REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
		Interim; 7/93-11/95

4. TITLE AND SUBTITLE
EQUATIONS FOR THE EXTENSION AND FLEXURE OF RELATIVELY THIN THERMOPIEZOELECTRIC PLATES SUBJECTED TO LARGE ELECTRIC FIELDS

5. FUNDING NUMBERS

$DAAH04-93-G-0214$

6. AUTHOR(S)

J.S. Yang and R.C. Batra

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

University of Missouri-Rolla, Rolla, MO

Virginia Polytechnic Institute and State University, Blacksburg, VA

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

$ARO\ 32109.32-MA-SM$

11. SUPPLEMENTARY NOTES

The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

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19960209 123

14. SUBJECT TERMS

Thin piezoelectric plates, thermoelasticity

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

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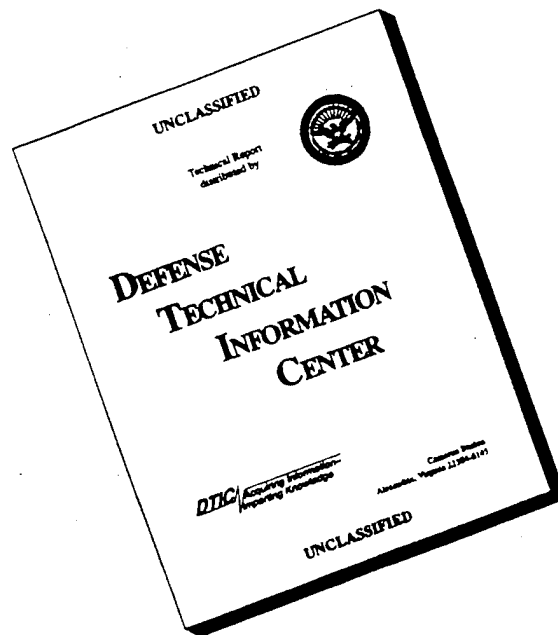
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Romesh C. Batra (Ed.)

Contemporary Research in Engineering Science

With 304 Figures

Springer, 1995.

Equations for the Extension and Flexure
of Relatively Thin Thermopiezoelectric Plates
Subjected to Large Electric Fields

J.S. Yang[1] and R.C. Batra[2]

[1]Department of Mechanical and Aerospace Engineering and Engineering Mechanics
University of Missouri-Rolla, Rolla, MO 65401*
[2]Clifton Garvin Professor, Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0219

ABSTRACT

A system of approximate two-dimensional equations for the extensional and flexural motion of thermopiezoelectric plates subject to large electric fields is derived from the three-dimensional equations of thermopiezoelectricity for small strains, small temperature variations, and strong electric fields. The two-dimensional equations are derived by introducing an appropriate expansion for the mechanical displacement, temperature field, and electric potential in the thickness-coordinate and integrating the balance laws and constitutive relations.

I. INTRODUCTION

Recently electroelastic equations for the extensional motion of very thin plates with fully electroded major surfaces subject to large driving voltages and undergoing small strain were derived [1] from the general nonlinear electroelastic equations [2,3]. These equations have been used to analyze of static and dynamic deformations of laminated beams and plates due to piezoelectric actuators [4-6]. A set of two-dimensional equations was subsequently obtained [7] for a higher order approximate description of the mechanical and nonlinear electrical behavior of the relatively thin electroelastic plates in extensional and flexural motion. The equations are valid for either the electroded or unelectroded plates and hence have been used [8] to analyze partially electroded actuators which have been shown to have a distributed shear stress rather than a singular shear stress of fully electroded actuators. This prevents the delamination of the piezoelectric layer from the base plate.

In the above analysis, only piezoelectric effect is considered, and thermal effect is not included. According to a recent review article by Rao and Sunar [9], the

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* Present address: Department of Mechanical Engineering, Aeronautical Engineering and Mechanics, Renselaer Polytechnic Institute, Troy, NY 12180.

composite intelligent structures have different response characteristics at different temperatures and the temperature variation in the piezoelectric materials can affect the overall performance of the control system, hence thermal effects are important in the precision distributed sensing and control of intelligent structures. According to Rao and Sunar, the applications of thermopiezoelectricity theory to practical engineering problems in general and vibration control of flexible structures in particular, are very few in the literature and the development of thermopiezoelectric sensors/actuators is important for advanced intelligent structures.

Mindlin [10] derived a set of two-dimensional plate equations based on the linear theory of thermopiezoelectricity for small strains, small temperature variations, and weak electric fields. Recent works on laminated thermopiezoelectric plates [11,12] are also for the linear theory and weak electric fields. It has been noted that piezoelectric materials are often operating under large driving voltages or strong electric fields [1] hence nonlinear terms in electric fields should be considered. Besides, in [11,12], only equations of the balance of linear momentum are derived but the electric charge equation and the heat equation are not included. The electric and thermal fields have to be obtained separately and hence these equations can not be applied to structures with partially electroded actuators.

A set of equations for electroded very thin thermopiezoelectric plates subject to large driving voltages or strong electric fields in extensional motion is derived in [13] based on the general nonlinear equations of thermopiezoelectricity [14]; the work in [13] generalizes that reported in [1] to include thermal effects. Here, a system of two-dimensional equations for the extensional and flexural motion of thermopiezoelectric plates subject to large electric fields is derived from the three-dimensional equations of thermopiezoelectricity for small strains, small temperature variations, and strong electric fields. The two-dimensional equations are derived by introducing an appropriate expansion for the mechanical displacement, temperature field, and electric potential in the thickness-coordinate and integrating the balance laws and constitutive relations through the thickness. The resulting equations are reduced to the uncoupled system of equations describing extensional motion and elementary flexure. These equations generalize those in [7] by including thermal fields. The equations are valid for either the electroded or the unelectroded plate and hence can be used to analyze either fully or partially electroded actuators.

II. BASIC EQUATIONS

It has been shown [13] that the thermoelectroelastic equations for infinitesimal strains, small temperature variations, and strong electric fields may be written as

$$\tau_{ij,i} = \rho \ddot{u}_j$$
$$D_{i,i} = 0$$
$$-Q_{i,i} = T\rho\dot{\eta}$$
$$\tau_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k - \lambda_{ij}\theta - \frac{1}{2}\hat{b}_{klij}E_kE_l$$

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$$D_i = e_{ikl}S_{kl} + \epsilon_{ik}E_k + p_i\theta + \frac{1}{2}\chi_{kji}E_kE_j \qquad (1)$$

$$\rho\eta = \lambda_{ij}S_{ij} + p_kE_k + \alpha\theta + \frac{1}{2}\gamma_{kj}E_kE_j$$

$$Q_i = -\kappa_{ij}\theta_{,j}$$

$$S_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$$

$$E_k = -\phi_{,k} .$$

Here a comma followed by an index l denotes partial differentiation with respect to the referential coordinate x_l, a dot over a variable denotes partial differentiation with respect to time and repeated indices are to be summed. The range of indices i,j,k,l is $1,2,3$. In Eq. (1) ρ is the mass density, τ_{ij} the stress tensor, u_i the mechanical displacement, S_{ij} the infinitesimal strain tensor, D_i the electric displacement vector, ϕ the electric potential, E_i the electric field, Q_i the heat flux, η the entropy density, T a uniform reference temperature, and θ the temperature variation. c_{ijkl}, e_{ikl}, ϵ_{ik} denote the elastic, piezoelectric and dielectric constants, λ_{ij} the thermal elastic constants, p_i the pyroelectric constants, α is related to specific heat and κ_{ij} is the heat conductivity tensor. $\hat{b}_{klij}$, χ_{kji}, and γ_{ij} are nonlinear material constants. We note that $\hat{b}_{klij}$, the effective electrostrictive constants, include the effect of the Maxwell electrostatic stress tensor [13]. Equation $(1)_1$ is the balance of linear momentum, $(1)_2$ the electrostatic charge equation, $(1)_3$ the heat equation, $(1)_{4-7}$ the constitutive equations, $(1)_8$ the strain-displacement relation, and $(1)_9$ the electric field-potential relation. When the electric fields are weak, quadratic terms on the right-hand side of equations $(1)_{4-6}$ are dropped and Eq. (1) then reduces to that for linear thermopiezoelectricity.

III. TWO-DIMENSIONAL EQUATIONS FOR THIN PLATES

A plan view and a cross-section of the thin plate with thickness $2h$ are shown in Figure 1 along with the coordinate system. The top and bottom surfaces may be identically electroded; the electrodes in a given region are assumed to have a potential difference or voltage across them.

We now obtain approximate two-dimensional mechanical equations of motion, charge equation of electrostatics and heat equation for the thin plate shown in Figure 1 by employing assumed expansion of u_j, ϕ, and θ in the thickness coordinate. Since only the lowest frequency approximations, i.e., equations of the extensional and flexural motion of plates are of interest here, we expand u_j in the form

$$u_j = u_j^{(0)}(x_a,t) + x_3u_j^{(1)}(x_a,t) + x_3^2u_j^{(2)}(x_a,t) \qquad (2)$$

in which $u_j^{(2)}$ is included to allow for the free thickness-strains accompanying elementary flexure [7]. We also adopt the convention that indices a,b,c,d take values $1,2$. Considering the behavior of ϕ across the electrodes when they are present,
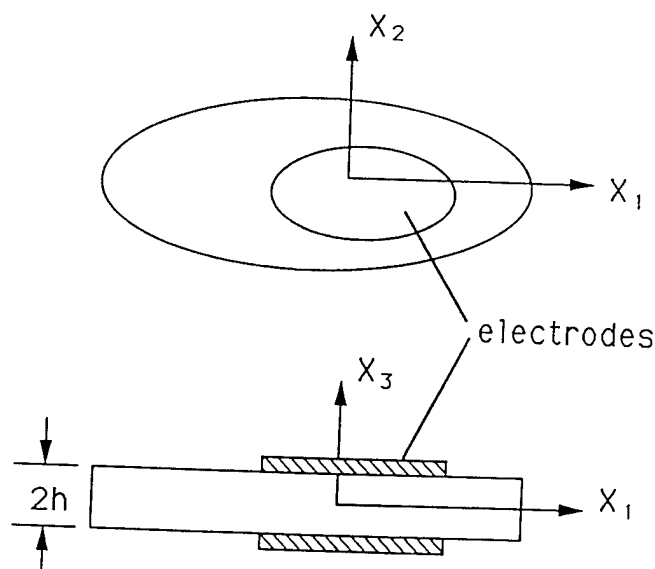
Fig. 1, Plan view and cross-section of a partially electroded thin plate.

we expand ϕ in the form

$$\phi = \phi^{(0)}(x_a,t) + \frac{x_3}{2h}\phi^{(1)}(x_a,t) + (\frac{x_3^2}{h^2} - 1)\phi^{(2)}(x_a,t) + \frac{x_3}{h}(\frac{x_3^2}{h^2} - 1)\phi^{(3)}(x_a,t) \quad (3)$$

in which only the $\phi^{(1)}$ term contributes to the voltage across the electrodes. Equation (3) is basically taken from [7], with the difference that here we have $\phi^{(0)}$ but the corresponding equation in [7] does not; $\phi^{(0)}$ is needed for a complete description of ϕ in the unelectroded region, although in the case treated in [8] it has no contribution. In an electroded region the two-dimensional plate electric potentials $\phi^{(0)}$ and $\phi^{(1)}$ must be independent of coordinates x_a in the plane of the plate. However, in an unelectroded region $\phi^{(0)}$ and $\phi^{(1)}$ are in general functions of coordinates x_a. Clearly, this expansion could be carried to higher order, but this is the lowest order that is needed to fully describe the response of a partially electroded thin plate. For the temperature field θ, we take [10]

$$\theta = \theta^{(0)}(x_a, t) + x_3\theta^{(1)}(x_a, t) \quad (4)$$

Next, we multiply equation $(1)_1$ by $\{1, x_3, x_3^2\}$, the electric charge equation $(1)_2$ by $\{1, x_3/2h, x_3^2/h^2 - 1, (x_3^2/h^2 - 1)x_3/h\}$, the heat equation $(1)_3$ by $\{1, x_3\}$, integrate the resulting equations from $-h$ to h and obtain the following nine two-dimensional

equations corresponding to the balance of linear momentum

$$\tau_{ak,a}^{(n)} - n\tau_{3k}^{(n-1)} + F_k^{(n)} = \rho \sum_{m=0}^{2} H_{mn}\ddot{u}_k^{(n)}, \quad n = 0, 1, 2 \tag{5}$$

four two-dimensional electric charge equations

$$\frac{1}{2h}D_{a,a}^{(0)} + d_3^{(0)} = 0$$

$$\frac{1}{2h}D_{a,a}^{(1)} - \frac{1}{2h}D_3^{(0)} + d_3^{(1)} = 0$$

$$\frac{1}{h^2}D_{a,a}^{(2)} - D_{a,a}^{(0)} - \frac{2}{h^2}D_3^{(1)} = 0 \tag{6}$$

$$\frac{1}{h^3}D_{a,a}^{(3)} - \frac{1}{h}D_{a,a}^{(1)} - \frac{3}{h^3}D_3^{(2)} + \frac{1}{h}D_3^{(0)} = 0$$

and two two-dimensional heat equations

$$Q_{a,a}^{(0)} + q_3^{(0)} = -T\rho\dot{\eta}^{(0)}$$

$$Q_{a,a}^{(1)} - Q_3^{(0)} + q_3^{(1)} = -T\rho\dot{\eta}^{(1)} \tag{7}$$

where

$$\{\tau_{kl}^{(n)}, D_k^{(n)}, Q_k^{(n)}, \eta^{(n)}\} = \int_{-h}^{h} x_3^n \{\tau_{kl}, D_k, Q_k, \eta\} dx_3$$

$$F_j^{(n)} = [x_3^n \tau_{3j}]_{-h}^h, \quad d_3^{(n)} = \frac{1}{2h}[x_3^n D_3]_{-h}^h, \quad q_3^{(n)} = [x_3^n Q_3]_{-h}^h$$

$$H_{00} = 2h, \quad H_{01} = H_{10} = 0, \quad H_{11} = \frac{2}{3}h^3 \tag{8}$$

$$H_{02} = H_{20} = \frac{2}{3}h^3, \quad H_{12} = H_{21} = 0, \quad H_{22} = \frac{2}{5}h^5$$

and we note that terms $F_j^{(n)}$, $d_3^{(n)}$ and $q_3^{(n)}$ arise from the integration by parts. Terms $d_3^{(2)}$ and $d_3^{(3)}$, which are analogous to $d_3^{(0)}$ and $d_3^{(1)}$, vanish because the x_3 dependence of coefficients of $\phi^{(2)}$ and $\phi^{(3)}$ vanish at $x_3 = \pm h$. We also note that in the balance of linear momentum equations (5) the three equations for $n = 2$ will not actually occur because they will be eliminated by allowing for the free development of $u_k^{(2)}$ accompanying anisotropic flexure [7]. In an electroded region in which $\phi^{(1)} = V \phi^{(0)}$, where V equals the voltage, is a constant and the first two equations of (6) are not needed to obtain the solution. They simply serve to define $d_3^{(0)}$ and $d_3^{(1)}$ in terms of the electric displacement $\mathbf{D}$. However, in an unelectroded region in which V is not prescribed these equations are required to obtain a solution. Similarly, if temperature is known on both major surfaces of the plate, $\theta^{(0)}$ and $\theta^{(1)}$ are determined directly from (4) and equation (7) is not needed to obtain the solution; it serves to determine $q_3^{(0)}$ and $q_3^{(1)}$ in terms of other variables. However, if heat flux is prescribed at either one of the major surfaces of the plate, equation (7) is needed to obtain a solution.

At this point we have nine two-dimensional stress equations in (5). However, since we are interested in obtaining only the uncoupled equations of anisotropic extension and elementary flexure, it is convenient to rewrite these as separate essentially extensional and essentially flexural equations. We note these equations are coupled due to anisotropy of the plate material. We further note that we can not complete the reduction without the plate constitutive relations, which are obtained in the next section. For the plate constitutive equations it is convenient to define Mindlin's plate strains, which is done simply by substituting the expansion (2) into the strain-displacement relation, Eq. $(1)_8$, and rearranging terms; the result is

$$S_{ij} = \sum_{n=0}^{2} x_3^n S_{ij}^{(n)} , \quad S_{ij}^{(n)} = \frac{1}{2}[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)(\delta_{3i}u_j^{(n+1)} + \delta_{3j}u_i^{(n+1)})] \quad (9)$$

We now note that we will not need $S_{ij}^{(2)}$ because of the reduction that is to be made. When elementary flexure and extension are to be uncoupled, from (5), following the steps in [7], we may separate the essentially extensional plate equations,

$$\tau_{ab,a}^{(0)} + F_b^{(0)} = 2\rho h \ddot{u}_b^{(0)} + \frac{2}{3}\rho h^3 \ddot{u}_b^{(2)}$$

$$\tau_{a3,a}^{(1)} - \tau_{33}^{(0)} + F_3^{(1)} = \frac{2}{3}\rho h^3 \ddot{u}_3^{(1)} \quad (10)$$

$$\tau_{ab,a}^{(2)} - 2\tau_{3b}^{(1)} + F_b^{(2)} = \frac{2}{3}\rho h^3 \ddot{u}_b^{(0)} + \frac{2}{5}\rho h^5 \ddot{u}_b^{(2)}$$

from the essentially flexural plate equations,

$$\tau_{a3,a}^{(0)} + F_3^{(0)} = 2\rho h \ddot{u}_3^{(0)} + \frac{2}{3}\rho h^3 \ddot{u}_3^{(2)}$$

$$\tau_{ab,a}^{(1)} - \tau_{3b}^{(0)} + F_b^{(1)} = \frac{2}{3}\rho h^3 \ddot{u}_b^{(1)} \quad (11)$$

$$\tau_{a3,a}^{(2)} - 2\tau_{33}^{(1)} + F_3^{(2)} = \frac{2}{3}\rho h^3 \ddot{u}_3^{(0)} + \frac{2}{5}\rho h^5 \ddot{u}_3^{(2)}$$

In the case of extension we must first allow for the free plate thickness-strains $S_{33}^{(0)}$ by setting $\tau_{33}^{(0)} = 0$. Then in order to eliminate the first order extensional equations completely we set $\tau_{3a}^{(1)} = 0$. This has the effect of eliminating the second order equations in (10) completely since all second order plate stress resultants $\tau_{ij}^{(2)}$ are irrelevant in the approximation, and may be ignored, as may the dynamic terms on the right-hand side of $(10)_3$ and $(11)_3$. Furthermore, in order to eliminate flexure from the extensional equations, from (11) it is clear that we must have $\tau_{a3}^{(0)} = 0$, $\tau_{ab}^{(1)} = 0$, $\tau_{33}^{(1)} = 0$. Collecting all the conditions on the stress resultants, we have

$$\tau_{3j}^{(0)} = 0 , \quad \tau_{ij}^{(1)} = 0 \quad (12)$$

the first of which will be used, in the next section, to reduce the general plate thermopiezoelectric constitutive equations to those that are suitable for anisotropic

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extension. Since we have eliminated flexure, therefore $\ddot{u}_3^{(0)} = 0$. Also since we are well below the lowest thickness resonant frequency of the plate, we may assume $\ddot{u}_j^{(1)} = 0$, $\ddot{u}_j^{(2)} = 0$, then all that remains out of (10) is

$$\tau_{ab,a}^{(0)} + F_b^{(0)} = 2\rho h \ddot{u}_b^{(0)} \tag{13}$$

which are the two equations for the extensional motion of thin plates.

For flexure, we must first allow for the plate thickness strain $S_{33}^{(1)}$ by setting $\tau_{33}^{(1)} = 0$. This has the additional effect of eliminating the second order flexural equation completely because, as already noted, all second order plate stress resultants are irrelevant in this approximation, and may be ignored. Then in order to eliminate extension from the flexural equations, from (10) it is clear that we must have $\tau_{ab}^{(0)} = 0$, $\tau_{a3}^{(1)} = 0$, $\tau_{33}^{(0)} = 0$. These conditions enable us to write

$$\tau_{3j}^{(1)} = 0 \ , \quad \tau_{ab}^{(0)} = 0 \ , \quad \tau_{33}^{(0)} = 0 \tag{14}$$

the first of which will be used, in the next section, to reduce the general plate thermopiezoelectric constitutive equations to those that are suitable for elementary flexure of thin plates. Since we have eliminated extension, therefore $\ddot{u}_b^{(0)} = 0$. Also, since we are well below the lowest thickness resonant frequency of the plate, we may assume that $\ddot{u}_j^{(1)} = 0$, $\ddot{u}_j^{(2)} = 0$. In order to complete the reduction to the elementary theory of flexure we must take the thickness shear strains $S_{3a}^{(0)}$ to vanish [7], with which $(9)_2$ yields

$$u_a^{(1)} = -u_{3,a}^{(0)} \tag{15}$$

which enables us to obtain a single equation in the one dependent variable $u_3^{(0)}$ in the elementary theory of the flexure of thin plates. Utilizing $\ddot{u}_b^{(1)} = 0$, which eliminates rotary inertia, we obtain $\tau_{3a}^{(0)}$ from $(11)_2$, which when substituted into $(11)_1$ yields

$$\tau_{ab,ab}^{(1)} + F_{b,b}^{(1)} + F_3^{(0)} = 2\rho h \ddot{u}_3^{(0)} \tag{16}$$

This is the equation for the elementary theory of the flexure motion of thin plates.

Thus, at this stage we have the extensional equations of motion (13) which require constitutive equations for $\tau_{ab}^{(0)}$, the flexural equations of motion (16) which need constitutive equations for $\tau_{ab}^{(1)}$, the four plate equations (6) for electrostatics which necessitate constitutive equations for $D_k^{(n)}$ ($n = 0, 1, 2, 3$), and two plate heat-conduction equations (7) which require constitutive relations for $Q_k^{(n)}$ ($n = 0, 1$) and $\eta^{(n)}$ ($n = 0, 1$). As noted earlier, the constitutive equations will be derived in the next section in terms of the nine dependent variables $u_j^{(0)}$, $\phi^{(n)}$ ($n = 0, 1, 2, 3$) and $\theta^{(n)}$ ($n = 0, 1$).

To the foregoing plate equations we must adjoin the appropriate initial and boundary conditions. For the equations of anisotropic extension (13), at an interface separating one region from another we have the well-known continuity condition

$$n_a[\tau_{ab}^{(0)}] = 0 \ , \quad [u_b^{(0)}] = 0 \tag{17}$$

where n_a denotes the unit normal directed from the "$-$" to the "$+$" side of the interface and, in which we have used the usual notation $[\![A]\!] = A^+ - A^-$ for the jump. At the edge either $n_a \tau_{ab}^{(0)}$ or $u_b^{(0)}$ or some combination thereof is prescribed. For the equation of elementary flexure, at an interface we have the continuity condition

$$n_a[\![\tau_{ab}^{(1)}]\!]n_b = 0 \;, \quad [\![\tau_{n3}^{(0)} + \frac{\partial \tau_{ns}^{(1)}}{\partial s}]\!] = 0 \;, \quad [\![\frac{\partial u_3^{(0)}}{\partial n}]\!] = 0 \;, \quad [\![u_3^{(0)}]\!] = 0 \qquad (18)$$

where

$$\tau_{n3}^{(0)} = n_b \tau_{b3}^{(0)} \;, \quad \tau_{ns}^{(1)} = n_a \tau_{ab}^{(1)} s_b \qquad (19)$$

and s_b denotes a unit vector tangent to an interface of separation in the counterclock direction. At an edge either $n_a \tau_{ab}^{(1)} n_b$ and $\tau_{n3}^{(0)} + \partial \tau_{ns}^{(1)}/\partial s$ or $\partial u_3^{(0)}/\partial n$ and $u_3^{(0)}$ or some combination thereof is prescribed. In addition, there are the well-known conditions across corners of discontinuous curves.

In the case of the two-dimensional plate equations of electrostatics (6), at an interface separating one region from another, we require the continuity conditions

$$[\![\phi^{(n)}]\!] = 0 \;, n = 0, 1, 2, 3 \qquad (20)$$

In addition, by multiplying the well-known three-dimensional electrostatic continuity conditions $[\![D_a n_a]\!] = 0$ by $\{1, x_3/2h, x_3^2/h^2 - 1, (x_3^2/h^2 - 1)x_3/h\}$, and integrating the resulting equations from $-h$ to h, we obtain the following two-dimensional plate continuity conditions

$$n_a[\![D_a^{(0)}]\!] = 0 \;, \quad n_a[\![D_a^{(1)}]\!] = 0$$
$$n_a[\![D_a^{(2)} - h^2 D_a^{(0)}]\!] = 0 \;, \quad n_a[\![D_a^{(3)} - h^2 D_a^{(1)}]\!] = 0 \qquad (21)$$

At an edge either $n_a D_a^{(0)}$, $n_a D_a^{(1)}$, $n_a(D_a^{(2)} - h^2 D_a^{(0)})$ and $n_a(D_a^{(3)} - h^2 D_a^{(1)})$ or $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$ or some combination thereof is prescribed.

For the heat equations (7), the continuity equations are [10]

$$[\![\theta^{(0)}]\!] = 0 \;, \quad [\![\theta^{(1)}]\!] = 0 \;, \quad n_a[\![Q_a^{(0)}]\!] = 0 \;, \quad n_a[\![Q_a^{(1)}]\!] = 0 \qquad (22)$$

At an edge either $\theta^{(0)}$ and $\theta^{(1)}$ or $n_a Q_a^{(0)}$ and $n_a Q_a^{(1)}$ or some combination thereof is prescribed.

IV. TWO-DIMENSIONAL CONSTITUTIVE EQUATIONS

In this section we obtain constitutive relations for the stress tensor, electric displacement, heat flux and entropy defined in (8). The resulting constitutive equations are then reduced to those appropriate for the uncoupled equations of anisotropic extension and elementary flexure by employing Eqs. $(12)_1$ and $(14)_1$, respectively. Clearly, the constitutive equations are obtained by substituting from $(1)_{4-7}$ into

$(8)_1$, employing (9) and the analogous equations for electrical equations and integrating through the thickness. From $(1)_9$ and (3) we find that the analogous electrical equations take the form

$$E_a = -\phi_{,a}^{(0)} - \frac{x_3}{2h}\phi_{,a}^{(1)} - (\frac{x_3^2}{h^2} - 1)\phi_{,a}^{(2)} - \frac{x_3}{h}(\frac{x_3^2}{h^2} - 1)\phi_{,a}^{(3)}$$

$$E_3 = -\frac{1}{2h}\phi^{(1)} - \frac{2x_3}{h^2}\phi^{(2)} - (\frac{3x_3^2}{h^3} - \frac{1}{h})\phi^{(3)}$$

(23)

As noted earlier, in this low order treatment, we ignore the second order equations completely, and we do not need the plate strains $S_{ij}^{(2)}$ in $(9)_1$, which means that for our purposes here, Eq. $(9)_1$ takes the form

$$S_{ij} = \sum_{n=0}^{1} x_3^n S_{ij}^{(n)}$$

(24)

which is analogous to (23).

For convenience, we introduce the usual compressed matrix notation for stresses and strains. In this convention the tensor indices ij or kl are replaced by p or q which take values 1, 2, 3, 4, 5, 6 as ij or kl take values 11, 22, 33, 23 or 32, 31 or 13, 12 or 21 respectively. Accordingly, we write constitutive equations $(1)_{4-6}$ in the compressed notation as

$$\tau_p = c_{pq}S_q - e_{kp}E_k - \lambda_p\theta - \frac{1}{2}\hat{b}_{klp}E_kE_l$$

$$D_i = e_{iq}S_q + \epsilon_{ik}E_k + p_i\theta + \frac{1}{2}\chi_{kji}E_kE_j$$

(25)

$$\rho\eta = \lambda_qS_q + p_kE_k + \alpha\theta + \frac{1}{2}\gamma_{kj}E_kE_j$$

where the sum from 1 to 6 on the repeated matrix indices is understood. We also note that in the matrix notation Eq. (24) takes the form

$$S_q = \sum_{n=0}^{1} x_3^n S_q^{(n)}$$

(26)

Similarly, from Eq. $(8)_1$ we obtain

$$\tau_p^{(0)} = \int_{-h}^{h} \tau_p dx_3 , \quad \tau_p^{(1)} = \int_{-h}^{h} x_3\tau_p dx_3 .$$

(27)

Now, substituting from (25) into (27) and recalling the definitions of $D_k^{(n)}$, $Q_k^{(n)}$ and $\eta^{(n)}$ in $(8)_1$, substituting from (26), (23), (4) and $(1)_7$, and integrating through the thickness, we obtain the following plate constitutive relations

$$\tau_p^{(0)} = 2hc_{pq}S_q^{(0)} + 2he_{ap}\phi_{,a}^{(0)} - \frac{4}{3}he_{ap}\phi_{,a}^{(2)} + e_{3p}\phi^{(1)} - 2h\lambda_p\theta^{(0)} - \frac{1}{2}\hat{b}_{ijp}N_{ij}^{(0)}$$

$$\tau_p^{(1)} = \frac{2}{3}h^3 c_{pq}S_q^{(1)} + \frac{h^2}{3}e_{ap}\phi_{,a}^{(1)} - \frac{4}{15}h^2 e_{ap}\phi_{,a}^{(3)} + \frac{4}{3}he_{3p}\phi^{(2)} - \frac{2}{3}h^3\lambda_p\theta^{(1)} - \frac{1}{2}\hat{b}_{ijp}N_{ij}^{(1)}$$

$$D_k^{(0)} = 2he_{kp}S_p^{(0)} - 2h\epsilon_{ka}\phi_{,a}^{(0)} + \frac{4}{3}h\epsilon_{ka}\phi_{,a}^{(2)} - \epsilon_{k3}\phi^{(1)} + 2hp_k\theta^{(0)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(0)}$$

$$D_k^{(1)} = \frac{2}{3}h^3 e_{kp}S_p^{(1)} - \frac{1}{3}h^2\epsilon_{ka}\phi_{,a}^{(1)} + \frac{4}{15}h^2\epsilon_{ka}\phi_{,a}^{(3)} - \frac{4}{3}h\epsilon_{k3}\phi^{(2)} + \frac{2}{3}h^3 p_k\theta^{(1)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(1)}$$

$$D_k^{(2)} = \frac{2}{3}h^3 e_{kp}S_p^{(0)} - \frac{2}{3}h^3\epsilon_{ka}\phi_{,a}^{(0)} + \frac{4}{15}h^3\epsilon_{ka}\phi_{,a}^{(2)} - \frac{1}{3}h^2\epsilon_{k3}\phi^{(1)}$$
$$- \frac{8}{15}h^2\epsilon_{k3}\phi^{(3)} + \frac{2}{3}h^3 p_k\theta^{(0)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(2)}$$

$$D_k^{(3)} = \frac{2}{5}h^5 e_{kp}S_p^{(1)} - \frac{1}{5}h^4\epsilon_{ka}\phi_{,a}^{(1)} + \frac{4}{35}h^4\epsilon_{ka}\phi_{,a}^{(3)} - \frac{4}{5}h^3\epsilon_{k3}\phi^{(2)} + \frac{2}{5}h^5 p_k\theta^{(1)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(3)}$$

$$\rho\eta^{(0)} = 2h\lambda_p S_p^{(0)} - 2hp_a\phi_{,a}^{(0)} + \frac{4}{3}hp_a\phi_{,a}^{(2)} - p_3\phi^{(1)} + 2h\alpha\theta^{(0)} + \frac{1}{2}\gamma_{ij}N_{ij}^{(0)}$$

$$\rho\eta^{(1)} = \frac{2}{3}h^3\lambda_p S_p^{(1)} - \frac{1}{3}h^2 p_a\phi_{,a}^{(1)} + \frac{4}{15}h^2 p_a\phi_{,a}^{(3)} - \frac{4}{3}hp_3\phi^{(2)} + \frac{2}{3}h^3\alpha\theta^{(1)} + \frac{1}{2}\gamma_{ij}N_{ij}^{(1)}$$

$$Q_i^{(0)} = -2h\kappa_{ij}(\theta_{,j}^{(0)} + \delta_{3j}\theta^{(1)})$$

$$Q_i^{(1)} = -\frac{2}{3}h^3\kappa_{ij}\theta_{,j}^{(1)}$$

(28)

where the nonlinear electrical terms $N_{ij}^{(n)}$ $(n = 0, 1, 2, 3)$ are defined by

$$N_{ab}^{(0)} = h[2\phi_{,a}^{(0)}\phi_{,b}^{(0)} + \frac{16}{105}\phi_{,a}^{(3)}\phi_{,b}^{(3)} - \frac{4}{3}(\phi_{,a}^{(2)}\phi_{,b}^{(0)} + \phi_{,a}^{(2)}\phi_{,b}^{(0)})$$
$$- \frac{2}{15}(\phi_{,a}^{(1)}\phi_{,b}^{(3)} + \phi_{,a}^{(3)}\phi_{,b}^{(1)}) + \frac{16}{15}\phi_{,a}^{(2)}\phi_{,b}^{(2)} + \frac{1}{6}\phi_{,a}^{(1)}\phi_{,b}^{(1)}]$$

$$N_{3b}^{(0)} = [\frac{8}{15}(\phi^{(3)}\phi_{,b}^{(2)} - \phi^{(2)}\phi_{,b}^{(3)}) + \frac{2}{3}(\phi^{(2)}\phi_{,b}^{(1)} - \phi^{(1)}\phi_{,b}^{(2)}) + \phi^{(1)}\phi_{,b}^{(0)}]$$

$$N_{33}^{(0)} = \frac{1}{30h}[48\phi^{(3)}\phi^{(3)} + 80\phi^{(2)}\phi^{(2)} + 15\phi^{(1)}\phi^{(1)}]$$

$$N_{ab}^{(1)} = h^2[\frac{16}{105}(\phi_{,a}^{(2)}\phi_{,b}^{(3)} + \phi_{,a}^{(3)}\phi_{,b}^{(2)}) - \frac{4}{15}(\phi_{,a}^{(3)}\phi_{,b}^{(0)} + \phi_{,b}^{(3)}\phi_{,a}^{(0)})$$
$$- \frac{2}{15}(\phi_{,a}^{(2)}\phi_{,b}^{(1)} + \phi_{,b}^{(2)}\phi_{,a}^{(1)}) + \frac{1}{3}(\phi_{,a}^{(0)}\phi_{,b}^{(1)} + \phi_{,b}^{(0)}\phi_{,a}^{(1)})]$$

$$N_{3b}^{(1)} = h[\frac{1}{6}\phi^{(1)}\phi_{,b}^{(1)} + \frac{4}{15}\phi^{(3)}\phi_{,b}^{(1)} - \frac{8}{15}\phi^{(2)}\phi_{,b}^{(2)} - \frac{2}{15}\phi^{(1)}\phi_{,b}^{(3)} - \frac{8}{105}\phi^{(3)}\phi_{,b}^{(3)} + \frac{4}{3}\phi^{(2)}\phi_{,b}^{(0)}]$$

$$N_{33}^{(1)} = [\frac{4}{3}\phi^{(2)}\phi^{(1)} + \frac{32}{15}\phi^{(2)}\phi^{(3)}]$$

$$N_{ab}^{(2)} = h^3[\frac{16}{315}\phi_{,a}^{(3)}\phi_{,b}^{(3)} - \frac{4}{15}(\phi_{,a}^{(2)}\phi_{,b}^{(0)} + \phi_{,b}^{(2)}\phi_{,a}^{(0)}) - \frac{2}{35}(\phi_{,a}^{(3)}\phi_{,b}^{(1)} + \phi_{,b}^{(3)}\phi_{,a}^{(1)})$$
$$+ \frac{16}{105}\phi_{,a}^{(2)}\phi_{,b}^{(2)} + \frac{1}{10}\phi_{,a}^{(1)}\phi_{,b}^{(1)} + \frac{2}{3}\phi_{,a}^{(0)}\phi_{,b}^{(0)}]$$

$$N_{3b}^{(2)} = h^2[\frac{8}{15}\phi^{(3)}\phi_{,b}^{(0)} + \frac{2}{5}\phi^{(2)}\phi_{,b}^{(1)} - \frac{2}{15}\phi^{(1)}\phi_{,b}^{(2)} + \frac{1}{3}\phi^{(1)}\phi_{,b}^{(0)} - \frac{8}{35}\phi^{(2)}\phi_{,b}^{(3)} - \frac{8}{105}\phi^{(3)}\phi_{,b}^{(2)}]$$

$$N_{33}^{(2)} = h[\frac{8}{15}\phi^{(3)}\phi^{(1)} + \frac{88}{105}\phi^{(3)}\phi^{(3)} + \frac{1}{6}\phi^{(1)}\phi^{(1)} + \frac{8}{5}\phi^{(2)}\phi^{(2)}]$$

(29)

$$N_{ab}^{(3)} = h^4 [\frac{16}{315}(\phi_{,a}^{(3)}\phi_{,b}^{(2)} + \phi_{,b}^{(3)}\phi_{,a}^{(2)}) - \frac{4}{45}(\phi_{,a}^{(0)}\phi_{,b}^{(3)} + \phi_{,b}^{(0)}\phi_{,a}^{(3)})$$
$$- \frac{2}{35}(\phi_{,a}^{(2)}\phi_{,b}^{(1)} + \phi_{,b}^{(2)}\phi_{,a}^{(1)}) + \frac{1}{5}(\phi_{,a}^{(0)}\phi_{,b}^{(1)} + \phi_{,b}^{(0)}\phi_{,a}^{(1)})]$$
$$N_{3b}^{(3)} = h^3 [\frac{1}{10}\phi^{(1)}\phi_{,b}^{(1)} + \frac{8}{35}(\phi^{(3)}\phi_{,b}^{(1)} - \phi^{(2)}\phi_{,b}^{(2)}) - \frac{2}{35}\phi^{(1)}\phi_{,b}^{(3)} - \frac{8}{105}\phi^{(3)}\phi_{,b}^{(3)} + \frac{4}{5}\phi^{(2)}\phi_{,b}^{(0)}]$$
$$N_{33}^{(3)} = h^2 [\frac{4}{5}\phi^{(2)}\phi^{(1)} + \frac{64}{35}\phi^{(2)}\phi^{(3)}]$$

Since we are interested in the low frequency range of extensional and flexural motion, we do not need the shear and thermal correction factors [10] in the constitutive relations. These constitutive relations are not yet in a useful form for our purposes because we have not yet imposed the extensional plate relaxation conditions $(12)_1$ and flexural plate relaxation conditions $(14)_1$, which in the compressed matrix notation take the forms

$$\tau_3^{(0)} = 0 , \quad \tau_4^{(0)} = 0 , \quad \tau_5^{(0)} = 0 \tag{30}$$

$$\tau_3^{(1)} = 0 , \quad \tau_4^{(1)} = 0 , \quad \tau_5^{(1)} = 0 \tag{31}$$

In view of (30) and (31) we introduce a matrix index convention which will be of considerable use in the sequel. We let subscripts u, v, w take the values 3, 4, 5 while subscripts r, s, t take the remaining values 1, 2, 6. This permits us to write (30) and (31) as

$$\tau_w^{(0)} = 0 , \quad \tau_w^{(1)} = 0 \tag{32}$$

We may now write $(28)_{1,2}$ in the form

$$\tau_r^{(0)} = 2hc_{rs}S_s^{(0)} + 2hc_{rv}S_v^{(0)} + 2he_{ar}\phi_{,a}^{(0)} - \frac{4}{3}he_{ar}\phi_{,a}^{(2)}$$
$$+ e_{3r}\phi^{(1)} - 2h\lambda_r\theta^{(0)} - \frac{1}{2}\hat{b}_{ijr}N_{ij}^{(0)}$$
$$\tau_w^{(0)} = 2hc_{ws}S_s^{(0)} + 2hc_{wv}S_v^{(0)} + 2he_{aw}\phi_{,a}^{(0)} - \frac{4}{3}he_{aw}\phi_{,a}^{(2)}$$
$$+ e_{3w}\phi^{(1)} - 2h\lambda_w\theta^{(0)} - \frac{1}{2}\hat{b}_{ijw}N_{ij}^{(0)}$$
$$\tau_r^{(1)} = \frac{2}{3}h^3 c_{rs}S_s^{(1)} + \frac{2}{3}h^3 c_{rv}S_v^{(1)} + \frac{h^2}{3}e_{ar}\phi_{,a}^{(1)} - \frac{4}{15}h^2 e_{ar}\phi_{,a}^{(3)}$$
$$+ \frac{4}{3}he_{3r}\phi^{(2)} - \frac{2}{3}h^3\lambda_r\theta^{(1)} - \frac{1}{2}\hat{b}_{ijr}N_{ij}^{(1)}$$
$$\tau_w^{(1)} = \frac{2}{3}h^3 c_{ws}S_s^{(1)} + \frac{2}{3}h^3 c_{wv}S_v^{(1)} + \frac{h^2}{3}e_{aw}\phi_{,a}^{(1)} - \frac{4}{15}h^2 e_{aw}\phi_{,a}^{(3)}$$
$$+ \frac{4}{3}he_{3w}\phi^{(2)} - \frac{2}{3}h^3\lambda_w\theta^{(1)} - \frac{1}{2}\hat{b}_{ijw}N_{ij}^{(1)} \tag{33}$$

Equations $(33)_{2,4}$ may be readily solved for $S_v^{(0)}$ and $S_v^{(1)}$, respectively, with the

result

$$S_v^{(0)} = -c_{vw}^{-1}c_{ws}S_s^{(0)} - e_{aw}c_{wv}^{-1}\phi_{,a}^{(0)} + \frac{2}{3}e_{aw}c_{wv}^{-1}\phi_{,a}^{(2)} - \frac{1}{2h}e_{3w}c_{wv}^{-1}\phi^{(1)}$$
$$+ \lambda_w c_{wv}^{-1}\theta^{(0)} + \frac{1}{4h}\hat{b}_{ijw}c_{wv}^{-1}N_{ij}^{(0)}$$

$$S_v^{(1)} = -c_{vw}^{-1}c_{ws}S_s^{(1)} - \frac{1}{2h}c_{vw}^{-1}e_{aw}\phi_{,a}^{(1)} + \frac{2}{5h}c_{vw}^{-1}e_{aw}\phi_{,a}^{(3)} - \frac{2}{h^2}c_{vw}^{-1}e_{3w}\phi^{(2)}$$
$$+ \lambda_w c_{wv}^{-1}\theta^{(1)} + \frac{3}{4h^3}\hat{b}_{ijw}c_{wv}^{-1}N_{ij}^{(1)}$$

(34)

where the matrix sums are over the indices 3, 4, 5 as a result of the convention. Substitution from (34) into $(33)_{1,3}$, yields

$$\tau_r^{(0)} = 2h\Gamma_{rs}S_s^{(0)} + 2h\psi_{cr}\phi_{,c}^{(0)} - \frac{4h}{3}\psi_{cr}\phi_{,c}^{(2)} + \psi_{3r}\phi^{(1)} - 2h\lambda_r^p\theta^{(0)} - \frac{1}{2}\hat{b}_{ijr}^p N_{ij}^{(0)}$$

$$\tau_r^{(1)} = \frac{2h^3}{3}\Gamma_{rs}S_s^{(1)} + \frac{h^2}{3}\psi_{cr}\phi_{,c}^{(1)} - \frac{4h^2}{15}\psi_{cr}\phi_{,c}^{(3)} + \frac{4h}{3}\psi_{3r}\phi^{(2)} - \frac{2}{3}h^3\lambda_r^p\theta^{(1)} - \frac{1}{2}\hat{b}_{ijr}^p N_{ij}^{(1)}$$

(35)

where

$$\Gamma_{rs} = c_{rs} - c_{rv}c_{vw}^{-1}c_{ws} , \quad \psi_{ks} = e_{ks} - e_{kw}c_{wv}^{-1}c_{vs}$$
$$\lambda_s^p = \lambda_s - \lambda_w c_{wv}^{-1}c_{vs} , \quad \hat{b}_{ijs}^p = \hat{b}_{ijs} - \hat{b}_{ijw}c_{wv}^{-1}c_{vs}$$

(36)

We note that the Γ_{rs} are the Voigt's anisotropic elastic plate constants and the $\hat{b}_{ijs}^p$ are the effective plate electrostrictive constants. The constitutive relations (35) are in the form required for the equations for anisotropic extension (13) and elementary flexure (16). At this point we note that the substitution of (15) into $(9)_2$ yields

$$S_{ab}^{(1)} = -u_{3,ab}^{(0)}$$

(37)

which is an important relation for the flexural equation. Writing $S_q^{(n)}$ in the constitutive equations $(28)_{3-8}$ as two separate terms, one containing $S_s^{(n)}$ and the other containing $S_v^{(n)}$ in accordance with our values for the indices, and substituting from (34) into the appropriate equations, we obtain

$$D_k^{(0)} = 2h\psi_{ks}S_s^{(0)} - 2h\zeta_{kc}\phi_{,c}^{(0)} + \frac{4h}{3}\zeta_{kc}\phi_{,c}^{(2)} - \zeta_{k3}\phi^{(1)} + 2h\bar{p}_k\theta^{(0)} + \frac{1}{2}\chi_{ijk}^p N_{ij}^{(0)}$$

$$D_k^{(1)} = \frac{2h^3}{3}\psi_{ks}S_s^{(1)} - \frac{h^2}{3}\zeta_{kc}\phi_{,c}^{(1)} + \frac{4h^2}{15}\zeta_{kc}\phi_{,c}^{(3)} - \frac{4h}{3}\zeta_{k3}\phi^{(2)} + \frac{2}{3}h^3\bar{p}_k\theta^{(1)} + \frac{1}{2}\chi_{ijk}^p N_{ij}^{(1)}$$

$$D_k^{(2)} = \frac{2h^3}{3}\psi_{ks}S_s^{(0)} - \frac{2}{3}h^3\zeta_{kc}\phi_{,c}^{(0)} + \frac{4h^3}{45}\hat{\zeta}_{kc}\phi_{,c}^{(2)} - \frac{h^2}{3}\zeta_{k3}\phi^{(1)}$$
$$- \frac{8h^2}{15}\epsilon_{k3}\phi^{(3)} + \frac{2}{3}h^3\bar{p}_k\theta^{(0)} + \frac{h^2}{6}\Delta_{ijk}N_{ij}^{(0)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(2)}$$

$$D_k^{(3)} = \frac{2h^5}{5}\psi_{ks}S_s^{(1)} - \frac{h^4}{5}\zeta_{kc}\phi_{,c}^{(1)} + \frac{4h^4}{175}\bar{\zeta}_{kc}\phi_{,c}^{(3)} - \frac{4h^3}{5}\zeta_{k3}\phi^{(2)}$$
$$+ \frac{2}{5}h^5\bar{p}_k\theta^{(1)} + \frac{3h^2}{10}\Delta_{ijk}N_{ij}^{(1)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(2)}$$

$$\rho\eta^{(0)} = 2h\lambda_s^p S_s^{(0)} - 2h\bar{p}_a\phi_{,a}^{(0)} + \frac{4}{3}h\bar{p}_a\phi_{,a}^{(2)} - \bar{p}_3\phi^{(1)} + 2h\bar{\alpha}\theta^{(0)} + \frac{1}{2}\bar{\gamma}_{ij}N_{ij}^{(0)}$$

$$\rho\eta^{(1)} = \frac{2}{3}h^3\lambda_s^p S_s^{(1)} - \frac{1}{3}h^2\bar{p}_a\phi_{,a}^{(1)} + \frac{4}{15}h^2\bar{p}_a\phi_{,a}^{(3)} - \frac{4}{3}h\bar{p}_3\phi^{(2)} + \frac{2}{3}h^3\bar{\alpha}\theta^{(1)} + \frac{1}{2}\bar{\gamma}_{ij}N_{ij}^{(1)}$$

$$(38)$$

where

$$\zeta_{kj} = \epsilon_{kj} + e_{kv}c_{vw}^{-1}e_{jw}, \quad \hat{\zeta}_{kc} = 3\epsilon_{kc} + 5e_{kv}c_{vw}^{-1}e_{cw}$$

$$\bar{\zeta} = 5\epsilon_{kc} + 7e_{kv}c_{vw}^{-1}e_{cw}, \quad \chi_{ijk}^p = \chi_{ijk} + \Delta_{ijk}$$

$$\Delta_{ijk} = \hat{b}_{ijw}c_{wv}^{-1}e_{kv}, \quad \bar{p}_k = p_k + e_{kv}c_{vw}^{-1}\lambda_w$$

$$\bar{\alpha} = \lambda_w c_{wv}^{-1}\lambda_v, \quad \bar{\gamma}_{ij} = \gamma_{ij} + \hat{b}_{ijw}c_{wv}^{-1}\lambda_v$$

$$(39)$$

The constitutive equations in (38) are in the form required for use in the plate electrostatic equation (6) and heat equation (7). However, in order to use the constitutive equations (35) in the balance equations (13) and (16), they must be converted from matrix form back into tensor form. This is accomplished simply by replacing r by ab and s by cd wherever they occur in (35). For example

$$\tau_{ab}^{(0)} = 2h\Gamma_{abcd}S_{cd}^{(0)} + 2h\psi_{cab}\phi_{,c}^{(0)} - \frac{4h}{3}\psi_{cab}\phi_{,c}^{(2)} + \psi_{3ab}\phi^{(1)} - 2h\lambda_{ab}^p\theta^{(0)} - \frac{1}{2}\hat{b}_{ijab}^p N_{ij}^{(0)}$$

$$\tau_{ab}^{(1)} = \frac{2h^3}{3}\Gamma_{abcd}S_{cd}^{(1)} + \frac{h^2}{3}\psi_{cab}\phi_{,c}^{(1)} - \frac{4h^2}{15}\psi_{cab}\phi_{,c}^{(3)} + \frac{4h}{3}\psi_{3ab}\phi^{(2)} - \frac{2}{3}h^3\lambda_{ab}^p\theta^{(1)}$$

$$- \frac{1}{2}\hat{b}_{ijab}^p N_{ij}^{(1)}$$

$$(40)$$

Equations obtained by substituting from constitutive equations (40) and (38) into (6), (7), (13) and (16), along with $(9)_2$ for $S_{ab}^{(0)}$ and (37) are nine equations in the nine dependent variables $u_j^{(0)}$, $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$, $\theta^{(0)}$ and $\theta^{(1)}$, which are nonlinear in the plate potentials $\phi^{(n)}$. By proper substitutions the boundary conditions can also be written in terms of the same nine variables in each region.

V. SUMMARY OF EQUATIONS

Balance laws

$$\tau_{ab,a}^{(0)} + F_b^{(0)} = 2\rho h\ddot{u}_b^{(0)}$$

$$\tau_{ab,ab}^{(1)} + F_{b,b}^{(1)} + F_3^{(0)} = 2\rho h\ddot{u}_3^{(0)}$$

$$\frac{1}{2h}D_{a,a}^{(0)} + d_3^{(0)} = 0$$

$$\frac{1}{2h}D_{a,a}^{(1)} - \frac{1}{2h}D_3^{(0)} + d_3^{(1)} = 0$$

$$\frac{1}{h^2}D_{a,a}^{(2)} - D_{a,a}^{(0)} - \frac{2}{h^2}D_3^{(1)} = 0$$

$$(41)$$

$$\frac{1}{h^3}D_{a,a}^{(3)} - \frac{1}{h}D_{a,a}^{(1)} - \frac{3}{h^3}D_3^{(2)} + \frac{1}{h}D_3^{(0)} = 0$$

$$Q_{a,a}^{(0)} + q_3^{(0)} = -T\rho\dot{\eta}^{(0)}$$

$$Q_{a,a}^{(1)} - Q_3^{(0)} + q_3^{(1)} = -T\rho\dot{\eta}^{(1)}$$

Strain-displacement relations

$$S_{ab}^{(0)} = \frac{1}{2}(u_{a,b}^{(0)} + u_{b,a}^{(0)})$$

$$S_{ab}^{(1)} = -u_{3,ab}^{(0)}$$

$$(42)$$

Constitutive equations

$$\tau_{ab}^{(0)} = 2h\Gamma_{abcd}S_{cd}^{(0)} + 2h\psi_{cab}\phi_{,c}^{(0)} - \frac{4h}{3}\psi_{cab}\phi_{,c}^{(2)} + \psi_{3ab}\phi^{(1)} - 2h\lambda_{ab}^p\theta^{(0)} - \frac{1}{2}\hat{b}_{ijab}^p N_{ij}^{(0)}$$

$$\tau_{ab}^{(1)} = \frac{2h^3}{3}\Gamma_{abcd}S_{cd}^{(1)} + \frac{h^2}{3}\psi_{cab}\phi_{,c}^{(1)} - \frac{4h^2}{15}\psi_{cab}\phi_{,c}^{(3)} + \frac{4h}{3}\psi_{3ab}\phi^{(2)} - \frac{2}{3}h^3\lambda_{ab}^p\theta^{(1)} - \frac{1}{2}\hat{b}_{ijab}^p N_{ij}^{(1)}$$

$$D_k^{(0)} = 2h\psi_{kcd}S_{cd}^{(0)} - 2h\zeta_{kc}\phi_{,c}^{(0)} + \frac{4h}{3}\zeta_{kc}\phi_{,c}^{(2)} - \zeta_{k3}\phi^{(1)} + 2h\bar{p}_k\theta^{(0)} + \frac{1}{2}\chi_{ijk}^p N_{ij}^{(0)}$$

$$D_k^{(1)} = \frac{2h^3}{3}\psi_{kcd}S_{cd}^{(1)} - \frac{h^2}{3}\zeta_{kc}\phi_{,c}^{(1)} + \frac{4h^2}{15}\zeta_{kc}\phi_{,c}^{(3)} - \frac{4h}{3}\zeta_{k3}\phi^{(2)} + \frac{2}{3}h^3\bar{p}_k\theta^{(1)} + \frac{1}{2}\chi_{ijk}^p N_{ij}^{(1)}$$

$$D_k^{(2)} = \frac{2h^3}{3}\psi_{kcd}S_{cd}^{(0)} - \frac{2}{3}h^3\zeta_{kc}\phi_{,c}^{(0)} + \frac{4h^3}{45}\hat{\zeta}_{kc}\phi_{,c}^{(2)} - \frac{h^2}{3}\zeta_{k3}\phi^{(1)}$$
$$\qquad - \frac{8h^2}{15}\epsilon_{k3}\phi^{(3)} + \frac{2}{3}h^3\bar{p}_k\theta^{(0)} + \frac{h^2}{6}\Delta_{ijk}N_{ij}^{(0)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(2)}$$

$$D_k^{(3)} = \frac{2h^5}{5}\psi_{kcd}S_{cd}^{(1)} - \frac{h^4}{5}\zeta_{kc}\phi_{,c}^{(1)} + \frac{4h^4}{175}\bar{\zeta}_{kc}\phi_{,c}^{(3)} - \frac{4h^3}{5}\zeta_{k3}\phi^{(2)}$$
$$\qquad + \frac{2}{5}h^5\bar{p}_k\theta^{(1)} + \frac{3h^2}{10}\Delta_{ijk}N_{ij}^{(1)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(2)}$$

$$\rho\eta^{(0)} = 2h\lambda_{cd}^p S_{cd}^{(0)} - 2h\bar{p}_a\phi_{,a}^{(0)} + \frac{4}{3}h\bar{p}_a\phi_{,a}^{(2)} - \bar{p}_3\phi^{(1)} + 2h\bar{\alpha}\theta^{(0)} + \frac{1}{2}\bar{\gamma}_{ij}N_{ij}^{(0)}$$

$$\rho\eta^{(1)} = \frac{2}{3}h^3\lambda_{cd}^p S_{cd}^{(1)} - \frac{1}{3}h^2\bar{p}_a\phi_{,a}^{(1)} + \frac{4}{15}h^2\bar{p}_a\phi_{,a}^{(3)} - \frac{4}{3}h\bar{p}_3\phi^{(2)} + \frac{2}{3}h^3\bar{\alpha}\theta^{(1)} + \frac{1}{2}\bar{\gamma}_{ij}N_{ij}^{(1)}$$

$$Q_i^{(0)} = -2h\kappa_{ij}(\theta_{,j}^{(0)} + \delta_{3j}\theta^{(1)})$$

$$Q_i^{(1)} = -\frac{2}{3}h^3\kappa_{ij}\theta_{,j}^{(1)}$$

$$(43)$$

Continuity conditions

$$n_a[\![\tau_{ab}^{(0)}]\!] = 0 \ , \quad [\![u_b^{(0)}]\!] = 0$$

$$n_a[\![\tau_{ab}^{(1)}]\!]n_b = 0 \ , \quad [\![\tau_{n3}^{(0)} + \frac{\partial\tau_{ns}^{(1)}}{\partial s}]\!] = 0 \ , \quad [\![\frac{\partial u_3^{(0)}}{\partial n}]\!] = 0 \ , \quad [\![u_3^{(0)}]\!] = 0$$

$$[\![\phi^{(n)}]\!] = 0 \ , n = 0, 1, 2, 3$$

$$n_a[\![D_a^{(0)}]\!] = 0 \ , \quad n_a[\![D_a^{(1)}]\!] = 0$$

$$(44)$$

$$n_a[D_a^{(2)} - h^2 D_a^{(0)}] = 0 , \quad n_a[\![D_a^{(3)} - h^2 D_a^{(1)}]\!] = 0$$

$$[\theta^{(0)}]\!] = 0 , \quad [\![\theta^{(1)}]\!] = 0 , \quad n_a[\![Q_a^{(0)}]\!] = 0 , \quad n_a[\![Q_a^{(1)}]\!] = 0$$

$$[\theta^{(0)}]\!] = 0 , \quad [\![\theta^{(1)}]\!] = 0 , \quad n_a[\![Q_a^{(0)}]\!] = 0 , \quad n_a[\![Q_a^{(1)}]\!] = 0$$

Definitions of material constants

$$\Gamma_{rs} = c_{rs} - c_{rv}c_{vw}^{-1}c_{ws} , \quad \psi_{ks} = e_{ks} - e_{kw}c_{wv}^{-1}c_{vs}$$

$$\lambda_s^p = \lambda_s - \lambda_w c_{wv}^{-1}c_{vs} , \quad \hat{b}_{ijs}^p = \hat{b}_{ijs} - \hat{b}_{ijw}c_{wv}^{-1}c_{vs}$$

$$\zeta_{kj} = \epsilon_{kj} + e_{kv}c_{vw}^{-1}e_{jw} , \quad \zeta_{kc} = 3\epsilon_{kc} + 5e_{kv}c_{vw}^{-1}e_{cw}$$

$$\bar{\zeta} = 5\epsilon_{kc} + 7e_{kv}c_{vw}^{-1}e_{cw} , \quad \chi_{ijk}^p = \chi_{ijk} + \Delta_{ijk} \tag{45}$$

$$\Delta_{ijk} = \hat{b}_{ijw}c_{wv}^{-1}e_{kv} , \quad \bar{p}_k = p_k + e_{kv}c_{vw}^{-1}\lambda_w$$

$$\bar{\alpha} = \lambda_w c_{wv}^{-1}\lambda_v , \quad \bar{\gamma}_{ij} = \gamma_{ij} + \hat{b}_{ijw}c_{wv}^{-1}\lambda_v$$

Ranges of indices

$$i,j,k,l : 1,2,3; \quad a,b,c,d : 1,2; \quad p,q : 1,2,3,4,5,6; \quad r,s,t : 1,2,6; \quad u,v,w : 3,4,5 \tag{46}$$

VI. CONCLUSIONS

A set of two-dimensional plate equations has been derived for the extension and flexure of a thermopiezoelectric plate under strong electric fields. The equations are nonlinear in the plate electric potentials and can be applied to fully or partially electroded actuators or sensors. These equations include the results of [1], [7] and [13] as special cases.

ACKNOWLEDGEMENTS

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