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On the Coupling of Pressure and Deviatoric Stress in Isotropic Hyperelastic Materials

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For isotropic elastic materials, we study the coupling of pressure and shear stress that results from the dependence of shear stress on volumetric strain and the dependence of pressure on shear strain.

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1 Introduction

The *Cauchy stress tensor* $\mathbf{T}$ can be decomposed into a spherical part,[1] $-p\mathbf{I}$, and a deviatoric part, $\mathbf{T}^*$:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}^*, \qquad t_i = -p + t_i^*. \tag{1.1}$$

The *pressure* p is given by

$$p \equiv -\tfrac{1}{3}\mathrm{tr}\,\mathbf{T} = -\tfrac{1}{3}(t_1 + t_2 + t_3), \tag{1.2}$$

and the *deviatoric stress tensor* $\mathbf{T}^*$ is given by

$$\mathbf{T}^* \equiv \mathbf{T} - \tfrac{1}{3}(\mathrm{tr}\,\mathbf{T})\mathbf{I}, \qquad t_i^* = t_i - \tfrac{1}{3}(t_1 + t_2 + t_3), \tag{1.3}$$

so that

$$\mathrm{tr}\,\mathbf{T}^* = t_1^* + t_2^* + t_3^* = 0. \tag{1.4}$$

The *principal stresses* t_i and *principal deviatoric stresses* t_i^* are the principal values of $\mathbf{T}$ and $\mathbf{T}^*$, respectively. For an isotropic elastic material the linear theory predicts that the pressure depends only on the volumetric strain, whereas $\mathbf{T}^*$, which is a tensorial measure of shear stress, depends only on the shear strain.

The object of this report is to study those aspects of the nonlinear elastic response of polycrystalline metals and ceramics which may be important in high velocity impacts, where large elastic increases in density are encountered. As is well-known, a nonlinear pressure/density relation is generally required for such applications. Here we focus on other nonlinear effects that are often ignored or treated incorrectly. In particular, we study the coupling of the pressure and deviatoric stress which results from the dependence of shear stress on volumetric strain and the dependence of pressure on shear strain.

We begin with a brief review of relevant results from the theory of hyperelastic materials.[2] Let $\mathbf{F}$ denote the *deformation gradient* relative to some fixed reference configuration. An *elastic* (or *Cauchy-elastic*) material is one for which $\mathbf{T}$ is a function of $\mathbf{F}$ only. A *hyperelastic* (or *Green-elastic*) material is an elastic material for which the *first Piola-Kirchhoff stress tensor* $(\det\mathbf{F})\mathbf{T}(\mathbf{F}^{-1})^T$ is the gradient of some real-valued function ε of $\mathbf{F}$, in

[1] Second-order tensors are denoted by boldface uppercase Roman letters. The *identity tensor* is denoted by $\mathbf{I}$, and tr denotes the *trace* function. The *deviatoric part* and the *transpose* of any tensor $\mathbf{A}$ are denoted by $\mathbf{A}^*$ and $\mathbf{A}^\mathsf{T}$, respectively. The *norm* of $\mathbf{A}$ is $\|\mathbf{A}\| \equiv \mathrm{tr}\,(\mathbf{A}^\mathsf{T}\mathbf{A})$.

[2] A standard reference is Truesdell and Noll [1].

which case ε is called the *strain energy* or the *stored energy* per unit reference volume.[3] We consider only hyperelastic materials here. For simplicity we also assume the material is isotropic. These conditions, together with the requirement that constitutive equations be properly invariant under changes of the frame of reference, yield various reduced forms (see below) for the strain energy and the Cauchy stress tensor.

The *left stretch tensor* $\mathbf{V}$ is the unique symmetric positive-definite tensor occurring in the *left polar decomposition* of the deformation gradient:

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \qquad \mathbf{V}^2 = \mathbf{B} \equiv \mathbf{F}\mathbf{F}^T, \tag{1.5}$$

where $\mathbf{R}$, the *rotation tensor*, is proper orthogonal, and $\mathbf{B}$, the *left Cauchy-Green tensor*, is symmetric positive-definite. The tensors $\mathbf{V}$ and $\mathbf{B}$ share a common set of principal axes, called the *principal axes of strain in the deformed state* or the *Eulerian strain axes*. The *principal stretches* λ_i $(i = 1, 2, 3)$ are the principal values of $\mathbf{V}$, and the *principal strains* are $\lambda_i - 1$. The principal values of $\mathbf{B}$ are $b_i = \lambda_i^2$. Let J denote the *Jacobian* of the deformation, and let $\tilde{\rho}$ denote the ratio of the densities ρ and ρ_0 in the deformed and reference configurations, respectively:

$$J \equiv \det \mathbf{F} = \det \mathbf{V} = \lambda_1 \lambda_2 \lambda_3, \qquad \tilde{\rho} \equiv \rho/\rho_0 = 1/J. \tag{1.6}$$

Relative to an undistorted reference configuration of an isotropic hyperelastic material, ε and $\mathbf{T}$ may be expressed as isotropic functions of $\mathbf{V}$ or $\mathbf{B}$; and the Eulerian strain axes are principal axes for $\mathbf{T}$. ε may also be expressed as a symmetric function of $\lambda_1, \lambda_2, \lambda_3$ or b_1, b_2, b_3. Then[4]

$$\mathbf{T} = \tilde{\rho}\,\mathbf{V}\frac{\partial \varepsilon}{\partial \mathbf{V}} = 2\tilde{\rho}\,\mathbf{B}\frac{\partial \varepsilon}{\partial \mathbf{B}}, \qquad t_i = \tilde{\rho}\lambda_i \frac{\partial \varepsilon}{\partial \lambda_i} = 2\tilde{\rho}\, b_i \frac{\partial \varepsilon}{\partial b_i}. \tag{1.7}$$

Note that the summation convention is not used here or elsewhere.

[3]This restriction on the constitutive equation is motivated by the restrictions imposed on thermoelastic materials by the second law of thermodynamics. A thermoelastic material reduces to a hyperelastic material for isentropic deformations, provided we take ε to be the internal energy per unit reference volume. The same thermoelastic material reduces to a (generally different) hyperelastic material for isothermal deformations, provided take ε to be the free energy per unit reference volume.

[4]If ε is a scalar-valued function of a symmetric tensor $\mathbf{A}$, then the gradient of ε at $\mathbf{A}$ is the symmetric tensor $\partial\varepsilon/\partial\mathbf{A}$ with the property $\mathrm{tr}\,[(\partial\varepsilon/\partial\mathbf{A})\mathbf{S}] = d/dt\ \varepsilon(\mathbf{A} + t\mathbf{S})|_{t=0}$, where $\mathbf{S}$ is any tensor such that $\mathbf{A} + t\mathbf{S}$ lies in the domain of ε for sufficiently small t. Furthermore, ε is an isotropic function of the symmetric tensor $\mathbf{A}$ iff ε is a symmetric function of the principal values a_i of $\mathbf{A}$, in which case $\partial\varepsilon/\partial\mathbf{A}$ is coaxial with $\mathbf{A}$ and has corresponding principal values $\partial\varepsilon/\partial a_i$.

2 The Logarithmic Strain Tensor

The *principal logarithmic strains* l_i are the logarithms of the principal stretches, and the *(Eulerian) logarithmic strain tensor* $\mathbf{L}$ is the tensor coaxial with $\mathbf{V}$ and $\mathbf{B}$ but with corresponding principal values equal to the principal logarithmic strains:

$$l_i \equiv \ln \lambda_i = \tfrac{1}{2} \ln b_i\,, \qquad \mathbf{L} = \ln \mathbf{V} = \tfrac{1}{2} \ln \mathbf{B}\,. \tag{2.1}$$

Then the strain energy function ε of an an isotropic hyperelastic material can be expressed as an isotropic function of $\mathbf{L}$ or, equivalently, as a symmetric function of the principal logarithmic strains. From $(1.7)_3$ and $(2.1)_1$ it follows that

$$t_i = \tilde{\rho}\, \frac{\partial \varepsilon}{\partial l_i}\,, \qquad \mathbf{T} = \tilde{\rho}\, \frac{\partial \varepsilon}{\partial \mathbf{L}}\,. \tag{2.2}$$

The logarithmic strain tensor has several interesting properties. By (2.1) and (1.6) we have

$$\operatorname{tr} \mathbf{L} = l_1 + l_2 + l_3 = \ln(\lambda_1 \lambda_2 \lambda_3) = \ln J = -\ln \tilde{\rho}\,. \tag{2.3}$$

Hence $\operatorname{tr} \mathbf{L}$ is a measure of the volumetric strain. The deviatoric part $\mathbf{L}^*$ of $\mathbf{L}$ has principal values

$$\begin{aligned} l_i^* &= l_i - \tfrac{1}{3}(l_1 + l_2 + l_3) = l_i - \tfrac{1}{3} \ln J \\ &= \ln \left(\frac{\lambda_i}{J^{1/3}} \right) = \frac{1}{3} \left[\ln \left(\frac{\lambda_i}{\lambda_j} \right) + \ln \left(\frac{\lambda_i}{\lambda_k} \right) \right], \end{aligned} \tag{2.4}$$

where i, j, k always denotes a permutation of $1, 2, 3$. Now consider another deformation which differs from the given one by a superimposed dilatation and rotation, so that this second deformation has left stretch tensor $a\mathbf{V}$ and principal stretches $a\lambda_1, a\lambda_2, a\lambda_3$ for some $a > 0$. Then $\mathbf{L}^*$ is the same for both deformations, since the factor a cancels out in (2.4). Hence the *deviatoric logarithmic strain tensor* $\mathbf{L}^*$ is independent of the dilatational part of the deformation gradient, i.e., it is independent of the volumetric stretch or strain as measured by (1.6) or (2.3). Therefore $\mathbf{L}^*$ is a tensorial measure of shear strain only.

From the results above it follows that ε, $\mathbf{T}$, $\mathbf{T}^*$, and p may be regarded as functions of the independent variables $\mathbf{L}^*$ and J (or $\tilde{\rho}$), with each function isotropic in $\mathbf{L}^*$ for fixed J (or $\tilde{\rho}$). Then it can be shown that $(2.2)_2$ is

equivalent to the conditions[5]

$$p = -\left(\frac{\partial \varepsilon}{\partial J}\right)_{\mathbf{L}^*} \quad \text{and} \quad J\mathbf{T}^* = \left(\frac{\partial \varepsilon}{\partial \mathbf{L}^*}\right)_J , \tag{2.5}$$

where the subscript denotes the variable held constant during differentiation. Equivalently,

$$p = \tilde{\rho}^2 \left(\frac{\partial \varepsilon}{\partial \tilde{\rho}}\right)_{\mathbf{L}^*} \quad \text{and} \quad \mathbf{T}^* = \tilde{\rho}\left(\frac{\partial \varepsilon}{\partial \mathbf{L}^*}\right)_{\tilde{\rho}} . \tag{2.6}$$

In terms of the *pressure response functions*

$$p = \hat{p}(J, \mathbf{L}^*) = \bar{p}(\tilde{\rho}, \mathbf{L}^*) , \tag{2.7}$$

equations $(2.5)_1$ and $(2.6)_1$ may be integrated to yield

$$\varepsilon = \int_J^1 \hat{p}(\xi, \mathbf{L}^*)\, d\xi + \hat{\varepsilon}(\mathbf{L}^*) = \int_1^{\tilde{\rho}} \bar{p}(\xi, \mathbf{L}^*)\xi^{-2}\, d\xi + \hat{\varepsilon}(\mathbf{L}^*) \tag{2.8}$$

for some isotropic function $\hat{\varepsilon}$ of $\mathbf{L}^*$ only. From (2.8) we see that *the pressure response function of an isotropic hyperelastic material determines the strain energy function to within an arbitrary function of* $\mathbf{L}^*$ *only*. Also note that *for an isotropic elastic material, the existence of a strain energy function places no restrictions on the pressure response function*. That is, given any isotropic function $\hat{p}$ of J and $\mathbf{L}^*$, there exist infinitely many strain energy functions ε, namely those given by $(2.8)_1$, for which the corresponding pressure response function is $p = \hat{p}(J, \mathbf{L}^*)$; then the response function for the deviatoric stress tensor is determined from $(2.5)_2$.

In practice we are usually faced with a different situation—namely, we have a limited amount of experimental data on the material response, and these data impose restrictions on the response functions for both the pressure and the deviatoric stress. We then need to determine any additional restrictions imposed on these response functions due to the existence of a strain energy function. We address this problem in the next section for small shear strains. The following result, which requires no restrictions on the magnitude of the shear strain, is an immediate consequence of (2.5) or (2.6).

[5]To prove $(2.5)_1$, for example, note that since l_i^* is independent of J, (2.4) yields $\partial l_i/\partial J = 1/(3J)$. Then

$$\frac{\partial \varepsilon}{\partial J} = \sum_{i=1}^{3} \frac{\partial \varepsilon}{\partial l_i}\frac{\partial l_i}{\partial J} = \frac{1}{3J}\sum_{i=1}^{3} \frac{\partial \varepsilon}{\partial l_i} .$$

But from (1.2), (2.2), and $\tilde{\rho} = 1/J$, we see that the expression on the right above is equal to $-p$.

For an isotropic hyperelastic material, the following three conditions are equivalent:

(i) *The strain energy decouples additively into a function of density only and a function of shear strain only, i.e.,* $\varepsilon = \tilde{\varepsilon}(\tilde{\rho}) + \hat{\varepsilon}(\mathbf{L}^*)$;

(ii) *The pressure depends only on the density, i.e.,* $p = \tilde{p}(\tilde{\rho})$;

(iii) $J\mathbf{T}^*$ *is independent of the density.*

If these conditions hold, then $\tilde{p}(\tilde{\rho}) = \tilde{\rho}^2 d\tilde{\varepsilon}/d\tilde{\rho}$ and $\mathbf{T}^* = \tilde{\rho}\,\partial\hat{\varepsilon}/\partial\mathbf{L}^*$. In particular, either (i) or (ii) implies $\mathbf{T}^*$ is proportional to the density ratio $\tilde{\rho}$. In Section §4 we show that these properties are generally inconsistent with experimental data.

In the remainder of this section we consider some relations involving the invariants of $\mathbf{L}^*$. These results are utilized in the next section. Let γ^* and δ^* denote the *second* and *third moments* of $\mathbf{L}^*$:

$$\begin{aligned} \gamma^* &\equiv \operatorname{tr}[(\mathbf{L}^*)^2] = \|\mathbf{L}^*\|^2 = (l_1^*)^2 + (l_2^*)^2 + (l_3^*)^2 \\ &= -2(l_1^* l_2^* + l_2^* l_3^* + l_3^* l_1^*) = 2[(l_i^*)^2 + l_i^* l_j^* + (l_j^*)^2] \\ &= \tfrac{1}{3}[(l_1^* - l_2^*)^2 + (l_2^* - l_3^*)^2 + (l_3^* - l_1^*)^2]\,, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \delta^* &\equiv \operatorname{tr}[(\mathbf{L}^*)^3] = (l_1^*)^3 + (l_2^*)^3 + (l_3^*)^3 \\ &= 3\det\mathbf{L}^* = 3l_1^* l_2^* l_3^* = -\sum_{m\neq n}(l_m^*)^2 l_n^*\,, \end{aligned} \tag{2.10}$$

where we have used the fact that

$$\operatorname{tr}\mathbf{L}^* = l_1^* + l_2^* + l_3^* = 0\,. \tag{2.11}$$

Both γ^* and δ^* are isotropic scalar measures of shear strain which are independent of the dilatational part of the deformation. Alternate expressions for γ^* follow from $(2.9)_6$ and the identities

$$l_i^* - l_j^* = l_i - l_j = \ln\left(\frac{\lambda_i}{\lambda_j}\right)\,. \tag{2.12}$$

Note that $\gamma^* \geq 0$ and that $\sqrt{\gamma^*} = \|\mathbf{L}^*\|$ is the norm of $\mathbf{L}^*$, which may be interpreted as the *equivalent shear strain*; it is zero iff the deformation gradient is a dilatation superimposed on a rotation:

$$\gamma^* = 0 \;\Leftrightarrow\; \|\mathbf{L}^*\| = 0 \;\Leftrightarrow\; \mathbf{L}^* = \mathbf{0} \;\Leftrightarrow\; \mathbf{V} = J^{1/3}\mathbf{I} \;\Leftrightarrow\; \mathbf{F} = J^{1/3}\mathbf{R}\,. \tag{2.13}$$

Of course, $\gamma^* = 0 \Rightarrow \delta^* = 0$, but the converse does not hold.[6]

Let $\mathbf{L}^{**}$ denote the deviatoric part of the square of the deviatoric part of $\mathbf{L}$:

$$\mathbf{L}^{**} \equiv [(\mathbf{L}^*)^2]^* = (\mathbf{L}^*)^2 - \tfrac{1}{3}\gamma^*\mathbf{I}. \tag{2.14}$$

Then the principal values l_i^{**} of $\mathbf{L}^{**}$ are given by

$$l_i^{**} = (l_i^*)^2 - \tfrac{1}{3}\gamma^* = \tfrac{1}{3}[(l_i^*)^2 + 2l_j^* l_k^*] = l_j^* l_k^* + \tfrac{1}{6}\gamma^*. \tag{2.15}$$

From (2.3), $(2.4)_1$, (2.9)–(2.11), and (2.15), we find that[7]

$$\frac{\partial\tilde{\rho}}{\partial l_i} = -\tilde{\rho}, \quad \frac{\partial\gamma^*}{\partial l_i} = 2l_i^*, \quad \frac{\partial\delta^*}{\partial l_i} = 3l_i^{**}. \tag{2.16}$$

As observed previously, any function of $\mathbf{L}$ can be regarded as a function of $\tilde{\rho}$ and $\mathbf{L}^*$. Therefore any scalar-valued isotropic function of $\mathbf{L}$ can be expressed as a function of $\tilde{\rho}$ and the second and third moments of $\mathbf{L}^*$. In particular, this applies to the strain energy ε and the pressure p:

$$\varepsilon = \varepsilon(\tilde{\rho}, \gamma^*, \delta^*), \qquad p = p(\tilde{\rho}, \gamma^*, \delta^*) = \tilde{\rho}^2 \left(\frac{\partial\varepsilon}{\partial\tilde{\rho}}\right)_{\gamma^*,\delta^*}, \tag{2.17}$$

where $(2.6)_1$ has been used. Then from (1.1), $(2.2)_1$, $(2.17)_3$, and (2.16), we obtain the following formula for the principal deviatoric stresses in an isotropic hyperelastic material:

$$t_i^* = 2\tilde{\rho}\left(\frac{\partial\varepsilon}{\partial\gamma^*}\right)_{\tilde{\rho},\delta^*} l_i^* + 3\tilde{\rho}\left(\frac{\partial\varepsilon}{\partial\delta^*}\right)_{\tilde{\rho},\gamma^*} l_i^{**}; \tag{2.18}$$

equivalently,

$$\mathbf{T}^* = 2\tilde{\rho}\left(\frac{\partial\varepsilon}{\partial\gamma^*}\right)_{\tilde{\rho},\delta^*} \mathbf{L}^* + 3\tilde{\rho}\left(\frac{\partial\varepsilon}{\partial\delta^*}\right)_{\tilde{\rho},\gamma^*} \mathbf{L}^{**}. \tag{2.19}$$

Note that the coefficients of $\mathbf{L}^*$ and $\mathbf{L}^{**}$ in (2.19) generally depend on all three of the variables $\tilde{\rho}$, γ^*, and δ^*.

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[6] If the principal stretches are ordered so that $\lambda_1 \le \lambda_2 \le \lambda_3$, then

$$\delta^* = 0 \;\Leftrightarrow\; l_2^* = 0 \;\Leftrightarrow\; l_1^* = -l_3^* \;\Leftrightarrow\; l_2 = \tfrac{1}{2}(l_1 + l_3) \;\Leftrightarrow\; \lambda_2 = \sqrt{\lambda_1\lambda_3}.$$

In particular, $\delta^* = 0$ for a simple shear (where $\lambda_1 = 1/\lambda_3$ and $\lambda_2 = 1$) and also for a simple shear with superimposed dilatation (where $\lambda_1 = J^{2/3}/\lambda_3$ and $\lambda_2 = J^{1/3}$).

[7] In deriving $(2.16)_{2,3}$ it is useful to first establish the following results: $\partial l_i^*/\partial l_i = \frac{2}{3}$, $\partial l_i^*/\partial l_j = -\frac{1}{3}$, $\partial l_i^{**}/\partial l_i = \frac{2}{3}l_i^*$, and $\partial l_i^{**}/\partial l_j = \frac{2}{3}l_k^*$.

3 Approximate Formulas

In this section we derive approximate formulas for the strain energy, pressure, and deviatoric stress that are valid for arbitrarily large changes in density and sufficiently small shear strains. From $(2.17)_1$, (2.9), and (2.10), it follows that

$$\begin{aligned} \varepsilon &= \tilde{\varepsilon}(\tilde{\rho}) + \varepsilon_2(\tilde{\rho})\,\gamma^* + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^3) \\ &= \tilde{\varepsilon}(\tilde{\rho}) + \varepsilon_2(\tilde{\rho})\,\gamma^* + \varepsilon_3(\tilde{\rho})\,\delta^* + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^4) \end{aligned} \tag{3.1}$$

for some functions $\tilde{\varepsilon}$, ε_2, and ε_3 of $\tilde{\rho}$ only. Here $\mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^n)$ denotes a function of $\tilde{\rho}$ and $\mathbf{L}^*$ which is of order n in $\mathbf{L}^*$; i.e., there is a function $M(\tilde{\rho})$ such that $\mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^n) \le M(\tilde{\rho})\,\|\mathbf{L}^*\|^n$ for $\mathbf{L}^*$ sufficiently close to $\mathbf{0}$. Similarly, from $(2.17)_2$ we have

$$\begin{aligned} p &= \tilde{p}(\tilde{\rho}) + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^2) \\ &= \tilde{p}(\tilde{\rho}) + p_2(\tilde{\rho})\,\gamma^* + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^3) \\ &= \tilde{p}(\tilde{\rho}) + p_2(\tilde{\rho})\,\gamma^* + p_3(\tilde{\rho})\,\delta^* + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^4) \end{aligned} \tag{3.2}$$

for some functions $\tilde{p}$, p_2, and p_3 of $\tilde{\rho}$ only. Since γ^* and δ^* are independent of $\tilde{\rho}$, $(2.17)_3$ and (3.1) imply

$$\tilde{p}(\tilde{\rho}) = \tilde{\rho}^2 \frac{d\tilde{\varepsilon}}{d\tilde{\rho}}, \quad p_2(\tilde{\rho}) = \tilde{\rho}^2 \frac{d\varepsilon_2}{d\tilde{\rho}}, \quad p_3(\tilde{\rho}) = \tilde{\rho}^2 \frac{d\varepsilon_3}{d\tilde{\rho}}. \tag{3.3}$$

We call $\tilde{p} = \tilde{p}(\tilde{\rho})$ the *hydrostatic pressure* at the density ratio $\tilde{\rho}$. It is the pressure the material would experience if the shear strain were zero; i.e., if the conditions in (2.13) hold, in which case (1.1) and (2.19) require that $\mathbf{T} = -p\mathbf{I} = -\tilde{p}(\tilde{\rho})\mathbf{I}$. In accordance with experimental data for most materials, we assume $d\tilde{p}/d\tilde{\rho} > 0$. Then the *bulk modulus* κ is positive:

$$0 < \kappa \equiv \tilde{\rho}\,\frac{d\tilde{p}}{d\tilde{\rho}} = \rho\,\frac{d\tilde{p}}{d\rho} = -J\,\frac{d\tilde{p}}{dJ}, \tag{3.4}$$

and $\tilde{\rho}$ may be regarded as a function of the hydrostatic pressure $\tilde{p}$. Hence any function of $\tilde{\rho}$, such as κ and the functions μ and ω below, may also be regarded as a function of $\tilde{p}$.

From (2.18) and (3.1), the principal deviatoric stresses satisfy

$$\begin{aligned} t_i^* &= 2\mu(\tilde{\rho})\,l_i^* + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^2) \\ &= 2\mu(\tilde{\rho})\,l_i^* + 3\,\omega(\tilde{\rho})\,l_i^{**} + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^3), \end{aligned} \tag{3.5}$$

where $\mu(\tilde{\rho})$, the *shear modulus* at density ratio $\tilde{\rho}$, is given by

$$\mu(\tilde{\rho}) = \tilde{\rho}\,\varepsilon_2(\tilde{\rho}) = \tilde{\rho}\left(\frac{\partial \varepsilon}{\partial \gamma^*}\right)_{\tilde{\rho},\delta^*}\Bigg|_{\gamma^*=\delta^*=0}, \tag{3.6}$$

and

$$\omega(\tilde{\rho}) = \tilde{\rho}\,\varepsilon_3(\tilde{\rho}) = \tilde{\rho}\left(\frac{\partial \varepsilon}{\partial \delta^*}\right)_{\tilde{\rho},\gamma^*}\Bigg|_{\gamma^*=\delta^*=0}. \tag{3.7}$$

The relations (3.5) are equivalent to

$$\begin{aligned} \mathbf{T}^* &= 2\mu(\tilde{\rho})\mathbf{L}^* + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^2) \\ &= 2\mu(\tilde{\rho})\mathbf{L}^* + 3\omega(\tilde{\rho})\mathbf{L}^{**} + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^3). \end{aligned} \tag{3.8}$$

From the constitutive relations (3.2) and (3.5) or (3.8), we see that in addition to the dependence of pressure on volumetric strain and the dependence of deviatoric stress on shear strain, the pressure will also generally depend on the shear strain, and the deviatoric stress will also generally depend on the volumetric strain. Furthermore, these latter dependencies are coupled. Indeed, from $(3.3)_{2,3}$, $(3.6)_1$, and $(3.7)_1$, it follows that the coefficients $p_2(\tilde{\rho})$ and $p_3(\tilde{\rho})$ in the expansion (3.2) for the pressure are related to μ and ω by

$$p_2(\tilde{\rho}) = \tilde{\rho}^2 \frac{d}{d\tilde{\rho}}\left(\frac{\mu}{\tilde{\rho}}\right) = \tilde{\rho}\,\frac{d\mu}{d\tilde{\rho}} - \mu = \kappa\left(\frac{d\mu}{d\tilde{p}} - \frac{\mu}{\kappa}\right), \tag{3.9}$$

$$p_3(\tilde{\rho}) = \tilde{\rho}^2 \frac{d}{d\tilde{\rho}}\left(\frac{\omega}{\tilde{\rho}}\right) = \tilde{\rho}\,\frac{d\omega}{d\tilde{\rho}} - \omega = \kappa\left(\frac{d\omega}{d\tilde{p}} - \frac{\omega}{\kappa}\right). \tag{3.10}$$

This coupling of the pressure and deviatoric stress may be made more explicit as follows. First, note that from (2.9) and either (3.8) or (3.5), we have

$$\|\mathbf{T}^*\|^2 = (t_1^*)^2 + (t_2^*)^2 + (t_3^*)^2 = [2\mu(\tilde{\rho})]^2\gamma^* + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{L}^*\|^3). \tag{3.11}$$

Then on solving (3.11) for γ^* and substituting into $(3.2)_2$, we obtain the following relation between the pressure p, the hydrostatic pressure $\tilde{p} = \tilde{p}(\tilde{\rho})$, the shear modulus μ, and the *equivalent shear stress* $\|\mathbf{T}^*\|$:

$$\begin{aligned} p &= \tilde{p}(\tilde{\rho}) + \frac{1}{4\mu^2}\left(\tilde{\rho}\,\frac{d\mu}{d\tilde{\rho}} - \mu\right)\|\mathbf{T}^*\|^2 + \mathcal{O}_{\tilde{\rho}}(\|\mathbf{T}^*\|^3) \\ &= \tilde{p} + \frac{1}{4\mu}\left(\frac{\kappa}{\mu}\frac{d\mu}{d\tilde{p}} - 1\right)\|\mathbf{T}^*\|^2 + \mathcal{O}_{\tilde{p}}(\|\mathbf{T}^*\|^3), \end{aligned} \tag{3.12}$$

where in $(3.12)_1$ the coefficient of $\|\mathbf{T}^*\|^2$ is regarded as a function of $\tilde{\rho}$, and in $(3.12)_2$ it is regarded as a function of $\tilde{p}$.

These results may also be expressed in terms of the deviatoric parts $\mathbf{E}^*$, $\mathbf{G}^*$, and $\mathbf{H}^*$ of the *(Eulerian) finite strain tensors*

$$\mathbf{E} \equiv \mathbf{V} - \mathbf{I}, \quad \mathbf{G} \equiv \tfrac{1}{2}(\mathbf{B} - \mathbf{I}), \quad \mathbf{H} \equiv \tfrac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}); \tag{3.13}$$

in particular, $\mathbf{H}$ is known as the *Almansi-Hamel* strain tensor. Some care must be taken here since, unlike $\mathbf{L}^*$, the tensors $\mathbf{E}^* = \mathbf{V}^*$, $\mathbf{G}^* = \frac{1}{2}\mathbf{B}^*$, and $\mathbf{H}^* = -\frac{1}{2}(\mathbf{B}^{-1})^*$ are not independent of the volumetric strain. Analogous to the definition (2.14) of $\mathbf{L}^{**}$, for any tensor $\mathbf{A}$ let $\mathbf{A}^{**} \equiv [(\mathbf{A}^*)^2]^* = (\mathbf{A}^*)^2 - \frac{1}{3}\mathrm{tr}\,[(\mathbf{A}^*)^2]\mathbf{I}$. Then it can be shown that

$$\begin{aligned} \mathbf{L}^* &\approx \tilde{\rho}^{1/3}\mathbf{E}^* - \tfrac{1}{2}\tilde{\rho}^{2/3}\mathbf{E}^{**} \\ &\approx \tilde{\rho}^{2/3}\mathbf{G}^* - \tilde{\rho}^{4/3}\mathbf{G}^{**} \approx \tilde{\rho}^{-2/3}\mathbf{H}^* + \tilde{\rho}^{-4/3}\mathbf{H}^{**}, \end{aligned} \tag{3.14}$$

$$\mathbf{L}^{**} \approx \tilde{\rho}^{2/3}\mathbf{E}^{**} \approx \tilde{\rho}^{4/3}\mathbf{G}^{**} \approx \tilde{\rho}^{-4/3}\mathbf{H}^{**}, \tag{3.15}$$

$$\gamma^* \approx \tilde{\rho}^{2/3}\|\mathbf{E}^*\|^2 \approx \tilde{\rho}^{4/3}\|\mathbf{G}^*\|^2 \approx \tilde{\rho}^{-4/3}\|\mathbf{H}^*\|^2, \tag{3.16}$$

where the error in each of these approximations is of order $\|\mathbf{L}^*\|^3$. Substitution of (3.14)–(3.16) into $(3.1)_1$, $(3.2)_2$, and $(3.8)_2$ yields alternate formulas for ε, p, and $\mathbf{T}^*$ to within an error of order $\|\mathbf{L}^*\|^3$. Also note that if the $\mathbf{E}^{**}$, $\mathbf{G}^{**}$, and $\mathbf{H}^{**}$ terms are omitted in (3.14), then the error in (3.14) is of order $\|\mathbf{L}^*\|^2$, and $(3.8)_1$ and $(3.2)_1$ yield

$$\begin{aligned} \mathbf{T}^* &\approx 2\mu(\tilde{\rho})\mathbf{L}^* \approx 2\tilde{\rho}^{1/3}\mu(\tilde{\rho})\mathbf{E}^* \\ &\approx 2\tilde{\rho}^{2/3}\mu(\tilde{\rho})\mathbf{G}^* \approx 2\tilde{\rho}^{-2/3}\mu(\tilde{\rho})\mathbf{H}^* \end{aligned} \tag{3.17}$$

and $p \approx \tilde{p}(\tilde{\rho})$, all to within an error of order $\|\mathbf{L}^*\|^2$. Thus when terms of second order in the shear strain are neglected, the pressure depends only on the density, whereas the deviatoric stress depends on both the shear strain and the density through the density dependence of the shear modulus.[8] As we see in the next section, for many materials the shear modulus changes substantially over the range of densities encounterd in high velocity impacts.

[8]The reader who prefers to work with the strain tensor $\mathbf{G}$, for example, may be tempted to use the term "shear modulus" for the coefficient $\tilde{\rho}^{2/3}\mu(\tilde{\rho})$ of $2\mathbf{G}^*$ in (3.17). However, as we see in the next section, this would be in conflict with the terminology used in much of the wave propagation literature.

4 Material Properties

Let U_L and U_S denote the *longitudinal* and *shear wave speeds* in an isotropic hyperelastic material in a state of dilatational strain (cf. (2.13)) under hydrostatic stress $\mathbf{T} = -\tilde{p}(\tilde{\rho})\mathbf{I}$. These wave speeds are related by the well-known formula (cf. [1, eq. (75.4)])

$$U_L^2 = \tfrac{4}{3}U_S^2 + \frac{d\tilde{p}}{d\rho} = \tfrac{4}{3}U_S^2 + U_B^2\,, \tag{4.1}$$

where

$$U_B \equiv \sqrt{\frac{d\tilde{p}}{d\rho}} \tag{4.2}$$

is referred to as the *bulk wave speed*. Then from (3.4) we have

$$\kappa = \rho\, U_B^2. \tag{4.3}$$

Similarly, it is customary to define the *shear modulus* μ (regarded as a function of ρ, $\tilde{\rho}$, or $\tilde{p}$) by

$$\mu \equiv \rho\, U_S^2. \tag{4.4}$$

Then $\frac{4}{3}\mu = \rho\, U_L^2 - \tilde{\rho}\, d\tilde{p}/d\tilde{\rho}$, and by use of the well-known formula for the longitudinal wave speed (cf. [1, eq. (74.7)]),

$$\rho\, U_L^2 = \frac{\partial t_i}{\partial l_i}\,, \tag{4.5}$$

it can be verified that the shear modulus as defined here agrees with the definition used in the previous section. The shear and bulk moduli as a function of density or hydrostatic pressure are usually obtained from measurements of ultrasonic longitudinal and shear wave speeds, together with the above relations. From (4.4), (3.9), and (3.4), we see that the coefficient p_2 of γ^* in the expansion (3.2) for p is also given by

$$p_2 = 2\rho\kappa U_S \frac{dU_S}{d\tilde{p}}\,. \tag{4.6}$$

Then from (3.9) and (4.6) it follows that

$$p_2 > 0 \;\Leftrightarrow\; \tilde{\rho}\frac{d\mu}{d\tilde{\rho}} > \mu \;\Leftrightarrow\; \frac{d\mu}{d\tilde{p}} > \frac{\mu}{\kappa} \;\Leftrightarrow\; \frac{dU_S}{d\tilde{p}} > 0\,. \tag{4.7}$$

In particular, $dU_S/d\tilde{p} > 0$ implies $d\mu/d\tilde{p} > 0$. Also, since $\gamma^* > 0$ for any nonzero shear strain, from (4.7) and $(3.2)_2$ we see that if $dU_S/d\tilde{p} > 0$, then

shear strain at constant density results in an increase in pressure, whereas shear strain at zero pressure results in bulking (since $\tilde{p}(\tilde{\rho}) < 0$ in this case), provided that terms of order three in the shear strain can be neglected. For most isotropic elastic materials, the shear wave speed does in fact increase with pressure.[9]

The elastic moduli and their pressure derivatives evaluated at $\tilde{p} = 0$ (equivalently, $\tilde{\rho} = 1$) are denoted with a zero subscript. Values for a few materials are listed in Table 1.[10]

Table 1: Elastic Moduli and Their Pressure Derivatives (κ_0 and μ_0 in GPa, $(d\mu/d\tilde{p})_0/\mu_0$ in GPa^{-1}, other quantities dimensionless).

Material	κ_0	μ_0	$\left.\frac{d\kappa}{d\tilde{p}}\right\|_0$	$\left.\frac{d\mu}{d\tilde{p}}\right\|_0$	$\left.\frac{1}{\mu_0}\frac{d\mu}{d\tilde{p}}\right\|_0$	$\left.\frac{d\mu}{d\tilde{p}}\right\|_0 - \frac{\mu_0}{\kappa_0}$
Al	76.0	26.1	4.4	1.8	0.069	1.5
Cu	137	47.7	5.5	1.4	0.028	1.0
U	113	84.4	5.9	3.0	0.036	2.3
W	310	160	3.9	2.3	0.014	1.8
TiB_2	237	246	2.0	9.0	0.036	7.9
ZnO	139	44.2	4.8	−0.69	−0.016	−1.0
fused silica	36.7	31.3	−6.3	−3.2	−0.10	−4.0

Table 1 shows that polycrystalline zinc oxide and fused silica are anomalous in that their shear modulus decreases with increasing pressure. For these materials the inequalities in (4.7) are reversed, in which case (3.2) implies shear induced compaction (for sufficiently small shear strains) at zero pressure. This is indeed a well-known phenomenon in fused silica. Observe that

$$\mu = \mu_0 + \left.\frac{d\mu}{d\tilde{p}}\right|_0 \tilde{p} + \mathcal{O}(\tilde{p}^2) \quad \text{and} \quad \frac{\mu - \mu_0}{\mu_0} = \left.\frac{1}{\mu_0}\frac{d\mu}{d\tilde{p}}\right|_0 \tilde{p} + \mathcal{O}(\tilde{p}^2)\,. \tag{4.8}$$

Thus $(d\mu/d\tilde{p})_0/\mu_0$ is a measure of the relative change in the shear modulus due to hydrostatic pressure.

[9]This statement requires some qualification in the adiabatic case, since the shear modulus and the shear wave speed should increase with pressure to some peak values and then decay to zero as the material melts. Also note that (4.7) remains valid if all inequalities are replaced by equality, in which case $\mu = \mu_0\tilde{\rho}$; thus for the typical case where $dU_S/d\tilde{p} \neq 0$, the shear modulus (and hence $\mathbf{T}^*$) cannot be proportional $\tilde{\rho}$.

[10]The data for Al, Cu, U, and W are taken from [2], TiB_2 from [3], ZnO from [4], and fused silica from [5].

Expansion of the hydrostatic pressure $\tilde{p}$ in powers of $\tilde{\rho}$ yields

$$\tilde{p}(\tilde{\rho}) = \kappa_0(\tilde{\rho}-1) + \frac{\kappa_0}{2}\left(\left.\frac{d\kappa}{d\tilde{p}}\right|_0 - 1\right)(\tilde{\rho}-1)^2 + \mathcal{O}((\tilde{\rho}-1)^3)\,. \tag{4.9}$$

Now consider the case where $\|\mathbf{L}^*\| = \mathcal{O}(\tilde{\rho}-1)$; i.e., where the shear strain and the volumetric strain are of the same order. Then

$$\gamma^* = a\,(\tilde{\rho}-1)^2 + \mathcal{O}((\tilde{\rho}-1)^3) \tag{4.10}$$

for some constant $a > 0$ which depends on the particular deformation in question; and from (3.2), (3.9), and (4.9), we have

$$\begin{aligned}\frac{p-\tilde{p}}{p} &= \frac{p-\tilde{p}}{\tilde{p}} + \mathcal{O}((\tilde{\rho}-1)^2) \\ &= a\left(\left.\frac{d\mu}{d\tilde{p}}\right|_0 - \frac{\mu_0}{\kappa_0}\right)(\tilde{\rho}-1) + \mathcal{O}((\tilde{\rho}-1)^2)\,,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\frac{p}{\kappa_0} &= (\tilde{\rho}-1) + \left[\frac{1}{2}\left(\left.\frac{d\kappa}{d\tilde{p}}\right|_0 - 1\right) + a\left(\left.\frac{d\mu}{d\tilde{p}}\right|_0 - \frac{\mu_0}{\kappa_0}\right)\right](\tilde{\rho}-1)^2 \\ &\quad + \mathcal{O}((\tilde{\rho}-1)^3)\,.\end{aligned} \tag{4.12}$$

Equation (4.11) gives an estimate for the relative increase in the pressure due to shear strain. Equation (4.12) is useful for comparing the second-order contributions to the total pressure: the first group of terms inside the square brackets is the contribution due volumetric strain; the second group is the contribution due to shear strain.

These results apply in particular to *uniaxial strain* where $\lambda_2 \equiv \lambda_3 \equiv 1$, in which case $\gamma^* = \frac{2}{3}(\ln\tilde{\rho})^2 \approx \frac{2}{3}(\tilde{\rho}-1)^2$, and hence $a = 2/3$ in (4.10)–(4.12). We will apply the above results to uniaxial strain of the polycrystalline titanium diboride tested in [3]. From the data in Table 1 we find that the contributions of volumetric strain and shear strain to the second-order term in the expansion (4.12) for the total pressure are 0.5 and 5.27, respectively; i.e., the second-order shear strain contribution to the total pressure is an order of magnitude greater than the second-order volumetric strain contribution. And from (4.11) we have $(p-\tilde{p})/\tilde{p} \approx 5.27\,(\tilde{\rho}-1)$. Of course, these results are valid only within the elastic range of this material. At the Hugoniot elastic limit (HEL), $\tilde{\rho}-1$ is approximately equal to 0.03 (cf. [6]), so that $(p-\tilde{p})/\tilde{p} \approx 0.16$. In other words, at the HEL the relative increase in pressure due to shear strain is about 16%. If not taken into account, this increase would lead to an overestimation of the shear stress at the HEL (obtained by subtracting p from the longitudinal stress).

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