcal{O}$-oriented layers of p and q:

$$\mathcal{O}\text{-block}(p,q) = \bigcap_{\mathcal{H}\in\mathcal{O}} \mathcal{H}\text{-layer}(p,q).$$

In other words, a point is in the $\mathcal{O}$-block of p and q if, for every $\mathcal{O}$-oriented hyperplane $\mathcal{H}$, the point is between $\mathcal{H}_p$ and $\mathcal{H}_q$.

In two dimensions, we may define $\mathcal{O}$-blocks in the same way: a planar layer is the "layer" between two parallel lines and the $\mathcal{O}$-block of two points is the intersection of all $\mathcal{O}$-oriented layers of these two points. This definition is equivalent to the definition of planar $\mathcal{O}$-blocks in Section 2, as illustrated by Figure 5.

We show some examples of three-dimensional $\mathcal{O}$-blocks in Figure 6. For the three-element orientation set in Figure 6(a), $\mathcal{O}$-blocks are parallelepipeds with $\mathcal{O}$-oriented facets. The orientation set in Figure 6(b) contains four planes and gives rise to more complex $\mathcal{O}$-blocks.

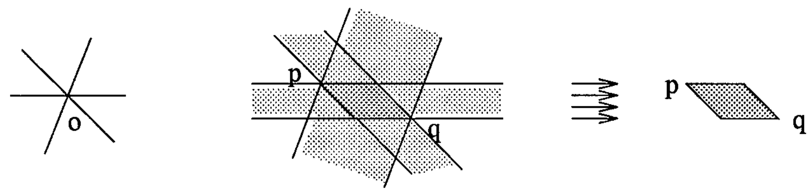

Figure 5: In two dimensions, the intersection of all the $\mathcal{O}$-oriented layers is the $\mathcal{O}$-block.

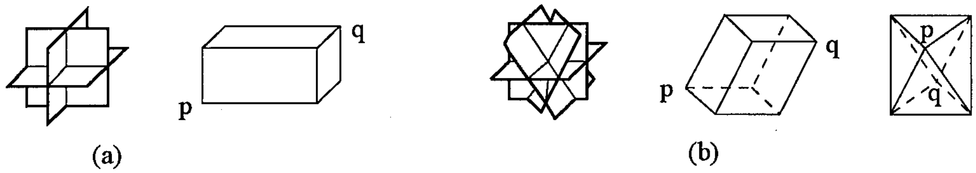

Figure 6: $\mathcal{O}$-blocks in three dimensions.

We define strong $\mathcal{O}$-convexity in higher dimensions in the same way as in two dimensions.

Strong Convexity *A set in $\mathcal{R}^d$ is* strongly $\mathcal{O}$-convex *if, for every two points of the set, their $\mathcal{O}$-block is contained in the set.*

We show some examples of strongly $\mathcal{O}$-convex polytopes in Figure 7. For the orientation set in Figure 7(a), strongly $\mathcal{O}$-convex polytopes are parallelepipeds with $\mathcal{O}$-oriented facets. The four-element orientation set of Figure 4(b) gives rise to more complex strongly $\mathcal{O}$-convex objects (Figure 7b); the facets of these objects are also $\mathcal{O}$-oriented, as we show in Section 6 (see Corollary 22).

4 Basic properties of strongly convex sets

We present some simple properties of strongly $\mathcal{O}$-convex sets in higher dimensions and compare them with properties of planar strongly $\mathcal{O}$-convex sets.

Let us recall the properties of planar strong $\mathcal{O}$-convexity presented in Section 2 (see Lemma 2). We readily generalize Properties 1–3 to higher dimensions: these properties hold in $\mathcal{R}^d$ and their proofs are the same as the proofs in $\mathcal{R}^2$. The most important of them is Property 2, which is a generalization of the intersection property for standard convex sets: the intersection of a collection strongly $\mathcal{O}$-convex sets is strongly $\mathcal{O}$-convex. Property 4 also holds in higher dimensions, as we demonstrate in Corollary 4.

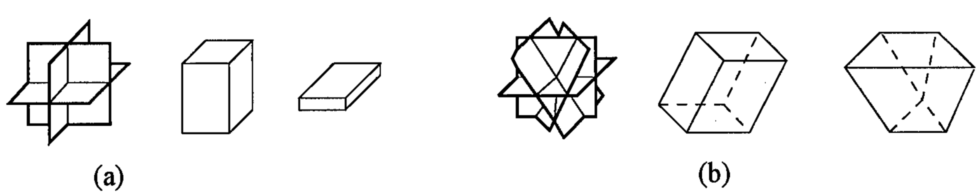

Figure 7: Strongly $\mathcal{O}$-convex sets.

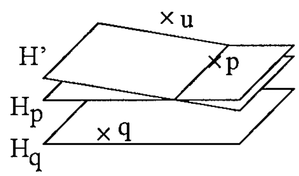

Figure 8: Proof of Lemma 3.

Property 5 holds only in one direction: if two orientation sets, $\mathcal{O}_1$ and $\mathcal{O}_2$, have identical closures, then strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity (see Corollary 4). The converse does not hold: strong convexity for $\mathcal{O}_1$ and $\mathcal{O}_2$ may be equivalent even if Closure($\mathcal{O}_1$) $\neq$ Closure($\mathcal{O}_2$) (see Example 1 in Section 5). We present a necessary and sufficient condition for the equivalence of strong $\mathcal{O}_1$-convexity and strong $\mathcal{O}_2$-convexity in Section 5.

For Property 6, we show that its analog holds in higher dimensions for *finite* orientation sets (Corollary 22): a polytope is strongly $\mathcal{O}$-convex if and only if it is convex and its facets are $\mathcal{O}$-oriented. For an infinite orientation set, a polytope may be strongly $\mathcal{O}$-convex even if its facets are not $\mathcal{O}$-oriented (see Section 6).

Since strongly $\mathcal{O}$-convex sets are defined in terms of $\mathcal{O}$-block visibility, we first compare $\mathcal{O}$-blocks for different orientation sets.

Lemma 3

1. *If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then, for every two points p and q, $\mathcal{O}_2$-block$(p,q) \subseteq \mathcal{O}_1$-block$(p,q)$.*

2. *If $\mathcal{O}_2$ is the closure of $\mathcal{O}_1$, then, for every two points p and q, $\mathcal{O}_1$-block(p,q) = $\mathcal{O}_2$-block(p,q).*

Proof.

(1) If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every $\mathcal{O}_1$-oriented layer is $\mathcal{O}_2$-oriented. Since the $\mathcal{O}$-block of two points is defined as the intersection of all $\mathcal{O}$-oriented layers, we conclude that, for every two points, their $\mathcal{O}_2$-block is a subset of their $\mathcal{O}_1$-block.

(2) If $\mathcal{O}_2$ is the closure of $\mathcal{O}_1$, then $\mathcal{O}_1 \subseteq \mathcal{O}_2$; therefore, for every two points p and q, $\mathcal{O}_2$-block$(p,q) \subseteq \mathcal{O}_1$-block$(p,q)$. We prove the converse inclusion by showing that, for every $\mathcal{O}_2$-oriented layer of p and q, $\mathcal{O}_1$-block(p,q) is a subset of this layer; that is, if a point u is not in the layer, then it is not in the $\mathcal{O}_1$-block(p,q) either.

Let $\mathcal{H}$-layer(p,q) be an $\mathcal{O}_2$-oriented layer, with boundary hyperplanes $\mathcal{H}_p$ (through p) and $\mathcal{H}_q$ (through q), and let u be a point outside of $\mathcal{H}$-layer(p,q). Without loss of generality, we assume that either $\mathcal{H}_p$ is between u and $\mathcal{H}_q$ (see Figure 8) or $\mathcal{H}_p = \mathcal{H}_q$. If $\mathcal{H}_p$ is $\mathcal{O}_1$-oriented, then $\mathcal{O}_1$-block$(p,q) \subseteq \mathcal{H}$-layer$(p,q)$ and, hence, $u \notin \mathcal{O}_1$-block(p,q). If $\mathcal{H}_p$ is not $\mathcal{O}_1$-oriented, then there is a sequence of $\mathcal{O}_1$-oriented hyperplanes through p convergent to $\mathcal{H}_p$. For some element $\mathcal{H}'$ of this sequence, q and u are "on different sides" of $\mathcal{H}'$ (Figure 8). The layer of p and q parallel to $\mathcal{H}'$ is $\mathcal{O}_1$-oriented and u is outside of this layer; therefore, we again have $u \notin \mathcal{O}_1$-block(p,q). □

Combining these properties of $\mathcal{O}$-blocks and the definition of strong $\mathcal{O}$-convexity, we immediately obtain the following results.

Corollary 4

1. *If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every strongly $\mathcal{O}_1$-convex set is strongly $\mathcal{O}_2$-convex.*

2. *If $\mathcal{O}_2$ is the closure of $\mathcal{O}_1$, then strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity.*

According to the second part of this result, *we may restrict our attention to the study of strong $\mathcal{O}$-convexity for closed orientation sets*, because strong convexity for every orientation set is equivalent to strong convexity for its closure.

We next show that $\mathcal{O}$-blocks are strongly $\mathcal{O}$-convex.

Lemma 5 *The $\mathcal{O}$-block of every two points is strongly $\mathcal{O}$-convex.*

Proof. We consider the $\mathcal{O}$-block of two points p and q. We have to show that, for every two points u and v in $\mathcal{O}$-block(p,q), we have $\mathcal{O}$-block$(u,v) \subseteq \mathcal{O}$-block$(p,q)$. We note that, for every $\mathcal{H} \in \mathcal{O}$, the points u and v are in the $\mathcal{H}$-layer of p and q; therefore, $\mathcal{H}$-layer$(u,v) \subseteq \mathcal{H}$-layer$(p,q)$. Since the $\mathcal{O}$-block of two points is the intersection of all their $\mathcal{O}$-oriented layers, we conclude that $\mathcal{O}$-block$(u,v) \subseteq \mathcal{O}$-block$(p,q)$. □

According to Property 3 of strong $\mathcal{O}$-convexity (see Lemma 2), strongly $\mathcal{O}$-convex sets are standard convex. We now present a condition for the equivalence of strong and standard convexity.

Lemma 6 *Every convex set is strongly $\mathcal{O}$-convex if and only if every straight line is strongly $\mathcal{O}$-convex.*

Proof. Every line is a convex set. Therefore, if every convex set is strongly $\mathcal{O}$-convex, then every line is strongly $\mathcal{O}$-convex.

Suppose, conversely, that every line is strongly $\mathcal{O}$-convex. Then, for every two points p and q, their $\mathcal{O}$-block is just the straight segment joining them: if the $\mathcal{O}$-block were a superset of this segment, then the line through p and q would not be strongly $\mathcal{O}$-convex. Therefore, the $\mathcal{O}$-block visibility is just standard visibility and strong $\mathcal{O}$-convexity is equivalent to standard convexity. □

5 Strongly convex flats

We now explore the properties of strongly $\mathcal{O}$-convex flats. A *flat*, also known as an *affine variety*, is a subset of $\mathcal{R}^d$ that is itself a lower-dimensional space. For example, points, straight lines, two-dimensional planes, and hyperplanes are flats.

First, we characterize strongly $\mathcal{O}$-convex flats in terms of *$\mathcal{O}$-oriented* flats, which are the intersections of $\mathcal{O}$-oriented hyperplanes. We show that, for a finite orientation set, a flat is strongly $\mathcal{O}$-convex if and only it is $\mathcal{O}$-oriented. For an infinite orientation set, every $\mathcal{O}$-oriented flat is strongly $\mathcal{O}$-convex, but the converse does not hold: a flat may be strongly $\mathcal{O}$-convex even if it is not $\mathcal{O}$-oriented.

Then, we consider the set $\tilde{\mathcal{O}}$ of all strongly $\mathcal{O}$-convex hyperplanes through o and describe strong $\mathcal{O}$-convexity with respect to this new orientation set $\tilde{\mathcal{O}}$. For finite $\mathcal{O}$, the orientation set $\tilde{\mathcal{O}}$ is identical to $\mathcal{O}$; however, if $\mathcal{O}$ is infinite, then $\tilde{\mathcal{O}}$ may be a superset of $\mathcal{O}$. We show that strong $\mathcal{O}$-convexity is equivalent to strong $\tilde{\mathcal{O}}$-convexity and use this result to derive a necessary and sufficient condition for the equivalence of strong convexity with respect to two different orientation sets. Finally, we establish the strong $\mathcal{O}$-convexity of the *affine hull* of a strongly $\mathcal{O}$-convex set, which is the minimal flat containing the set.

We begin by defining the notion of a flat. Analytically, a *k-dimensional flat* in d dimensions is a subset of $\mathcal{R}^d$ that is represented in Cartesian coordinates as a system of $(d-k)$ independent linear equations. The whole space $\mathcal{R}^d$ is also considered to be a flat. For example, in three dimensions, there are four types of flats: points, lines, planes, and the whole space $\mathcal{R}^3$. Two flats are *parallel* if they are translations of each other (note that parallel flats are of the same dimension). We use the following properties of flats in our exploration.

Proposition 7 (Properties of flats)

1. *The intersection of a collection of flats is either empty or a flat.*
2. *The intersection of a k-dimensional flat η and a hyperplane is empty, η, or a $(k-1)$-dimensional flat.*

We now define *$\mathcal{O}$-oriented* flats.

Definition 3 ($\mathcal{O}$-oriented flats) *A flat is $\mathcal{O}$-*oriented *if it is the intersection of several $\mathcal{O}$-oriented hyperplanes. $\mathcal{O}$-oriented hyperplanes themselves and the whole space $\mathcal{R}^d$ are also $\mathcal{O}$-oriented flats.*

In particular, the lines formed by the intersections of $\mathcal{O}$-oriented hyperplanes are called *$\mathcal{O}$-oriented lines.*

Since every $\mathcal{O}$-oriented hyperplane is parallel to one of the hyperplanes of the orientation set $\mathcal{O}$, every $\mathcal{O}$-oriented flat is parallel to some flat formed by the intersection of several elements of $\mathcal{O}$. If the point o is the intersection of several elements of $\mathcal{O}$, then every point in $\mathcal{R}^d$ is an $\mathcal{O}$-oriented flat.

For example, the intersections of the four planes of the orientation set given in Figure 4(b) form six different lines through o and every $\mathcal{O}$-oriented line for this orientation set is parallel to one of these six lines. The point o is also the intersection of the elements of this set $\mathcal{O}$; thus, all points are $\mathcal{O}$-oriented.

Lemma 8 *Every $\mathcal{O}$-oriented flat is strongly $\mathcal{O}$-convex.*

Proof. If points p and q are in an $\mathcal{O}$-oriented flat, then the $\mathcal{O}$-block of p and q is contained in this flat, since the $\mathcal{O}$-block is a subset of every $\mathcal{O}$-oriented hyperplane through p and q and the flat is equal to the intersection of these hyperplanes. □

Can a flat be strongly $\mathcal{O}$-convex if it is not $\mathcal{O}$-oriented? If $\mathcal{O}$ is a finite or closed countably infinite set, the answer to this question is negative: only $\mathcal{O}$-oriented flats are strongly $\mathcal{O}$-convex (see Theorem 10). If $\mathcal{O}$ is not closed, then strong $\mathcal{O}$-convexity is equivalent to strong convexity with respect to the closure of $\mathcal{O}$ (Corollary 4). In this case, all hyperplanes in

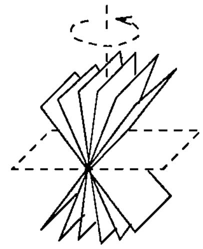

Figure 9: Construction of the orientation set $\mathcal{O}_{\mathrm{sc}}$.

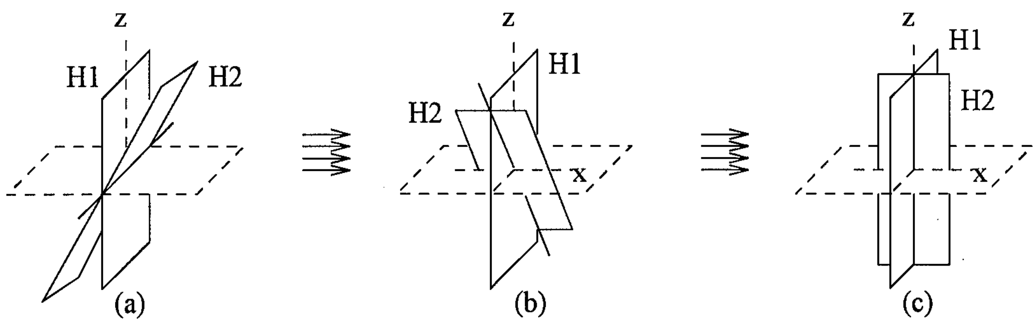

Figure 10: Construction to show that all lines are $\mathcal{O}_{\mathrm{sc}}$-lines.

the closure of $\mathcal{O}$ and all intersections of these hyperplanes are strongly $\mathcal{O}$-convex flats, even though some of them are not $\mathcal{O}$-oriented. For closed uncountable $\mathcal{O}$, points and lines are strongly $\mathcal{O}$-convex only if they are $\mathcal{O}$-oriented (see Theorem 11), whereas higher-dimensional flats may be strongly $\mathcal{O}$-convex even if they are not $\mathcal{O}$-oriented, as we show in the following example.

Example 1: A strongly $\mathcal{O}$-convex flat may not be $\mathcal{O}$-oriented.
Let $\mathcal{O}_{\mathrm{sc}}$ be the orientation set in three dimensions that includes all planes through o whose angle with the "horizontal" plane is at least $\pi/3$ (where any plane through o can serve as the horizontal plane). We illustrate the construction of the orientation set $\mathcal{O}_{\mathrm{sc}}$ in Figure 9, where the horizontal plane is shown by dashed lines. The set contains the (uncountably many) planes shown by solid lines and all the rotations of these planes around the vertical axis. The index "sc" stands for "standard convexity," as we show that strong $\mathcal{O}_{\mathrm{sc}}$-convexity is equivalent to standard convexity.

We now demonstrate that every line through o is the intersection of two elements of $\mathcal{O}_{\mathrm{sc}}$. An informal proof of this claim is illustrated in Figure 10, where H_1 and H_2 are elements of $\mathcal{O}_{\mathrm{sc}}$ and the horizontal plane is shown by dashes. In Figure 10(a), the intersection of H_1 and H_2 is a horizontal line. Now suppose that we rotate H_2 around the vertical axis z, until it reaches the position shown in Figure 10(b). We then rotate H_2 around the horizontal axis x, until it becomes as shown in Figure 10(c). At all times H_2 remains an element of $\mathcal{O}$. The intersection of H_1 and H_2 is always a line, whose position continuously changes from horizontal to vertical. Since every rotation around the vertical axis z maps $\mathcal{O}_{\mathrm{sc}}$ into itself, we conclude that every line through o can be formed by the intersection of two elements of $\mathcal{O}_{\mathrm{sc}}$.

Since translations of elements of $\mathcal{O}_{\mathrm{sc}}$ are $\mathcal{O}_{\mathrm{sc}}$-oriented planes, we conclude that every line is the intersection of two $\mathcal{O}_{sc}$-oriented planes; therefore, every line is strongly $\mathcal{O}_{sc}$-convex. According to Lemma 6, if every line is strongly convex, then strong convexity is equivalent to standard convexity; therefore, every plane is strongly $\mathcal{O}_{sc}$-convex. We have shown that all planes are strongly $\mathcal{O}_{sc}$-convex, whereas some planes are not $\mathcal{O}_{sc}$-oriented.

Note that, if we define $\mathcal{O}'_{sc}$ as the set of planes through o whose angle with some vertical plane is at least $\pi/3$, then strong convexity for $\mathcal{O}'_{sc}$ is also equivalent to standard convexity. This example demonstrates that the notions of strong convexity for different closed orientation sets can be equivalent, which means that Property 5 of strongly $\mathcal{O}$-convex sets (see Lemma 2) does not hold in three dimensions. □

In the following result, we characterize strongly $\mathcal{O}$-convex flats in terms of $\mathcal{O}$-oriented flats.

Theorem 9 *For a closed orientation set $\mathcal{O}$, a flat η is strongly $\mathcal{O}$-convex if and only if, for every two points of η, there is an $\mathcal{O}$-oriented flat through them that is contained in η.*

Proof. Suppose that, for every $p, q \in \eta$, there is an $\mathcal{O}$-oriented flat $H \subseteq \eta$ through p and q. Since H is strongly $\mathcal{O}$-convex (Lemma 8), $\mathcal{O}$-block$(p, q) \subseteq H \subseteq \eta$. Thus, for every two points of η, their $\mathcal{O}$-block is in η; therefore, η is strongly $\mathcal{O}$-convex.

Suppose, conversely, that η is strongly $\mathcal{O}$-convex and consider two points, p and q, of η. Let H be the intersection of all $\mathcal{O}$-oriented hyperplanes through p and q; then, H is an $\mathcal{O}$-oriented flat. We show, by contradiction, that $H \subseteq \eta$.

Suppose that H is *not* in η. Now, $H \cap \eta$ is a strongly $\mathcal{O}$-convex flat whose dimension is less than the dimension of H. Let u be the middle point of the straight segment joining p and q. Since $\mathcal{O}$-block$(p, q) \subseteq H \cap \eta$ and the dimension of $H \cap \eta$ is less than the dimension of H, we conclude that, for every ball S_u centered at u, $H \cap S_u \not\subseteq \mathcal{O}$-block$(p, q)$. Therefore, for every ball S_u centered at u, there is an $\mathcal{O}$-oriented layer of p and q that does not contain $H \cap S_u$.

If a layer of p and q does not contain $H \cap S_u$, then each boundary hyperplane of this layer intersects S_u and does not contain H. Thus, we can select a sequence of $\mathcal{O}$-oriented hyperplanes through p that do not contain H such that the distances from these hyperplanes to u converge to zero. Selecting a convergent subsequence of this sequence and taking its limit, we get an $\mathcal{O}$-oriented hyperplane through p and q that does not contain H, which contradicts the definition of H. (Recall that we have defined H as the intersection of *all* $\mathcal{O}$-oriented hyperplanes through p and q.) □

We have demonstrated that every $\mathcal{O}$-oriented flat is strongly $\mathcal{O}$-convex (Lemma 8). We next show that, for finite and closed countably infinite orientation sets, only $\mathcal{O}$-oriented flats are strongly $\mathcal{O}$-convex.

Theorem 10 *If $\mathcal{O}$ is a closed countable set, a flat is strongly $\mathcal{O}$-convex if and only if it is $\mathcal{O}$-oriented.*

Proof. By Lemma 8, an $\mathcal{O}$-oriented flat is strongly $\mathcal{O}$-convex. To prove the converse, suppose that $\mathcal{O}$ is countable and consider a strongly $\mathcal{O}$-convex flat η that is *not* $\mathcal{O}$-oriented.

We denote the dimension of η by k. For every $\mathcal{O}$-oriented flat contained in η, it dimension is at most $(k-1)$.

Let p be some point of η. The set of $\mathcal{O}$-oriented hyperplanes through p is countable. The intersections of these hyperplanes form countably many $\mathcal{O}$-flats. Therefore, there are only countably many $\mathcal{O}$-oriented flats through p contained in η. Since the dimension of these flats is at most $(k-1)$, they do not cover η. Thus, there is a point q in η such that no $\mathcal{O}$-flat through p and q is contained in η. Therefore, by Theorem 9, η is not strongly $\mathcal{O}$-convex. $\square$

For lines and points, the analogous result holds even when an orientation set is uncountable: for closed $\mathcal{O}$, only $\mathcal{O}$-oriented lines and points are strongly $\mathcal{O}$-convex.

Theorem 11 *If $\mathcal{O}$ is a closed orientation set, then a line or point is strongly $\mathcal{O}$-convex if and only if it is $\mathcal{O}$-oriented.*

Proof. Every $\mathcal{O}$-oriented flat is strongly $\mathcal{O}$-convex; it remains to prove the "only if" part. We first prove it for a point and then for a line.

Suppose that a point p is strongly $\mathcal{O}$-convex. If $p = q$, then the $\mathcal{H}$-oriented layer of p and q is just the hyperplane through p parallel to $\mathcal{H}$. Therefore, the $\mathcal{O}$-block of p and q is the intersection of all $\mathcal{O}$-oriented hyperplanes through p. Since p is strongly $\mathcal{O}$-convex, this $\mathcal{O}$-block is contained in p. Therefore, p is the intersection of $\mathcal{O}$-hyperplanes and, hence, it is $\mathcal{O}$-oriented.

Now suppose that a line l is strongly $\mathcal{O}$-convex and let p and q be two distinct points of l. By Theorem 9, there is an $\mathcal{O}$-oriented flat through p and q contained in l. Since the only flat through p and q contained in l is l itself, we conclude that l is $\mathcal{O}$-oriented. $\square$

For a given orientation set $\mathcal{O}$, we define $\widetilde{\mathcal{O}}$ as the set of all strongly $\mathcal{O}$-convex hyperplanes through o. For example, consider the three-dimensional orientation set $\mathcal{O}_{\text{sc}}$, described in Example 1, which contains the planes whose angle with the horizontal plane is at least $\pi/3$. We have shown that all planes are strongly convex for $\mathcal{O}_{\text{sc}}$; thus, $\widetilde{\mathcal{O}}_{\text{sc}}$ contains all planes through o.

We consider the notion of strong $\widetilde{\mathcal{O}}$-convexity, which is strong convexity with respect to the orientation set $\widetilde{\mathcal{O}}$. Observe that, since every $\mathcal{O}$-oriented hyperplane is strongly $\mathcal{O}$-convex, we have $\mathcal{O} \subseteq \widetilde{\mathcal{O}}$; therefore, every strongly $\mathcal{O}$-convex set is strongly $\widetilde{\mathcal{O}}$-convex (Corollary 4). We next show that the converse also holds: every strongly $\widetilde{\mathcal{O}}$-convex set is strongly $\mathcal{O}$-convex.

Theorem 12

1. *Strong $\mathcal{O}$-convexity is equivalent to strong $\widetilde{\mathcal{O}}$-convexity. Moreover, for every orientation set $\mathcal{O}_1$, if strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}$-convexity, then $\mathcal{O}_1 \subseteq \widetilde{\mathcal{O}}$.*

2. *Strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity if and only if $\widetilde{\mathcal{O}}_1 = \widetilde{\mathcal{O}}_2$.*

Proof.
(1) We prove the equivalence by demonstrating that, for every two points p and q, we have $\mathcal{O}$-block$(p,q) = \widetilde{\mathcal{O}}$-block$(p,q)$. Without loss of generality, we assume that $\mathcal{O}$ is closed (Lemma 3).

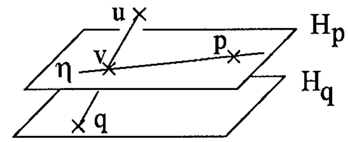

Figure 11: Proof of Theorem 12.

Since $\mathcal{O}$ is a subset of $\tilde{\mathcal{O}}$, we immediately conclude that $\tilde{\mathcal{O}}$-block$(p,q) \subseteq \mathcal{O}$-block$(p,q)$ (Lemma 3). To prove the converse inclusion, we show that, for every $\tilde{\mathcal{O}}$-oriented layer of p and q, $\mathcal{O}$-block(p,q) is a subset of this layer; that is, if a point u is not in the layer, then it is not in the $\mathcal{O}$-block(p,q) either.

Let $\mathcal{H}$-layer(p,q) be an $\tilde{\mathcal{O}}$-oriented layer, with boundary hyperplanes $\mathcal{H}_p$ (through p) and $\mathcal{H}_q$ (through q), and let u be a point outside of $\mathcal{H}$-layer(p,q) (see Figure 11). Since $\mathcal{H}_p$ and $\mathcal{H}_q$ are $\tilde{\mathcal{O}}$-oriented, they are strongly $\mathcal{O}$-convex.

First, suppose that $\mathcal{H}_p = \mathcal{H}_q$; that is, q is in $\mathcal{H}_p$. Then, the $\mathcal{O}$-block of p and q is a subset of $\mathcal{H}_p$, because $\mathcal{H}_p$ is strongly $\mathcal{O}$-convex; therefore, u is not in $\mathcal{O}$-block(p,q).

Next, suppose that $\mathcal{H}_p$ and $\mathcal{H}_q$ are distinct hyperplanes. Without loss of generality, we assume that $\mathcal{H}_p$ is between u and $\mathcal{H}_q$ (Figure 11). Then, the segment joining q and u intersects $\mathcal{H}_p$; we denote the point of their intersection by v. Since $\mathcal{O}$ is closed, we conclude, by Theorem 9, that there is an $\mathcal{O}$-oriented flat η through p and v that is contained in $\mathcal{H}_p$. By the definition of $\mathcal{O}$-flats, η is the intersection of several $\mathcal{O}$-oriented hyperplanes; since $u \notin \eta$, one of these hyperplanes, say $\mathcal{H}_1$, does not contain u. Then, q is not in $\mathcal{H}_1$ either and u and q are "on different sides" of $\mathcal{H}_1$. Therefore, u is not in the $\mathcal{H}_1$-layer of p and q. Since the $\mathcal{H}_1$-layer is $\mathcal{O}$-oriented, we conclude that $u \notin \mathcal{O}$-block(u,v).

Finally, we have to show that, if strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}$-convexity, then $\mathcal{O}_1 \subseteq \tilde{\mathcal{O}}$. If the two convexities are equivalent, then $\tilde{\mathcal{O}}$ contains all the strongly $\mathcal{O}_1$-convex hyperplanes through o. Since every $\mathcal{O}_1$-oriented hyperplane is strongly $\mathcal{O}_1$-convex, we conclude that $\mathcal{O}_1 \subseteq \tilde{\mathcal{O}}$.

(2) Since strong $\mathcal{O}_1$-convexity is equivalent to strong $\tilde{\mathcal{O}}_1$-convexity and the same holds for $\mathcal{O}_2$, we conclude that, if $\tilde{\mathcal{O}}_1 = \tilde{\mathcal{O}}_2$, then strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity. On the other hand, if strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity, then every hyperplane is strongly $\mathcal{O}_1$-convex if and only if it is strongly $\mathcal{O}_2$-convex; therefore, by definition, $\tilde{\mathcal{O}}_1 = \tilde{\mathcal{O}}_2$. □

We conclude from the first part of Theorem 12 that $\tilde{\mathcal{O}}$ is the maximal orientation set for which strong convexity is equivalent to strong $\mathcal{O}$-convexity. The second part of Theorem 12 is a generalization of Property 5 of planar strong convexity (see Lemma 2). It provides a necessary and sufficient condition for the equivalence of strong convexity for two different orientation sets.

We thus have shown that, for every orientation set $\mathcal{O}$, there is a unique maximal set for which strong convexity is equivalent to strong $\mathcal{O}$-convexity. On the other hand, there may *not* be a unique minimal orientation set for which strong convexity is equivalent to strong $\mathcal{O}$-convexity. For example, suppose that $\mathcal{O}$ is an orientation set in two dimensions that contains all lines through o. Strong convexity with respect to some orientation set $\mathcal{O}_1$ is equivalent

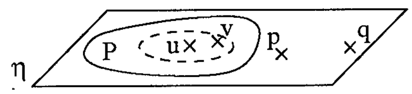

Figure 12: Proof of Lemma 15.

to strong $\mathcal{O}$-convexity if and only if the closure of $\mathcal{O}_1$ is $\mathcal{O}$ (see Part 5 of Lemma 2). The collection of all such orientation sets does not have a unique minimal element. In fact, it does not have any minimal elements. For every set $\mathcal{O}_1$ whose closure is $\mathcal{O}$, we can construct a proper subset of $\mathcal{O}_1$ whose closure is also $\mathcal{O}$, by removing some line from $\mathcal{O}_1$.

Corollary 13 *For every $\mathcal{O}$, the set $\tilde{\mathcal{O}}$ is closed.*

Proof. Let $\tilde{\mathcal{O}}_{\rm cl}$ be the closure of $\tilde{\mathcal{O}}$. By Lemma 3, strong $\tilde{\mathcal{O}}_{\rm cl}$-convexity is equivalent to strong $\tilde{\mathcal{O}}$-convexity. Therefore, by Theorem 12, $\tilde{\mathcal{O}}_{\rm cl} \subseteq \tilde{\mathcal{O}}$, which implies that $\tilde{\mathcal{O}}_{\rm cl} = \tilde{\mathcal{O}}$. □

We now establish the strong $\mathcal{O}$-convexity of the affine hull of a strongly $\mathcal{O}$-convex set. The *affine hull* η of a set P is the minimal flat that contains P. In other words, it is the intersection of all flats that contain P (recall that the intersection of flats is a flat). For example, the affine hull of a straight segment is a line, the affine hull of a triangle is a two-dimensional plane, and the affine hull of a ball is the whole space.

Next, we define the *relative interior* of a set P in its affine hull η. Since η is a lower-dimensional space, we can speak of the interior of P within this space; this interior is called the relative interior of P. For example, suppose that P is a triangle in $\mathcal{R}^3$ and η is the plane that contains this triangle. The interior of the triangle in $\mathcal{R}^3$ is empty. On the other hand, its relative interior includes all points except the sides of the triangle, since only the sides make the boundary of the triangle within the two-dimensional space η. We use the following property of relative interiors in our exploration [Grünbaum *et al.*, 1967].

Proposition 14 *If P is a convex set and η is the affine hull of P, then the relative interior of P in η is nonempty.*

The next result gives an important property of the affine hulls of strongly $\mathcal{O}$-convex sets, which we use in characterizing strongly $\mathcal{O}$-convex sets in terms of halfplane intersections (see Lemma 19).

Lemma 15 *The affine hull of a strongly $\mathcal{O}$-convex set is strongly $\mathcal{O}$-convex.*

Proof. Let P be a strongly $\mathcal{O}$-convex set and η be the affine hull of P (see Figure 12). Since P is convex, the relative interior of P in η is nonempty. Therefore, we can choose an interior point u in P and a ball $S_u \subseteq P$ centered at u. (Note that S_u is a ball in the space η rather than in $\mathcal{R}^d$; this ball is shown by the dashed circle in Figure 12.)

We have to show that, for every two points p and q of η, the $\mathcal{O}$-block of these two points is in η. Let v be a point in S_u such that the line through u and v is parallel to the line through p and q (Figure 12). The $\mathcal{O}$-block of u and v is in P; therefore, it is in η. The $\mathcal{O}$-block of p and q is a scaled version of $\mathcal{O}$-block(u, v); therefore, it is also in η. □

6 Strongly convex halfspaces

We now study the properties of strongly $\mathcal{O}$-convex halfspaces and show that their role in strong $\mathcal{O}$-convexity is similar to the role of halfspaces in standard convexity. We present, in Theorems 17 and 18, strong-convexity analogs of the supporting-planes and halfspace-intersection properties of convex sets (see Section 1). We characterize strongly $\mathcal{O}$-convex sets in terms of supporting hyperplanes and in terms of halfspace intersections.

We begin by characterizing strongly $\mathcal{O}$-convex halfspaces in terms of their boundaries.

Theorem 16 *A halfspace is strongly $\mathcal{O}$-convex if and only if its boundary is a strongly $\mathcal{O}$-convex hyperplane.*

Proof. Let P be a halfspace and $\mathcal{H}$ be its boundary hyperplane. Suppose that $\mathcal{H}$ is strongly $\mathcal{O}$-convex. We show that P is strongly $\mathcal{O}$-convex by demonstrating that it is strongly $\tilde{\mathcal{O}}$-convex. (Recall that, by Theorem 12, strong $\mathcal{O}$-convexity is equivalent to strong $\tilde{\mathcal{O}}$-convexity). Thus, we have to show that, for every two points p and q of P, the $\tilde{\mathcal{O}}$-block of these points is in P. Since $\mathcal{H}$ is strongly $\mathcal{O}$-convex, it is $\tilde{\mathcal{O}}$-oriented; therefore, $\tilde{\mathcal{O}}$-block(p, q) is a subset of the $\mathcal{H}$-layer of p and q. This layer is parallel to the boundary $\mathcal{H}$ of P; therefore, it is contained in P. Since $\tilde{\mathcal{O}}$-block(p, q) is contained in the $\mathcal{H}$-layer of p and q, we conclude that $\tilde{\mathcal{O}}$-block(p, q) is in P.

Now suppose, conversely, that the boundary $\mathcal{H}$ of a halfspace P is *not* strongly $\mathcal{O}$-convex. Then, there are points p and q in $\mathcal{H}$ such that $\mathcal{O}$-block(p, q) is not in $\mathcal{H}$. The $\mathcal{O}$-block is centrally symmetric with respect to the middle point of the straight segment joining p and q; therefore, it is not contained in either of the halfspaces with boundary $\mathcal{H}$. Thus, p and q are in P and their $\mathcal{O}$-block is not in P; therefore, P is *not* strongly $\mathcal{O}$-convex. □

We next describe supporting hyperplanes of strongly $\mathcal{O}$-convex sets. A hyperplane *supports* a set if it "touches" the set in some of its boundary points and does not cut the set in two parts. For example, if we put a three-dimensional object on a table, then the surface of the table is a plane that supports the object. To put it more formally, a hyperplane $\mathcal{H}$ supports a set P if the intersection of $\mathcal{H}$ and the boundary of P is nonempty and P is contained in one of the two halfspaces whose boundary is $\mathcal{H}$.

We can describe standard convex sets in terms of supporting hyperplanes: a closed set with a nonempty interior is convex if and only if, for every point of its boundary, there is a supporting hyperplane through this point. We now generalize this property to strongly $\mathcal{O}$-convex sets.

Theorem 17 *A closed set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if, for every point in the boundary of the set, there is a strongly $\mathcal{O}$-convex hyperplane through this point that supports the set.*

Proof. Let P be a closed set with a nonempty interior. Suppose that, for every point r of P's boundary, there is a strongly $\mathcal{O}$-convex hyperplane through r that supports the set. Then, for every boundary point r, there is a strongly $\mathcal{O}$-convex halfspace with boundary through r that contains P. Clearly, he intersection of all such halfspaces is the set P. By

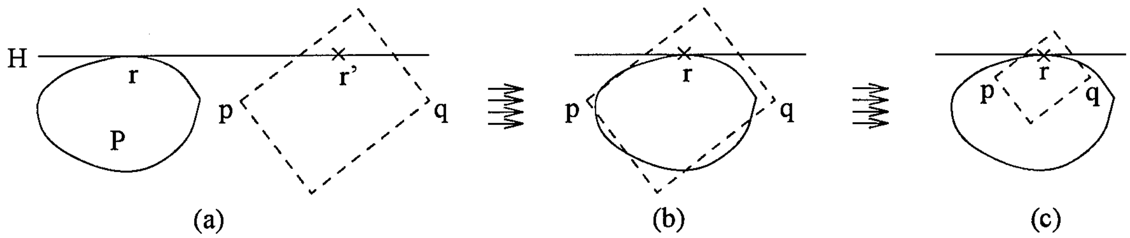

Figure 13: Proof of Theorem 17.

Theorem 16, these halfspaces are strongly $\mathcal{O}$-convex; therefore, their intersection P is also strongly $\mathcal{O}$-convex.

Suppose, conversely, that P is strongly $\mathcal{O}$-convex and let r be a point in the boundary of P. Since P is convex, its boundary in some neighborhood of r can be viewed as a graph of some convex function f.

First, we consider the case when r corresponds to a *regular point* of the function f, which means that the function is differentiable at this point. Then, there is exactly one supporting hyperplane $\mathcal{H}$ through r. We have to prove that this hyperplane is strongly $\mathcal{O}$-convex. For convenience, we view $\mathcal{H}$ as a horizontal plane and P as being below $\mathcal{H}$ (see Figure 13a). We prove that $\mathcal{H}$ is strongly $\mathcal{O}$-convex by contradiction.

Suppose that $\mathcal{H}$ is *not* strongly $\mathcal{O}$-convex. Then, the halfspace with boundary $\mathcal{H}$ that contains P is not strongly $\mathcal{O}$-convex either (Theorem 16). Therefore, there are points p and q in this halfspace such that $\mathcal{O}$-block(p, q) is not in the halfspace (Figure 13a). Without loss of generality, we assume that p and q are *not* in $\mathcal{H}$ (if p or q is in $\mathcal{H}$, we can move these points down "a little bit," in such a way that a part of $\mathcal{O}$-block(p, q) remains above $\mathcal{H}$).

Let us choose some point $r' \in \mathcal{O}$-block$(p, q) \cap \mathcal{H}$ and translate $\mathcal{O}$-block(p, q) in such a way that r' becomes identical to r (Figure 13b). Next, we scale $\mathcal{O}$-block(p, q) in such a way that the point r' of the $\mathcal{O}$-block remains identical to the point r of the set P (Figure 13c). Since the function f is differentiable at r, for a sufficiently small scaled version the $\mathcal{O}$-block, the points p and q are below the graph of the function; that is, they are in P (Figure 13c). On the other hand, a part of the scaled version of $\mathcal{O}$-block(p, q) is above $\mathcal{H}$ and, hence, outside P. Since a translation and a scaled version of an $\mathcal{O}$-block is an $\mathcal{O}$-block, we conclude that there are two points of P such that the $\mathcal{O}$-block of these points is not in P, contradicting the assumption that P is strongly $\mathcal{O}$-convex.

Next we consider the case when r is *not* a regular point; that is, f is not differentiable at r. Then, there may be more than one supporting hyperplane through r. We have to show that at least one of these hyperplanes is strongly $\mathcal{O}$-convex.

Since f is a convex function, it is a function of locally bounded variation. Functions of bounded variation are differentiable "almost everywhere," which means that the set of nonregular points is of measure zero. Therefore, there is a sequence of regular points in the graph of f convergent to r. The supporting hyperplane through each of these points is strongly $\mathcal{O}$-convex.

We can select a convergent subsequence from this sequence of supporting hyperplanes; let $\mathcal{H}$ be the limit of this subsequence. Then, $r \in \mathcal{H}$ and, since the set $\tilde{\mathcal{O}}$ of strongly $\mathcal{O}$-convex hyperplanes is closed (Corollary 13), $\mathcal{H}$ is strongly $\mathcal{O}$-convex. It remains to show that $\mathcal{H}$

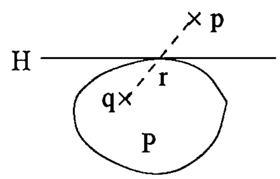

Figure 14: Proof of Theorem 18.

supports P. If $\mathcal{H}$ does *not* support P, then, since P is convex, $\mathcal{H}$ intersects the interior of P. Let u be an interior point of P that belongs to $\mathcal{H}$ and $S_u \subseteq P$ be an open ball centered at u. Then, some hyperplane of the convergent subsequence intersects S_u and, therefore, this hyperplane does not support P, yielding a contradiction. □

To see that the analogous result does not hold for sets with an empty interior, let us consider an $\mathcal{O}$-oriented plane H (say, in three dimensions) and a nonconvex set P contained in H. Then, for every point in P's boundary, H is a supporting plane through this point; however, P is not strongly $\mathcal{O}$-convex since it is not convex.

Our next goal is to generalize the halfspace-intersection property of convex sets: every closed convex set is the intersection of the halfspaces that contain it. We first show that an analogous result holds for strongly $\mathcal{O}$-convex sets with a *nonempty interior.*

Theorem 18 *A closed set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if it is the intersection of strongly $\mathcal{O}$-convex halfspaces.*

Proof. The intersection of strongly $\mathcal{O}$-convex sets is strongly $\mathcal{O}$-convex; therefore, if a set P is the intersection of strongly $\mathcal{O}$-convex halfspaces, then P is strongly $\mathcal{O}$-convex.

Suppose, conversely, that P is a strongly $\mathcal{O}$-convex set with a nonempty interior. To demonstrate that P is the intersection of strongly $\mathcal{O}$-convex halfspaces, we show that, for every point $p \notin P$, there is a strongly $\mathcal{O}$-convex halfspace that contains P and does not contain p.

Let q be an interior point of P and r be a point of the intersection of the straight segment joining p and q with P's boundary (see Figure 14). Note that, since P is closed, $r \neq p$. By Theorem 17, there is a strongly $\mathcal{O}$-convex hyperplane $\mathcal{H}$ through r that supports P. (We show this hyperplane by a solid line in Figure 14.) Since q is an interior point of P, we conclude that $q \notin \mathcal{H}$; therefore, $p \notin \mathcal{H}$. Thus, P and p are "on different sides" of $\mathcal{H}$, which means that the halfspace with boundary $\mathcal{H}$ that contains P does not contain p. □

This result can be readily generalized to nonclosed sets if we use *open halfspaces*, that is, halfspaces that do not contain their boundaries. A set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if it is the intersection of strongly $\mathcal{O}$-convex open halfspaces.

We next characterize strongly $\mathcal{O}$-convex sets with empty interiors in terms of the intersections of lower-dimensional strongly $\mathcal{O}$-convex halfspaces. For a given set P, we consider the affine hull η of P. Since η is a lower-dimensional space, we can speak of halfspaces within this space; we call them *η-halfflats.* For example, a ray is a one-dimensional halfflat and a halfplane (say, in $\mathcal{R}^3$) is a two-dimensional halfflat.

If P is a strongly $\mathcal{O}$-convex set, then the relative interior of P in its affine hull η is nonempty (Proposition 14). Using this observation and Theorem 18, we demonstrate that P is the intersection of strongly $\mathcal{O}$-convex η-halfflats.

Lemma 19 *Let P be a closed strongly $\mathcal{O}$-convex set and η be the affine hull of P. Then, P is the intersection of strongly $\mathcal{O}$-convex η-halfflats.*

Proof. Let k be the dimension of η. We treat η as an independent k-dimensional space and define the orientation set $\mathcal{O}_\eta$ in this space as follows: a $(k-1)$-dimensional flat $H \subseteq \eta$ is $\mathcal{O}_\eta$-oriented if it is the intersection of η with some $\mathcal{O}$-oriented hyperplane. Note that, if a hyperplane intersects η and does not contain η, then its intersection with η is a $(k-1)$-dimensional flat (Proposition 7). Therefore, every $\mathcal{O}$-oriented hyperplane that intersects and does not contain η gives rise to an $\mathcal{O}_\eta$-oriented $(k-1)$-dimensional flat.

We next observe that, for every two points p and q of η, a set is an $\mathcal{O}_\eta$-oriented layer of p and q if and only if it is the intersection of an $\mathcal{O}$-oriented layer of p and q with η. This observation implies that, for every two points p and q of η, we have $\mathcal{O}_\eta$-block$(p,q) =$ $\mathcal{O}$-block$(p,q) \cap \eta$. Since η is strongly $\mathcal{O}$-convex (Lemma 15), $\mathcal{O}$-block(p,q) is in η; therefore, $\mathcal{O}_\eta$-block$(p,q) = \mathcal{O}$-block(p,q). We conclude from this equality that a set contained in η is strongly $\mathcal{O}_\eta$-convex if and only if it is strongly $\mathcal{O}$-convex.

Since P is strongly $\mathcal{O}$-convex, it is convex; therefore, its relative interior in η is nonempty (Proposition 14). On the other hand, since strong $\mathcal{O}_\eta$-convexity is equivalent to strong $\mathcal{O}$-convexity, P is strongly $\mathcal{O}_\eta$-convex. Therefore, by Theorem 18, P is the intersection of strongly $\mathcal{O}$-convex η-halfflats. $\square$

We next describe a condition under which all strongly $\mathcal{O}$-convex sets, even those with an empty interior, are formed by the intersections of strongly $\mathcal{O}$-convex halfspaces. We show that, if every strongly $\mathcal{O}$-convex flat is the intersection of strongly $\mathcal{O}$-convex hyperplanes, then every strongly $\mathcal{O}$-convex halfflat is the intersection of strongly $\mathcal{O}$-convex halfspaces, in which case all strongly $\mathcal{O}$-convex sets are formed by the intersections of strongly $\mathcal{O}$-convex halfspaces.

Theorem 20 *Every closed strongly $\mathcal{O}$-convex set is the intersection of strongly $\mathcal{O}$-convex halfspaces if and only if every strongly $\mathcal{O}$-convex flat is the intersection of strongly $\mathcal{O}$-convex hyperplanes.*

Proof. Suppose that every closed strongly $\mathcal{O}$-convex set is the intersection of strongly $\mathcal{O}$-convex halfspaces and consider a strongly $\mathcal{O}$-convex flat η. We note that, if a halfspace contains η, then either η is wholly contained in the interior of the halfspace or η is wholly in the boundary of the halfspace. We consider the collection C of all the strongly $\mathcal{O}$-convex halfspaces whose boundaries contain η.

Clearly, the intersection of this collection C of halfspaces is equal to the intersection of the collection of *all* the strongly $\mathcal{O}$-convex halfspaces that contain η. Since η is a closed strongly $\mathcal{O}$-convex set, this intersection is exactly η. Since η is wholly contained in the boundary of every halfspace in C, we conclude that η is the intersection of the boundaries of the halfspaces in C. By Theorem 16, the boundaries of strongly $\mathcal{O}$-convex halfspaces are strongly $\mathcal{O}$-convex hyperplanes; therefore, η is the intersection of strongly $\mathcal{O}$-convex hyperplanes.

Now suppose, conversely, that every strongly $\mathcal{O}$-convex flat is the intersection of strongly $\mathcal{O}$-convex hyperplanes. To prove that every closed strongly $\mathcal{O}$-convex set is the intersection of strongly $\mathcal{O}$-convex halfspaces, we use the definition and properties of the lower-dimensional orientation set $\mathcal{O}_\eta$ presented in the proof of Lemma 19.

We consider a strongly $\mathcal{O}$-convex set P with the affine hull η. By Lemma 19, P is the intersection of strongly $\mathcal{O}$-convex η-halfflats. We demonstrate that P is the intersection of strongly $\mathcal{O}$-convex halfspaces by proving that every strongly $\mathcal{O}$-convex η-halfflat Q is the intersection of strongly $\mathcal{O}$-convex halfspaces.

Let H be the boundary of Q in η (H is a $(k-1)$-dimensional flat). We have shown in the proof of Lemma 19 that strong $\mathcal{O}_\eta$-convexity is equivalent to strong $\mathcal{O}$-convexity. Since Q is strongly $\mathcal{O}$-convex, it is strongly $\mathcal{O}_\eta$-convex; therefore, its boundary H is also strongly $\mathcal{O}_\eta$-convex (Theorem 9) and, hence, H is strongly $\mathcal{O}$-convex. Therefore, H is the intersection of strongly $\mathcal{O}$-convex hyperplanes. At least one of these hyperplanes, say $\mathcal{H}$, does not contain η. We then readily see that the η-halfflat Q is the intersection of η and a halfspace with boundary $\mathcal{H}$.

Finally, we note that, since η is strongly $\mathcal{O}$-convex, it is the intersection of strongly $\mathcal{O}$-convex hyperplanes and every strongly $\mathcal{O}$-convex hyperplane is the intersection of two strongly $\mathcal{O}$-convex halfspaces. Thus, η is the intersection of strongly $\mathcal{O}$-convex halfspaces. Since Q is the intersection of η with a strongly $\mathcal{O}$-convex halfspace, we conclude that Q is the intersection of strongly $\mathcal{O}$-convex halfspaces. $\square$

We next show that, for closed countable orientation sets and for all orientation sets in three dimensions, every strongly $\mathcal{O}$-convex flat is the intersection of strongly $\mathcal{O}$-convex hyperplanes.

If $\mathcal{O}$ is a finite or closed countably infinite orientation set, then every strongly $\mathcal{O}$-convex flat is $\mathcal{O}$-oriented (Corollary 10). Therefore, every strongly $\mathcal{O}$-convex flat is the intersection of $\mathcal{O}$-oriented hyperplanes, which are strongly $\mathcal{O}$-convex.

In three dimensions, there are only three types of flats: planes, lines, and points. A strongly $\mathcal{O}$-convex plane in three dimensions is a strongly $\mathcal{O}$-convex hyperplane. Strongly $\mathcal{O}$-convex lines and points are $\mathcal{O}$-oriented (Theorem 11); therefore, they are formed by intersections of $\mathcal{O}$-oriented hyperplanes. Thus, every strongly $\mathcal{O}$-convex flat in three dimensions is the intersection of strongly $\mathcal{O}$-convex hyperplanes, even for uncountable $\mathcal{O}$.

Applying Theorem 20 to these two special cases, we obtain the following results.

Corollary 21

1. *If $\mathcal{O}$ is a closed countable orientation set, then every closed strongly $\mathcal{O}$-convex set is the intersection of strongly $\mathcal{O}$-convex halfspaces.*

2. *In three dimensions, every closed strongly $\mathcal{O}$-convex set is the intersection of strongly $\mathcal{O}$-convex halfspaces.*

To summarize, we have demonstrated that a strongly $\mathcal{O}$-convex set can be characterized in terms of halfspace intersection if at least one of the following three conditions holds: the interior of the set is nonempty, the orientation set $\mathcal{O}$ is finite or countably infinite, or the space is three-dimensional. If none of these conditions hold, we can characterize a strongly $\mathcal{O}$-convex set through the intersection of halfflats of the set's affine hull.

If an orientation set $\mathcal{O}$ is finite, then the intersection of strongly $\mathcal{O}$-convex halfspaces is a convex polytope with $\mathcal{O}$-oriented facets. Thus, the following result describes strongly $\mathcal{O}$-convex sets for finite $\mathcal{O}$; this result is analogous to Property 6 of planar strong $\mathcal{O}$-convexity (see Lemma 2).

Corollary 22 *For a finite orientation set $\mathcal{O}$, a set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if it is a convex polytope whose facets are $\mathcal{O}$-oriented.*

If $\mathcal{O}$ is an infinite orientation set, a polytope may be strongly $\mathcal{O}$-convex even if its facets are not $\mathcal{O}$-oriented. For example, if $\mathcal{O}$ is a (countable or uncountable) set whose closure contains all hyperplanes through o, then strong $\mathcal{O}$-convexity is equivalent to standard convexity and, hence, every convex polytope is strongly $\mathcal{O}$-convex.

7 Concluding Remarks

We described a generalization of standard convexity in higher dimensions, called strong $\mathcal{O}$-convexity, and demonstrated that a number of the major properties of strongly $\mathcal{O}$-convex sets are similar to properties of standard convex sets.

We also established three important properties of strongly $\mathcal{O}$-convex sets: the characterization of strongly $\mathcal{O}$-convex flats in terms of $\mathcal{O}$-flats (Theorem 9), the strong $\mathcal{O}$-convexity of the affine hull of a strongly $\mathcal{O}$-convex set (Lemma 15), and a condition of the equivalence of strong convexity for two different orientation sets (Theorem 12).

The presented work is just a beginning; it leaves many unanswered questions, which we are currently trying to address. First, we have not studied the computational aspects of strong convexity, such as finding strongly $\mathcal{O}$-convex hulls. Second, we are exploring an alternative generalization of convexity, called *restricted-orientation convexity* [Rawlins, 1987], in higher dimensions [Fink and Wood, 1995a, Fink and Wood, 1995b]. Third, we plan to explore other generalizations of convexity. For example, the notion of NESW convexity [Rawlins, 1987, Rawlins and Wood, 1989] can be generalized to higher dimensions.

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