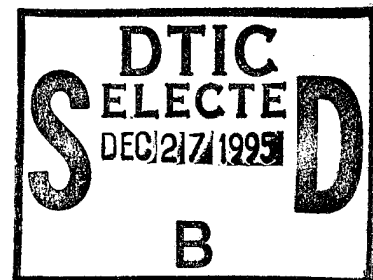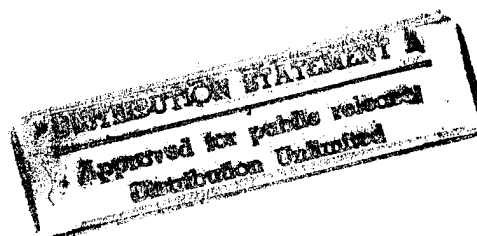*NO AD# MTR*

A THEORY OF VISCOPLASTICITY BASED ON
INFINITESIMAL TOTAL STRAIN

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--   1  OF    1
--                     ***DTIC DOES NOT HAVE THIS ITEM***
-- 1 -  AD NUMBER: D428940
-- 5 -  CORPORATE AUTHOR: RENSSELAER POLYTECHNIC INST  TROY NY DEPT OF
--      MECHANICAL ENGINEERING AERONAUTICAL ENGINEERING AND MECHANICS
-- 6 -  UNCLASSIFIED TITLE:    A THEORY OF VISCOPLASTICITY BASED ON
--      INFINITESIMAL TOTAL STRAIN,
--10 -  PERSONAL AUTHORS: CERNOCKY,E. P. ;KREMPL,E. ;
--11 -  REPORT DATE:     APR    , 1978
--12 -  PAGINATION:    45P
--14 -  REPORT NUMBER:   RPI CS 78-3
--15 -  CONTRACT NUMBER:  N00014-76-C-0231
--20 -  REPORT CLASSIFICATION:  UNCLASSIFIED
--22 -  LIMITATIONS (ALPHA):   APPROVED FOR PUBLIC RELEASE; DISTRIBUTION
--      UNLIMITED. AVAILABILITY: DEPARTMENT OF THE NAVY, OFFICE OF NAVAL
--      RESEARCH, STRUCTURAL MECHANICS PROGRAM, ARLINGTON, VA. 22217.
--33 -  LIMITATION CODES:  1

-- END             Y  FOR NEXT ACCESSION             END
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ABSTRACT

A viscoplasticity theory based upon a nonlinear viscoelastic solid, linear in the rates of the strain and stress tensors but nonlinear in the stress tensor and the infinitesimal strain tensor, is being investigated for isothermal, homogeneous motions. A general anisotropic form and a specific isotropic formulation are proposed. A yield condition is not part of the theory and the transition from linear (elastic) to nonlinear (inelastic) behavior is continuous. Only total strains are used and the constant volume hypothesis is not employed. In this paper Poisson's ratio is assumed to be constant. The proposed equation can represent: initial linear elastic behavior; initial elastic response in torsion (tension) after arbitrary prestrain (prestress) in tension (torsion); linear elastic behavior for pure hydrostatic loading; initial elastic slope upon large instantaneous changes in strain rate; stress (strain)-rate sensitivity; creep and relaxation; defined behavior in the limit of very slow and very fast loading. Stress-strain curves obtained at different loading rates will ultimately have the same "slope" and their spacing is nonlinearly related to the loading rate.

The above properties of the equation are obtained by qualitative arguments based on the characteristics of the solutions of the resulting nonlinear first-order differential equations. In some instances numerical examples are given.

For metals and isotropy we propose a simple equation whose coefficient functions can be determined from a tensile test [Eqs.(31), (35), (37), (38)]. Specializations suitable for materials other than metals are possible.

The paper shows that this nonlinear viscoelastic model can represent essential features of metal deformation behavior and reaffirms our previous assertion that metal deformation is basically rate-dependent and can be represented by piecewise nonlinear viscoelasticity. For cyclic loading the proposed model must be modified to account for history dependence in the sense of plasticity.

1. Introduction

The description of inelastic behavior, specifically of metals, has in recent years attracted considerable attention. This interest is caused by demands of technology and by the availability of powerful computation methods in the design office. In addition recent developments in material test techniques show that the classical idealizations of real material behavior are not adequate.

There were for a long time three almost separate disciplines of metal "plasticity". At low homologous temperature rate (time)-dependence was considered in dynamic plasticity, see for example Cristescu [1], whereas rate (time)-independence was assumed in static plasticity, Hill [2], Prager [3] and others. At high homologous temperature creep is important and separate creep theories were developed, Odquvist [4], Rabotnov [5], Hoff [6]. These theories were then combined with rate (time)-independent plasticity for the representation of metallic material deformation behavior, Leckie [7], Corum et al. [8], under quasi-static conditions.

A growing body of evidence suggests that inelastic deformation of metals is basically rate-dependent, Rice [9], Perzyna [10], Kratochvil [11], Miller [12], Hart [13], Eisenberg, Lee and Phillips [14], Phillips and Ricciuti [15]. Recent investigations also aim to give a general representation of plasticity which combines the above approaches and at the same time improves upon the capability of reproducing real metal deformation behavior, since the capabilities of the classical theories were shown to be in need of improvement, Krempl [16].

A review was made of the experimental foundations of static plasticity theory emphasizing the experiments designed to differentiate between incremental, physical, and deformation theories of static plasticity, Edelman [17]. In almost all instances creep at room temperature was mentioned as a problem which had to be avoided. "Creep causes great difficulty in full interpretation at high values

of strain. It is still a confusing factor in the range of small strains investigated. However, no correction was made for time effects", Drucker and Stockton [18]. Recent experiments on Type 304 stainless steel, copper, brass, and an aluminum alloy, Moon and Krempl [19], Hart et al. [20], showed that rate sensitivity, creep, and relaxation were present at room temperature. These facts attest further to the rate-dependent deformation of structural metals at room temperature.

In previous papers, Krempl [21,22], an operational definition of history dependence in the sense of plasticity, of aging, and of rate-dependence was given. As a consequence of these definitions we asserted that viscoplasticity cannot be distinguished from nonlinear viscoelasticity while a material is loaded, and we postulated that viscoplasticity is piecewise, nonlinear viscoelasticity. Rather than postulating state variables and their growth laws, the introduction of new origins and the possible updating of the material parameters provides for the necessary representation in constitutive equations of the internal microstructural changes, Krempl [21].

Here we propose a relatively simple nonlinear viscoelasticity law based upon small total strain; it is linear in the stress rate and strain rate tensors but nonlinear in the stress and strain tensors. The anisotropic form exhibits key characteristics of metal deformation behavior. Subsequently an isotropic formulation is given which exhibits creep, relaxation, and rate-sensitivity in a unified way. Only total strains are employed, the constant volume assumption is not used, and the model can predict linear elastic response under hydrostatic loading. The axial and torsional equations exhibit similar solution characteristics. In uniaxial deformation the specific isotropic equations proposed herein reduce to the previously proposed uniaxial equations, Cernocky and Krempl [23], Liu and Krempl [24], which were shown to represent many features of rate-dependent metal deformation behavior. The attempt here is not to present

general theories but rather a relatively simple model complex enough
to reproduce qualitatively key features of metal deformation as long as there is
no cyclic loading involved. The modifications of this nonlinear viscoelastic
model to fully represent metallic behavior for cyclic loading are not a subject
of this paper. They were in principle given previously, Krempl [21], and will
be developed for the constitutive equation of this paper in a future publication.

2. General Properties of the Anisotropic Model

We consider only homogeneous motions and propose for small strain $\underset{\sim}{\varepsilon}$ and strain
rate $\dot{\underset{\sim}{\varepsilon}}$ and associated stress $\underset{\sim}{\sigma}$ and stress rate $\dot{\underset{\sim}{\sigma}}$ the constitutive equation

$$\underset{\approx}{M}[\underset{\sim}{\sigma},\underset{\sim}{\varepsilon}]\dot{\underset{\sim}{\varepsilon}} + \underset{\sim}{G}[\underset{\sim}{\varepsilon}] = \underset{\sim}{\sigma} + \underset{\approx}{K}[\underset{\sim}{\sigma},\underset{\sim}{\varepsilon}]\dot{\underset{\sim}{\sigma}} . \tag{1}$$

In the above square brackets denote function of the quantities inside the
brackets and a dot designates differentiation with respect to time.

The fourth order tensors $\underset{\approx}{M}$ and $\underset{\approx}{K}$ linearly transform $\dot{\underset{\sim}{\varepsilon}}$ and $\dot{\underset{\sim}{\sigma}}$, respectively.
They are required to be symmetric, positive definite linear transformations for
all values of their arguments such that

$$M_{ijk\ell}B_{ij}B_{k\ell} > 0 \quad \text{and} \quad K_{ijk\ell}B_{ij}B_{k\ell} > 0 \tag{2}$$

for all nonzero tensors $\underset{\sim}{B}$. Because of this requirement the inverses of $\underset{\approx}{M}$ and $\underset{\approx}{K}$
exist.

The function $\underset{\sim}{G}[\underset{\sim}{\varepsilon}]$ is constructed so that $\underset{\sim}{G}[\underset{\sim}{0}] = \underset{\sim}{0}$, and we usually require
it to be odd so that

$$\underset{\sim}{G}[-\underset{\sim}{\varepsilon}] = - \underset{\sim}{G}[\underset{\sim}{\varepsilon}] . \tag{3}$$

Further, we usually require that $\underset{\sim}{G}$ be a bijective function over all real tensor
values. In this case we ensure that (1) is an equation of state, i.e., given
any three tensor variables in (1) the fourth tensor variable is uniquely
determined.

For zero $\dot{\underset{\sim}{\epsilon}}$ and zero $\dot{\underset{\sim}{\sigma}}$, $\underset{\sim}{\sigma} = \underset{\sim}{0}$ and $\underset{\sim}{\epsilon} = \underset{\sim}{0}$ as well as $\underset{\sim}{\sigma} = \underset{\sim}{G}[\underset{\sim}{\epsilon}]$ are solutions to (1). The function $\underset{\sim}{G}[\underset{\sim}{\epsilon}]$ represents for given strain $\underset{\sim}{\epsilon}$ the locus of $\underset{\sim}{\sigma}$ for which both the stress rate and strain rate are zero; the origin is one of these points.

A generalized creep test[*] is performed by setting $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{0}$ and $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^O$ for $t \geq t_o$ where $\underset{\sim}{\sigma}^O$ is a constant tensor. Equation (1) reduces to

$$\underset{\approx}{M}\dot{\underset{\sim}{\epsilon}} = \underset{\sim}{\sigma}^O - \underset{\sim}{G}[\underset{\sim}{\epsilon}] \tag{4}$$

which must be solved subject to the initial condition $\underset{\sim}{\epsilon}(t_o) \neq \underset{\sim}{0}$ to obtain $\underset{\sim}{\epsilon}^C = \underset{\sim}{\epsilon}[t] - \underset{\sim}{\epsilon}[t_o]$, the strain accumulated in the creep test.

Similarly, for a generalized relaxation test $\dot{\underset{\sim}{\epsilon}} = \underset{\sim}{0}$ and $\underset{\sim}{\epsilon} = \underset{\sim}{\epsilon}^O$ for $t \geq t_o$ such that

$$\underset{\approx}{K}\dot{\underset{\sim}{\sigma}} = \underset{\sim}{G}[\underset{\sim}{\epsilon}^O] - \underset{\sim}{\sigma} \ . \tag{5}$$

Again (5) has to be solved for a suitable initial condition $\underset{\sim}{\sigma}(t_o) \neq \underset{\sim}{0}$.

Although both tests follow different paths the stress rate will be zero at the stress $\underset{\sim}{\sigma} = \underset{\sim}{G}[\underset{\sim}{\epsilon}^O]$ and the creep rate may become zero at a strain which satisfies $\underset{\sim}{\sigma}^O = \underset{\sim}{G}[\underset{\sim}{\epsilon}]$[**]. We conclude that the relaxation test (5) will always reach equilibrium whereas the creep test may not. However, if $\underset{\sim}{G}$ is bijective for all real $\underset{\sim}{\epsilon}$ then both tests terminate on the $\underset{\sim}{G}[\underset{\sim}{\epsilon}]$ curve.

If we multiply (4) by $\dot{\underset{\sim}{\epsilon}}$ and contract we obtain

$$(\sigma^O_{ij} - \underset{\sim}{G}_{ij}[\epsilon_{k\ell}])\dot{\epsilon}_{ij} \geq 0 \tag{6}$$

and similarly from (5)

$$(G_{ij}[\epsilon^O_{k\ell}] - \sigma_{ij})\dot{\sigma}_{ij} \geq 0 \tag{7}$$

because of the positive definiteness of $\underset{\approx}{M}$ and $\underset{\approx}{K}$.

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[*] By generalized creep test we mean a test condition where all the components of $\underset{\sim}{\sigma}$ are constant; some may be zero.

[**] For a given $\underset{\sim}{G}$ such an $\underset{\sim}{\epsilon}$ may not be found.

In the case where $\underset{\sim}{G}$ is bijective so that the inverse of $\underset{\sim}{G}$ exists, for every tensor $\bar{\underset{\sim}{\varepsilon}}$ = constant the function $\underset{\sim}{G}$ may be utilized to construct a surface. Then in the case of $\underset{\sim}{\sigma} = \underset{\sim}{G}[\underset{\sim}{\varepsilon}]$, we obtain

$$\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij} = G_{ij}^{-1}[\sigma_{k\ell}]G_{ij}^{-1}[\sigma_{k\ell}] \; . \tag{8}$$

For each constant value of $\bar{\underset{\sim}{\varepsilon}}$, (8) represents a surface in stress space which may be isotropic or anisotropic. For zero strain (8) degenerates into the origin. With increasing strain the surface defined in (8) "increases in size", since the left-hand side represents the square of the magnitude of the strain tensor. Equation (8) represents for each constant $\bar{\underset{\sim}{\varepsilon}}$ the surface for which there are zero rates of stress and strain.

Relation Between $\underset{\approx}{M}$ and $\underset{\approx}{K}$

Using the chain rule we may rewrite Eq. (1) as

$$\left(\underset{\approx}{M}[\underset{\sim}{\sigma},\underset{\sim}{\varepsilon}] - \underset{\approx}{K}[\underset{\sim}{\sigma},\underset{\sim}{\varepsilon}] \frac{\partial\underset{\sim}{\sigma}}{\partial\underset{\sim}{\varepsilon}}\right) \dot{\underset{\sim}{\varepsilon}} = \underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\varepsilon}] \tag{9}$$

or as

$$\left(\underset{\approx}{M}[\underset{\sim}{\sigma},\underset{\sim}{\varepsilon}] \frac{\partial\underset{\sim}{\varepsilon}}{\partial\underset{\sim}{\sigma}} - \underset{\approx}{K}[\underset{\sim}{\sigma},\underset{\sim}{\varepsilon}]\right) \dot{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\varepsilon}] \; . \tag{10}$$

We are interested in the response of (1) with initial conditions such that $\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\varepsilon}] = \underset{\sim}{0}$, i.e., we want to compute the response of (1) upon leaving the "equilibrium stress-strain curve" [15] at any point $\underset{\sim}{\varepsilon}$.

For this case (9) and (10) represent linear, homogeneous equations in the strain and stress rates, respectively. The rates can be <u>arbitrarily</u> imposed and $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ are also arbitrary. Consequently, the expressions in the parentheses must vanish when $\underset{\sim}{\sigma} = \underset{\sim}{G}[\underset{\sim}{\varepsilon}]$ and we obtain

$$\underset{\approx}{K}^{-1}\underset{\approx}{M}\Big|_{\underset{\sim}{\sigma}=\underset{\sim}{G}[\underset{\sim}{\varepsilon}]} = \frac{\partial\underset{\sim}{\sigma}}{\partial\underset{\sim}{\varepsilon}} \tag{11}$$

and

$$(\underset{\approx}{K}^{-1}\underset{\approx}{M})^{-1}\Big|_{\underset{\sim}{\sigma}=\underset{\sim}{G}[\underset{\sim}{\varepsilon}]} = \frac{\partial\underset{\sim}{\varepsilon}}{\partial\underset{\sim}{\sigma}} \; . \tag{12}$$

If we select $\underset{\approx}{K}^{-1}\underset{\approx}{M}\Big|_{\sigma=G[\epsilon]} = \underset{\approx}{c}$ where $\underset{\approx}{c}$ represents the fourth-order tensor of the elastic constants, then (11) and (12) show that all curves depart from the equilibrium stress-strain curve $\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}] = \underset{\sim}{0}$ with elastic "slope". (Note, this includes the origin. Also it is impossible to depart from $\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}] = \underset{\sim}{0}$ by a creep or relaxation test.)

Following Cernocky and Krempl [23] we now impose

$$\underset{\approx}{K}^{-1}\underset{\approx}{M} = \underset{\approx}{c} \tag{11a}$$

for <u>all values</u> of $\underset{\sim}{\sigma}$ and $\underset{\sim}{\epsilon}$, since this relation leads to several useful properties in the model. Note, however, that $\underset{\approx}{K}$ and $\underset{\approx}{M}$ remain nonlinear functions; only their combination according to (11a) is constant.

A consequence of condition (11a) is the ability to model realistically the subsequent response of a metal in torsion after a preload in tension. Such experiments are reported in the plasticity literature. Most of the experiments show, Edelman [17], that in the presence of arbitrary axial preloading the <u>initial</u> response in torsion is purely elastic.

Appendix I demonstrates that (1) subject to (11a) and additional specified restrictions on $\underset{\approx}{M}$, $\underset{\approx}{K}$ and $\dfrac{\partial \underset{\sim}{\sigma}}{\partial \underset{\sim}{\epsilon}}$ can reproduce the initial elastic response for various material symmetries including isotropy, transverse isotropy, and orthotropy. We therefore have demonstrated that the rate-dependent Eq.(1) can reproduce a key result which is normally considered to be in the domain of rate (time)-independent incremental plasticity theory. Note that we have used total strains only.

Formally, we can at any time split the strains into elastic and inelastic strains which are, however, rate-dependent. To demonstrate this we rewrite (1) using (11a) and obtain

$$\underset{\sim}{\epsilon}[t] = \int_{0}^{t} \underset{\approx}{M}^{-1}(\underset{\sim}{\sigma}[\tau] - \underset{\sim}{G}[\underset{\sim}{\epsilon}[\tau]])\, d\tau + \underset{\approx}{c}^{-1}\underset{\sim}{\sigma} \tag{13}$$

or

$$\underset{\sim}{\epsilon}[t] \equiv \underset{\sim}{\epsilon}^{in}[t] + \underset{\sim}{\epsilon}^{el}[t]$$

where $\left.\begin{array}{c}\\\\\\\\\end{array}\right\}$ (14)

$$\underset{\sim}{\epsilon}^{in}[t] \equiv \int\limits_{o}^{t} \underset{\approx}{M}^{-1}(\underset{\sim}{\sigma}[\tau] - \underset{\sim}{G}[\underset{\sim}{\epsilon}[\tau]]) \, d\tau$$

and where we have assumed that for $t \le 0$, $\underset{\sim}{\sigma} = \underset{\sim}{0}$ and $\underset{\sim}{\epsilon} = \underset{\sim}{0}$.

Behavior at and near $\underset{\sim}{\sigma} = \underset{\sim}{0}$ and $\underset{\sim}{\epsilon} = \underset{\sim}{0}$

We assume that $\underset{\sim}{G}[\underset{\sim}{\epsilon}]$ is linear in $\underset{\sim}{\epsilon}$ in the neighborhood of the origin $\epsilon_{ij}\epsilon_{ij} \le \bar{\xi} \ll 1$ where $\bar{\xi}$ is a suitable constant, and we investigate the two possibilities

$$\left.\frac{d\underset{\sim}{G}}{d\underset{\sim}{\epsilon}}\right|_{\epsilon_{ij}\epsilon_{ij} \le \bar{\xi}} = \underset{\approx}{c} \tag{15a}$$

and

$$\left.\frac{d\underset{\sim}{G}}{d\underset{\sim}{\epsilon}}\right|_{\epsilon_{ij}\epsilon_{ij} \le \bar{\xi}} = \underset{\approx}{\hat{c}} \tag{15b}$$

where $\underset{\approx}{\hat{c}}$ and $\underset{\approx}{c}$ are both constant tensors.

Using (11a) and (15a), Eq.(9) can be rewritten as

$$\underset{\approx}{K}\left(\underset{\approx}{c} - \frac{\partial\underset{\sim}{\sigma}}{\partial\underset{\sim}{\epsilon}}\right) \underset{\sim}{\dot{\epsilon}} = \underset{\sim}{\sigma} - \underset{\approx}{c}\underset{\sim}{\epsilon} \ . \tag{15c}$$

The initial material response predicted by (15c) is independent of the strain rate and is that of a linear elastic material. Alternatively when (15b) is used instead and when $\underset{\approx}{K}$ is assumed to be constant in the neighborhood of the origin, then the initial response predicted by (1) is that of a linear aniso-tropic viscoelastic solid.

The model proposed in (1) subject to (11a) reproduces initial linear elastic response, and initial linear shear response after an arbitrary prestress in tension for special material symmetries. Also in the neighborhood of $\underset{\sim}{\sigma} = \underset{\sim}{0}$ and $\underset{\sim}{\epsilon} = \underset{\sim}{0}$ linear elastic or linear viscoelastic behavior can be modeled.

A Particular Dependence of $\underset{\approx}{M}$ and $\underset{\approx}{K}$ Upon $\underset{\sim}{\sigma}$ and $\underset{\sim}{\epsilon}$

Thus far the dependence of $\underset{\approx}{K}$ and $\underset{\approx}{M}$ upon $\underset{\sim}{\sigma}$ and $\underset{\sim}{\epsilon}$ has not been stipulated and the previous results are valid for all $\underset{\approx}{K}$ and $\underset{\approx}{M}$. If we make these functions depend upon the difference $\{\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}]\}$ then additional desirable properties can be modeled. Specifically, Eq. (9) can be rewritten as

$$\underset{\approx}{K}[\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}]]\left\{\underset{\approx}{c} - \frac{\partial \underset{\sim}{\sigma}}{\partial \underset{\sim}{\epsilon}}\right\} \underset{\sim}{\dot{\epsilon}} = \underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}] \tag{16}$$

and from (4) and (5) we obtain, for the generalized creep and relaxation test, respectively

$$\underset{\sim}{\dot{\epsilon}} = \underset{\approx}{c}^{-1}\underset{\approx}{K}^{-1}[\underset{\sim}{\sigma}^{\circ} - \underset{\sim}{G}[\underset{\sim}{\epsilon}]]\{\underset{\sim}{\sigma}^{\circ} - \underset{\sim}{G}[\underset{\sim}{\epsilon}]\} \tag{17}$$

and

$$\underset{\sim}{\dot{\sigma}} = \underset{\approx}{K}^{-1}[\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}^{\circ}]]\{\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}^{\circ}]\}. \tag{18}$$

Suppose that a generalized constant strain rate test with $\underset{\sim}{\dot{\epsilon}} = \underset{\sim}{\mu}$ is conducted with the hypothetical material represented by (16) and that we observe $\left.\dfrac{d\underset{\sim}{\sigma}}{d\underset{\sim}{\epsilon}}\right|_{\underset{\sim}{\dot{\epsilon}}=\underset{\sim}{\mu}}$ to be constant; then we must conclude that $\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}] = \underset{\sim}{A}$ where $\underset{\sim}{A}$ is a constant tensor and we can construct surfaces

$$\overset{*}{\epsilon}_{ij}\overset{*}{\epsilon}_{ij} = \underset{\sim}{\sigma}_{ij}^{-1}[\underset{\sim}{G}[\underset{\sim}{\epsilon}] + \underset{\sim}{A}]\underset{\sim}{\sigma}_{ij}^{-1}[\underset{\sim}{G}[\underset{\sim}{\epsilon}] + \underset{\sim}{A}] \tag{19}$$

where $\overset{*}{\underset{\sim}{\epsilon}}$ is some constant strain field. For $\underset{\sim}{A} = \underset{\sim}{0}$ we obtain the surface of Eq. (8).

Therefore in a creep or relaxation test started from any point on a curve for $\underset{\sim}{\dot{\epsilon}} = \underset{\sim}{\mu}$ on which $\dfrac{d\underset{\sim}{\sigma}}{d\underset{\sim}{\epsilon}}$ is observed to be constant, the initial creep rate or the initial relaxation rate is independent of the actual value of $\underset{\sim}{\sigma}$ and $\underset{\sim}{\epsilon}$, depending only upon $\{\underset{\sim}{\sigma} - \underset{\sim}{G}[\underset{\sim}{\epsilon}]\}$.

We now derive a specific and simple isotropic version of (1). This version permits the identification of the coefficient functions from experimental

results and the simulation of real experiments by numerical integration of the resulting first-order nonlinear differential equations.

3. An Isotropic Formulation

A specific isotropic formulation can be obtained by using the following requirements which are deemed suitable for metals:

- The isotropic equation is to be derived from (1).

- The tensors $\underset{\approx}{M}$ and $\underset{\approx}{K}$ must be tensors of constants times a respective scalar-valued function of the invariants of the stress and strain tensors.

- For hydrostatic stress (strain) states the classical linear elastic relation must be obtained.

- The constant volume assumption is not imposed upon this theory, because a recent literature survey has not produced experimental evidence to support this assumption in the small strain range, Hewelt and Krempl [25].

- The isotropic equation must reduce to the uniaxial formulation [23] when the uniaxial deformation field is imposed.

With these stipulations and the symmetry of $\underset{\sim}{\dot{\sigma}}$ and $\underset{\sim}{\dot{\epsilon}}$ in mind we set

$$M_{ijab} = \delta_{ia}\delta_{jb}M_1 + M_2\delta_{ij}\delta_{ab} \tag{20}$$

$$K_{ijab} = \delta_{ia}\delta_{jb}K_1 + K_2\delta_{ij}\delta_{ab} \tag{21}$$

where M_i and K_i ($i = 1,2$) are isotropic scalar-valued functions of invariants of the stress and strain tensors. When $\underset{\sim}{s}$ represents the deviatoric stress, $\underset{\sim}{e}$ the deviatoric strain and $\underset{\sim}{G}^d$ the deviatoric component of $\underset{\sim}{G}$, $G^d_{\sim ij} = G_{ij} - \frac{1}{3}G_{aa}\delta_{ij}$, we may rewrite (1) using (20), (21) as

$$M_1\dot{e}_{ij} - K_1\dot{s}_{ij} = s_{ij} - G^d_{ij}[\underset{\sim}{\epsilon}] \tag{22}$$

$$(M_1 + 3M_2)\dot{\epsilon}_{aa} - (K_1 + 3K_2)\dot{\sigma}_{aa} = \sigma_{aa} - G_{aa}[\underset{\sim}{\epsilon}] . \tag{23}$$

The previously proposed uniaxial constitutive equation is [23,24]

$$m[\sigma,\epsilon]\dot{\epsilon} - k[\sigma,\epsilon]\dot{\sigma} = \sigma - g[\epsilon] \qquad (24)^{*}$$

where σ is the axial stress and ϵ is the axial strain in the uniaxial deformation field. We stipulate that in this deformation field Eqs.(22) and (23) must reduce to (24). This requires that when the strain tensor is

$$[\underset{\sim}{\epsilon}]_{ij} = \epsilon \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix} \qquad (25)$$

then the tensor $\underset{\sim}{G}$ must assume the specific form

$$\left[\underset{\sim}{G}[\underset{\sim}{\epsilon}]\right]_{ij} = g[\epsilon] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (26)$$

where ν is Poisson's ratio which for this paper is considered to be a positive constant less than 1/2. In addition, the coefficient functions M_i and K_i must assume the specific forms:

$$M_1 = \frac{m[\sigma,\epsilon]}{1+\nu} \qquad (27)$$

$$M_2 = \frac{\nu\, m[\sigma,\epsilon]}{(1+\nu)(1-2\nu)} \qquad (28)$$

$$K_1 = k[\sigma,\epsilon] \qquad (29)$$

$$K_2 = 0. \qquad (30)$$

To generalize our construction of M_i and K_i and meet the requirements of isotropy and those embodied in (27) - (30) we assume for all deformation fields the forms presented in (27) - (30) except that we replace the stress therein by a suitable invariant $\hat{\varphi}$ of the stress tensor, and we replace the strain therein by a suitable invariant φ of the strain tensor. These invariants are restricted by the requirements that in a uniaxial deformation field, $\hat{\varphi}$ must reduce to the

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* From [23] m[ ] and k[ ] are restricted to be positive and bounded.

axial stress σ and φ must reduce to the axial strain ϵ. An example of a choice suitable for metals is

$$\varphi = \varphi_1 = \left(\frac{3}{2} e_{ij} e_{ij}\right)^{\frac{1}{2}} / (1 + \nu) \tag{31}$$

and

$$\hat{\varphi} = \hat{\varphi}_1 = \left(\frac{3}{2} s_{ij} s_{ij}\right)^{\frac{1}{2}} . \tag{32}$$

Additional pairs of $\hat{\varphi}, \varphi$ appear in Table I. We remark that the $\hat{\varphi}, \varphi$ pairs are related by the restriction that when on the surface defined by (8), i.e., when $\sigma_{ij} = G_{ij}[\underset{\sim}{\epsilon}]$, then $\hat{\varphi} = g[\varphi]$; this constrains $\hat{\varphi}$ for a given choice of φ.

From (27) - (32) we obtain the constitutive representation

$$\frac{m[\hat{\varphi}, \varphi]}{1 + \nu} \dot{e}_{ij} - k[\hat{\varphi}, \varphi] \dot{s}_{ij} = s_{ij} - G_{ij}^{d}[\underset{\sim}{\epsilon}] \tag{33}$$

$$\frac{m[\hat{\varphi}, \varphi]}{(1 - 2\nu)} \dot{\epsilon}_{aa} - k[\hat{\varphi}, \varphi] \dot{\sigma}_{aa} = \sigma_{aa} - G_{aa}[\underset{\sim}{\epsilon}] . \tag{34}$$

Equations (25), (26) severely restrict the representation of the functions $G_{ij}[\underset{\sim}{\epsilon}]$; a specific construction of $\underset{\sim}{G}$ appears in Appendix II. We note that the argument of $\underset{\sim}{G}^{d}$ and G_{aa} is the total strain tensor $\underset{\sim}{\epsilon}$, and not its respective deviatoric and hydrostatic components. Therefore, Eqs. (33) and (34) must be regarded as coupled equations.

Alternatively (33) and (34) may be combined to obtain

$$m[\hat{\varphi}, \varphi] \dot{\psi}_{ij} + G_{ij}[\underset{\sim}{\epsilon}] = \sigma_{ij} + k[\hat{\varphi}, \varphi] \dot{\sigma}_{ij} \tag{35}$$

where ψ_{ij} is defined by

$$\psi_{ij} \equiv \frac{\epsilon_{ij}}{(1 + \nu)} + \frac{\nu \epsilon_{aa}}{(1 + \nu)(1 - 2\nu)} \delta_{ij} \tag{36}$$

Also from (11a) we require that $m[\ ]$ and $k[\ ]$ always be related through the ratio

$$\frac{m[\ ]}{k[\ ]} = E \tag{37}$$

where E is the modulus of elasticity. (Appendix III discusses the consequences of $m/k \neq E$.)

In view of (11a) and the discussion associated with (16) we now define the invariant

$$\Gamma \equiv \left\{ (\sigma_{ij} - G_{ij}[\underset{\sim}{\epsilon}])(\sigma_{ij} - G_{ij}[\underset{\sim}{\epsilon}]) \right\}^{\frac{1}{2}} . \tag{38}$$

Individually we now select m[ ] and k[ ] to be functions of Γ alone. In the uni-axial deformation field, this corresponds to $m = m[\sigma - g[\epsilon]]$ and $k = k[\sigma - g[\epsilon]]$ in (24)[*].

Using the chain rule (35) may be rewritten using (37) as

$$\left(E \frac{\partial \psi_{ij}}{\partial \epsilon_{km}} - \frac{\partial \sigma_{ij}}{\partial \epsilon_{km}} \right) \dot{\epsilon}_{km} = \frac{\sigma_{ij} - G_{ij}[\underset{\sim}{\epsilon}]}{k[\Gamma]} . \tag{39}[**]$$

It is easily seen that departure from the curves $\sigma_{ij} = G_{ij}[\underset{\sim}{\epsilon}]$ is linear elastic. Furthermore (39) predicts initial elastic shear response in the presence of axial prestress (prestrain) and predicts initial elastic axial response in the presence of shear prestress (prestrain); (see Appendix I for details). Figure 1 demonstrates the initial elastic shear response predicted by (39) at various axial prestrains; for ease in the numerical integration and since (A9) clearly shows that the _initial_ elastic response is obtained for any k-function, k was chosen to be constant.

Limiting Behavior at Large Times

Following the methods developed for the uniaxial case, Cernocky and Krempl [23], we now transform (35) subject to (37) into an equivalent integral expression and obtain with $\underset{\sim}{\sigma}(0) = \underset{\sim}{0}$ and $\underset{\sim}{\epsilon}(0) = \underset{\sim}{0}$

$$\sigma_{ij} = G_{ij}[\underset{\sim}{\epsilon}] + \int_o^t \left\{ E \frac{\partial \psi_{ij}}{\partial \epsilon_{km}} - \frac{\partial G_{ij}}{\partial \epsilon_{km}} \right\} \left\{ \exp - \int_\tau^t \frac{dx}{k[\Gamma[x]]} \right\} \dot{\epsilon}_{km} d\tau . \tag{40}$$

Provided the limits of $\dfrac{\partial G_{ij}[\epsilon[t]]}{\partial \epsilon_{km}}$, $\dot{\epsilon}_{km}$ and $k[\Gamma[t]]$ are bounded and finite as $t \to \infty$, Eq. (40) can be used to determine the response for large times

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[*] Further motivations for this modification appear in Part 4 and in [23].

[**] $\dfrac{\partial \psi_{ij}}{\partial \epsilon_{km}}$ appears in (B5) of Appendix II.

following the procedure of Cernocky and Krempl [23]. We obtain

$$\lim_{t \to \infty} \left\{ \sigma_{ij} - G_{ij}[\underset{\sim}{\epsilon}[t]] \right\} = \left\{ \left(E \frac{\partial \psi_{ij}}{\partial \epsilon_{km}} - \frac{\partial G_{ij}}{\partial \epsilon_{km}} \right) \dot{\epsilon}_{km} k[\Gamma] \right\}_{t=\infty} \tag{41}$$

and

$$\lim_{t \to \infty} \frac{\partial \sigma_{ij}}{\partial \epsilon_{km}} = \frac{\partial G_{ij}[\underset{\sim}{\epsilon}[\infty]]}{\partial \epsilon_{km}} \ . \tag{42}$$

Equation (41) says that ultimately the "slopes" of stress-strain curves are equal to the "slopes" of the $\underset{\sim}{G}$-curves.

Performing the limits in (41) and (42) may appear unrealistic and in violation of the small strain assumption. In reality this is not so since the solutions of (40) are rapidly asymptotic to these limits. This has been demonstrated in the uniaxial case, Liu and Krempl [24], Cernocky and Krempl [23], and will be reaffirmed by some of the examples to be given later, see specifically Fig.3. Therefore (41) and (42) may be used as approximate relations when time is finite.

If we assume that G_{ij} is approximated by (B8) and that $\varphi > X_f^*$, then from (41) and (42) with (B9) we have for large times

$$(\sigma_{ij} - G_{ij}) \approx (E - E_s) \frac{\partial \psi_{ij}}{\partial \epsilon_{km}} \dot{\epsilon}_{km}[\infty] k[\Gamma|_{t \to \infty}] \tag{43}$$

and

$$\frac{\partial \sigma_{ij}}{\partial \epsilon_{km}} \approx E_s \frac{\partial \psi_{ij}}{\partial \epsilon_{km}} \ , \tag{44}$$

respectively. In the above E_s denotes the constant slope of the uniaxial stress-strain curve in the plastic range. Again (43) and (44) may be used as approximate expressions for finite time.

Consider now a uniaxial tensile test with strain ϵ and performed with constant strain rate α and let $\sigma_{11} = \sigma$, then from (40) using (B1)

$$\sigma - g(\epsilon) = \alpha \int_o^t (E - g'[\epsilon[s]]) \left(\exp - \int_s^t \frac{dx}{k[\sigma[x] - g[\epsilon[x]]]} \right) ds \tag{45}$$

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* For $\varphi > X_f$, $g(\varphi)$ in Appendix II is assumed to be approximately linear.

Let $X = \lim_{t \to \infty} (\sigma - g[\epsilon[t]])$; then from (45) or (43)

$$\frac{X}{k[X]} = (E - E_s)\alpha . \tag{46}$$

Similarly, from (40) the response in a shear test with constant shear-strain

rate $\dot{\epsilon}_{12} = \dot{\epsilon}_{21} = \gamma$, where $\sigma_{12} = \sigma_{21}$, is obtained to be

$$(\sigma_{12} - G_{12}[\epsilon_{12};\epsilon_{21}]) = \gamma \int_o^t \left\{ \frac{E}{1+\nu} - \frac{\partial G_{12}[\epsilon_{12}[s];\epsilon_{21}[s]]}{\partial \epsilon_{12}[s]} - \frac{\partial G_{12}[\epsilon_{12}[s];\epsilon_{21}[s]]}{\partial \epsilon_{21}[s]} \right\}$$

$$\left\{ \exp - \int_s^t \frac{dx}{k[\sqrt{2}\,(\sigma_{12}[x] - G_{12}[\epsilon_{12}[x];\epsilon_{21}[x]])]} \right\} ds . \tag{47}$$

We let $Y = \lim_{t \to \infty} (\sigma_{12} - G_{12})$ and obtain, using (47) or (43)

$$\frac{Y}{k[\sqrt{2}\,Y]} = \frac{E - E_s}{1 + \nu} \gamma \tag{48}$$

where X and Y denote the respective heights of the axial and shear responses

above the axial and shear equilibrium curves. If $\alpha = \gamma$ we see from (46) and (48)

that Y and X are different. If $\underset{\sim}{G}$ does not permit the approximation (B8) then

the right-hand sides of (46) and (48) also depend upon the choice of φ. In this

case Eq. (41) applies.

For the invariant φ_1 given in (31), (47) reduces to

$$\left(\sigma_{12} - \frac{1}{\sqrt{3}} g\left[\frac{\epsilon_{12}\sqrt{3}}{(1+\nu)} \right] \right) = \frac{\gamma}{(1+\nu)} \int_o^t \left\{ E - g'\left[\frac{\sqrt{3}}{(1+\nu)} \epsilon_{12}[s] \right] \right\}$$

$$\left(\exp - \int_s^t \frac{dx}{k\left[\sqrt{2}\left(\sigma_{12}[x] - \frac{1}{\sqrt{3}} g\left[\frac{\epsilon_{12}[x]\sqrt{3}}{(1+\nu)} \right] \right) \right]} \right) ds . \tag{47a}$$

The solutions of (45) and (47a) for various $\alpha = \gamma$-values and a specific

choice of g[ ] and k[ ] are given in Fig.3. The results were obtained with a

computer program developed by Liu and Krempl [24]. Note the nonlinear spacing

of the curves at various constant strain rate values in the axial and shear

tests. This is due to the dependence of the k-function upon $\{\underset{\sim}{\sigma} - \underset{\sim}{G}\}$ through the invariant Γ; see the discussion in Cernocky and Krempl [23].

Limiting Behavior for Very Large and Very Small Constant Strain Rates

The case of uniaxial deformation under limiting magnitude of loading rates is considered in detail in Cernocky and Krempl [23]. We will consider here the limiting loading rate cases for shear. We let $\dot{\epsilon}_{12} = \dot{\epsilon}_{21} = \gamma$ so that (47) applies, and we define the transformation

$$\hat{\sigma}_{12}[\epsilon_{12}, \epsilon_{21}; \gamma] = \sigma(t)\Big|_{\substack{\epsilon_{12} = \epsilon_{21} = \gamma t \\ \text{all other } \epsilon_{ij} = 0}} \tag{49}$$

where $\hat{\sigma}_{12}$ is the stress response as a function of strain and parametric dependence upon γ is indicated. Proceeding we obtain

$$\hat{\sigma}_{12} = G_{12} + \int_{o}^{\epsilon_{12}} \left\{\frac{1}{2} E/(1+\nu) - \frac{\partial G_{12}}{\partial x_{12}}\right\} \exp\left\{- 1/\gamma \int_{x_{12}}^{\epsilon_{12}} \frac{dz_{12}}{k[\Gamma[z_{12}]]}\right\} dx_{12}$$
$$+ \int_{o}^{\epsilon_{21}} \left\{\frac{1}{2} E/(1+\nu) - \frac{\partial G_{12}}{\partial x_{21}}\right\} \exp\left\{- 1/\gamma \int_{x_{21}}^{\epsilon_{21}} \frac{dz_{21}}{k[\Gamma[z_{21}]]}\right\} dx_{21} \tag{50}$$

and where in this case

$$\Gamma[z_{21}] = \Gamma[z_{12}] = \sqrt{2} \ (\hat{\sigma}_{12} - G_{12}[z_{12}, z_{21}]) . \tag{51}$$

We consider limiting slow loading rate and we let $\gamma \to 0$ in (50) to obtain

$$\hat{\sigma}_{12}[\epsilon_{12}, \epsilon_{21}; 0] = G_{12} . \tag{52}$$

If (31) is used with (B1) then G_{12} in (52) is given by

$$G_{12} = \frac{1}{\sqrt{3}} \ g\left[\frac{\sqrt{3} \ \epsilon_{12}}{1+\nu}\right] . \tag{53}$$

Appropriate representations for G_{12} using the other φ of Table I are easily obtained, see Fig.2.

Similarly for limiting fast loading rate we let $\gamma \to \infty$ and we obtain from (50)

$$\hat{\sigma}_{12}[\varepsilon_{12}, \varepsilon_{21}; \infty] = \frac{E}{(1+\nu)} \varepsilon_{12} \qquad (54)$$

Equation (52) shows again that $\underset{\sim}{G}$ represents the equilibrium stress-strain curves. The linear elastic response at high strain rate is due to the assumption (37). Analogous results apply when any component of the strain (or stress) tensor is applied at a limitingly fast or slow loading rate. Therefore, (39) or (40) predicts linear elastic behavior in the limit of very fast loading.

If instead of (37) Eq. (C1) applies, then a nonlinear response in (54) is permitted, so that (54) would become for all φ which satisfy (B3)

$$\hat{\sigma}_{12}[\varepsilon_{12}, \varepsilon_{21}; \infty] = \frac{\varepsilon_{12}}{(1+\nu)} \frac{\overset{*}{g}[\varphi]}{\varphi} \quad . \qquad (55)$$

Instantaneous Change in Strain Rate

Suppose the strain rate is changed instantaneously at some point $\underset{\sim}{\sigma}^1 \# \underset{\sim}{0}$, $\underset{\sim}{\varepsilon}^1 \# \underset{\sim}{0}$ and corresponding time t_o. Let $\underset{\sim}{\dot{\varepsilon}}^1$ and $\underset{\sim}{\dot{\varepsilon}}^2$ be the strain rates for $t < t_o^-$ and $t > t_o^+$, respectively. Then we derive from (39)

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{km}} \dot{\varepsilon}_{km}^1 \bigg|_{t=t_o^-} - \frac{\partial \sigma_{ij}}{\partial \varepsilon_{km}} \dot{\varepsilon}_{km}^2 \bigg|_{t=t_o^+} = E \frac{\partial \psi_{ij}}{\partial \varepsilon_{km}} (\dot{\varepsilon}_{km}^1 - \dot{\varepsilon}_{km}^2) \quad . \qquad (56)$$

The change in slope is therefore independent of $\underset{\sim}{\sigma}^1$ and $\underset{\sim}{\varepsilon}^{1**}$. As a specific example we consider

$$\underset{\sim}{\dot{\varepsilon}}^2 = b\underset{\sim}{\dot{\varepsilon}}^1 \qquad (57)$$

where b is some constant. Then

** Note that this property is true even if the function k depends upon $\hat{\varphi}, \varphi$.

$$\left\{ \frac{\partial \sigma_{ij}}{\partial \epsilon_{km}}\bigg|_{t=t_o^+} - \left(\frac{1}{b}\frac{\partial \sigma_{ij}}{\partial \epsilon_{km}}\bigg|_{t=t_o^-} + E\frac{\partial \psi_{ij}}{\partial \epsilon_{km}}\left(1 - \frac{1}{b}\right)\right) \right\} \dot{\epsilon}_{km}^1 = 0 . \tag{58}$$

Let us now consider the case where the change in strain rate takes place when (42) and the approximation (44) hold, i.e., if the strain rate is changed in the "plastic region". Under this condition (58) may be rewritten as

$$\left\{ \frac{\partial \sigma_{ij}}{\partial \epsilon_{km}}\bigg|_{t=t_o^+} - \left(E\frac{\partial \psi_{ij}}{\partial \epsilon_{km}}\left(1 - \frac{1}{b}\left(1 - \frac{E_s}{E}\right)\right)\right) \right\} \dot{\epsilon}_{km}^1 = 0 . \tag{59}$$

We assume that $E_s/E \ll 1$ as is true for most metals. Then we see that for $|b| \gg 1$, i.e., <u>very</u> <u>large</u> positive or negative changes in strain rate, the slope at $t = t_o^+$ is approximately elastic. On the other hand if we reverse the strain rate, $b = -1$, then the "slope" at $t = t_o^+$ is approximately twice the elastic "slope"[*].

Creep and Relaxation

Before a creep and relaxation test can be started from some value of the stress and strain tensor we must reach this stress and strain tensor by another test. Let this other test be terminated at time $t = t_o$, and up to this time we impose an arbitrary constant strain rate $\varkappa_{ij}$, so that $\dot{\epsilon}_{ij}\big|_{t \leq t_o} = \varkappa_{ij}$. We assume that $g'[\varphi] \leq E$ so that (B7) is positive. Then from (40)

$$\{\sigma_{ij} - G_{ij}[\underset{\sim}{\epsilon}]\}\varkappa_{ij} \geq 0 \tag{60}$$

and with this result from (39)

$$\left(E\frac{\partial \psi_{ij}}{\partial \epsilon_{km}} - \frac{\partial \sigma_{ij}}{\partial \epsilon_{km}}\right)\varkappa_{ij}\varkappa_{km} \geq 0 . \tag{61}$$

for $t \leq t_o$.

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[*] These properties are shared by the anisotropic model as well; compare (16) with (39). However, in this case we have not shown that (42) holds true at large times.

Because $\underset{\sim}{\mu}$ can be arbitrary Eq. (61) asserts that the "slope" $\dfrac{d\underset{\sim}{\sigma}}{d\underset{\sim}{\epsilon}}$ obtained in a constant strain rate test cannot exceed the "elastic slope". From (60) we deduce that in a tensile or shear test with positive (negative) constant strain rate the corresponding components of $\underset{\sim}{\sigma} - \underset{\sim}{G}$ are positive (negative). Both statements require (B7) to be positive.

At $t = t_o$ the creep or relaxation test commences and from (35) using (37) and (38) for $t > t_o$

$$\dot{\psi}_{ij} = \frac{\sigma_{ij}^o - G_{ij}[\underset{\sim}{\epsilon}]}{Ek[\overline{\Gamma}]} \tag{62}$$

for the case of creep; for the case of relaxation

$$\dot{\sigma}_{ij} = \frac{G_{ij}[\underset{\sim}{\epsilon}^o] - \sigma_{ij}}{k[\overline{\Gamma}]} \tag{63}$$

where the superscript ^{o} denotes the quantity which is kept constant during a specific test.

Using the interpretation of (60) given earlier, we can now state that the axial and shear creep rates following the respective test with constant positive (negative) strain rate are positive (negative). However, the relaxation rates have the opposite sign of the corresponding creep rates. Since k is positive, see Cernocky and Krempl [23], the sign of a particular component of the creep (relaxation) rate is always determined by the sign of the appropriate component of $\{\underset{\sim}{\sigma} - \underset{\sim}{G}\}$. The initial loading determines therefore the sign of a particular component of the creep and relaxation rates. Moreover, the creep (relaxation) rates are zero only if the corresponding components of $\{\underset{\sim}{\sigma} - \underset{\sim}{G}\}$ are zero.

If the invariant φ_1 or any other deviatoric φ together with (B1) is used then $E\underset{\sim}{\psi} - \underset{\sim}{G} = \underset{\sim}{0}$ for a hydrostatic state of strain and $\underset{\sim}{\sigma} = \underset{\sim}{G}$ from (40). For this state of strain and for deviatoric φ there is no creep and no relaxation. However, if a nondeviatoric invariant φ was to be employed instead then creep and relaxation can occur for a hydrostatic state of strain.

4. <u>Discussion</u>

In the preceding the properties of a nonlinear anisotropic or isotropic constitutive equation based on total strain were investigated. It has been shown that this model can represent many qualitative features of metal deformation behavior in a unified way, including

- Initial linear elastic behavior

- Initial elastic response in torsion (tension) after arbitrary prestrain (prestress) in tension (torsion)

- Linear elastic, rate-independent behavior for pure hydrostatic stress (strain)

- Initial elastic "slope" upon <u>large</u> instantaneous changes in strain rate in the "plastic region" under any state of stress

- Strain (stress)-rate sensitivity of the stress-strain curves

- Defined behavior in the limit of very slow and very fast loading rates

- Nearly rate-independent behavior for small strain rates and a proper choice of the material function k

- Stress-strain curves obtained at different constant strain rates will ultimately have the same slope.

- The spacing of the stress-strain diagrams can be highly nonlinear. The stresses at a given strain for stress-strain curves obtained with strain rates differing by several orders of magnitude can be much less than an order of magnitude different.

- Creep and relaxation are included in a natural way

- Relaxation will ultimately terminate, but both primary and secondary creep are possible.

- The creep rates have the same sign as the strain rates used to arrive at the creep stress level. The relaxation rates have the opposite sign.

- Initial creep and relaxation rates in tests started from a point in the "plastic range" of a constant strain-rate tensile test depend only upon the strain rate, and not on the particular values of stress and strain.

In the above we have used stress-strain curves, strain rate, creep, and relaxation rate in a scalar sense. It is implied that the tensor equations of the paper are specialized to suitable homogeneous deformations such as the tensile or shear (torsion) test.

We have kept the equations as simple as possible, and the two remaining coefficient functions in (35) subject to (37) and (38) can in principle be determined from uniaxial tensile tests alone. Here we have assumed that Poisson's ratio is constant. A forthcoming paper will deal with variable Poisson's ratio to remove this restriction. The proper choice of the invariant φ in $\underset{\sim}{G}$ will determine the relation between the axial and shear responses as demonstrated in Fig. 2. The isotropic formulation given in (35) is of course only one of many that can be derived from (1). Equation (1) is itself a very specific choice.

But even the specific choice of (35) offers many possibilities. We have emphasized the application to metals, i.e., conditions (37) and (38) together with a deviatoric φ in $\underset{\sim}{G}$. However, if a nondeviatoric φ is used in $\underset{\sim}{G}$ while keeping (37) and (38) we can model creep and relaxation under pure hydrostatic stress (strain). Replacing (37) by (C1) offers other possibilities. Equation (35) could be applied to materials other than metals.

The nonlinear viscoelastic solid proposed herein is not a valid model for metals if cyclic loadings are involved. Specifically, we contend that (1) or (35) subject to (37) and (38) needs modifications whenever any one tensor component $\{\sigma_{ij} - G_{ij}\}$ changes sign. Equivalently we need modifications when a loading path would penetrate the surface defined by (8). These modifications

will be discussed in a subsequent paper and are stated in principle by

Krempl [21]. Note that the model holds for some nonproportional loading paths,
see specifically Fig.1. (Further it can be seen from (17), (18) or (62), (63)
that a creep or relaxation test does not penetrate the surface defined in (8).)

We may therefore consider a combined creep and relaxation test of the
following character. Through proportional loading we reach a shear stress and
an axial strain which are subsequently kept constant. We have therefore creep
in torsion simultaneously occurring with axial relaxation.

Considering (62) and (63) we see that the two tests influence each other
through the invariant φ in $\underset{\sim}{G}$ and through the invariant Γ in k. If the condi-
tions are such that only primary creep occurs we can compute from (62) and (63)
the final value of the shear creep strain as influenced by the shear creep
stress and the axial relaxation strain since this value does not depend upon
the function k.

Figure 4 shows a graph illustrating this relationship for a particular
choice of $\underset{\sim}{G}$ and the φ of (31). Particular relaxation and creep curves for this
test are given in Figs.5a and b, respectively, for a constant k. These curves
reflect only the influence of the invariant φ. A dependence of k upon Γ would
certainly alter the detailed variation of the curves with time, but would not
influence the qualitative behavior.

No experiments duplicating the above calculations appear to be available
for metals. The trend predicted by our equations has been observed by Lai and
Findlay [28] on polyurethane.

The present model was established as a rate-dependent model. It can
predict almost rate-independent behavior for limited ranges of strain rates
through the function k. If k is small then the exponential term in (40) can
become very small. If in addition the strain rates are small then the integral
in (40) may be small relative to $\underset{\sim}{G}$.

Although our approach differs conceptually from others proposed in the literature, certain common elements can be identified.

Equations (16) and (35), (37) and (38) show that the behavior predicted by these equations for a given stress (strain) rate tensor is determined by the value of $\{\underset{\sim}{\sigma} - \underset{\sim}{G}\}$. Because of this feature our equation is related to the overstress model which was previously proposed by Malvern [29], see also Perzyna [30].

The curve $\underset{\sim}{G}$ can be interpreted as the "equilibrium" stress-strain curve of Eisenberg et al. [14] and (8) could be considered an equilibrium surface. Concentric to this surface are the ones defined in Eq. (19). Equation (42) shows that a constant strain-rate test can ultimately reach $\underset{\sim}{\sigma} - \underset{\sim}{G} = \underset{\sim}{A}$ where $\underset{\sim}{A}$ is constant. On the surfaces $\underset{\sim}{A} = \text{const}$, the inelastic strain rate is constant, see Eq. (14); it is zero for $\underset{\sim}{A} = \underset{\sim}{0}$. Therefore, the surfaces $\underset{\sim}{A} = \text{const}$ could for a given $\underset{\sim}{\varepsilon}$ be interpreted as the Ω-surfaces proposed by Rice [9] and $\underset{\sim}{G}[\underset{\sim}{\varepsilon}]$ for a given $\underset{\sim}{\varepsilon}$ would be identified as the rest stress, see Rice [9]. Further if we consider the approximation of rate independence discussed earlier then the surfaces $\underset{\sim}{A} = \text{const}$ are close together as proposed by Rice, see Fig. 2 in [9]. Equation (18) clearly shows that the creep strain rate is dependent on $\underset{\sim}{\sigma} - \underset{\sim}{G}$ as discussed by Eisenberg et al. [14], p.1249.

The concept of a rest stress or back stress is also employed in the basically rate-dependent formulations of Miller [31] and Krieg et al. [32]. In their approach the inelastic strain rate is zero when the applied stress reaches the rest stress. This property is shared by the present model.

The above shows that our theory contains elements of other approaches. The connecting link is the $\{\underset{\sim}{\sigma} - \underset{\sim}{G}\}$ dependence of our final equations.

This dependence together with the specialization (11a) or (37) assures that the solutions depend on $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ only through $\{\underset{\sim}{\sigma} - \underset{\sim}{G}\}$. As a consequence the solutions have properties representative of actual metal deformation behavior. These properties include:

- Initial elastic response upon departure from $\underset{\sim}{\sigma} = \underset{\sim}{G}[\underset{\sim}{\epsilon}]$, Eqs. (9), (10), (11a) and (39).

- The existence of a "steady-state" condition for constant strain rate, Eq. (41). If in (41) k would depend on φ, $\hat{\varphi}$ instead of Γ then $k[\hat{\varphi}, \varphi]\big|_{t=\infty}$ would have to be constant for $\{\underset{\sim}{\sigma} - \underset{\sim}{G}\}$ to be constant*. In this case the $\{\sigma - G\}$-curves for various constant strain rates would be linearly spaced, see (41) and [23]. A creep or relaxation test started from the steady-state condition would in this case be linear in stress, see (62) and (63). If k is made to depend on Γ then both nonlinear spacing of the stress-strain curves at various constant strain rates results and the creep and relaxation curves originating from the steady-state position of the stress-strain curve depend nonlinearly on stress in accordance with the qualitative behavior of metals.

There are other desirable properties as a consequence of the $\{\underset{\sim}{\sigma} - \underset{\sim}{G}\}$ dependence of the equations. They are discussed in [23] for the uniaxial case and carry over to the multiaxial case. For details in the uniaxial case the reader is referred to Cernocky and Krempl [23] and Liu and Krempl [24].

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* In making this argument we assume $\dfrac{\partial G_{ij}}{\partial \epsilon_{km}}$ to be constant. For realistic cases $\left\|\dfrac{\partial G_{ij}}{\partial \epsilon_{km}}\right\| \ll E$ so that small changes of $\dfrac{\partial G_{ij}}{\partial \epsilon_{km}}$ with $\underset{\sim}{\epsilon}$ have little effect on $\underset{\sim}{\sigma} - \underset{\sim}{G}$.

Acknowledgement

TABLE I

INVARIANTS φ and $\hat{\varphi}$* **

Invariant Subscript	φ	$\hat{\varphi}$
1	$\dfrac{(1.5\, e_{ij}e_{ij})^{1/2}}{(1+\nu)}$	$(1.5\, s_{ij}s_{ij})^{1/2}$
2	$(\psi_{ij}\psi_{ij})^{1/2}$	$(\sigma_{ij}\sigma_{ij})^{1/2}$
3	$\dfrac{(1.5\, e_{ij}e_{ij}\psi_{ab}\psi_{ab})^{1/4}}{(1+\nu)^{1/2}}$	$(1.5\, s_{ij}s_{ij}\sigma_{ab}\sigma_{ab})^{1/4}$
4	$\left(\dfrac{\epsilon_{ij}\epsilon_{ij}}{(1+2\nu^2)}\right)^{1/2}$	$\left(\dfrac{(1+\nu)^2 s_{ij}s_{ij}+\frac{1}{3}(1-2\nu)^2(\sigma_{kk})^2}{1+2\nu^2}\right)^{1/2}$
5	$\left[\dfrac{(1.5\, e_{ij}e_{ij})+\beta(e_{jj})^2}{(1+\nu)^2+\beta(1-2\nu)^2}\right]^{1/2}$ $\beta \geq 0$	$\left[\dfrac{(1.5\, s_{ij}s_{ij})(1+\nu)^2+\beta(1-2\nu)^2(\sigma_{jj})^2}{(1+\nu)^2+\beta(1-2\nu)^2}\right]^{1/2}$

* From (B1) and $\underset{\sim}{\sigma}=\underset{\sim}{G}[\underset{\sim}{\epsilon}]$ we get $\hat{\varphi}_i = g[\varphi_i]$ $i = 1 \ldots 5$ and this represents an "effective" stress-strain diagram.

** Only positive roots are intended.

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FIGURE CAPTIONS

Figure 1   Torsional response for various constant axial prestrain values
              at a shear strain rate of 10^{-4} s^{-1}. The pure shear and the
              pure tensile stress-strain curves at the same strain rates are
              also shown. Note the <u>initial</u> linear elastic shear response
              is independent of the <u>tensile</u> prestrain (prestrain values
              greater than .04 are only for the illustration of the mathe-
              matical properties of the solutions). $\nu = .3$.

Figure 2   The influence of the choice of the invariant φ upon the pure
              shear response in the limit of very slow loading, see Eq. (52).
              The tensile curve is independent of the choice of φ. In all
              the shear curves $\nu = .3$.

Figure 3   Uniaxial stress-strain curves at various strain rates, Fig.3a,
              and shear stress vs. <u>engineering</u> shear strain ($2\varepsilon_{12}$) curves

              at the same rates, Fig.3b. The solutions correspond to the
              integration of (45) and (47a). Note the nonlinear spacing of
              the curves which is due to the dependence of k upon Γ.

Figure 4   Simultaneous axial relaxation and shear creep. Final value of
              total shear strain in primary creep at various constant shear
              stress values plotted vs axial constant prestrain for a
              specific g-function and $\varphi = \varphi_1$; $\nu = .3$.

Figure 5a   Axial relaxation curves with shear creep at constant shear
              stress occurring simultaneously. Material properties corre-
              spond to Fig.4. k = 1 hr.

Figure 5b   Complement to Fig.5a. Total shear strain curves in shear creep
              with axial relaxation at constant axial strain occurring
              simultaneously. Material properties are those used in Figs.4
              and 5a.

STRESS $*10^3$ PSI

STRAIN $*10^{-1}$

Pure Tensile Curve

Shear Curves

Pure Shear Curve

Constant Axial
Prestrain

.008
.016
.024

.040

.100
.250
.400
∞

$\varphi = \varphi_1$; $k = 16$ s; strain rate $= 10^{-4}$ s^{-1}

1 psi $\approx 6.894 \cdot 10^3$ Pa

Fig.1

PURE SHEAR CURVES A-F CORRESPOND TO DIFFERENT

INVARIANTS φ FROM TABLE I

CURVE: A – φ_5 with β = 19

         B – φ_2

         C – φ_5 with β = 3.62

         D – φ_3

         E – φ_4

         F – φ_1

Tension

A

B C D E F

1 psi ≈ 6.894 · 10^3 Pa

STRAIN *10^{-1}

STRESS *10^3 PSI

0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14 0.16 0.18

0.00 10.00 20.00 30.00 40.00 50.00 60.00

Fig.2

AXIAL STRESS (MPA)

AXIAL STRESS (KSI)

AXIAL STRAIN (%)

1     $g[\epsilon]$

2     $\dot{\epsilon} = 10^{-5} \ s^{-1}$

3     $\dot{\epsilon} = 10^{-3} \ s^{-1}$

4     $\dot{\epsilon} = 10^{-1} \ s^{-1}$

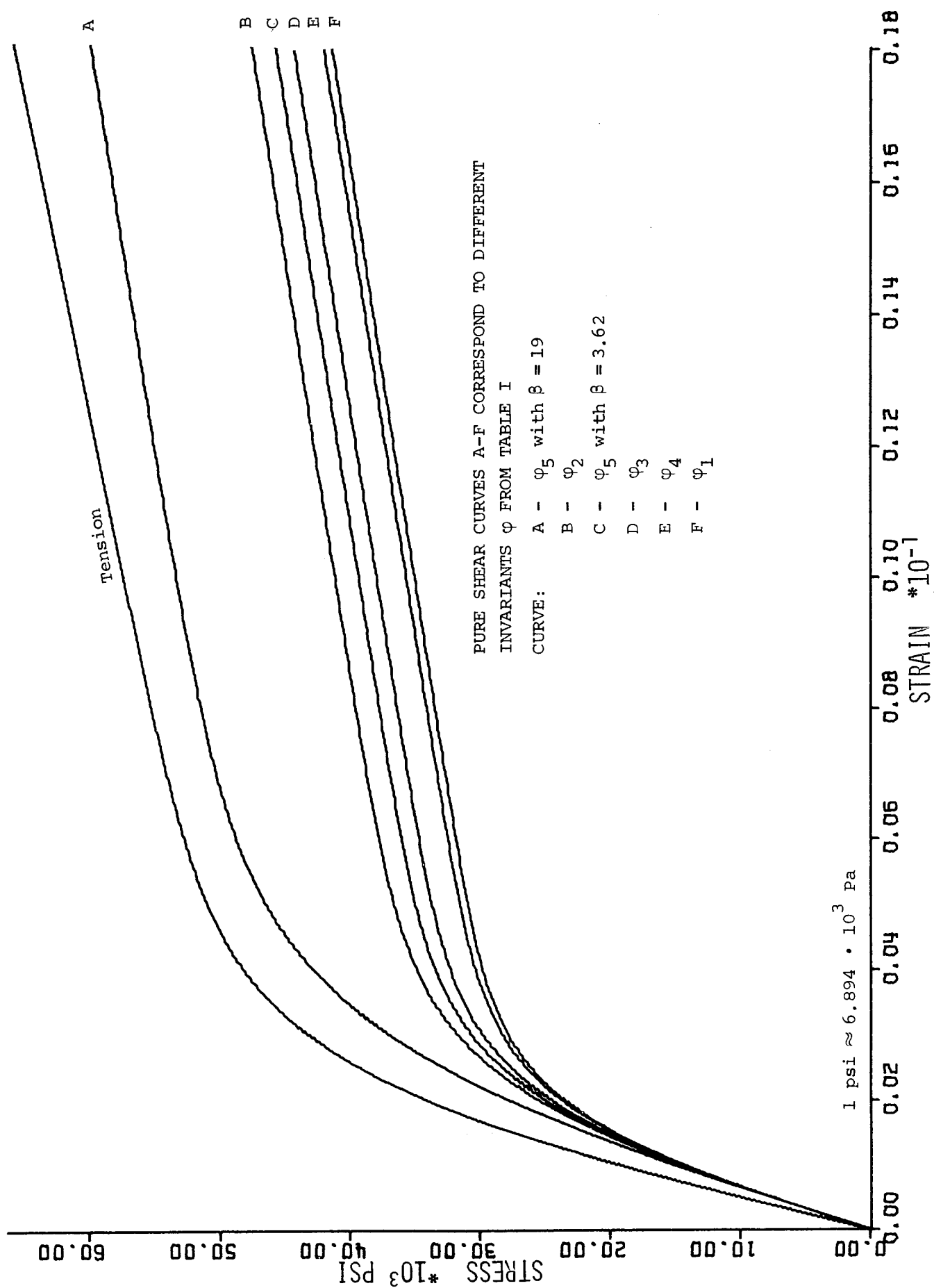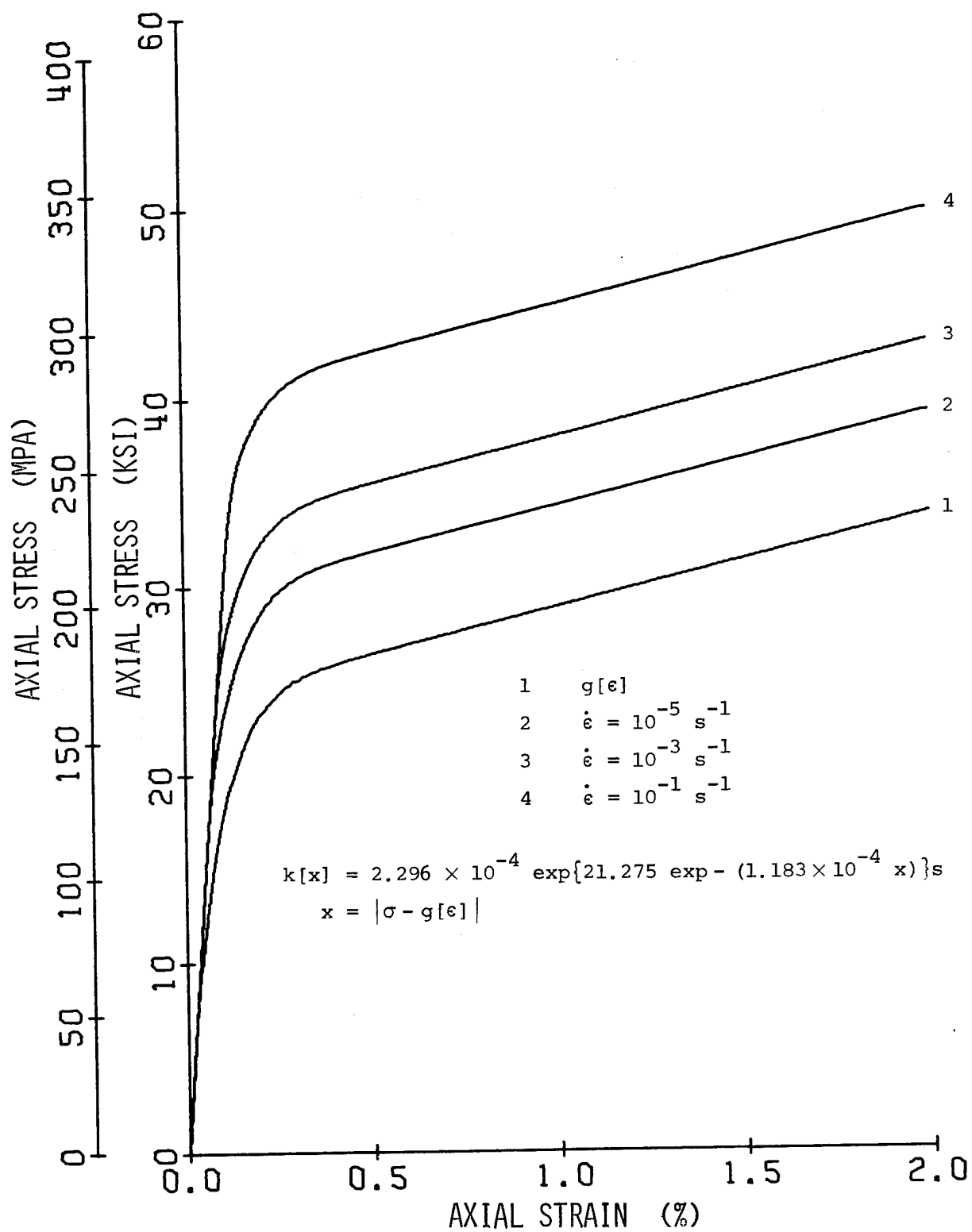$$k[x] = 2.296 \times 10^{-4} \ \exp\{21.275 \ \exp-(1.183 \times 10^{-4} \ x)\}s$$

$$x = |\sigma - g[\epsilon]|$$

Fig.3a

Fig. 3b

Fig.4

AXIAL STRESS $*10^3$ PSI

$\epsilon_{11} = .002 \quad \sigma_{12} = 0$

$\epsilon_{11} = .004 \quad \sigma_{12} = 15$ ksi

$\epsilon_{11} = .004 \quad \sigma_{12} = 18$ ksi

$\epsilon_{11} = .002 \quad \sigma_{12} = 15$ ksi

1 psi $\approx 6.894 \cdot 10^3$ Pa

TIME   HRS

Fig.5a

SHEAR STRAIN *10^{-1}

$\epsilon_{11} = .004 \quad \sigma_{12} = 18$ ksi

$\epsilon_{11} = .004 \quad \sigma_{12} = 15$ ksi

$\epsilon_{11} = .002 \quad \sigma_{12} = 15$ ksi

$\epsilon_{11} = 0 \quad \sigma_{12} = 15$ ksi
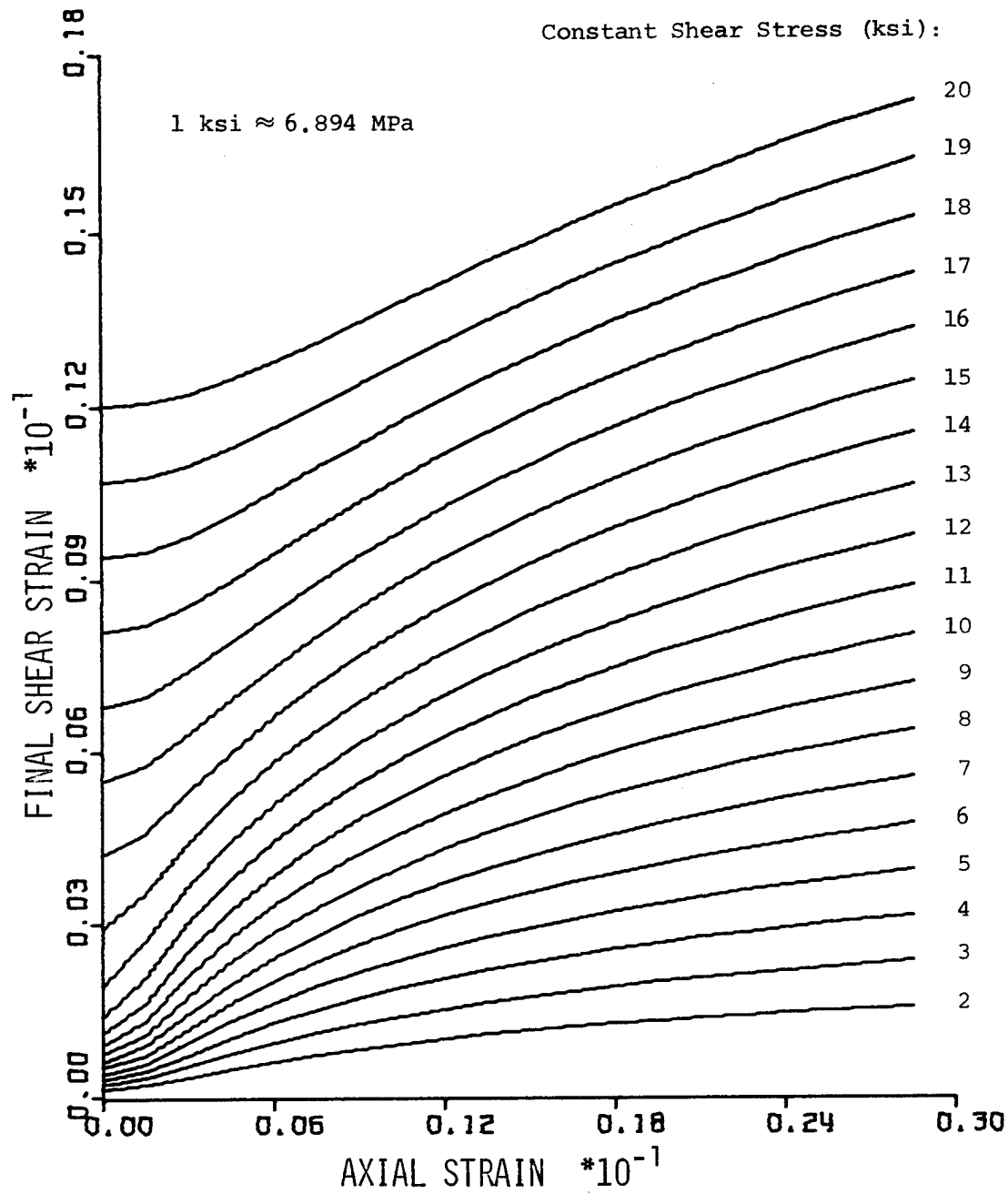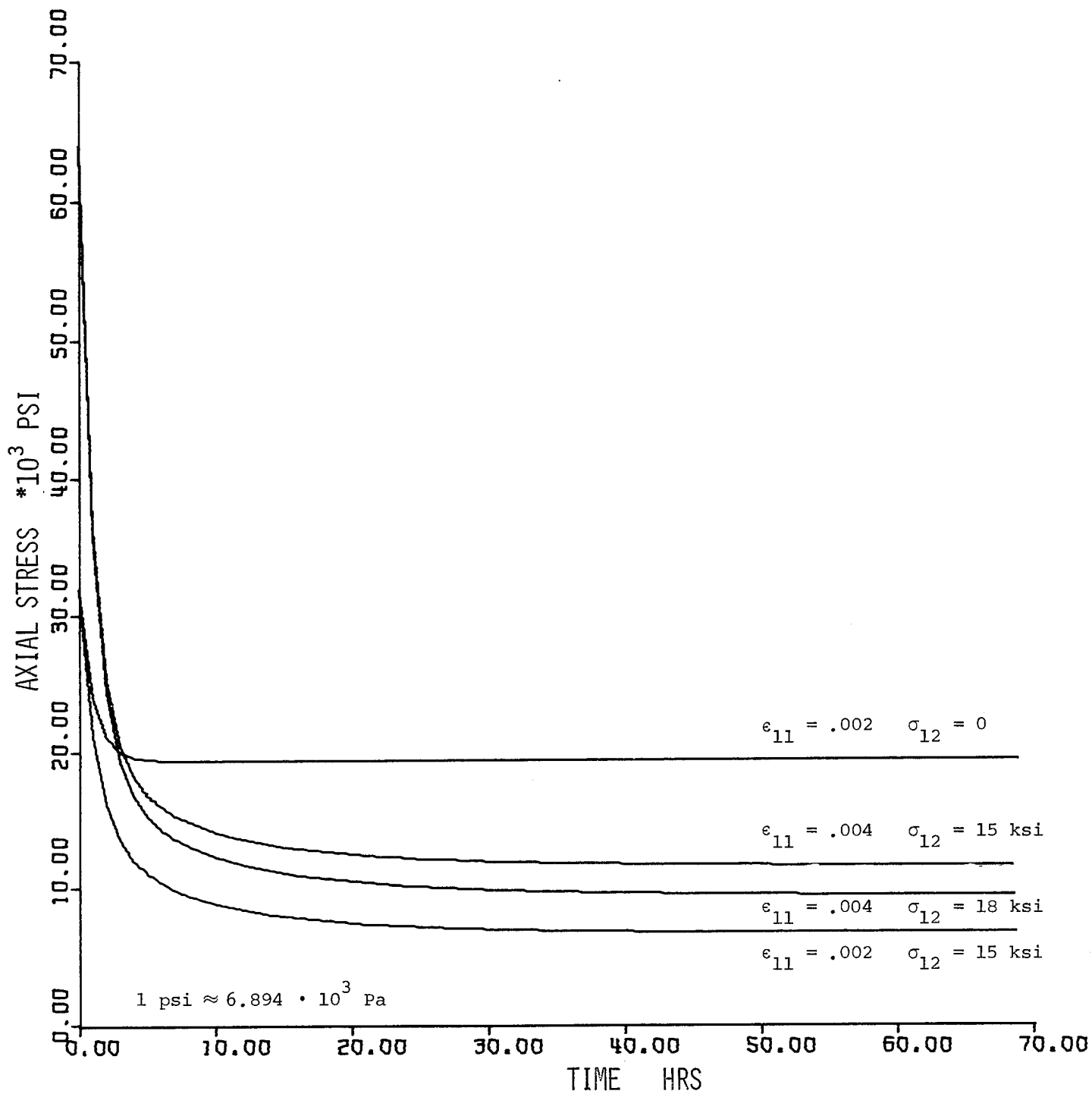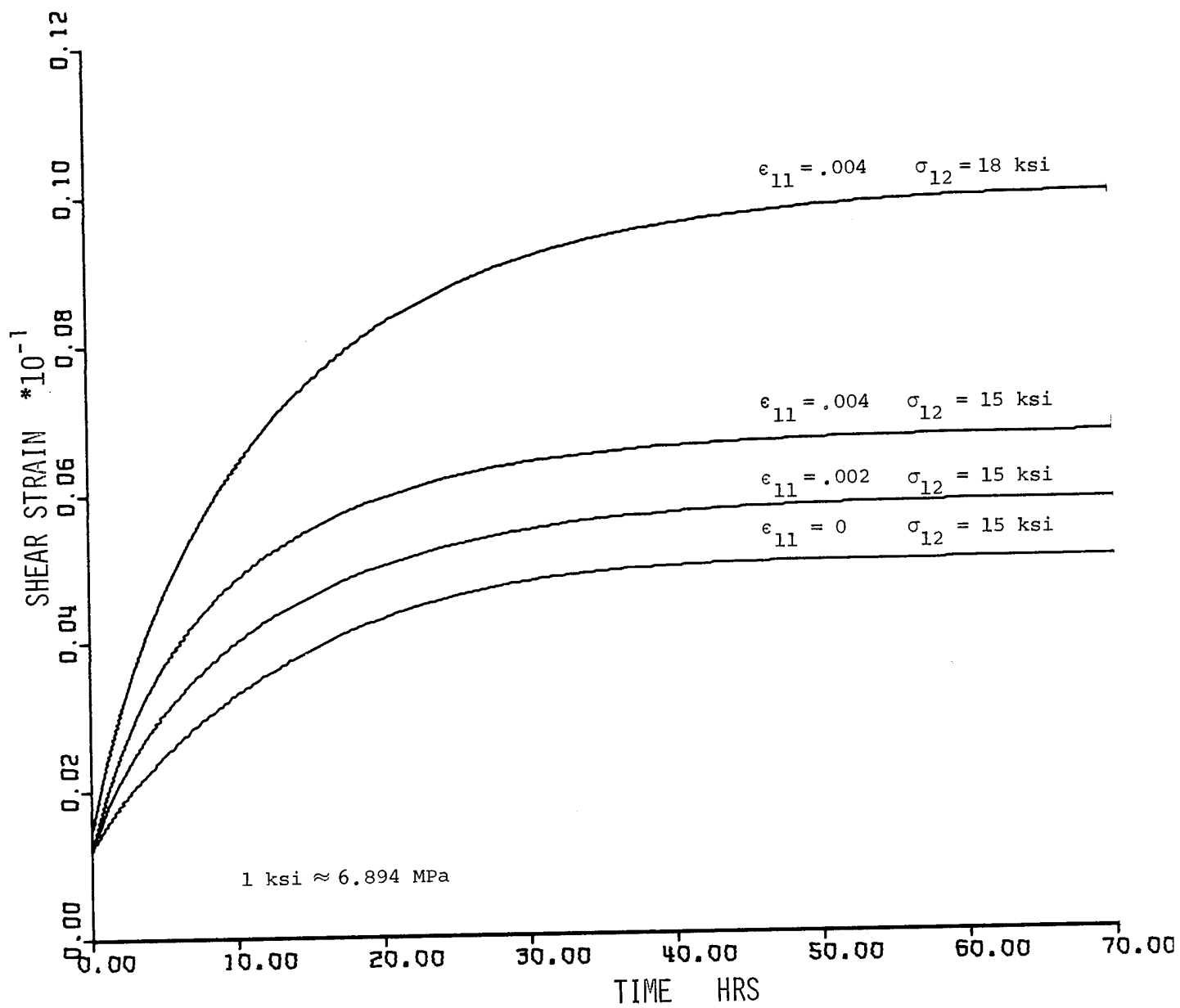
1 ksi $\approx$ 6.894 MPa

TIME   HRS

Fig.5b

APPENDIX I

TORSIONAL RESPONSE IN THE PRESENCE OF AXIAL PRESTRESS. THE PREDICTION OF (1)

For the thin-walled tube usually employed in plasticity experiments an axial prestress

$$[\underset{\sim}{\sigma}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A1}$$

results in a strain tensor

$$[\underset{\sim}{\epsilon}] = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & \epsilon_{23} \\ 0 & \epsilon_{23} & \epsilon_{33} \end{bmatrix}. \tag{A2}$$

In the case of isotropy $\epsilon_{23} = 0$ and $\epsilon_{22} = \epsilon_{33}$. The strain matrix (A2) can be arrived at by purely kinematical consideration and by assuming that the state of stress and the state of deformation are homogeneous.

We have to assume that $\underset{\sim}{G}$, $\underset{\approx}{M}$ and $\underset{\approx}{K}$ in (1) are constructed in such a way that (A1) gives rise to (A2). Now let $\underset{\sim}{\sigma}$ in (A1) be constant for all $t \geq t_o > 0$.

At time $t = t_o$ the stress increment

$$[d\underset{\sim}{\sigma}] = \begin{bmatrix} 0 & d\sigma_{12} & d\sigma_{13} \\ d\sigma_{12} & 0 & 0 \\ d\sigma_{13} & 0 & 0 \end{bmatrix} \tag{A3}$$

with $d\sigma_{12} = d\sigma_{13}$ is imposed which can result in the strain increment (depending upon the material symmetries some of the $d\underset{\sim}{\epsilon}$ components may be zero; in the case of isotropy $d\epsilon_{23} = 0$)

$$[d\underset{\sim}{\epsilon}] = \begin{bmatrix} d\epsilon_{11} & d\epsilon_{12} & d\epsilon_{13} \\ d\epsilon_{12} & d\epsilon_{22} & d\epsilon_{23} \\ d\epsilon_{13} & d\epsilon_{23} & d\epsilon_{33} \end{bmatrix} \tag{A4}$$

The components of the tensors in (A1) - (A4) are referred to a rectangular co-ordinate system with the e_1-direction along the axis of the usually employed thin-walled tube.

To obtain useful results we have to restrict $\underset{\approx}{M}$, $\underset{\approx}{K}$ and $\dfrac{d\underset{\sim}{\sigma}}{d\underset{\sim}{\epsilon}}$. Specifically the components of the above tensors with an index appearing only once or with two identical indices and the other two indices different will be set equal to zero[*]. We then obtain from (9)

$$M_{1111}\dot{\epsilon}_{11} + M_{1122}\dot{\epsilon}_{22} + M_{1133}\dot{\epsilon}_{33} = \sigma_{11} - G_{11}$$

$$\left.\begin{aligned} M_{2211}\dot{\epsilon}_{11} + M_{2222}\dot{\epsilon}_{22} + M_{2233}\dot{\epsilon}_{33} = 0 \\ \\ M_{3311}\dot{\epsilon}_{11} + M_{3322}\dot{\epsilon}_{22} + M_{3333}\dot{\epsilon}_{33} = 0 \end{aligned}\right\} \qquad \text{(A5-1)}$$

$$\dot{\epsilon}_{23} = 0 \qquad \text{(A5-2)}$$

$$\left(M_{1212} - 2K_{1212}\frac{\partial\sigma_{12}}{\partial\epsilon_{12}}\right)\dot{\epsilon}_{12} = 0 \qquad \text{(A5-3)}$$

$$\left(M_{1313} - 2K_{1313}\frac{\partial\sigma_{13}}{\partial\epsilon_{13}}\right)\dot{\epsilon}_{13} = 0 . \qquad \text{(A5-4)}$$

Because of the fact that $\dot{\epsilon}_{12}$ and $\dot{\epsilon}_{13}$ are arbitrary in (A5-3) and (A5-4), respectively we see that the shear response is elastic provided (11a) holds. Equations (A5-1) are constraint equations which must and can be satisfied.

The material symmetries to which we have restricted ourselves include the usual cases of isotropy, transverse isotropy and orthotropy. These are material symmetries that are pertinent to the thin-walled tube tests of plasticity.

From the above we see that (1) together with (11a) and the imposed sym-metry restrictions reproduces the initial elastic shear response under a tensile prestress observed in many experiments. We note that while the initial shear

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[*] We are therefore not considering the most general case.

response remains unaltered by the presence of large axial prestress (prestrain), as shown in Fig.1 the <u>subsequent</u> nonlinear shear response is affected by the magnitude of the prestress (prestrain) value.

Isotropic Case

From (39) we obtain for $d\epsilon_{23} = 0$ and $\epsilon_{23} = 0$

$$\left(E \frac{\partial \psi_{ij}}{\partial \epsilon_{11}} - \frac{\partial \sigma_{ij}}{\partial \epsilon_{11}}\right)\dot{\epsilon}_{11} + 2\left(E \frac{\partial \psi_{ij}}{\partial \epsilon_{12}} - \frac{\partial \sigma_{ij}}{\partial \epsilon_{12}}\right)\dot{\epsilon}_{12} +$$

$$2\left(E \frac{\partial \psi_{ij}}{\partial \epsilon_{13}} - \frac{\partial \sigma_{ij}}{\partial \epsilon_{13}}\right)\dot{\epsilon}_{13} = \frac{\sigma_{ij} - G_{ij}[\underset{\sim}{\epsilon}]}{k[\Gamma]} \quad . \tag{A6}$$

From (A6) using (A1), (A3), (B1) and (B5)

$$\dot{\epsilon}_{11} = \frac{\sigma_{11} - G_{11}[\underset{\sim}{\epsilon}]}{Ek[\Gamma]} \tag{A7}$$

as well as

$$- \frac{\partial \sigma_{12}}{\partial \epsilon_{11}} \dot{\epsilon}_{11} + 2\left(\frac{E}{2(1+\nu)} - \frac{\partial \sigma_{12}}{\partial \epsilon_{12}}\right) \dot{\epsilon}_{12} = 0 \tag{A8}$$

where we have used $d\sigma_{12} = d\sigma_{13}$, $\dot{\epsilon}_{12} = \dot{\epsilon}_{13}$ and $\frac{\partial \sigma_{12}}{\partial \epsilon_{12}} = \frac{\partial \sigma_{13}}{\partial \epsilon_{13}}$. Since $\dot{\epsilon}_{12}$ can be arbitrarily imposed and since $\dot{\epsilon}_{11}$ is given by (A7) we conclude from (A7) and (A8)

$$\frac{\partial \sigma_{12}}{\partial \epsilon_{12}} = \frac{\partial \sigma_{13}}{\partial \epsilon_{13}} = \frac{E}{2(1+\nu)} \tag{A9}$$

and

$$\frac{\partial \sigma_{13}}{\partial \epsilon_{11}} = \frac{\partial \sigma_{12}}{\partial \epsilon_{11}} = 0 . \tag{A10}$$

Note that no restrictions were put on $\frac{\partial \underset{\sim}{\sigma}}{\partial \underset{\sim}{\epsilon}}$ in the isotropic case. The derivation for torsional prestress and subsequent axial loading is similar with analogous results.

APPENDIX II

CONSTRUCTION OF ISOTROPIC G_{ij} FUNCTIONS AND INVARIANTS

When $\epsilon_{ij} = \epsilon_{ji} \neq 0$ and all other ϵ_{ab} are zero, the component functions $G_{ij} = G_{ji}$ are required to have the general form of a stress-strain curve, and all other G_{ab} must equal zero. Further, $\underset{\sim}{G}$ must be isotropic and must reduce to the linear isotropic elastic relationship in strain for an initial interval of strain. Also we require that $\underset{\sim}{G}$ reduce appropriately, Eq. (26), to the uniaxial case when a uniaxial deformation field, Eq. (25), is imposed. Consequently, an acceptable representation of G_{ij} is

$$G_{ij}[\underset{\sim}{\epsilon}] = \psi_{ij} \frac{g[\varphi]}{\varphi}$$

with

$$\psi_{ij} = \frac{1}{(1+\nu)} \left(\epsilon_{ij} + \frac{\nu}{(1-2\nu)} \epsilon_{kk} \delta_{ij} \right)$$

$\Bigg\}$ (B1)

and for $\varphi < \overline{\xi}$ where $\overline{\xi}$ is some appropriate number $\ll 1$

$$\left. \frac{g[\varphi]}{\varphi} \right|_{\varphi \leq \overline{\xi}} \approx g'[0] .$$ (B2)

Usually $g'[0] = E$, the modulus of elasticity. The function $g[x]$ represents the uniaxial stress-strain curve for the axial strain x in the limit of very slow loading. For its idealized mathematical representations the methods proposed by Liu et al. [26] or Cernocky and Krempl [27] may be used. In these two cases $g[x]$ is analytic, monotonic, and is $\mathcal{O}[x]^{*}$ both as $x \to 0$ and as $x \to \infty$.

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* $\mathcal{O}[x]$ denotes the Landau order symbol; see p.128 of R.G. Bartle, The Elements of Real Analysis, Wiley 1976.

Various constructions of the invariant φ and its partner invariant $\hat{\varphi}$ appear in Table I. Certainly an infinite variety of invariant constructions is possible. We note that two of the representation pairs of $(\varphi, \hat{\varphi})$, that is $(\varphi_1, \hat{\varphi}_1)$ and $(\varphi_3, \hat{\varphi}_3)$, are deviatoric; they vanish under a hydrostatic strain and stress field, respectively. All φ and $\hat{\varphi}$ in Table I reduce to the uniaxial strain and stress, respectively, in the uniaxial deformation field. In addition, all of the listed φ satisfy the differential equation

$$\frac{\partial \varphi}{\partial \epsilon_{ij}} \epsilon_{ij} = \varphi. \tag{B3}$$

Differentiation of G_{ij} with respect to strain yields

$$\frac{\partial G_{ij}}{\partial \epsilon_{ab}} = \frac{g[\varphi]}{\varphi} \frac{\partial \psi_{ij}}{\partial \epsilon_{ab}} + \psi_{ij} \frac{\varphi g'[\varphi] - g[\varphi]}{\varphi^2} \frac{\partial \varphi}{\partial \epsilon_{ab}}$$

and

$$\left. \frac{\partial G_{ij}}{\partial \epsilon_{ab}} \right|_{\underset{\sim}{\epsilon}=\underset{\sim}{0}} = g'[0] \frac{\partial \psi_{ij}}{\partial \epsilon_{ab}}$$

$\left.\right\}$ (B4)

where

$$\frac{\partial \psi_{ij}}{\partial \epsilon_{ab}} = \frac{1}{(1+\nu)} \left(\frac{1}{2} (\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}) + \frac{\nu}{(1-2\nu)} \delta_{ij}\delta_{ab} \right) \tag{B5}$$

so that we obtain $\frac{\partial \psi_{ij}}{\partial \epsilon_{ab}} \epsilon_{ab} = \psi_{ij}$. Then for the strain invariants which satisfy (B3) we obtain the condition

$$\frac{\partial G_{ij}}{\partial \epsilon_{ab}} \epsilon_{ij}\epsilon_{ab} = \frac{g'[\varphi]}{(1+\nu)} (\epsilon_{ij}\epsilon_{ij} + \frac{\nu}{(1-2\nu)} (\epsilon_{jj})^2) \geq 0 \tag{B6}$$

provided $g'[\varphi] \geq 0$ which is usually the case for metals. Also from (B3) - (B6) we obtain

$$\left(E \frac{\partial \psi_{ij}}{\partial \epsilon_{ab}} - \frac{\partial G_{ij}}{\partial \epsilon_{ab}} \right) \epsilon_{ij}\epsilon_{ab} = (E - g'[\varphi]) \frac{1}{(1+\nu)} \left(\epsilon_{ij}\epsilon_{ij} + \frac{\nu}{(1-2\nu)} (\epsilon_{jj})^2 \right) \geq 0 \tag{B7}$$

provided $E \geq g'[\varphi]$.

If the Y[X] functions of Cernocky and Krempl [27] are used for the representation of g[φ] then the following approximate expressions result for $\varphi > X_f$ for the second kernel form and for appropriately selected constants

$$G_{ij}[\epsilon] \approx \psi_{ij}\left(E_s + \frac{1}{\varphi}\ \frac{(E - E_s)(RX_f - \lambda_o)}{RF[RX_f - \lambda_o]}\right) \tag{B8}$$

and

$$\frac{\partial G_{ij}}{\partial \epsilon_{ab}} \approx E_s\ \frac{\partial \psi_{ij}}{\partial \epsilon_{ab}} \tag{B9}$$

where

E - elastic modulus

E_s - tangent modulus in the "plastic range"

R, X_f, λ_o - parameters of the uniaxial stress-strain curve as defined in [27]

F[x] - any of the base functions listed in [27].

The selection of the Y[X]-functions for the representation of g[φ] requires that the actual uniaxial stress-strain diagram can be approximated by an initial linear range, followed by a strong nonlinear monotonic curvature which terminates with another linear region. Although we have found these representations for g[φ] very useful, the theory presented herein is valid for any suitable g-function.

We note that the particular choice of the suitable invariant φ does not influence the character of the constitutive equations governing uniaxial deformation; this is because in the uniaxial field φ is made to reduce to the axial strain. However, the selection of a particular invariant is a constitutive assumption, and the particular construction of φ does influence the nature of all deformations other than those of the tensile test. Specifically, in a pure torsion deformation, the shear stress- shear strain curve will differ from a pure tensile stress- axial strain curve at corresponding strain values, and this difference will depend upon the particular choice of φ. This is clear in

Fig. 2 in which we have plotted the solution for a pure tensile loading at limitingly small strain rate and then have plotted pure shear deformation curves on the same strain scale and at the same limiting loading rate for each of the five invariants φ in Table I. The influence of the choice of φ upon the shear stress-strain diagram is apparent. The results shown in Fig. 2 apply similarly for other and more complicated deformations. Consequently, full understanding of the uniaxial and shear deformation characters of the material will strongly motivate the construction of appropriate invariants φ.

We note that if the φ which appears in the hydrostatic equation (34) is deviatoric, then when (37) applies, Eq. (35) predicts linear elastic rate-independent response to a pure hydrostatic loading. This prediction is deemed suitable for metals. Conversely if φ is not deviatoric in (34) then the theory predicts a nonlinear rate-dependent response under pure hydrostatic loading.

The deviatoric and hydrostatic relations (33) and (34) are combined into (35) because the same stress and strain invariants have been used in both of (33) and (34). We may however elect to use different constructions of φ in the respective deviatoric and hydrostatic constitutive equations. This increases the capability of the model. For example, a deviatoric φ may be used in Eq. (34), while a nondeviatoric φ may be used in the deviatoric relation (33). We recall $\hat{\varphi}$ is determined according to the condition $\hat{\varphi} = g[\varphi]$ if $\underset{\sim}{\sigma} = \underset{\sim}{G}$.

APPENDIX III

Instead of postulating $\dfrac{m[\;]}{k[\;]} = E$ as in (38) we set

$$\frac{m[\;]}{k[\;]} = \overset{*}{g}{}'[\varphi] = \frac{d\overset{*}{g}[\varphi]}{d\varphi} \quad \text{and} \quad \overset{*}{g}{}'[0] = E \qquad (C1)$$

where φ is a suitable invariant of the strain tensor. The assertions given below can be verified if E is replaced by $\overset{*}{g}{}'[\varphi]$ in the appropriate equations.

Equation (C1) has the following consequences:

- The "elastic strain rate" is no longer linear, i.e., from (33) and (34), respectively,

$$\dot{e}_{ij} = \left(\frac{s_{ij} - G^{d}_{ij}}{m[\;]}\right)(1+\nu) + \frac{\dot{s}_{ij}(1+\nu)}{\overset{*}{g}{}'[\varphi]} \qquad (C2)$$

$$\dot{\epsilon}_{aa} = \left(\frac{\sigma_{aa} - G_{aa}}{m[\;]}\right)(1-2\nu) + \frac{\dot{\sigma}_{aa}(1-2\nu)}{\overset{*}{g}{}'[\varphi]} \qquad (C3)$$

- If $g'[\varphi] > \overset{*}{g}{}'[\varphi]$ then (60), (61) change sign, see Cernocky and Krempl [23]

- Initial linear elastic response in shear may not result in the presence of axial prestress, see Eqs.(A8) and (A9)

- The spacing of stress-strain curves obtained at various strain rates is strongly affected by $\overset{*}{g}$, see (41) and (43). Note that (44) still holds; see [23].

- The change in slope upon an instantaneous strain-rate change, Eqs.(56) – (59) is no longer independent of $\underset{\sim}{\epsilon}$

- The creep rate depends not only on $\sigma^{o}_{ij} - G_{ij}[\underset{\sim}{\epsilon}]$ but also on $\overset{*}{g}{}'[\varphi]$, Eq.(62).

- In the limit of very fast strain rates a nonlinear stress-strain curve may be obtained, see Eq.(55).

- For a hydrostatic state of strain the stress response may become nonlinear and rate-dependent under suitable selections of the invariants in $\overset{*}{g}$ and $\underset{\sim}{G}$. If a nondeviatoric invariant is used for $\overset{*}{g}$ then the stress response to a hydrostatic strain may be nonlinear and rate dependent for nonzero strain rates. In the limit of very slow loading, however, the response is linear if a deviatoric invariant is used in $\underset{\sim}{G}$.

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1. REPORT NUMBER RPI CS 78-3	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* A Theory of Viscoplasticity Based on Infinitesimal Total Strain		5. TYPE OF REPORT & PERIOD COVERED Topical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR*(s)* E.P. Cernocky and E. Krempl		8. CONTRACT OR GRANT NUMBER*(s)* N00014-76-C-0231
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mechanical Engineering, Aeronautical Engineering & Mechanics Rensselaer Polytechnic Institute, Troy, NY 12181		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 064-571
11. CONTROLLING OFFICE NAME AND ADDRESS Dept. of the Navy, Office of Naval Research Structural Mechanics Program Arlington, VA 22217		12. REPORT DATE May 1978
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)* Office of Naval Research - Resident Representative 715 Broadway - 5th Floor New York, NY 10003		15. SECURITY CLASS. *(of this report)*
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

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18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Viscoplasticity, creep, relaxation, rate-effects, constitutive equation

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A viscoplasticity theory based upon a nonlinear viscoelastic solid, linear in the rates of the strain and stress tensors but nonlinear in the stress tensor and the infinitesimal strain tensor, is being investigated for isothermal, homogeneous motions. A general anisotropic form and a specific isotropic formulation are proposed. A yield condition is not part of the theory and the transition from linear (elastic) to nonlinear (inelastic) behavior is continuous. Only total strains are used and the constant volume hypothesis is not

employed. In this paper Poisson's ratio is assumed to be constant. The proposed equation can represent: initial linear elastic behavior; initial elastic response in torsion (tension) after arbitrary prestrain (prestress) in tension (torsion); linear elastic behavior for pure hydrostatic loading; initial elastic slope upon large instantaneous changes in strain rate; stress (strain)-rate sensitivity; creep and relaxation; defined behavior in the limit of very slow and very fast loading. Stress-strain curves obtained at different loading rates will ultimately have the same "slope" and their spacing is nonlinearly related to the loading rate.

The above properties of the equation are obtained by qualitative arguments based on the characteristics of the solutions of the resulting nonlinear first-order differential equations. In some instances numerical examples are given.

For metals and isotropy we propose a simple equation whose coefficient functions can be determined from a tensile test [Eqs. (31), (35), (37), (38)]. Specializations suitable for materials other than metals are possible.

The paper shows that this nonlinear viscoelastic model can represent essential features of metal deformation behavior and reaffirms our previous assertion that metal deformation is basically rate-dependent and can be represented by piecewise nonlinear viscoelasticity. For cyclic loading the proposed model must be modified to account for history dependence in the sense of plasticity.