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COMPACTLY SUPPORTED TIGHT AFFINE SPLINE FRAMES IN $L_2(\mathbb{R}^d)$

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ABSTRACT

The theory of [RS2] is applied to yield compactly supported tight affine frames (wavelets) in $L_2(\mathbb{R}^d)$ from box splines. The wavelets obtained are smooth piecewise-polynomials on a simple mesh; furthermore, they exhibit a wealth of symmetries, and have a relatively small support. The number of "mother wavelets", however, increases with the increase of the required smoothness.

Two bivariate constructions, of potential practical value, are highlighted. In both, the wavelets are derived from four-direction mesh box splines that are refinable with respect to the dilation matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

AMS (MOS) Subject Classifications: Primary 42C15 41A15 41A63, Secondary 42C30

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Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$

AMOS RON AND ZUOWEI SHEN

1. Introduction

Given a finite set $\Psi \subset L_2(\mathbb{R}^d)$, and a dilation matrix s, the **affine system generated by** Ψ is defined as the collection

$$X := \{D^k E^\alpha \psi : \ \psi \in \Psi, \ k \in \mathbb{Z}, \ \alpha \in \mathbb{Z}^d\}, \tag{1.1}$$

where

$$E^\alpha : f \mapsto f(\cdot + \alpha)$$

is the **shift operator** and

$$D^k : f \mapsto |\det s|^{k/2} f(s^k \cdot)$$

is the **dilation operator**. A system $X \subset L_2(\mathbb{R}^d)$ is a **fundamental tight frame with frame bound** 1 if the map

$$T^* : L_2(\mathbb{R}^d) \to \ell_2(X) : f \mapsto (\langle f, x\rangle)_{x \in X}$$

is unitary (but not necessarily onto: a tight frame with frame bound 1 whose corresponding T^* is onto is necessarily orthonormal). In what follows, all systems that we treat are affine and fundamental, and all tight frames that are considered have frame bound 1, hence "a tight frame" should always be understood as "a fundamental tight affine frame with frame bound 1". A tight frame can be used for the atomic decompositions of functions exactly in the same way orthonormal bases are used, i.e.,

$$T^* : f \mapsto T^* f$$

transform f into discrete information, and its adjoint

$$T : \ell_2(X) \to L_2 : c \mapsto \sum_{x \in X} c(x) x$$

can then be used to recover f from its discrete transformation (i.e., $TT^* f = f$). In order for T^* to exhibit good space-frequency localization, the functions Ψ should be local in the space domain (ideally compactly supported), should be smooth (which leads to good decay in the frequency domain), and should provide positive approximation order. We refer to [D1,2], [HW] and [RS1,2] for further discussions on frames and tight frames.

While tight frames should ideally be generated by few smooth compactly supported functions with simple structure, the only examples of compactly supported multivariate tight frames in the literature that we are aware of are bivariate orthonormal bases that were derived in [CD] from the univariate Daubechies' wavelets, [D3], as well as tensor products of Daubechies' wavelets. We briefly review those constructions now.

Let $L_2 := L_2(\mathbb{R}^2)$. Let h be the refinement mask of a Daubechies' scaling function ϕ, [D3] (whose shifts are known to be orthonormal), and let ψ be the corresponding wavelet. Define the bivariate mask

$$\tau(\omega_1, \omega_2) := h(\omega_1).$$

Given a dilation matrix s, a bivariate scaling function (or distribution) Φ can then be defined by

$$\widehat{\Phi} : \omega \mapsto \prod_{j=1}^{\infty} \tau(s^{*-j}\omega).$$

Cohen and Daubechies employed in [CD] two different dilation matrices:

$$s := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad s_1 := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{1.2}$$

and obtained therefore two different scaling functions, say Φ and Φ_1, respectively. They observed essential differences between the two so-obtained functions. The matrix s satisfies $s^2 = 2I$, and therefore the resulted Φ is clearly the tensor product (referred to as "separable" in [CD])

$$\Phi(x) = \phi(x_2)\phi(x_1 - x_2),$$

and therefore the shifts of Φ are necessarily orthonormal. Furthermore, the standard wavelet construction then yields the wavelet

$$\Psi(x) = \psi(x_2)\phi(x_1 - x_2).$$

While the refinable function Φ is separable, the refinable function Φ_1 and the wavelet constructed from it are not separable in any sense. Nonetheless, it is proved in [CD] that the shifts of Φ_1 are orthonormal. It is further proved in [CD] that, unfortunately, Φ_1 cannot be C^1, regardless of the order of the univariate scaling function which is used. Therefore, it is necessary to develop some other algorithms to construct nonseparable compactly supported tight frames with high smoothness.

It will not be entirely correct to say that the [CD] constructions and tensor product constructions comprise all known multivariate affine tight frames: according to [CS], tight frames can be constructed by appending to Ψ some of their translates (=:oversampling); that, of course, only increases the number of elements in each of the above constructions, while preserving any deficiences (such as lack of symmetries and low smoothness in the non-separable case, and parallelogram supports in the separable case) that the orthonormal system may have had.

The theory established in [RS2], however, makes the construction of useful simple nonseparable compactly supported tight frames with high smoothness and a variety of symmetries an easy task. In fact, one can essentially construct tight frames with the aid of the shifts of any refinable function, and therefore it is possible to impose a simple structure on the wavelets by selecting a refinable function with such desired structure. Indeed, we constructed in [RS2], for every positive integer m, a tight frame for $L_2(\mathbb{R})$ that is generated by m wavelets, each of which is a spline of degree $m-1$, support $[0, m]$, and smoothness C^{m-2}. Further, all knots of the spline-wavelets are half-integers, and each spline is either symmetric or anti-symmetric. (The case $m = 4$ of this construction is discussed in §4).

Since the variety of possible constructions of multivariate (and univariate) tight frames based on the [RS2] theory is unlimited, we carefully selected for the present article those constructions which, in our opinion, may be used in practical applications. We kept in mind that different applications may require different properties from the wavelet system; for example, in data compression applications the number of different wavelets used (which accounts to the oversampling rate) should be minimized, while in finite element applications many elements with small supports may be preferred. Our two favorite bivariate constructions are detailed in §2 and §3: in §2, the system is generated by many highly symmetric wavelets of small support (which, actually, are not so "many" for practical smoothness requirements); in §3, the system is generated by fewer larger elements. In each construction, the wavelets are splines, i.e., smooth piecewise-polynomials, always of compact support, with various symmetries. Moreover, the relevant grid is the four-direction (i.e., quincunx, see below) mesh, hence all the wavelets have a "round" octagonal support. On the the other hand, similarly to our univariate construction, the number of elements used, as well as the volume of the support of each, increase together with the increase in the required smoothness. The ease in constructing tight spline frames is not limited to two dimensions: in §4 a general inductive algorithm for constructing multidimensional tight frames is provided. The algorithm works particularly well with box splines, in fact, with *every* box spline. We call the box spline wavelets obtained by this algorithm *boxlets.*

We have chosen to carry the constructions in §2,3 with respect to the dilation matrix s (cf. 1.2). It is possible, though, to carry these same constructions with respect to s_1 with only one limitation: all four directions in the definition of the box spline must appear with the same multiplicity. In any event, the use of s_1 instead of s does not yield different systems, and the reason is very simple: for a four-direction mesh, not only that $s\mathbb{Z}^2 = s_1\mathbb{Z}^2$, but also, in case the box spline ϕ has equi-multiplicities, $\phi(s\cdot) = \phi(s_1\cdot)$.

2. Bivariate tight frames generated by $m^2 - 1$ C^{3m-5} spline wavelets

In fact, the construction here is a bit more general than the title indicates: it includes tight frames generated by m^2+m-1 wavelets of smoothness C^{3m-4}. Of course, we assume $m > 1$.

Let ϕ be the box spline

$$\widehat{\phi}(\omega) = \prod_{j=1}^{4}\left(\frac{1-e^{-i\xi_j\cdot\omega}}{i\xi_j\cdot\omega}\right)^{m_j}, \tag{2.1}$$

where

$$(\xi_1,\xi_2,\xi_3,\xi_4) := \left(\begin{pmatrix}1\\0\end{pmatrix}\ \begin{pmatrix}0\\1\end{pmatrix}\ \begin{pmatrix}1\\1\end{pmatrix}\ \begin{pmatrix}1\\-1\end{pmatrix}\right),$$

and with $m_1 = m_3$, and $m_2 = m_4$. It is well-known (cf. [BHR]) that the box spline ϕ satisfies the following properties:

(a) It is a piecewise-polynomial of local degree $2(m_1+m_2-1)$, on the four-direction mesh (that is obtained by adding the diagonals to each square of integer vertices).

(b) It is globally $C^{\rho-2}$ with $\rho := \min\{2m_1 + m_2, 2m_2 + m_1\}$; (in fact, its ($\rho$−2)-order derivatives are all Lip_1, hence its Hölder continuity is $\rho - 1$). Further, it provides approximation order ρ.

(c) It is supported in the octagon

$$\{\sum_{j=1}^{4} t_j \xi_j : \ 0 \le t_j \le m_j, \ j = 1, \ldots, 4\}.$$

Further, it is essentially positive on its support.

In the sequel, we will introduce a variety of octagonal domains similar to the support of ϕ above. For reasons of efficiency, we therefore denote

$$[a_1, a_2, a_3, a_4] := \{\sum_{j=1}^{4} t_j \xi_j : \ 0 \le t_j \le a_j, \, j = 1, \ldots, 4\}.$$

Note that each $[a, b, c, d]$ is an octagon whose area is

$$\mathrm{area}([a, b, c, d]) = ab + ac + ad + bc + bd + 2cd.$$

We prefer the above box spline over other variants because it is refinable not only with respect to the dilation matrix $2I$, but also with respect to the dilation matrix

$$s := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.2}$$

This well-known fact was rarely exploited before because the shifts of the 4-direction box spline do not form a Riesz basis (or a frame). Here, that is no more an obstacle, since, as we already mentioned, the [RS2] theory does not impose any a priori assumption on the refinable function ϕ.

A straightforward computation shows that

$$\widehat{\phi}(s\cdot) = \tau_0 \widehat{\phi}$$

with the function

$$\tau_0(\omega) := \left(\frac{1 + e^{-i\omega_1}}{2}\right)^{m_1} \left(\frac{1 + e^{-i\omega_2}}{2}\right)^{m_2} = \theta(\omega) \cos^{m_1}(\omega_1/2) \cos^{m_2}(\omega_2/2),$$

with

$$\theta(\omega) := e^{-i(m_1\omega_1 + m_2\omega_2)/2}.$$

We note that ϕ is **refinable**, which means that the **mask** τ_0 is 2π-periodic (and hence $\phi(s^{-1}\cdot)$ can be written as a linear combination of the integer translates of ϕ).

Our construction invokes the following theorem, which is a special case of Corollary 6.7 of [RS2].

Theorem 2.3. *Let ϕ be any box spline that is refinable with respect to the dilation matrix s of (2.2), and has a (2π-periodic) refinement mask τ_0. Let n be a positive integer, and let $(\tau_j)_{j=1}^n$ be n 2π-periodic essentially bounded measurable functions. Assume that, for a.e. ω, and for $\nu := (\pi, \pi)$,*

$$\sum_{j=0}^{n} |\tau_j|^2 = 1, \quad \sum_{j=0}^{n} \tau_j \overline{E^\nu \tau_j} = 0.$$

Then the wavelets $\Psi := (\psi_j)_{j=1}^n$, defined by

$$\widehat{\psi}(s\cdot) := \tau_j \widehat{\phi}, \quad j = 1, \ldots, n,$$

generate a tight frame (i.e., a fundamental affine tight frame with frame bound 1) for L_2.

It should be understood that the dilation matrix s that is involved in the refinement equation is the same dilation matrix s that is used to generate the affine system from Ψ.

To construct now the wavelets with the aid of the above box spline ϕ, we first define the following *univariate* 4π-periodic functions:

$$y_j(t) := y_j^{[m]}(t) := \sqrt{\binom{m}{j}} \cos^{m-j} t/2 \sin^j t/2, \ 0 \le j \le m. \tag{2.4}$$

(Warning: m in $y_j^{[m]}$ is an index, not a power!) Note that, for any fixed m,

$$\sum_{j=0}^{m} |y_j|^2 = 1, \quad \sum_{j=0}^{m} y_j \overline{y_j(\cdot + \pi)} = 0. \tag{2.5}$$

We next define the tensor product bidimensional mask system

$$(\tau_n : \ n \in N := \{0, 1, \ldots, m_1\} \times \{0, 1, \ldots, m_2\})$$

with

$$\tau_n(\omega) := \theta(\omega)\, y_{n_1}^{[m_1]}(\omega_1)\, y_{n_2}^{[m_2]}(\omega_2).$$

Each τ_n is 2π-periodic.

We then observe that $\tau_0 := \tau_{0,0}$ in the above tensor system is the refinement mask of ϕ. Moreover, the convex hull of the spectrum of each τ_n, $n \in N$, is independent of n (i.e., in down-to-earth language: the 2π-integers that lie in the convex hull of the frequencies of the exponentials whose linear combination form τ_n, are the same for all n). This means that the $(m_1 + 1)(m_2 + 1) - 1$ functions, defined by

$$\Psi := \{\psi_n : \widehat{\psi}_n(s\cdot) := \tau_n \widehat{\phi}, \quad n \in N \backslash 0\}, \tag{2.6}$$

are all supported in the support of ϕ, i.e., in the octagon $[m_1, m_2, m_1, m_2]$. The functions Ψ are nonseparable piecewise-polynomials, with the same smoothness as the box spline ϕ used.

The tensor product structure of the masks (together with the fact that $|\theta| = 1$) clearly implies (cf. (2.5)) that

$$\sum_{n\in N} |\tau_n(\omega)|^2 = (\sum_{j=0}^{m_1} |y_j^{[m_1]}(\omega_1)|^2)(\sum_{j=0}^{m_2} |y_j^{[m_2]}(\omega_2)|^2) = 1,$$

and by the same token, for $\nu = (\pi, \pi)$ (with $c := \theta\,\overline{E^\nu \theta}$)

$$\sum_{n\in N} \tau_n(\omega)\overline{E^\nu \tau_n(\omega)} = c(\sum_{j=0}^{m_1} y_j^{[m_1]}(\omega_1)\overline{y_j^{[m_1]}(\omega_1+\pi)})(\sum_{j=0}^{m_2} y_j^{[m_2]}(\omega_2)\overline{y_j^{[m_2]}(\omega_2+\pi)}) = 0 \cdot 0 = 0.$$

Therefore, Theorem 2.3 implies the following:

Theorem 2.7. *The wavelets Ψ constructed in (2.6) generate a tight affine frame with frame bound* 1.

Remark. Each one of these wavelets is supported in the support of ϕ, and is a $C^{\rho-2}$, $\rho := \min\{2m_1+m_2, 2m_2+m_1\}$, piecewise-polynomial of local degree $2(m_1+m_2-1)$, with respect to the mesh $s^{-1}M$, with M the standard four-direction mesh. The mesh $s^{-1}M$ consists of all lines of the form

$$x_1 + x_2 = j,\ x_1 - x_2 = j,\ x_1 = j/2,\ x_2 = j/2,$$

where j varies over $\mathbb{Z}$, and (x_1, x_2) is the generic point in the space domain. □

Example. We take $m_1 = m_2 = 1$. In this case ϕ is the well-known C^1 piecewise-quadratic Zwart element, supported in the octagon $[1,1,1,1]$. The above construction yields three C^1 piecewise-quadratic wavelets each supported in $[1,1,1,1]$, and each exhibiting various symmetries (cf. Figures 2.1 and 2.2). The support of each wavelet is of area 7; this small support comes despite of fact that we oversample by a factor of 3.

It is hard to compare the above construction to literature counterparts, since, as we mentioned before, the latter hardly exist. Here are two possible comparisons. In the first, we take the separable construction of [CD], using Daubechies scaling function that provides the same approximation order (i.e., 3) as the Zwart element. The result is a single wavelet supported in a parallelogram of area 25, and whose shifts are orthonormal. The smoothness of this wavelet is about half of the above tight frame (in terms of Hölder exponents: 2 vs. 1.1; cf. [D2: pp. 232-239]), and it lacks symmetries.

Another comparison may be with "plain" tensor product of the same Daubechies' scaling function. In this case, we obtain 3 mother wavelets each with a square support of area 25, and with the same smoothness and (lack of) symmetries. The shifts of these wavelets now fill an entire dyadic level. To fill in an entire dyadic level using integer shifts of our elements, we need the three elements of support area of 7 each, and 6 elements of half-size support, with total area of supports 42. □

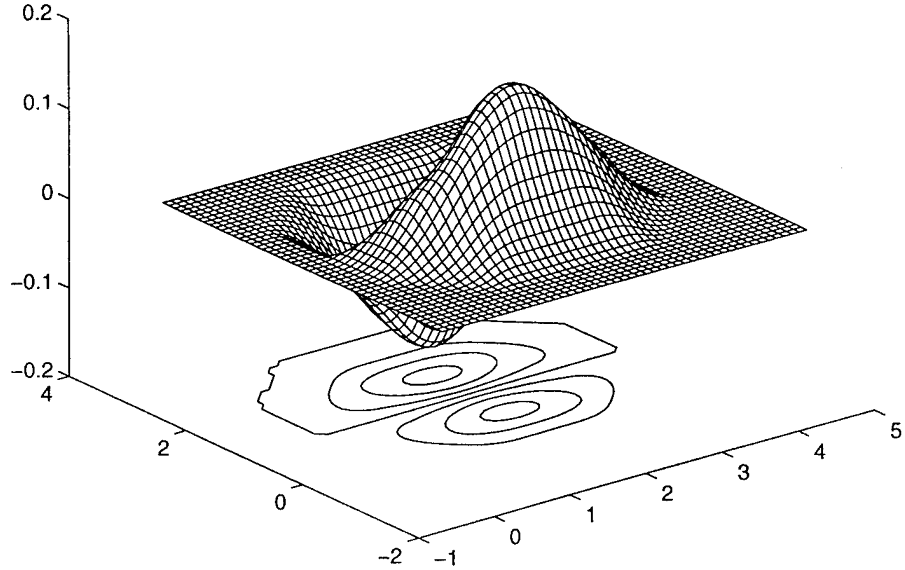

Figure 2.1. The C^1 piecewise-quadratic wavelet $\psi_{0,1}$. The wavelet $\psi_{1.0}$ is obtained by rotation.

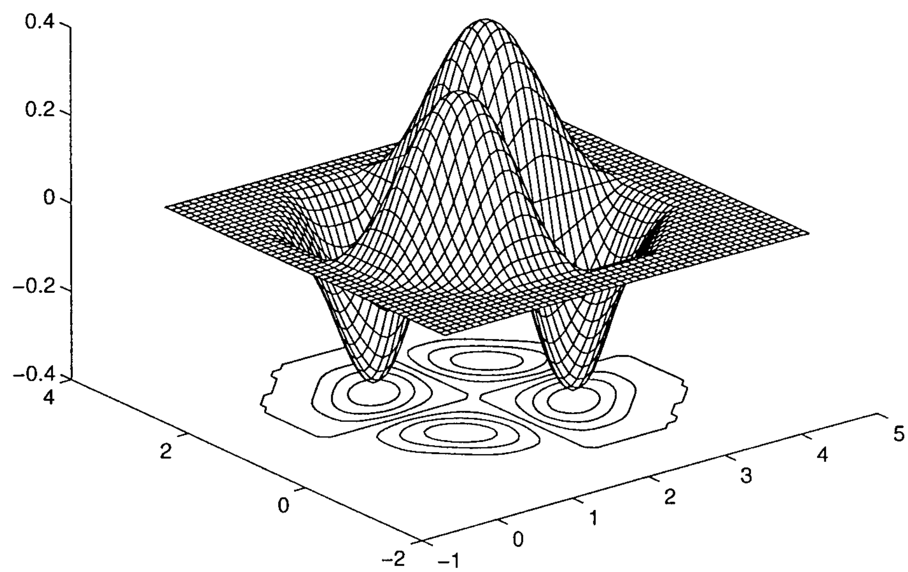

Figure 2.2. The C^1 piecewise-quadratic wavelet $\psi_{1,1}$. Note the octagonal support

Example. We take $m_1 = 1$, $m_2 = 2$. In this case ϕ is globally C^2 (and is C^3 on half of the mesh lines), piecewise-quartic. The construction now yields 5 wavelets, all supported in the octagon $[1,2,1,2]$ whose area is 15. They all have the same (global) smoothness and local degree as the box spline.

Example 2.8. We take $m_1 = m_2 = 2$. Then we obtain 8 elements supported each in the octagon $[2,2,2,2]$ whose vertices (up to a $(1,1)$-shift) are $(\pm 3, \pm 1) \cup (\pm 1, \pm 3)$ and whose area is 28. Each spline wavelet is C^4 (with Hölder exponent 5). The local polynomial degree is 6.

3. Tight frames generated by $2m$ C^{3m-2} spline wavelets

Here, m assumes any positive integer value. Again, the construction is more general than the title indicates: it includes the construction of tight frames generated by $2m+1$ C^{3m-1} wavelets.

The wavelets constructed in this section are selected from the same box spline spaces used in the previous section. Given a four-direction box spline with corresponding multiplicities (m_1, m_2, m_1, m_2), the construction selects appropriate mother wavelets as the s-dilate of certain functions in the span of the shifts of the box spline. This means that the wavelets here are comparable to the wavelets of the previous section in terms of local polynomial degrees, underlying meshes, and smoothness.

However, the number of wavelets associated with the previous construction grows quadraticly with the required smoothness. Though the three examples that followed show that for practical smoothness requirements the number of wavelets is "within reason", it is possible to construct tight spline frames whose number of generators grow only linearly with the required smoothness. The associated wavelets, on the other hand, have larger, somewhat less symmetric, support.

It is convenient to carry out the construction here not with the aid of the box spline ϕ of (2.1), but, rather, with the following "averaged" spline φ:

$$\widehat{\varphi}(\omega) := \left(\frac{1+e^{-i\omega_1}}{2}\right)^{m_1} \widehat{\phi}(\omega).$$

In standard box spline terminology, φ is a box spline with direction set $(2\xi_1, \xi_2\, \xi_3, \xi_4)$ (cf. (2.1)) with corresponding multiplicities (m_1, m_2, m_1, m_2). Also, direct computation yields that $\widehat{\varphi}$ satisfies

$$\begin{aligned}\widehat{\varphi}(s\omega) =& \left(\frac{1+e^{-i(\omega_1+\omega_2)}}{2}\right)^{m_1} \left(\frac{1+e^{-i\omega_2}}{2}\right)^{m_2} \widehat{\varphi}(\omega)\\ =& \vartheta(\omega) \cos^{m_1}((\omega_1+\omega_2)/2) \cos^{m_2}(\omega_2/2)\, \widehat{\varphi}(\omega)\end{aligned},$$

where

$$\vartheta(\omega) := e^{-i(m_1, m_1+m_2)\cdot\omega/2}.$$

Recalling now the definition of $y_j^{[m]}$ in (2.4), we define m_1+m_2+1 2π-periodic functions as follows:

$$\tau_j(\omega) := \vartheta(\omega)\, y_j^{[m_1]}(\omega_1+\omega_2)\, y_0^{[m_2]}(\omega_2), \quad j = 0, \ldots, m_1,$$

and

$$\tau_{m_1+j}(\omega) := \theta(\omega) y_j^{[m_2]}(\omega_2), \quad j = 1, \ldots, m_2,$$

where

$$\theta(\omega) := e^{-im_2\omega_2/2}.$$

We note that the functions $e^{-im_1(\omega_1+\omega_2)/2} y_j^{[m_1]}(\omega_1+\omega_2)$ are periodic with respect to shifting by (π, π), hence are periodic with respect to the lattice $s^{-1}2\pi\mathbb{Z}^2$. From that, together with the fact, (2.5), that

$$\sum_{j=0}^{m_1} |y_j^{[m_1]}|^2 = 1,$$

one immediately concludes that for $\varepsilon \in \{0,1\}$, and $\nu = (\pi, \pi)$,

$$\sum_{j=0}^{m_1} \tau_j(\omega)\overline{E^{\varepsilon\nu}\tau_j(\omega)} = \theta(-\varepsilon\pi)\, y_0^{[m_2]}(\omega_2)\, \overline{y_0^{[m_2]}(\omega_2 + \varepsilon\pi)}.$$

This leads further to the conclusions that, firstly,

$$\sum_{j=0}^{m_1+m_2} |\tau_j|^2 = \sum_{j=0}^{m_2} |y_j^{[m_2]}|^2 = 1,$$

and, secondly, with $\nu := (\pi, \pi)$,

$$\sum_{j=0}^{m_1+m_2} \tau_j(\omega)\overline{E^{\nu}\tau_j}(\omega) = \theta(-\pi) \sum_{j=0}^{m_2} y_j^{[m_2]}(\omega_2)\overline{y_j^{[m_2]}(\omega_2 + \pi)} = 0.$$

In summary, the mask vector

$$(\tau_j)_{j=0}^{m_1+m_2}$$

satisfies the conditions of Theorem 2.3.

Consequently, since τ_0 in the above construction is the refinment mask of φ, we conclude from Theorem 2.3 that the $m_1 + m_2$ wavelets defined by

$$\widehat{\psi}_j(s\cdot) := \tau_j \widehat{\varphi}, \quad j = 1, \ldots, m_1 + m_2,$$

generate a tight frame.

Corollary 3.1. *Let φ be the box spline whose direction set is $(2\xi_1, \xi_2, \xi_3, \xi_4)$ with corresponding multiplicities (m_1, m_2, m_1, m_2):*

$$\widehat{\varphi} = \left(\frac{1 + e^{-i\omega_1}}{2}\right)^{m_1} \widehat{\phi}(\omega),$$

with ϕ the box spline of (2.1). Then, with $(\tau_j)_{j=1}^{m_1+m_2}$ the above constructed masks, the wavelets $\Psi := (\psi_j)_{j=1}^{m_1+m_2}$, defined by $\widehat{\psi}_j(s\cdot) := \tau_j\widehat{\varphi}$, generate a tight frame (with frame bound 1) for L_2. These wavelets share the same smoothness, local degree, and mesh with the wavelets of Theorem 2.7.

Discussion. The box spline φ in the above corollary is supported in the octagon $[2m_1, m_2, m_1, m_2]$ whose area is $2m_1^2 + 7m_1m_2 + m_2^2$. As to the wavelets, the "large" wavelets, i.e., the first m_1 elements, have the same support as φ. The other m_2 elements have the octagonal support $[m_1, m_2, m_1, m_2]$, whose area is $m_1^2 + 5m_1m_2 + m_2^2$.

Remark. The general construction detailed in the next section shows that there are many possible modifications of the above construction.

Example 3.2. We consider the case $m_1 = m_2 = 1$. Figure 3.1 shows the box spline φ. It is a C^1 piecewise-quadratic that provides approximation order 3, and it is supported in the octagon $[2, 1, 1, 1]$, with area 10. Two wavelets are constructed here. Direct computation yields that the larger support wavelet, ψ_1 (cf. Figure 3.2), whose support is identical to that of φ, satisfies

$$\widehat{\psi}_1(\omega) = \tan(\omega_1/2)\widehat{\varphi}(\omega), \tag{3.3}$$

while the smaller support ψ_2 (cf. Figure 3.3) has the form

$$\widehat{\psi}_2 = \frac{1 - e^{-i(\omega_1-\omega_2)/2}}{2i}\widehat{\varphi}(\frac{\omega_1+\omega_2}{2}, \frac{\omega_1-\omega_2}{2}).$$

The corresponding supports are $[2, 1, 1, 1]$ and $[1, 1, 1, 1]$ with corresponding areas 10 and 7. □

Example. We consider $m_1 = 1$, $m_2 = 2$. The box spline φ is then C^2 piecewise-quartic and provides approximation order 4. Its support is the octagon $[2, 2, 1, 2]$ whose area is 20. The mother wavelet set consists of three elements, two "small" and one "large". The large element ψ_1 is defined exactly as in (3.3) (only that φ has been changed), and has the support of ϕ. The other wavelets are

$$\widehat{\psi}_2 = \frac{1 - e^{-i(\omega_1-\omega_2)}}{2\sqrt{2}i}\widehat{\varphi}(\frac{\omega_1+\omega_2}{2}, \frac{\omega_1-\omega_2}{2}),$$

and

$$\widehat{\psi}_3 = \frac{-1 + 2e^{-i\frac{\omega_1-\omega_2}{2}} - e^{-i(\omega_1-\omega_2)}}{4}\widehat{\varphi}(\frac{\omega_1+\omega_2}{2}, \frac{\omega_1-\omega_2}{2}),$$

Both supported in the same octagon $[1, 2, 1, 2]$, whose area is 15. □

Example. The case $m_1 = m_2 = 2$ yields four wavelets, two supported in $[4, 2, 2, 2]$, and the other two supported in $[2, 2, 2, 2]$. The area of these octagons is 40 and 28 respectively, hence their total area is 136. The smoothness of the box spline, as well as its local degree is the same as in Example 2.8. □

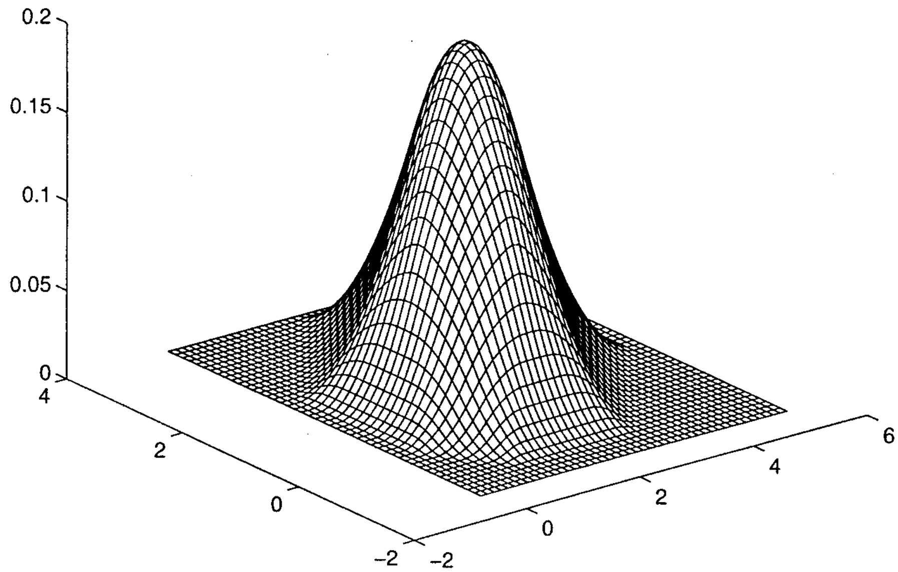

Figure 3.1. The C^1 piecewise-quadratic box spline that is used in Example 3.2.

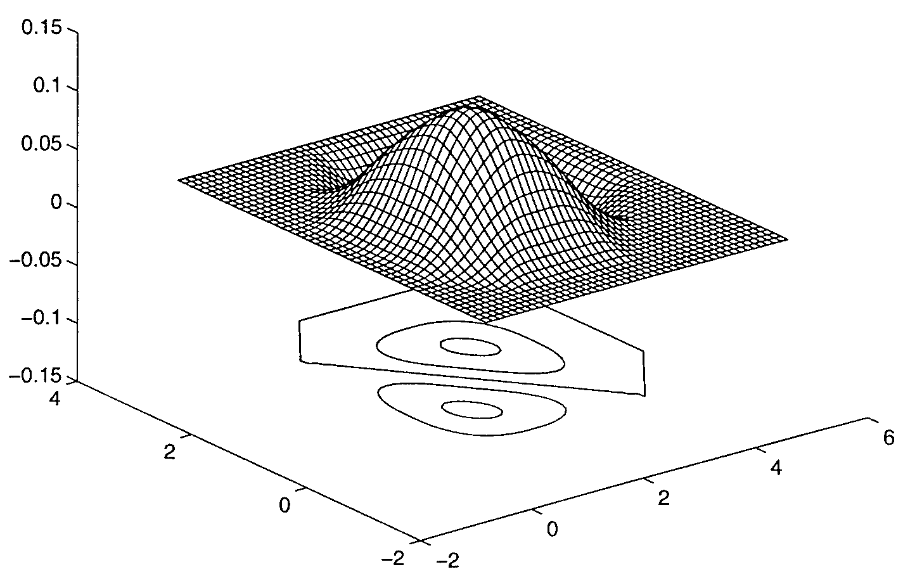

Figure 3.2. The C^1 piecewise-quadratic wavelet with larger support ψ_1.

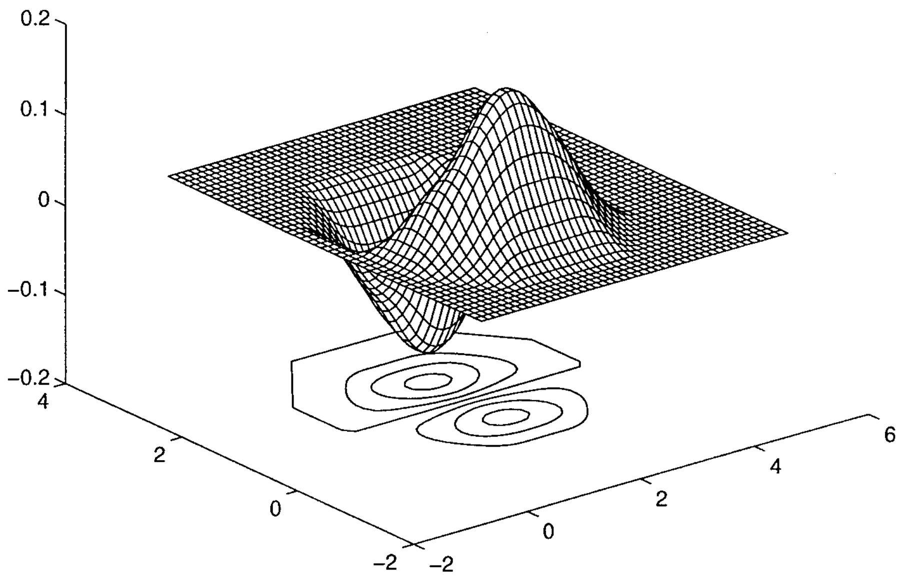

Figure 3.3. The C^1 piecewise-quadratic wavelet with smaller support ψ_2.

4. A general inductive algorithm for constructing tight frames

The method described in the last section for the construction of box spline wavelets on a four-direction mesh can be significantly generalized. Since we are unable to predict, at the time when this article is written, what specific variants may be implemented in practice, we decided to simply outline the highest level of generalization that we are able to observe.

The setup here is as follows: we hold a dilation matrix s (i.e., an integer matrix whose inverse is contractive) and two d-variate functions (or, more generally, distributions), say ϕ_1, ϕ_2, that are refinable with respect to s. The corresponding masks, that are assumed to be 2π-periodic (that is a part of the *definition* of refinability), are also assumed to be bounded; in practical situations the masks are trigonometric polynomials, hence certainly bounded. The basic assumption is that we already know how to derive a tight frame from the scaling function ϕ_1, and we would like to use that known frame in order to obtain a new, improved (in terms of smoothness, for example) tight frame. Specifically, we would like to extract the new frame from the (necessarily refinable) convolution

$$\phi := \phi_1 * \phi_2.$$

This idea (of convolving the given scaling function with a suitably chosen distribution) has been used in the spline and wavelet theory many times. For example, Daubechies obtained her univariate refinable functions whose shifts are orthonormal by convolving a univariate B-spline ϕ_1 (whose shifts are stable but not orthonormal), with a suitably chosen distribution ϕ_2. In her construction, the new ϕ provides the same approximation order as the B-spline ϕ_1, but is significantly less smooth; (the smoothness of the scaling function and the orthonormality of its shifts are then transferred to the constructed wavelets.) The fact that one cannot obtain by convolution the orthonormality of the shifts, while simultaneously improving, or at least retaining, the smoothness of the original ϕ_1, is one of main reasons the construction of the affine orthonormal bases is fairly involved.

The theory of [RS2] allows us to pay less attention to the properties of the scaling function: "bad" properties of the latter may not at all be inherited by the wavelets! Specifically, the smoothness of the wavelets and the tightness of the frame they generate do not compete any more. In fact, we can separate the construction of the smooth tight frame into two steps: in the first a basic low-smoothness frame is constructed, and then the smoothness of the frame is improved without hampering its tightness: in terms of ϕ_1, ϕ_2 above, ϕ_1 is the low- (or intermediate-) smooth scaling function that yields a tight affine system, and ϕ_2 is the convolutor that should improve the smoothness.

We specify now the exact conditions that are imposed on ϕ_1, ϕ_2. In these conditions, we use the notation

$$\Gamma := 2\pi(s^{*-1}\mathbb{Z}^d/\mathbb{Z}^d).$$

For example, if $s = 2I$, Γ is the group $\{0, \pi\}^d$, with addition modulo 2π. Note that the order of Γ is $|\det s|$.

(4.1) Conditions assumed on ϕ_1, ϕ_2.

(a) For $j = 1, 2$, ϕ_j is a refinable distribution with bounded mask τ_j.

(b) There exists a collection T_1 of 2π-periodic bounded functions that satisfy, for every $\nu \in \Gamma$, and a.e. on $\mathbb{R}^d$,

$$\tau_1\overline{E^\nu\tau_1} + \sum_{\tau\in\mathrm{T}_1} \tau\overline{E^\nu\tau} = \delta_\nu.$$

(c) There exist another collection of 2π-periodic bounded functions T_2, that satisfy

$$|\tau_2|^2 + \sum_{\tau\in\mathrm{T}_2} |\tau|^2 = 1.$$

Discussion. The fact that we assume ϕ_2 to be merely a distribution seems to be practically important as we will see in the examples presented at the end of this section. On the other hand, ϕ_1, in all examples we carry in mind, is a *function*. In fact, if ϕ_1 *is* a function, Corollary 6.7 of [RS2] "almost" implies that, under the condition (b) of the conditions (4.1), the system generated by the wavelets that are defined via their Fourier transforms by

$$\{\widehat{\psi}(s^*\cdot) := \tau\widehat{\phi}_1 : \ \tau \in \mathrm{T}_1\}$$

is a tight frame for $L_2(\mathbb{R}^d)$. The only missing condition is a very mild smoothness requirement of ϕ_1, that all box splines, for example, satisfy (cf. (4.6) of [RS2]). Note that significantly weaker conditions are assumed on T_2.

With the assumptions of (4.1) in hand, we will be able to construct a tight frame based *not* on the convolution $\phi_1 * \phi_2$, but, rather, on the convolution with larger support

$$\phi := \frac{\phi_2(s^{-1}\cdot) * \phi_1}{|\det s|}.$$

Note that $\widehat{\phi} = \widehat{\phi}_2(s^* \cdot)\widehat{\phi}_1$, and hence that ϕ is refinable with mask

$$\tau_\phi := \tau_2(s^* \cdot)\, \tau_1.$$

We now introduce the new mask collection

$$\mathrm{T} := \mathrm{T}_1 \cup (\tau_1 \mathrm{T}_2(s^* \cdot)),$$

i.e., we apply s^* dilation to each of the masks in T_2, multiply each by τ_1, and then append the new collection to T_1. Clearly,

$$\#\mathrm{T} = \#\mathrm{T}_1 + \#\mathrm{T}_2.$$

Lemma 4.2. *Each of the masks in* T *above is* 2π*-periodic and bounded. Furthermore, for every* $\nu \in \Gamma$,

$$\tau_\phi \overline{\tau_\phi} + \sum_{\tau \in \mathrm{T}} \tau \overline{E^\nu \tau} = \delta_\nu.$$

Proof. We note that the masks in $\mathrm{T}_2(s^* \cdot)$ are not only $2\pi \mathbb{Z}^d$-periodic, but also $2\pi s^{*-1}\mathbb{Z}^d$-periodic. Therefore, for every $\nu \in \Gamma$, by (c) of (4.1),

$$\tau_\phi \overline{E^\nu \tau_\phi} + \sum_{\tau \in \mathrm{T}_2(s^* \cdot)} (\tau_1 \tau)\overline{E^\nu(\tau_1 \tau)} = (\tau_1 \overline{E^\nu \tau_1})\left(|\tau_2(s^* \cdot)|^2 + \sum_{\tau \in \mathrm{T}_2(s^* \cdot)} |\tau|^2\right) = \tau_1 \overline{E^\nu \tau_1}.$$

This implies that, by (b) of (4.1), that, for every $\nu \in \Gamma$,

$$\tau_\phi \overline{E^\nu \tau_\phi} + \sum_{\tau \in \mathrm{T}} \tau \overline{E^\nu \tau} = \tau_1 \overline{E^\nu \tau_1} + \sum_{\tau \in \mathrm{T}_1} \tau \overline{E^\nu \tau} = \delta_\nu.$$

□

As we already mentioned above, the properties proved with respect to T in the above lemma almost imply that the wavelet system $\Psi := (\psi)_{\psi \in \Psi}$ define by

$$\{\widehat{\psi}(s^* \cdot) := \tau \widehat{\phi} : \ \tau \in \mathrm{T}\}$$

generate a tight frame: we only a need a mild smoothness condition of ϕ. Rather than quoting the complicated condition (4.6) of [RS2], we assume in the following corollary smoothness conditions that are slightly stronger but more transparent.

Corollary 4.3. *In the above notations, if the refinable ϕ is a box spline, or if its Fourier transform decay at ∞ like $O(|\cdot|^{-1/2-\delta})$ for $\delta > 0$, then the wavelets Ψ constructed as above generate a fundamental affine tight frame with frame bound 1 for $L_2(\mathbb{R}^d)$.*

(4.4) Boxlets. We assume that the dilation matrix s satisfies $s^k = 2I$, for some positive integer k. In this case, there is a natural way for choosing the smoothing factor ϕ_2: starting with a **cycle** $\Xi := (\xi_1, \ldots, \xi_k)$ of s (i.e., $s\xi_j = \xi_{j+1}$, $j = 1, \ldots, k-1$, hence necessarily $s\xi_k = 2\xi_1$), we define ϕ_2 as the box spline with directions Ξ:

$$\widehat{\phi}_2(\omega) = \prod_{\xi \in \Xi} \frac{1 - e^{-i\xi\cdot\omega}}{i\xi \cdot \omega}.$$

Then, ϕ_2 is refinable with mask

$$\tau_2(\omega) = \frac{1 + e^{-i\xi_1\cdot\omega}}{2}.$$

Since we tacitly assume Ξ to be integer, τ_2 is 2π-periodic. Selecting T_2 is then trivial: T_2 is taken to be the singleton

$$\mathrm{T}_2 := \{\frac{1 - e^{-i\xi_1\cdot\omega}}{2}\}.$$

The satisfaction of (c) in (4.1) is then automatic, and, assuming that T_1, ϕ_1 are already given, and that the refinement mask of ϕ_1 is τ_1, our general inductive step reads here as follows:

(a) The new refinable function is $\phi := \frac{\phi_1 * \phi_2(s^{-1}\cdot)}{|\det s|}$.

(b) The refinement mask of ϕ is

$$\tau_\phi(\omega) := \tau_1(\omega)\,\frac{1 + e^{-is\xi_1\cdot\omega}}{2} = \tau_1(\omega)\,\frac{1 + e^{-i\xi_2\cdot\omega}}{2}.$$

(c) The new wavelet mask set T is obtained by appending to T_1 the single mask

$$\tau(\omega) := \tau_1(\omega)\,\frac{1 - e^{-i\xi_2\cdot\omega}}{2}.$$

Boxlets correspond to the choice $s = 2I$ for the dilation matrix. The cycles of $2I$ are, of course, singletons, hence the inductive process allows us to insert one direction per step. We may begin the inductive process with any box spline ϕ_1 whose (integer) direction set is a basis for $\mathbb{R}^d$ (that box spline is then, up to a normalization constant, the support function of some parallelepiped). As the initial wavelet masks T_1, we take the tensor product construction (thus, we obtain, up to a linear transformation, the multivariate Haar wavelets). We select any sequence $\Xi = (\xi_1, \ldots, \xi_m)$, and, after m insertions obtain the final box spline

$$\widehat{\phi}(\omega) = \widehat{\phi}_1(\omega) \prod_{j=1}^{m} \frac{1 - e^{-i2\xi_j\cdot\omega}}{2i\xi_j \cdot \omega}.$$

There are $2^d - 1 + m$ boxlets in this construction. The first $2^d - 1$ ones correspond to the initial tensor product masks (which, we stress, are applied to the smooth ϕ). The other masks, which we index by Ξ, have a "triangular structure":

$$\tau_{\xi_j}(\omega) := \tau_1(\omega)\, \frac{1 - e^{-i2\xi_j\cdot\omega}}{2} \prod_{\ell=1}^{j-1} \frac{1 + e^{-i2\xi_\ell\cdot\omega}}{2},$$

with τ_1 the refinement mask of the initial ϕ_1.

Assuming that m is relatively small, most of the boxlets have small support. For example, if we wish to construct C^2 boxlets, then, independently of the spatial dimension d, we may do with $m = 3$. In this case, there are $2^d - 1$ wavelets with "small" support, one with "large" support, and two wavelets with intermediate size of supports. □

We remark that, in the above construction, we may start with any refinable (box spline) ϕ_1, provided that a derivation of a tight frame from ϕ_1 is available.

Example. We show that the construction of §3 is a special case of the construction of this section. Here, ϕ_1 is a four-direction box spline (as defined in (2.1)) with multiplicities $(0, m_2, 0, m_2)$, and ϕ_2 is a four-direction box spline with multiplicities $(m_1, 0, m_1, 0)$ (so that each box spline has only two active directions). The function ϕ_2 is refinable with respect to the dilation matrix s employed in §3, and with mask (up to an exponential factor) $\cos^{m_1}(\omega_1/2)$. Since this mask is univariate, one can use our univariate construction from [RS2] to obtain the T_2 masks

$$\mathrm{T}_2 := \{y_j^{[m_1]}(\omega_1) : \ j = 1, \ldots, m_1\}.$$

As to ϕ_1, the box spline ϕ_1 is refinable with mask (up to an exponential factor) $\cos^{m_2}(\omega_2/2)$, and hence the same univariate construction can be repeated (as indeed we did). Otherwise, we may assume by induction that T_1, the wavelet masks associated with ϕ_1, are already given.

Note that the above discussion shows that the construction of §3 could have been made gradual: starting with four-direction box spline with multiplicities $(0, m_2, 0, m_2)$, we may have appended the four-direction box spline with multiplicities $(m_1, 0, m_1, 0)$ step by step: The number of wavelets will be then unchanged, $(m_1 + m_2)$ but a certain saving in the size of the larger support wavelets can be achieved in this way. □

Univariate C^2 cubic splines. In the final example of this section, we employ the general algorithm of this section in the construction of a univariate compactly supported tight frame generated by C^2 cubic splines. Prior to doing that, we recall that [RS2] already provides such a construction. Its construction yields four generators $(\psi_j)_{j=1}^4$, each supported in the interval $[-2, 2]$ whose Fourier transforms satisfy

$$\widehat{\psi}_j(2\omega) = \sqrt{\binom{4}{j}}\, \frac{\cos^{4-j}(\omega/2)\sin^{4+j}(\omega/2)}{(\omega/2)^4}.$$

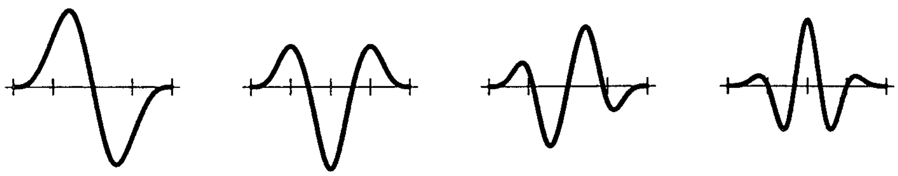

Figure 4.1. The four C^2 piecewise-cubic wavelets that were constructed in [RS2]. All of them are supported in $[-2,2]$.

The graphs of these three functions are given in Figure 4.1.

Here, we construct another C^2 piecewise-cubic tight frame, generated by 3 wavelets, albeit with larger support. In this construction, we choose ϕ_1 as the Haar wavelet, and ϕ_2 as the quadratic B-spline. Thus:

$$\widehat{\phi}_1(\omega) = \frac{1-e^{-i\omega}}{i\omega}, \quad \widehat{\phi}_2 = (\widehat{\phi}_1)^3,$$

and the corresponding masks are

$$\tau_1(\omega) = \frac{1+e^{-i\omega}}{2}, \quad \tau_2 = \tau_1^3.$$

We now choose T_1 to be the singleton

$$\mathrm{T}_1 := \{\frac{1-e^{-i\omega}}{2i}\}$$

(that corresponds to the Haar wavelet), and choose T_2 to consist of two masks:

$$\mathrm{T}_2 := \{\left(\frac{1-e^{-i\omega}}{2i}\right)^3, \frac{\sqrt{3}(1-e^{-2i\omega})}{4i}\}.$$

It is obvious that T_1 satisfies (b) of (4.1). It is less obvious, but still can be checked directly, that T_2 satisfies (c) of (4.1). Thus, by our algorithm, the following three mother wavelets generate a tight frame for $L_2(\mathbb{R})$:

$$\widehat{\psi}_1(\omega) = \frac{1-e^{-i\omega/2}}{2i}\widehat{\phi}(\omega/2),$$

$$\widehat{\psi}_2(\omega) = \frac{1+e^{-i\omega/2}}{2}\frac{\sqrt{3}(1-e^{-2i\omega})}{4i}\widehat{\phi}(\omega/2),$$

and

$$\widehat{\psi}_3(\omega) = \frac{1+e^{-i\omega/2}}{2}\left(\frac{1-e^{-i\omega}}{2i}\right)^3\widehat{\phi}(\omega/2),$$

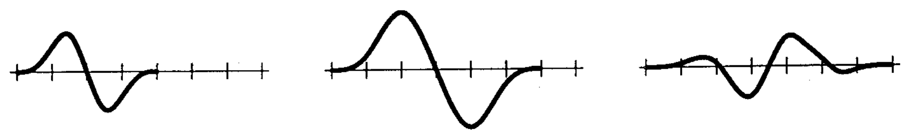

Figure 4.2. The three C^2 piecewise-cubic wavelets that were constructed here.

where

$$\widehat{\phi}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)\left(\frac{1-e^{-i2\omega}}{i2\omega}\right)^3.$$

The supports of these wavelets are $[0,4]$, $[0,6]$, $[0,7]$, respectively, and their graphs are shown in Figure 4.2. We note that ψ_2 and ψ_3 are splines with *integer* knots; i.e., in standard wavelet terminology, they belong to V_0.

Acknowledgments

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References

[BHR] C. de Boor, K. Höllig and S.D. Riemenschneider, Box splines, Springer Verlag, New York, (1993).

[CD] A. Cohen and I. Daubechies, Non-separable bidimensional wavelets bases, Rev. Math. Iberoamericana, Vol **9** (1993), 51–137.

[CS] C.K. Chui and X. Shi, Inequalities on matrix-dilated Littlewood-Paley functions and oversampled affine operators, CAT Report #337, Texas A&M University, 1994, SIAM J. Math. Anal., to appear.

[D1] I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, IEEE Trans. Inform. Theory **36** (1990), 961–1005.

[D2] I. Daubechies, Ten lectures on wavelets, CBMF conference series in applied mathematics, Vol **61**, SIAM, Philadelphia, 1992.

[D3] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure and Appl. Math. **41** (1988), 909-996.

[HW] C. Heil and D. Walnut, Continuous and discrete wavelet transforms, SIAM Review **31** (1989), 628–666.

[RS1] A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$: Canad. J. Math. **47** (1995), 1051–1094. Ftp site: `anonymous@ftp.cs.wisc.edu/Approx` file `frame1.ps`

[RS2] A. Ron and Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$, the analysis of the analysis operator CMS-TSR # 96–02 , University of Wisconsin - Madison, Madison WI 53706, 1995. Ftp site: `anonymous@ftp.cs.wisc.edu/Approx` file `affine.ps`