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A Systematic Approach to Vibration Analysis for a Gas Turbine Engine

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The vibrational phenomena of the M1 gas turbine engine is analyzed from a fundamental/elementary approach based on first principles. Insight was gained on the basic dynamic forces under play in many components and assemblies of an actual turbine engine, with particular emphasis on torsional-spring mass systems. Since very little, if any, analysis of this particular type of engine has been conducted, it was necessary to gain familiarity with the vibratory signatures of this application in order to assess the possible operating modes/normalities of a "typical" fielded engine. Lumped parameter methods are very good for analyzing individual components and for a snapshot into further areas of investigation. After spectrum analyses will be performed, a detailed investigation can then follow up, premised upon many principles introduced in this report.

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1. INTRODUCTION

Gas turbine engine vibration is influenced by mechanical unbalances of various engine components in addition to other mechanical faults occurring in gear boxes, bearings, pumps, gears per se, compressor rotors and stators, as well as shafts (Zabriskie 1974; Allwood and Christie 1991; Kerfoot, Hauck, and Palm 1973). The vibration sensitivity to each foregoing component varies widely and cannot be understood unless each individual element is analyzed separately and then eventually synthesized collectively. It is now more and more necessary to predict accurately both the local and global performance of gas turbine engines for diagnostic purposes. Although in-depth analysis can be made of each component via finite element analysis, these methods are very time consuming and tedious and require extensive resources. A quick look at the problem from a lumped parameter perspective is sometimes sufficient to understand the diagnostic situation at hand. In particular, it behooves us to access fundamental concepts quickly in order to appreciate the accelerations and forces of the various components in a systematic but yet methodical approach.

Hence, the purpose of this report is to view the vibrational phenomena of the gas turbine engine of an M1 tank from a fundamental and elementary approach, to gain familiarity with the physical components and their relation to each other, using free-body analysis and fundamentals of engineering concepts. It is with certainty that a unique insight can be gained by this approach that can never be achieved via complex numerical techniques, at least not at the outset.

It is quite clear, however, that in-depth distributed analysis can naturally ensue for a particular component or assembly if the fundamental approach reveals that, in fact, a problem exists for further investigation.

2. APPROACH

In appreciation for a vibration analysis of the gas turbine engine, we should first establish elementary principles and analogies for various parts of the overall structure. This can only be done through an examination of first principles applied to elementary example components.

Let us first examine the mass-spring system as follows:

Newton's Method

Assuming the restoring force is in the elemental spring, we have

$$\Sigma F_X = -kX, \quad (1)$$

such that the force is acting from right to left.

(1) If we displace the mass, m, an amount X positive from left to right, then $\dot{X}$, $\ddot{X}$, F are also assumed positive.

(2) Using Newton's Second Law

$$\Sigma F = m\ddot{X}, \quad (2)$$

$$\Rightarrow -kX = m\ddot{X}, \quad (3)$$

or

$$\ddot{X} + \frac{k}{m}X = 0. \quad (4)$$

The solution is given in the Appendix as

$$X(t) = X_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t, \quad (5)$$

where $\omega = \left[\frac{k}{m}\right]^{\frac{1}{2}}$ rad/s (the natural circular frequency).

If we employ the energy method, then

(1) The energy stored in the spring is

$$V = \left(\int_0^X [mg + kX] \, dX - mgX \right) = \frac{1}{2} kX^2. \tag{6}$$

(2) The kinetic energy of the mass is

$$T = \frac{1}{2} m\dot{X}^2. \tag{7}$$

(3) Since there is no dissipation of energy,

$$T + V = \text{Constant} \tag{8}$$

$$\therefore \qquad \frac{1}{2} m\dot{X}^2 + \frac{1}{2} kX^2 = C. \tag{9}$$

Hence, if we differentiate equation (9), we have

$$\dot{X} \left[m\ddot{X} + kX \right] = 0. \tag{10}$$

Since $\dot{X} \neq 0$ for all time, then equation (10) reverts to equation (9).

Typical of gas turbine engines are rotating shafts, such as quill shafts, sun gear shafts, and shaft assemblies transmitting engine torque from various stage rotors to gear shafts. Hence, consider a disk of mass M and radius R at the end of a "weightless" shaft fixed at the other end. The disk vibrates such that the restoring torque is $k_\phi \, \phi$ in which the torsional spring constant becomes

$$k_\phi = \frac{GI_\rho}{L}. \tag{11}$$

For a circular cross-section, shaft of radius r,

$$I_\rho = \frac{1}{2}\pi r^4 \text{ (polar moment of inertia)} \tag{12}$$

G – Shear modulus of shaft material.

If we displace the disk through an angle ϕ, the restoring torque becomes $k_\phi \phi$ opposite in direction to ϕ.

Using Newton's Second Law,

$$\Sigma T_0 = J_0 \ddot{\phi} \tag{13}$$

such that

$$J_0 = \frac{1}{2} MR^2 \text{ (mass moment of the disk about its center of mass),} \tag{14}$$

and

$$\Sigma T_0 = -k_\phi \phi \text{ (torque acting on disk which is opposite positive direction of } \phi\text{).} \tag{15}$$

We obtain,

$$-k_\phi \phi = J_0 \ddot{\phi}, \tag{16}$$

or

$$J_0 \ddot{\phi} + k_\phi \phi = 0 = \frac{1}{2} MR^2 \ddot{\phi} + \frac{GI_\rho}{L} \phi = 0, \tag{17}$$

or

$$\ddot{\phi} + \frac{\pi G r^4}{MR^2 L} \phi = 0. \tag{18}$$

The solution of Equation (18) is given in the Appendix as

$$\phi = \phi_0 \cos\omega t + \frac{\phi_0}{\omega}\sin\omega t, \tag{19}$$

where

$$\omega = \left[\frac{\pi G r^4}{M R^2 L}\right]^{\frac{1}{2}} \text{ rad/s.} \tag{20}$$

Considering other turbine engine components analogizing a pendulum such as the connection of the inlet housing assembly to the air diffuser housing, we have,

$$\Sigma M_0 = -mgL\sin\theta, \tag{21}$$

displacing mass counterclockwise as positive such that $\dot{\theta}$, $\ddot{\theta}$, and M are positive counterclockwise.

Using Newton's Second Law,

$$\Sigma M_0 = I_0\ddot{\theta} = -mgL\sin\theta, \tag{22}$$

where

$$I_0 = mL^2, \tag{23}$$

(moment of inertia of its mass with respect to axis counterclockwise normal to the view). Thus,

$$mL^2\ddot{\theta} + mgL\sin\theta = 0, \tag{24}$$

or for θ small,

$$\sin\theta \approx 0.$$

Therefore,

$$\ddot{\theta} + \frac{g}{L}\theta = 0. \tag{25}$$

Equation (25) was solved in the Appendix. The solution is given by,

$$\theta = \theta_0 \cos \omega t + \frac{\theta_0}{\omega} \sin \omega t, \tag{26}$$

where

$$\omega = \left[\frac{g}{L}\right]^{\frac{1}{2}} \text{rad/s (natural circular frequency).} \tag{27}$$

If we apply the energy method to the same component, then let us assume that the change in potential energy in the system is due to mass m moving upward (gaining energy) in a gravitational field.

$$V = mg(L - L\cos\theta). \tag{28}$$

The instantaneous velocity of the mass is

$$v = L\dot{\theta}. \tag{29}$$

Thus the kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(L\dot{\theta})^2. \tag{30}$$

Since there is assumed no loss in energy (no damping),

$$T + V = \text{constant}, \tag{31}$$

or

$$\frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta) = C. \tag{32}$$

If we differentiate this expression with respect to time, we obtain

$$mL^2\dot{\theta}\ddot{\theta} + mgL\dot{\theta}\sin\theta = 0, \quad (33)$$

or

$$\dot{\theta}\left[mL^2\ddot{\theta} + mgL\sin\theta\right] = 0. \quad (34)$$

Since $\dot{\theta} \neq 0$ for all time (no motion), then

$$mL^2\ddot{\theta} + mgL\sin\theta = 0, \quad (35)$$

or for θ small,

$$\ddot{\theta} + \frac{g}{L}\theta = 0, \quad (36)$$

whose solution is given as for Equation (25).

To analyze the inlet guide vane (IGV) slotted lever associated with the IGV system, we can analogize the situation through a mass spring system by the use of Rayleigh's Method to include the effect of the mass of the spring.

If we let,

Z - which locates a point on the spring
u - which is the definition of the spring at Z,

then,

$$\text{Kinetic energy} = T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\int_0^{\ell}(\rho dz)\,\dot{u}^2 \quad (37)$$

$$\text{Stored energy} = V = \frac{1}{2}kX^2 \quad (38)$$

$$\text{we assume} \quad u = \frac{z}{\ell}X \quad (39)$$

then,

$$T = \frac{1}{2} M\dot{X}^2 + \frac{1}{2}\int_0^L \left(\rho \dot{X}^2\right) \frac{z^2 dz}{\ell^2} = \frac{1}{2} M\dot{X}^2 + \frac{1}{2}\left(\frac{\rho \ell^3}{3}\right)\dot{X}^2 \tag{40}$$

$$= \frac{1}{2}\left[M + \frac{m}{3}\right]\dot{X}^2. \tag{41}$$

Also, since $m = \rho\ell$, it follows that $\omega = \left[\frac{K}{M + \frac{m}{3}}\right]^{\frac{1}{2}}$, the Rayleigh frequency. (42)

Thus, one-third of the mass of the spring is added to the mass M to account for the effect of the spring mass.

Furthermore, if we could extend the IGV slotted lever analogy to include a forced-simple mass spring system where $F_o \sin \Omega t$ is the force and k is the spring constant, then

$$\Sigma F_X = m \ddot{X}, \tag{43}$$

$$\Sigma F_X = F_0 \sin\Omega t - kX, \tag{44}$$

so

$$m\ddot{X} + kX = F_0 \sin\omega t, \tag{45}$$

or

$$\ddot{X} + \frac{K}{m} X = \frac{F_0}{m} \sin\Omega T = \frac{F_0}{k}\frac{k}{\omega} \sin\Omega t, \tag{46}$$

or

$$\ddot{X} = \delta_s \omega^2 \sin \Omega t, \tag{47}$$

where

$$\delta_s = \frac{F_0}{k}, \tag{48}$$

or

$$\ddot{X} + \omega^2 X = \delta_s \omega^2 \sin\Omega t. \tag{49}$$

The steady-state solution as given in the Appendix is

$$X_s = \delta_s \frac{1}{1 - r^2} \sin\Omega t. \tag{50}$$

In the system shown in Figure 1,

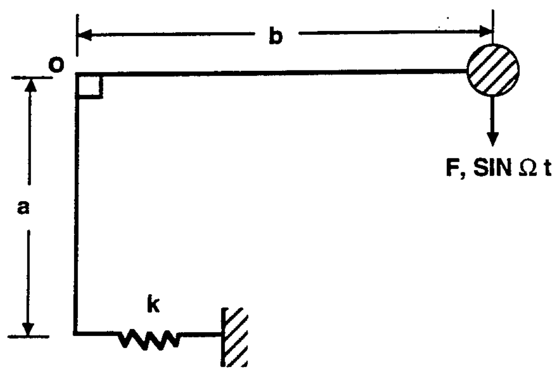

Figure 1. Torque-spring mass system.

a = 10 in, b = 15 in, mg = 5.0 lb, and k = 50 lb-in. We can then determine (a) the natural frequency, and (b) the steady-state solution, i.e.,

$$J_0 \ddot{\theta} = \Sigma \left(\text{Torque}\right)_0 \text{ such that } \quad J_0 = mb^2 \tag{51}$$

$$\Rightarrow mb^2\ddot{\theta} + ka^2\theta = F_1 b \sin\Omega t \tag{52}$$

$$\therefore \ddot{\theta} \frac{ka^2}{mb^2} \theta = \frac{F_1 b}{mb_2} \sin\Omega t \tag{53}$$

Noting that

$$\omega^2 = \frac{ka^2}{mb^2} = \frac{50 \times (10)^2}{\frac{5}{386} \cdot (15)^2} = 1717.33 \tag{54}$$

$$\left(\text{Note:} \quad m = \frac{W}{g} \sim \left[386\,\text{in/s}^2\right] \right), \tag{55}$$

where, $g \sim 386$ in/s^2

$$\Rightarrow \omega = 41.44 \text{ rad/s}; \quad F_1 = 25 \text{ lb}; \quad \Omega = 50 \text{ rad/s}. \tag{56}$$

Thus,

$$\text{AMPL} = \theta_0 = \theta_2 \frac{1}{1 - r^2} = \frac{F_1 b}{ka^2} \frac{1}{1 - \left(\frac{\Omega}{\omega}\right)^2} = \frac{25 \times 15}{50\,(10)^2} \frac{1}{1 - \left(\frac{50}{41.44}\right)^2}. \tag{57}$$

or

$$\theta_0 = 0.164 \text{ radian} = 9.428 \text{ degrees}.$$

$$\left(\text{Note:} \quad r = \frac{\text{Forced frequency}}{\text{Free vibration frequency}} \right).$$

To analyze the torque distributed to the quill shaft in the Accessory Gearbox Module, let us consider a 2-in-thick steel disc having a radius of 8 inches subjected to an oscillatory torque of 1,000 sin 100π t in/lb. The steel shaft from which the disc is suspended is fixed at its upper end, has a 2-in diameter, and may be considered weightless. We can easily determine:

(1) The natural frequency of the system,

(2) The steady-state angular amplitude of motion of the disc,

(3) The maximum oscillatory shear stress in the shaft.

Thus,

$$\text{Mass of Disk} = \rho V = \frac{0.3}{386.4} \times \pi (8)^6 \times 2. \tag{58}$$

Mass Moment of Inertia = 0.3122 $lb_f\, s^2 \cdot in$, i.e.,

$$J = \frac{1}{2} M R^2 = \frac{1}{2} x\ 0.3122\,(8)^2 = 9.99\ lb_f\ in\ s^2. \tag{59}$$

Shaft Spring Constant:

$$K_\theta = \frac{G I_\rho}{L} = \frac{12 \times 10^6 \times 1.57}{15} = 1.256 \times 10^6\ in/lb/rad, \tag{60}$$

where

Polar Moment of Inertia is $$I_\rho = \frac{\pi d^4}{32} = \frac{\pi (2)^4}{32} = 1.57\ in^4, \tag{61}$$

since

$$\omega^2 = \frac{K_\theta}{J} = \frac{1.256 \times 10^6}{9.99} = 0.126 \times 10^6 \tag{62}$$

$$\Rightarrow\ \omega = 354\ rad/s \quad \text{or}\quad f = \frac{\omega}{2\pi} = \frac{35.4}{2\pi} = 56\ cps\ (Hz). \tag{63}$$

Also,

$$\theta_0 = \frac{T_0}{K_q} \frac{1}{1 - r^2}, \tag{64}$$

where

$$\Omega = 100\pi = 314.16\ rad/s \tag{65}$$

and

$$r = \frac{\Omega}{\omega} = \frac{314.16}{354} = 0.887 \tag{66}$$

$$\Rightarrow \theta_0 = \frac{1{,}000}{1.256 \times 10^6} \cdot \frac{1}{1 - (0.887)^2} = 3{,}734 \times 10^{-6}. \tag{67}$$

Thus,

$$\text{Shear stress} = \tau = \frac{T_0 C}{I_\rho} = \frac{(k_q \theta_0) \times 1}{1.57} \text{ lb/in}^2, \tag{68}$$

where T_0 = torque, and

C = distance from neutral axis to extreme fiber.

Hence,

$$\tau = \frac{1.256 \times 10^6 \times 3{,}734 \times 10^{-6} \times 1}{1.57} = 2{,}987 \text{ lb/in}^2. \tag{69}$$

The situation of the tie rods fastening the forward assembly to the rear assembly can be typified as a cantilever beam with a 200-lb weight at its tip. An exciting force equal to $F = 500 \sin 20 \pi t$ acts on the mass along its vertical centerline. We can then determine

(1) The natural frequency of the system,

(2) The steady-state amplitude of the motion of the mass, and

(3) The maximum bending stress.

Hence, we have,

$$M = \frac{200}{386.4} = 0.517 \text{ lb}_m = \frac{W}{g} \tag{70}$$

and

$$K = \frac{3EI}{L^3} = \frac{3 \times 30 \times 10^6 \times 0.667}{(24)^3} \text{ lb/in} \tag{71}$$

with

$$I = \frac{1}{12} bh^3 = \frac{1}{12} \times 1 \times (2)^3 = 0.667 \text{ in}^4 \tag{72}$$

and

$$K = 4{,}342 \text{ lb/in}. \tag{73}$$

Therefore,

$$\omega^2 = \frac{K}{M} = \frac{4{,}342}{0.517} = 8{,}388 \tag{74}$$

$$\Rightarrow \omega = 91.59 \text{ rad/s}, \qquad f = \frac{\omega}{2\pi} = 14.5 \text{ Hz}. \tag{75}$$

Thus,

$$X_0 = \frac{F_0}{k} \frac{1}{1 - r^2} = \frac{500}{4{,}342} \cdot \frac{1}{1 - \left(\frac{20\pi}{91.59}\right)^2} = 0.217 \text{ in}, \tag{76}$$

where

$$r = \frac{\omega_F}{\omega}$$

and the steady-state displacement = $0.217 \sin 20 \pi t$.

If the bending stress at the support $= \sigma = \frac{MC}{I}$, (77)

$$\Rightarrow M = (KX)L = (4{,}342 \times 0.21T) \times 24 = 22{,}613 \text{ in/lb} \tag{78}$$

whereas

$$\sigma_o = \frac{22{,}613 \times 1}{0.667} = 33{,}900 \text{ lb/in}^2 = \frac{T_0 C}{I}. \tag{79}$$

The IGV operation occurs when fuel from the electromechanical fuel system enters the IGV actuator and causes the piston to move. The accompanied movement might be typified by a mass-spring-damper system such that

Exciting Force - $F_0 \cos \Omega t$

F_0 - Amplitude

Ω - Forcing Frequency - rad/s

such that

$$M\ddot{X} + c\dot{X} + kX = F_0 \cos \Omega t, \tag{80}$$

or

$$\ddot{X} + \frac{C}{M}\dot{X} + \frac{k}{M}X = \frac{F_0}{M}\cos \Omega t. \tag{81}$$

The transient solution is (see the Appendix)

$$X_c = C_2 \ell^{-\xi \omega t} \cos(\omega_0 t - \phi). \tag{82}$$

This response approaches zero after a brief time, therefore, only the steady-state response exists, i.e.,

$$X_p = X_0 \cos(\Omega t - \phi_1), \tag{83}$$

where

$$X_0 - \text{dynamic amplitude} = \frac{\delta_1}{\left[\left(1 - r^2\right)^2 + \left(2\xi r\right)^2\right]^{\frac{1}{2}}} \tag{84}$$

and

$$r = \frac{\Omega}{\omega} = \frac{\text{Forcing frequency}}{\text{Undamped natural frequency}} \tag{85}$$

and

$$\xi = \frac{c}{\omega} \text{ in this case } \xi = \frac{c}{2m\omega} \tag{86}$$

also,

$$\delta_1 = \frac{F_0}{k} \quad \left(\text{Static displacement of M due to a force } F_0\right). \tag{87}$$

Hence,

$$\phi_1 = \tan^{-1} \frac{2\xi r}{1 - r^2}, \tag{88}$$

which is the phase angle between the force and displacement.

$$\text{At resonance } r = 1, \ \frac{X_0}{\delta_1} = \frac{1}{2\xi} \Rightarrow \phi_1 = 90° \tag{89}$$

$$\text{As } r \rightarrow \infty, \ \frac{X_0}{\delta_1} \rightarrow 0, \quad \Rightarrow \quad \phi_1 \rightarrow 139°. \tag{90}$$

Identifying the dynamic force amplitude transmitted into the support (Figure 2),

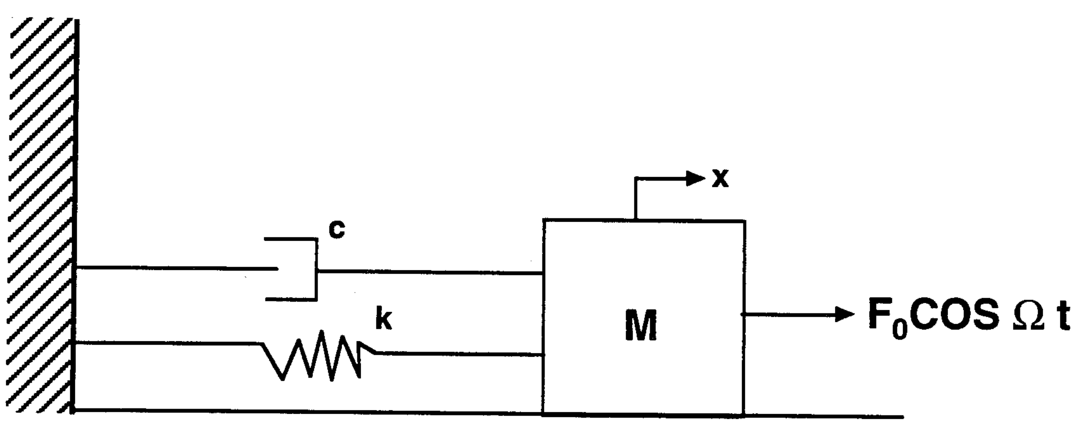

Figure 2. Mass-spring damper system for IGV actuator.

we have

$$F_T = C\dot{X} + kX = F_{TO} \cos(\Omega t - \phi_1 + \phi_2). \quad (91)$$

Since

$$X = \frac{\delta_1 \cos(\Omega t - \phi_1)}{\left[(1 - r^2)^2 + (2\xi r)^2\right]^{\frac{1}{2}}} \quad (92)$$

$$\Rightarrow F_{TO} = F_0 \left[\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}\right]^{\frac{1}{2}}, \quad (93)$$

hence,

$$T = \text{The transmissibility} = \frac{F_{TO}}{F_0} = \left[\frac{1 + (2\xi r)^2}{\left(1 - r^2\right)^2 + (2\xi r)^2}\right]^{\frac{1}{2}}. \quad (94)$$

"Isolation" of various components for analysis within assemblies is common for all gas turbine engines. If we consider in a generic sense any of the turbine engine modules which are modeled as a spring-mass-damper system, then we have

$X_1 = X_{10} \cos \Omega t$ = displacement of support or "box"

X_2 = Absolute displacement of mass to be isolated.

Equation of motion of mass m is

$$\ddot{X}_2 + \frac{c}{m}\dot{X}_2 + \frac{k}{m}X_2 = \frac{k}{m}X_1 + \frac{c}{m}\dot{X}_1 = X_{10}\omega^2\left[1 + (2\xi r)^2\right]^{\frac{1}{2}}\cos(\Omega t + \phi), \quad (95)$$

where the solution is

$$X_2 = X_{20} \cos\left(\Omega t - \phi + \phi_2\right) = X_{10}\left[\frac{1 + (2\xi r)^2}{\left(1 - r^2\right)^2 + (2\xi r)^2}\right]^{\frac{1}{2}} \cdot \cos\left(\Omega t - \phi + \phi_2\right). \quad (96)$$

The ratio of the amplitude of the isolated mass m to the amplitude of the support or "box" is

$$\frac{X_{20}}{X_{10}} = \left[\frac{1 + (2\xi r)^2}{\left(1 - r^2\right)^2 + (2\xi r)^2}\right]^{\frac{1}{2}} \tag{97}$$

such that this ratio plots exactly as a "transmissibility" curve.

Considering the rotating system assembly and its associated platform, if the assembly and platform are displaced and released, then the displacement vs. time might look like the following (Figure 3) (assuming a total weight of 80 lb).

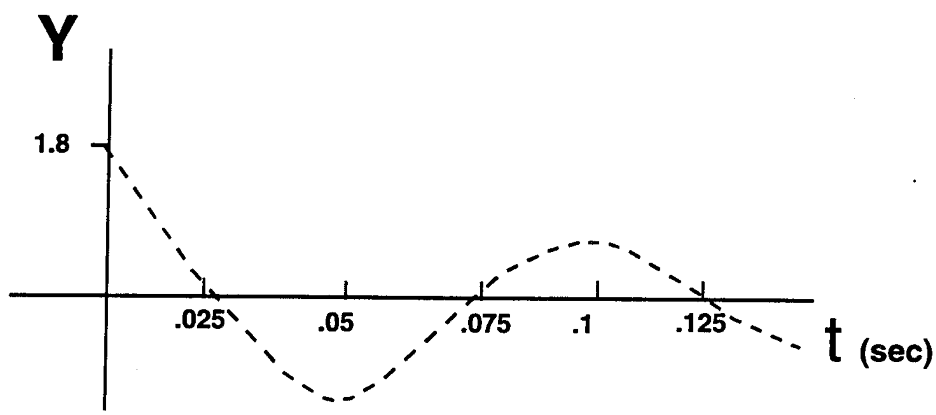

Figure 3. Displacement vs. time for rotating system assembly.

What is the damped natural circular frequency? What is the damping ratio ξ? What is the undamped natural circular frequency?

Let us proceed as follows:

$$\delta = \ln \frac{X_n}{X_{n+1}} = \ln \frac{1.8}{1.0} + 0.5877 = 2\pi\xi \tag{98}$$

so that $\xi \doteq 0.0935$.

From X vs. t, one cycle takes 0.1 s

$$\therefore \quad f_d = \frac{1 \text{ cycle}}{0.1 \text{ s}} = 10 \text{ cps} \tag{99}$$

$$\Rightarrow \quad \omega_d = 2\pi f_d = 62.8 \text{ rad/s} = 20\pi \text{ rad/s}. \tag{100}$$

But,

$$\omega_d = \omega\left[1 - \xi^2\right]^{\frac{1}{2}} \tag{101}$$

$$\therefore \omega = \frac{\omega_d}{\left[1 - \xi^2\right]^{\frac{1}{2}}} = \frac{62.8}{\left[1 - (0.0935)^2\right]^{\frac{1}{2}}} \doteq 62.8 \text{ rad/s}. \tag{102}$$

If an exciting force 10 sin 30 π t acts on the assembly and platform, then, what is the steady-state displacement and what force is transmitted into the support as shown in Figure 4?

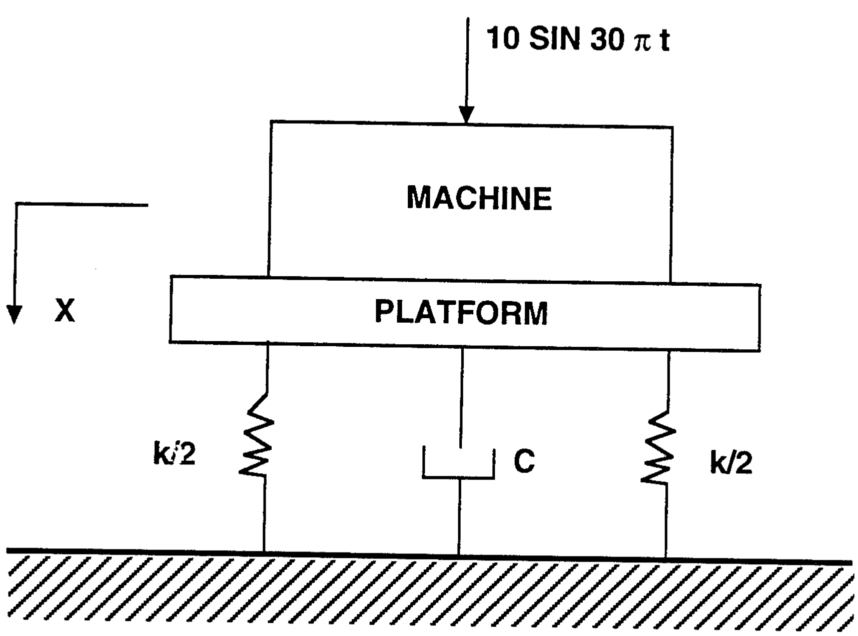

Figure 4. Machine-platform assembly of IGV assembly-casing in turbine engine.

Since

$$X = \frac{\delta_1}{\left[\left(1 - r^2\right)^2 + (2\xi r)^2\right]^{\frac{1}{2}}} \tag{103}$$

and

$$\delta_1 = \frac{F_0}{K}, \qquad F_0 = 10 \text{ lb} \tag{104}$$

and

$$\text{since } \omega^2 = \frac{k}{M} \quad \text{then } k = M\omega^2 = \frac{80}{386} \times (62.8)^2 \tag{105}$$

$$\Rightarrow k = 817.37 \text{ lb/in} \tag{106}$$

and

$$\delta_1 = \frac{10}{817.37} = 0.0122 \text{ in; } r = \frac{\Omega}{\omega} = \frac{30\pi}{20\pi} = 1.5\,, \tag{107}$$

hence,

$$X = \frac{0.0122}{\left[\left(1 - (1.5)^2\right)^2 + (2 \times 0.0935 \times 1.5)^2\right]^{\frac{1}{2}}} = 0.0095 \text{ in}. \tag{108}$$

Also, the transmitted force is

$$F_t = F_0 \left[\frac{1 + (2\xi r)^2}{\left(1 - r^2\right)^2 + (2\xi r)^2}\right]^{\frac{1}{2}} = 6.5 \text{ lb}. \tag{109}$$

If we look again at the IGV operation returning us to the mass-string-damper system as in Figure 2, but with a rotating unbalanced external force due to perhaps wear/tear, we have

$$M\ddot{X} + C\dot{X} + kX = me\,\Omega^2 \cos\Omega t \tag{110}$$

(Note that the horizontal component = me Ω^2 cos Ω t.)

or

$$\ddot{X} + \frac{C}{M}\dot{X} + \frac{k}{M}X = \frac{me\Omega^2}{M}\cos\Omega t. \tag{111}$$

m - rotating mass; e - eccentricity - inches

Ω - angular velocity - rad/s

The steady-state solution is

$$X_\rho = \frac{m}{M} e \frac{r^2}{\left[\left(1 - r^2\right)^2 + \left(2\xi r\right)^2\right]^{\frac{1}{2}}} \cos\left(\Omega t - \phi_1\right), \tag{112}$$

where

$$r = \frac{\Omega}{\omega}, \qquad \omega = \left[\frac{k}{m}\right]^{\frac{1}{2}} \tag{113}$$

$$\xi = \frac{c}{2M\omega}, \qquad \phi_1 = \tan^{-1}\frac{2\xi r}{1 - r^2}. \tag{114}$$

Taking into account Force Transmissibility, we have

$$F_T = kX + C\dot{X}.$$

And noting that

$$X = \frac{m}{M} e \frac{r^2}{\left[\left(1 - r^2\right)^2 + \left(2\xi r\right)^2\right]^{\frac{1}{2}}} \cos\left(\Omega t - \phi\right), \tag{115}$$

$$\Rightarrow \; X = \frac{m\,e\,\Omega^2}{M\omega^2}\,\frac{1}{\left[\left(1 - r^2\right)^2 + \left(2\,\xi\,r\right)^2\right]^{\frac{1}{2}}}\cos\left(\Omega t - \phi\right). \tag{116}$$

Now, since

$$F_T = F_{TO}\cos\left(\Omega t - \phi\right)$$

$$\Rightarrow \; F_{TO} = m e \Omega^2 \left[\frac{1 + \left(2\xi r\right)^2}{\left(1 - r^2\right)^2 + \left(2\xi r\right)^2}\right]^{\frac{1}{2}}. \tag{117}$$

Also,

$$T_1R = \frac{F_{TO}}{m e \Omega^2} = \left[\frac{1 + \left(2\xi r\right)^2}{\left(1 - r^2\right)^2 + \left(2\xi r\right)^2}\right]^{\frac{1}{2}}. \tag{118}$$

As a specific application of the unbalanced force phenomena, consider a typical gas turbine engine rotating gear shaft weighing 40 lb with an unbalanced torque applied of 0.5 in/lb. Some tests indicate a natural frequency of 1,000 rpm and a damping ratio of 0.2. Let us determine the steady-state amplitude when operating at 1,200 rpm and the amount of unbalance force transmitted into the support.

$$\text{Letting } r = \frac{\Omega}{\omega} = \frac{1{,}200}{1{,}000} = 1.2, \; r^2 = 1.44, \; \xi = 0.2 \qquad 2\xi r = 0.48 \tag{119}$$

$$\Rightarrow \; \Omega = 1{,}200 \times \frac{2\pi}{60} = 125.66 \text{ rad/s} \tag{120}$$

and

$$m e = \frac{0.5 \text{ in-lb}}{386.4 \text{ in/s}^2} = 0.00129 \text{ in/lb}, \tag{121}$$

where

$$g = 386.4 \text{ in/s}^2 = 32.2 \text{ ft/s}^2 \tag{122}$$

such that

$$M = \frac{40}{386.4} = 0.1035 \text{ lb.} \tag{123}$$

Steady-state displacement gives us

$$X = \frac{me}{M} \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2\xi r)^2\right]^{\frac{1}{2}}} = \frac{0.00129}{0.1035} \frac{1.44}{\left[(1 - 1.44)^2 + (0.48)^2\right]^{\frac{1}{2}}} \tag{124}$$

$$\therefore \quad X = 0.0276 \text{ in.} \tag{125}$$

Since F_T = Force transmitted,

$$\Rightarrow \; F_T = me\Omega^2 \left[\frac{1 + (2\xi r)^2}{\left(1 - r^2\right)^2 + (2\xi r)^2}\right]^{\frac{1}{2}} = 34.7 \text{ lb.} \tag{126}$$

If we have a periodic but nonsinusoidal excitation of an engine component analogous to our spring-mass system, then,

$$F(t) = A_0 + \sum_{s=1}^{\infty} A_s \sin s\,\Omega t + \sum_{s=1}^{\infty} B_s \cos \Omega t, \tag{127}$$

where

$$A_0 = \frac{\Omega}{2\pi} \int_0^{\frac{2\pi}{\Omega}} F(t)dt; \quad A_s = \frac{\Omega}{\pi} \int_0^{\frac{2\pi}{\Omega}} F(t) \sin s\Omega t\,dt, \tag{128}$$

$$B_s = \frac{\Omega}{\pi} \int_0^{\frac{2\pi}{\Omega}} F(t) \cos s\Omega t\,dt. \tag{129}$$

For the previous case,

$$A_0 = \frac{\Omega}{2\pi}\int_0^{\frac{\pi}{\Omega}} 2F_0\,dt + \int_{\frac{\pi}{\Omega}}^{\frac{2\pi}{\Omega}} 0 \cdot dt = F_0 \tag{130}$$

$$A_s = \frac{\Omega}{2\pi}\int_0^{\frac{\pi}{\Omega}} 2F_0 \sin s\Omega t\,dt + \int_{\frac{\pi}{\Omega}}^{\frac{2\pi}{\Omega}} 0 \cdot dt = \frac{2F_0}{\pi s} \text{ for s odd} \tag{131}$$

$$B_s = \frac{\Omega}{2\pi}\int_0^{\frac{\pi}{\Omega}} 2F_0 \cos s\Omega t\,dt + \int_{\frac{\pi}{\Omega}}^{\frac{2\pi}{\Omega}} 0 \cdot dt = 0 \tag{132}$$

$$\therefore \quad F(t) = F_0 + \sum_{1,3,5,7}^{\infty} \frac{2F_0}{\pi s} \sin s\Omega t\,. \tag{133}$$

Then,

$$\ddot{X} + \omega^2 X = \frac{F_0}{m} + \sum_{1,3,5,7}^{\infty} \frac{2F_0}{\pi s m} \sin s\Omega t, \tag{134}$$

whose solution is

$$X = \frac{X_s}{2} + \sum_{1,3,5,7}^{\infty} \frac{X_s}{\pi r\left(1-r_s^2\right)} \sin s\Omega t, \tag{135}$$

where

$$X_s = \frac{2F_0}{k},\; r_s = \frac{s\Omega}{\omega},\; \text{and } \omega^2 = \frac{k}{m}. \tag{136}$$

Most of the vibrational problems associated with gas turbine engines involve torsional analyses of components, viz., rotating gear shafts, rotating turbine blades, rotors, nozzles, bearings, spacers, shims, etc. In many instances, two-degree-of-freedom studies are required for consideration of vibrational diagnostics.

Looking at a typical disk-shaft configuration, let us consider the following generic situation of two disks in torsion on a shaft:

$$\Sigma \text{ Torque}_0 = J\ddot{\theta} \quad \therefore \quad -k_1\theta_1 - k_2\left(\theta_1 - \theta_2\right) = J_1\ddot{\theta}_1 .$$

Hence,

$$J_1\ddot{\theta}_1 + \left(k_1 + k_2\right)\theta_1 + k_2\theta_2 = 0 . \tag{137}$$

Also,

$$\Sigma T = J_2\ddot{\theta}_2 \quad \therefore \quad k_2\left(\theta_1 - \theta_2\right) = J_2\ddot{\theta}_2 .$$

Hence,

$$J_2\ddot{\theta}_2 + k_2\theta_2 - k_2\theta_1 = 0. \tag{138}$$

Assuming
$$\begin{cases} \theta_1 = A_1 \cos(\omega t + \psi) \\ \theta_2 = A_2 \cos(\omega t + \psi). \end{cases}$$

We obtain from (137) and (138)

$$\begin{bmatrix} \left(\dfrac{k_1 + k_2}{J_1} - \omega^2\right) & -\dfrac{k_2}{J_1} \\ -\dfrac{k_2}{J_2} & \left(\dfrac{k_2}{J_2} - \omega^2\right) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 . \tag{139}$$

For $k_1 = k_2 = k$ and $J_1 = 2J_2$, then the $|\text{Det}| = 0$ yielding the frequency equation,

$$\left(\frac{2k}{2J_2} - \omega^2\right)\left(\frac{k}{J_2} - \omega^2\right) - \frac{k^2}{2J_2^2} = 0, \tag{140}$$

or

$$\left(\omega_0^2 - \omega^2\right)^2 - \frac{1}{2}\,\omega_0^4 = 0, \tag{141}$$

where

$$\omega_0^2 = \frac{k}{J_2}, \tag{142}$$

solving

$$\omega_1^2 = \left[1 - \frac{\sqrt{2}}{2}\right]\omega_0^2 = 0.293\,\omega_0^2 \tag{143}$$

$$\omega_2^2 = \left[1 + \frac{\sqrt{2}}{2}\right]\omega_0^2 = 1.707\,\omega_0^2. \tag{144}$$

The mode shapes may be obtained from

$$-\omega^2 A_1 + \left(\omega_0^2 - \omega^2\right) A_2 = 0, \tag{145}$$

or

$$\frac{A_1}{A_2} = \frac{\omega_0^2 - \omega^2}{\omega_0^2}. \tag{146}$$

For $\omega^2 = \omega_1^2 = 0.293\ \omega_0^2 \quad \Rightarrow \quad \dfrac{A_{11}}{A_{21}} = \dfrac{(1 - 0.293)\,\omega_0^2}{\omega_0^2} = 0.707\,.$

For $\omega^2 = \omega_2 = 1.707\ w_0^2 \quad \Rightarrow \quad \dfrac{A_{12}}{A_{22}} = \dfrac{(1 - 1.707)\ \omega_0^2}{\omega_0^2} = -0.707\,.$

Consider a two-degree-of-freedom system having a forcing function $F_1 \sin \Omega t$ acting on mass m_1. The coupled equations may be written in the matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \sin \Omega t \\ 0 \end{Bmatrix}. \tag{147}$$

For a steady-state solution, assume

$$X_1 = X_1 \sin \Omega t \quad \text{and} \quad X_2 = X_2 \sin \Omega t. \tag{148}$$

The two equations above become:

$$\begin{bmatrix} \left(k_{11} - m_1\Omega^2\right) & k_{12} \\ k_{12} & \left(k_{22} - m_2\Omega^2\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}. \tag{149}$$

Using Cramer's Rule,

$$X_1 = \frac{\begin{vmatrix} F_1 & k_{12} \\ 0 & \left(k_{22} - m_2\omega^2\right) \end{vmatrix}}{\begin{vmatrix} \left(k_{11} - m_1\Omega^2\right) & k_{12} \\ k_{12} & \left(k_{22} - m_2\Omega^2\right) \end{vmatrix}} = \frac{F_1\left(k_{22} - m_2\Omega^2\right)}{D} \tag{150}$$

such that D = determinant of coefficient = $\left(k_{11} - m_1\Omega^2\right)\left(k_{22} - m_2\Omega^2\right) - k_{12}^2$ and

$$X_2 = \frac{-F_1 k_{12}}{D}. \tag{151}$$

If we let ω_1 and ω_2 be the two natural frequencies, then D may be written as

$$D = m_1 m_2 \left(\omega_1^2 \cdot \Omega^2\right)\left(\omega_2^2 \cdot \Omega^2\right), \tag{152}$$

where ω_1 and ω_2 are found by setting D = 0.

An alternative analysis or formulation using transfer matrices can be employed for obtaining relations that govern the motion of a discrete system composed of lumped masses connected by massless elastic parts. The properties and conditions are expressed by state vectors at sections or points immediately adjacent to the sides of a discrete mass. Specifically, a state vector is a column matrix which contains the components of the displacements, forces, and moments at a point or station adjacent to a mass. Such a state vector can then be transferred to another location by a transfer matrix, there being two types. A point transfer matrix transfers a state vector from a location on one side of a mass to the other side, at the same designated station and thus is a transfer at a point. A field transfer matrix transfers a state vector across a spatial distance or field of the system from a station at one mass to a station at another mass.

The relations resulting from the use of transfer matrices lead to a solution in which the natural frequencies and mode shapes may be determined from the characteristic equation if the number of degrees of freedom is small or otherwise by a numerical procedure such as Holzer's method.

Consider a general mass-spring system that is restricted to move in the horizontal direction only. The letters "L" and "R" are used to denote the left-hand and right-hand sides, respectively, of a mass station.

For spring k_j, the following relation can be written:

$$X_j^L = X_{j-1}^R + \frac{F_{j-1}^R}{k_j}, \tag{153}$$

$$F_j^L = F_{j-1}^R. \tag{154}$$

In matrix form, these would be written as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_j^L = \begin{bmatrix} 1 & \frac{1}{k} \\ 0 & 1 \end{bmatrix}_j \begin{Bmatrix} X \\ F \end{Bmatrix}_{j-1}^R . \tag{155}$$

Or, more concisely,

$$\{v\}_j^L = [F]_j \{v\}_{j-1}^R , \tag{156}$$

where

$$\{v\}_j^L = \begin{Bmatrix} X \\ F \end{Bmatrix}_j^L \qquad \{v\}_{j-1}^R = \begin{Bmatrix} X \\ F \end{Bmatrix}_{j-1}^R \tag{157}$$

are state vectors for the displacements and internal forces at stations j and j – 1, respectively. The scalar matrix,

$$[F]_j = \begin{bmatrix} 1 & \frac{1}{k} \\ 0 & 1 \end{bmatrix}_j , \tag{158}$$

is the field transfer matrix which relates $\{v\}_j^L$ to $\{v\}_{j-1}^R$.

For mass m_j, the following equations apply:

$$X_j^R = X_j^L , \tag{159}$$

$$F_j^R - F_j^L = m_j \ddot{X}_j . \tag{160}$$

Now for harmonic motion of m_j, $X_j = A_j \sin(\omega t + \phi)$ and $\ddot{X}_j = -\omega^2 A_j \sin(\omega t + \phi)$, so that

$$\ddot{X}_j = -\omega^2 x_j. \tag{161}$$

Equation (163) then becomes

$$F_j^{\,R} = -m_j \omega^2 x_j^{\,L} + F_j^{\,L}. \tag{162}$$

In matrix form, Equations (162) and (165) would be expressed as

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_j^R = \begin{bmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{bmatrix}_j \begin{Bmatrix} X \\ F \end{Bmatrix}_j^L, \tag{163}$$

or

$$\{v\}_j^R = [P]_j \{v\}_j^L. \tag{164}$$

Here $\{v\}_j^R$ and $\{v\}_j^L$ are state vectors for the displacements and internal forces to the right and left of mass m_j, respectively, and

$$[P]_j = \begin{bmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{bmatrix}_j \tag{165}$$

is the scalar point transfer matrix, which relates $\{v\}_j^R$ and $\{v\}_j^L$. Substituting Equation (159) into Equation (164) gives

$$\{v\}_j^R = [P]_j [F]_j \{v\}_{j-1}^R = [Q]_j \{v\}_{j-1}^R, \tag{166}$$

where (see Equations [158] and [165])

$$[Q]_j = [P]_j [F]_j, \tag{167}$$

$$[Q]_j = \begin{bmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{bmatrix}_j \begin{bmatrix} 1 & \frac{1}{k} \\ 0 & 1 \end{bmatrix}_j \tag{168}$$

$$\text{i.e., } [Q]_j = \begin{bmatrix} (1 \times 1 + 0 \times 0) & \left(1 \times \frac{1}{k} + 0 \times 1\right) \\ \left(-m\omega^2 \times 1 + 1 \times 0\right) & \left(-m\omega^2 \times \frac{1}{k} + 1 \times 1\right) \end{bmatrix}_j = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & \left(1 - \frac{m\omega^2}{k}\right) \end{bmatrix}_j . \tag{169}$$

This procedure can be continued. Thus

$$\{v\}_n^R = [Q]_n \{v\}_0^R, \tag{170}$$

where

$$[Q]_n = [P]_n [F]_n [P]_{n-1} [F]_{n-1} \cdots [P]_1 [F]_1 . \tag{171}$$

Equation (170) expresses the state vector $\{v\}_n^R$ at the nth station in terms of the state vector $\{v\}_0^R$ at the initial station.

Noting that Equation (171) represents two algebraic equations, and that usually a boundary condition, such as X = 0 or F = 0, would be known at each end of the system, then the equations can be solved to yield the natural frequencies and principal modes of vibration. The precise manner in which this would be carried out is not readily apparent, and will be illustrated below. Two methods are suitable. The first yields the nth-order characteristic equation, which can then be solved for the n natural frequencies. Each natural frequency can then be substituted into individual stages of the determinations to give the amplitude ratios and thus the modal pattern. This procedure is feasible only if the number of degrees of freedom is small.

The second method is to follow a numerical procedure such as Holzer's method. By assuming a value for ω^2 and assigning a value to the unknown boundary condition at station 0, then the unknown parameters of x and F can be determined at Station 1 by the related matrix multiplication. This process can be continued from one station to the next until the final station n is reached and the boundary condition there is checked, yielding the error. A new value of ω^2 would then be assumed and the entire procedure would be repeated, resulting in a new error value. The process can be continued until the error is brought to zero—or rather, to acceptable small value.

To illustrate the above methods, consider the mass-spring system shown in the Figure 5. Let us now use transfer matrices to determine the principal modes and natural frequencies by the first method described above. Assume $k_1 = k_2 = k_3 = k$ and $m_1 = m$, $m_2 = 2m$.

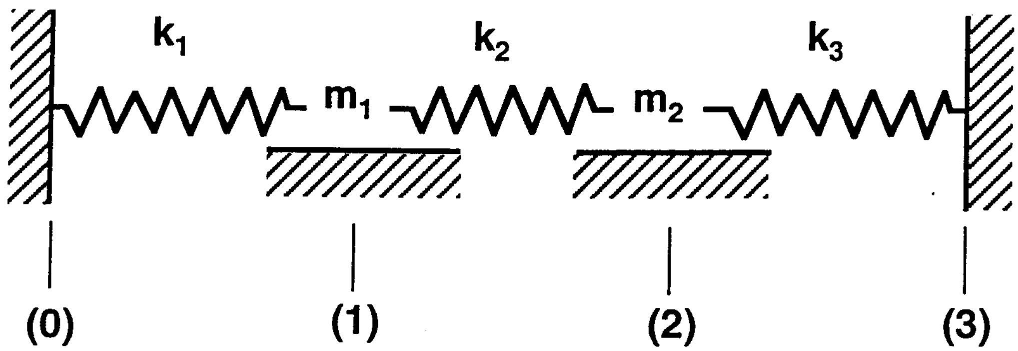

Figure 5. Mass-spring system for transfer matrices.

The boundary conditions for Sta. 0^R are X = 0, F = F_0, and for Sta. 3^R are x = 0, F = F_3, where F_3 is unknown. The first transfer matrix would be written for Sta. 1^R in terms of Sta. 0^R as

a.)
$$\begin{Bmatrix} X \\ F \end{Bmatrix}_1^R = \begin{bmatrix} 1 & \dfrac{1}{k_1} \\ -m_1\omega^2 & \left(1 - \dfrac{m_1\omega^2}{k_1}\right) \end{bmatrix}_1 \begin{Bmatrix} X \\ F \end{Bmatrix}_0^R \tag{172}$$

$$= \begin{bmatrix} 1 & \dfrac{1}{k} \\ -m\omega^2 & \left(1 - \dfrac{m\omega^2}{k}\right) \end{bmatrix}_1 \begin{Bmatrix} 0 \\ F \end{Bmatrix}_0^R \; . \tag{173}$$

Next, the transfer matrix for Sta. 2 is written as follows:

b.)

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \dfrac{1}{k_2} \\ -m_2\omega^2 & \left(1 - \dfrac{m_2\omega^2}{k_2}\right) \end{bmatrix}_2 \begin{Bmatrix} X \\ F \end{Bmatrix}_1^R$$

$$= \begin{bmatrix} 1 & \dfrac{1}{k} \\ -2m\omega^2 & \left(1 - \dfrac{2m\omega^2}{k}\right) \end{bmatrix}_2 \begin{bmatrix} 1 & \dfrac{1}{k} \\ -m\omega^2 & \left(1 - \dfrac{m\omega^2}{k}\right) \end{bmatrix}_1 \begin{Bmatrix} 0 \\ F \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} 1 \times \dfrac{1}{k} + \dfrac{1}{k}\left(1 - \dfrac{m\omega^2}{k}\right) \\ -2m\omega^2 \times \dfrac{1}{k} + \left(1 - \dfrac{2m\omega^2}{k}\right)\left(1 - \dfrac{m\omega^2}{k}\right) \end{bmatrix}_2 \begin{Bmatrix} 0 \\ F \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} \dfrac{2}{k} - \dfrac{m\omega^2}{k^2} \\ \left(1 - \dfrac{5m\omega^2}{k} + \dfrac{2m^2\omega^4}{k^2}\right) \end{bmatrix}_2 \begin{Bmatrix} 0 \\ F \end{Bmatrix}_0^R . \qquad (174)$$

The computations in the first column of certain of the transfer matrices shown by the vertical line are omitted as they are not needed, due to the zero in the first row of the state vector for Sta. 0^R. Continuing,

c.)

$$\begin{Bmatrix} X \\ F \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k_3} \\ -m_3\omega^2 & \left(1 - \frac{m_3\omega^2}{k_3}\right) \end{bmatrix}_3 \begin{Bmatrix} X \\ F \end{Bmatrix}_2^R$$

$$= \begin{bmatrix} 1 & \frac{1}{k} \\ 0 & 1 \end{bmatrix}_3 \begin{bmatrix} \left(\frac{2}{k} - \frac{m\omega^2}{k^2}\right) \\ \left(1 - \frac{5m\omega^2}{k} + \frac{2m^2\omega^4}{k^2}\right) \end{bmatrix}_2 \begin{Bmatrix} 0 \\ F \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} 1 \times \left(\frac{2}{k} - \frac{m\omega^2}{k^2}\right) + \frac{1}{k} \times \left(1 - \frac{5m\omega^2}{k} + 2m^2\frac{\omega^4}{k^2}\right) \\ 0 \times \left(\frac{2}{k} - m\frac{\omega^2}{k^2}\right) + 1 \times \left(1 - \frac{5m\omega^2}{k} + 2m^2\omega^4 k^2\right) \end{bmatrix}_3 \begin{Bmatrix} 0 \\ F \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} \frac{3}{k} - \frac{6m\omega^2}{k^2} + \frac{2m^2\omega^4}{k^3} \\ \left(1 - \frac{5m\omega^2}{k} + \frac{2m^2\omega^4}{k^2}\right) \end{bmatrix}_3 \begin{Bmatrix} 0 \\ F_0 \end{Bmatrix}_0^R . \tag{175}$$

From the first algebraic equation of this final matrix, we have

$$0 = \left(\frac{3}{k} - \frac{6m\omega^2}{k^2} + \frac{2m^2\omega^4}{k^3}\right) F_0 , \tag{176}$$

hence

$$\frac{m^2\omega^4}{k^2} - \frac{3m\omega^2}{k} + \frac{3}{2} = 0, \tag{177}$$

which represents the characteristic equation and has the roots

$$\frac{m^2\omega^4}{k^2} = \frac{3 - \sqrt{3}}{2}, \quad \frac{3 + \sqrt{3}}{2}. \tag{178}$$

This defines the natural frequencies as

$$\omega_1^2 = \frac{3 - \sqrt{3}}{2}\frac{k}{m}, \quad \omega_2^2 = \frac{3 + \sqrt{3}}{2}\frac{k}{m}. \tag{179}$$

Setting $X_1 = 1$ and also substituting ω_1^2 into Equation (173) gives

$$F_0 = k, \quad F_1 = \frac{\sqrt{3} - 1}{2}k. \tag{180}$$

Then substituting these into Equation (174) gives

$$X_2 = \frac{1 + \sqrt{3}}{2}. \tag{181}$$

Similarly, setting $X_1 = 1$ and substituting ω_2^2 into Equation (173) gives

$$F_0 = k, \quad F_1 = \frac{-\sqrt{3} - 1}{2}k \tag{182}$$

and substituting these into Equation (174) yields

$$X_2 = \frac{1 - \sqrt{3}}{2}.$$

Thus the principal modes are as follows:

$$\omega_1^2 = \frac{3 - \sqrt{3}}{2}\frac{k}{m} = 0.6334\frac{k}{m}, \quad A_1 = 1, \quad A_2 = \frac{1 + \sqrt{3}}{2} = 1.366 \tag{183}$$

$$\omega_2^2 = \frac{3 + \sqrt{3}}{2}\frac{k}{m} = 2.366\frac{k}{m}, \quad A_1 = 1, \quad A_2 = \frac{1 - \sqrt{3}}{2} = -0.361. \tag{184}$$

For a torsional system composed of disks on a massless shaft, the analysis and formulation of the transfer matrices are identical to that of the rectilinear mass-spring system. Thus considering the torsional systems and corresponding free-body diagrams shown in Figure 6, the transfer-matrix relations for the torsional shaft (spring) K_j are

$$\begin{Bmatrix} \theta \\ M_t \end{Bmatrix}_j^L = \begin{bmatrix} 1 & \frac{1}{k} \\ 0 & 1 \end{bmatrix}_j \begin{Bmatrix} \theta \\ M_t \end{Bmatrix}_{j-1}^R, \tag{185}$$

or in general form

$$\{v\}_j^L = [F]_j \{v\}_{j-1}^R. \tag{186}$$

For disk I_j, the matrix relations for harmonic motion are

$$\begin{Bmatrix} \theta \\ M_t \end{Bmatrix}_j^R = \begin{bmatrix} 1 & 0 \\ -I\omega^2 & 1 \end{bmatrix}_j \begin{Bmatrix} \theta \\ M_t \end{Bmatrix}_j^L, \tag{187}$$

having the general form

$$\{v\}_j^R = [P]_j \{v\}_j^L. \tag{188}$$

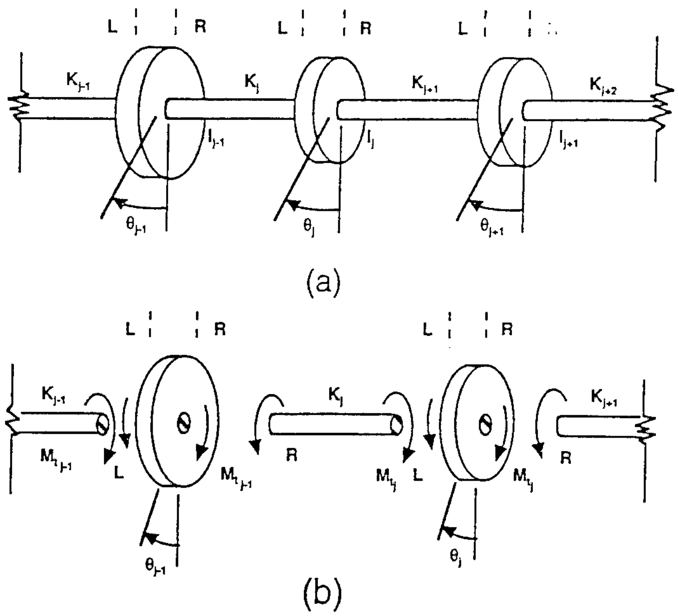

Figure 6. Torsional systems associated with gear box of M1 turbine engine.

Substituting Equation (186) into Equation (188) gives

$$\{v\}_j^R = [P]_j [F]_j \{v\}_{j-1}^R$$

$$= [Q]_j \{v\}_{j-1}^R , \qquad (189)$$

where

$$[Q]_j = \begin{bmatrix} 1 & \frac{1}{K} \\ -I\omega^2 & \left(1 - \frac{I\omega^2}{K}\right) \end{bmatrix}_j . \qquad (190)$$

If Equation (189) is applied to successive stations of the torsional system, then

$$\{v\}_n^R = [Q]_{n!} \{v\}_0^R , \qquad (191)$$

where

$$[Q]_{n!} = [P]_n [F]_n [P]_{n-1} [F]_{n-1} \cdots [P]_1 [F]_1 . \quad (192)$$

Equations (185) through (192) are the same as those for the mass-spring system except that m has been replaced by I, X has been replaced by 0, and the k now represents a torsional spring constant.

Let us now consider a torsional system shown in Figure 7 and let us employ transfer matrices to determine the principal modes and natural frequencies by the second method described above (the Holzer procedure).

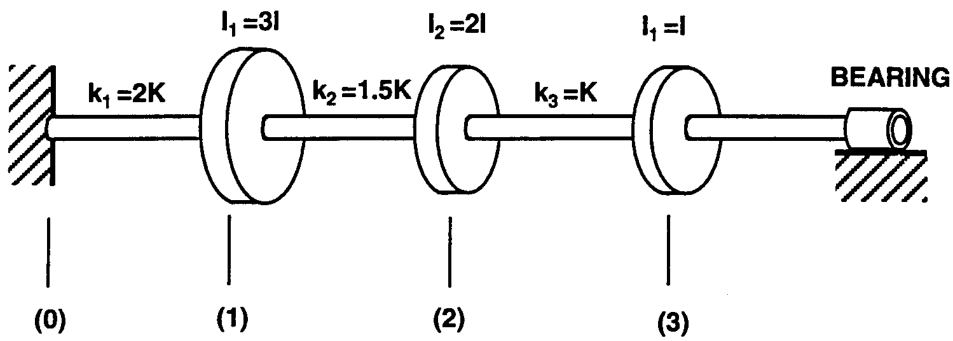

Figure 7. Torsional system of M1 gas turbine engine with bearing.

The boundary conditions for Sta. 0^R are $\theta = 0$, $M_t = M_{to} = 1$ and for Sta. 3^R are $\theta = \theta$, $M_t = 0$, where θ_3 is unknown. Assuming the first trial frequency value as $\omega^2 = 1.0$ (K I), and referring to Equation 202, the first transfer matrix for Sta. 1^R in terms of Sta. 0^R would be written as

a.)

$$\begin{Bmatrix} \theta \\ M_t \end{Bmatrix}_1^R = \begin{bmatrix} 1 & \frac{1}{2k} \\ -3I \times \frac{k}{I} & \left(1 - \frac{3I}{2k} \times \frac{k}{I}\right) \end{bmatrix}_1 \begin{Bmatrix} 0 \\ M_{to} \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} \frac{1}{2k} \\ -\frac{1}{2} \end{bmatrix}_t \begin{Bmatrix} 0 \\ t \end{Bmatrix}_0^R . \quad (193)$$

As in the preceding example, the elements of the first row of the transfer matrix shown by the vertical line are not included as they are not needed, due to the zero in the first row of the state vector for Sta. 0^R. This condition also applies to subsequent transfer matrices in this example.

Next, the transfer matrix for Sta. 2^R is written,

b.)

$$\begin{Bmatrix} \theta \\ M_t \end{Bmatrix}_2^R = \begin{bmatrix} 1 & \frac{1}{1.5k} \\ -2I\frac{k}{I} & \left(1 - \frac{2I}{1.5k} \times \frac{k}{I}\right) \end{bmatrix}_2 \begin{bmatrix} \frac{1}{2K} \\ -\frac{1}{2} \end{bmatrix}_1 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} 1 & \frac{1}{1.5k} \\ -2k & -\frac{1}{3} \end{bmatrix}_2 \begin{bmatrix} \frac{1}{2k} \\ -\frac{1}{2} \end{bmatrix}_1 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} 1 \times \frac{1}{2k} + \frac{1}{1.5k} \times \left(-\frac{1}{2}\right) \\ -2k \times \frac{1}{2k} - \frac{1}{3} \times \left(-\frac{1}{2}\right) \end{bmatrix}_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} \frac{1}{6k} \\ -\frac{5}{6} \end{bmatrix}_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}_0^R . \qquad (194)$$

Next

c.)

$$\begin{Bmatrix} \theta \\ M_t \end{Bmatrix}_3^R = \begin{bmatrix} 1 & \frac{1}{k} \\ -1 \times \frac{k}{I} & \left(1 - \frac{I}{k} \times \frac{k}{I}\right) \end{bmatrix}_3 \begin{bmatrix} \frac{1}{6}K \\ -\frac{5}{6} \end{bmatrix}_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} 1 & \frac{1}{k} \\ -k & 0 \end{bmatrix}_3 \begin{bmatrix} \frac{1}{6k} \\ -\frac{5}{6} \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} 1 \times \frac{1}{6k} + \frac{1}{k} \times \left(-\frac{5}{6}\right) \\ -k \times \frac{1}{6k} + 0 \times \left(-\frac{5}{6}\right) \end{bmatrix}_3 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}_0^R$$

$$= \begin{bmatrix} -\frac{2}{3k} \\ -\frac{1}{6} \end{bmatrix}_3 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}_0^R . \qquad (195)$$

From the second algebraic equation of this matrix, $M_{t3} = -\frac{1}{6}$ representing the error since M_t should be zero.

A new value of ω^2 is now chosen and the process is repeated, resulting in a new error. This is continued until the M_{t3} error is reduced to an acceptable value, and the corresponding ω^2 is then a correct natural frequency value. Plotting the error vs. the ω^2 value aids in interpolating for the correct frequency values. Such continued calculations are not shown here, but the correct (acceptable) ω^2 values can be verified as

$$\omega_1^2 = 0.2168\,\frac{k}{I},\quad \omega_2^2 = 1.0965\,\frac{k}{I},\quad \omega_3^2 = 2.1035\,\frac{k}{I}. \qquad (196)$$

The corresponding errors are

$$M_{t3}^{(1)} = 0.00000, \quad M_{t3}^{(2)} = 0.0013, \quad M_{t3}^{(3)} = 0.00023. \tag{197}$$

The principal-mode amplitude ratios can be determined by substituting the natural frequency values into the transfer matrix equations for each station.

3. REFERENCES

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APPENDIX:

TYPICAL SINGLE-DEGREE-OF-FREEDOM DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

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1. FREELY VIBRATING—NO DAMPING

$$\ddot{X} + \omega^2 X = 0$$

<u>Solution</u>

$$X = X_0 \cos \omega t = \frac{v_0}{\omega} \sin \omega t$$

where

X_0 – initial displacement – in

v_0 – initial velocity – in/s

ω – natural circular frequency – rad/s

2. FREELY VIBRATING—WITH VISCOUS DAMPING

$$\ddot{X} + 2\xi\omega\dot{X} + \omega^2 X = 0$$

<u>Solution</u>

$$X = e^{-\xi\omega t} A_1 \cos\left(\omega_0 t - \phi\right)$$

$$\omega_0 = \omega \left[1 - \xi^2\right]^{\frac{1}{2}}$$

$$A_1 = \left[X_0^2 + \left(\frac{v_0 + X_0\xi\omega}{\omega_0}\right)^2\right]^{\frac{1}{2}}$$

$$\phi = \tan^{-1}\left(\frac{v_0 + \xi\omega X_0}{X_0\omega_0}\right)$$

X_0 – initial displacement – in

v_0 – intial velocity – in/s

ω_0 – damped natural circular frequencies – rad/s.

3. CONSTANT FORCED VIBRATION—NO DAMPING

$$\ddot{X} + \omega^2 X = G$$

$$X = X_0 \cos \omega t + \frac{v_0}{\omega} \sin\omega t + \frac{G}{\omega^2},$$

where G = a constant independent of time.

4. FORCED VIBRATION WITH NO DAMPING

$$\ddot{X} + \omega^2 X = G(t) = G_0 \cos \Omega t$$

Solution

$$X = X_0 \cos\omega t + \frac{v_0}{\omega} \sin\omega t + \frac{G_0}{\omega^2 - \Omega^2} \left[\cos\Omega t - \cos\omega t\right]$$

Ω – forcing frequency – rad/s

Steady-state solution

$$X_s = \frac{G_0}{\omega^2 - \Omega^2} \cos\Omega t = \frac{G_0}{w^2} \frac{1}{1 - r^2} \cos\Omega t,$$

where

$$r = \frac{\Omega}{\omega}.$$

Subcases (steady-state solution)

(1) $G_0 = G_1 \omega^2$

$$x_s = \frac{G_1}{1 - r^2} \cos\Omega t$$

(2) $G_0 = G_2 \Omega^2$

$$x_3 = G_2 \frac{r^2}{1 - r^2} \cos\Omega t$$

5. FORCED VIBRATION WITH DAMPING

$$\ddot{X} + 2\xi\omega\dot{X} + \omega^2 X = G(t) = G_0 \cos\Omega t$$

Solution

$$X = A_1 e^{\xi\omega t} \cos\left(\omega_0 t - \phi\right) + \frac{\frac{G_0}{\omega^2}\cos\left(\Omega t - \phi_1\right)}{\left[\left(1 - r^2\right)^2 + (2\xi r)^2\right]^{\frac{1}{2}}}$$

$$\omega_0 = \omega\left[1 - \xi^2\right]^{\frac{1}{2}}$$

$$A_1 = \left[X_0^2 + \left(\frac{v_0 + \xi\omega X_0}{x_0\omega_0}\right)^2\right]^{\frac{1}{2}}$$

$$\phi = \tan^{-1}\left(\frac{v_0 + \xi\omega X_0}{X_0\omega_0}\right)$$

$$r = \frac{\Omega}{\omega}$$

$$\phi_1 = \tan^{-1}\left(\frac{2\xi r}{1 - r^2}\right).$$

Steady-state solution

$$X_s = \frac{\frac{G_0}{\omega^2}}{\left[\left(1 - r^2\right)^2 + \left(2\xi r\right)^2\right]^{\frac{1}{2}}} \cos\left(\Omega t - \phi_1\right)$$

Subcases (steady-state solutions)

(1) $G_0 = G_1 \omega^2$

$$X_s = \frac{G_1 \cos\left(\Omega t - \phi_1\right)}{\left[\left(1 - r^2\right)^2 + \left(2\xi r\right)^2\right]^{\frac{1}{2}}} .$$

(2) $G_0 = G_2 \Omega^2$

$$X_s = \frac{G_2 r^2 \cos\left(\Omega t - \phi_1\right)}{\left[\left(1 - r^2\right)^2 + \left(2\xi r\right)^2\right]^{\frac{1}{2}}} .$$

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GLOSSARY - NOMENCLATURE AND UNITS

k - spring constant or spring modulus (linear–lb/in) (torsional–in/lb/rad)

m - mass–lb/in/s^2

mg - w–weight–lb

J - polar mass moment of inertia–lb/in/s^2

I - area moment of inertia–in^4

g - gravitational constant–386.4 in/s^2

C - viscous damping constant–(linear–$\frac{lb/s}{in}$) (torsional–$\frac{in/lb/s}{rad}$)

c - damping ratio = C/C_c

C_c - critical viscous damping constant—that amount of damping, which if introduced into the system just prevents the system from vibrating

X – displacement–in

$\dot{X}$ - velocity–in/s

$\ddot{X}$ - acceleration– in/s^2

θ - angular displacement–radian

$\dot{\theta}$ - angular velocity–rad/s

$\ddot{\theta}$ - angular acceleration–rad/s^2

t - time–seconds

ω - natural circular frequency–rad/s

Ω - forced circular frequency–rad/s

$r = \frac{\Omega}{\omega}$ ratio of forced to natural circular frequency

$f = \frac{\omega}{2\pi}$ frequency of vibration in cycles/s

$$\delta = \frac{2\pi\xi}{\left[1 - \xi^2\right]^{\frac{1}{2}}} = \text{logarithmic decrement} = \ln \frac{X_m}{X_{M+1}}$$

I_ρ = Polar area moment of inertia–in^4

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