MASTER COPY , KEEP THIS COPY FOR REPRODUCTION PURPOSES

# REPORT DOCUMENTATION PAGE

*Form Approved*
*OMB No. 0704-0188*



**1. AGENCY USE ONLY** *(Leave blank)*

**2. REPORT DATE**
December 12, 1995

**3. REPORT TYPE AND DATES COVERED**
Final

**4. TITLE AND SUBTITLE**
Analysis and Optimization of Elastic Materials

**5. FUNDING NUMBERS**
DAAL03-89-G-0082

**6. AUTHOR(S)**
Rouben Rostamian
William Hager

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**
Department of Mathematics and Statistics
University of Maryland Baltimore County
Baltimore, MD 21228-5398

**8. PERFORMING ORGANIZATION REPORT NUMBER**

**9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)**
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

**10. SPONSORING/MONITORING AGENCY REPORT NUMBER**
ARO 27022.12-MA-SM

**11. SUPPLEMENTARY NOTES**
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

**12a. DISTRIBUTION/AVAILABILITY STATEMENT**
Approved for public release; distribution unlimited.

**12b. DISTRIBUTION CODE**

**13. ABSTRACT** *(Maximum 200 words)*

The objective of this research is to investigate the propagation of waves in stratified elastic media. The focus is on the design of elastic coatings which due to their layered nature deflect the incident energy of the waves and therefore can affect the reflectivity properties of solid surfaces in interesting ways.

19960209 105

**14. SUBJECT TERMS**
Stratified elastic materials, wave propagation, Optimal design, non-reflective coatings.

**15. NUMBER OF PAGES**
13

**16. PRICE CODE**

| 17. SECURITY CLASSIFICATION OF REPORT | 18. SECURITY CLASSIFICATION OF THIS PAGE | 19. SECURITY CLASSIFICATION OF ABSTRACT | 20. LIMITATION OF ABSTRACT |
|---|---|---|---|
| UNCLASSIFIED | UNCLASSIFIED | UNCLASSIFIED | UL |

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

# Analysis and Optimization of Elastic Materials

## Final Report

by

Rouben Rostamian and William W. Hager

December 10, 1995

US Army Research Office

Contract Number DAAL03–89–G–0082

Department of Mathematics and Statistics
University of Maryland Baltimore County
Baltimore, Maryland 21228

# 1 Foreword

This is the final report on the ARO research contract DAAL03–89–G–0082-DAAL03–89–G–0082 issued to the University of Maryland Baltimore County on 9/1/89 for an initial period of 3 years. Upon the request of one of the PIs (Rouben Rostamian) and approval by the program director, the length of the contract was extended by a period of two years. This was to accommodate Rostamian's leave of absence from the university for a period of 26 months, where he served as the program director of the Applied Mathematics Program at the National Science Foundation. The research was resumed and brought to a completion upon his return from NSF.

# 2 Project description

The general objective of the research was to investigate the propagation of waves in stratified elastic media. Of particular interest were problem on the design of elastic coatings which due to their layered nature deflect the incident energy of the waves and therefore can affect the reflectivity properties of solid surfaces in interesting ways.

# 3 Summary of the most important results

Based on some preliminary investigations, we made the following statement in the original proposal submitted to ARO (page 1):

> Although no one coating will eliminate all reflected or transmitted waves, we can optimize a coating by minimizing the maximum energy of a reflection or transmission coefficient over a range of frequencies.

Surprisingly, it turned out that we can do much better than what we had hoped for. The major result of our investigation is that it is possible to eliminate reflected waves of a prescribed bandwidth from a solid object completely. (See section 4 of this report.)

We have also investigated the effects of air bubbles embedded in an absorbant coating layer affixed to a solid surface. We have developed techniques and implemented algorithms for predicting the strength per unit weight of a bubbly elastic material.

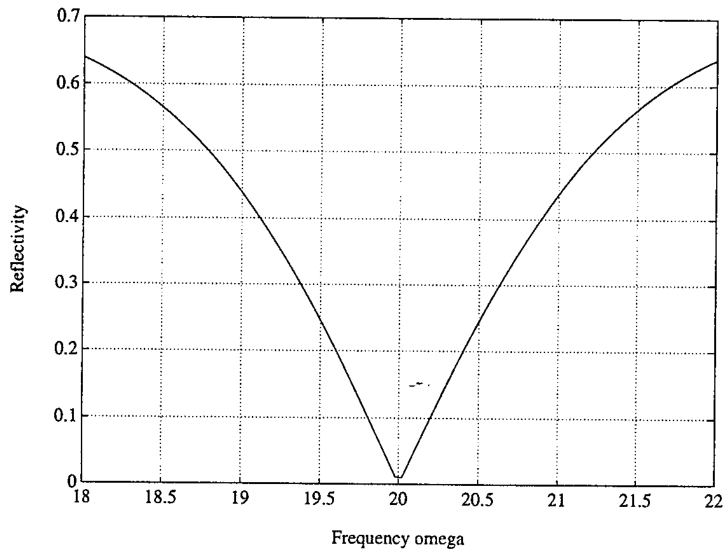


Figure 1: The frequency response of a one-layer system.

# 4 Wave propagation in layered media

Consider a two elastic half-spaces which sandwich between them an elastic slab. The elastic properties of the two half-spaces are supposed to be known while those of the slab are unspecified. All three media are isotropic.

Consider a steady-state harmonically oscillating wave impinging form infinity perpendicular to the surface of the slab. The ratio of the reflected energy over the incident energy is called *the reflectivity* of the slab.

It is not difficult to show that for any given frequency $\omega$, there exists a set of elastic moduli for the slab such that the reflectivity is zero. Figure 1 shows the frequency response of such a slab that is designed to annihilate waves of frequency $\omega = 20$. We see in that graph that when $\omega$ deviates from the design frequency, the reflectivity rapidly becomes non-zero.

To extend the range of frequencies over which the reflectivity remains small, we replace the original slab with two distinct slabs, each of which being half as thick as the original. The additional freedom gained in choosing two sets of elastic properties instead of one, allows us to successfully annihilate two arbitrarily prescribed frequencies, as shown in Figure 2. If we split the original slab into a larger number of distinct slabs of increasingly smaller

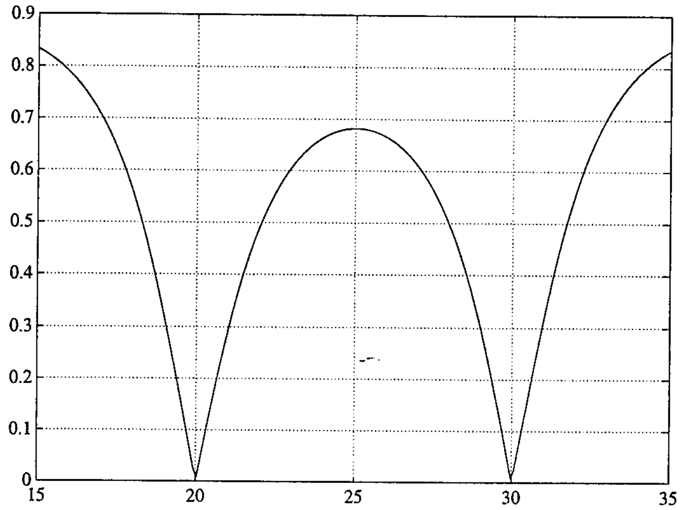


Figure 2: The frequency response of a two-layer system.

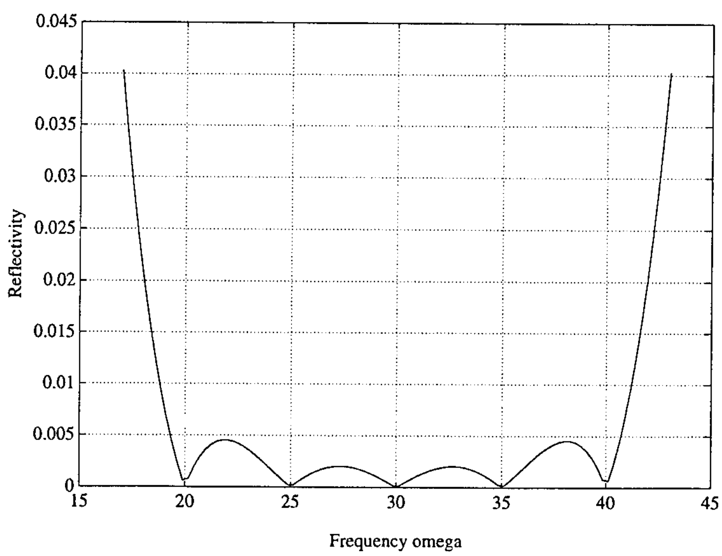


Figure 3: The frequency response of a five-layer system.

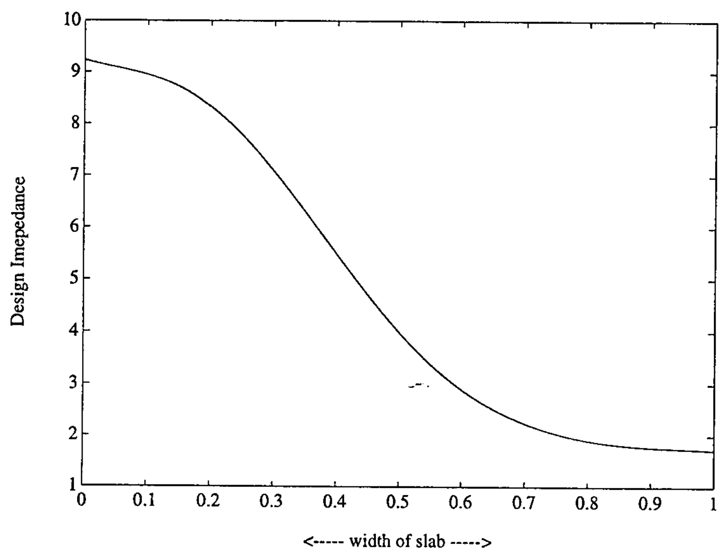


Figure 4: The frequency response of the variable-properties slab shown in figure 5.

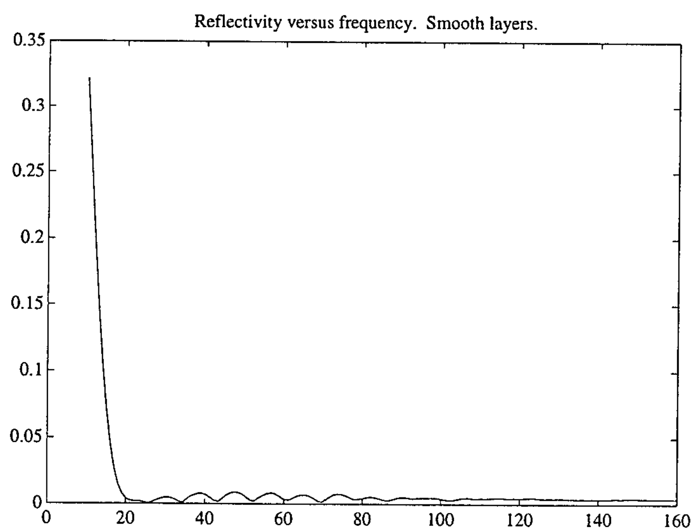


Figure 5: The impedance of a variable-properties slab that annihilates incident waves of frequencies within the 20–160 range.

thickness, we can impose additional frequencies which are annihilated. Figure 3 shows the frequency response of a system with five slabs and five totally annihilated frequencies.

We can carry this process to its limit by splitting the original slab into infinite number of layers of infinitesimal thickness. The elastic properties of such a slab become continuously varying functions within the slab. Figure 4 shows the impedance of such a slab, designed to annihilated incident waves in the 20–160 frequency range. The frequency response of the corresponding system is shown in Figure 5. The apparent deviation of the frequency response from zero within this range is due to the accumulation of numerical errors.

# 5 Optimization: Theory and algorithms

The basis of numerical computations that lead to the optimal impedance profile of Figure 4 lies in an optimization and control problem that can be described as follows.

Let $r(\omega)$ denote the reflectivity of the slab corresponding to an incident wave of frequency $\omega$. If we regard the mechanical properties of the material in slab as unknown, then the reflectivity depends on these mechanical properties as well as on $\omega$. Since ultimately the mechanical properties is a continuous function of the depth within the slab (of total thickness $T$), we regard the reflectivity as function $r : [\omega_1, \omega_2] \times C^0[0,T] \to \mathbb{C}$ and write $r = r(\omega, \kappa)$. The interval $[\omega_1, \omega_2]$ is an arbitrarily prescribed frequency band.

We wish to minimize the worst-case reflectivity over the frequency band. Hence we arrive at the min-max problem:

$$\min_{\kappa \in K \subset C^0[0,T]} \max_{\omega \in [\omega_1, \omega_2]} |r(\omega, \kappa)|,$$

where $K$ is a suitably chosen subset of $C^0[0,T]$ that incorporates in it certain physical constraints, such as the requirement that density cannot be negative or that the elastic moduli should be such that the corresponding elastic material is strongly elliptic.

The computational task for determining the optimal solution of this constrained, infinite-dimensional min-max problem is non-trivial. The papers [11], [12], and [13] concern a new family of dual optimization algorithms motivated by the goal of solving the min-max problem. By dualizing certain difficult constraints in an optimization, much faster convergence is obtain than in competing algorithms. The papers [9] and [10] analyze the stability of optimal control problems relative to a parameter. These stability results are applied to the analysis of various numerical algorithms, penalty and multiplier methods, sequential quadratic programming, and Euler discretizations, for solving the optimal design min-max problem.

## 6 Perforated composites

The presence of the air bubbles within an elastic material reduces the effective elastic moduli and enhances movement. That, combined with viscoelastic effects can help reduce reflections in an active way, by converting mechanical energy to heat, as opposed to simply re-routing the energy in a different direction.

The investigation of the "bubbly" composites is a subject of interest of its own, independently of the issue of the design of absorbant coatings. We have made some preliminary investigation on the dependence of the strength of bubbly elastic materials per unit weight basis. There are two opposing effect in determining the strength of a bubbly material per unit weight: The introduction of air bubbles in the material decreases the effective density decreases and therefore the material becomes more lightweight. Concurrently, the bubbles reduce the elastic strength of the material. It is not *a priori* obvious how the strength per unit mass will behave. A typical result is shown in figure 6. The three curves labeled $Q1111$, $Q1122$, and $Q1212$ depict the three effective elastic moduli for an initially elastic material which has been perforated by a periodic array of uniformly distributed circular holes. The horizontal axis indicates the relative size of the holes compared to the distance between the centers of adjacent holes. The left extreme corresponds to holes that are small and far apart and the right extreme corresponds to holes that are large and closely spaced (but do not overlap.) The gap between the curves labeled $Q1111$ and $2Q1212+Q1122$ represents the degree of anisotropy introduced do to oriented structure of the array of the perforations.

As expected, the curves are all monotone decreasing, indicating the reduction in the strength of the solid as we remove more and more material from it.

In figure 7 we have normalized the effective moduli by computing them per unit weight of the perforated material. Here we see that the curves also monotonically decline, but at a slower rate. In particular, $Q1212$ which measures one of the shear moduli, is effectively a constant. We have not yet completed our investigation of the perforated media. In particular, we would hope to

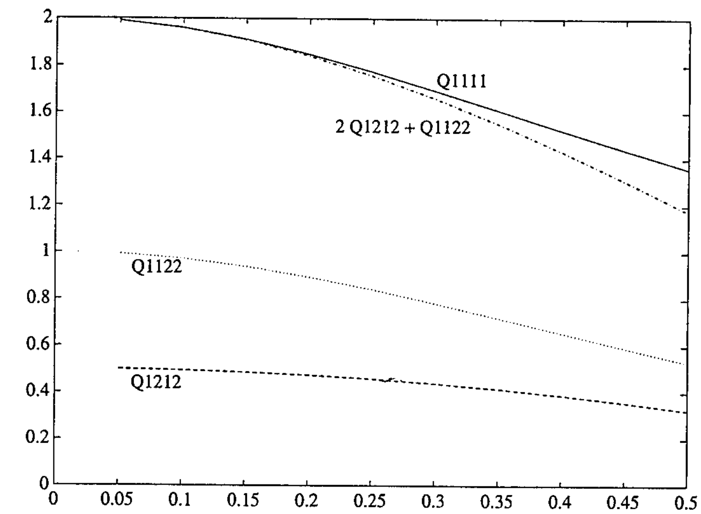


Figure 6: Effective elastic moduli in a perforated material as a function of size of perforations.

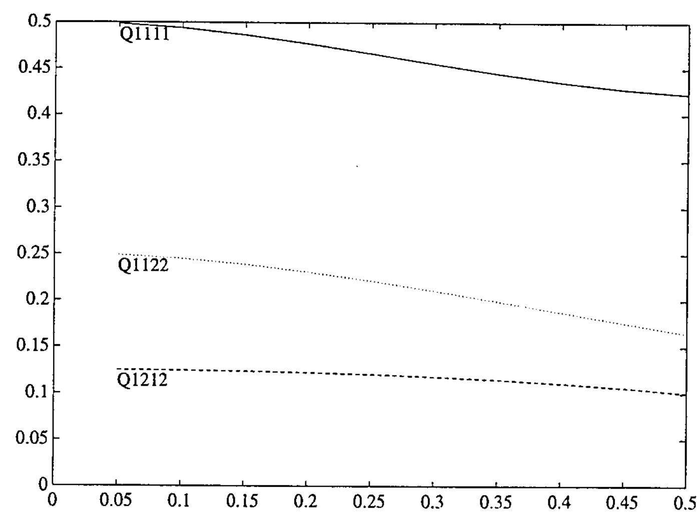


Figure 7: Effective elastic moduli *per unit mass* in a perforated material as a function of size of perforations.

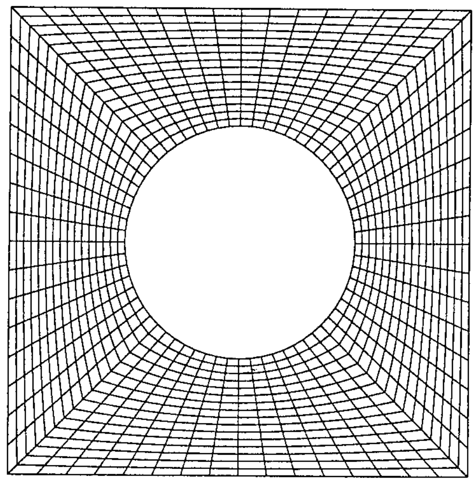

Figure 8: The grid for finite-elements computation of the unit-cell problem for a material perforated with a rectangular array of round holes.

find a case where the normalized effective moduli reach a maximum for a certain hole size. This will correspond to a light-density and high-strength material.

The effective moduli for the perforated materials were computed by applying the method of *homogenization*. In the core of the computation is the analysis of the *unit cell* problem. Figures 8 and 9 depict a representative finite-elements mesh that we used to numerically solve the unit cell problem and which in turn lead to figures6 and 7.

# 7 Combustion

The mathematical techniques developed in this project for the analysis of layered elastic media have found an unanticipated application in the investigation of combustion of inhomogeneous solids found in some modern artillery. In a joint investigation with W. Oberle of the Weapons Technology Directorate of the Aberdeen Proving Ground, we are currently investigating the effective chemical characteristics of finely layered explosives.

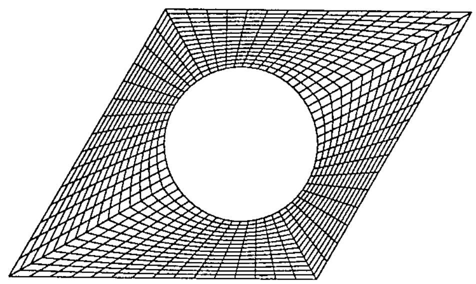

Figure 9: The grid for finite-elements computation of the unit-cell problem for a material perforated with a hexagonal array of round holes.

# 8 List of all participating personnel

The research for this project was conducted under the direction of the principal investigators: William Hager and Rouben Rostamian. In addition, the following students and colleagues made contributions to the project.

**Ping Wang:** Doctoral student working under the supervision of R. Rostamian. Graduated in 1990. Thesis topic: *Homogenization of perforated elastic media.* He developed mathematical techniques to compute the effective elastic properties of perforated solids. The major contributions of his thesis has been published in [6] and [7].

**Wen-Jong Shyong:** Doctoral student working under the supervision of R. Rostamian. Graduated in 1990. Thesis topic: *Homogenization in elasticity*. He developed computational techniques and implemented algorithms to compute the effective elastic properties of perforated solids. Figures 6 and 7 were produced by him.

**Dongxing Wang:** Masters student working under the supervision of W. Hager. Was instrumental in the comptational implemention of the numerical algorithms that lead to Figures 1–5. He is a co-author with the PIs on the forthcoming paper [15].

## Bibliography

1. W. Hager and R. Rostamian: Reflection and refraction of elastic waves for stratified materials, *Wave Motion*, **10**(1988), 333–348.

2. W. Hager and R. Rostamian: Invariant imbedding and the reflection of elastic waves, in Proceedings of IFIP, Conference on Control System Governed by Partial Differential Equations, Santiago de Compostela, July 1987. In Lecture Notes in Control and Information Sciences, vol. 114, 144–155, Springer-Verlag, 1989.

3. A. Damlamian, W. Hager and R. Rostamian: Homogenization for degenerate equations of elasticity, *Asymptotic Analysis*, 1(1988), 283–302.

4. W. Hager and R. Rostamian: Plane waves, optimization, and homogenization in elasticity, 183–197, in Smart Materials, Structures, and Mathematical Issues, C. Rogers, ed., Proceedings of the U.S. Army Research Office workshop on Smart Materials, Structures, and Mathematical Issues, Technomics Publishing Co., 1989.

5. W. Hager and R. Rostamian: Total absorption in elastic media, Proceedings of the Seventh U.S. Army Conference on Mathematics, West Point, 1990.

6. Ping Wang: Homogenization of an inhomogeneous elastic body containing gas bubbles, *Applicable Analysis*, 44(1992), 1–20. (Based on doctoral dissertation research conducted under supervision of R. Rostamian.)

7. Ping Wang: The effective moduli and the rate of homogenization in degenerate elastostatics, *Applicable Analysis*, **52** (1994), 45–52. (Based on doctoral dissertation research conducted under supervision of R. Rostamian.)

8. W. J. Shyong: Homogenization of linearly elastic materials, PhD Thesis, University of Maryland Baltimore County, 1991. (Doctoral dissertation conducted under supervision of R. Rostamian.)

9. W. Hager: Optimality, stability, and convergence in nonlinear control, *Applied Mathematics and Optimization*, **31** (1995) 297–326.

10. W. Hager: Lipschitzian stability in nonlinear control and optimization, *SIAM Journal on Control and Optimization*, **31** (1993), 569–603.

11. W. Hager: Analysis and implementation of a dual algorithm for constrained optimization, *Journal of Optimization Theory and Applications*, **79** (1993), 427–462.

12. W. Hager: Application of the dual active set algorithm to quadratic networks, *Computational Optimization and Applications*, 1 (1993), 349–373.

13. W. Hager: The dual active set algorithm, in *Advances in Optimization and Parallel Computing*, P. M. Pardalos, ed., North Holland, Amsterdam, (1992) 137–142.

14. W. Hager and R. Rostamian: Total absorption and nonreflective surfaces in linear elasticity, manuscript (copies included with this report.)

## Articles in preparation

15. W. Hager, R. Rostamian, and Dongxing Wang: Total absorption in elastic media. Part 1: Normal incidence, and Part 2: Oblique incidence (in preparation.)

16. W. Hager and R. Rostamian: Wave reflection for anisotropic materials, (in preparation.)